Chaotic Business Cycles within a Kaldor-Kalecki Framework

Giuseppe Orlando

Abstract This chapter, after providing some background on business cycles, Kaldor's original model and related literature, presents an original specification Orlando (Math Comput Simul 125:83–98, 2016) which adds to the cyclical behaviour some peculiar characteristics such as an asymmetric investment and consumption function, lagged investments and integration of economic shocks. A further section proves the chaotic behaviour of the model and adds some insights derived from recurrence quantification analysis. The final part draws some concluding remarks and makes some suggestions for future research. This work investigates chaotic behaviours within a Kaldor-Kalecki framework. This can be achieved by an original specification of the functions describing the investments and consumption as variants of the hyperbolic tangent function rather than the usual arctangent. Therefore fluctuations of economic systems (i.e. business cycles) can be explained by the shape of the investment and saving functions which, in turn, are determined by the behaviour of economic agents. In addition it is explained how the model can accommodate those cumulative effects mentioned by Kaldor which may have the effect of translating the saving and investment functions. This causes the so-called shocks which may be disruptive to the economy or that may have the effect of helping the system to recover from a crisis.

Keywords Numerical chaos · Applications to economics · Economic dynamics

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1 Introduction

The seminal work of Kaldor (1940) on the business cycle is one of the most fruitful for researching on non-linear dynamics in economics. In that paper the author's intention, contrary to the traditional Keynesian multiplier-accelerator concept, was to explain from a macroeconomic viewpoint the fundamental reasons for cyclical phenomena. After several years the Kaldor model re-gained the attention of some scholars interested in non-linear phenomena in economics such as Morishima (1959), Yasui (1953) and Ichimura (1955a, b) that were the first in investigating the existence, stability and uniqueness of limit cycles in a nonlinear trade cycle model. Among other notable contributors were Hicks (1950), Goodwin (1951) and particularly Kalecki (1966). The latter divided the investment process into three steps where the first is the decision, the second the time needed for the production and the last is the delivery of the capital good. In such a way the dynamic of capital stock in the economy is described by a non-linear difference-differential equation which exhibits a complex behaviour (including chaos) and, as a result, oscillations of capital induce fluctuations of other economic variables. Chiarella (1990) showed how the model could adjust to adaptive expectation of inflation. Krawiec and Szydlowski (1999, 2001) analyzing the Kaldor-Kalecki model of business cycle found a Hopf bifurcation leading to a limit cycle. Last but not least Pham et al. (2017) observed that the presence of time delay, such as those hypothesized by Kalecki, could induce unexpected oscillations, therefore time-delay systems may be suitable to introducing chaotic dynamics.

Further we applied recurrence plots (RPs) and their quantitative description provided by recurrence quantification analysis (RQA) to detect relevant changes in the dynamic regime of business time series. RQA aims at a direct and quantitative appreciation of the amount of deterministic structure of time series and has been proven to be an efficient and relatively simple tool in non-linear analysis of a wide class of signals. The technique allows for the identification of sudden phase-changes possibly pointing to underlying phenomena. Therefore RQA may be suitable for studying business cycles as well as identifying possible signals of changes in the economy.

The remainder of the chapter is organized as follows: Sect. 2 contains the definition of business cycle and summarizes the literature on recurrence quantification analysis and its applications to economics and finance. Section 3 presents some results on the applicability of RQA to economic time series. Section 4 illustrates the model along with its parameters and peculiarities. Section 5 shows the numerical analysis performed and the results obtained. Finally Sect. 6 consolidates the ideas and presents concluding remarks.

2 Literature Review

2.1 On the Business Cycles

A definition of business cycles can be found in Burns and Mitchell (1946)

> Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle.

and it has been used for:

1. Creation of composite leading, coincident, and lagging indices based on the consistent pattern of comovement among various variables over the business cycle (e.g. Shishkin 1961).
2. The identification within the business cycles of separate phases or regimes.

The National Bureau of Economic Research (NBER) defines a recession as "a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales. A recession begins just after the economy reaches a peak of activity and ends as the economy reaches its trough. Between trough and peak, the economy is in an expansion. Expansion is the normal state of the economy; most recessions are brief and they have been rare in recent decades." Figure 1 illustrates the business cycle where recession (trough) follows expansion (peak) and Fig. 2 shows the financial and business cycle in the United States.

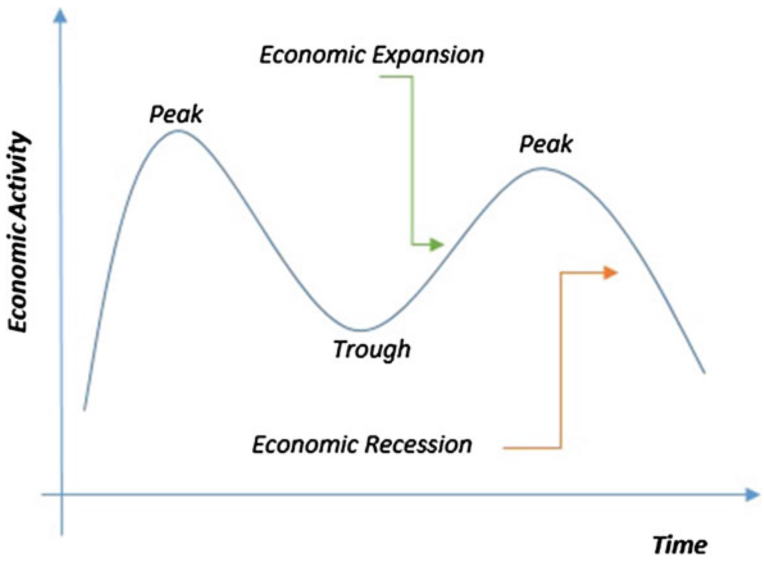


Fig. 1 The business cycle

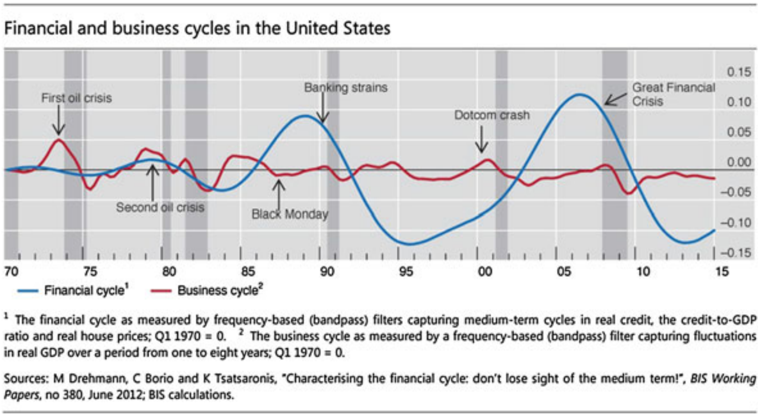


Fig. 2 BIS 85th annual report 2015

Schumpeter (1954) mentioned four stages linking together production, stock exchange, confidence, demand, interest rates and prices:

1. Expansion (increase in production and prices and low interest rates)
2. Crisis (stock exchanges crash and multiple bankruptcies of firms occur)
3. Recession (drops in prices and in production and high interest rates)
4. Recovery (stocks recover because of the fall in prices and incomes)

In addition he suggested that each business cycles has its own typology according to the periodicity so a number of cycles were named after their discoverers: see Korotayev and Sergey 2010: Kitchin (1923) or inventory cycle (3–5 years long), Juglar (1862) cycle (7–11 years long), Kuznets (1930) cycle (15–25 years long) and Kondratiev (1935) technological cycle (45–60 years long).

Theories on business cycles expound on the volatility of economies and may differ (for a review see Hillinger and Sebold-Bender 1992; Zarnowitz 1992; Mullineux 1984) depending on:

1. Their ability to explain cycle without having to rely on outside forces/shocks and they are called respectively endogenous and exogenous business cycle theories.
2. The assumption of a general equilibrium framework (neoclassical theories) or the assumption of market imperfections and/or disequilibrium (Keynesian theories).
3. Attributing cycles to real shocks or monetary shocks or too much or too little investment or consumption.
4. Explaining business cycles from actions of individuals (micro-founded theories) economic units or using aggregate variables (macro theories).

2.2 On the Recurrence Quantification Analysis

Recurrence Quantification Analysis (RQA) in economics started relatively recently see Zbilut (2005), Crowley and Schultz (2010), Karagianni and Kyrtsou (2011), Chen (2011), Moloney and Raghavendra (2012). Fabretti and Ausloos (2005) found cases where RQA could detect a warning before a crash. In accordance to that Addo et al. (2013) assert "the usefulness of recurrence plots in identifying, dating and explaining financial bubbles and crisis". Strozzi et al. (2007) claim that determinism and laminarity change "more clearly than standard deviation and then they provide an alternative measure of volatility". Finally, to quote Piskun and Piskun (2011), laminarity (LAM) "is the most suitable measure, sensitive to critical events on markets".

The ability of RQA to predict catastrophic changes stems from the fact that RQA is based upon the change in correlation structure of the observed phenomenon that is known to precede the actual event in many different systems, ranging from physiology Zimatore et al. (2011) and geophysics Zimatore et al. (2017) to economy Crowley (2008). Gorban et al. (2010) found out that even before crisis correlation increases as well as variance (and volatility) increases too. In particular their dataset composed of thirty largest companies from British stock market within the period 2006–2008 supports the hypothesis of increasing correlations during a crisis and, therefore, that correlation (or equivalently determinism) increases when the market goes down (respectively decreases when it recovers).

3 Recurrence Plot

As mentioned by Eroglu et al. (2014) "recurrence plots (RPs) have been shown to be a powerful technique to uncover statistically many characteristic properties" of

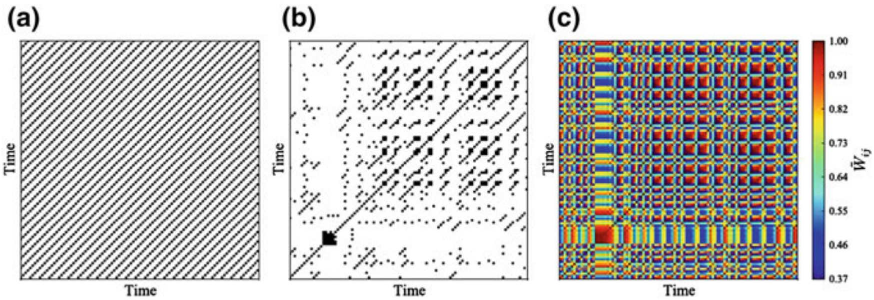


Fig. 3 (Color online) Recurrence plot of logistic map for **a** r = 3.5 (periodic regime) and **b** r = 4.0 (chaotic regime) and **c** weighted recurrence plot of logistic map for r = 4.0 (chaotic regime). *Source* Eroglu et al. (2014)

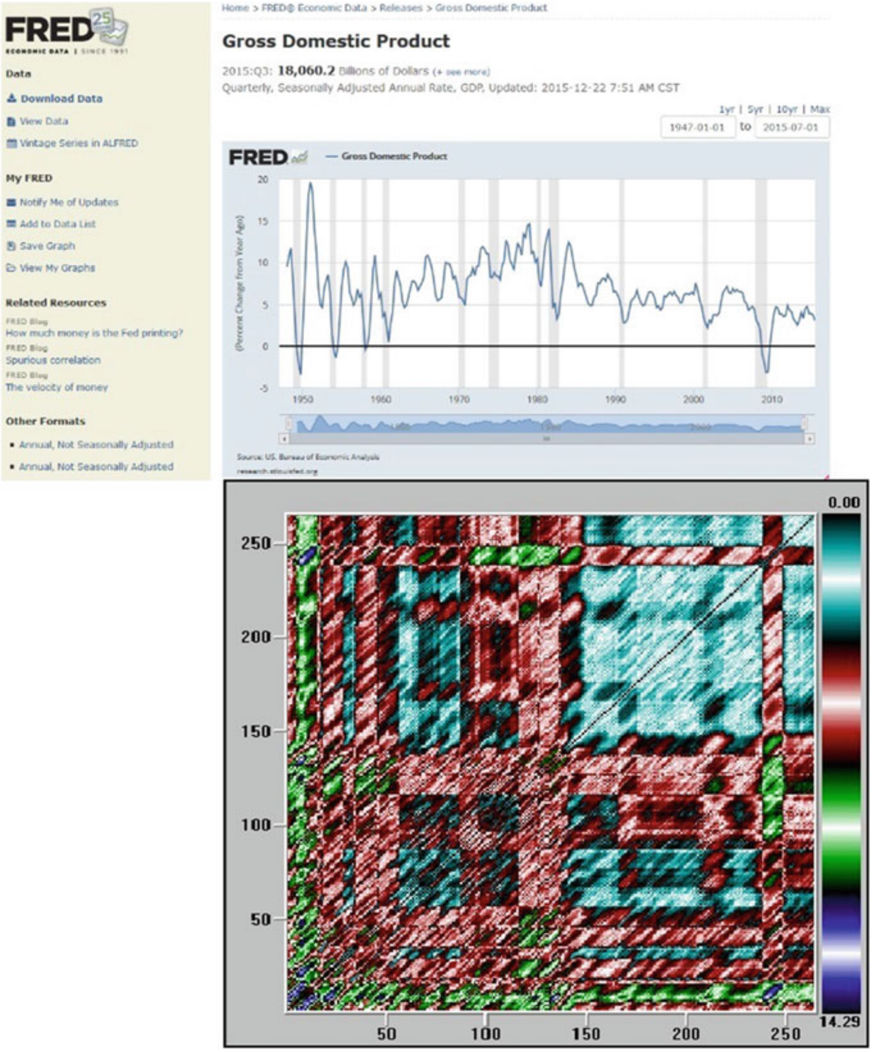


Fig. 4 Changes in US GDP (above) and its unthreshold RP or Distance Matrix (DM) (below). Period: 01-01-1947–01-07-2015. Compare the RP of the logistic Fig. 3c with the RP of business cycles. *Source* St. Louis Fed, FRED database

complex dynamical systems. "In a given m-dimensional phase space, two points are considered to be recurrent if their state vectors lie in a neighbourhood characterized by a threshold ε". Therefore in a RP, "elements $R_{i,j} \equiv 1$ (recurrence) are usually said to be black dots, whereas $R_{i,j} \equiv 0$ (no recurrence) are usually called white dots" (see Fig. 3a and b).

In order to have some indications whether business cycles are chaotic and to study recessions from the point of view of a phase transition of non-linear phenomena we applied RQA on time series extracted from Federal Reserve Economic Data

(FRED)—St. Louis Fed. In Fig. 4 the recurrence plot of USA GDP% variation is shown right below the FRED graph. Greyed areas correspond to periods of economic recessions as reckoned by FRED. For the unthreshold RP or Distance Matrix (DM) it is possible to observe the anticipating transitions to turbulent phases. The noteworthy results consist in a correspondence between vertical lines in DM (i.e. chaos to chaos transitions) and grey lines (recession periods) as well as in the apparent resemblance between the RP of the chaotic logistic Fig. 3c with the RP of business cycles Fig. 4.

4 The Model

The discretized Kaldor model is

$$\begin{aligned} Y_{t+1} - Y_t &= \alpha(I_t - S_t) = \alpha[I_t - (Y_t - C_t)] \\ K_{t+1} - K_t &= I_t - \delta K_t \end{aligned} \tag{1}$$

where Y, I, S, K define respectively income, investment, saving and capital, α is the "speed" by which the output responds to excess investment and δ represents the depreciation rate of capital. It is worth mentioning that a key feature for Kaldor is that $I = I(Y, K)$ and $S = S(Y, K)$ are non-linear functions of income and capital.

The author's original variant is (for a detailed description see Orlando 2016)

$$\begin{aligned} Y_{t+1} - Y_t &= \alpha \left[f_1 \left(g \left(\frac{Y_{t-1}-Y_{t-2}}{Y_{t-2}} - \frac{K_{t-1}-K_{t-2}}{K_{t-2}} \right) \right) + f_2 \left(g \left(\frac{Y_t-Y_{t-1}}{Y_{t-1}} - \frac{C_t-C_{t-1}}{C_{t-1}} \right) \right) - Y_t \right] \\ K_{t+1} - K_t &= f_1 \left(g \left(\frac{Y_{t-1}-Y_{t-2}}{Y_{t-2}} - \frac{K_{t-1}-K_{t-2}}{K_{t-2}} \right) \right) - \delta K_t \end{aligned} \tag{2}$$

Where the parameters $\alpha, \delta, \tau_1, \tau_2, \rho, \hat{c}, k$ have the following meaning:

1. α is the savings adjustment speed with regard to investment. Its reciprocal in physics is called delay and measures the time necessary for the adjustment.
2. δ is a percentage that determines the fixed capital which is lost during the productive process (due to obsolescence or actual consumption).
3. τ determines the f function knee; it is, therefore, a measure of the reactivity of the function to the variation in its argument.
4. ρ measures the maximum possible level (in capital terms) of investment. This value changes according to the economic system (pre-industrial, industrial, post-industrial) and the type of investment (i.e. high or low capital intensity).
5. $\hat{c} = 1 - c$ multiplied by actual income, determines the minimum level of consumption, therefore it is also called the base level. Its complement to 1, is c and represents the average level of consumption.
6. The k parameter changes according to the economic development.

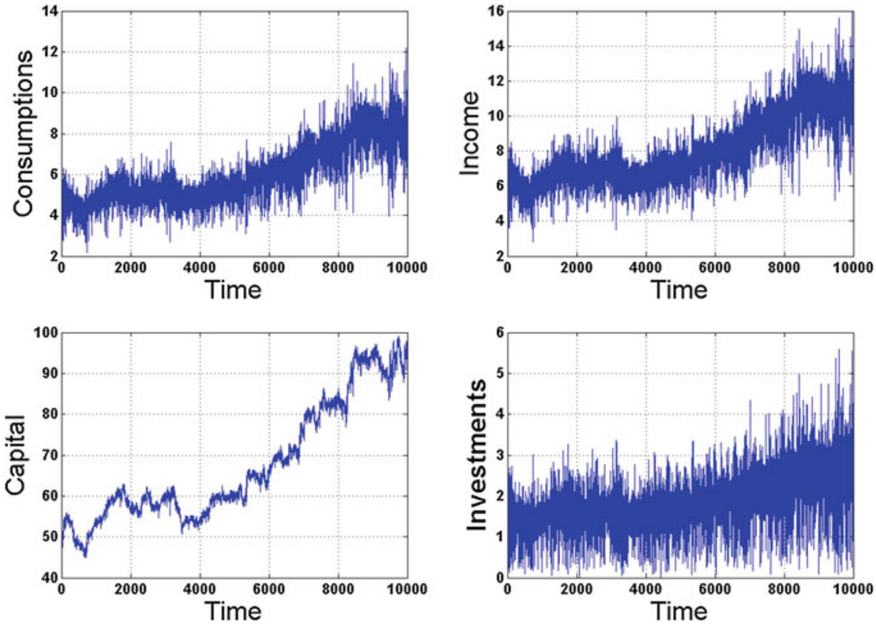


Fig. 5 A simulation displaying a steady growth

In Figs. 5 and 6 are two examples of the system's dynamic which differ only because a different initial starting point.

4.1 Shocks in the Economy

Kaldor explained that there are some factors which affect consumption and saving and which have the effect of shifting the functions in one direction or the other. In the model this translation can be easily achieved and operates when the capital or the income changes negatively (with the ultimate effect to help the system recover from a crisis).

4.2 Consumption, Saving and Economic Recessions

The idea that an increase in the disposable income (which is the basis of fiscal stimulus such as tax rebates that are supposed to encourage consumption, and hence aggregate demand), automatically translates into an increase in the aggregate demand, can be fallacious as it neglects to consider the state of health of the economy and

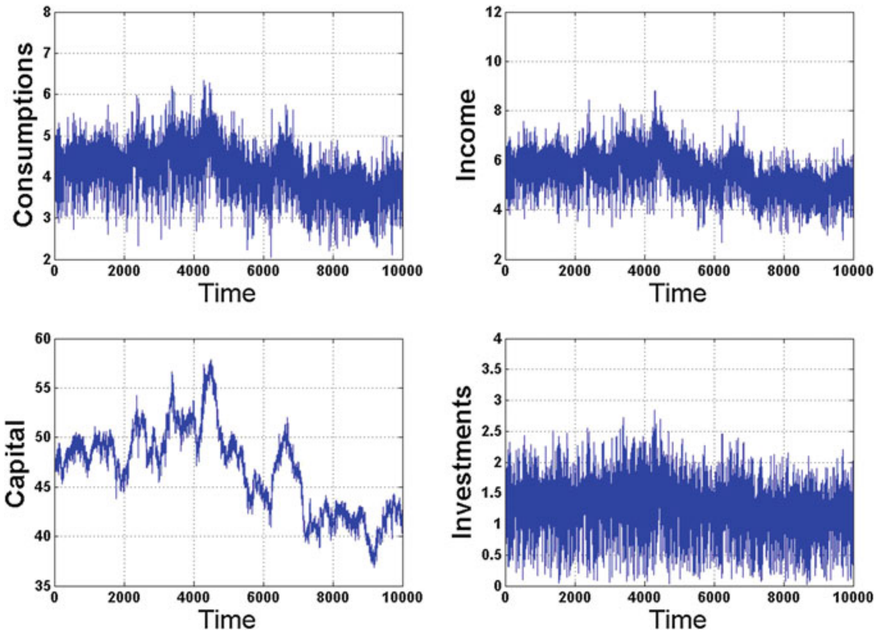


Fig. 6 A simulation displaying a steady fall

therefore the confidence in it. In fact, if confidence in the economy is low, it could be that people may reduce their consumption during the recession years: consumers will continue the process of deleveraging (they use the money to pay off debt and save more) because of uncertainty in the future.

For the above mentioned reasons, in the model, the change in consumption is linked to the change of income as follows

$$w = \frac{\triangle Y}{Y} - \frac{\triangle C}{C} \tag{3}$$

which we believe describes correctly the behaviour of consumption (Figs. 7 and 8).

4.3 Modelling Investment and Consumption

In the usual set-up, modified versions of Kaldor's approach adopt the trigonometric investment function arctg (see Mircea et al. 1963; Kaddar and Alaoui 2009; Agliari et al. 2007; Januario et al. 2005, 2009; Bischi et al. 2001 etc.). We have decided, instead, to use a different functional form: the hyperbolic tangent. The reasons why this specific functional form has been chosen are that there is no particular

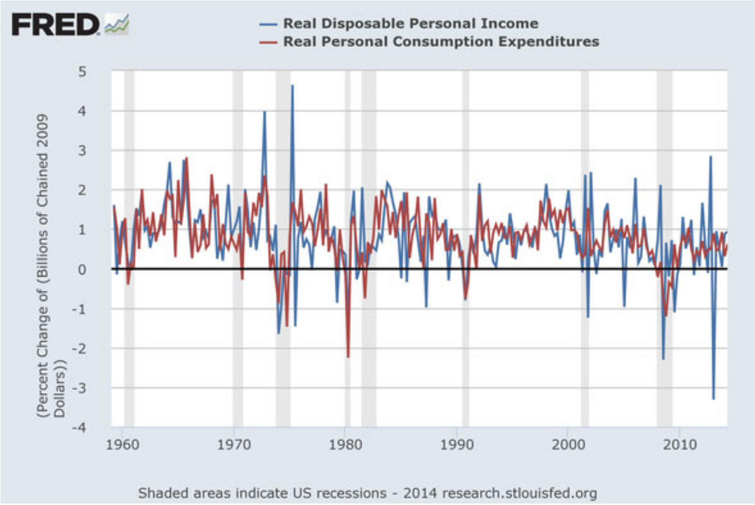


Fig. 7 Changes in US real disposable personal income (blue—DSPIC96) and real personal consumption expenditures (red—PCECC96) 1959 (Q1)—2014 (Q2). *Source* St. Louis Fed, FRED database

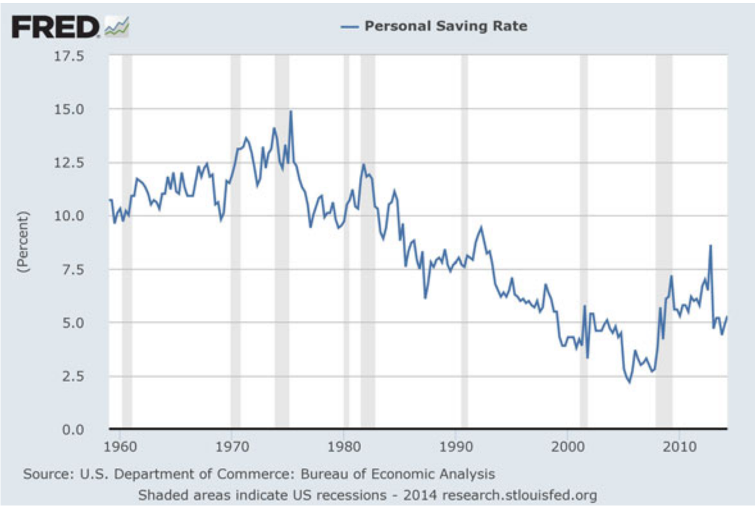


Fig. 8 US personal savings rate (PSAVERT) 1959 (Q1)—2014 (Q2). *Source* St. Louis Fed. FRED Database

justification to prefer the arctg,[1] whilst there are several to prefer the tanh. A good reason, for example, is that across multiple sciences growth and decay are better modelled by such exponential-like functions as tanh instead of trigonometric function such as arctg. For some applications to physics (such as radioactive decay, capacitor discharge, damped oscillations, etc.), to chemistry and biology (such as first order reaction rates and population and cancer growth), to actuarial science and finance (e.g. Gompertz-Makeham law of mortality, compound interest, etc.) Stewart (2010), Benson (2008), Reger et al. (2010), Purves et al. (2003), Wheldon (1988), etc. Moreover the arctg tends to its asymptotes quite slowly compared to the hyperbolic tangent whilst we wanted to design a framework in which economic agents can adjust quickly (how quickly depends on some parameters as τ) to changes. Given this framework the suggested model could link up, for example, to the classic Solow-Swan growth model in which labour and knowledge are represented by exponential functions.

Finally it should be noted that while consumption and investment are similarly function of a difference (i.e., respectively, between the growth rates of income and capital and the growth rates of income and consumption) their timing differs. In fact, *à la* Kalecki, we suggest that the investment process has different timing than does consumption hence the difference in the considered time lags (see Eq. 2).

5 Numerical Analysis

Up to now the sensitive dependence on initial conditions and the irregular trend of variables over time has only been shown graphically. Naturally the experimental evidence is not sufficient to prove the chaoticity of a system. Therefore we must use some numeric instruments in order to have a better insight into the nature of the system. Specifically, we will report the results obtained by the spectral analysis as well as the calculation of the correlation integral, the Lyapunov exponents, the Kolmogorov entropy and the embedding dimension.

5.1 Spectral Analysis

Spectral analysis has the aim of determining the spectral content of a time series by decomposing a given time series into different harmonic series with different frequencies and, by doing that, identifies the contribution of each series to the overall signal Stoica and Moses (2005).

Spectral analysis may aid in identifying chaos for a given time series as well as helping in discovering hidden periodicities in data. In the following three pictures of

[1]On some issues related to the arctg see for example Bradford and Davenport (2002), Collicott (2012), Walter (2010), Gonnet and Scholl (2009).

logistic map are presented in order to illustrate how the power spectra change with the parameter μ. The first box (top left) shows the cobweb diagram, the second (bottom left) is for the orbits, the third (top right) depicts the power spectrum obtained with a rectangular (sometimes called boxcar or Dirichlet) window and the last box shows the power spectrum with a Hamming window. Figures 9, 10 and 11 illustrate, respectively, very regular orbits with a single frequency peak, regular orbits with two frequency peaks and irregular orbits (chaos) with several frequency peaks.

Even though the technique is not conclusive if the system has "many hidden degrees of freedom of which the observer is unaware" (Moon 1987 p. 45) "chaotic time series are known to have aperiodic cycles of many lengths, so it seems reasonable to assume that if they are present in the candidate time series they should have been observed" (McBurnett 1996 p. 50). In any case, as the proposed model is by construction deterministic, the spectral analysis can definitively help in understanding whether the system shows chaotic dynamics.

Following this reasoning we have run a simulation in order to show that for the generated time series there is no peak that clearly dominates all other peaks (power spectrum for C, K, I, Y with rectangular and Hamming windows are shown in Figs. 12 and 13, respectively).

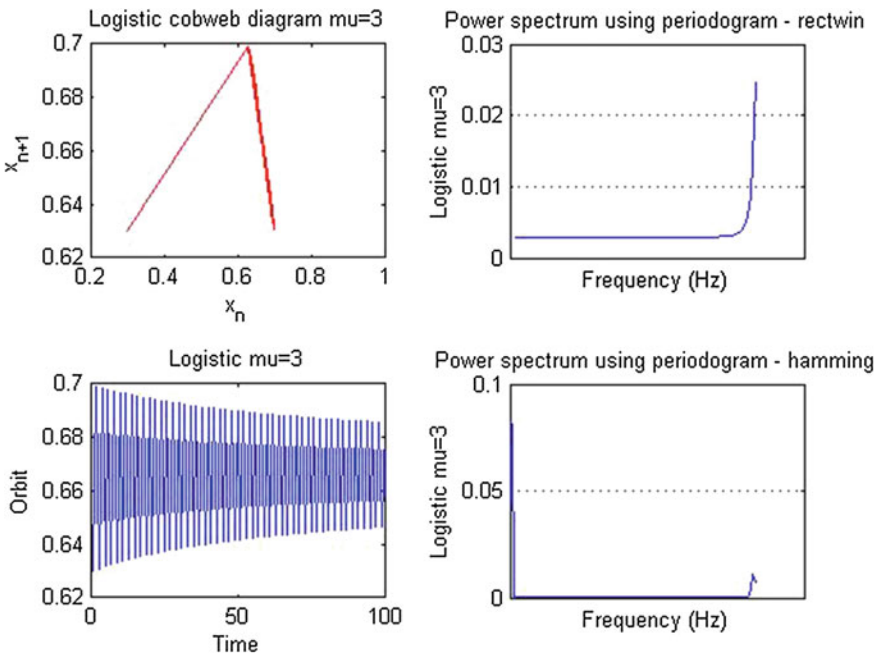


Fig. 9 Logistic map, $\mu = 3$ cobweb diagram and periodogram

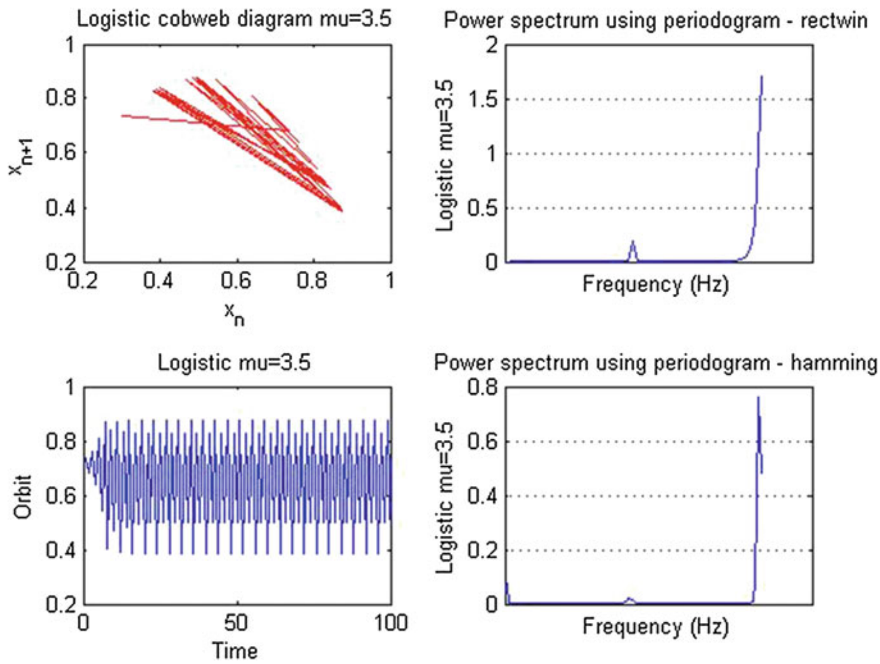


Fig. 10 Logistic map, $\mu = 3.5$ cobweb diagram and periodogram

5.2 *Embedding Dimension*

"The embedding dimension is the smallest dimension required to embed an object (a chaotic attractor for instance). In other words, this is the minimum dimension of the space in which you reconstruct a phase portrait starting from your measurements and in which the trajectory does not cross itself, that is, in which the determinism is verified. Of course, this is a statistical measure, meaning that you may have some "rare" self-crossings. When a global model is attempted, this is the minimum dimension your model must have" Letellier (2013).

Cao (1997) has suggested an algorithm based on the work of Kennel et al. (1992) for estimating the embedding dimension (see Takens 1981; Adachi 1993; Whitney 1992) through E1(d) and E2(d) functions, where d denotes the dimension.[2] The function E1(d) stops changing when d is greater than or equal to the embedding dimension staying close to 1. The function E2(d), instead, is used to distinguish deterministic from stochastic signals. If the signal is deterministic, there exist some d such

[2]Which has the following advantages: (a) does not require any subjective parameters except for the time-delay for the embedding; (b) does not strongly depend on the number of data points; (c) is able to distinguish between deterministic and stochastic signals; (d) it is computationally efficient and works well in presence of high-dimensional attractors.

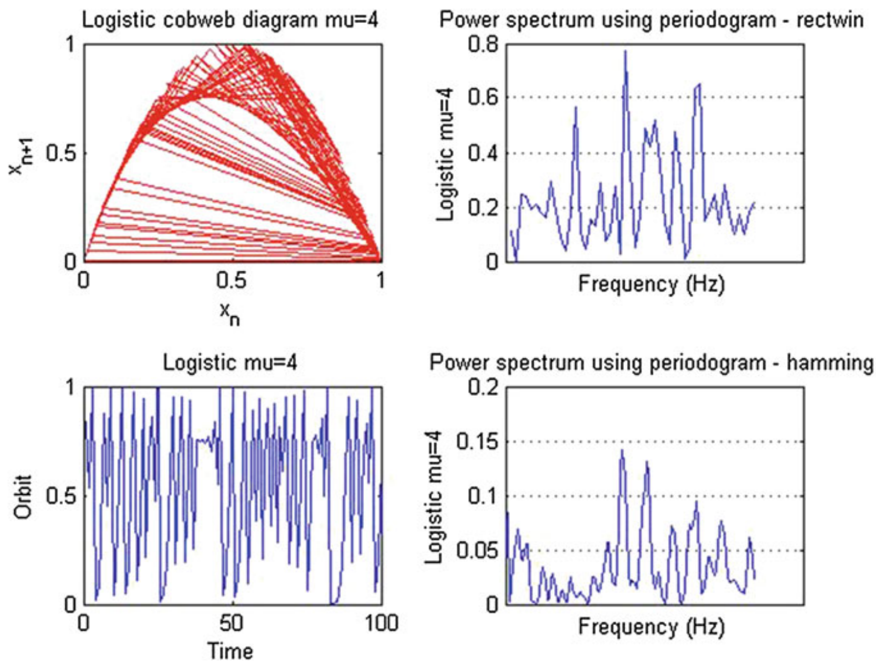


Fig. 11 Logistic map, $\mu = 4$ cobweb diagram and periodogram

that E2(d)! = 1 whilst if the signal is stochastic E2(d) is approximately 1 for all the values (see also Arya 1993; Arya et al. 1998).

For example, in Figs. 14 and 15, analogously to Cao (2002) we report the embedding dimension for the FX British Pound/US Dollar showing the values of E1 and E2 for a time series of 1,008 data points. In "looking at the results of the quantity E2 whose values are very close to 1 with some oscillations when the dimensions are small, this implies that the time series is likely a random time series, comparing with the case of random colored noise shown in Cao (1997). Given the oscillation behaviour away from 1 when the embedding dimensions are small, the time series should contain some determinism although the determinism may be weak".[3]

By applying the same analysis to our model, it is possible to observe a similar behaviour in the following figures where the E1 and E2 are calculated on consumption (Fig. 16), income (Fig. 17), capital (Fig. 18) and investment (Fig. 19). In fact we can observe that E2, on the four macroeconomic variables that by construction are not deterministic, is not 1 for all values but approaches it for $d \geq 10$.

[3]By contrast in the proposed model determinism is ensured by construction.

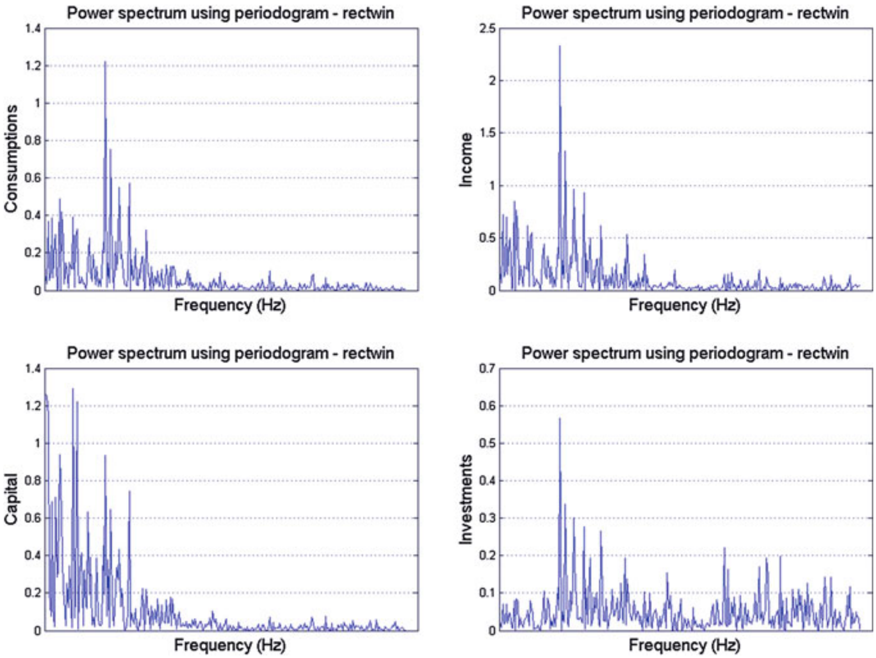


Fig. 12 Power spectrum with rectangular window

5.3 Correlation Integral

This kind of correlation is called spatial correlation and it aims to measure the degree of "relationship" between the different points on the strange attractor. The function which performs this task is called *correlation integral* $C(r, m)$ (see Lorenz 1993) which represents a direct arithmetic average of the pointwise mass function Theiler (1990) and it is defined as

$$C(r, m) = \lim_{m \to \infty} \lim_{r \to 0} \frac{1}{\vartheta} ln \frac{C(r, m)}{C(r, m+1)} \tag{4}$$

with m embedding dimension and r radius (i.e. the space contraction or stretching).

In the following Figs. 20 and 21 we can observe that its trend (or its logarithm) is a function of r the radius and when it depicts a regular growth it confirms that the system is deterministic.

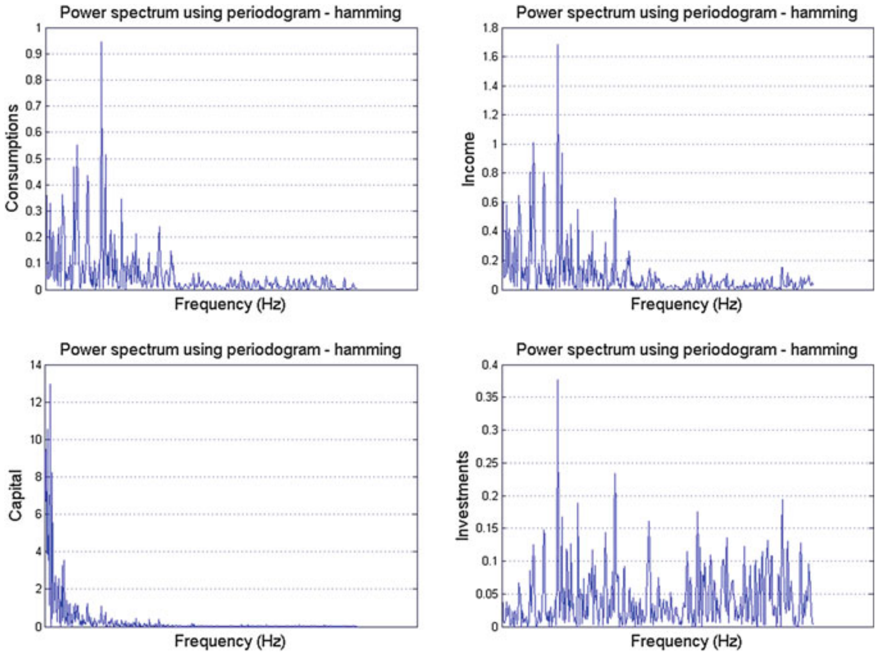


Fig. 13 Power spectrum with Hamming window

Fig. 14 Embedding dimension for USD/EUR FX rates (tao = 1, data points = 260)

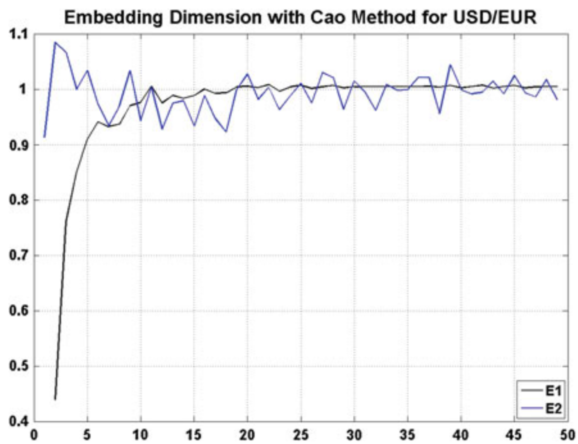


Fig. 15 Embedding dimension for USD/GBP FX rates (tao = 1, data points = 260)

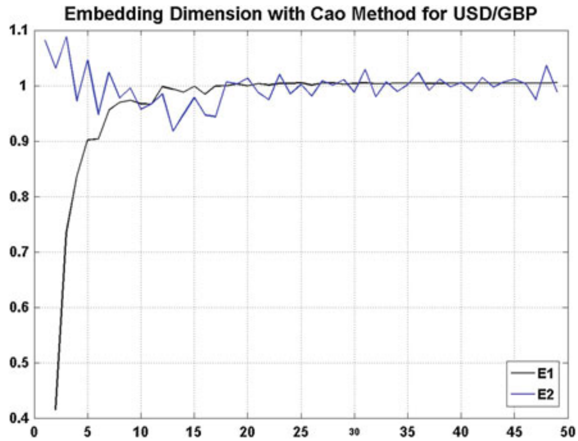


Fig. 16 Embedding dimension for consumption (tao = 1, data points = 10,000)

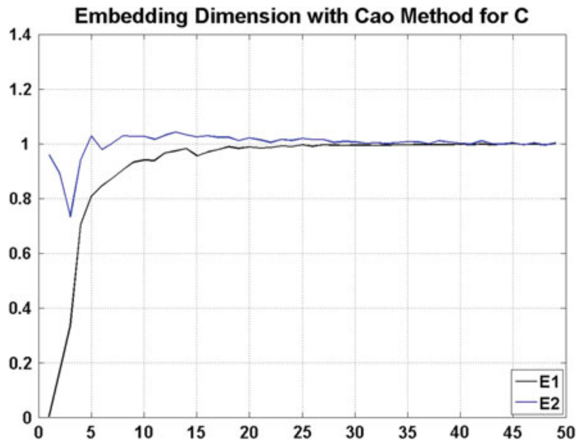
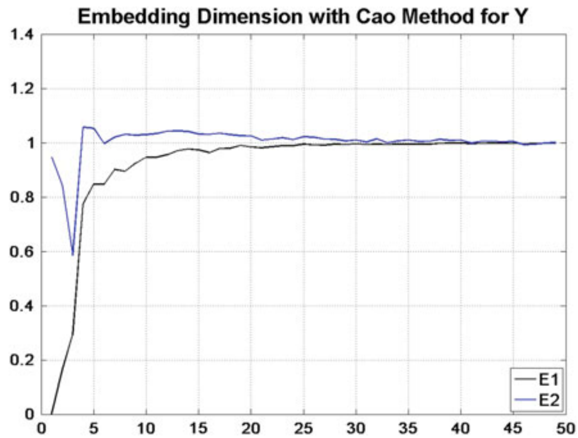


Fig. 17 Embedding dimension for income (tao = 1, data points = 10,000)



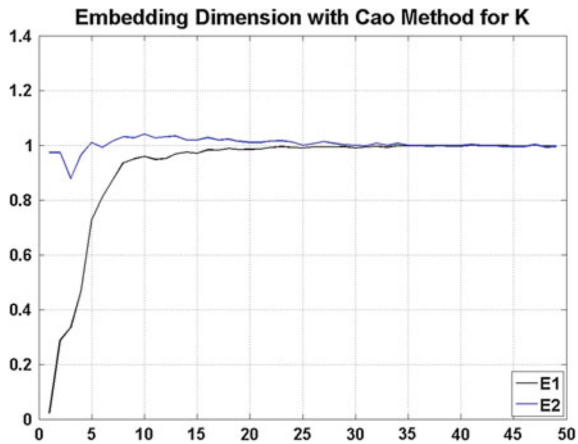


Fig. 18 Embedding dimension for capital (tao = 1, data points = 10,000)

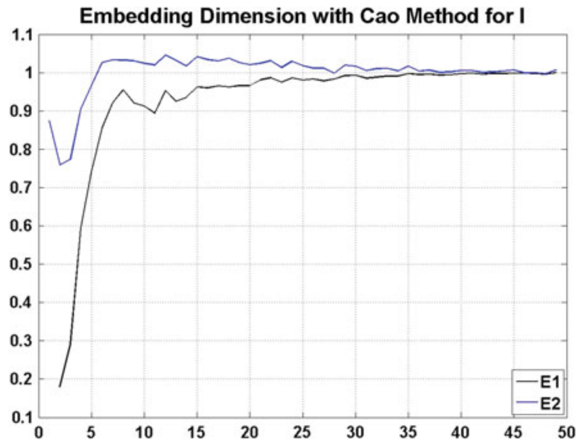


Fig. 19 Embedding dimension for investments (tao = 1, data points = 10,000)

5.4 Correlation Dimension

Another useful notion is the correlation dimension defined as

$$D^C(m) = \lim_{r \to 0} \frac{ln(C(r, m))}{ln(r)} \tag{5}$$

hence

$$D^C(m)ln(r) \approx ln(C(r, m)) \text{ i.e. } r^{D^C(m)} \approx C(r, m). \tag{6}$$

The correlation dimension is intended to measure the information content "where the limit of small size is taken to ensure invariance over smooth coordinate changes. This small-size limit also implies that dimension is a local quantity and that any global definition of fractal dimension will require some kind of averaging" Theiler

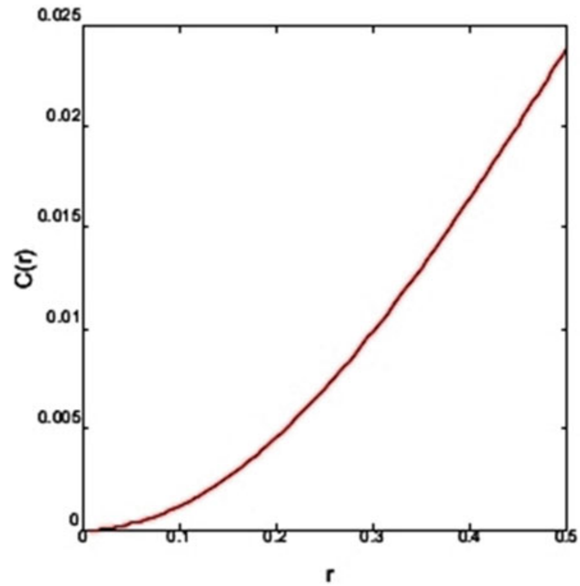


Fig. 20 Correlation integral trend versus *r*

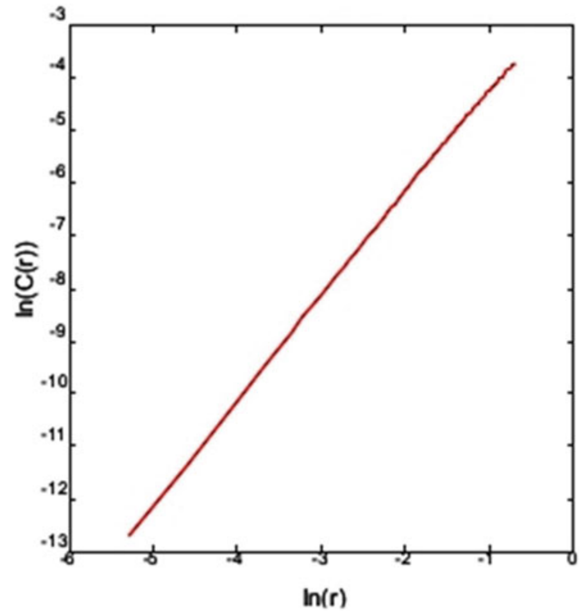


Fig. 21 Log-log plot

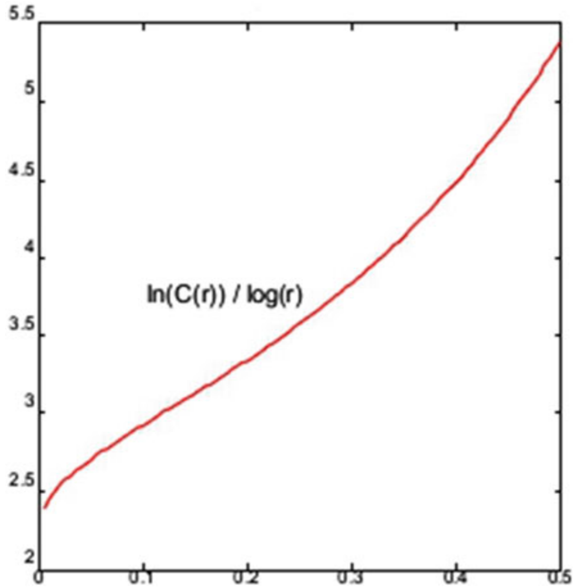


Fig. 22 Correlation dimension when $r \to 0$

(1990). Moreover it is shown that the correlation dimension D^C is an approximation of Hausdorff's dimension D^H with $D^C \leq D^H$. In Fig. 22 it can be seen that the dimension of correlation is noninteger. As D^C is a "more relevant measure of the attractor than D^H because it is sensitive to the *dynamical* process of the coverage of the attractor" (Grassberger and Procaccia 1983c p. 348), we can say that the system is fractal (see Grassberger and Procaccia 1983a; Grassberger 1986).

5.5 Lyapunov Exponents

"Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in the phase space. Since nearby orbits correspond to nearly identical states, exponential orbital divergence means that systems whose initial differences that may not be possible to resolve will soon behave quite differently, i.e. predictive ability is rapidly lost" (Sivakumar and Berndtsson 2010 p. 424). Dynamical systems have a spectrum of Lyapunov exponents, one for each dimension of the phase space and, similarly to the largest eigenvalue of a matrix, the largest Lyapunov exponent determines the dominant behaviour of a system. The sensible dependence on initial conditions, hence, can be restated as follows:

$$\|\delta x(t)\| \approx e^{\lambda t}\|\delta x_0\| \tag{7}$$

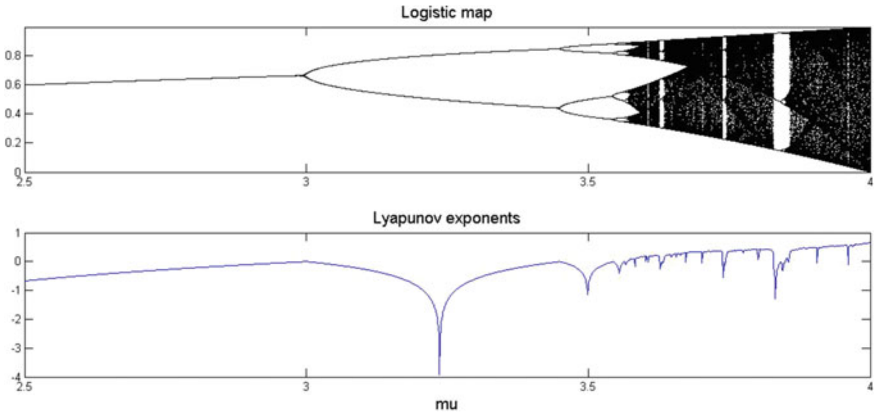


Fig. 23 Logistic map with Lyapunov exponents

Table 1 Lyapunov exponents

	Min	Max	Mean
Consumption	6.22	11.399	10.885
Income	12.8338	19.6440	13.3534
Capital	7.3165	14.594	12.999
Investment	5.511	11.969	11.049

Calculated Lyapunov's exponents for 10,000 points time series of *C, Y, K* and *I*

"where λ, the mean rate of separation of trajectories of the system, is called the leading Lyapunov exponent. In the limit of infinite time the Lyapunov exponent is a global measure of the rate at which nearby trajectories diverge, averaged over the strange attractor" Cvitanovic et al. (2012).

Figure 23 shows the Lyapunov exponents of the Logistic Map for comparison (see Schuster 1988).

Lyapunov exponents are used to measure the rate at which nearby trajectories of a dynamical system diverge (see for example Sivakumar and Berndtsson 2010; Cvitanovic et al. 2012). A dynamic dissipative system is chaotic if its biggest Lyapunov exponent is a positive number (see Lorenz 1993). In our simulations, by adopting the Wolf algorithm (see Wolf et al. 1985; Wolf 1986), we have found out that the biggest Lyapunov exponent has always been positive (see Table 1).

5.6 Entropy

To supplement the above mentioned analysis it could be useful to refer to the entropy *K* of Kolmogorov-Sinai (see Farmer 1982; Schuster 1988). We know it must converge to a positive finite value in order for the time series to be defined as chaotic because "Kolmogorov entropy is the mean rate of information created by the system. It is important in characterizing the average predictability of a system of which it represents the sum of the positive Lyapunov exponents. The Kolmogorov entropy quantifies the average amount of new information on the system dynamics brought by the measurement of a new value of the time series. In this sense, it measures the rate of information produced by the system, being zero for periodic or quasiperiodic (i.e. completely predictable) time series, and infinite for white noise (i.e. unpredictable by definition), and between the two for chaotic system" (Sivakumar and Berndtsson 2010 p. 424). In fact, according to the Pesin's theorem, Pesin (1977), the sum of all the positive Lyapunov exponents gives an estimate of the Kolmogorov-Sinai entropy. So, if $K > 0$, then the biggest Lyapunov exponent is bigger than zero and the system is chaotic. In order to measure K we used the approximation K_2 as defined by Grassberger and Procaccia (1983b) and we found that it was positive (e.g. 21.34561).

5.7 Symplectic Geometry Method

In addition to the embedding dimension, the symplectic geometry method (see Lei et al. 2002; Xie et al. 2005; Lei and Meng 2011) is used as a consistency check to verify the appropriate embedding dimension from a scalar time series. Symplectic similarity transformation is nonlinear and has measure-preserving properties i.e. time series remain unchanged when performing symplectic similarity transformation. For this reason symplectic geometry spectra (SGS) is preferred to singular value decomposition (SVD) (which is by nature a linear method that can bring distorted and misleading results Palus and Dvorak 1992).

Figures 24 and 25 show respectively the embedding dimension of the Logistic (which is deterministic) and the Gauss white noise and they can be be used as reference for comparing the embedding dimension obtained for consumption (Fig. 26), income (Fig. 27), capital (Fig. 28) and investment (Fig. 29) depicted.

5.8 Correlation Integral and Embedding Dimension

As we have repeatedly shown the system behaves stochastically but by construction is deterministic. In Table 2 we report the correlation integral versus the embedding dimension for our variables. As it can be observed the correlation integral does not increase with the embedding dimension confirming the validity of this analysis and that the system is deterministic (see Lorenz 1993 p. 213).

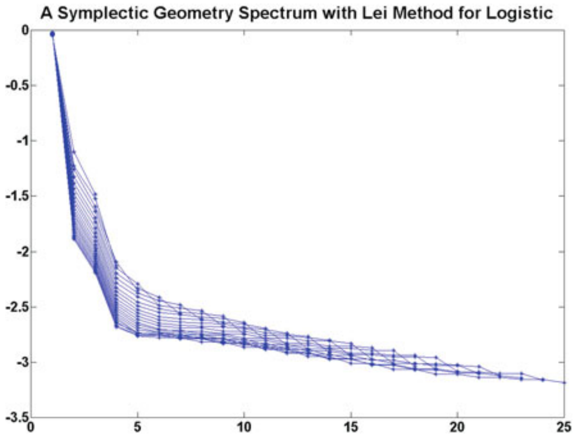


Fig. 24 Embedding dimension symplectic geometry method for logistic, $\mu = 3.9$ (data points = 1,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

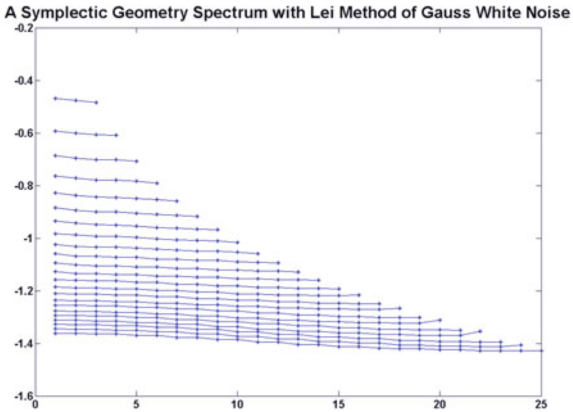


Fig. 25 Embedding dimension symplectic geometry method for consumption (data points = 10,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

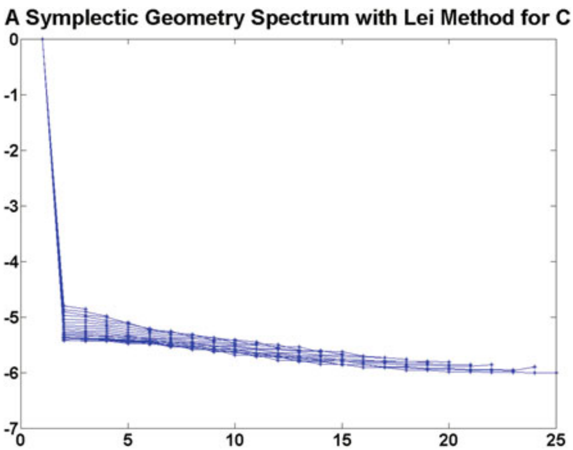


Fig. 26 Embedding dimension symplectic geometry method for capital (data points = 10,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

Fig. 27 Embedding dimension symplectic geometry method for income (data points = 10,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

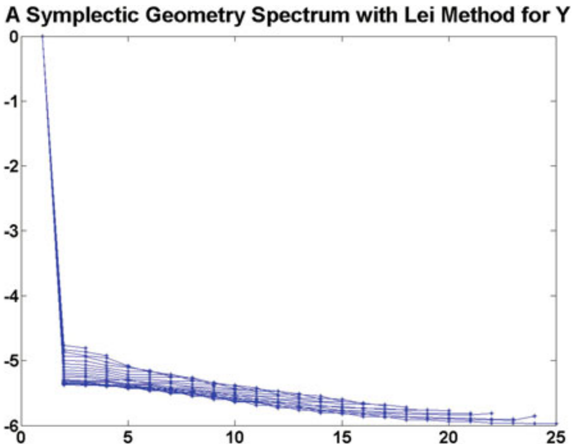


Fig. 28 Embedding dimension symplectic geometry method for capital (data points = 10,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

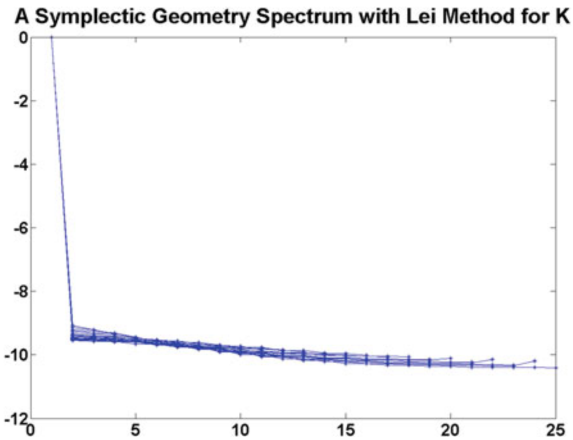


Fig. 29 Embedding dimension symplectic geometry method for investments (data points = 10,000, abscissa is d, ordinate is $log(\frac{\sigma_i}{tr(\sigma_i)})$)

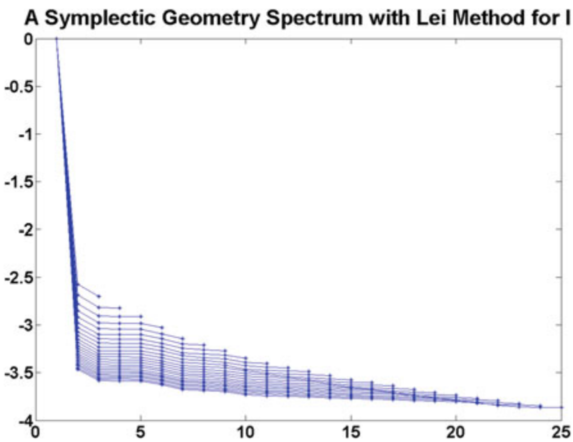


Table 2 Correlation integral versus embedding dimension

Embedding dimension								
Correlation integral for		2	3	4	5	6	7	8
Consumptions	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Income	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Capital	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Investments	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797

Correlation integral versus embedding dimension for 10,000 points time series of C, Y, K and I. As shown the correlation integral is quite stable for $m = 2, \ldots, 8$.

6 Conclusions

We share the view that "economists will be led, as natural scientists have been led, to seek in nonlinearities an explanation of the maintenance of oscillation" Goodwin (1951). In this paper, with the help of RP, we have first shown that business cycles may be chaotic in nature and then we have proposed a non-linear model has a chaotic behaviour. But while the latter result has been achieved by many, we suggested a different functional form (i.e. the hyperbolic tangent) instead of the usual arctangent. Moreover the originality of this work lies in the specification of consumption and investment as a function of the difference, respectively, between the growth rates of income and capital and the growth rates of income and consumption. This has been obtained by considering, *à la* Kalecki, that the investment process has different timing than does consumption, hence the difference in the considered time lags. Last but not least the model can accommodate such external perturbations as shocks by a translation of the argument of the function f.

In future we will research the calibration of the model to real economy as well as study further its features with the help of RQA (which, possibly, may result to identifying some leading indicators of economic crashes). Moreover here we stress that such non-linear behaviour has some implications which could be potentially considered as being of some interest, e.g. what should be the rules set by a regulator (such as the central bank) within a chaotic framework?

For example if the system is not predictable, not reachable and then not observable but nevertheless controllable (see Romieras et al. 1992; Grebogi and Laib 1997; Calvo and Cartwright 1998; Pettini 2005), can one set up a system of controls that is able to drive the economy?

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