Chapter 17
When Is a Generic Argument a Proof?

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Abstract We discuss whether a generic argument can be considered a proof. Two positions on this question have recently been published which focus on the fussiness of an argument as a deciding criterion. We take a third view that takes into account psychological and social factors. Psychologically, for a generic argument to be a proof it must result in a convincing deductive reasoning process occurring in the mind of the reader. Socially, for a generic argument to be a proof it must conform to the social conventions of the context. For classroom settings, we suggest two kinds of evidence that should be reflected in written work in order for a generic argument to be accepted as a proof. These kinds of evidence reveal the linkage between the psychological and social factors.

Keywords Generic arguments · Proof · Social perspectives · Psychological perspectives · Evidence

Introduction

A generic argument "involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class. The account involves the characteristic properties and structures of a class, while doing so in terms of the names and illustration of one of its representatives" (Balacheff 1988, p. 219). The "characteristic representative of its class" is called a *generic example* (Mason and Pimm 1984). Such arguments have been discussed in the mathematics education literature since Mason and Pimm introduced the terms "generic example" and

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"generic proof", and this discussion has frequently included debate about if and when generic argument are acceptable mathematical proofs. For example, Leron and Zaslavsky (2013) write,

> The main weakness of a generic proof is, obviously, that it does not really prove the theorem. The "fussiness" of the full, formal, deductive proof is necessary to ensure that the theorem's conclusion infallibly follows from its premises. (p. 27)

Yopp et al. (2015) contest this,

> We … propose that students can make and judge the viability of generic example arguments and that in certain situations … these arguments can be accepted as proof. (p. 10)

In this chapter we take up this debate, and argue that the main criterion used by Leron and Zaslavsky as well as Yopp, Ely, and Johnson-Leung to distinguish proofs from non-proofs, "fussiness", is not in fact the criterion that is critical in determining whether an argument using a generic example is a mathematical proof. We take an alternative view that takes into account also psychological and social factors.

Fussiness

Movshovitz-Hadar (1988) also maintains that a generic argument cannot be a mathematical proof.

> The proof of a generic example should not be confused with a fully general proof. It only suggests the full proof through a generalizable concrete example. From the purely logical point of view there is no replacement for the formal proof. (p. 18)

In mathematics education there is wide agreement that formal proofs are needed. The NCTM (2000) Standards say that students should understand that a proof is an argument "consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM 2000, p. 56). The need for logically rigorous deductions based on previously established propositions is the "fussiness" called for by Leron and Zaslavsky and which Yopp, Ely, and Johnson-Leung claim generic argument can achieve "in certain situations".

But is fussiness actually a characteristic of mathematical proofs? Not if it is absolute and complete fussiness.

> For many mathematical investigations, full mathematical formalization and complete formal proof, even if possible in principle, may be impossible in practice. They may require time, patience, and interest beyond the capacity of any human mathematician. Indeed, they can exceed the capacity of any available or foreseeable computing system. (Hersh 1993, p. 390)

Aberdein (2012) elaborates Epstein's (2012) model of proof based on mathematical practice. He characterizes mathematical proofs as an argument with two parallel structures, one argumentational and the other inferential. The inferential structure has absolute and complete fussiness. Every step is a deduction based on and justified by previous propositions. It is fully formal. But it is never actually presented. As Hersh points out this is usually impossible. Instead, using the argumentational structure “mathematicians attempt to convince each other of the soundness of the inferential structure” (Aberdein 2012, p. 362).

> This account both conserves and transcends the conventional view of mathematical proof. The inferential structure is held to strict standards of formal rigour, without which the proof would not qualify as mathematical. However, the step-by-step compliance of the proof with these standards is itself a matter of argument, and susceptible to challenge. Hence much actual mathematical practice takes place in the argumentational structure. (p. 363)

mathematical proofs are not “logically rigorous deductions of conclusions from hypotheses” as the NCTM asserts. Instead, they are arguments that such deductions exist. As Hardy observed long ago, proofs do not prove in the formal sense, they point.

> If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call gas, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. (Hardy 1928, p. 18, his emphasis)

Furthermore proofs are not completely fussy.

> Mathematical arguments, just like arguments in our daily lives, leave much unsaid. And of what is said, much is only hints or sketches, with lots explicitly left to the reader. (Epstein 2012, p. 269)

Psychological Factors

Whether or not generic argument are fussy, we believe this is not the criterion that determines if they are proofs, as proofs are not completely fussy either. To be proofs, generic argument must fulfill the function of argumentative structures, to *point* to the inferential structure, to affect psychology, to stimulate the imagination. As Fischbein (1982) states, “there are frequent situations in mathematics in which a formal conviction, derived from a formally certain proof, is NOT associated with the subtle feeling of ‘It must be so’, ‘I feel it must be so’” (p. 11). So, for some readers this stimulation of the reader’s mind might be more difficult to achieve using fussy formal proofs than with generic arguments that qualify as proofs.

As examples of generic argument, we include the three following arguments for the claim: “Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$”.

Argument 1

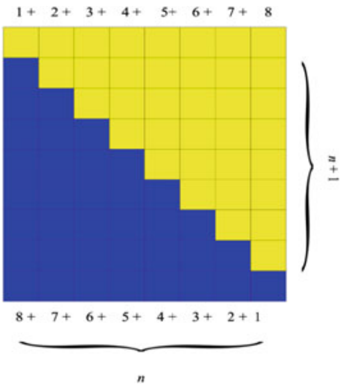
Consider the sum $1+2+3+4+5+6+7+8+9+10$. Write this sum, and the reverse, and add them:

$$\begin{array}{l} 1+2+3+4+5+6+7+8+9+10 \\ \underline{10+9+8+7+6+5+4+3+2+1} \\ 11+11+11+11+11+11+11+11+11+11 = 10 \times 11 \end{array}$$

Because the sum was added to itself, dividing 10×11 by 2 gives the sum.

Argument 2

Consider the sum: $1+2+3+4+5+6+7+8$



$$\Rightarrow 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Argument 3

n is either odd or even.

First, consider an odd n, for example 7. Then the sum is $1+2+3+4+5+6+7$. You can rearrange this is to 3 pairs: $1+7, 2+6, 3+5$, all adding up to 8, with the 4 in the middle left out. So the sum is $3 \times 8 + 4$, or in general $\left(\frac{n-1}{2}\right)(n+1) + \left(\frac{n+1}{2}\right)$, which simplifies to $\left(\frac{n(n+1)}{2}\right)$.

Next, consider an even n, for example 8. Then the sum is $1+2+3+4+5+6+7+8$. You can rearrange this is to 4 pairs: $1+8, 2+7, 3+6, 4+5$ all adding up to 9. So the sum is 4×9, or in general $\left(\left(\frac{n}{2}\right)(n+1)\right)$, which simplifies to $\left(\frac{n(n+1)}{2}\right)$.

At the level of school mathematics, Stylianides (2007b) provides a conceptualization of the meaning of proof that was built taking into account the four major elements of any argument:

> The four elements are the argument's *foundation* (i.e., what constitutes its basis: definitions, axioms, etc.), *formulation* (i.e., how it is developed: as a logical deduction, as a generalization from particular cases, etc.), *representation* (i.e., how it is expressed: using everyday language, algebraically, etc.), and *social dimension* (i.e., how it plays out in the social context of the community wherein it is created). (Stylianides 2007a, p. 2)

In Stylianides' terminology the term "formulation" is ambiguous. Does it refer to the reasoning going on in the mind of the author, or in the mind of the reader, or is it independent of any mind? Our examples illustrate this ambiguity. For instance, as Argument 1's authors we were thinking of 10 as a generic example, and the argument as a proof, as it would work in exactly the same way for any number. But a reader might assume we chose 10 as an example that is sufficiently large to be a typical number, so that if it works for 10 it would probably work for other numbers. In Balacheff's (1988) terminology, the reader sees 10 as a "crucial experiment", not a generic example. Or in Aberdein's (2012) terms we intended our argument to point to a deductive inferential structure, but the reader might not follow that pointer. Thus the argument has two possible "formulations". In one the argument uses logical deduction on a generic example, and we would consider it a proof. In the other, it is a generalization from a particular case that has been chosen to be typical, but is not seen as general. This ambiguity means "formulation" cannot be independent of a mind; the determination of the formulation of an argument depends on a psychological process occurring in a reader (who might be the author of the argument). To be a proof we believe a generic argument must be truly generic, and that depends on a psychological process that might be different for different readers. Something similar can be seen with the other two arguments above.

Social Factors

Even if a generic argument is psychologically a proof, in that it points to a deductive inference structure, it may still not be socially acceptable as a proof. Stylianides (2007a) argues that "the convincing power of an argument is by itself not enough to capture the social dimension of proof in school settings" (p. 12). We believe that what is psychologically convincing and what is socially convincing are mostly different. "A proof becomes a proof after the social act of 'accepting it as a proof'. This is true of mathematics as it is in physics, linguistics, and biology" (Manin 1977, p. 48). And it is also true in classrooms.

> An argument that could count as proof in a classroom community should be accepted as proof by the community – and, thus, it should be convincing to the students – on the basis of socially accepted rules of discourse that are compatible with those of wider society. (Stylianides 2007a, p. 15)

Hence, the arguments above might be differently considered from psychological and social perspectives. It might be a proof in a school community or in ancient Greece, but might not be a proof in the university mathematical community.

Limiting our view of proofs by just taking into account the "fussiness" of an argument would not allow us to acknowledge that proof and proving exist also in social settings, like schools, with their own criteria for proof.

As we noted, at the level of the mathematicians' community the criteria are broadly known by all members belonging to that community. But what happens at school level? Do school students know "a priori" what a generic argument is? Do they really understand this kind of arguments (either those they write or read)? How can a mathematics (school) teacher know his/her students actually understand generic arguments? What would be the classroom criteria for agreeing if and when an argument is a proof? Can those class rules be determined? How?

We believe that one key point to take into account here is clear rules guiding the classroom work in the context of proof, which should also include the case of generic arguments. We return to this in the next section.

Implications for Education: Connecting Psychological and Social Factors

Many authors have pointed out the importance of working with generic examples (e.g., Balacheff 1988; Kempen and Biehler 2015; Malek and Movshovitz-Hadar 2011; Mason and Pimm 1984). In any case, we may say that generic arguments are powerful tools as they can make proof construction accessible to students at any level.

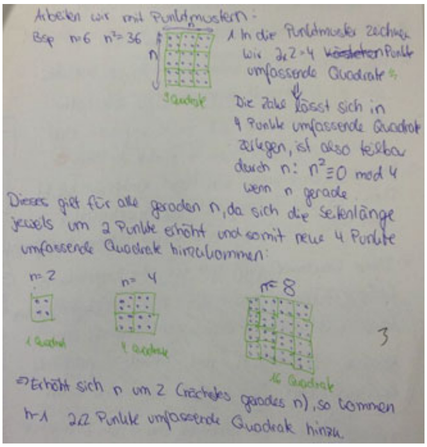
As we have outlined above, determining if a generic argument is a proof or not cannot be done solely on the basis of a characteristic of the text itself, like 'fussiness'. This is the 'absolutist' perspective Stylianides et al. (2016) refer to when discuss perspectives that can be considered in relation to the function(s) of a proof. We adopt what they call a 'subjectivist' perspective, in which psychological processes occurring in readers and the author of the text are considered, and in addition we consider the standards for proof in the community. We suggest an intertwined relationship between the psychological and the social factors as a way to include the use and understanding of generic arguments in classroom settings.

In mathematics teaching, the teacher is setting and the students are learning the classroom standards for proof in part through the acceptance or rejection of arguments. If generic arguments are to be accepted in classrooms (as Yopp et al. 2015, among others, have advocated) then it is important to provide teachers and students with a framework in which to decide if a generic argument is a proof or not. There also needs to be a framework for the social acceptance of proofs that overcomes the limitation that each individual has access only to their own psychological processes.

In the following sections we outline some criteria that could form such a framework. This framework seeks to establish a bridge between the psychological and social factors considered above. This interconnection between these two factors is relevant in the sense that it can promote the kind of explicitness necessary in classrooms when discussing whether or not a generic argument is a proof. And as Selden (2012) notes, "Understanding and constructing such proofs entails a major transition for students but one that is often supported by relatively little explicit instruction" (p. 392). Even though the author refers to proofs in general, we believe that in this context, making this framework explicit in classrooms when working with generic arguments might help, first, students to be more aware of what their considerations are when involved in generic arguments writing, and second, it might also help teachers to have in some way access to the students' psychology.

The Need for Further Examples

Fischbein (1982) observed that once a mathematics statement has been proven, there should be no need for further examples. While this is not categorically true, as examples can have purposes beyond verification (see Lockwood et al. 2012), a significant difference between a generic example and a *specific* example is that additional *specific* examples add nothing to a generic example. A generic example provides a model for the generation of endless specific examples, removing the need to actually produce them. However, multiple specific examples can be important in the formulation of an argument, as a systematic variation of examples can be used to reveal the structure of a generic example (see Fig. 17.1).



The statement being proven is that the square of an even number is always divisible by 4. The text reads "*We work with a dot pattern: E.g.,* $n = 6$ $n^2 = 36$
1st. In the dot pattern we draw $2 \times 2 = 4$ *dots making squares*
⇒ *The number can be divided into squares encompassing 4 dots, and so is* <u>*divisible by n*</u>*:*
$n^2 \equiv 0 \bmod 4$ *if* n *even.*
This is true for all even n*, because as the side length increases by 2 dots, so new squares encompassing 4 dots are added.*
As n *increases by 2 (next even* n*),* $n-1$ *squares encompassing* 2×2 *dots are added.*"

In the underlined portion, we believe that when the student wrote divisible by n she meant divisible by 4.

Fig. 17.1 A generic proof using several examples to show how the structure applies to other cases

In the classroom, the teacher can ask whether the students believe they need more examples to verify a statement, and based on their need for more examples and the use they make of those examples (to provide empirical evidence or to reveal structure) the teacher can determine if the students understood the examples as generic or specific. On the other hand, in written work (for instance in written tests) one might need to have other kind of evidence. We will return to this point later (see *Reconstructing psychology from text*).

Understanding

An important difference between a evidence and an argument with a specific example, or even a set of specific examples, is that a generic argument can be explanatory. It is difficult to define exactly what makes a proof explanatory. Steiner (1978) suggests that

> An explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response. (p. 143)

When applied to a proof using a generic example, this suggests that for an example to be (psychologically) a generic example, it must be possible to see that as the example varies in some ways the theorem remains true. For example, in Argument 1, the final number in the sum can be larger or smaller, but the numbers must be consecutive and must begin with 1 for the argument to work. The argument can reveal to a reader the properties of the example that are important and in this way they explain the theorem. This means that a reader can answer questions like “Why divide by 2?”, “Why multiply by n + 1?” when given either Argument 1 or Argument 2. Not all arguments are equally explanatory. Argument 3 makes it more difficult to understand the division by 2, because the reason for it is slightly different in the two cases. In contrast, however, a simple example only shows that the formula works for that example, not *why*.

Reconstructing Psychology from Text

In written work, the teacher does not have the immediate opportunity to ask the student questions to determine their thinking. And it is sometimes necessary to decide if a written text is a generic argument that qualifies as a proof without asking further questions (for example when evaluating examinations). In educational settings, it is desirable for both teachers and students to have clear criteria for the evidence that should be included in written work. This evidence should be sufficient to allow the teacher to determine whether the students' attention is on particular/specific examples or a general structure. We suggest that two kinds of evidence should be included in written work:

(1) Evidence of awareness of generality;
(2) Mathematical evidence of reasoning.

Evidence of awareness of generality can be revealed by phrases such as "the same reasoning can be used for the other cases", or "it also applies to the other cases involved", or as in the example seen in Fig. 17.1, the student includes: "this is true for all even *n*". This kind of evidence shows that the author is actually aware that she is working with an argument that is general enough to be valid in all cases involved in her statement. This evidence must be part of the awareness the students should have when presenting a generic argument. The main reason of considering this as relevant evidence is the need to be sure whether or not the students are aware that they are not only dealing with empirical evidence, but that their work shows general structures through the use of their examples. If this is the case, they should include this as part of their written work.

Mathematical evidence of reasoning, reveals the form of the reasoning behind the argument. This kind of evidence mainly points to the mathematical reasons for why the same structure can be extrapolated for other cases from the example(s) given, and it is based not only on the conditions of the problem given but also on the ground knowledge the community shares at that point (the social aspect). For example, in Fig. 17.1, the student is using certain assumptions which seem to be accepted in the context of her class: the square of a number "n" is a square of dots with "n" rows and "n" columns of dots; a number is divisible by 4 if you can make groups of 4 dots without having any dot without grouping it, etc. And based on these assumptions and her data (she is only working with even numbers), she provides the mathematical reasons of why the conclusion holds through the use of her examples.

Both kinds of evidence are relevant when working with generic arguments. One might think of a student including the first kind of evidence in her written work, but if she does not provide the reasons (mathematical evidence of reasoning) that support her "apparent" awareness of generality, then the student's argument could not qualify as a proof in this classroom environment. Or vice versa, if a student works on a general well-structured argument through the use of examples, but if she does not see it as general (she is not aware of this generality), then it is (psychologically) not a proof for that student. In any case, it is a challenge for teachers to determine whether or not an argument based on an example (or a set of examples) is a generic proof without having sufficient evidence of both kinds.

In this context, the following questions can be considered to guide students when writing generic arguments:

(1) Did you state that the argument can be applied to all other cases in discussion? Can it?

(2) Did you describe the reasons (sufficient deductive evidence) behind the generic argument? That is, have you identified the underlying structure in the example or examples, and shown why it occurs in every case?

We think that having in mind these two kinds of evidence, the students can be more aware of what is expected from them when writing a generic argument so it is clearer that they regard it whether or not as a proof, and it might help a reader reconstruct the author's psychology.

These classroom criteria for the kind of evidence that should be provided must be known by the classroom community. They are also part of the social context for generic arguments in the classroom. Indeed, these criteria establish a link between the psychological and social factors since they indicate both what the author finds convincing using deductive reasoning and whether the argument conforms to the classroom conventions.

For example, in Argument 1, it is not clear enough that the author is aware that the argument applies to any natural number. A reader might have this awareness of generality, but it is a mistake to assume an author has the same awareness without having evidence. Argument 1[a] shows a variation of Argument 1 that includes evidence of awareness of generality.

Argument 1[a]

Consider the sum $1+2+3+4+5+6+7+8+9+10$. Write this sum, and the reverse, and add them:

$$\begin{array}{l} 1+2+3+4+5+6+7+8+9+10 \\ \underline{10+9+8+7+6+5+4+3+2+1} \\ 11+11+11+11+11+11+11+11+11+11 = 10\times 11 \end{array}$$

Because the sum was added to itself, dividing 10×11 by 2 gives the sum.

The same reasoning can be used for any natural number n, and not only for the case of 10.

The addition of the final line provides evidence of awareness, but the form of the reasoning behind the argument is still not clear. Argument 1[b] shows a variation of Argument 1[a] that also includes mathematical evidence that reveals the form of the reasoning behind the argument.

Argument 1[b]

Consider the sum $1+2+3+4+5+6+7+8+9+10$. Write this sum, and the reverse, and add them:

$$\begin{array}{l} 1+2+3+4+5+6+7+8+9+10 \\ \underline{10+9+8+7+6+5+4+3+2+1} \\ 11+11+11+11+11+11+11+11+11+11 = 10\times 11 \end{array}$$

> Adding down always gives 11 (one more than the last term in the sum) because in the first case we are adding $10 + 1$ (the last term plus 1), and then we are adding a number that is one more (1 becomes 2) to a number that is one less (10 becomes 9). The numbers are consecutive, so the increase in the top row is the same as the decrease in the bottom row. There is one sum adding down for every number in the top row, which is 10 in this case. So we multiply the highest number in the sum (10) by one more than the highest number (11). The product is two times bigger than it should be, because we added $(1 + 2 + \cdots + 9 + 10)$ twice, so to find the real sum we divide the product (10×11) by 2. The same reasoning can be used for any natural number n, and not only for the case of 10.

With the addition of these last lines the status of Argument 1 becomes less ambiguous. As readers trying to reconstruct the psychology of the author of Argument 1, the evidence included in Argument 1[b] gives us a basis to believe that the author was aware of the generality of the argument, and used deductive reasoning to arrive at the conclusion. Thus this argument meets the criteria for a generic argument to be a proof in the classroom. It not only makes the reader aware of the general character of the argument, but also reflects the author's awareness of this generality.

Kempen and Biehler (2015) call the text we added in Argument 1[b] "narrative reasoning" and say such a text should accompany an argument using generic examples in order for it to be considered a proof.

> It is the narrative reasoning that follows the generic examples, which makes a generic proof a valid general argument. So it gets possible to stress the differences between purely empirical examples and valid general arguments. (p. 137)

However, we do not suggest that the *only* way to provide mathematical evidence is with the use of narratives. Some students might feel confident using written words to express their ideas of generality, but others might struggle with linguistic formulations and be better able to use other representations to express the same idea. Argument 1[c] (in Fig. 17.2) shows an alternative way to present Argument 1 [b], without the use of much written language.

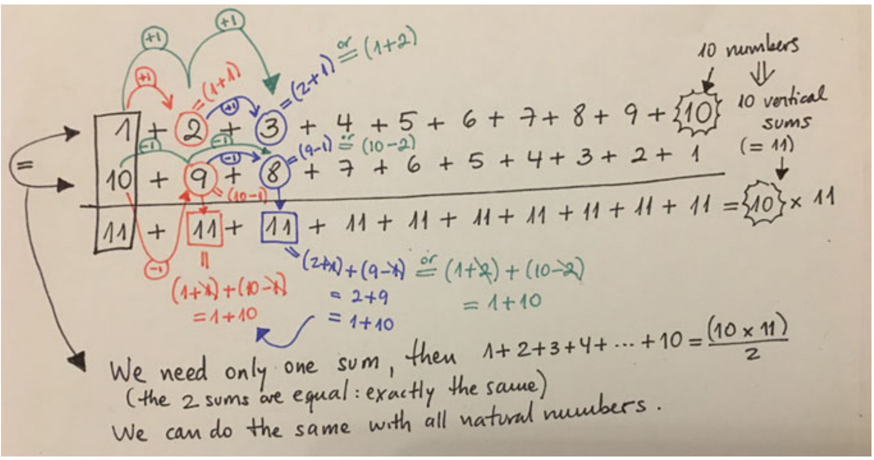


Fig. 17.2 Argument 1[c] in which markings and labels replace most of the verbal narrative

In both cases (Arguments 1[b] and 1[c]) the general structure of the argument has been pointed out, and there is evidence of awareness of generality. Hence these two examples of generic arguments can be considered proofs, according to the criteria suggested for the context of written work in a classroom.

Conclusion

Examples can be used when providing an argument that might qualify as a proof. Depending on whether or not the general structure through the use of examples has been pointed out in an argument (with the use of narratives, or other representations), students can count on a more accessible way to present proofs in classroom settings.

In this article we have argued that the criterion of fussiness is inadequate to decide if generic arguments are proofs. Instead we suggest two other requirements, one psychological and one social. Psychologically, for a generic argument to be a proof it must result in a general deductive reasoning process occurring in the mind of the reader that convinces the reader that there exists a fully deductive inference structure behind the argument. Socially, for a generic argument to be a proof it must conform to the social conventions of the context. In school classrooms or in ancient Greece a generic argument might be acceptable proof. In a university classroom exactly the same argument might not be. Searching for properties of an argument that make it a proof is insufficient because being a proof depends on psychological and social factors independent of the argument. In school classrooms the social conventions are partly based on mathematical criteria, but also on the need in schools for students to convince the teacher that the relevant awareness and reasoning occurred.

Mason and Pimm (1984) raised several questions about generic examples when introducing the concept:

> How can you expose the genericity of an example to someone who sees only its specificity? Apart from stressing and ignoring, and repeating the general statement over and over, how can the necessary act of perception, of seeing the general in the particular, be fostered?
>
> How can you discern the extent of the generality perceived by someone else when looking at a particular example together?
>
> Why do we offer students examples in class, and what are they supposed to make of them? If examples are always examples of something, how can students become aware of that which the examples are supposed to be exemplifying? (pp. 287–288)

We hope that we have addressed in part these questions, and that we have contributed to the ongoing conversation on the role of generic examples in proof and proving.

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