# Backstepping Control for Combined Function Projective Synchronization Among Fractional Order Chaotic Systems with Uncertainties and External Disturbances

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**Abstract** In the present chapter the combined function projective synchronization among fractional order chaotic systems in the presence of uncertain parameters and external disturbances using backstepping control method is investigated. The chaotic attractors of the systems are found for fractional-order time derivative, which is described in Caputo sense. A new lemma of Caputo derivatives is used to design the controller based on Lyapunov stability theory. During the combined function projective synchronization among the non-identical fractional order systems, the Lorenz, Rossler and Chen systems are taken to illustrate the effectiveness of the considered method. Numerical simulation and graphical results for different particular cases clearly exhibit that the method with this new procedure is easy to implement and reliable for synchronization of non-identical fractional order chaotic systems.

**Keywords** Fractional order chaotic system · Backstepping method
Lyapunov stability theory · Combined function projective synchronization

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## 1 Introduction

In last few decades, much attention has been devoted to the study of the fractional calculus and their numerous applications in the area of mathematics, physics and engineering. Fractional differential equations which are generalizations of classical differential equations describe the memory effect, which is the major advantage over integer-order derivatives. It has been extensively applied for modelling of many real problems such in viscoelasticity (Koeller 1984) dielectric polarization (Sun et al. 1984), electromagnetic waves (Heaviside 1971), quantitative finance (Laskin 2000), quantum evolution of complex system (Kunsezov et al. 1999), chaos control of dynamical systems (Chen and Yu 2003; Azar and Vaidyanathan 2015) and the control of fractional order dynamic systems (Hartley and Lorenzo 2002) etc.

It is evident from literature survey that during last few decades the nonlinear phenomena occurring in various areas of scientific fields have gained immense popularity amongst the scientists and engineers who have delivered tireless efforts towards the development of the models using non-linear differential equations. Introduction of fractional calculus in nonlinear models had given a new dimension to the existing problems. The interesting phenomena of nonlinear dynamics are the possibility of chaos. Most of the nonlinear systems reveal chaotic behaviour which is deterministic and has a periodic long-term behaviour, and also exhibit sensitive dependence on initial conditions. A periodic long-term behaviour means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time approaches to infinity. Deterministic means that the system has no random or noisy inputs. This irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces. Sensitivity means that a small change in the initial state will lead to progressively larger changes in later system. Hence, an arbitrarily small perturbation of the current trajectory may lead to different future behaviour. The concept of chaos has been used to explain how systems subject to known laws of physics may be predictable in the short term but are apparently random on a longer time scale.

The nonlinear chaotic dynamic system of fractional order has taken care by mathematical and physical communities in the last few years. The chaotic dynamics of fractional order systems are important topics, which are mainly devoted to the chaos synchronization problem in nonlinear dynamical systems. Synchronization of chaos refers to a process wherein two or more identical or non-identical chaotic systems have a common behaviour due to a coupling, which appears to be structurally stable. In other words, synchronization, an important achievement in the research of chaos, means that the trajectories of two systems will converge and they will remain in step with each other. Pecora and Carroll (1990), first introduced a method about synchronization between the drive (master) and response (slave) systems of two identical or non identical systems with different initial conditions, which has important applications in ecological system, physical system, chemical system, modelling brain activity, system identification, pattern recognition phenomena and secure communications etc. Different types of synchronization schemes had already been handled by various researchers for the synchronization of chaotic systems, such as complete synchronization, anti-synchronization lag synchronization, hybrid synchronization, projective synchronization, and function projective synchronization (Agrawal et al. 2012; Yu and Liu 2003; Zhang and Sun 2004; Rosenblum et al. 1997; Srivastava et al. 2013a; Si et al. 2012; Zhou and Zhu 2011) etc. using different types of control scheme such as linear and non linear feedback synchronization, adaptive control, active control, sliding mode control etc.

In the present chapter a new way for combined function projective synchronization among fractional order chaotic systems in the presence of parametric uncertainties and external disturbances is described using backstepping control method. Function projective synchronization (FPS), the generalization of projective synchronization (PS), is one of the synchronization methods where two identical (or different) chaotic systems can synchronize up to a scaling function matrix with different initial values. From literature survey, it is seen that many researchers and scientists have worked on function projective synchronization of fractional order chaotic systems (Yu and Li 2010; Chen and Li 2007; Yadav et al. 2017a). In combination synchronization (Runzi et al. 2011; Yadav et al. 2017b), two or more master systems and one slave system are synchronized. This synchronization scheme has advantages over the usual drive response synchronization, such as being able to provide greater security in secure communication. The influences of the uncertainties during synchronization have been considered late. In the real world applications, such as in secure communication (Vaidyanathan and Volos 2016), the receiver plants will definitely suffer from the various uncertainties including parameter perturbation or external disturbance, which will no doubt influence the accuracy of the communication. Therefore, the synchronization between fractional order chaotic systems with uncertainties and disturbances are tough jobs for researchers. There are possibilities of destroying synchronization with the effects of those parameters (Srivastava et al. 2013b). The synchronization between chaotic systems with uncertainties and disturbances are not easy jobs for researchers since there are always possibilities of destroying synchronization under the effects of those parameters especially for fractional order systems. There are few results about the chaotic systems with uncertainties (Jawaadaa et al. 2012; Chen et al. 2012). Recently, Park (2006), Wu et al. (2009) have shown that the back stepping method is very simple, reliable and powerful for controlling the chaotic behavior and synchronization of chaotic systems. Wang and Ge (2001) proposed the adaptive synchronization of uncertain chaotic systems via backstepping design. In the same year, Lu and Zhang (2001) controlled the Chen's chaotic attractors using backstepping design based on parameters identification. Tan et al. (2003) synchronize the chaotic systems using backstepping design and again in the same year Yu and Zhang (2003) controlled the uncertain behavior of chaotic systems using backstepping design. These have motivated the authors to study on the combined function projective synchronization of fractional order chaotic systems with the presence of parametric uncertainties and external disturbances using backstepping control method. To the best of authors' knowledge the combined function projective synchronization among fractional order chaotic systems in the presence of parametric uncertainties and external disturbances using backstepping control method are few in numbers. Numerical simulation results are displayed graphically which clearly exhibit that the backstepping design control method is effective, easy to implement and reliable for combined function projective synchronizations of two nonlinear fractional order uncertain chaotic systems.

This chapter has been organized as follows. In Sect. 2, problem formulation of the combined function projective synchronization scheme of two different chaotic master systems, and one chaotic response system are presented. Section 3 contains some preliminaries, definition and lemma. In Sect. 4, the system descriptions of Lorenz, Rossler and Chen systems are given. Combined function projective synchronization among fractional order chaotic systems with uncertainties and external disturbances using backstepping control method are discussed in Sect. 5. In Sect. 6, the conclusion of the research work is presented.

## 2 Problem Formulation

Consider two uncertain fractional order chaotic systems as the master system as

$$D_t^q x = (A_1 + \Delta A_1)x + f_1(x) + d_1, \tag{1}$$

$$D_t^q y = (A_2 + \Delta A_2)y + f_2(y) + d_2, \quad 0 < q < 1 \tag{2}$$

and another uncertain fractional order chaotic system as the slave system as

$$D_t^q z = (A_3 + \Delta A_3)z + f_3(z) + d_3 + u(t), \tag{3}$$

where $x = [x_1, x_2, \ldots x_n]^T \in R^n$, $y = [y_1, y_2, \ldots y_n]^T \in R^n$ and $z = [z_1, z_2, \ldots z_n]^T \in R^n$ are the state vectors, $A_1$, $A_2$, $A_3 \in R^{n \times n}$ are constant matrices with proper dimensions, $f_1, f_2, f_3 : R^n \to R^n$ are the nonlinear functions of the systems, $\Delta A_1, \Delta A_2$, $\Delta A_3 \in R^{n \times n}$ are parametric uncertainties of chaotic systems with $|\Delta A_1| \le \delta_1$, $|\Delta A_2| \le \delta_2$, $|\Delta A_3| \le \delta_3$, where $\delta_1$, $\delta_2$, $\delta_3$ are positive constants and $d_1$, $d_2$, $d_3$ are the external disturbances of uncertain chaotic systems with $|d_1| \le \rho_1$, $|d_2| \le \rho_2$, $|d_3| \le \rho_3$, where $\rho_1$, $\rho_2$, $\rho_3 > 0$ and $u(t) \in R^n$ is the control input vector of the uncertain chaotic system (3). Now controller $u(t)$ is to be designed in such a way that the master and slave systems are synchronized through the proper definitions of errors.

If the synchronization error is defined by $e = z - k(y + x)$, where $k$ is the scaling function, then the corresponding error dynamics can be obtained as

$$\begin{aligned} D_t{}^q e = {} & (A_3 + \Delta A_3)e - k[(A_1 + \Delta A_1 - A_3 - \Delta A_3)x + (A_2 + \Delta A_2 - A_3 - \Delta A_3)y \\ & + f_1(x) + f_2(y) + d_1 + d_2] + f_3(z) + d_3 + u(t) \end{aligned} \tag{4}$$

Therefore, for combined function projective synchronization we use backstepping control method to design the control functions in such a way that the origin becomes asymptotically stable equilibrium point of the error dynamics i.e., $\lim_{t \to \infty} \| z - k(y + x) \| = 0$. The demonstration of backstepping control method is given in Sect. 5.

# 3 Some Preliminaries, Definition and Lemma

## *3.1 Fractional Calculus*

Fractional calculus is a generalization of integration and differentiation of integer order operator to a non-integer integro-differential operator denoted by ${}_aD_t^q$ and defined by

$${}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & R(q) > 0 \\ 1, & R(q) = 0 \\ \int_a^t (d\tau)^{-q}, & R(q) < 0, \end{cases}$$

where $q$ is the fractional order which may be a complex number and $R(q)$ denotes the real part of $q$ and $a$ is the fixed lower terminal and $t$ is the moving upper terminal.

**Definition 1** (Kilbas et al. 2006) The Caputo derivative for fractional order $q$ is defined as

$${}_a^cD_t^q \phi(t) = \frac{1}{\Gamma(n - q)} \int_a^t \frac{\phi^{(n)}(\tau)}{(t - \tau)^{q + 1 - n}} d\tau,\ t > a,$$

where $q \in R^+$ on the half axis $R^+$ and $n = \min\{k \in \mathrm{N}/k > q\}, q > 0$.

**Lemma 1** (Aguila-Camacho et al. 2014) *Let $x(t) \in R$ be a continuous and derivable function. Then for any time instant $t \geq t_0$,*

$$\frac{1}{2} {}_{t_0}^cD_t^q x^2(t) \leq x(t) {}_{t_0}^cD_t^q x(t), \forall q \in (0, 1].$$

# 4 Systems' Description

## 4.1 Fractional Order Lorenz System

The Lorenz attractor is an example of a non linear dynamical system corresponding to the long term behaviour of the Lorenz oscillation. The Lorenz oscillator is a three dimensional dynamical system that exhibits lemniscates type shaped chaotic flow which shows how the state of dynamical system evolves over time in a complex and non-repeating pattern. The Lorenz equations deal with the stability of fluid flows in the atmosphere. In addition to its interest in the field of non linear mathematics, the Lorenz model has important implications for climate and weather predictions. The case is also applicable for simplified models for lasers (Lorenz 1963) and dynamos (Knobloch 1981).

The fractional order Lorenz system (Wu and Shen 2009; Grigorenko and Grigorenko 2003) is given by

$$
\begin{aligned}
\frac{d^q x_1}{dt^q} &= a_1(y_1 - x_1),\\
\frac{d^q y_1}{dt^q} &= x_1(c_1 - z_1) - y_1,\\
\frac{d^q z_1}{dt^q} &= x_1 y_1 - b_1 z_1,
\end{aligned}
\tag{5}
$$

where $a_1$ is the Prandtl number, $c_1$ is the Rayleigh number and $b_1$ is the size of the region approximated by the system. The phase portraits of Lorenz system is shown through Fig. 1 for the parameters' values $a_1 = 10,\ b_1 = 8/3,\ c_1 = 28$ and initial condition $(0.2,\ 0,\ 2)$. The lowest value of fractional order $q$ for which the system remains chaotic is 0.99 (Wu and Shen 2009). The chaotic attractors in the $x_1 - y_1 - z_1$ space, $x_1 - y_1$, $x_1 - z_1$, $y_1 - z_1$ planes are shown in Fig. 1 for order of derivative $q = 0.993$.

The fractional order Lorenz system with uncertain parameters and external disturbances is defined as

$$
\begin{aligned}
\frac{d^q x_1}{dt^q} &= a_1(y_1 - x_1) + 0.11 z_1 - \cos(10d),\\
\frac{d^q y_1}{dt^q} &= x_1(c_1 - z_1) - y_1 - 0.14 x_1 - 2\cos(10d),\\
\frac{d^q z_1}{dt^q} &= x_1 y_1 - b_1 z_1 + 0.23 y_1 - 3\sin(10d),
\end{aligned}
\tag{6}
$$

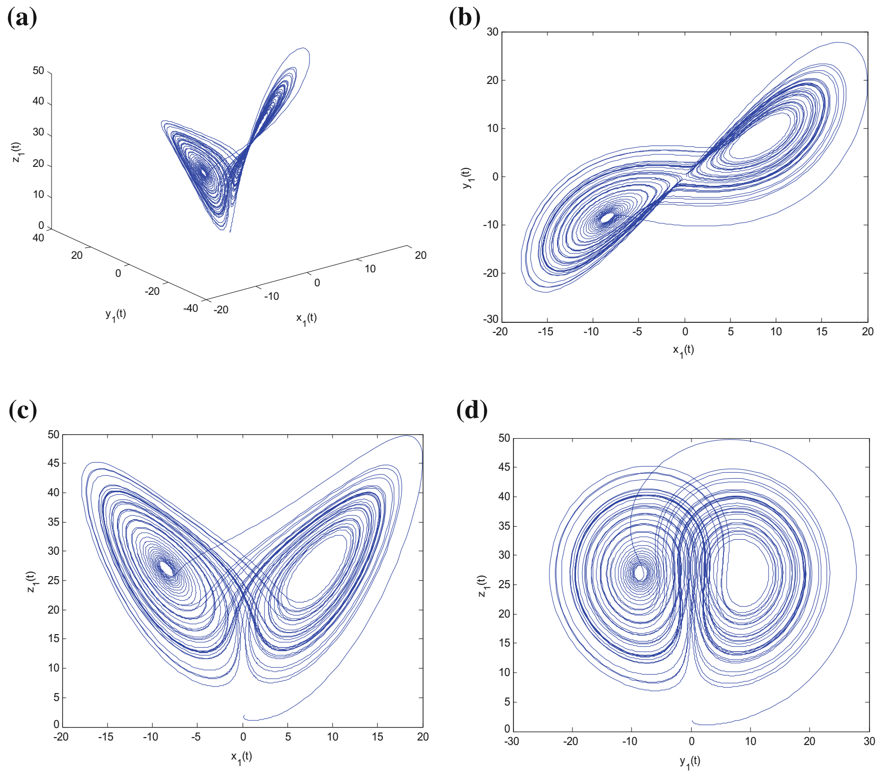


**Fig. 1** Phase portraits of the Lorenz system at $q = 0.993$: **a** $x_1 - y_1 - z_1$ space, **b** $x_1 - y_1$ plane, **c** $x_1 - z_1$ plane, **d** $y_1 - z_1$ plane

where uncertain parameter $\Delta A_1 = \begin{bmatrix} 0 & 0 & 0.11 \\ -0.14 & 0 & 0 \\ 0 & 0.23 & 0 \end{bmatrix}$ and disturbance term $d_1 = \begin{bmatrix} -\cos(10d) \\ -2\cos(10d) \\ -3\sin(10d) \end{bmatrix}$. Figure 2 shows the phase portraits of the fractional order Lorenz system with uncertainties and disturbances in $x_1 - y_1 - z_1$ space, $x_1 - y_1$, $x_1 - z_1$, $y_1 - z_1$ planes for the order of the derivative $q = 0.993$.

## *4.2 Fractional Order Rossler Systems*

The fractional order Rossler system (Yan and Li 2007; Zhou and Cheng 2008) is given by

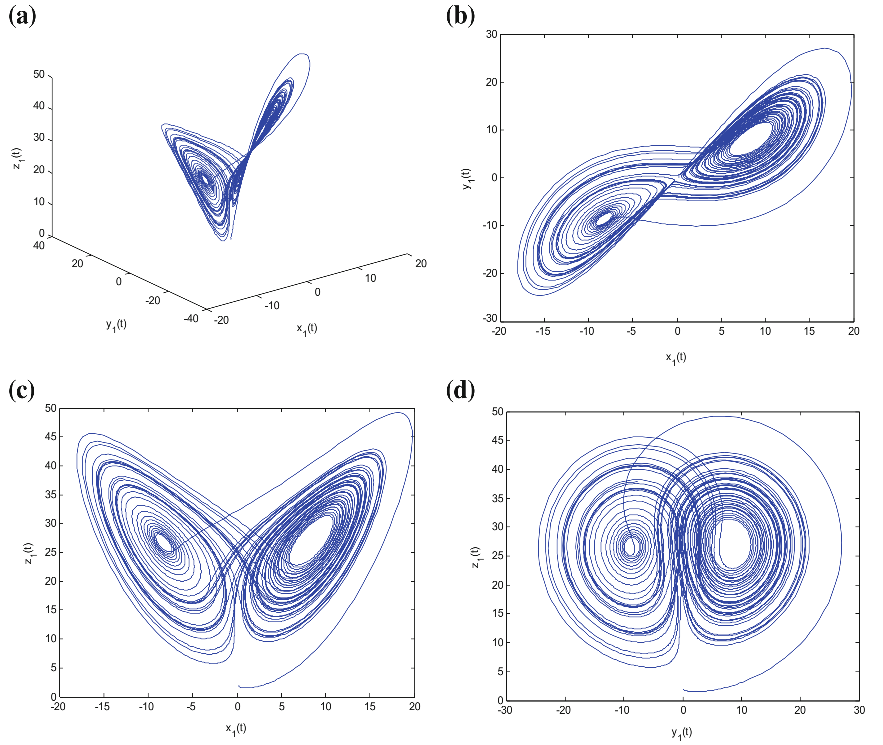


**Fig. 2** Phase portraits of the Lorenz system with uncertain parameters and external disturbances at $q = 0.993$: **a** $x_1 - y_1 - z_1$ space, **b** $x_1 - y_1$ plane, **c** $x_1 - z_1$ plane, **d** $y_1 - z_1$ plane

$$\begin{aligned} \frac{d^q x_2}{dt^q} &= -y_2 - z_2, \\ \frac{d^q y_2}{dt^q} &= x_2 + a_2 y_2, \\ \frac{d^q z_2}{dt^q} &= b_2 + x_2 z_2 - c_2 z_2, \end{aligned} \tag{7}$$

For the parameters' values $a_2 = 0.2,\ b_2 = 0.2,\ c_2 = 5.7$ and $q = 0.96$, the system (7) is chaotic. The phase portraits of Rossler system for order of derivative $q = 0.98$ are shown through Fig. 3.

The fractional order Rossler system with uncertain parameters and external disturbances is defined as

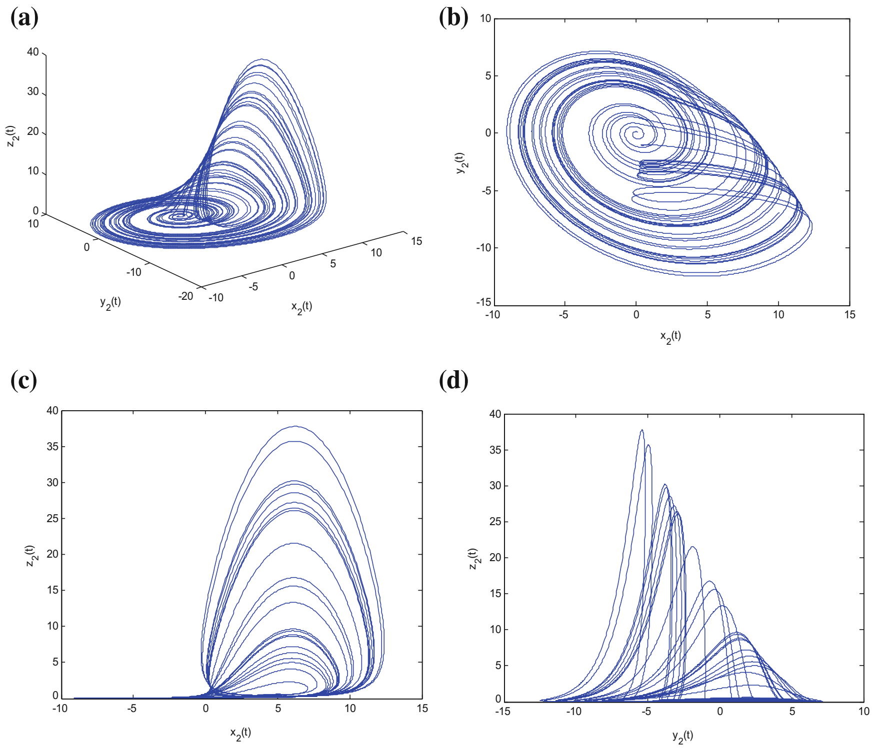


**Fig. 3** Phase portraits of the Rossler system at $q = 0.98$: **a** $x_2 - y_2 - z_2$ space, **b** $x_2 - y_2$ plane, **c** $x_2 - z_2$ plane, **d** $y_2 - z_2$ plane

$$\begin{aligned}
\frac{d^q x_2}{dt^q} &= -y_2 - z_2 - 0.01x_2 - 0.1\sin(20d), \\
\frac{d^q y_2}{dt^q} &= x_2 + a_2 y_2 - 0.02z_2 - 0.3\cos(20d), \\
\frac{d^q z_2}{dt^q} &= b_2 + x_2 z_2 - c_2 z_2 - 0.15y_2 - 0.04\sin(20d),
\end{aligned} \tag{8}$$

where uncertain parameter $\Delta A_2 = \begin{bmatrix} -0.01 & 0 & 0 \\ 0 & 0 & -0.02 \\ 0 & -0.15 & 0 \end{bmatrix}$ and disturbance term $d_2 = \begin{bmatrix} -0.1\sin(20d) \\ -0.3\cos(20d) \\ -0.04\sin(20d) \end{bmatrix}$. The phase portraits of fractional order Rossler system with uncertainties and disturbances in $x_2 - y_2 - z_2$ space, $x_2 - y_2$, $x_2 - z_2$, $y_2 - z_2$ planes are depicted through Fig. 4 at $q = 0.98$.

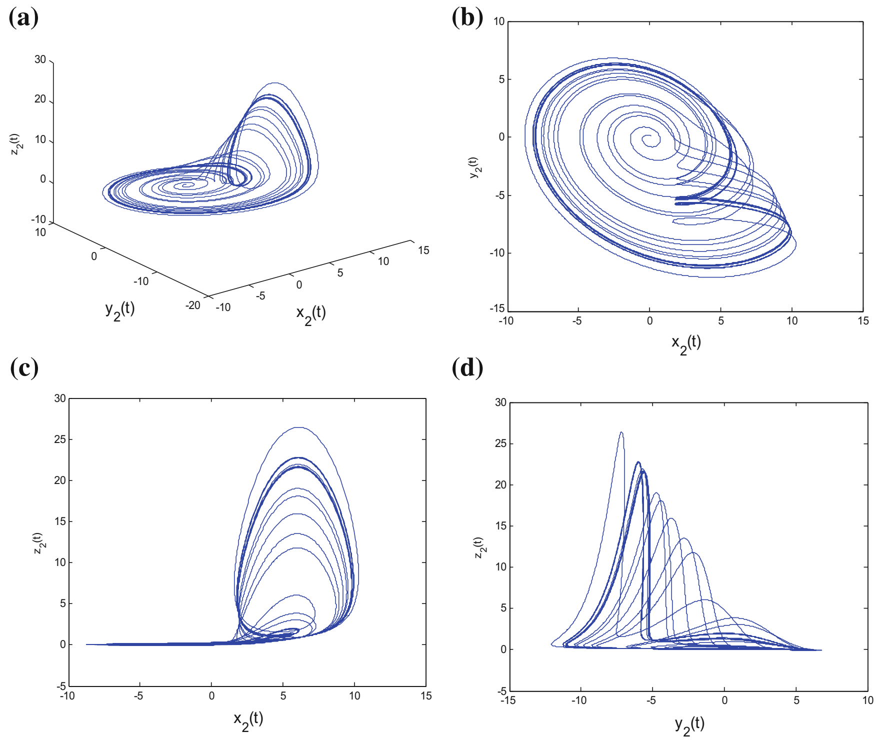


**Fig. 4** Phase portraits of the Rossler system with uncertain parameters and external disturbances at $q = 0.98$: **a** $x_2 - y_2 - z_2$ space, **b** $x_2 - y_2$ plane, **c** $x_2 - z_2$ plane, **d** $y_2 - z_2$ plane

## 4.3 Fractional Order Chen System

The fractional order Chen system (Lu and Chen 2006) is defined as

$$\begin{aligned}
\frac{d^q x_3}{dt^q} &= a_3(y_3 - x_3), \\
\frac{d^q y_3}{dt^q} &= (c_3 - a_3)x_3 - x_3 z_3 + c_3 y_3, \\
\frac{d^q z_3}{dt^q} &= x_3 y_3 - b_3 z_3,
\end{aligned} \tag{9}$$

For the parameters' values $a_3 = 35,\ b_3 = 3,\ c_3 = 28,\ q = 0.7$ and initial condition $(3,\ 4,\ 6)$, the system (9) shows the chaotic behaviour. The phase portraits of Chen system at $q = 0.90$ are depicted through Fig. 5.

Fractional order Chen system with uncertain parameters and external disturbances is defined as

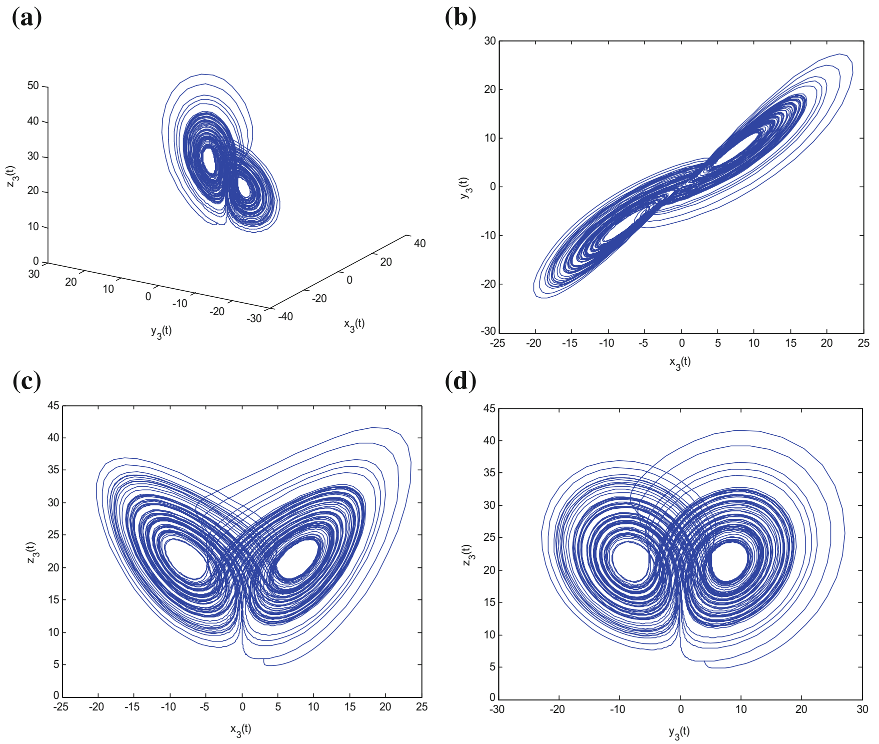


**Fig. 5** Phase portraits of the Chen system at $q = 0.90$: **a** $x_3 - y_3 - z_3$ space, **b** $x_3 - y_3$ plane, **c** $x_3 - z_3$ plane, **d** $y_3 - z_3$ plane

$$\begin{aligned}
\frac{d^q x_3}{dt^q} &= a_3(y_3 - x_3) - 0.2z_3 + 0.1\sin(100d), \\
\frac{d^q y_3}{dt^q} &= (c_3 - a_3)x_3 - x_3 z_3 + c_3 y_3 - 0.4z_3 - 0.2\cos(100d), \qquad (10) \\
\frac{d^q z_3}{dt^q} &= x_3 y_3 - b_3 z_3 + 0.1x_3 - \sin(100d),
\end{aligned}$$

where uncertain parameter $\Delta A_3 = \begin{bmatrix} 0 & 0 & -0.2 \\ 0 & 0 & -0.4 \\ 0.1 & 0 & 0 \end{bmatrix}$ and disturbance term $d_3 = \begin{bmatrix} 0.1\sin(100d) \\ -0.2\cos(100d) \\ -\sin(100d) \end{bmatrix}$. The phase portraits of fractional order Chen system with uncertainties and disturbances in $x_3 - y_3 - z_3$ space, $x_3 - y_3$, $x_3 - z_3$, $y_3 - z_3$ planes are depicted through Fig. 6 at $q = 0.90$.

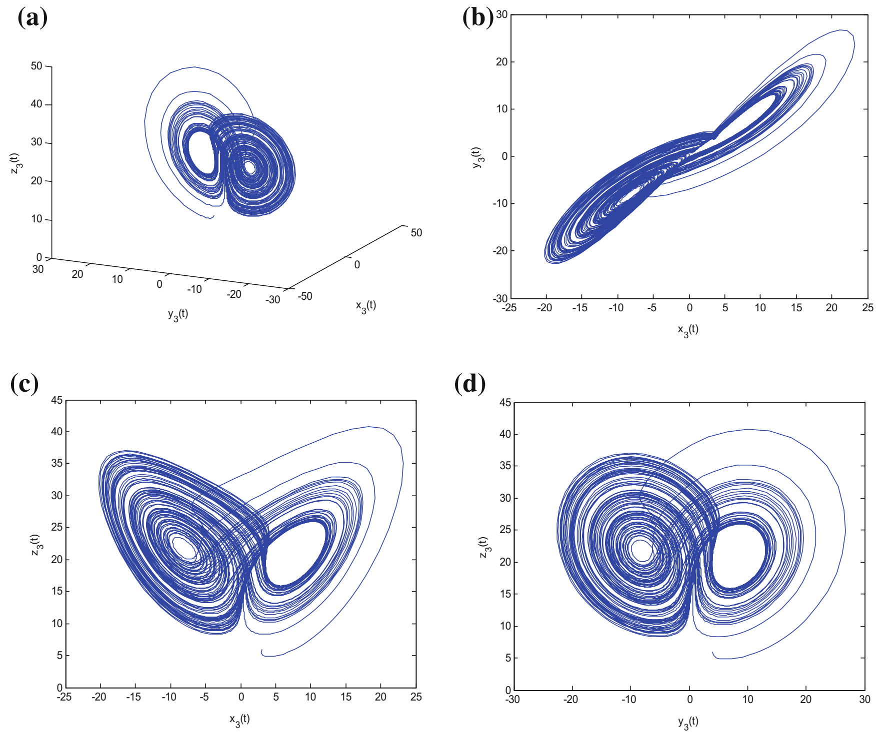


**Fig. 6** Phase portraits of the Chen system with uncertain parameters and external disturbances at $q = 0.90$: **a** $x_3 - y_3 - z_3$ space, **b** $x_3 - y_3$ plane, **c** $x_3 - z_3$ plane, **d** $y_3 - z_3$ plane

## 5 Combined Function Projective Synchronization Among Fractional Order Chaotic Systems with Uncertainties and External Disturbances Using Backstepping Control Method

For the study of combined function projective synchronization among fractional order chaotic systems with uncertain parameters and external disturbances, two systems viz., Lorenz system (6) and Rossler system (8) are considered as drive system-I and drive system-II and Chen system (10) is considered as response system. The response system with the control functions is defined as

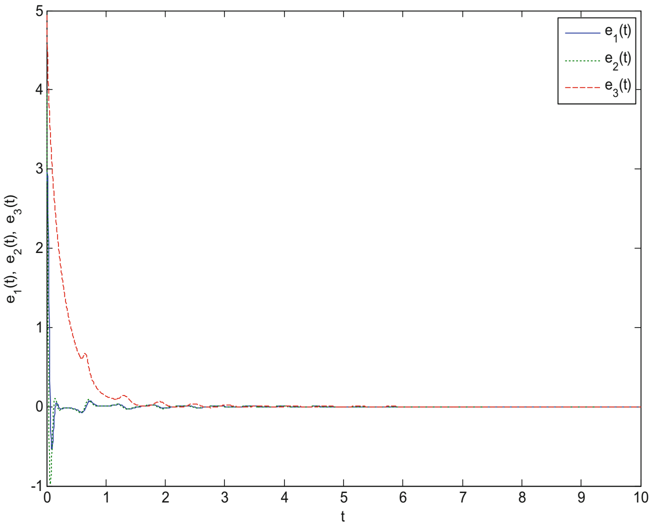


**Fig. 7** Evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ for fractional order $q = 0.96$

$$
\begin{aligned}
\frac{d^q x_3}{dt^q} &= a_3(y_3 - x_3) - 0.2z_3 + 0.1\sin(100d) + u_1(t),\\
\frac{d^q y_3}{dt^q} &= (c_3 - a_3)x_3 - x_3z_3 + c_3y_3 - 0.4z_3 - 0.2\cos(100d) + u_2(t),\\
\frac{d^q z_3}{dt^q} &= x_3y_3 - b_3z_3 + 0.1x_3 - \sin(100d) + u_3(t),
\end{aligned} \tag{11}
$$

where $u(t) = [u_1(t),\ u_2(t),\ u_3(t)]^T$ is the control functions to be deigned later. Defining the error functions as

$$e_1 = x_3 - k_1(x_2 + x_1)$$

$$e_3 = y_3 - k_2(y_2 + y_1)$$

$$e_3 = z_3 - k_3(z_2 + z_1),$$

we obtain the error system as

$$
\begin{aligned}
\frac{d^q e_1}{dt^q} &= a_3(e_2 - e_1) - 0.2e_3 + \phi_1 + u_1(t),\\
\frac{d^q e_2}{dt^q} &= (c_3 - a_3)e_1 + c_3e_2 - 0.4e_3 + \phi_2 + u_2(t),\\
\frac{d^q e_3}{dt^q} &= -b_3e_3 + 0.1e_1 + \phi_3 + u_3(t),
\end{aligned} \tag{12}
$$

where

$$\phi_1 = -0.2k_3(z_2+z_1) + a_3k_2(y_2+y_1) - k_1[a_3(x_2+x_1) - y_2 - z_2 - 0.01x_2 - 0.1\sin(20d) + a_1(y_1-x_1) + 0.11z_1 - \cos(10d)] + 0.1\sin(100d)$$

$$\phi_2 = (c_3-a_3)k_1(x_2+x_1) - k_2[-c_3(y_2+y_1) + x_2 + a_2y_2 - 0.02z_2 - 0.3\cos(20d) + x_1(c_1-z_1) - y_1 - 0.14x_1 - 2\cos(10d)] - 0.4k_3(z_2+z_1) - x_3z_3 - 0.2\cos(100d)$$

$$\phi_3 = 0.1k_1(x_2+x_1) - k_3[b_3(z_2+z_1) + b_2 + x_2z_2 - c_2z_2 - 0.15y_2 - 0.04\sin(20d) + x_1y_1 - b_1z_1 + 0.23y_1 - 3\sin(10d)] + x_3y_3 - \sin(100d).$$

Now the control functions would be properly designed using backstepping approach for combination function projective synchronization among fractional order chaotic systems in presence of uncertain parameters and external disturbances.

**Theorem 1** *If the control functions are chosen as*

$$u_1(t) = 0.2e_3 - \phi_1,$$

$$u_2(t) = -c_3w_1 - w_2 - c_3w_2 - \phi_2,$$

$$u_3(t) = -0.1w_1 + 0.4w_2 - \phi_3,$$

*where* $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_3$, *the systems (6) and (8) will be synchronized with the system (10).*

*Proof* To achieve control functions, we use active backstepping procedure through following three steps.

**Step I**: Considering $w_1 = e_1$, the fractional derivative of $w_1$ is

$$\frac{d^qw_1}{dt^q} = \frac{d^qe_1}{dt^q} = a_3(e_2 - w_1) - 0.2e_3 + \phi_1 + u_1(t), \tag{13}$$

where $e_2 = \alpha_1(w_1)$ is regarded as an virtual controller. To stabilize $w_1$-subsystem, we define the Lyapunov function $V_1$ as

$$V_1 = \frac{1}{2}w_1^2.$$

Fractional derivative of $V_1$ is
$\frac{d^qV_1}{dt^q} = \frac{1}{2}\frac{d^qw_1^2}{dt^q} \le w_1\frac{d^qw_1}{dt^q}$ (Using Lemma 1)

$$i.e., \qquad \leq w_1[a_3(\alpha_1 - w_1) - 0.2e_3 + \phi_1 + u_1(t)].$$

Taking $\alpha_1(w_1) = 0$ and $u_1(t) = 0.2e_3 - \phi_1$, we get $\frac{d^q V_1}{dt^q} \leq -a_3 w_1^2 < 0$, negative definite, which implies that $w_1$-subsystem (13) is asymptotically stable. For the virtual control function $\alpha_1(w_1)$, we define a variable $w_2$ between $e_2$ and $\alpha_1(w_1)$ as

$$w_2 = e_2 - \alpha_1(w_1).$$

Then, $(w_1, w_2)$ subsystem is obtained as

$$\begin{aligned} \frac{d^q w_1}{dt^q} &= a_3(w_2 - w_1), \\ \frac{d^q w_2}{dt^q} &= (c_3 - a_3)w_1 + c_3 w_2 - 0.4e_3 + \phi_2 + u_2(t). \end{aligned} \tag{14}$$

Let $e_3 = \alpha_2(w_1, w_2)$ is an virtual controller.

**Step II**: In this step to stabilize $(w_1, w_2)$-subsystem (14), define the Lyapunov function $V_2$ as

$$V_2 = V_1 + \frac{1}{2}w_2^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2.$$

Now

$$\begin{aligned} \frac{d^q V_3}{dt^q} &= \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} \\ &\leq w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q} \end{aligned}$$

$$i.e., \qquad \leq -a_3 w_1^2 + w_2[c_3 w_1 + c_3 w_2 - 0.4\alpha_2(w_1, w_2) + \phi_2 + u_2(t)],$$

If $\alpha_2(w_1, w_2) = 0$ and $u_2(t) = -c_3 w_1 - w_2 - c_3 w_2 - \phi_2$, then $\frac{d^q V_2}{dt^q} \leq -a_3 w_1^2 - w_2^2 < 0$ makes the subsystem (14) asymptotically stable.

Considering $w_3 = e_3 - \alpha_2(w_1, w_2)$, we get the following $(w_1, w_2, w_3)$-subsystem as

$$\begin{aligned} \frac{d^q w_1}{dt^q} &= a_3(w_2 - w_1), \\ \frac{d^q w_2}{dt^q} &= -a_3 w_1 - w_2 - 0.4w_3 \\ \frac{d^q w_3}{dt^q} &= -b_3 w_3 + 0.1w_1 + \phi_3 + u_3(t), \end{aligned} \tag{15}$$

**Step III**: In order to stabilize $(w_1, w_2, w_3)$-subsystem (15), choosing the Lyapunov function as

$$V_3 = V_2 + \frac{1}{2}w_3^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{1}{2}w_3^2,$$

we get

$$\begin{aligned} \frac{d^q V_3}{dt^q} &= \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} + \frac{1}{2}\frac{d^q w_3^2}{dt^q} \\ &\le w_1 \frac{d^q w_1}{dt^q} + w_2 \frac{d^q w_2}{dt^q} + w_3 \frac{d^q w_3}{dt^q}, \\ \textit{i.e.}, \quad &\le -a_3 w_1^2 - w_2^2 - 0.4 w_2 w_3 - b_3 w_3^2 + w_3[0.1 w_1 + \phi_3 + u_3(t)]. \end{aligned}$$

If $u_3(t) = -0.1w_1 + 0.4w_2 - \phi_3$, then $\frac{d^q V_3}{dt^q} \le -a_3 w_1^2 - w_2^2 - b_3 w_3^2 < 0$ negative definite. In view of $w_1 = e_1,\ w_2 = e_2 - \alpha_1(w_1) = e_2,\ w_3 = e_3 - \alpha_2(w_1, w_2) = e_3,$ the error states will converge to zero after a finite period of time, and thus the combined function projective synchronization among Lorenz, Rossler and Chen systems in the presence of uncertain parameters and external disturbances will be achieved.

## 5.1 Numerical Simulation and Results

In numerical simulation, the parameters of Lorenz system, Rossler system and Chen system are taken as $a_1 = 10,\ b_1 = 8/3,\ c_1 = 28;\ a_2 = 0.2,\ b_2 = 0.2,\ c_2 = 5.7$ and $a_3 = 35,\ b_3 = 3,\ c_3 = 28$ respectively. Time step size is taken as 0.005. The initial condition of two master systems and one slave system are taken as $(0.1,\ 0.1,\ 0.1)$, $(0.2,\ 0,\ 2)$ and $(3,\ 4,\ 6)$ respectively. Thus according to definition of error functions, the initial errors are $(2.85,\ 3.96,\ 4.95)$.

During the combined function projective synchronization the scaling functions are taken as periodic function as

$$k_1 = a_{11}\cos(a_{12}x_1) + a_{13}$$

$$k_2 = a_{21}\cos(a_{22}y_1) + a_{23}$$

$$k_3 = a_{31}\cos(a_{32}z_1) + a_{33}.$$

For the values of parameters $a_{11} = 0.4, a_{12} = 0.1, a_{13} = 0.1,\ a_{21} = 0.1, a_{22} = 0.2,$ $a_{23} = 0.3,\ a_{31} = 0.3, a_{32} = 0.3, a_{33} = 0.2$ it is seen from Fig. 7 that the error functions asymptotically converge to zero as time becomes large for the order of the derivatives $q = 0.96$, which shows that the master systems (6) and (8) are synchronized with the slave system (10).

# 6 Conclusion

The contribution of the present chapter is the investigation of the combined function projective synchronization among different fractional order chaotic systems with uncertainties and external disturbances using backstepping method. Based on Lyapunov stability theory, the synchronization with function scaling factor of chaotic systems through the proper design of control functions is achieved. The components of error state tend to zero as time becomes large help to get the time requires for combined synchronization among the systems. Numerical simulation results demonstrate that the method is reliable, convenient and effective for the combined function projective synchronization even for fractional order chaotic systems.

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