# Synchronization Phenomena in Coupled Dynamical Systems with Hidden Attractors

**C. K. Volos, Viet-Thanh Pham, Ahmad Taher Azar, I. N. Stouboulos and I. M. Kyprianidis**

**Abstract** Recently, Leonov and Kuznetsov introduced a new class of nonlinear dynamical systems, which is called systems with hidden attractors, in contrary to the well-known class of systems with self-excited attractors. In this class, dynamical systems with infinite number of equilibrium points, with stable equilibria, or without equilibrium are classified. Since then, the study of chaotic systems with hidden attractors has become an attractive research topic because this new class of dynamical systems could play an important role not only in theoretical problems but also in engineering applications. In this direction, the proposed chapter presents the bidirectional and unidirectional coupling schemes between two identical dynamical chaotic systems with no-equilibrium points. As it is observed, when the value of the coupling coefficient is increased in both coupling schemes, the coupled systems undergo a transition from desynchronization mode to complete synchronization. Also, the simulation results reveal the richness of the coupled system's dynamical behavior, especially in the bidirectional case, showing interesting nonlinear dynamics, with a transition between periodic, quasiperiodic and chaotic behavior as the coupling coefficient increases, as well as synchronization phenomena, such as complete and anti-phase synchronization. Various tools of nonlinear theory for the

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study of the proposed coupling method, such as bifurcation diagrams, phase portraits and Lyapunov exponents have been used.

**Keywords** Complete synchronization · Anti-phase synchronization
Chaos · Hidden attractors · Bifurcation diagram · Lyapunov exponent

## 1 Introduction

In the past three decades, the phenomenon of synchronization between coupled nonlinear systems and especially of systems with chaotic behavior has attracted the interest of the research community because it is an interesting phenomenon with a broad range of applications, such as in various complex physical, chemical and biological systems (Holstein-Rathlou et al. 2001; Mosekilde et al. 2002; Pikovsky et al. 2003; Szatmári and Chua 2008; Tognoli and Kelso 2009; Wang et al. 2009; Liu and Chen 2010), in secure and broadband communication system (Kocarev et al. 1992; Cuomo et al. 1993; Wu and Chua 1993; Feki et al. 2003; Sheng-Hai and Ke 2004; Dimitriev et al. 2006; Jafari et al. 2010) and in cryptography (Annovazzi-Lodi et al. 1997; Baptista 1998; Grassi and Mascolo 1999; Dachselt and Schwarz 2001; Klein et al. 2005; Alvarez and Li 2006; Volos et al. 2006; Banerjee 2010).

The concept of synchronization of two or more systems with chaotic behavior is the phenomenon in which the coupled systems can adjust a given of their motion property to a common behavior (equal trajectories or phase locking), due to forcing or coupling (Luo 2013). However, having two chaotic systems being synchronized, it is a major surprise, due to the exponential divergence of the nearby trajectories of the systems. Nevertheless, nowadays the phenomenon of synchronization of coupled chaotic oscillators is well-studied theoretically and proven experimentally (Ouannas et al. 2017a, b; Azar and Vaidyanathan 2015a, b, c, 2016; Vaidyanathan et al. 2015a, b, c, 2017a, b, c; Boulkroune et al. 2016a, b; Vaidyanathan and Azar 2015a, b, c, d, 2016a, b, c, d, e, f; Ouannas et al. 2016a, b).

Synchronization theory has begun studying in the 1980s and early 1990s by Fujisaka and Yamada (1983), Pikovsky (1984), Pecora and Carroll (1990). Onwards, a great number of research works based on synchronization of nonlinear systems has risen and many synchronization schemes depending on the nature of the coupling schemes and of the interacting systems have been presented. Complete or full chaotic synchronization (Maritan and Banavar 1994; Kyprianidis and Stouboulos 2003a, b; Woafo and Enjieu Kadji 2004; Kyprianidis et al. 2006a, 2008), phase synchronization (Dykman et al. 1991; Parlitz et al. 1996), lag synchronization (Rosenblum et al. 1997; Taherion and Lai 1999), generalized synchronization (Rulkov et al. 1995), antisynchronization (Kim et al. 2003; Liu et al. 2006), anti-phase synchronization (Cao and Lai 1998; Astakhov et al. 2000; Zhong et al. 2001; Blazejczuk-Okolewska et al. 2001; Kyprianidis et al. 2006b; Tsuji et al. 2007), projective synchronization (Mainieri and Rehacek 1999; Ouannas et al. 2017c), anticipating (Voss 2000), inverse lag synchronization (Li 2009) and fractional order synchronization (Tolba et al. 2017; Azar et al. 2017a, b; Pham et al. 2017c, d; Ouannas et al. 2017d, e, f, g, h, i, j, k) are the most interesting types of synchronization, which have been investigated numerically and experimentally by many research groups.

However, the most interesting and the most studied case of synchronization is the *Complete or Full synchronization*. In this case the interaction between two coupled identical nonlinear circuits leads to a perfect coincidence of their chaotic trajectories, i.e.

$$x_1(t) = x_2(\mathrm{t}), \text{ as } t \to \infty. \tag{1}$$

Also, in 1998, another interesting type of synchronization between mutually coupled identical autonomous nonlinear systems was observed. In this new type of synchronization, which is called *Anti-phase synchronization*, each one of the uncoupled systems produces chaotic attractors (Wang et al. 2017). This synchronization phenomenon is observed when the coupled system is in a phase locked (periodic) state, depending on the coupling factor and it can be characterized by a π-phase delay. So, the periodic signals ($x_1$ and $x_2$) of each coupled circuits have a time lag $\tau$, which is equal to $T/2$, where $T$ is the period of the signals $x_1$ and $x_2$.

$$x_1(t) = x_2(t + \tau), \text{ where } \tau = T/2. \tag{2}$$

The anti-phase synchronization was also observed by Volos et al. (2013) in the case of two mutually coupled identical non-autonomous Duffing-type systems, which as it is known, have symmetry, because the transformation:

$$S\colon (x,\ y,\ t) \to (-x,\ -y,\ t + T/2) \tag{3}$$

leaves Duffing's system equations invariant.

It is well-known that chaotic dynamical systems exhibit high sensitivity on initial conditions or system's parameters and if they are identical and start from almost the same initial conditions, they follow trajectories which rapidly become uncorrelated. That is why many techniques exist to obtain chaotic synchronization. So, many of these techniques for coupling two or more nonlinear chaotic systems can be mainly divided into two classes: *unidirectional coupling* and *bidirectional* or *mutual coupling* (Gonzalez-Miranda 2004). In the first case, only the first system, the master system, drives the second one, the slave system, while in the second case, each system's dynamic behavior influences the dynamics of the other.

Recently, a great interest for dynamical systems with hidden attractors has been raised. The term *hidden attractor* is referred to the fact that in this class of systems the attractor is not associated with an unstable equilibrium and thus often remains undiscovered because it may occur in a small region of parameter space and with a small basin of attraction in the space of initial conditions (Kuznetsov et al. 2010; Leonov et al. 2011a, b, 2012; Pham et al. 2014a, b). In 2010, for the first time, a chaotic hidden attractor was discovered in the most well-known nonlinear circuit, in Chua's circuit, which is described by a three-dimensional dynamical system (Kuznetsov et al. 2010).

The problem of analyzing hidden oscillations arose for the first time in the second part of Hilbert's 16th problem (1900) for two-dimensional polynomial systems. The first nontrivial results were obtained in Bautin's works (Bautin 1939, 1952), which were devoted to constructing nested limit cycles in quadratic systems and showed the necessity of studying hidden oscillations for solving this problem. Later, in the middle of the 20th century, Kapranov studied (Kapranov 1956) the qualitative behavior of Phase-Locked Loop (PLL) systems, which are used in telecommunications and computer architectures, and estimated stability domains. In that work, Kapranov assumed that in PLL systems there were self-excited oscillations only. However, in 1961, (Gubar 1961) revealed a gap in Kapranov's work and showed analytically the possibility of the existence of hidden oscillations in two-dimensional system of PLL, thus, from a computational point of view, the system considered was globally stable, but, in fact, there was only a bounded domain of attraction.

Also, in the same period, the investigations of the widely known Markus-Yamabe (1960) and Kalman (1957) conjectures on absolute stability have led to the finding of hidden oscillations in automatic control systems with a unique stable stationary point and with a nonlinearity, which belongs to the sector of linear stability (Bernat and Llibre 1996; Fitts 1966; Leonov and Kuznetsov 2013).

Furthermore, systems with hidden attractors have received attention due to their practical and theoretical importance in other scientific branches, such as in mechanics (unexpected responses to perturbations in a structure like a bridge or in an airplane wing) (Lauvdal et al. 1997). So, the study of these systems is an interesting topic of a significant importance.

So, from the introduction of dynamical systems with hidden attractors a great number of systems belonging in this category has been reported. All these systems can be classified in three families of systems depending on the kind of systems' equilibria (Pham et al. 2017a). The first family is the systems without equilibrium points. The works of Nosé (1984) and Hoover (1985) in 1984–1985 have led the study of the aforementioned family of dynamical systems. Since then, many 3D or 4D dynamical systems of this family have been studied (Jafari et al. 2013; Wei 2011; Wang et al. 2012a; Wang and Chen 2013; Wei et al. 2014; Maaita et al. 2015; Tahir et al. 2015; Pham et al. 2016a, b; Wang et al. 2016; Zuo and Li 2016). The second family is the systems with stable equilibria (Wang and Chen 2012b; Molaie et al. 2013; Wei and Wang, 2013; Kingni et al. 2014; Lao et al. 2014; Pham et al. 2017b), while the third is the systems with an infinite number of equilibria (Jafari and Sprott 2013; Li and Sprott 2014; Gotthans and Petržela 2015; Gotthans et al. 2016; Pham et al. 2016c, d, e).

In the present chapter, the study of various synchronization phenomena between bidirectionally or unidirectionally coupled dynamical systems with hidden attractors is presented. For this reason, a no-equilibrium chaotic system, introduced by Pham et al. has been used (Pham et al. 2014c). Especially, in the case of the mutually coupled systems, except of the complete chaotic synchronization, the existence of anti-phase synchronization is also confirmed from the simulation results.

The rest of the chapter is organized as follows. Section 2 provides the mathematical model as well as the dynamics and properties of the proposed system with hidden attractors. Section 3 describes the coupling schemes of two identical no-equilibrium chaotic systems, while the simulation results of the coupled systems are thoroughly presented in Sect. 4. Finally, conclusions are drawn in Sect. 5

## 2 Description and Dynamics of the System Without Equilibrium

In 2013, Jafari and Sprott have introduced nine simple chaotic flows with a line equilibrium by using an exhaustive computer search (Jafari and Sprott 2013). These systems belong to the family of systems with hidden attractors because it is impossible to verify the chaotic attractor by choosing an arbitrary initial condition in the vicinity of the unstable equilibria.

As an example, the first of these systems, which is described by the following system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x + yz \\ \dot{z} = -x - axy - bxz \end{cases} \quad (4)$$

where $a$, $b$ are real positive parameters, has a line of equilibria $E(0, 0, z)$.

In the third equation of system (5) (Pham et al. 2014c) added a real parameter $c$ in order to obtain the following new system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x + yz \\ \dot{z} = -x - axy - bxz + c \end{cases} \quad (5)$$

which possesses no equilibrium points. So, it belongs to the family of dynamical systems without equilibrium.

Next, in order to discover system's (5) dynamics well-known tools of nonlinear theory, such as phase portrait, bifurcation diagram and Lyapunov spectrum, are used. For this reason the proposed system is integrated numerically using the classical fourth-order Runge-Kutta integration algorithm. For each set of parameters used in this work, the time step is always $\Delta t = 0.002$ and the calculations are

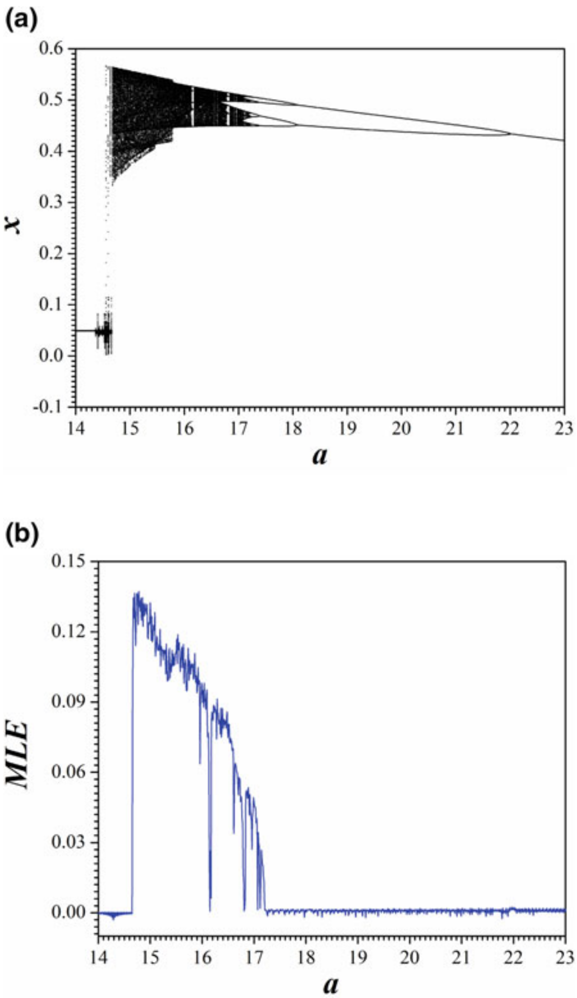


**Fig. 1** **a** Bifurcation diagram of system (5) for decreasing values of $a$ and **b** the graph of the maximal Lyapunov exponent plotted in the range of $14 \leq a \leq 23$, with $b = 1$, $c = 0.001$ and initial conditions $(x_0, y_0, z_0) = (0, 0.5, 0.5)$

performed using variables and parameters in extended precision mode. For each parameter settings, the system is integrated for a sufficiently long time and the transient is discarded.

To study the type of scenario giving rise to chaos by considering the parameter $a$ in system (5), as the main control parameter, the bifurcation diagram in Fig. 1a is obtained, while the other parameters remain fixed as $b = 1$ and $c = 0.001$ and the initial conditions are chosen as $(x_0, y_0, z_0) = (0, 0.5, 0.5)$. The bifurcation diagram is obtained by plotting the variable $x$ when the trajectory cuts the plane $y = 0$ with $dy/dt < 0$, as the control parameter $a$ is decreased in tiny steps in the range of $14 \leq a \leq 23$. From the bifurcation diagram of Fig. 1a it is possible to verify that the system (5) is driven to chaos through a period-doubling route as the control

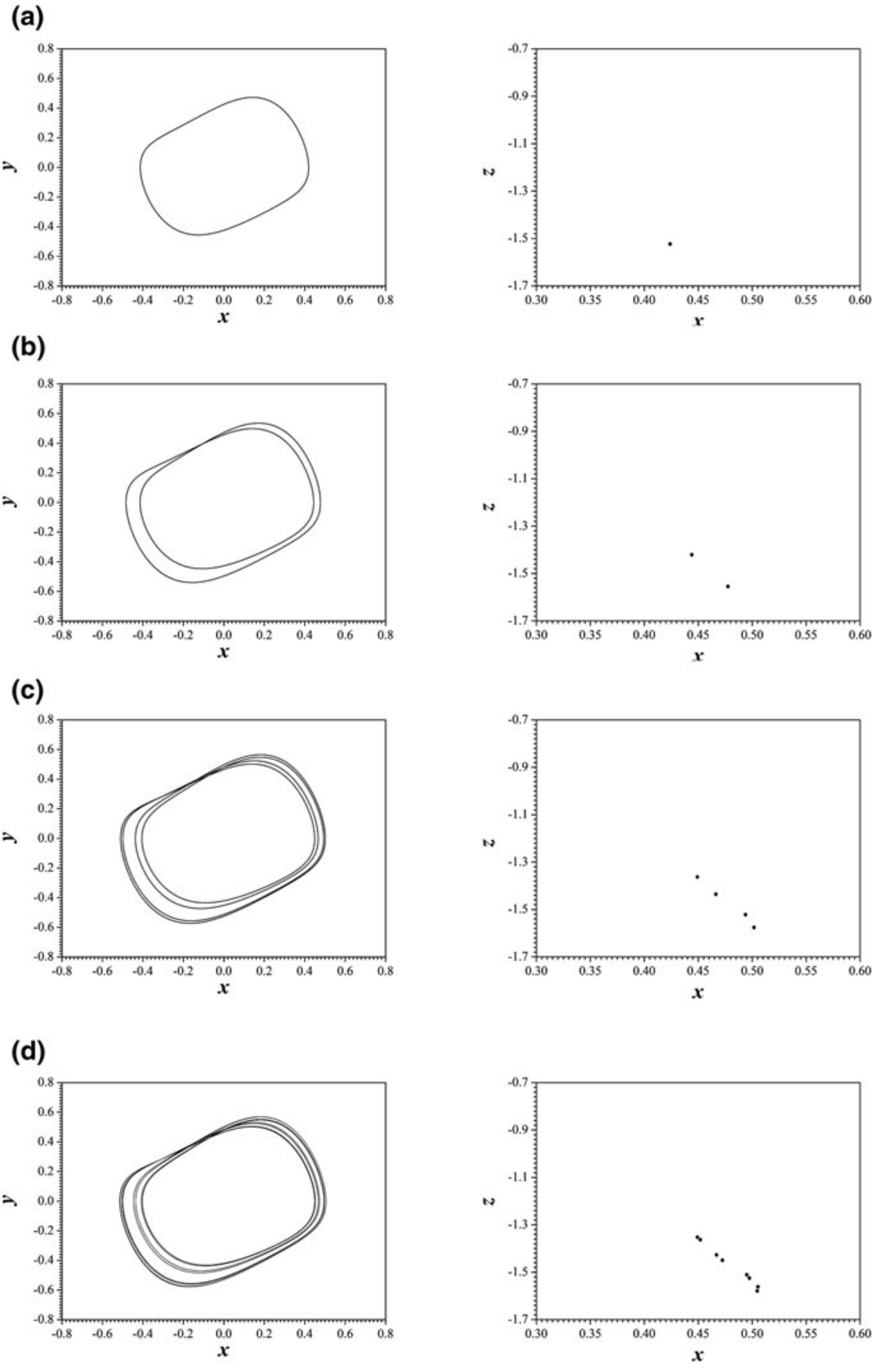


**Fig. 2** Simulation phase portraits and Poincaré maps of system (5) for **a** $a = 23$ (period-1), **b** $a = 19$ (period-2), **c** $a = 17.5$ (period-4), **d** $a = 17.3$ (period-8), **e** $a = 15$ (chaos), **f** $a = 14.5$ (period-1), with $b = 1$, $c = 0.001$ and initial conditions $(x_0, y_0, z_0) = (0, 0.5, 0.5)$

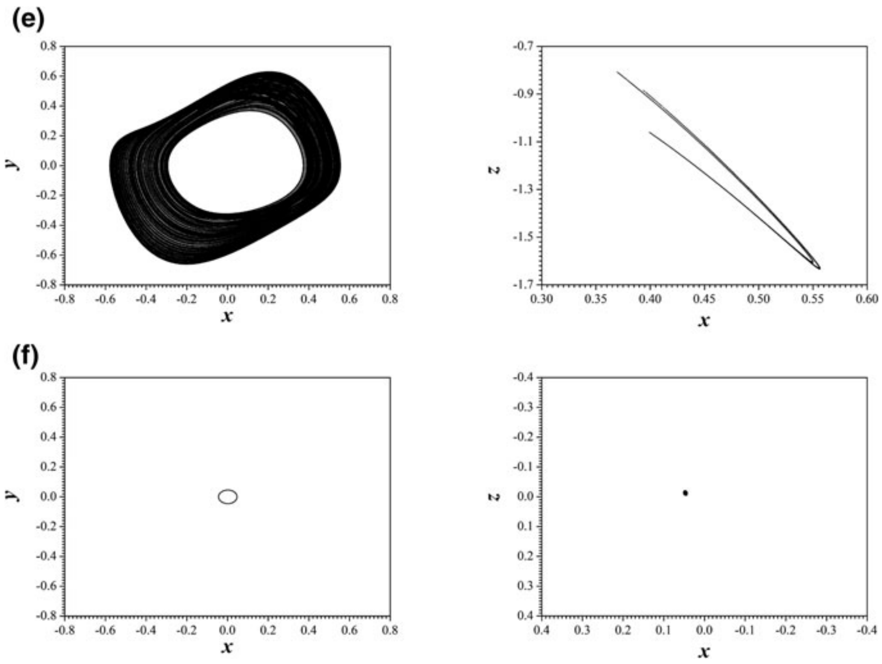


**Fig. 2** (continued)

parameter is decreased and through a crisis is resulted to a period-1 steady state. Furthermore, the corresponding spectrum of the three Lyapunov exponents is shown in Fig. 1b. It can be seen that the bifurcation diagram well coincides with the spectrum of the Lyapunov exponents. Figure 2 depicts a series of phase portraits of $y$ versus $x$ and the respective Poincaré maps of $z$ versus $x$, for various values of the parameter $a$, showing the route to chaos.

## 3 The Coupling Schemes

Generally, there are various methods of coupling between coupled nonlinear systems available in the literature. However, two are the most interesting. In the first method due to Pecora and Carroll (1990), a stable subsystem of a chaotic system could be synchronized with a separate chaotic system under certain suitable conditions. In the second method, chaos synchronization between two nonlinear systems is achieved due to the effect of coupling without requiring to construct any stable subsystem (Chua et al. 1992; Kyprianidis et al. 2005; Volos et al. 2006).

This second method can be divided into two classes: *drive-response* or *unidirectional coupling* and *bidirectional* or *mutual coupling*. In the first case, one system drives another one called the response or slave system. The system of two unidirectional coupled identical systems is described by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = F(x_1) \\ \dot{x}_2 = F(x_2) \ + C(x_1 - x_2) \end{cases} \tag{6}$$

where $F(\boldsymbol{x})$ is a vector field in a phase space of dimension $n$ and $C$ a matrix of constants, which describes the nature and strength of the coupling between the oscillators. It is obvious from (6) that only the first system influences the dynamic behavior of the other.

In the second case, both the coupled systems are connected and each one influences the dynamics of the other. This is the reason for which this method is called mutual (or bidirectional). The coupled system of two mutually coupled chaotic oscillators is described by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = F(x_1) \ + C(x_2 - x_1) \\ \dot{x}_2 = F(x_2) \ + C(x_1 - x_2) \end{cases} \tag{7}$$

In the last twenty years, many research groups approached the coupling methods between coupled chaotic systems, with the intention to study not only the cases of synchronization but also the various desynchronization phenomena. In this direction, the desynchronization in connection with a parameter mismatch between two coupled electronic oscillators has been studied (Astakhov et al. 1998). Furthermore, in (Yanchuk et al. 2001), the bifurcation sequence associated with desynchronization of a pair of coupled identical Rössler systems as the coupling parameter being reduced, has been followed. Starting with the transverse destabilization of a periodic orbit embedded in the fully synchronized chaotic state, this sequence proceeds via a torus bifurcation and regimes of anti-phase periodic and chaotic dynamics to asynchronous chaos.

## 4 Simulation Results

In this chapter, the study of the dynamic behavior of the bidirectionally and unidirectionally coupled systems with hidden attractors has been investigated numerically by employing the fourth order Runge-Kutta algorithm. Due to the fact that each one of the three system's variables and especially the variables $y$ and $z$ holds different order of nonlinearity the synchronization phenomena as well as the threshold for complete synchronization can be dependent on the selection of coupling variable. For this reason, in this work, the variable $y$ has been preferred as the coupling variable because a great variety of phenomena can be observed.

So, the system of differential equations that describes the bidirectionally coupled systems' dynamics is:

$$\begin{cases} \dot{x}_1 = -y_1 \\ \dot{y}_1 = -x_1 + y_1 z_1 + \xi(y_2 - y_1) \\ \dot{z}_1 = -x_1 - a x_1 y_1 - b x_1 z_1 + c \\ \dot{x}_2 = -y_2 \\ \dot{y}_2 = -x_2 + y_2 z_2 + \xi(y_1 - y_2) \\ \dot{z}_2 = -x_2 - a x_2 y_2 - b x_2 z_2 + c \end{cases} \tag{8}$$

The first three equations of system (8) describe the first of the two coupled identical systems with hidden attractors, while the other three describe the second one. Also, the parameter $\xi$ is the coupling coefficient and it is present in the equations of both systems, since the coupling between them is mutual.

In the case of unidirectionally coupled systems (5) the following system of differential equations is produced.

$$\begin{cases} \dot{x}_1 = -y_1 \\ \dot{y}_1 = -x_1 + y_1 z_1 \\ \dot{z}_1 = -x_1 - a x_1 y_1 - b x_1 z_1 + c \\ \dot{x}_2 = -y_2 \\ \dot{y}_2 = -x_2 + y_2 z_2 + \xi(y_1 - y_2) \\ \dot{z}_2 = -x_2 - a x_2 y_2 - b x_2 z_2 + c \end{cases} \tag{9}$$

The coupling coefficient $\xi$ is present only in the second coupled system, since only the first system affects the dynamics of the second.

The parameters of the system are chosen as: $a = 15$, $b = 1$, $c = 0.001$. With these values each one of the coupled systems with hidden attractors are in chaotic mode.

So, by solving the coupled systems' Eqs. (8) and (9) the bifurcation diagrams of the signal's difference $(x_2 - x_1)$ versus the coupling factor $\xi$ are produced. In details, these diagrams are produced by increasing the coupling factor $\xi$, from $\xi = 0$ (uncoupled systems) with step $\Delta\xi = 0.0002$, in two different ways. In the first, the initial conditions in each iteration have the same values $(x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}) = (0, 0.5, 0.5, 0.1, 0.6, 0.6)$, while in the second case the initial conditions in each iteration have different values. This occurs because the last values of the state variables in the previous iteration become the initial values for the next iteration. The second type of bifurcation diagram is more close to the experimental observation of coupled systems' dynamic behavior in many scientific fields, such as electronics, economy, biology etc.

## *4.1 Same Initial Conditions in Each Iteration*

In the first case, as it is mentioned, the initial conditions have the same values in each iteration and the bifurcations diagrams in the cases of bidirectional and unidirectional coupling schemes have been produced (Figs. 3a and 10).

The bifurcation diagram of the bidirectionally coupling system (8) shows that the coupled system undergoes from full desynchronization, for $\xi < 0.048$, where each system is in a chaotic state and lays on its own manifold, to complete chaotic synchronization, for $\xi \geq 0.39$, where their manifolds coincide, through an intermediate region where the system shows a more complex dynamic behavior. This is a typical transition from full desynchronization to complete synchronization. Simulation phase portraits of $x_2$ versus $x_1$ of the bidirectionally coupled systems (8) are depicted in Fig. 4, for various values of the coupling coefficient.

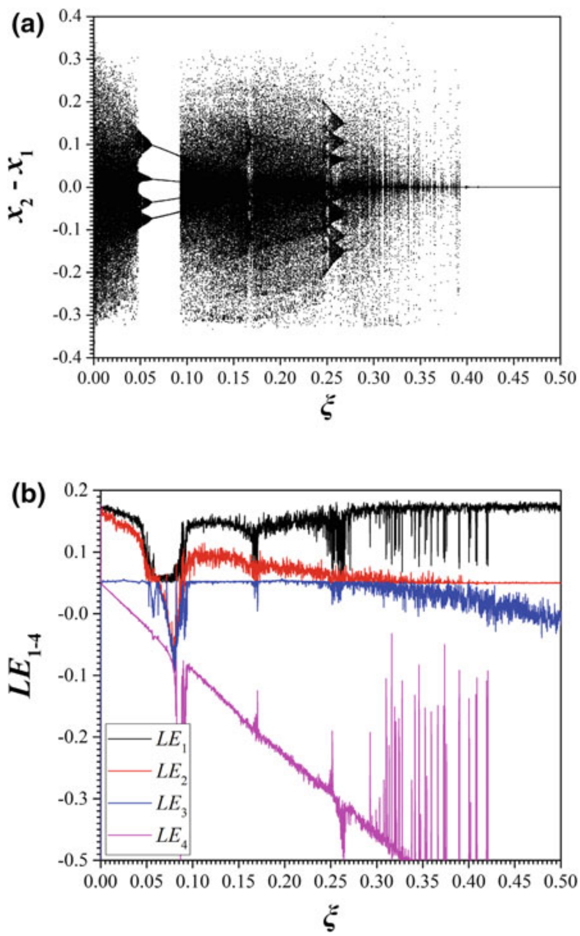


**Fig. 3** **a** Bifurcation diagram of $(x_2 - x_1)$ versus $\xi$ and **b** the spectrum of Lyapunov exponents of the bidirectionally coupling system (8), with the same initial conditions in each iteration. The parameters are $a = 15$, $b = 1$, $c = 0.001$ and initial conditions $(x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}) = (0, 0.5, 0.5, 0.1, 0.6, 0.6)$

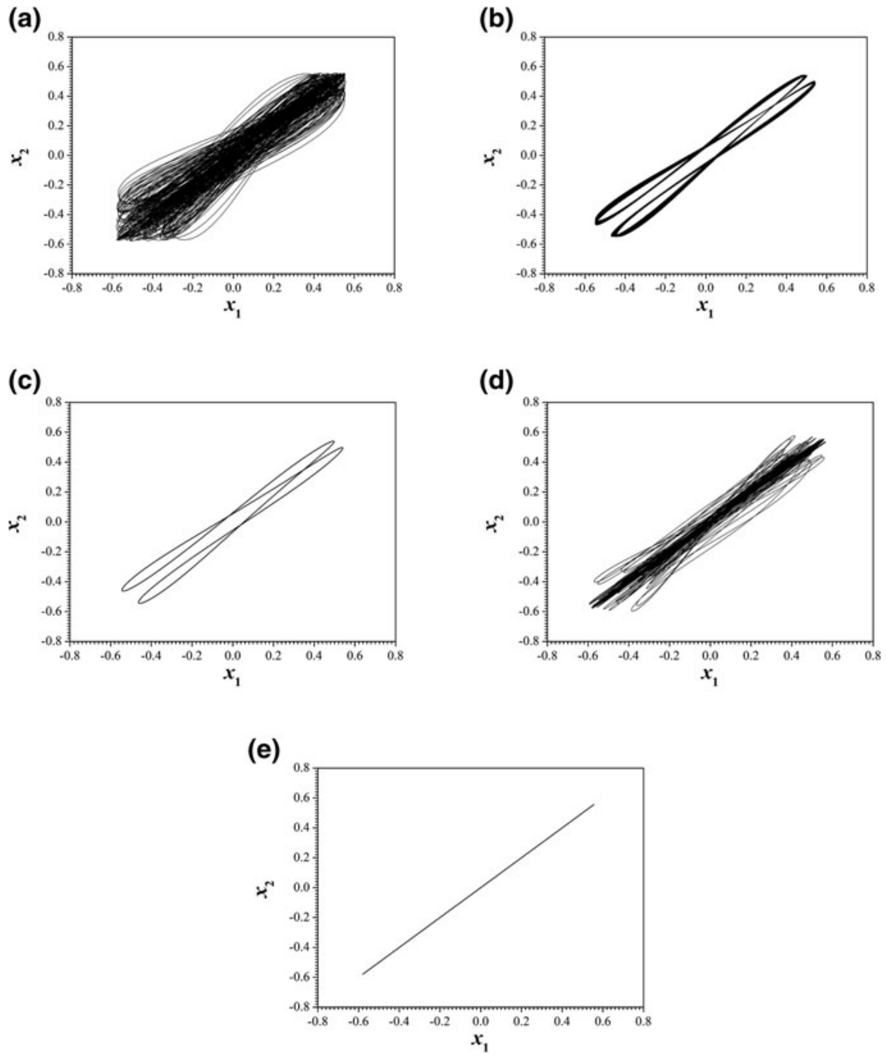


**Fig. 4** Simulation phase portraits of $x_2$ versus $x_1$ of the bidirectionally coupled system (8) with the same initial conditions in each iteration, for **a** $\xi = 0.01$ (chaotic state), **b** $\xi = 0.055$ (quasiperiodic state), **c** $\xi = 0.075$ (period-4 steady state), **d** $\xi = 0.38$ (chaotic state), **e** $\xi = 0.5$ (complete chaotic synchronization). The parameters are $a = 15$, $b = 1$, $c = 0.001$ and initial conditions $(x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}) = (0, 0.5, 0.5, 0.1, 0.6, 0.6)$

The intermediate region of the bifurcation diagram of Fig. 3a is more complicated and it can be divided in three discrete regions:

- Region I: $0.048 < \xi \leq 0.063$ (Quasiperiodic state). This type of behavior is confirmed from the spectrum of Lyapunov exponents (Fig. 3b), which i.e. for

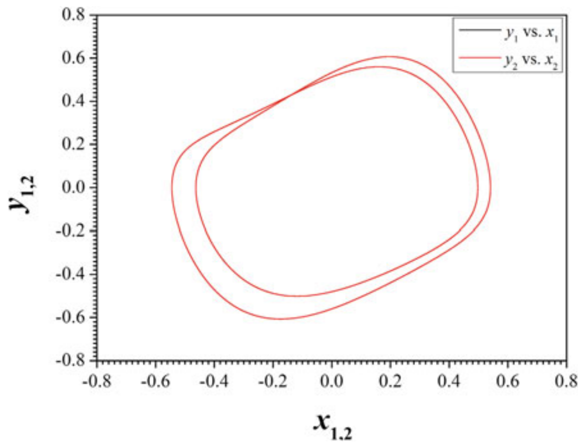


**Fig. 5** Simulation phase portrait of $y_{1,2}$ versus $x_{1,2}$, for $\xi = 0.075$ (anti-phase synchronization)

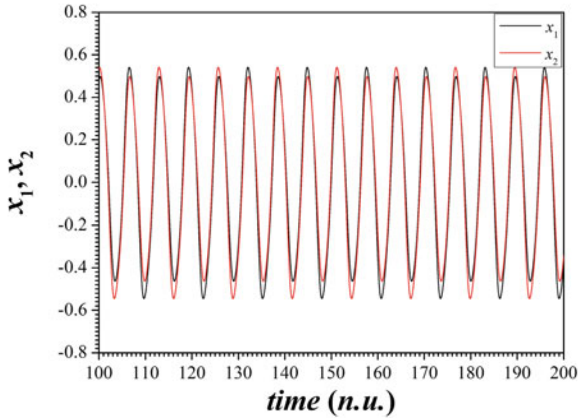


**Fig. 6** Time-series of $x_1$, $x_2$, for $\xi = 0.075$

$\xi = 0.055$ are $LE_1 = 0.0094$, $LE_2 = 0.0063$, $LE_3 = -0.2398$, $LE_4 = -0.7223$, $LE_5 = -0.7590$, $LE_6 = -0.7894$.

- Region II: $0.063 < \xi \leq 0.093$ (Period-4 steady state). In this region the coupled system shows the phenomenon of anti-phase synchronization. This occurs because each one of the coupled circuits remains in the same periodic state. Figures 5 show the simulation phase portraits of $y_{1,2}$ versus $x_{1,2}$, for $\xi = 0.075$, respectively. In this figure the coincidence of circuits' attractors in the phase plain is presented. Furthermore, in Fig. 6, the time-series of the state variables $x_1$ and $x_2$ of the coupled circuits are shown. It is obvious that the two signals $x_1$ and $x_2$ are identical with a time lag.

To quantify this time lag we have used the well-known *Similarity Function S* (Rosenblum et al. 1997).

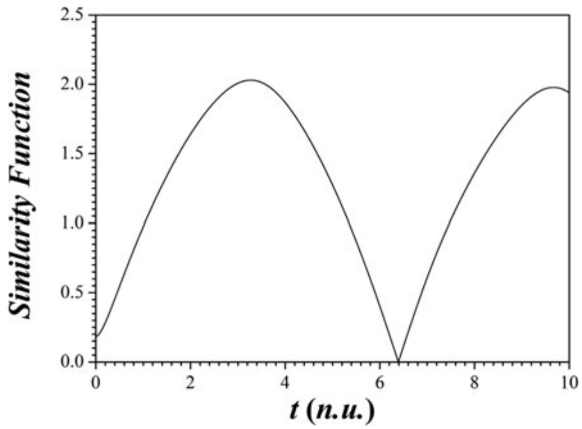


**Fig. 7** The similarity function ($S$) versus time ($t$), for $\xi = 0.075$. $S_{min} = 0$ means lag with time shift of $\tau_{min} = 6.39 = T/2$. So, the phenomenon of anti-phase synchronization is confirmed

$$S(t) = \sqrt{\frac{\langle [x_2(t+\tau) - x_1(t)]^2 \rangle}{\left[ \langle (x_1(t))^2 \rangle \cdot \langle (x_2(t))^2 \rangle \right]^{1/2}}} \tag{11}$$

Let $S_{min}$ be the minimum value of the Similarity function $S(\tau)$ and let $\tau_{min}$ be the amount of time lag, when $S_{min}$ is achieved. The time lag $\tau_{min}$ between the variables $x_1$ and $x_2$ is found, when the conditions $S_{min} = 0$ and $\tau_{min} \neq 0$ are fulfilled. The calculation of the similarity function for $\xi = 0.075$ (Fig. 7) shows that the expected time lag $\tau_{min} = 6.39$ n.u., is equal to $T/2$, where $T$ is the period of $x_1$ and $x_2$.

Furthermore, the same time lag is found for every value of coupling coefficient ($\xi$) in the Region II. So, the value of time lag remains always the same in this region and equals to the half of the period of the external voltage source. Moreover the fact that the difference of $[x_1(t) - x_2(t + T/2)]$ is equal to zero (Fig. 8), confirms that the coupled system demonstrates $\pi$ phase delay, which is defined as anti-phase

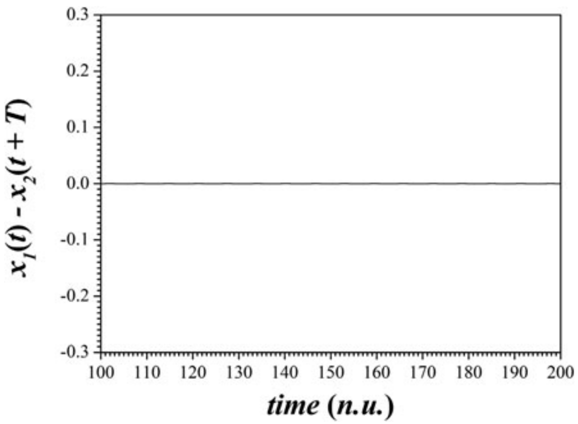


**Fig. 8** Time-series of $x_1(t) - x_2(t + T/2)$, for $\xi = 0.075$

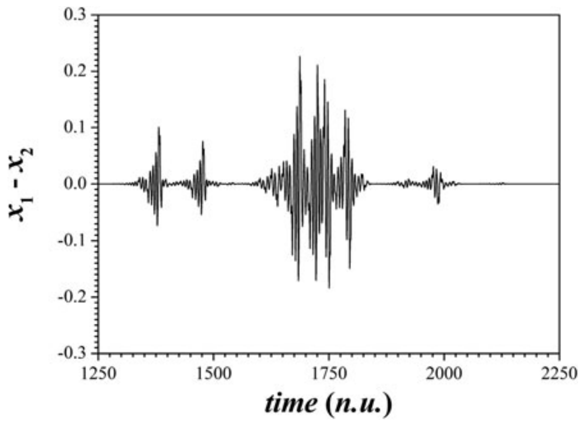


**Fig. 9** Time-series of $(x_1 - x_2)$, for $\xi = 0.38$

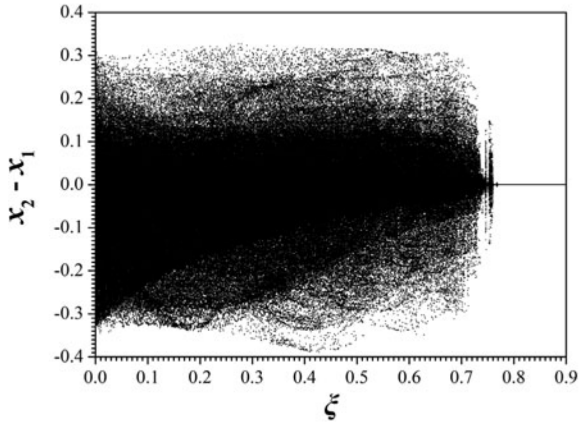


**Fig. 10** **a** Bifurcation diagram of $(x_2 - x_1)$ versus $\xi$ of the unidirectionally coupled system (9), with the same initial conditions in each iteration. The parameters are $a = 15$, $b = 1$, $c = 0.001$ and initial conditions $(x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}) = (0, 0.5, 0.5, 0.1, 0.6, 0.6)$

synchronization or π-lag synchronization. Finally, Fig. 9 shows the time-series of $(x_1 - x_2)$ in the case of system's intermittent behavior for $\xi = 0.38$.

- Region III: $0.093 < \xi \leq 0.39$ (Hyperchaotic state). This type of behavior is confirmed from the two positive Lyapunov exponents in Fig. 3b. For example the Lyapunov exponents for this type of behavior, for a value of the coupling coefficient $\xi = 0.2$, are $LE_1 = 0.113$, $LE_2 = 0.0521$, $LE_3 = 0$, $LE_4 = -0.2788$, $LE_5 = -0.7416$, $LE_6 = -0.8867$. Especially, in the region $0.31 < \xi \leq 0.39$ the system has an intermittent behavior as it is observed from the time-series of $x_1 - x_2$ for $\xi = 0.38$.

The bifurcation diagram of Fig. 10, in the case of unidirectionally coupling system (9), shows that the coupled system undergoes from full desynchronization, for $\xi < 0.76$ (Fig. 11a) directly to complete chaotic synchronization (Fig. 11b).

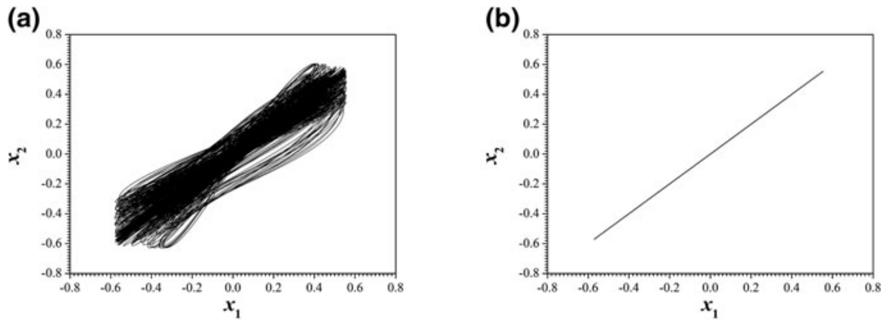


**Fig. 11** Simulation phase portraits of $x_2$ versus $x_1$ of the unidirectionally coupled system (9) with the same initial conditions in each iteration, for **a** $\xi = 0.4$ (chaotic state) and **b** $\xi = 0.8$ (complete chaotic synchronization). The parameters are $a = 15$, $b = 1$, $c = 0.001$ and initial conditions $(x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}) = (0, 0.5, 0.5, 0.1, 0.6, 0.6)$

This occurred because only the first system affects the dynamics of the second. So, there is no any complex behavior and the value of the synchronization threshold ($\xi = 0.76$) is significant higher than in the case of bidirectional coupling ($\xi = 0.39$).

## 4.2 *Different Initial Conditions in Each Iteration*

In the second case of study, the initial conditions have different values in each iteration and the bifurcation diagrams in the cases of bidirectional and unidirectional coupling schemes have been produced (Figs. 12a and 14).

The bifurcation diagram in the case of bidirectionally coupling scheme (8) shows that the coupled system undergoes from full desynchronization, for $\xi < 0.048$, to complete chaotic synchronization, for $\xi \geq 0.277$, through an intermediate region where the system shows a more complex dynamic behavior than in the respective case of bidirectional coupling of the previous case. Simulation phase portraits of $x_2$ versus $x_1$ of the bidirectionally coupled systems (8) are depicted in Fig. 13, for various values of the coupling coefficient.

In the intermediate region of the bifurcation diagram of Fig. 12a, the coupled system can be characterized by three different dynamical behavior:

- *Quasiperiodic state*. This type of behavior is observed in three different distinct regions ($\xi \in (0.054, 0.061]$, $\xi \in (0.1670, 0.1684]$ and $\xi \in (0.2620, 0.2664]$) and is confirmed by the spectrum of Lyapunov exponents of Fig. 12b.
- *Periodic state*. In the following five regions of the bifurcation diagram of Fig. 13 the system is in a periodic state. In more details:

  1. For $\xi \in (0.0491, 0.0518]$ the system is in a period-12 steady state.
  2. For $\xi \in (0.061, 0.093]$ the system is in a period-4 steady state.

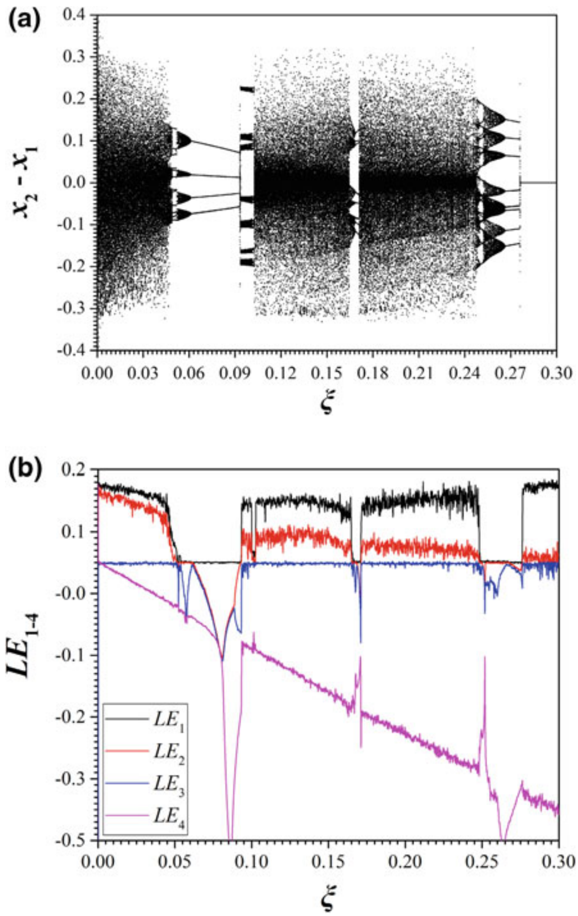


**Fig. 12** **a** Bifurcation diagram of $(x_2 - x_1)$ versus $\xi$ and **b** the spectrum of Lyapunov exponents of the bidirectionally coupling system (9), with different initial conditions in each iteration. The parameters are $a = 15$, $b = 1$, $c = 0.001$

3. For $\xi \in (0.1684, 0.1710]$ the system is in a period-8 steady state.
4. For $\xi \in (0.250, 0.252]$ the system is in a period-22 steady state.
5. For $\xi \in (0.2664, 0.2760]$ the system is in a period-8 steady state.

In all these windows of periodic behavior the coupled system shows the phenomenon of anti-phase synchronization. By calculating the Similarity function $S(\tau)$ in each case we find that the expected time lag $\tau_{min}$ is equal to $T/2$, where $T$ is the period of $x_1$ and $x_2$.

- *Hyperchaotic state*. In the rest of this intermediate region the system displays an hyperchaotic behavior, as it is observed from the respective phase portraits of Fig. 13a, f, and i, as well as from the spectrum of the Lyapunov exponents of Fig. 12b.

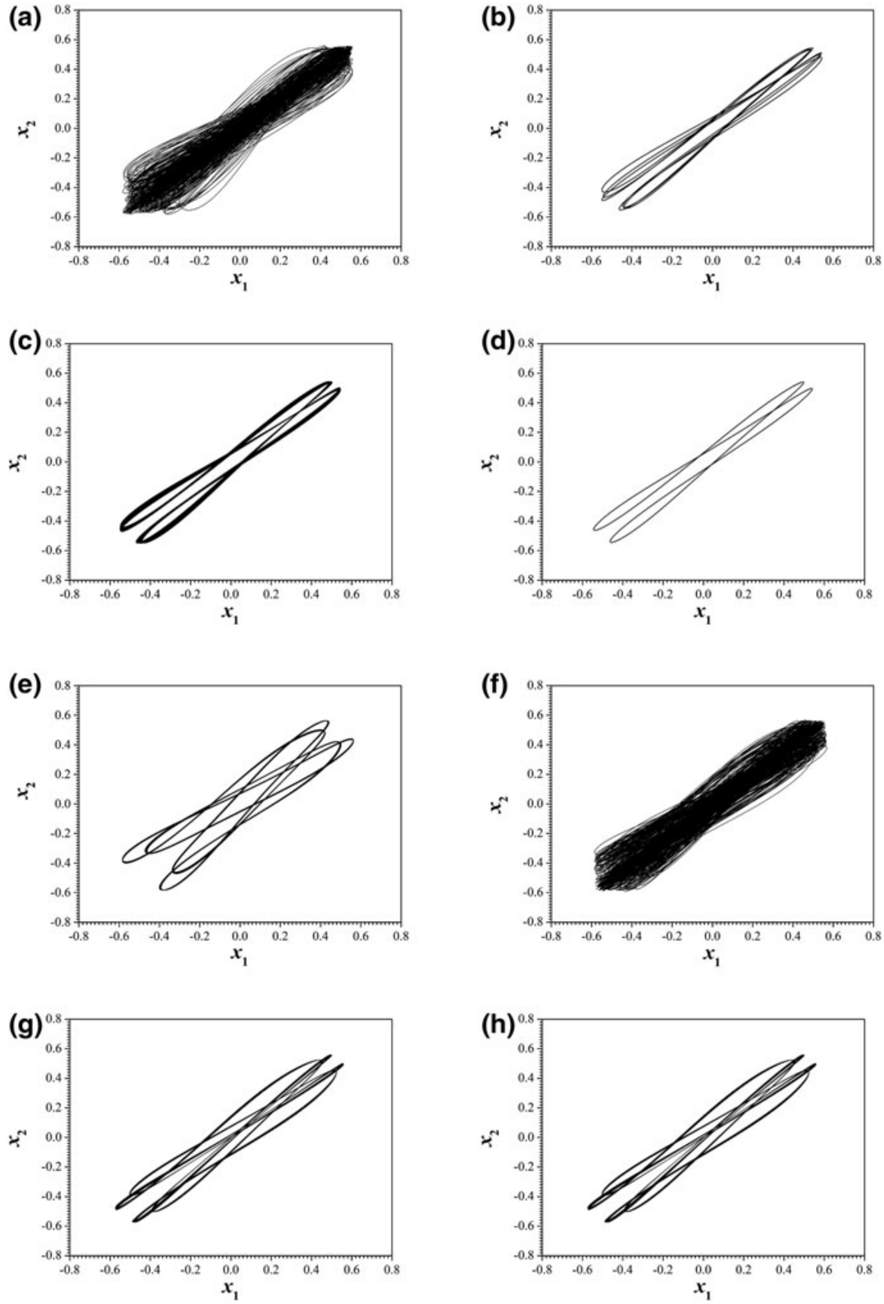


**Fig. 13** Simulation phase portraits of $x_2$ versus $x_1$ of the bidirectionally coupled system (9) with different initial conditions in each iteration, for **a** $\xi = 0.03$ (hypechaotic state), **b** $\xi = 0.05$ (periodic state), **c** $\xi = 0.055$ (quasiperiodic state), **d** $\xi = 0.075$ (periodic state), **e** $\xi = 0.10$ (hyperchaotic state), **f** $\xi = 0.14$ (hyperchaotic state), **g** $\xi = 0.1678$ (quasiperiodic state), **h** $\xi = 0.17$ (periodic state), **i** $\xi = 0.2$ (chaotic state), **j** $\xi = 0.251$ (periodic state) **k** $\xi = 0.263$ (quasiperiodic state), **l** $\xi = 0.27$ (periodic state), **m** $\xi = 0.28$ (complete chaotic synchronization). The parameters are $a = 15$, $b = 1$, $c = 0.001$

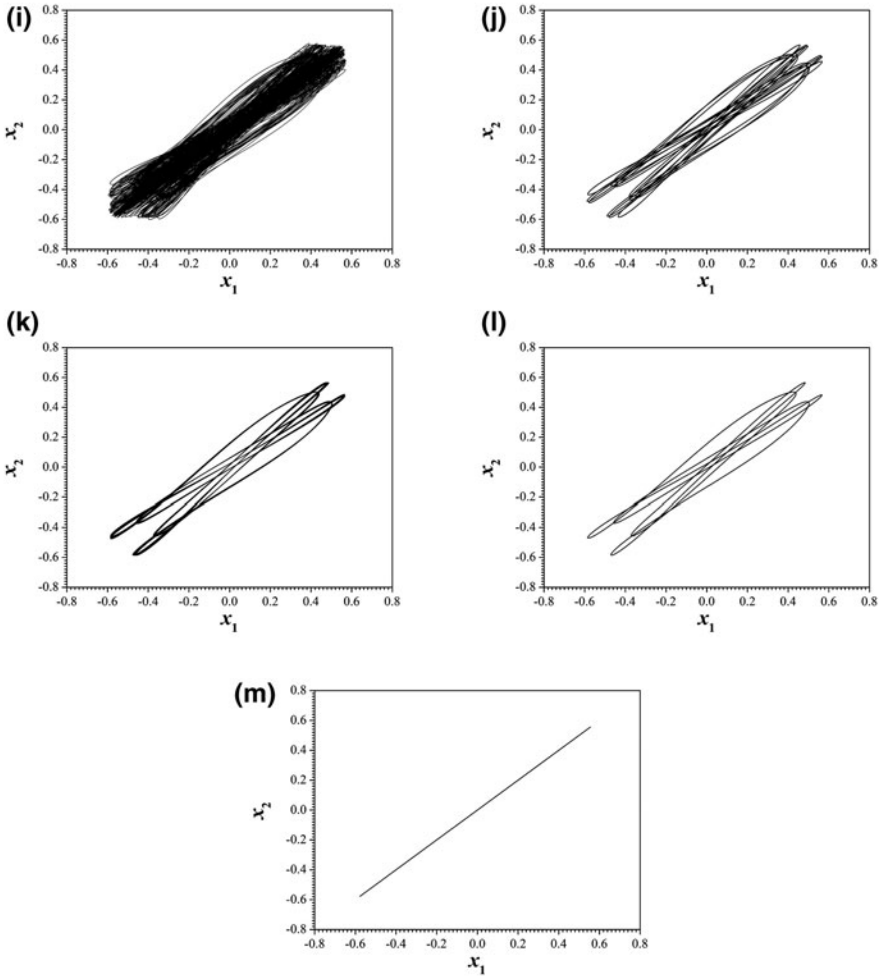


**Fig. 13** (continued)

Finally, from the bifurcation diagram (Fig. 14) in the case of unidirectionally coupling system (9) we can conclude that the coupled system undergoes from full desynchronization, for $\xi < 0.55$ directly to complete chaotic synchronization, without appearing any complex dynamical behavior, while the value of the synchronization threshold ($\xi = 0.55$) is significant higher than in the case of bidirectional coupling ($\xi = 0.\ 0.277$).

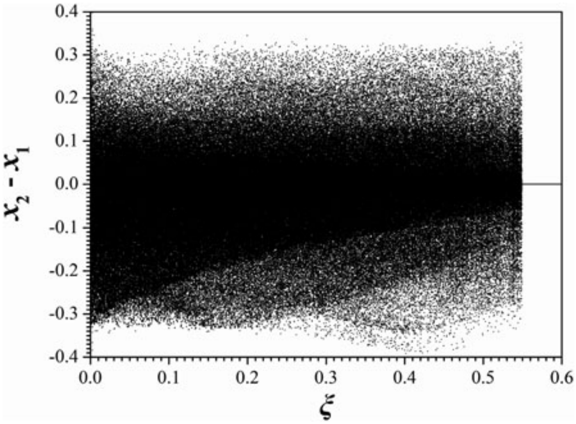


**Fig. 14** **a** Bifurcation diagram of $(x_2 - x_1)$ versus $\xi$ of the unidirectionally coupled system (10), with different initial conditions in each iteration. The parameters are $a = 15$, $b = 1$, $c = 0.001$

# 5 Conclusion

In the present chapter, a gallery of various synchronization phenomena between resistively coupled identical nonlinear systems with hidden attractors was presented. For this reason, two coupling schemes were adopted. The first one was the well-known bidirectional coupling while the second one was the unidirectional coupling. In each coupling scheme two different study cases related with systems' initial conditions were also adopted. The initial conditions in each iteration had the same values in the first case, while in the second one the initial conditions in each iteration had different values.

In more details, in the bidirectional coupling scheme, with the same initial conditions in each iteration, the coupled systems undergone from full desynchronization, where each system was in a chaotic state to complete chaotic synchronization, through an intermediate region where the coupled systems were in a periodic states showing the phenomenon of anti-phase synchronization. In the case of unidirectionally coupled systems, the coupled system undergone from full desynchronization directly to complete chaotic synchronization, without showing any other complex dynamics.

Similarly, the coupling schemes (bidirectional and unidirectional), with different initial conditions in each iteration, appeared the same route from desynchronization to complete chaotic synchronization. However, in the bidirectional coupling scheme, a more complex dynamics was arisen as the system had more periodic windows where the phenomenon of anti-phase synchronization was presented.

As a future work, a more exhaustive study of coupling schemes between identical dynamical systems with other types of hidden attractors will be done.

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