Self-Excited Attractors in Jerk Systems: Overview and Numerical Investigation of Chaos Production

Wafaa S. Sayed, Ahmed G. Radwan and Salwa K. Abd-El-Hafiz

Abstract Chaos theory has attracted the interest of the scientific community because of its broad range of applications, such as in secure communications, cryptography or modeling multi-disciplinary phenomena. Continuous flows, which are expressed in terms of ordinary differential equations, can have numerous types of post transient solutions. Reporting when these systems of differential equations exhibit chaos represents a rich research field. A self-excited chaotic attractor can be detected through a numerical method in which a trajectory starting from a point on the unstable manifold in the neighborhood of an unstable equilibrium reaches an attractor and identifies it. Several simple systems based on jerk-equations and different types of nonlinearities were proposed in the literature. Mathematical analyses of equilibrium points and their stability were provided, as well as electrical circuit implementations of the proposed systems. The purpose of this chapter is double-fold. First, a survey of several self-excited dissipative chaotic attractors based on jerk-equations is provided. The main categories of the included systems are explained from the viewpoint of nonlinearity type and their properties are summarized. Second, maximum Lyapunov exponent values are explored versus the different parameters to identify the presence of chaos in some ranges of the parameters.

Keywords Jerk equation · Maximum Lyapunov exponent · Phase portraits
Time series

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1 Introduction

Nonlinear dynamical systems with chaotic or strange attractors are characterized by the sensitivity to initial conditions, which is a required property for many applications (Layek 2015; Schöll 2001; Strogatz 2014). Chaos theory, dating back to Lorenz (1963), has attracted the interest of the scientific community and took part in many engineering applications such as dynamical modeling, pseudo-random number generation for secure communication and cryptography applications (Abd-El-Hafiz et al. 2014, 2015, 2016; Abdelhaleem et al. 2014; Barakat et al. 2013; Chien and Liao 2005; Frey 1993; Kocarev and Lian 2011; Lau and Tse 2003; Radwan and Abd-El-Hafiz 2013, 2014; Radwan et al. 2012, 2015a, b; Radwan AG et al. 2014; Sayed et al. 2015a, b, 2017a; Tolba et al. 2017) and control and synchronization (Azar and Vaidyanathan 2015, 2016; Azar et al. 2017; Henein et al. 2016; Martínez-Guerra et al. 2015; Radwan et al. 2013, 2017; Radwan A et al. 2014; Sayed et al. 2016, 2017b). Consequently, chaotic systems have been implemented in several numerical and electronic forms (Petras 2011; Radwan et al. 2003, 2004, 2007; Radwan 2013; Sayed et al. 2017d; Zidan et al. 2012).

Continuous flows expressed in terms of ordinary differential equations can have numerous types of post-transient solutions. An attractor is defined as the set of points approached by the orbit as the number of iterations increases to infinity representing its long term behavior. For a continuous system of differential equations, the equilibrium points are defined to be those points at which all time derivatives equal zero. The linear stability of each of the obtained points can be determined by Routh-Hurwitz stability criterion (Sprott 1994). The eigenvalues of the linearized Jacobian matrix are calculated. If all eigenvalues have negative real part, then the system is stable near the equilibrium point. If any eigenvalue has a real part that is positive, then the point is unstable. If the matrix has at least one eigenvalue with positive real part, at least one with negative real part, and no eigenvalues with zero real part, then the point is called a saddle (Alligood et al. 1996).

Furthermore, nearby trajectories diverge on strange attractors, giving rise to the butterfly effect in chaotic dynamical systems. This divergence is exponential and may be quantified using characteristic exponents known as Lyapunov exponents (Addison 1997). The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations. They are arranged in a descending order and if the maximum Lyapunov exponent is positive, then the system is chaotic. The sum of Lyapunov exponents represents the average contraction rate of volumes in phase space. The sum is less than zero in dissipative dynamical systems, as the post-transient solutions lie on attractors with zero phase volume. Dissipative systems exhibit chaos for most initial conditions in a specified range of parameters. On the other hand, a conservative system exhibits periodic and quasi-periodic solutions for most values of parameters and initial conditions, and can exhibit chaos for special values only. Consequently, dissipative systems usually appear in most applications of chaos theory such as chaos-based communication, physical and financial modeling. Conservative systems have another different set of applications that study the development of chaos in some kinds of systems. They are useful in describing certain dynamical systems where there is no dissipation, or it is so slight that it can be ignored, e.g., models of the solar system. Another important classification of chaotic attractors is either self-excited or hidden. A self-excited attractor has a basin of attraction that is associated with or excited from unstable equilibria. On the other hand, a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. From a computational point of view, a self-excited chaotic attractor can be detected through a numerical method in which a trajectory started from a point on the unstable manifold in the neighborhood of an unstable equilibrium reaches an attractor and identifies it. Hidden attractors cannot be found by this method (Leonov and Kuznetsov 2013).

Introducing novel chaotic systems requires a system of, at least, three differential equations involving, at least, one nonlinear term. A system of three or more first order ordinary differential equations that contain one or more nonlinear term(s) is constructed with the tendency to be as simple as possible. Some systematic numerical search methods have been developed for detecting the presence of chaotic solutions for new systems that contain multiple parameters. These parameters mainly appear as the coefficients of each term in the system of differential equations. Those numerical methods aim at setting many coefficients to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits (Sprott 1994). These systems, with the least number of existing coefficients and nonlinear terms, should exhibit chaotic properties of aperiodic bounded long-time evolution and sensitive dependence on initial conditions for some ranges of parameters. Reporting the parameter ranges for which systems of differential equations exhibit chaos or a strange attractor represents a rich research field.

Many researches focused on coming up with novel chaotic systems that, in the simplest form, involve a differential equation of at least third order $\dddot{x} = G(\ddot{x}, \dot{x}, x)$ and a nonlinearity. Differential equations of this form are called jerk equations because they involve third derivatives. The word "jerk" refers to the rate of change of acceleration, i.e., the derivative of acceleration with respect to time, the second derivative of velocity, and the third derivative of position. The mathematically simple jerk equation, which is equivalent to a system of three first-order ordinary non-linear differential equations, was shown to have solutions that exhibit chaotic behavior (Gottlieb 1996). Moreover, the simple circuit implementation of jerk-based systems suggests their utilization in secure communications and broadband signal generation. Systems involving a fourth or higher derivative are accordingly called hyperjerk systems.

Several simple systems based on the jerk-equation and different types of nonlinearities were proposed in the literature (Elwakil et al. 2000; Sayed et al. 2017c; Sprott 1994, 1997, 2000a, b, 2011; Vaidyanathan 2015; Vaidyanathan et al. 2014, 2015b, c). Mathematical analysis of equilibrium points and their stability were provided, as well as electrical circuit implementation of the proposed systems. Jerk-based chaotic systems express a third order ordinary differential equation as a system of three simultaneous first-order ordinary differential equations. Hence, they are considered as one of the simplest types of continuous chaotic systems. Consequently, they have been utilized in many applications including control and synchronization.

This chapter focuses on dissipative chaotic systems with self-excited attractors because they are not easily driven away from chaotic behavior when correctly adjusting the parameters and varying the initial conditions. All the reviewed systems are based on jerk equations and were shown to be chaotic for specific values of the parameters in the original papers which introduced them. The main properties of the selected systems, which have different types of nonlinear terms, are reviewed. The associated phase portraits and Maximum Lyapunov Exponent (MLE) values are tabulated in Sect. 2. Section 3 explores the responses in wider ranges of parameters to investigate the possibility of chaos production using MLE as a chaotic measure. Section 4 summarizes the contributions of the chapter.

2 Review of Some Self-Excited Jerk-Based Attractors

Early researches presented different variations on chaotic systems such that their equations look simpler or more "elegant" (Sprott 1994). Sprott (2000a) discussed several systems of the general form $\dddot{x} + A\ddot{x} + \dot{x} = f(x)$, where $f(x)$ is a nonlinear function satisfying some conditions that guarantee boundedness. The equation is redefined as $\dot{x} = y$, $\dot{y} = z$ and $\dot{z} = -Az - y + f(x)$. An electrical circuit implementation has been suggested for cases in which $f(x)$ is a piecewise linear function. Several cases in which $f(x)$ is a piecewise linear, quadratic, cubic, sinusoidal or hyperbolic tangent nonlinear function are illustrated in Fig. 1.

The systems presented in (Sprott 2000a) are listed as fourteen systems in Tables 1, 2, 3 and 4. Systems (1) to (14) are self-excited attractors that posses an unstable equilibrium point at the zero of $f(x)$. These systems are elementary, both in the sense of having the algebraically simplest autonomous Ordinary Differential Equation (ODE) and in the form of the nonlinearity. The first five systems, which are discussed in Table 1, represent the simplest cases with piece-wise nonlinearity. Their governing equations are the easiest to implement on electronic platforms. They represent a class of chaotic electrical circuit that is simple to construct, analyze, and scale over a wide

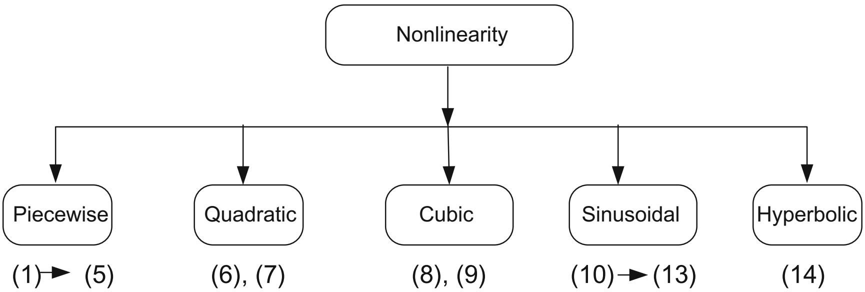


Fig. 1 Categorization of the reviewed dissipative chaotic systems with self-excited attractors from the viewpoint of the type of nonlinearity

Table 1 Systems with piece-wise linear function

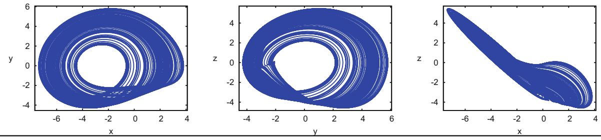
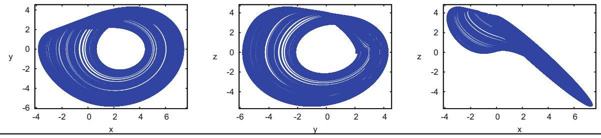
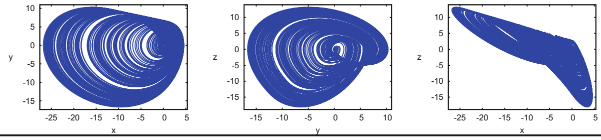
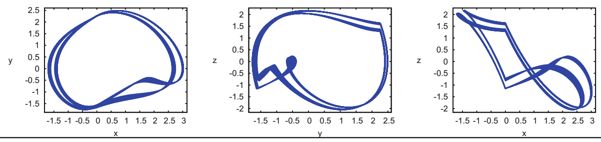
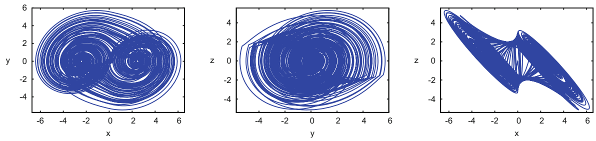
	Equations	Attractor
(1)	$f(x) = B\|x\| - C$ $B = 1$ $\mathrm{MLE} = 0.036$	
(2)	$f(x) = -B\|x\| + C$ $B = 1$ $\mathrm{MLE} = 0.036$	
(3)	$f(x) = -B\max(x, 0) + C$ $B = 6$ $\mathrm{MLE} = 0.093$	
(4)	$f(x) = Bx - Csgn(x)$ $B = 1.2$ $\mathrm{MLE} = 0.657$	
(5)	$f(x) = -Bx + Csgn(x)$ $B = 1.2$ $\mathrm{MLE} = 0.162$	

Table 2 Systems with quadratic nonlinearity

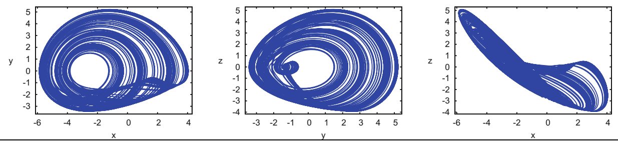
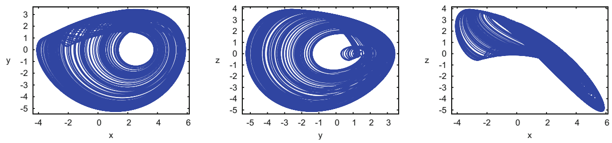
	Equations	Attractor
(6)	$f(x) = B(\frac{x^2}{C} - C)$ $B = 0.58$ $\mathrm{MLE} = 0.078$	
(7)	$f(x) = -B(\frac{x^2}{C} - C)$ $B = 0.58$ $\mathrm{MLE} = 0.078$	

Table 3 Systems with cubic nonlinearity

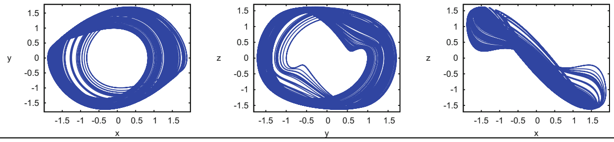
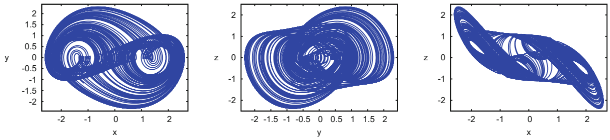
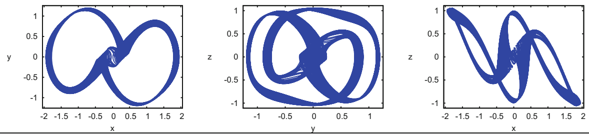
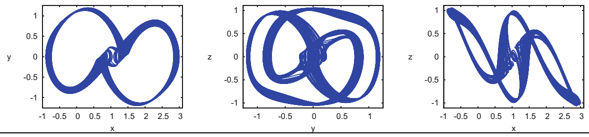
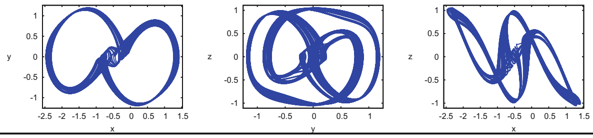
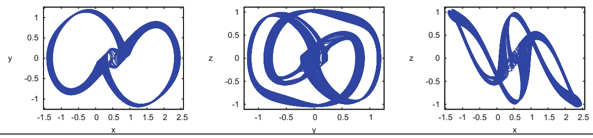
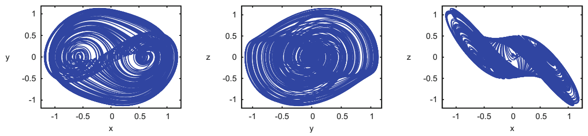
Equations	Attractor
(8) $f(x) = Bx(\frac{x^2}{C} - 1)$ $B = 1.6$ $\text{MLE} = 0.103$	
(9) $f(x) = -Bx(\frac{x^2}{C} - 1)$ $B = 0.9$ $\text{MLE} = 0.126$	

Table 4 Systems with sinusoidal or hyperbolic nonlinearity

Equations	Attractor
(10) $f(x) = B\sin(Cx)/C$ $B = 2.7$ $\text{MLE} = 0.069$	
(11) $f(x) = -B\sin(Cx)/C$ $B = 2.7$ $\text{MLE} = 0.069$	
(12) $f(x) = B\cos(Cx)/C$ $B = 2.7$ $\text{MLE} = 0.069$	
(13) $f(x) = -B\cos(Cx)/C$ $B = 2.7$ $\text{MLE} = 0.069$	
(14) $f(x) = -B[x - 2\tanh(Cx)/C]$ $B = 2.2$ $\text{MLE} = 0.221$ Hyperbolic tangent	

range of frequencies. In addition, it does not involve analog multiplication and uses only resistors, capacitors, diodes and operational amplifiers. The rest of the systems range between quadratic, cubic, sinusoidal and hyperbolic nonlinearities. Although they are more complicated, they are still good candidates for detailed quantitative analysis of bifurcation theory and other chaotic properties to be compared with simulation or implementation results.

For all the systems, the parameter $A = 0.6$, while C can be arbitrarily chosen as it acts as a scaling factor for the size of the attractor diagram. Each table provides the nonlinear function $f(x)$ and the specific value of the parameter B that produces chaos. In addition, the attractor diagrams or phase portraits are shown with the corresponding positive value of MLE, base-e, both indicating chaotic behavior. The diagrams have been plotted using Economics and Finance (E&F) chaos software (Diks et al. 2008) at the specified parameter values and the MLE values were given in (Sprott 2000a). Systems with similar equations exhibit similar attractor diagrams, e.g., systems (1) and (2) and systems (10) to (13).

Several other papers presented jerk-based chaotic attractors (Elwakil et al. 2000; Sprott 1994, 1997, 2000b, 2011). General three dimensional autonomous ordinary differential equations with quadratic nonlinearities were examined in (Sprott 1994). The resulting simple chaotic systems are composed of either five terms and two nonlinearities or six terms and one nonlinearity. Systems with cubic nonlinearities were presented in (Sprott 1997) and employed in (Vaidyanathan et al. 2015a). A very simple jerk-based system with piecewise nonlinearity generated by a signum function was presented in (Elwakil et al. 2000).

Several recent researches presented new jerk-based systems as part of their work (Sayed et al. 2017c; Vaidyanathan 2015; Vaidyanathan et al. 2014, 2015b, c). A six-term three dimensional novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities was presented in (Vaidyanathan et al. 2014). An adaptive backstepping controller was designed to stabilize the system with two unknown parameters. In addition, synchronization of two systems with two unknown parameters was achieved. Moreover, an electronic circuit realization of the novel jerk chaotic system using Spice was presented. A four-dimensional novel hyperchaotic hyperjerk system was proposed in (Vaidyanathan et al. 2015b) associated with control, synchronization and electronic circuit realization. A six-term three-dimensional jerk chaotic system with two exponential nonlinearities was presented in (Vaidyanathan et al. 2015c). A seven-term three-dimensional novel jerk chaotic system with two quadratic nonlinearities was presented in (Vaidyanathan 2015). Adaptive backstepping control of the proposed system and synchronization of two identical entities with unknown parameters were also proposed. Generalized forms of two well-known discrete-time chaotic maps were utilized as the nonlinear function of the jerk-equation in (Sayed et al. 2017c). The two maps are the scaled tent map with piece-wise nonlinearity and the scaled logistic map with quadratic nonlinearity.

Fully digital implementations of four different systems in the third order jerk-equation based chaotic family using Euler approximation were presented in (Mansingka et al. 2013). The systems ranged between absolute value, signum, quadratic and cubic nonlinearities. The high performance metrics of the digitally implemented systems as pseudo-random number generators were verified and shown to be suitable for communication systems and hardware encryption applications.

3 Sensitivity to Parameter Variations

This section provides some extra results and simulations for a selected set of the systems summarized in the previous section. The results mainly focus on plotting the phase portraits or strange attractors at values of parameters around those specified in the original paper (Sprott 2000a). A simulation-based procedure for specifying parameter ranges of chaos production around the specified values is discussed through plotting MLE versus the different parameters.

In general, the basin of attraction is the set of initial conditions which leads to a particular post-transient solution. Parameter values can control whether chaotic behavior is exhibited or not. While the parameter values that drive the system into chaos are called parameter basin of attraction of the chaotic attractor, the initial points that converge to a chaotic orbit are called its basin of attraction. There are two reasons for the importance of parameter basin of attraction. First, to test the robustness of the solution or its sensitivity to small parameter changes. Second, to have an estimation of the allowed parameter space and which values produce chaos.

Plots of phase portraits and MLE versus parameters have been generated by the aid of E&F software (Diks et al. 2008). In addition, some of the calculations of Lyapunov exponents were carried out by Lyapunov Exponent Toolbox (LET) (Siu 1998) and a MATLAB-based program for dynamical system investigation (MATDS) (Govorukhin 2004). Calculations of Lyapunov exponents have been carried out for $10,000$ iterations up to accuracy of four decimal places. These choices are made to ensure reaching a post-transient value of Lyapunov exponents. Lyapunov exponent calculations at the specified parameters and initial conditions should satisfy the two conditions for chaos production. First, their summation should be less than zero since they are dissipative dynamical systems. Second, the MLE should be positive which accounts for chaotic behavior.

Several numerical simulations are performed to get the ranges of parameters, rather than specific values only, that produce chaos. The procedure makes use of the calculated Lyapunov exponents in determining approximate ranges of parameters that produce chaos. The systems are shown to satisfy the condition of dissipative systems by using the Lyapunov exponent calculation function of MATDS. MLE is plotted versus different system parameters using E&F chaos software. For the parameter values specified in (Sprott 2000a), the neighborhood of each parameter value is explored while fixing the other parameters. The approximate ranges of parameters that exhibit positive values of MLE are recorded. Visualizing phase portraits is also used as a check of chaos production.

The parameter values that correspond to maximum chaos (largest MLE) are specified. In addition, other values of parameters are shown to drive the response out of chaos and generate other types of solutions. The flows of most dynamical systems with respect to parameter variation exhibit the following pattern of different types of solutions: stable or fixed followed by periodic then quasi-periodic and afterwards chaotic and finally unstable or divergent, whether in the direction of increasing the parameter or vice versa. Lyapunov exponents can be used to determine the type of the attractor as follows, where three dimensional phase space is assumed for simplicity. Lyapunov exponents with signs $(+, 0, -)$ correspond to chaos or strange attractor, $(-, -, -)$ to fixed point, $(0, -, -)$ to limit cycle and $(0, 0, -)$ to quasi-periodic torus (Addison 1997).

3.1 A Dissipative Self-Excited Attractor with Quadratic Nonlinearity: System (6)

System (6) in Table 2 represents a sample for quadratic nonlinearity. Figure 2 shows the ranges of parameters A, B and C that can produce chaos in system (6). Wider ranges were investigated using E&F chaos software, but the chaotic range is focused as shown in the figure. Figure 2a shows that for approximately $0.6 \leq A \leq 0.675$, the value of MLE varies but remains positive through almost the whole interval. This indicates that the system exhibits chaotic behavior in this range of the parameter A.

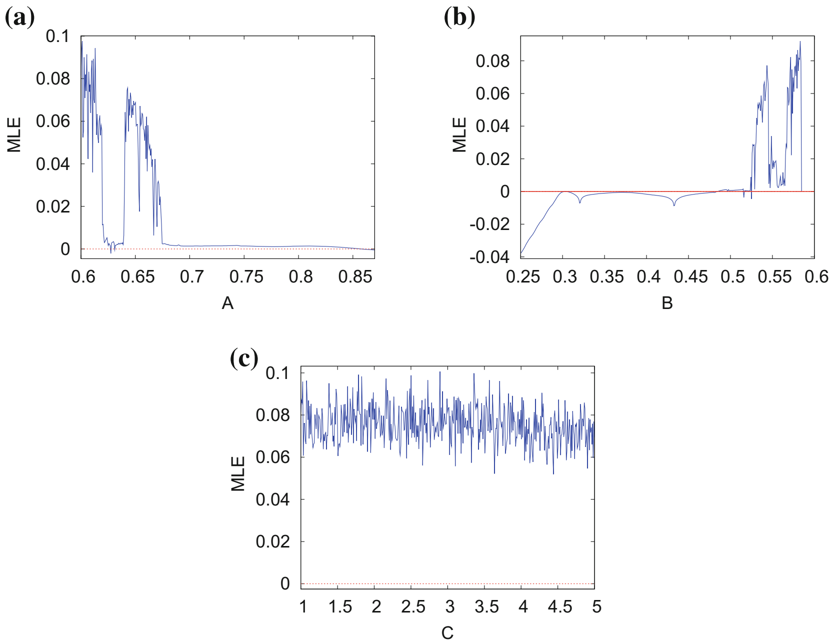


Fig. 2 Ranges of parameters that produce chaos for system (6) **a** MLE versus A, **b** MLE versus B, and **c** MLE versus C

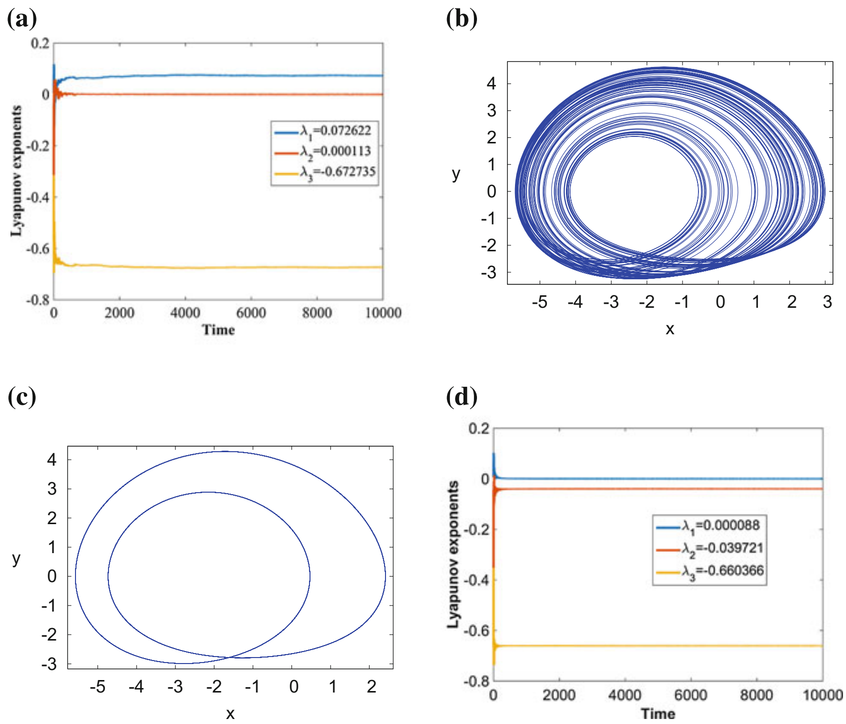


Fig. 3 **a** Time evolution of Lyapunov exponents at $A = 0.6$ and $B = 0.58$, **b** Phase portrait of the chaotic attractor at $A = 0.62$ and $B = 0.55$, **c** Non-chaotic phase portrait, and **d** Time evolution of Lyapunov exponents at $A = 0.7$ and $B = 0.58$ for system (6)

For A slightly less than 0.6, MLE diverges corresponding to unstable system, while for A slightly greater than 0.675, MLE is around zero or negative corresponding to periodic responses.

Regarding the effect of the parameter B, Fig. 2b shows that for approximately $0.525 \leq B \leq 0.585$, the value of MLE varies but remains positive. For B slightly greater than 0.585, MLE diverges corresponding to unstable system, while for B slightly less than 0.525, MLE is around zero or negative corresponding to periodic responses.

Figure 2c shows that the value of C does not affect the type of the behavior, which conforms to its description in (Sprott 2000a) as a scaling factor for the attractor size.

The values chosen in (Sprott 2000a) to produce chaos are $A = 0.6$ and $B = 0.58$ corresponding to the attractor diagrams shown in Table 2. Figure 3a shows the time evolution of Lyapunov exponents at these values of parameters, which satisfy the conditions for chaotic behavior. The MLE value approaches the value given in Table 2 as time advances. Furthermore, it is shown in Fig. 3b that other combinations of A and B, which belong to the intervals specified in this section, can yield a chaotic attractor too. Values of parameters outside the specified ranges drive the system out of chaos and can yield periodic responses as shown in Fig. 3c. Such responses exhibit $(0, -, -)$ values for the three Lyapunov exponents as shown in Fig. 3d indicating a limit cycle.

3.2 A Dissipative Self-Excited Attractor with Cubic Nonlinearity: System (8)

The range of the parameter A for system (8), with cubic nonlinearity, to exhibit chaotic behavior is limited to roughly about $0.6 \leq A < 0.65$ as shown in Fig. 4a with the largest MLE occurring at $A = 0.6$. For A slightly less than 0.6, the response diverges, while for A slightly greater than 0.65, periodic responses start to appear.

For the parameter B, chaos is produced in the approximate interval $1.5 < B < 1.65$, preceded by periodic responses and followed by divergent ones as shown in Fig. 4b. The conditions on the values of Lyapunov exponents at the parmeter values given in Table 3 can be illustrated similar to the previous case. Furthermore, Fig. 4c shows the phase portrait of a chaotic attractor at other parameter values that belong to the intervals defined in this section.

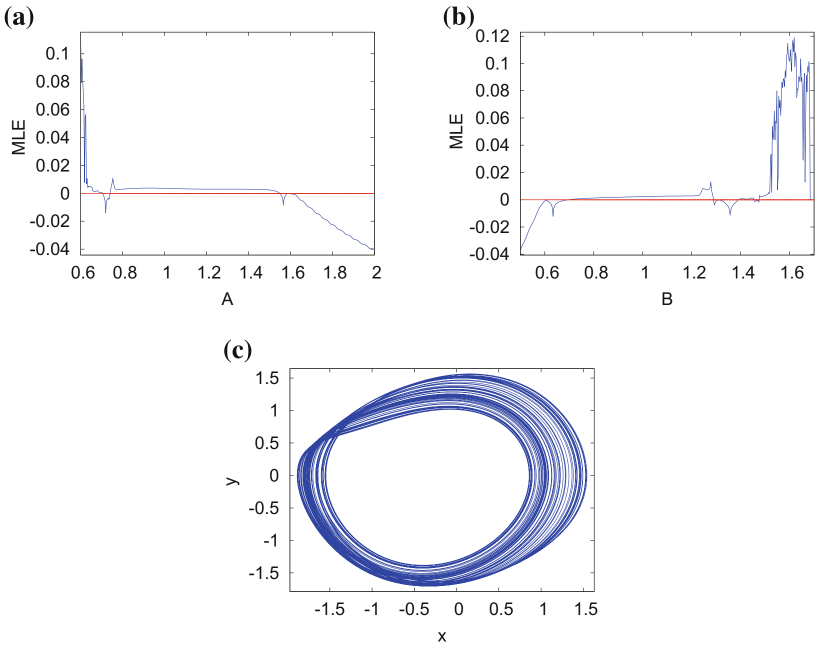


Fig. 4 Ranges of parameters that produce chaos for system (8) **a** MLE versus A, **b** MLE versus B, and **c** Phase portrait at $A = 0.62$ and $B = 1.64$

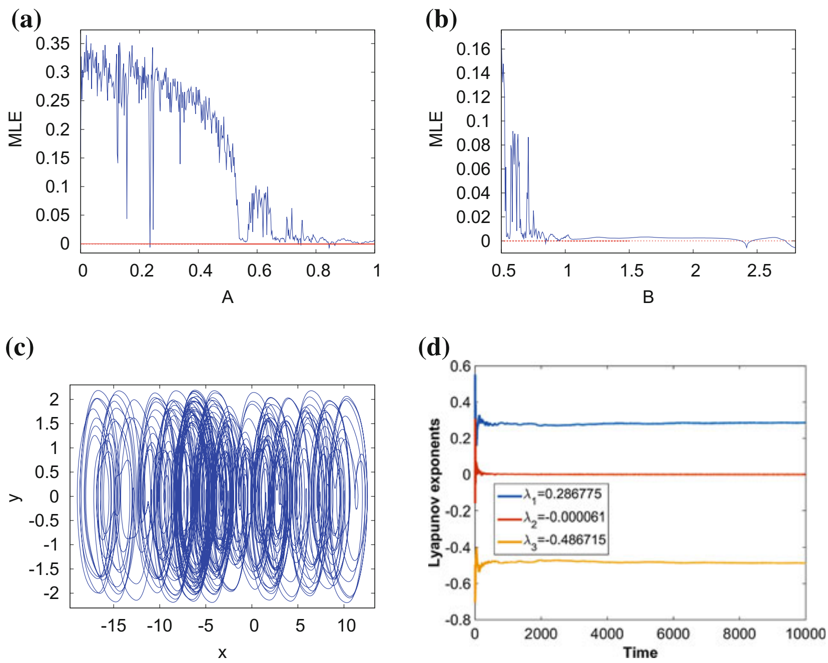


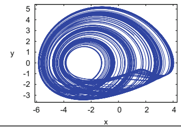
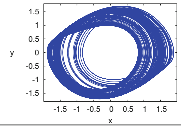
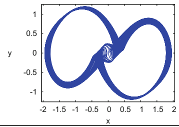
Fig. 5 Ranges of parameters that produce chaos for system (10) **a** MLE versus A, **b** MLE versus B, **c** Phase portrait, and **d** Time evolution of Lyapunov exponents at $A = 0.2$ and $B = 2.7$

3.3 A Dissipative Self-Excited Attractor with Sinusoidal Nonlinearity: System (10)

System (10) is studied as a sample of systems with sinusoidal nonlinearity. Various values that belong to the approximate intervals $0 < A < 0.7$ and $0.5 < B < 2.75$ correspond to chaotic behavior as shown in Fig. 5a and b. The intervals are not continuous, i.e., some exceptional values that correspond to non-chaotic behavior are found in between. The ranges of parameters which correspond to chaotic responses for the systems with sinusoidal nonlinearity are wider than the previous systems. Values other than those stated in (Sprott 2000a) can produce chaos with larger values of MLE. This is illustrated through the phase portrait and Lyapunov exponents in Fig. 5c and d, respectively.

Table 5 summarizes the main results obtained in this section for three continuous dissipative chaotic systems with self-excited attractors and various types of nonlinearities. A combination of the parameter values, which produce chaos, was given in (Sprott 2000a) as a single value for each parameter rather than a range. Table 5 shows the attractor digram of each system at the specified parameter values and wider ranges of parameters that produce chaos, which were not mentioned in the original

Table 5 Summary of the results obtained for the selected systems

System	Equations	Attractor	Ranges of Parameters	Category and Comments
(6)	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -Az - y + B(\frac{x^2}{C} - C)$		$0.6 \le A \le 0.675$ $0.525 \le B \le 0.585$ C scaling only (Figures 2 and 3)	– Dissipative. – Self-excited. – Quadratic nonlinearity.
(8)	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -Az - y + Bx(\frac{x^2}{C} - 1)$		$0.6 \le A < 0.65$ $1.5 < B < 1.65$ C scaling only (Figure 4)	– Dissipative. – Self-excited. – Cubic nonlinearity.
(10)	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -Az - y + B\sin(Cx)/C$		$0 < A < 0.7$ $0.5 < B < 2.75$ (Figure 5)	– Dissipative. – Self-excited. – Sinusoidal nonlinearity. – Relatively wide ranges of parameters produce chaos.

paper. Moreover, the main category to which each system belongs and comments on its behavior are included.

4 Conclusions

A review of dissipative jerk-based continuous chaotic systems with self-excited attractors has been presented. The systems posses various types of nonlinearities: piecewise, quadratic, cubic, sinusoidal and hyperbolic. The parameter values and chaotic properties of each system have been validated through phase portraits and MLE values. Using numerical simulations, wider ranges of parameters that correspond to chaotic behavior have been defined and shown to exhibit positive value of MLE. In addition, either periodic or divergent responses corresponding to values of parameters outside these ranges have been included.

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