Chapter 2
Basic Applied Functional Analysis

To measure data and solutions spatially, we recall a number of useful definitions and results on Lebesgue and standard Sobolev spaces. Then, we introduce more specialized Sobolev spaces, which are better suited to measuring solutions to electromagnetics problems, in particular, the divergence and the curl of fields. This also allows one to measure their trace at interfaces between two media, or on the boundary. Last, we construct *ad hoc* function spaces, adapted to the study of time- and space-dependent electromagnetic fields.

For bibliographical references on the general results, we refer the reader to [3, 4, 62, 91–93, 114, 124, 125, 157, 166, 185, 199, 207]. For some of the more specialized results, we provide references along the way.

2.1 Function Spaces for Scalar Fields

Unless otherwise specified, the function spaces will be defined on a subset of $\mathbb{R}^n$ (possibly $\mathbb{R}^n$ itself). The definitions and properties that we list hereafter can depend on the category of subsets of $\mathbb{R}^n$ on which they are given. We shall consider three categories: (C1) open subsets, (C2) open subsets with Lipschitz boundary, and (C3) bounded, open connected subsets with Lipschitz boundary, also called *domains*. The last category will include an important subcategory, the *curved polyhedra*, that is, domains with a piecewise smooth, curved boundary.

An element $\alpha = (\alpha_1, \cdots, \alpha_n)$ of $\mathbb{N}^n$ is called a multi-index, with $|\alpha| = \sum_{j=1}^n \alpha_j$. The partial derivative of order α is further denoted by

$$\partial_\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Let $d\boldsymbol{x} = dx_1 dx_2 \cdots dx_n$ denote the Lebesgue measure in $\mathbb{R}^n$.

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Category (C1) Open subsets of $\mathbb{R}^n$.

Consider a set Ω that belongs to the category (C1).

Let us begin with the Lebesgue spaces $L^p(\Omega)$, for $1 \le p \le \infty$. One usually considers complex-valued functions, but all definitions are easily extended to real-valued function spaces. Details on Banach and Hilbert spaces, and also on the duality and interpolation theories, can be found in Sect. 4.1.

Definition 2.1.1 The space $L^p(\Omega)$ is composed of all complex-valued, Lebesgue-measurable functions f on Ω, and such that

$$\begin{cases} \text{for } 1 \le p < \infty & \|f\|_{L^p(\Omega)} := \left\{ \int_\Omega |f|^p \, d\boldsymbol{x} \right\}^{1/p} < \infty \\ \text{for } p = \infty & \|f\|_{L^\infty(\Omega)} := \operatorname{esssup}_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| < \infty \end{cases}.$$

Endowed with the norm $\| \cdot \|_{L^p(\Omega)}$, $L^p(\Omega)$ is a Banach space and, for $1 \le p < \infty$, is separable.

Let $p \in [1, \infty]$, $f_1 = f_2$ in $L^p(\Omega)$ mean that $f_1, f_2 \in L^p(\Omega)$ and $f_1 = f_2$ almost everywhere in Ω. One can then define the spaces of functions that are *locally* in L^p in the following way. If[1] $f \mathbf{1}_K$ belongs to $L^p(\Omega)$ for every compact subset K of Ω, then f is locally in $L^p(\Omega)$, and one writes

$$f \in L^p_{loc}(\Omega).$$

One then has a stability result of the multiplication by elements of $L^\infty(\Omega)$.

Proposition 2.1.2 *Let $1 \le p \le \infty$. The multiplication is a continuous bilinear mapping from $L^\infty(\Omega) \times L^p(\Omega)$ to $L^p(\Omega)$.*

Given $1 \le p \le \infty$, one defines its *conjugate exponent* p' as $1/p + 1/p' = 1$. The Hölder inequality yields the next result.

Proposition 2.1.3 *Let $1 \le p \le \infty$ and p' be its conjugate exponent. Then, given $(f, g) \in L^p(\Omega) \times L^{p'}(\Omega)$, one has $fg \in L^1(\Omega)$.*

One can build dual spaces of the Lebesgue spaces.

Proposition 2.1.4 *Let $1 \le p < \infty$ and p' be its conjugate exponent. Then, the dual space of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$: $(L^p(\Omega))' = L^{p'}(\Omega)$. On the other hand, $L^1(\Omega) \subset (L^\infty(\Omega))'$ but $(L^\infty(\Omega))' \ne L^1(\Omega)$.*

Emphasis is then laid on the $L^2(\Omega)$ space, which is, in addition, a separable Hilbert space.

Proposition 2.1.5 *The space $L^2(\Omega)$ is a separable Hilbert space, endowed with the scalar product*

$$(f|g) := \int_\Omega f \, \overline{g} \, d\boldsymbol{x}.$$

[1] Given any subset S of $\mathbb{R}^n$, $\mathbf{1}_S$ denotes the *indicator function* of S.

Let us recall now some basic ideas about distributions, including the definition of differentiation in the sense of distributions. We begin with the space $\mathcal{D}(\Omega)$ of infinitely differentiable functions,[2] with compact support on Ω. Classically, this function space is not reduced to $\{0\}$. In practice, one can use the convergence of sequences to define the topology. Let $(f_k)_k$ be a sequence of elements of $\mathcal{D}(\Omega)$: it converges in $\mathcal{D}(\Omega)$ to f if, and only if:

(i) there exists a compact subset K of Ω such that $\mathrm{supp}(f_k) \subset K$, for large enough k;
(ii) for all multi-indices α, $(\partial_\alpha f_k)_k$ converges uniformly in K to $\partial_\alpha f$.

Definition 2.1.6 A linear and continuous form T defined on $\mathcal{D}(\Omega)$ is called a distribution. The space of distributions is denoted by $\mathcal{D}'(\Omega)$.

Let $T \in \mathcal{D}'(\Omega)$ and $f \in \mathcal{D}(\Omega)$: the action of T on f is written with the help of duality brackets, that is,

$$\langle T, f\rangle.$$

According to the topology on $\mathcal{D}(\Omega)$, T is continuous, provided that

$$\forall (f_k)_k, f \in \mathcal{D}(\Omega) \text{ such that } f_k \to f \text{ in } \mathcal{D}(\Omega), \quad \langle T, f_k\rangle \to \langle T, f\rangle.$$

A few examples will be provided in the sequel (2.1), (2.5), (2.6). As a dual space, $\mathcal{D}'(\Omega)$ can be equipped in a "natural" way with a topology, called the weak-star topology.

Definition 2.1.7 Let $(T_k)_k$ be a sequence of elements of $\mathcal{D}'(\Omega)$: it converges in $\mathcal{D}'(\Omega)$ to T if, and only if, for all f in $\mathcal{D}(\Omega)$, $\langle T_k, f\rangle \to \langle T, f\rangle$.

One can easily prove the imbedding

$$L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega), \tag{2.1}$$

by identifying elements f of $L^1_{loc}(\Omega)$ with distributions, still denoted by f, according to

$$\forall g \in \mathcal{D}(\Omega), \quad \langle f, g\rangle = \int_\Omega f\, g\, d\boldsymbol{x}. \tag{2.2}$$

Since, for $p \in [1, \infty]$, one has $L^p(\Omega) \subset L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$, one can also consider elements of $L^p(\Omega)$ or $L^p_{loc}(\Omega)$ as distributions. In particular, given $f \in L^2(\Omega)$, one has $\langle f, g\rangle = (f|\overline{g})$ for all $g \in \mathcal{D}(\Omega)$.

Let us recall a property that will be used throughout this book, namely…

[2] The space $\mathcal{D}(\Omega)$ can also be denoted by $C_c^\infty(\Omega)$, where the index $_c$ stands for compact support.

Proposition 2.1.8 *Let f_1 and f_2 be two elements of $L^1_{loc}(\Omega)$. The relation $\langle f_1, g\rangle = \langle f_2, g\rangle$ for all $g \in \mathcal{D}(\Omega)$ implies that $f_1 = f_2$ almost everywhere in Ω.*

Now, one can introduce the notion of differentiation in the sense of distributions.

Definition 2.1.9 Let $T \in \mathcal{D}'(\Omega)$. Its j-th partial derivative $(j = 1, \cdots, n)$ is defined by

$$\forall f \in \mathcal{D}(\Omega), \quad \langle \frac{\partial T}{\partial x_j}, f\rangle = -\langle T, \frac{\partial f}{\partial x_j}\rangle.$$

One thus has...

Proposition 2.1.10 *The mapping $T \mapsto \partial_j T$ is linear and continuous from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)$.*

Since $L^2(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$ (by identification, cf. (2.2)), it is therefore possible to differentiate its elements in the sense of distributions. We define below the first Sobolev space in a long series.

Definition 2.1.11 Let $H^1(\Omega) := \{f \in L^2(\Omega) \ : \ \partial_j f \in L^2(\Omega), \ j = 1, \cdots, n\}$, where differentiation is understood in the sense of distributions (Definition 2.1.9). An associated norm is

$$\|f\|_{H^1(\Omega)} := \left\{ \int_\Omega (|f|^2 + |\,\mathbf{grad}\, f|^2)\, d\boldsymbol{x} \right\}^{1/2}.$$

It is a separable Hilbert space, endowed with the scalar product

$$(f, g)_{H^1(\Omega)} := \int_\Omega (f\,\overline{g} + \mathbf{grad}\, f \cdot \overline{\mathbf{grad}\, g})\, d\boldsymbol{x}.$$

It is also possible to give an equivalent definition of $H^1(\Omega)$.

Proposition 2.1.12 *Let $f \in L^2(\Omega)$. Then, f belongs to $H^1(\Omega)$ if, and only if, there exist $C_1, \cdots, C_n \geq 0$, such that, for $j = 1, \cdots, n$,*

$$\forall g \in \mathcal{D}(\Omega), \quad \left| (f | \frac{\partial g}{\partial x_j}) \right| \leq C_j \|g\|_{L^2(\Omega)}.$$

Now, let α be a multi-index. From Definition 2.1.9, one recursively deduces...

Definition 2.1.13 Let $T \in \mathcal{D}'(\Omega)$; its partial derivative of order α is defined by

$$\forall f \in \mathcal{D}(\Omega), \quad \langle \partial_\alpha T, f\rangle = (-1)^{|\alpha|}\langle T, \partial_\alpha f\rangle.$$

When $\alpha = (0, \cdots, 0)$, there is no differentiation involved!

This allows us to consider Sobolev spaces of integer order m, $m \geq 2$.

Definition 2.1.14 Let $m \in \mathbb{N}$: $H^m(\Omega) := \{f \in L^2(\Omega) \; : \; \partial_\alpha f \in L^2(\Omega), \; \forall \alpha \in \mathbb{N}^n, \; |\alpha| \leq m\}$. The canonical norm is

$$\|f\|_{H^m(\Omega)} := \left\{ \int_\Omega \sum_{\alpha \in \mathbb{N}^n, \; |\alpha| \leq m} |\partial_\alpha f|^2 \, d\boldsymbol{x} \right\}^{1/2} . \tag{2.3}$$

It is a separable Hilbert space, endowed with the scalar product

$$(f, g)_{H^m(\Omega)} := \int_\Omega \sum_{\alpha \in \mathbb{N}^n, \; |\alpha| \leq m} \partial_\alpha f \, \overline{\partial_\alpha g} \, d\boldsymbol{x}.$$

Finally, $|\cdot|_{H^m(\Omega)}$ denotes the semi-norm

$$|f|_{H^m(\Omega)} := \left\{ \int_\Omega \sum_{\alpha \in \mathbb{N}^n, \; |\alpha| = m} |\partial_\alpha f|^2 \, d\boldsymbol{x} \right\}^{1/2} . \tag{2.4}$$

Remark 2.1.15 If $m = 1$, the two definitions of $H^1(\Omega)$ coincide, whereas if $m = 0$, one has $H^0(\Omega) = L^2(\Omega)$.

Then, one can introduce fractional-order Sobolev spaces, that is, with order $s \in \mathbb{R}_+ := [0, \infty[$. Let us consider the case $\Omega = \mathbb{R}^n$, for which one can use the Fourier transform from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Classically, for $f \in L^2(\mathbb{R}^n)$, the Fourier transform of f is $\hat{f}$, given by

$$\forall \boldsymbol{k} \in \mathbb{R}^n, \quad \hat{f}(\boldsymbol{k}) = (2\pi)^{-n} \int_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) \exp(-\imath \boldsymbol{k} \cdot \boldsymbol{x}) \, d\boldsymbol{x} \, .$$

In particular, one has $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|f\|_{L^2(\mathbb{R}^n)}$.

Definition 2.1.16 Let $s \in \mathbb{R}_+$: $H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \; : \; (1 + |\cdot|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n)\}$, with norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \left\{ \|\hat{f}\|^2_{L^2(\mathbb{R}^n)} + \|(1 + |\cdot|^2)^{s/2} \hat{f}\|^2_{L^2(\mathbb{R}^n)} \right\}^{1/2} .$$

It is a Hilbert space, endowed with the scalar product

$$(f, g)_{H^s(\mathbb{R}^n)} := (\hat{f}, \hat{g})_{L^2(\mathbb{R}^n)} + \left((1 + |\cdot|^2)^{s/2} \hat{f}, (1 + |\cdot|^2)^{s/2} \hat{g} \right)_{L^2(\mathbb{R}^n)} .$$

Obviously, when $s \in \mathbb{N}$, $H^s(\mathbb{R}^n)$ coincides algebraically and topologically with the space of Definition 2.1.14 (case $\Omega = \mathbb{R}^n$).

When Ω is an open subset of $\mathbb{R}^n$, let us define $H^s(\Omega)$ for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ by interpolation.

Definition 2.1.17 Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$, and write $s = m + \sigma$, with $(m, \sigma) \in \mathbb{N}\times]0, 1[$. The Hilbert space $H^s(\Omega)$ is the interpolated space

$$H^s(\Omega) := [H^{m+1}(\Omega), H^m(\Omega)]_{1-\sigma}.$$

Its norm and scalar product are denoted by $\|\cdot\|_{H^s(\Omega)}$ and $(\cdot,\cdot)_{H^s(\Omega)}$.

Remark 2.1.18 The above Definition is motivated by the fact that, when $\Omega = \mathbb{R}^n$, the definitions via the Fourier transform and the interpolation theory coincide algebraically and topologically.

One can then define the spaces of functions that are *locally* in H^s in the following way. If f belongs to $H^s(\omega)$ for every open subset ω of every compact subset of Ω, then f is locally in $H^s(\Omega)$, and one writes

$$f \in H^s_{loc}(\Omega).$$

One has the continuous imbeddings, for $t > s > 0$,

$$\mathcal{D}(\Omega) \subset H^t(\Omega) \subset H^s(\Omega) \subset L^2(\Omega). \tag{2.5}$$

To extend the scale of Sobolev spaces to negative fractional order, let us build dual spaces of the Sobolev spaces $H^s(\Omega)$, $s \geq 0$. As a matter of fact, one *instead* considers the dual spaces of

$$H^s_0(\Omega) := \text{closure of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega), \text{ for } s \geq 0.$$

As a closed subspace of $H^s(\Omega)$, $H^s_0(\Omega)$ is a separable Hilbert space. The motivation is twofold:

- By a density argument, one can replace elements of $H^s_0(\Omega)$ with elements of $\mathcal{D}(\Omega)$.
- When the boundary of Ω is bounded and appropriately smooth, $H^s_0(\Omega)$ can be characterized as a subspace of $H^s(\Omega)$, the elements of which fulfill some homogeneous boundary conditions (see Theorem 2.1.62 and Remark 2.1.64.)

NB. It holds that $H^s_0(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, for all $s \geq 0$.

Definition 2.1.19 For $s \geq 0$, the dual space of $H^s_0(\Omega)$ is called $H^{-s}(\Omega)$.

The action of elements of $H^{-s}(\Omega)$ on elements of $H^s_0(\Omega)$ is denoted with the help of duality brackets: $\langle\cdot,\cdot\rangle_{H^s_0(\Omega)}$.

Its canonical norm is denoted by $\|\cdot\|_{H^{-s}(\Omega)}$:

$$\|f\|_{H^{-s}(\Omega)} := \sup_{v \in H^s_0(\Omega), v \neq 0} \frac{\langle f, v\rangle_{H^s_0(\Omega)}}{\|v\|_{H^s(\Omega)}}.$$

Endowed with $\|\cdot\|_{H^{-s}(\Omega)}$, $H^{-s}(\Omega)$ is a Banach space. Furthermore, as the dual of a (separable) Hilbert space, $H^{-s}(\Omega)$ can be made into a (separable) Hilbert space, with a scalar product $(\cdot,\cdot)_{H^{-s}(\Omega)}$ such that $\|f\|^2_{H^{-s}(\Omega)} = (f, f)_{H^{-s}(\Omega)}$ for all $f \in H^{-s}(\Omega)$.

Proposition 2.1.20 *Let $m \in \mathbb{N}$. The space $H^{-m}(\Omega)$ is made up of distributions of the form*

$$\sum_{\alpha\in\mathbb{N}^n,\ |\alpha|\le m} \partial_\alpha f_\alpha, \text{ with } f_\alpha \in L^2(\Omega).$$

Identifying $L^2(\Omega)$ with its dual space, one has the continuous imbeddings, for $t > s > 0$,

$$L^2(\Omega) \subset H^{-s}(\Omega) \subset H^{-t}(\Omega) \subset \mathcal{D}'(\Omega). \tag{2.6}$$

In order to deal with functions that are defined on a *proper* subset of the actual domain of interest, one has (unfortunately) to introduce a final class of Sobolev space…

Definition 2.1.21 Let $s \ge 0$. The space $\widetilde{H}^s(\Omega)$ is composed of elements f of $H^s(\Omega)$ such that the continuation of f by zero outside Ω belongs to $H^s(\mathbb{R}^n)$. The dual space of $\widetilde{H}^s(\Omega)$ is denoted by $\widetilde{H}^{-s}(\Omega)$.

Now, let us consider functions that are defined up to the boundary, i.e., on $\overline{\Omega}$. To that aim, we need some additional assumptions, which are summarized below.

Category (C2) Open subsets of $\mathbb{R}^n$, with a *Lipschitz boundary*.

Definition 2.1.22 Let Ω be an open subset of $\mathbb{R}^n$, with boundary Γ. the boundary Γ is said to be *Lipschitz* if, and only if:

- at each point $\boldsymbol{x}$ of Γ, there exists a *Lipschitz-continuous* mapping (defined on a hypercube of $\mathbb{R}^{n-1}$ with values in $\mathbb{R}$), the graph of which locally represents Γ in a neighborhood of $\boldsymbol{x}$;
- at each point $\boldsymbol{x}$ of Γ, Ω is *locally on one side only* of Γ.

Similarly, the boundary is said to be C^k (respectively $C^{k,1}$) for $k \in \mathbb{N}^*$, when all local mappings are of regularity C^k (respectively $C^{k,1}$).[3]

[3]Classically, for $k \in \mathbb{N}$, $\beta \in]0, 1]$, $\mathcal{O} \subset \mathbb{R}^n$, $C^{k,\beta}(\mathcal{O})$ is the Hölder space defined by

$$C^{k,\beta}(\mathcal{O}) := \{f \in C^k(\mathcal{O})\ :\ \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|=k} \sup_{\boldsymbol{x}\ne\boldsymbol{y}} \frac{|\partial_\alpha f(\boldsymbol{x}) - \partial_\alpha f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\beta} < \infty\},$$

where $C^k(\mathcal{O}) := \{f \in C^0(\mathcal{O})\ :\ \partial_\alpha f \in C^0(\mathcal{O}),\ \forall\alpha \in \mathbb{N}^n,\ |\alpha| \le k\}$.

Lipschitz-continuity coincides with $C^{0,1}$ continuity.

Remark 2.1.23 When Γ is Lipschitz, it is, in particular, a Lipschitz submanifold of $\mathbb{R}^n$. On the one hand, the interior Ω_i and the exterior Ω_e of a cube belong to the category (C2). On the other hand, a set with a boundary including cusps, cuts or slits does not…

It is then *a priori* possible to define the *unit outward normal vector* to the boundary of an open set of category (C2), where, by outward, it is understood that the vector points out of Ω.

Definition 2.1.24 In an open set Ω of category (C2), one denotes by $\boldsymbol{n}$ the unit outward normal vector to its boundary Γ.

Proposition 2.1.25 *In an open set Ω of category (C2), the unit outward normal vector field $\boldsymbol{n}$ is defined almost everywhere on Γ, and furthermore, $n_i \in L^\infty(\Gamma)$, $i = 1, \cdots, n$.*

Remark 2.1.26 In an open subset of $\mathbb{R}^n$ with $C^{k,1}$ boundary ($k \in \mathbb{N}^*$), it holds that $n_i \in C^{k-1,1}(\Gamma)$, $i = 1, \cdots, n$.

In such open sets of $\mathbb{R}^n$, it is possible to establish very convenient *density* results. Let us first introduce a set of smooth functions.

Definition 2.1.27 The space $C_c^\infty(\overline{\Omega})$ is composed of the restrictions to $\overline{\Omega}$ of C^∞ functions with compact support in $\mathbb{R}^n$.

Proposition 2.1.28 *Let $s \geq 0$. In an open set Ω of category (C2), $C_c^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$.*

It is because Ω is locally on *only* one side of its boundary that one can define elements of $C_c^\infty(\overline{\Omega})$ as restrictions. This property allows one to establish the previous Proposition. Another closely related result is…

Proposition 2.1.29 *Let $s \geq 0$. In an open set Ω of category (C2), $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$.*

These results are also related to *restriction* and *continuation* properties that we recall below.

Proposition 2.1.30 *Let $s \geq 0$, and let Ω be an open set of category (C2).*

Then, the restriction operator $u \mapsto u_{|\Omega}$ is continuous from $H^s(\mathbb{R}^n)$ to $H^s(\Omega)$.

Proposition 2.1.31 *Let $s \geq 0$, and let Ω be an open set of category (C2) with a bounded boundary.*

Then, there exists a continuous (linear) continuation operator E from $H^s(\Omega)$ to $H^s(\mathbb{R}^n)$, independent of s, such that, for all $u \in H^s(\Omega)$, $(Eu)_{|\Omega} = u$.

Remark 2.1.32 If, in addition, Ω is bounded, one can choose a closed ball $\mathcal{O}$ containing Ω such that for all $u \in H^s(\Omega)$, Eu is supported in $\mathcal{O}$.

<u>**Category**</u> (<u>**C3**</u>): *bounded*, open and *connected* subsets of $\mathbb{R}^n$ with a Lipschitz boundary. A set of category (C3) will be called a *domain* later on.

NB. Ω_i belongs to the category (C3), but Ω_e does not.

Let us review some practical instances of open sets Ω of the category (C3), in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In $\mathbb{R}^2$, open sets bounded by a polygonal boundary automatically fall into this category: these are called *polygons*.

This is also the case for *curvilinear polygons*, defined as follows. An open subset Ω of $\mathbb{R}^2$ of the category (C3) has a C^2 curvilinear polygonal boundary Γ if, for all points G of Γ, there exists $r_G > 0$ and a diffeomorphism χ_G, such that χ_G is a piecewise, C^2-diffeomorphism that maps the neighborhood $\overline{\Omega} \cap B(G, r_G)$ of G to a neighborhood of the origin O, included in the plane sector $P_G := \{(r\cos\theta, r\sin\theta) : r \geq 0,\ \theta \in [0; \omega_G]\}$ of opening $\omega_G \in]0; 2\pi[$, G being sent to O.

In the same spirit, one can define *spherical curvilinear polygons*, as open subsets of the sphere $\mathbb{S}^2$ that fulfill the same property (existence of a piecewise, C^2-diffeomorphism) at all boundary points.

All of the above belong to the class of *curvilinear polygons*. Loosely speaking, the boundary of a curvilinear polygon is a manifold with corners.

In $\mathbb{R}^3$, one can consider a set Ω with a boundary Γ, made of a finite set of *planes faces*, i.e., a polyhedral boundary. Note that, contrary to the sets of $\mathbb{R}^2$, there actually exist bounded open sets with a polyhedral boundary, which do not fulfill the second requirement, stating that at each point of Γ, Ω is locally on one side of Γ. An example is pictured below: let Ω_0 be an open set, interior to the "two sugarcubes". In any neighborhood of the point C, which is located at the intersection of boundary edges, Ω_0 is not only on one side of its boundary.

One can also define *curved polyhedra*. Let us consider an open subset Ω of $\mathbb{R}^3$ of the category (C3): Ω has a C^2 curved polyhedral boundary Γ if, for all points G of Γ, there exists $r_G > 0$ and a diffeomorphism χ_G, such that χ_G is a piecewise, C^2-diffeomorphism that maps the neighborhood $\overline{\Omega} \cap B(G, r_G)$ of G to a neighborhood of the origin O, included in the cone $C_G := \left\{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x}/|\boldsymbol{x}| \in S_G\right\}$, with S_G a spherical curvilinear polygon of $\mathbb{S}^2$, G being sent to O.

Subsets of $\mathbb{R}^3$ of the category (C3) with a polyhedral boundary, or with a curved polyhedral boundary, are called *curved polyhedra*.

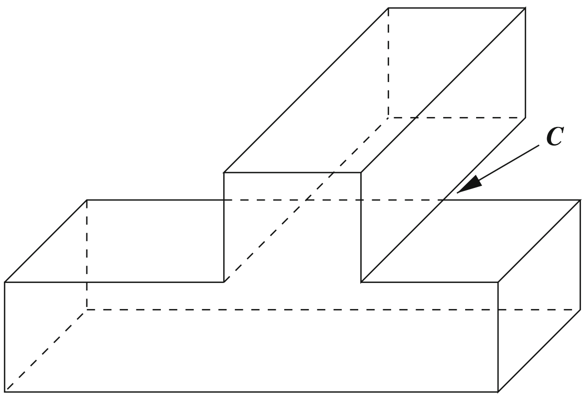

Fig. 2.1 The "two sugarcubes"

Finally, let us mention briefly *axisymmetric domains* of $\mathbb{R}^3$, which are generated by the rotation of a polygon around one of its edges (these will be of use in Chap. 9). More precisely, the rotation occurs around a line, the so-called axis, that contains this edge.

Remark 2.1.33 In general, an axisymmetric domain is not a curved polyhedron, because the rotation of each of the two edges that intersect the axis generates a cone with a circular base, unless there is a right angle at the corresponding vertex.

Loosely speaking again, we note that the boundary of a curved polyhedron or of an axisymmetric domain is a manifold with corners and edges.

The sets of curvilinear polygons, curved polyhedra and axisymmetric domains form three important subcategories of (C3), in the sense that it is possible to get more precise, and often more explicit, results than for the "general" domains of (C3).

In open sets that belong to the category (C3), one can nevertheless establish many useful results.

Let us begin with a result that is sometimes called the Lions' Lemma.

Theorem 2.1.34 *In a domain Ω, it holds that, algebraically and topologically,*

$$L^2(\Omega) = \{f \in H^{-1}(\Omega) \ : \ \partial_j f \in H^{-1}(\Omega),\ j = 1, \cdots, n\};$$
$$L^2(\Omega) = \{f \in L^2_{loc}(\Omega) \ : \ \partial_j f \in H^{-1}(\Omega),\ j = 1, \cdots, n\}.$$

Let us continue with the definition of equivalent norms on $H^m_0(\Omega)$, which stems from the famous *Poincaré inequalities*.

Theorem 2.1.35 *Let $m \geq 1$. Given a domain Ω, there exists a constant C_m, which depends only on Ω, such that*

$$\forall f \in H^m_0(\Omega), \quad \|f\|_{H^m(\Omega)} \leq C_m\, |f|_{H^m(\Omega)}.$$

NB. It is enough to assume that Ω belongs to the category (C2), and that it is bounded in one direction ($\exists\, \boldsymbol{e} \in \mathbb{R}^n$ such that $-\infty < \inf_{\boldsymbol{x}\in\Omega} \boldsymbol{x}\cdot\boldsymbol{e} < \sup_{\boldsymbol{x}\in\Omega} \boldsymbol{x}\cdot\boldsymbol{e} < +\infty$), to prove the claim in Theorem 2.1.35.

Accordingly,

Corollary 2.1.36 *Let $m \geq 1$. Given a domain Ω, $\|\cdot\|_{H^m(\Omega)}$ and $|\cdot|_{H^m(\Omega)}$ are equivalent norms on $H^m_0(\Omega)$.*

In $H^m(\Omega)$, one can further prove the so-called *Poincaré-Wirtinger inequality*.

Theorem 2.1.37 *Let $m \geq 1$. Given a domain Ω, there exists a constant C'_m, which depends only on Ω, such that*

$$\forall f \in H^m(\Omega), \quad \|f\|_{H^m(\Omega)} \leq C'_m \left\{ |f|^2_{H^m(\Omega)} + \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|<m} \left| \int_\Omega \partial_\alpha f\, d\boldsymbol{x} \right|^2 \right\}^{1/2}.$$

In practice, one uses the Poincaré-Wirtinger inequality in the subspace

$$H^1_{zmv}(\Omega) := \{f \in H^1(\Omega) \,:\, (f|1) = 0\}.$$

From now on, the index $_{zmv}$ generically indicates that one considers the subspace made of zero mean value fields, such as $L^2_{zmv}(\Omega)$, $H^1_{zmv}(\Omega)$, etc.

In a domain Ω, one can prove (cf. [196]) that the Definition 2.1.17 of the fractional-order spaces $H^s(\Omega)$ coincides algebraically and topologically with the definition below, where the norm is explicit.

Definition 2.1.38 Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$, and write $s = m + \sigma$, with $(m, \sigma) \in \mathbb{N} \times]0, 1[$. The space $H^s(\Omega)$ is composed of elements f of $H^m(\Omega)$, such that

$$|f|_{H^s(\Omega)} := \left\{ \sum_{\alpha \in \mathbb{N}^n,\ |\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial_\alpha f(\boldsymbol{x}) - \partial_\alpha f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2\sigma}} d\boldsymbol{x}\, d\boldsymbol{y} \right\}^{1/2} < \infty. \tag{2.7}$$

Let

$$\|f\|_{H^s(\Omega)} := \left\{ \|f\|^2_{H^m(\Omega)} + |f|^2_{H^s(\Omega)} \right\}^{1/2}. \tag{2.8}$$

Endowed with the norm $\|\cdot\|_{H^s(\Omega)}$, $H^s(\Omega)$ is a Banach space.

It is a Hilbert space, endowed with the scalar product

$$(f, g)_{H^s(\Omega)} := (f, g)_{H^m(\Omega)} + \sum_{\alpha \in \mathbb{N}^n,\ |\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial_\alpha f(\boldsymbol{x}) - \partial_\alpha f(\boldsymbol{y}))\overline{(\partial_\alpha g(\boldsymbol{x}) - \partial_\alpha g(\boldsymbol{y}))}}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2\sigma}} d\boldsymbol{x}\, d\boldsymbol{y}.$$

Remark 2.1.39 One can compare the semi-norms $(|\cdot|_{H^s(\Omega)})_{s\in]0,1[}$ to the semi-norm $|\cdot|_{H^1(\Omega)}$, provided Ω is a domain. Following [60], one can prove that

$$\exists C_1, C_2 > 0,\ \forall f \in H^1(\Omega), \quad C_1|f|_{H^1(\Omega)} \le \lim_{s\to 1}(1-s)|f|_{H^s(\Omega)} \le C_2|f|_{H^1(\Omega)}.$$

For the comparison to hold, one must include the $(1-s)$ multiplicative factor in the limit.

Remark 2.1.40 One can also introduce the series of Sobolev spaces based on $L^p(\Omega)$, with $1 \le p \le \infty$. This results in the well-known $W^{s,p}(\Omega)$, for $s \ge 0$. Then, 2 (respectively 1/2) is replaced by p (respectively $1/p$) in (2.3), (2.4), (2.7) and (2.8). When $1 < p < \infty$, these function spaces are separable, reflexive Banach spaces and, for $p = 2$, they are Hilbert spaces: in this case, one has $W^{s,2}(\Omega) = H^s(\Omega)$ algebraically and topologically. Afterwards, one defines the dual spaces $W^{-s,p'}(\Omega)$ of $W^{s,p}_0(\Omega)$ (the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$), with the conjugate exponent p' s.t. $1/p + 1/p' = 1$. Also, one can identify $W^{1,\infty}(\Omega)$ with $C^{0,1}(\overline{\Omega})$, the space of Lipschitz-continuous functions on $\overline{\Omega}$. However, since most problems in this book are accurately resolved with the help of the $(H^s(\Omega))_{s\in\mathbb{R}}$ series of spaces, we shall concentrate on them.

One can establish *imbedding* results: *continuous* imbeddings, also called *Sobolev imbeddings*, and *compact* imbeddings.

Proposition 2.1.41 *In a domain Ω, it holds that, algebraically and topologically, for $s > n/2$:*

- $H^s(\Omega) \subset C^k(\overline{\Omega})$, *for $k \in \mathbb{N}$ such that $k < s - n/2$;*
- $H^s(\Omega) \subset C^{k,\beta}(\overline{\Omega})$, *for $k \in \mathbb{N}$ such that $k < s-n/2 < k+1$, and $\beta = s-n/2-k$.*

We recall that the scale of Sobolev spaces is defined "recursively" by differentiation. Let us note that differentiation loses *exactly* one order, in the following manner.

Proposition 2.1.42 *Let Ω be a domain. Then:*

- $\partial_i : H^s(\Omega) \to H^{s-1}(\Omega)$ *is continuous, for $s \in \mathbb{R} \setminus \{1/2\}$.*
- $\partial_i : H^{1/2}(\Omega) \to \widetilde{H}^{-1/2}(\Omega)$ *is continuous.*

As far as compact imbeddings (denoted by $\subset_c$) are concerned, one has the results below.

Proposition 2.1.43 *In a domain Ω, it holds that*

$$H^{s'}(\Omega) \subset_c H^{s''}(\Omega),\ \text{for } s', s'' \in \mathbb{R},\ s' > s''.$$

Let us now categorize the series of Sobolev spaces $H^s(\Omega)$, $H^s_0(\Omega)$ and $\widetilde{H}^s(\Omega)$, for $s \geq 0$. In the process, some useful results are derived.

Proposition 2.1.44 *In a domain Ω, it holds that*

- $H^s_0(\Omega) = H^s(\Omega)$, *for all $1/2 \geq s \geq 0$;*
- $H^s_0(\Omega)$ *is strictly included in $H^s(\Omega)$, for all $s > 1/2$;*
- $\widetilde{H}^s(\Omega) = [H_0^{s+1/2}(\Omega), H_0^{s-1/2}(\Omega)]_{1/2}$, *for all $s \geq 0$, such that $s + 1/2 \in \mathbb{N}$.*

By direct computations, one can bound integrals that appear in the definition of fractional-order Sobolev spaces, cf. (2.7).

Definition 2.1.45 Let Ω be a domain, with boundary Γ.
The *distance to the boundary ρ_Γ* is defined by:

$$\rho_\Gamma(\boldsymbol{x}) := \inf_{\boldsymbol{y}\in\Gamma} |\boldsymbol{x} - \boldsymbol{y}|.$$

Lemma 2.1.46 *In a domain Ω, one has $\rho_\Gamma \in W^{1,\infty}(\Omega)$.*
Let $\sigma \in [0, 1[$. There exist two constants $C_\sigma \geq c_\sigma > 0$ such that

$$\forall \boldsymbol{x} \in \Omega, \quad c_\sigma \rho_\Gamma(\boldsymbol{x})^{-2\sigma} \leq \int_{\mathbb{R}^n\setminus\overline{\Omega}} \frac{d\boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2\sigma}} \leq C_\sigma \rho_\Gamma(\boldsymbol{x})^{-2\sigma}.$$

This result has two important consequences. The first one is an alternate definition of $\widetilde{H}^s(\Omega)$. The second one concerns the equivalence between piecewise–H^s and H^s fields (see Definition 2.1.48 and Corollary 2.1.49 hereafter).

Proposition 2.1.47 *Let $s \geq 0$, and write $s = m + \sigma$, with $\sigma \in [0, 1[$. In a domain Ω, one can define $\widetilde{H}^s(\Omega)$ by*

$$\widetilde{H}^s(\Omega) := \{f \in H_0^s(\Omega) \ : \ \frac{\partial_\alpha f}{\rho_\Gamma^\sigma} \in L^2(\Omega), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| = m\}.$$

Furthermore, one has:

- $\widetilde{H}^s(\Omega) = H_0^s(\Omega)$*, for all $s \geq 0$, such that $s + 1/2 \notin \mathbb{N}$;*
- $\widetilde{H}^s(\Omega)$ *is strictly included in* $H_0^s(\Omega)$*, for all $s \geq 0$, such that $s + 1/2 \in \mathbb{N}$.*

The last statement contains a justification of the need for the spaces $\widetilde{H}^s$ (apart from a purely mathematical interest!). As a matter of fact, they are needed when the exponent is equal to $s = 1/2$ in many situations, especially when one considers functions, which are defined on a part of the boundary. For instance, the characteristic function χ_Ω belongs to $H^{1/2}(\Omega) = H_0^{1/2}(\Omega)$, whereas it is readily checked that $\chi_\Omega \notin \widetilde{H}^{1/2}(\Omega)$, according to Corollary 2.1.49 below. Before that, let us introduce the notion of the *partition* of a domain.

Definition 2.1.48 Let Ω be a domain. A *partition* of Ω, $\mathcal{P} := (\Omega_p)_{1 \leq p \leq P}$, is such that:

- Ω_p is a domain, for $1 \leq p \leq P$;
- $\Omega_p \cap \Omega_q = \emptyset$ for $p \neq q$;
- $\overline{\Omega} = \cup_{1 \leq p \leq P} \overline{\Omega_p}$.

We also introduce the corresponding set $\mathcal{F}$ of *interfaces* (here, only the manifolds of dimension $n - 1$ are kept), indexed by pairs of indices: an element Σ_{pq} of $\mathcal{F}$ is characterized by $1 \leq p \neq q \leq P$ such that $\Sigma_{pq} = \partial\Omega_p \cap \partial\Omega_q$, and $\mathcal{N}_I$ denotes the set of pairs of indices that correspond to an interface.

Finally, for $s \in [0, +\infty]$, $PH^s(\Omega, \mathcal{P})$ is the set of *piecewise–H^s functions* (with the notation $H^\infty = C^\infty$), with respect to the partition $\mathcal{P}$:

$$PH^s(\Omega, \mathcal{P}) := \{f \in L^2(\Omega) \ : \ f_{|\Omega_p} \in H^s(\Omega_p), \ 1 \leq p \leq P\}.$$

Corollary 2.1.49 *Let Ω be a domain, and $\mathcal{P} := (\Omega_p)_{1 \leq p \leq P}$ a partition of Ω:*

- *If $s \in [0, 1/2[$, $H^s(\Omega) = PH^s(\Omega, \mathcal{P})$;*
- *If $s \geq 1/2$, $H^s(\Omega)$ is a* strict *subset of $PH^s(\Omega, \mathcal{P})$.*

Let us now focus on functions defined on the boundary Γ of a domain Ω.

Remark 2.1.50 Before we proceed, let us remark that all results below, which deal with function spaces defined on the boundary or with trace mappings, are also valid for *exterior domains*, that is, open sets $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$, Ω_0 being a domain of $\mathbb{R}^n$.

Let $d\Gamma$ denote the usual Lebesgue measure on the surface Γ. Introduce...

Definition 2.1.51 The space $L^2(\Gamma)$ is composed of all complex-valued, Lebesgue-measurable functions f on Γ such that

$$\|f\|_{L^2(\Gamma)} := \left\{ \int_\Gamma |f|^2 \, d\Gamma \right\}^{1/2} < \infty.$$

Endowed with the norm $\|\cdot\|_{L^2(\Gamma)}$, $L^2(\Gamma)$ is a Banach space. In addition, it is a Hilbert space, endowed with the scalar product

$$(f, g)_{L^2(\Gamma)} := \int_\Gamma f \, \overline{g} \, d\Gamma.$$

One can then further define, for suitable s, some Sobolev spaces on Γ.

Definition 2.1.52 Let $s \in]0, 1[$.

The space $H^s(\Gamma)$ is composed of elements f of $L^2(\Gamma)$ such that

$$|f|_{H^s(\Gamma)} := \left\{ \int_\Gamma \int_\Gamma \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n-1+2s}} d\Gamma(\boldsymbol{x}) \, d\Gamma(\boldsymbol{y}) \right\}^{1/2} < \infty.$$

Let

$$\|f\|_{H^s(\Gamma)} := \left\{ \|f\|^2_{L^2(\Gamma)} + |f|_{H^s(\Gamma)^2} \right\}^{1/2}.$$

Endowed with the norm $\|\cdot\|_{H^s(\Gamma)}$, $H^s(\Gamma)$ is a Banach space.

The dual space of $H^s(\Gamma)$ is called $H^{-s}(\Gamma)$. Its canonical norm is denoted by $\|\cdot\|_{H^{-s}(\Gamma)}$.

Let us now focus on H^s Sobolev spaces on (a part of) the boundary, for $s \in]0, 1[$. First, we note that they can indeed be defined on an open subset Γ' of the boundary, using the above Definition, with Γ' instead of Γ.

Definition 2.1.53 Let Ω be a domain with boundary Γ, and let Γ' denote an open subset of Γ with $meas_\Gamma(\Gamma') > 0$ such that its boundary is a Lipschitz submanifold of Γ (of dimension $n-2$). We denote by $\widetilde{H}^{1/2}(\Gamma')$ the space composed of elements of $H^{1/2}(\Gamma')$ such that their continuation by zero belongs to $H^{1/2}(\Gamma)$. Its dual space is denoted by $\widetilde{H}^{-1/2}(\Gamma')$.

Let us consider the practical case of a curved polyhedron Ω, with $s = 1/2$.

Definition 2.1.54 Let Ω be a curved polyhedron, with a boundary Γ made of smooth faces, labeled $(\Gamma_j)_{1\leq j\leq N_\Gamma}$. The restriction to a face Γ_j of the normal vector $\boldsymbol{n}$ (respectively an element f of $L^2(\Gamma)$) is denoted by $\boldsymbol{n}_j$ (respectively f_j).

Let Ω be a polyhedral domain. When two faces possess a common edge, it is denoted by $e_{ij} = \overline{\Gamma}_i \cap \overline{\Gamma}_j$, and one can choose a unit vector $\boldsymbol{\tau}_{ij}$ parallel to e_{ij}.

Furthermore, one can introduce $\boldsymbol{\tau}_i(j) = \boldsymbol{\tau}_{ij} \times \boldsymbol{n}_i$, so that $(\boldsymbol{\tau}_i(j), \boldsymbol{\tau}_{ij}, \boldsymbol{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$. The set of pairs (i, j) such that $\overline{\Gamma}_i \cap \overline{\Gamma}_j$ is an edge is denoted by $\mathcal{N}_E$.

NB. When $\overline{\Gamma}_i \cap \overline{\Gamma}_j$ is not empty (for $i \neq j$), it is either an edge or a vertex.

Definition 2.1.55 Let Ω be a curved polyhedron, with the notations of Definition 2.1.54. Let $H_-^{1/2}(\Gamma)$ be the function space

$$H_-^{1/2}(\Gamma) := \{f \in L^2(\Gamma) \ : \ f_j \in H^{1/2}(\Gamma_j),\ 1 \leq j \leq N_\Gamma\}.$$

Let $(i, j) \in \mathcal{N}_E$. Given $f \in H_-^{1/2}(\Gamma)$, one writes $f_i \overset{1/2}{=} f_j$ if, and only if,

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|f_i(\boldsymbol{x}) - f_j(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^3} d\Gamma(\boldsymbol{x})\, d\Gamma(\boldsymbol{y}) < \infty.$$

One can prove (cf. [65])

Proposition 2.1.56 *Let Ω be a curved polyhedron, with the notations of Definition 2.1.54. Let $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$ share only a common vertex. Then, for all $f \in H_-^{1/2}(\Gamma)$, it holds that*

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^3} d\Gamma(\boldsymbol{x})\, d\Gamma(\boldsymbol{y}) < \infty.$$

One infers from this Proposition an alternative definition of the space $H^{1/2}(\Gamma)$...

Corollary 2.1.57 *Let Ω be a curved polyhedron, with the notations of Definitions 2.1.54 and 2.1.55. One has*

$$H^{1/2}(\Gamma) := \{f \in H_-^{1/2}(\Gamma) \ : \ f_i \overset{1/2}{=} f_j,\ \forall (i, j) \in \mathcal{N}_E\}.$$

Remark 2.1.58 To summarize, the values on two adjacent faces of elements of $H^{1/2}(\Gamma)$ are not correlated, provided that the two faces share only a vertex. On the other hand, it is clear that they are correlated, when they share an edge. The correlation is explained below, in the particular case when the element vanishes on one face. For more general results on compatibility conditions for elements of $H^s(\Gamma)$, see [44, 123].

Proposition 2.1.59 *Let Ω be a curved polyhedron, and let Γ_1 be a face of its boundary. The space $\widetilde{H}^{1/2}(\Gamma_1)$ is equal to*

$$\widetilde{H}^{1/2}(\Gamma_1) = \{f \in H^{1/2}(\Gamma_1) \ : \ \frac{f}{\sqrt{\rho_{\partial\Gamma_1}}} \in L^2(\Gamma_1)\},$$

where $\rho_{\partial\Gamma_1}$ is the distance to the boundary $\partial\Gamma_1$.

Let us consider again any domain Ω with boundary Γ, and let Γ' be an open subset of Γ, with $meas_\Gamma(\Gamma') > 0$, such that its boundary is a Lipschitz submanifold of Γ: one can define the space $\widetilde{H}^{1/2}(\Gamma')$ as in Definition 2.1.53. Moreover, one notices that if $f \in H^{-1/2}(\Gamma)$, its restriction to Γ', denoted by $f_{|\Gamma'}$, naturally belongs to $\widetilde{H}^{-1/2}(\Gamma')$, according to

$$\forall g \in \widetilde{H}^{1/2}(\Gamma'), \quad \langle f_{|\Gamma'}, g\rangle_{\widetilde{H}^{1/2}(\Gamma')} = \langle f, \tilde{g}\rangle_{H^{1/2}(\Gamma)}, \tag{2.9}$$

where $\tilde{g}$ is the continuation of g by zero to the whole boundary Γ.

On the other hand, one has the result below.[4]

Proposition 2.1.60 *Let Ω be a domain with boundary Γ, let Γ' be an open subset of Γ, with $0 < meas_\Gamma(\Gamma') < meas_\Gamma(\Gamma)$, such that its boundary is a Lipschitz submanifold of Γ, and let $\Gamma'' = int(\Gamma \setminus \Gamma')$.*

Let $f \in H^{-1/2}(\Gamma)$. Then, one has $f_{|\Gamma'} \in H^{-1/2}(\Gamma')$ if, and only if, $f_{|\Gamma''} \in H^{-1/2}(\Gamma'')$. In this case, one can write

$$\forall g \in H^{1/2}(\Gamma), \quad \langle f, g\rangle_{H^{1/2}(\Gamma)} = \langle f_{|\Gamma'}, g_{|\Gamma'}\rangle_{H^{1/2}(\Gamma')} + \langle f_{|\Gamma''}, g_{|\Gamma''}\rangle_{H^{1/2}(\Gamma'')}.$$

Moreover, for some $C > 0$, which depends only on Γ and Γ':

$$\|f_{|\Gamma'}\|_{H^{-1/2}(\Gamma')} \le C \left(\|f\|_{H^{-1/2}(\Gamma)} + \|f_{|\Gamma''}\|_{H^{-1/2}(\Gamma'')}\right).$$

The next result establishes the existence of traces of elements of $H^s(\Omega)$ on the boundary Γ, for suitably chosen s (see [111] for the special case $s = 1$).

Definition 2.1.61 Let Ω be a domain. Let f be a smooth function defined on $\overline{\Omega}$. Its trace $f_{|\Gamma}$ on the boundary Γ is denoted by $\gamma_0 f$, and γ_0 is called the trace mapping.

Theorem 2.1.62 *Let Ω be a domain, and let $s \in]1/2, 1]$. The mapping γ_0 has a unique continuous extension, from $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$, which is surjective.*

In addition, the following characterization holds:

$$H^s_0(\Omega) = \{f \in H^s(\Omega) \ : \ f_{|\Gamma} = 0\}.$$

Remark 2.1.63 Since we assume only Lipschitz regularity of the boundary, one cannot define the trace mapping of the normal derivative $f \mapsto \mathbf{grad}\, f \cdot \boldsymbol{n}_{|\Gamma}$ from $H^2(\Omega)$ to $H^{1/2}(\Gamma)$. Indeed, assume that Ω is a curved polyhedron, and consider $f \in H^2(\Omega)$. One sees easily that, for $1 \le j \le N_\Gamma$, $\mathbf{grad}\, f \cdot \boldsymbol{n}_{|\Gamma_j}$ belongs to $H^{1/2}(\Gamma_j)$. But the values on two adjacent faces (sharing an edge) are uncorrelated. According to Corollary 2.1.57, $\gamma_1 f$ does not belong to $H^{1/2}(\Gamma)$. However, one can still define a trace mapping of the normal derivative with values in $H^{-1/2}(\Gamma)$ (see

[4] Given any subset S of $\mathbb{R}^n$, $int(S)$ denotes the *interior* of S.

Corollary 2.2.20 in the next section). On the other hand, if the boundary is $C^{1,1}$, then this trace mapping actually goes from $H^2(\Omega)$ to $H^{1/2}(\Gamma)$.

Remark 2.1.64 In the same spirit, one can also characterize the spaces $H^s_0(\Omega)$ for $s > 1$, provided Ω is a curvilinear polygon, a curved polyhedron or an axisymmetric domain. It holds that (cf. [91])

$$H^s_0(\Omega) = \{f \in H^s(\Omega) \ : \ \frac{\partial^k f}{\partial n^k}_{|\Gamma} = 0, \ \forall k \in \mathbb{N}, \ k < s - 1/2\}.$$

Above, the definition of the trace of the normal derivative of order k is

$$\frac{\partial^k f}{\partial n^k} = k! \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \frac{1}{\alpha!} \partial_\alpha f \, n^\alpha,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $n^\alpha = n_1^{\alpha_1} \cdots n_n^{\alpha_n}$. For instance, for $s \in]3/2, 5/2[$, one has

$$H^s_0(\Omega) = \{f \in H^s(\Omega) \ : \ f_{|\Gamma} = 0, \ \mathbf{grad}\, f \cdot \boldsymbol{n}_{|\Gamma} = 0\}.$$

Definition 2.1.65 Let Ω be a domain with boundary Γ. Let Γ' be an open subset of Γ such that its boundary is a Lipschitz submanifold of Γ, with $meas_\Gamma(\Gamma') > 0$. Introduce

$$C^\infty_{\Gamma'}(\overline{\Omega}) := \{f \in C^\infty(\overline{\Omega}) \ : \ f = 0 \text{ in a neighborhood of } \Gamma'\}.$$

Then, one can define, for $s \in]1/2, 3/2[$,

$$H^s_{0,\Gamma'}(\Omega) := \text{closure of } C^\infty_{\Gamma'}(\overline{\Omega}) \text{ in } H^s(\Omega)\,;$$

furthermore, it holds that

$$H^s_{0,\Gamma'}(\Omega) = \{f \in H^s(\Omega) \ : \ f_{|\Gamma'} = 0\}.$$

Also, one can prove another *Poincaré inequality*, set in $H^1_{0,\Gamma'}(\Omega)$.

Proposition 2.1.66 *Let Ω be a domain with boundary Γ. Let Γ' be an open subset of Γ, with $meas_\Gamma(\Gamma') > 0$. Then, there exists a constant C_1, which depends only on Ω and Γ' such that*

$$\forall f \in H^1_{0,\Gamma'}(\Omega), \quad \|f\|_{H^1(\Omega)} \le C_1 \, |f|_{H^1(\Omega)}.$$

Whenever applicable, we shall use the subscript $_{per}$ to label subspaces composed of elements with periodic traces.

Finally, let us conclude with a classical result, which uses traces on parts of the boundary, and which can be seen as a complement to Corollary 2.1.49.

Definition 2.1.67 Let Ω be a domain partitioned into $\mathcal{P} := (\Omega_p)_{p=+,-}$. Let $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$ be the *interface* separating Ω_+ and Ω_-. Denote by $\boldsymbol{n}_+$ (respectively $\boldsymbol{n}_-$) the unit outward normal vector field to $\partial\Omega_+$ (respectively $\partial\Omega_-$). Denote by $\boldsymbol{n}_\Sigma$ a unit normal vector field to Σ, and define

$$\delta_\Sigma^+ := \begin{cases} +1 \text{ if } \boldsymbol{n}_+ = \boldsymbol{n}_\Sigma \text{ on } \Sigma \\ -1 \text{ if } \boldsymbol{n}_+ = -\boldsymbol{n}_\Sigma \text{ on } \Sigma \end{cases}, \qquad \delta_\Sigma^- := \begin{cases} +1 \text{ if } \boldsymbol{n}_- = \boldsymbol{n}_\Sigma \text{ on } \Sigma \\ -1 \text{ if } \boldsymbol{n}_- = -\boldsymbol{n}_\Sigma \text{ on } \Sigma \end{cases}.$$

Given $f \in PH^s(\Omega, \mathcal{P})$ for $s > 1/2$, the *jump* of f through Σ is equal to

$$[f]_\Sigma := \delta_\Sigma^+ \gamma_{0,+} f + \delta_\Sigma^- \gamma_{0,-} f.$$

The jump is understood as a difference, because $\delta_\Sigma^+ = -\delta_\Sigma^-$.

Proposition 2.1.68 *Let Ω be a domain partitioned into $\mathcal{P} := (\Omega_p)_{1\le p\le P}$, and let $\mathcal{F}$ denote the set of interfaces. For $s \in]1/2, 1]$, it holds that*

$$H^s(\Omega) = \{f \in PH^s(\Omega, \mathcal{P}) \ : \ [f]_{\Sigma_{pq}} = 0, \ \forall (p,q) \in \mathcal{N}_I\}.$$

NB. To handle the case $s = 1/2$, one needs some *ad hoc* theory, see, for instance, Corollary 2.1.57.

2.2 Vector Fields: Standard Function Spaces

In this section, since electromagnetic fields are considered, unless otherwise specified, we stand explicitly in $\Omega = \mathbb{R}^3$, or in an open subset Ω of $\mathbb{R}^3$.

In what follows, we use ξ defined on Ω, and such that

$$\xi \in \mathbb{L}^\infty(\Omega) \quad \text{and} \quad \xi^{-1} \in \mathbb{L}^\infty(\Omega), \quad \text{i.e.,} \tag{2.10}$$
$$(\xi)_{i,j} \in L^\infty(\Omega) \text{ and } (\xi^{-1})_{i,j} \in L^\infty(\Omega),\ 1 \le i, j \le 3.$$

2.2.1 *Elementary Results*

Let us introduce our first space of vector fields,

$$\boldsymbol{D}(\Omega) := \{\boldsymbol{g} \ : \ g_j \in \mathcal{D}(\Omega),\ j = 1, 2, 3\}.$$

Looking at Eqs. (1.6–1.9), one sees that Sobolev spaces like $H^1(\Omega)$ are not explicitly required, since the first-order differential operators that appear are not the gradient, but rather the curl and divergence. More precisely, all partial derivatives of the electromagnetic fields are used, but they appear in *linear combinations*, if one recalls that

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}, \quad \mathbf{curl}\, \boldsymbol{v} = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix},$$

together with the formula $\operatorname{div}(\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v} - \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w}$.

For any smooth vector field $\boldsymbol{v}$, the pointwise inequalities hold:

$$|\operatorname{div} \boldsymbol{v}(\boldsymbol{x})|^2 \leq \left(\sum_{1 \leq i \leq 3} \left| \frac{\partial v_i}{\partial x_i}(\boldsymbol{x}) \right| \right)^2 \leq 3 |\, \mathbf{Grad}\, \boldsymbol{v}(\boldsymbol{x})|^2, \tag{2.11}$$

$$|\, \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x})|^2 \leq 2 \sum_{1 \leq i, j \leq 3,\ i \neq j} \left| \frac{\partial v_i}{\partial x_j}(\boldsymbol{x}) \right|^2 \leq 2 |\, \mathbf{Grad}\, \boldsymbol{v}(\boldsymbol{x})|^2, \tag{2.12}$$

$$\text{with } (\mathbf{Grad}\, \boldsymbol{v}(\boldsymbol{x}))_{i,j} = \frac{\partial v_i}{\partial x_j}(\boldsymbol{x}),\ 1 \leq i, j \leq 3,\ |\, \mathbf{Grad}\, \boldsymbol{v}(\boldsymbol{x})|^2 = \sum_{1 \leq i, j \leq 3} \left| \frac{\partial v_i}{\partial x_j}(\boldsymbol{x}) \right|^2.$$

This being remarked, let us note that the Sobolev space $H^1(\Omega)$ is useful, and especially the space of its traces $H^{1/2}(\Gamma)$, since it is of fundamental importance in the definition and characterization of traces of the electromagnetic fields.

Definition 2.2.1 Let $1 \leq p \leq \infty$. The spaces $\boldsymbol{L}^p(\Omega) := \{\boldsymbol{v} : v_i \in L^p(\Omega),\ i = 1, 2, 3\}$ are Banach spaces. They are separable, with the exception of $\boldsymbol{L}^\infty(\Omega)$.

In particular, $\boldsymbol{L}^2(\Omega)$ is a Hilbert space, endowed with the scalar product

$$(\boldsymbol{v}|\boldsymbol{w}) := \int_\Omega \boldsymbol{v} \cdot \overline{\boldsymbol{w}}\, d\boldsymbol{x}.$$

Definition 2.2.2 Let $s \in \mathbb{R}_+$. The spaces below are separable Hilbert spaces:

- $\boldsymbol{H}^s(\Omega) := \{\boldsymbol{v} : v_i \in H^s(\Omega),\ i = 1, 2, 3\}$.
- $\boldsymbol{H}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$, where the curl is taken in the sense of distributions. The canonical norm is

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} := \left\{ \int_\Omega (|\boldsymbol{v}|^2 + |\, \mathbf{curl}\, \boldsymbol{v}|^2)\, d\boldsymbol{x} \right\}^{1/2}. \tag{2.13}$$

- $\boldsymbol{H}(\mathbf{curl}\,\xi, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathbf{curl}\,\xi\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$, where the curl of $\xi\boldsymbol{v}$ is taken in the sense of distributions. The canonical norm is

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)} := \left\{\int_\Omega (|\boldsymbol{v}|^2 + |\,\mathbf{curl}\,\xi\boldsymbol{v}|^2)\,d\boldsymbol{x}\right\}^{1/2}. \tag{2.14}$$

- $\boldsymbol{H}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\}$, where the divergence is taken in the sense of distributions. The canonical norm is

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} := \left\{\int_\Omega (|\boldsymbol{v}|^2 + |\mathrm{div}\,\boldsymbol{v}|^2)\,d\boldsymbol{x}\right\}^{1/2}. \tag{2.15}$$

- $\boldsymbol{H}(\mathrm{div}\,\xi, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\,\xi\boldsymbol{v} \in L^2(\Omega)\}$, where the divergence of $\xi\boldsymbol{v}$ is taken in the sense of distributions. The canonical norm is

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div}\,\xi,\Omega)} := \left\{\int_\Omega (|\boldsymbol{v}|^2 + |\mathrm{div}\,\xi\boldsymbol{v}|^2)\,d\boldsymbol{x}\right\}^{1/2}. \tag{2.16}$$

- $\boldsymbol{L}^2(\Gamma) := \{\boldsymbol{v} \ : \ v_i \in L^2(\Gamma), \ i = 1, 2, 3\}$.
- $\boldsymbol{H}^s(\Gamma) := \{\boldsymbol{v} \ : \ v_i \in H^s(\Gamma), \ i = 1, 2, 3\}$.

Let $s \in]0, 1/2[$. The spaces below are separable Hilbert spaces:

$$\boldsymbol{H}_{-s}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \mathrm{div}\,\boldsymbol{v} \in H^{-s}(\Omega)\}.$$

The canonical norm is

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_{-s}(\mathrm{div},\Omega)} := \left\{\int_\Omega |\boldsymbol{v}|^2\,d\boldsymbol{x} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{H^{-s}(\Omega)}\right\}^{1/2}.$$

Using (2.11) and (2.12) together with Proposition 2.1.28, one immediately gets the imbedding results below.

Proposition 2.2.3 *The space* $\boldsymbol{H}^1(\Omega)$ *is continuously imbedded in* $\boldsymbol{H}(\mathbf{curl}, \Omega)$ *and in* $\boldsymbol{H}(\mathrm{div}, \Omega)$.

NB. Let us point out that one has to be careful with "reverse" imbeddings, since $\boldsymbol{H}(\mathrm{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$ is only imbedded in $\boldsymbol{H}^1_{loc}(\Omega)$ in general (see [9]).

One then has the convenient properties below.

Proposition 2.2.4 *Under the assumptions (2.10) on* ξ*, one has:*

- $\boldsymbol{v}$ *belongs to* $\boldsymbol{H}(\mathbf{curl}\,\xi, \Omega)$ *if, and only if,* $\xi\boldsymbol{v}$ *belongs to* $\boldsymbol{H}(\mathbf{curl}, \Omega)$*;*
- $\boldsymbol{v}$ *belongs to* $\boldsymbol{H}(\mathrm{div}\,\xi, \Omega)$ *if, and only if,* $\xi\boldsymbol{v}$ *belongs to* $\boldsymbol{H}(\mathrm{div}, \Omega)$.

This Proposition allows us to simply derive useful results for elements of $\boldsymbol{H}(\mathbf{curl}\,\xi, \Omega)$ (respectively $\boldsymbol{H}(\mathrm{div}\,\xi, \Omega)$), *via* those obtained for elements of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ (respectively $\boldsymbol{H}(\mathrm{div}, \Omega)$).

Recall that (see Proposition 2.1.12), an element v of $L^2(\Omega)$ belongs to $H^1(\Omega)$ if, and only if, there exists $C_{grad} \geq 0$ such that,

$$\forall \boldsymbol{g} \in \boldsymbol{D}(\Omega), \quad |(v|\mathrm{div}\,\boldsymbol{g})| \leq C_{grad}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}.$$

One can prove similar results.

Proposition 2.2.5 *Let $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.*

- *$\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ if, and only if, there exists $C_{curl} \geq 0$ such that*

$$\forall \boldsymbol{g} \in \boldsymbol{D}(\Omega), \quad |(\boldsymbol{v}|\,\mathbf{curl}\,\boldsymbol{g})| \leq C_{curl}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}.$$

- *$\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ if, and only if, there exists $C_{div} \geq 0$ such that*

$$\forall g \in \mathcal{D}(\Omega), \quad |(\boldsymbol{v}|\,\mathbf{grad}\,g)| \leq C_{div}\|g\|_{L^2(\Omega)}.$$

One can then introduce the closures of $\boldsymbol{D}(\Omega)$, respectively, in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$.

Definition 2.2.6 Consider:

- $\boldsymbol{H}_0(\mathbf{curl}, \Omega) :=$ closure of $\boldsymbol{D}(\Omega)$ in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ according to the norm (2.13);
- $\boldsymbol{H}_0(\mathrm{div}, \Omega) :=$ closure of $\boldsymbol{D}(\Omega)$ in $\boldsymbol{H}(\mathrm{div}, \Omega)$ according to the norm (2.15).

NB. It holds that $\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^n) = \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^n)$ and $\boldsymbol{H}_0(\mathrm{div}, \mathbb{R}^n) = \boldsymbol{H}(\mathrm{div}, \mathbb{R}^n)$.

In the spirit of Proposition 2.2.4, one can define $\boldsymbol{H}_0(\mathbf{curl}\,\xi, \Omega)$ and $\boldsymbol{H}_0(\mathrm{div}\,\xi, \Omega)$.

Definition 2.2.7 Under the assumptions (2.10) on ξ, introduce:

$$\boldsymbol{H}_0(\mathbf{curl}\,\xi, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \xi\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)\}\,;$$
$$\boldsymbol{H}_0(\mathrm{div}\,\xi, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \ : \ \xi\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega)\}.$$

Let us mention a continuation result.

Proposition 2.2.8 *Let Ω be an open set of category (C2) with a bounded boundary. Then, there exists a continuous (linear) continuation operator $\mathtt{E}$ from $\boldsymbol{H}(\mathbf{curl}, \Omega)$ to $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$, respectively $\boldsymbol{H}(\mathrm{div}, \Omega)$ to $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^3)$, such that, for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, respectively $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, one has $(\mathtt{E}\boldsymbol{v})_{|\Omega} = \boldsymbol{v}$.*

Remark 2.2.9 If, in addition, Ω is bounded, one can choose a closed ball $\mathcal{O}$ containing Ω such that for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, respectively $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, $\mathtt{E}\boldsymbol{v}$ is supported in $\mathcal{O}$.

Before carrying on with traces, let us consider some simple, but crucial, results about the mappings **grad** and **curl**. The proof is given hereafter, since it is a good example of the simplicity and of the range of the theory of distributions…

Proposition 2.2.10 *One has the following:*

1. *The mapping* **grad** *is continuous from* $H^1(\Omega)$ *to* $\boldsymbol{H}(\mathbf{curl}, \Omega)$*;*
2. *the mapping* **grad** *is continuous from* $H_0^1(\Omega)$ *to* $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$.
3. *The mapping* **curl** *is continuous from* $\boldsymbol{H}(\mathbf{curl}, \Omega)$ *to* $\boldsymbol{H}(\mathrm{div}, \Omega)$*;*
4. *the mapping* **curl** *is continuous from* $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ *to* $\boldsymbol{H}_0(\mathrm{div}, \Omega)$.

Proof

1. Given v in $H^1(\Omega)$, let us check first that $\boldsymbol{w} = \mathbf{grad}\, v$ belongs to $\boldsymbol{H}(\mathbf{curl}, \Omega)$. By definition, one has $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$. If $\boldsymbol{w}$ were smooth, then $\mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}(\mathbf{grad}\, v) = 0$ would follow. Unfortunately, this is not the case. Nevertheless, one can consider $\mathbf{curl}\, \boldsymbol{w}$ in the sense of distributions, to reach, for all $\boldsymbol{g} \in \boldsymbol{D}(\Omega)$

$$\langle \mathbf{curl}\, \boldsymbol{w}, \boldsymbol{g}\rangle = \langle \boldsymbol{w}, \mathbf{curl}\, \boldsymbol{g}\rangle = \langle \mathbf{grad}\, v, \mathbf{curl}\, \boldsymbol{g}\rangle = -\langle v, \mathrm{div}\, (\mathbf{curl}\, \boldsymbol{g})\rangle = 0.$$

   (Above, the first equality is left to the reader.)
   In other words, $\mathbf{curl}\, \boldsymbol{w} = 0$ in the sense of distributions. As a consequence, since 0 belongs to $\boldsymbol{L}^2(\Omega)$, considered as a subspace of $\boldsymbol{D}'(\Omega) := (\mathcal{D}'(\Omega))^3$, one finds that $\mathbf{curl}\, \boldsymbol{w}$ is in $\boldsymbol{L}^2(\Omega)$. Thus, $\boldsymbol{w}$ is an element of $\boldsymbol{H}(\mathbf{curl}, \Omega)$.
   Also, one has

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} = |v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)},$$

   which establishes the continuity of the **grad** mapping from $H^1(\Omega)$ to $\boldsymbol{H}(\mathbf{curl}, \Omega)$.
2. According to item 1, given v in $H_0^1(\Omega)$ and $\boldsymbol{w} = \mathbf{grad}\, v$, one has $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$. Therefore, one has only to check that $\boldsymbol{w}$ actually belongs to $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$. By definition of $H_0^1(\Omega)$, there exists a sequence $(v_k)_k$ of elements of $\mathcal{D}(\Omega)$, which converges to v in $\|\cdot\|_{H^1(\Omega)}$-norm. According to item 1, $(\boldsymbol{w}_k)_k$, with $\boldsymbol{w}_k = \mathbf{grad}\, v_k$, converges to $\boldsymbol{w}$ in $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$-norm. Moreover, all $\boldsymbol{w}_k$ belong to $\boldsymbol{D}(\Omega)$, so $\boldsymbol{w}$ belongs to its closure in $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$-norm, which is precisely equal to $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$.
3. The proof is similar to that of item 1.
4. The proof is similar to that of item 2. ■

We conclude this subsection with the introduction of a number of Hilbert function spaces with curl-free or divergence-free elements.

Definition 2.2.11 Define

$$\boldsymbol{H}(\mathrm{div}\, 0, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \ :\ \mathrm{div}\, \boldsymbol{v} = 0\}\,;$$

$$\boldsymbol{H}_0(\mathrm{div}\, 0, \Omega) := \boldsymbol{H}(\mathrm{div}\, 0, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}, \Omega)\,;$$

$$\boldsymbol{H}(\mathbf{curl}\, 0, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \ :\ \mathbf{curl}\, \boldsymbol{v} = 0\}\,;$$

$$\boldsymbol{H}_0(\mathbf{curl}\, 0, \Omega) := \boldsymbol{H}(\mathbf{curl}\, 0, \Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

Under the assumptions (2.10) on ξ, define

$$\boldsymbol{H}(\mathrm{div}\,\xi 0,\Omega) := \{\boldsymbol{v}\in \boldsymbol{H}(\mathrm{div}\,\xi,\Omega)\ :\ \mathrm{div}\,\xi\boldsymbol{v}=0\}\,;$$
$$\boldsymbol{H}_0(\mathrm{div}\,\xi 0,\Omega) := \boldsymbol{H}(\mathrm{div}\,\xi 0,\Omega)\cap \boldsymbol{H}_0(\mathrm{div}\,\xi,\Omega)\,;$$
$$\boldsymbol{H}(\mathbf{curl}\,\xi 0,\Omega) := \{\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)\ :\ \mathbf{curl}\,\xi\boldsymbol{v}=0\}\,;$$
$$\boldsymbol{H}_0(\mathbf{curl}\,\xi 0,\Omega) := \boldsymbol{H}(\mathbf{curl}\,\xi 0,\Omega)\cap \boldsymbol{H}_0(\mathbf{curl}\,\xi,\Omega).$$

2.2.2 Traces of Vector Fields

In order to define properly the trace on Γ of elements of $\boldsymbol{H}(\mathbf{curl},\Omega)$ or of $\boldsymbol{H}(\mathrm{div},\Omega)$, it is convenient to have integration-by-parts formulas at one's disposal. As a matter of fact, one can proceed by *duality*, with respect to the spaces $\boldsymbol{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively, that is, those trace spaces that originate from $\boldsymbol{H}^1(\Omega)$ and $H^1(\Omega)$.

From now on, let Ω be a domain. As far as notations are concerned, one notices that in a domain, which is bounded by definition, the index $_c$ (for compact support) of the set $C_c^\infty(\overline{\Omega})$ of Definition 2.1.27 can be dropped.

Let us begin with *density* results (cf. [117, Chapter I] and Amrouche, 2011, Private communication).

Proposition 2.2.12 *One has the following:*

- $\boldsymbol{C}^\infty(\overline{\Omega})$ *is dense in* $\boldsymbol{H}(\mathbf{curl},\Omega)$*;*
- $\boldsymbol{C}^\infty(\overline{\Omega})$ *is dense in* $\boldsymbol{H}(\mathrm{div},\Omega)$*;*
- *for* $s\in]0,1/2[$, $\boldsymbol{C}^\infty(\overline{\Omega})$ *is dense in* $\boldsymbol{H}_{-s}(\mathrm{div},\Omega)$.

With the help of Proposition 2.2.4, one easily infers other results.

Corollary 2.2.13 *Under the assumptions* (2.10) *about* ξ*, one concludes that:*

- $\xi^{-1}\boldsymbol{C}^\infty(\overline{\Omega})$ *is dense in* $\boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)$*;*
- $\xi^{-1}\boldsymbol{C}^\infty(\overline{\Omega})$ *is dense in* $\boldsymbol{H}(\mathrm{div}\,\xi,\Omega)$.

One can define the unit outward normal vector $\boldsymbol{n}=n_1\boldsymbol{e}_1+n_2\boldsymbol{e}_2+n_3\boldsymbol{e}_3$ to its boundary, *almost everywhere* (cf. Proposition 2.1.25).

It is well-known that it holds that, for two functions f and g of $C^1(\overline{\Omega})$,

$$\int_\Omega \{f\,\frac{\partial g}{\partial x_i}+\frac{\partial f}{\partial x_i}\,g\}\,d\boldsymbol{x}=\int_\Gamma f\,g\,n_i\,d\Gamma,\quad i=1,2,3. \tag{2.17}$$

What can be deduced from this formula?

- First, if $\boldsymbol{f}$ belongs to $\boldsymbol{C}^1(\overline{\Omega})$,

  all three $(f_i)_{i=1,2,3}$ belong to $C^1(\overline{\Omega})$; as a consequence,

$$\int_\Omega \{f_i \, \frac{\partial g}{\partial x_i} + \frac{\partial f_i}{\partial x_i} \, g\} \, d\boldsymbol{x} = \int_\Gamma f_i \, g \, n_i \, d\Gamma, \quad i = 1, 2, 3.$$

Summing over i yields

$$\int_\Omega \{\boldsymbol{f} \cdot \mathbf{grad} \, g + \mathrm{div} \, \boldsymbol{f} \, g\} \, d\boldsymbol{x} = \int_\Gamma \boldsymbol{f} \cdot \boldsymbol{n} \, g \, d\Gamma. \tag{2.18}$$

- Second, given two elements $\boldsymbol{f}$ and $\boldsymbol{g}$ of $\boldsymbol{C}^1(\overline{\Omega})$,

  the following formulas are satisfied:

$$\int_\Omega \boldsymbol{f} \cdot \mathbf{curl} \, \boldsymbol{g} \, d\boldsymbol{x} = \int_\Omega \left\{ f_1(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3}) + f_2(\frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1}) + f_3(\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}) \right\} d\boldsymbol{x}$$

$$\int_\Omega \mathbf{curl} \, \boldsymbol{f} \cdot \boldsymbol{g} \, d\boldsymbol{x} = \int_\Omega \left\{ (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}) g_1 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}) g_2 + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}) g_3 \right\} d\boldsymbol{x}.$$

Taking the difference yields,

$$\begin{aligned}
\int_\Omega \{\boldsymbol{f} \cdot \mathbf{curl} \, \boldsymbol{g} - \mathbf{curl} \, \boldsymbol{f} \cdot \boldsymbol{g}\} \, d\boldsymbol{x} = & \int_\Omega \Big\{ (f_1 \, \frac{\partial g_3}{\partial x_2} + \frac{\partial f_1}{\partial x_2} \, g_3) - (f_1 \, \frac{\partial g_2}{\partial x_3} + \frac{\partial f_1}{\partial x_3} \, g_2) \\
& + (f_2 \, \frac{\partial g_1}{\partial x_3} + \frac{\partial f_2}{\partial x_3} \, g_1) - (f_2 \, \frac{\partial g_3}{\partial x_1} + \frac{\partial f_2}{\partial x_1} \, g_3) \\
& + (f_3 \, \frac{\partial g_2}{\partial x_1} + \frac{\partial f_3}{\partial x_1} \, g_2) - (f_3 \, \frac{\partial g_1}{\partial x_2} + \frac{\partial f_3}{\partial x_2} \, g_1) \Big\} d\boldsymbol{x} \\
\overset{(2.17)}{=} & \int_\Gamma \{ f_1 (g_3 \, n_2 - g_2 \, n_3) + f_2 (g_1 \, n_3 - g_3 \, n_1) \\
& + f_3 (g_2 \, n_1 - g_1 \, n_2) \} \, d\Gamma \\
= & - \int_\Gamma \boldsymbol{f} \cdot (\boldsymbol{g} \times \boldsymbol{n}) \, d\Gamma.
\end{aligned}$$

NB. The left-hand side is skew-symmetric with respect to $(\boldsymbol{f}, \boldsymbol{g})$: one can therefore replace the right-hand side with

$$\int_\Gamma (\boldsymbol{f} \times \boldsymbol{n}) \cdot \boldsymbol{g} \, d\Gamma.$$

As a conclusion, it follows that

$$\int_{\Omega} \{\boldsymbol{f} \cdot \mathbf{curl}\, \boldsymbol{g} - \mathbf{curl}\, \boldsymbol{f} \cdot \boldsymbol{g}\}\, d\boldsymbol{x} = \int_{\Gamma} (\boldsymbol{f} \times \boldsymbol{n}) \cdot \boldsymbol{g}\, d\Gamma. \tag{2.19}$$

One can infer a first *generalized integration-by-parts formula* from (2.19), using the density results of Definition 2.2.6 and Proposition 2.2.12.

Theorem 2.2.14 *Let* $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \times \boldsymbol{H}(\mathbf{curl}, \Omega)$*:*

$$(\boldsymbol{f} | \mathbf{curl}\, \boldsymbol{g}) - (\mathbf{curl}\, \boldsymbol{f} | \boldsymbol{g}) = 0. \tag{2.20}$$

Similarly, second and third *generalized integration-by-parts formulas* can be proven, again using density results (namely, the definition of $H_0^1(\Omega)$, and Proposition 2.2.12) and (2.18).

Theorem 2.2.15 *Let* $(\boldsymbol{f}, g) \in \boldsymbol{L}^2(\Omega) \times H_0^1(\Omega)$*:*

$$(\boldsymbol{f} | \mathbf{grad}\, g) + \langle \operatorname{div} \boldsymbol{f}, g \rangle_{H_0^1(\Omega)} = 0. \tag{2.21}$$

Let $(f, g) \in H^1(\Omega) \times H_0^1(\Omega)$*:*

$$(\mathbf{grad}\, f | \mathbf{grad}\, g) + \langle \Delta f, g \rangle_{H_0^1(\Omega)} = 0. \tag{2.22}$$

Thanks to (2.18), one can prove some results concerning the *normal trace* of elements of $\boldsymbol{H}(\operatorname{div}, \Omega)$ (cf. [117, Chapter I]).

Remark 2.2.16 As remarked previously, the results that deal with function spaces defined on the boundary or with trace mappings are also valid for *exterior domains* $\Omega = \mathbb{R}^3 \setminus \overline{\Omega}_0$, with Ω_0 being a domain.

Definition 2.2.17 Let $\boldsymbol{f}$ be a smooth vector function defined on $\overline{\Omega}$. Its normal trace $\boldsymbol{f} \cdot \boldsymbol{n}_{|\Gamma}$ on the boundary Γ is denoted by $\gamma_n \boldsymbol{f}$, and γ_n is called the normal trace mapping.

Theorem 2.2.18 *The mapping* γ_n *has a unique continuous extension, from* $\boldsymbol{H}(\operatorname{div}, \Omega)$ *to* $H^{-1/2}(\Gamma)$*, which is surjective.*

In addition, the following characterization holds:

$$\boldsymbol{H}_0(\operatorname{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}, \Omega) \;:\; \boldsymbol{v} \cdot \boldsymbol{n}_{|\Gamma} = 0\}.$$

Note that, according to this framework, one can define as a by-product[5] the trace mapping of the normal derivative.

[5] Evidently, a direct construction is also possible!

Definition 2.2.19 Let f be a smooth scalar function defined on $\overline{\Omega}$. Its trace of the normal derivative $(\partial_n f)_{|\Gamma} := \mathbf{grad}\, f \cdot \boldsymbol{n}_{|\Gamma}$ on the boundary Γ is denoted by $\gamma_1 f$, and γ_1 is called the trace mapping of the normal derivative of scalar fields.

Consider the space

$$E(\Delta, L^2(\Omega)) := \{\phi \in H^1(\Omega) \ : \ \Delta\phi \in L^2(\Omega)\},$$

endowed with the graph norm (see Definition 4.1.5). Given any element f of $E(\Delta, L^2(\Omega))$, its gradient $\mathbf{grad}\, f$ belongs to $\boldsymbol{H}(\mathrm{div}, \Omega)$, so its normal trace is well-defined. Then, since it is easily proven that $C^\infty(\overline{\Omega})$ is dense in $E(\Delta, L^2(\Omega))$, one finds that $\gamma_1 f$ actually coincides with $\gamma_n(\mathbf{grad}\, f)$. One can finally prove...

Corollary 2.2.20 *The mapping γ_1 has a unique continuous extension, from $E(\Delta, L^2(\Omega))$ to $H^{-1/2}(\Gamma)$, which is surjective.*

It is important to note that the normal traces of elements of $\boldsymbol{H}(\mathrm{div}, \Omega)$ do not belong, in general, to $L^2(\Gamma)$, but to a larger space. This is a reversed situation, compared to the trace of elements of $H^1(\Omega)$. This means that, unless otherwise specified, the normal trace is not (locally) integrable on Γ.

Remark 2.2.21 Consider ξ that fulfills (2.10). With respect to the norm (2.16), the closure of $\xi^{-1}\boldsymbol{D}(\Omega)$ in $\boldsymbol{H}(\mathrm{div}\,\xi, \Omega)$, $\boldsymbol{H}_0(\mathrm{div}\,\xi, \Omega)$, is equal to

$$\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,\xi, \Omega) \ : \ \xi\boldsymbol{v} \cdot \boldsymbol{n}_{|\Gamma} = 0\}.$$

To conclude on the normal trace, we give the result of (Amrouche, 2011, Private communication) regarding elements of $\boldsymbol{H}_{-s}(\mathrm{div}, \Omega)$.

Theorem 2.2.22 *Let $s \in]0, 1/2[$. The mapping γ_n has a unique continuous extension, from $\boldsymbol{H}_{-s}(\mathrm{div}, \Omega)$ to $H^{-1/2}(\Gamma)$, which is surjective.*

Thanks to (2.19), one can now prove some results concerning the *tangential trace* of elements of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ (cf. [117, Chapter I]).

Definition 2.2.23 Let $\boldsymbol{f}$ be a smooth vector function defined on $\overline{\Omega}$. Its tangential trace $\boldsymbol{f} \times \boldsymbol{n}_{|\Gamma}$ on the boundary Γ is denoted by $\gamma_\top \boldsymbol{f}$, and $\gamma_\top$ is called the tangential trace mapping.

Theorem 2.2.24 *The mapping $\gamma_\top$ has a unique continuous extension, from $\boldsymbol{H}(\mathbf{curl}, \Omega)$ to $\boldsymbol{H}^{-1/2}(\Gamma)$.*

In addition, the following characterization holds:

$$\boldsymbol{H}_0(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \ : \ \boldsymbol{v} \times \boldsymbol{n}_{|\Gamma} = 0\}.$$

Again, unless otherwise specified, tangential traces of elements of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ are not (locally) integrable on Γ.

Remark 2.2.25 Consider ξ that fulfills (2.10). With respect to the norm (2.14), the closure of $\xi^{-1}\boldsymbol{D}(\Omega)$ in $\boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)$, $\boldsymbol{H}_0(\mathbf{curl}\,\xi,\Omega)$, is equal to

$$\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)\;:\;\xi\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}=0\}.$$

If one introduces Γ', an open subset of Γ, with $meas_\Gamma(\Gamma')>0$, such that its boundary is a Lipschitz submanifold of Γ, then one can characterize [109] the restriction to Γ' of the normal (respectively tangential) trace of elements of $\boldsymbol{H}(\mathrm{div},\Omega)$ (respectively $\boldsymbol{H}(\mathbf{curl},\Omega)$), in the same way and with the same notations as (2.9). Indeed, one finds that:

- given $\boldsymbol{f}\in\boldsymbol{H}(\mathrm{div},\Omega)$, $\boldsymbol{f}\cdot\boldsymbol{n}_{|\Gamma'}$ belongs to $\widetilde{H}^{-1/2}(\Gamma')$, according to

$$\forall g\in\widetilde{H}^{1/2}(\Gamma'),\;\langle\boldsymbol{f}\cdot\boldsymbol{n}_{|\Gamma'},g\rangle_{\widetilde{H}^{1/2}(\Gamma')}=\langle\boldsymbol{f}\cdot\boldsymbol{n},\tilde{g}\rangle_{H^{1/2}(\Gamma)}\,;\tag{2.23}$$

- given $\boldsymbol{f}\in\boldsymbol{H}(\mathbf{curl},\Omega)$, $\boldsymbol{f}\times\boldsymbol{n}_{|\Gamma'}$ belongs to $\widetilde{\boldsymbol{H}}^{-1/2}(\Gamma')$, according to

$$\forall\boldsymbol{g}\in\widetilde{\boldsymbol{H}}^{1/2}(\Gamma'),\quad\langle\boldsymbol{f}\times\boldsymbol{n}_{|\Gamma'},\boldsymbol{g}\rangle_{\widetilde{\boldsymbol{H}}^{1/2}(\Gamma')}=\langle\boldsymbol{f}\times\boldsymbol{n},\tilde{\boldsymbol{g}}\rangle_{\boldsymbol{H}^{1/2}(\Gamma)}\,.\tag{2.24}$$

Remark 2.2.26 Results similar to (2.23) (respectively (2.24)) hold for fields of $\boldsymbol{H}(\mathrm{div}\,\xi,\Omega)$ (respectively $\boldsymbol{H}(\mathbf{curl}\,\xi,\Omega)$), under the assumptions (2.10) about ξ.

Definition 2.2.27 Let Ω be a domain with boundary Γ. Let Γ' be an open subset of Γ such that its boundary is a Lipschitz submanifold of Γ, with $meas_\Gamma(\Gamma')>0$. Introduce

$$\boldsymbol{C}^\infty_{\Gamma'}(\overline{\Omega}):=\{\boldsymbol{f}\in\boldsymbol{C}^\infty(\overline{\Omega})\;:\;\boldsymbol{f}=0\text{ in a neighborhood of }\Gamma'\}.$$

Then, one can define

$$\boldsymbol{H}_{0,\Gamma'}(\mathbf{curl},\Omega):=\text{closure of }\boldsymbol{C}^\infty_{\Gamma'}(\overline{\Omega})\text{ in }\boldsymbol{H}(\mathbf{curl},\Omega)\,;$$
$$\boldsymbol{H}_{0,\Gamma'}(\mathrm{div},\Omega):=\text{closure of }\boldsymbol{C}^\infty_{\Gamma'}(\overline{\Omega})\text{ in }\boldsymbol{H}(\mathrm{div},\Omega).$$

Furthermore, it holds that

$$\boldsymbol{H}_{0,\Gamma'}(\mathbf{curl},\Omega)=\{\boldsymbol{f}\in\boldsymbol{H}(\mathbf{curl},\Omega)\;:\;\boldsymbol{f}\times\boldsymbol{n}_{|\Gamma'}=0\}\,;$$
$$\boldsymbol{H}_{0,\Gamma'}(\mathrm{div},\Omega)=\{\boldsymbol{f}\in\boldsymbol{H}(\mathrm{div},\Omega)\;:\;\boldsymbol{f}\cdot\boldsymbol{n}_{|\Gamma'}=0\}.$$

As a consequence of Proposition 2.1.60, we note that if $\boldsymbol{f}\in\boldsymbol{H}_{0,\Gamma'}(\mathbf{curl},\Omega)$, then $\boldsymbol{f}\times\boldsymbol{n}_{|\Gamma''}\in\boldsymbol{H}^{-1/2}(\Gamma'')$, where $\Gamma''=int(\Gamma\backslash\Gamma')$ (here, $meas_\Gamma(\Gamma')<meas_\Gamma(\Gamma)$). Similarly, if $\boldsymbol{f}\in\boldsymbol{H}_{0,\Gamma'}(\mathrm{div},\Omega)$, then $\boldsymbol{f}\cdot\boldsymbol{n}_{|\Gamma''}\in H^{-1/2}(\Gamma'')$.

Once the existence of the trace mappings has been established, it is possible to consider some other *generalized integration-by-parts formulas* (2.18) and (2.19). Note that those formulas are closely intertwined with the characterization of subspaces composed of trace-free elements. We recall that, according to Proposition 2.1.44, for $s \in]0, 1/2[$, one has $H^s_0(\Omega) = H^s(\Omega)$.

Theorem 2.2.28 *Let* $(\boldsymbol{f}, g) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H^1(\Omega)$*:*

$$(\boldsymbol{f} | \mathbf{grad}\, g) + (\mathrm{div}\, \boldsymbol{f} | g) = \langle \boldsymbol{f} \cdot \boldsymbol{n}, g \rangle_{H^{1/2}(\Gamma)}. \tag{2.25}$$

Given $s \in]0, 1/2[$*, let* $(\boldsymbol{f}, g) \in \boldsymbol{H}_{-s}(\mathrm{div}, \Omega) \times H^1(\Omega)$*:*

$$(\boldsymbol{f} | \mathbf{grad}\, g) + \langle \mathrm{div}\, \boldsymbol{f},\ g \rangle_{H^s_0(\Omega)} = \langle \boldsymbol{f} \cdot \boldsymbol{n}, g \rangle_{H^{1/2}(\Gamma)}. \tag{2.26}$$

Let $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{H}(\mathbf{curl}, \Omega) \times \boldsymbol{H}^1(\Omega)$*:*

$$(\boldsymbol{f} | \mathbf{curl}\, \boldsymbol{g}) - (\mathbf{curl}\, \boldsymbol{f} | \boldsymbol{g}) = \langle \boldsymbol{f} \times \boldsymbol{n}, \boldsymbol{g} \rangle_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{2.27}$$

Let us conclude this study of fields of $\boldsymbol{H}(\mathrm{div}\, \xi, \Omega)$ and $\boldsymbol{H}(\mathbf{curl}\, \xi, \Omega)$—one has possibly $\xi = \mathbb{I}_3$—with results dealing with jumps of the normal and tangential traces. We begin with the jump of normal traces.

Definition 2.2.29 Let Ω be a domain partitioned into $\mathcal{P} := (\Omega_p)_{p=+,-}$. Let $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$ be the *interface* separating Ω_+ and Ω_-. We use the same notations as in Definition 2.1.67. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{f}_{|\Omega_p} \in \boldsymbol{H}(\mathrm{div}, \Omega_p)$ for $p = +, -$, the *normal jump* of $\boldsymbol{f}$ through Σ is equal to

$$[\boldsymbol{f} \cdot \boldsymbol{n}_\Sigma]_\Sigma := \delta^+_\Sigma(\gamma_{n,+}\boldsymbol{f} + \gamma_{n,-}\boldsymbol{f}).$$

Here, the normal jump is understood as a difference! Indeed, on the interface, it holds that $\boldsymbol{n}_- = -\boldsymbol{n}_+$.

Proposition 2.2.30 *Let* Ω *be a domain partitioned into* $\mathcal{P} := (\Omega_p)_{p=+,-}$*, and let* $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$*. Under the assumptions (2.10) about* ξ*, it holds that*

$$\boldsymbol{H}(\mathrm{div}\, \xi, \Omega) = \{\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)\ :\ \boldsymbol{f}_{|\Omega_p} \in \boldsymbol{H}(\mathrm{div}\, \xi, \Omega_p),\ p = +, -,$$
$$[\xi \boldsymbol{f} \cdot \boldsymbol{n}_\Sigma]_\Sigma = 0 \text{ in } \widetilde{H}^{-1/2}(\Sigma)\}.$$

We then consider the jump of tangential traces.

Definition 2.2.31 Let Ω be a domain partitioned into $\mathcal{P} := (\Omega_p)_{p=+,-}$. Let $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$ be the *interface* separating Ω_+ and Ω_-. We use the same notations as in Definition 2.1.67. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{f}_{|\Omega_p} \in \boldsymbol{H}(\mathbf{curl}, \Omega_p)$ for $p = +, -$, the *tangential jump* of $\boldsymbol{f}$ through Σ is equal to

$$[\boldsymbol{f} \times \boldsymbol{n}_\Sigma]_\Sigma := \delta^+_\Sigma(\gamma_{\top,+}\boldsymbol{f} + \gamma_{\top,-}\boldsymbol{f}).$$

Once more, the tangential jump is understood as a difference.

Proposition 2.2.32 *Let Ω be a domain partitioned into $\mathcal{P} := (\Omega_p)_{p=+,-}$, and let $\Sigma = \partial\Omega_+ \cap \partial\Omega_-$. Under the assumptions (2.10) about ξ, it holds that*

$$\boldsymbol{H}(\mathbf{curl}\,\xi, \Omega) = \{\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \; : \; \boldsymbol{f}_{|\Omega_p} \in \boldsymbol{H}(\mathbf{curl}\,\xi, \Omega_p), \; p = +, -, \\ [\xi\boldsymbol{f} \times \boldsymbol{n}_\Sigma]_\Sigma = 0 \text{ in } \widetilde{\boldsymbol{H}}^{-1/2}(\Sigma)\}.$$

2.3 Practical Function Spaces in the $(t, \boldsymbol{x})$ Variable

To solve some time-dependent problems, in particular, the time-dependent Maxwell equations, one needs to introduce function spaces depending both on the time variable t and on the space variable $\boldsymbol{x}$. Indeed, in that case, the unknowns, i.e., the electromagnetic fields, depend on the $(t, \boldsymbol{x})$ variable. Obviously, one can consider distributions in *space and time*, that is, on $\mathbb{R} \times \mathbb{R}^3$. However, one generally distinguishes between the variables t and $\boldsymbol{x}$, since they do not play the same role. Classically, one deals with the values of a field at a *given time t*. Hence, for a function f depending on both $\boldsymbol{x}$ and t, we are interested in $\boldsymbol{x} \mapsto f(t_0, \boldsymbol{x})$, for a given t_0.

More precisely, let $T_- \in [-\infty, +\infty[$ and $T_+ \in]-\infty, +\infty]$ with $T- < T_+$ respectively denote the initial and final times, and let Ω denote the subset of $\mathbb{R}^3$ of interest. With respect to distributions in *space and time*, the corresponding space of distributions is simply $\mathcal{D}'(]T_-, T_+[\times\Omega)$. A classical result that allows one to go back and forth from distributions in the $(t, \boldsymbol{x})$ variable to continuous functions of the variable t, with values in function spaces of the variable $\boldsymbol{x}$, is that

the tensor product space $\mathcal{D}(]T_-, T_+[) \otimes \mathcal{D}(\Omega)$ is dense in $\mathcal{D}(]T_-, T_+[\times\Omega)$.

Next, consider the function

$$\begin{aligned} f : \;]T_-, T_+[\times\Omega &\to \mathbb{R} \\ (t, \boldsymbol{x}) &\mapsto f(t, \boldsymbol{x}) . \end{aligned}$$

For any time $t \in]T_-, T_+[$, one can introduce the function $f(t)$

$$\begin{aligned} f(t) : \Omega &\to \mathbb{R} \\ \boldsymbol{x} &\mapsto f(t, \boldsymbol{x}), \end{aligned}$$

so that the function f can be identified with the function

$$\begin{aligned}]T_-, T_+[&\to \{\Omega \to \mathbb{R}\} \\ t &\mapsto f(t). \end{aligned}$$

In what follows, we will define the function spaces in the $(t, \boldsymbol{x})$ variable, which will be useful for the weak formulations in the subsequent chapters. For that, it will be sufficient to define two types of function space and one class of vector distribution. To fix ideas, consider that $T_- = 0$ and $T_+ = T < +\infty$. Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$, and let X, Y and H respectively be two Banach spaces and a Hilbert space of the space variable $\boldsymbol{x}$. Finally, let $\mathcal{L}(X, Y)$ be the space of continuous, linear mappings from X to Y.[6]

Definition 2.3.1 Given an interval I of $\mathbb{R}$, $C^m(I; X)$ is the set of functions of class C^m in I, valued into X. Endowed with the norm

$$\|f\|_{C^m(I;X)} := \sum_{k=0}^{m} \sup_{t\in I} \|\frac{d^k f}{dt^k}(t)\|_X,$$

this is a Banach space.

Definition 2.3.2 The space $L^p(0, T; X)$ is the set of Lebesgue-measurable functions valued into X, and such that

$$\begin{cases} \text{for } 1 \leq p < \infty & \|f\|_{L^p(0,T;X)} := \left\{\int_0^T \|f(t)\|_X^p \, dt\right\}^{1/p} < \infty \\ \text{for } p = \infty & \|f\|_{L^\infty(0,T;X)} := \operatorname{esssup}_{t\in]0,T[} \|f(t)\|_X < \infty. \end{cases}$$

Endowed with the norm $\|\cdot\|_{L^p(0,T;X)}$, $L^p(0, T; X)$ is a Banach space.

In addition, if $X = H$ and $p = 2$, the space $L^2(0, T; H)$ is a Hilbert space endowed with the scalar product

$$(f, g)_{L^2(0,T;H)} := \int_0^T (f(t), g(t))_H \, dt.$$

Remark 2.3.3 According to the Fubini theorem, one can easily verify that

$$L^2(0, T; L^2(\Omega)) = L^2(]0, T[\times\Omega).$$

Hence, if f belongs to $L^2(0, T; L^2(\Omega))$, one can define its partial derivative with respect to the variable t in the sense of distributions, in $\mathcal{D}'(]0, T[\times\Omega)$, and consider elements such that $\partial_t f \in L^2(0, T; L^2(\Omega))$, which allows us to define $H^1(0, T; L^2(\Omega))$, and so on.

We recall a number of classical, elementary results below.

Proposition 2.3.4 *Let X' be the dual space of X.*

- *For all $f \in L^1(0, T; X)$, there exists one, and only one, $F \in X$ such that*

$$\forall g \in X',\ \langle g, F\rangle_X = \int_0^T \langle g, f(t)\rangle_X \, dt\,;\ F \textit{ is denoted by } \int_0^T f(t)\, dt\,;$$

[6]See Sect. 4.1, Definition 4.1.1, for details on continuous linear mappings.

- *For all $g \in L^1(0, T; X')$, there exists one, and only one, $G \in X'$ such that*

$$\forall f \in X,\ \langle G, f\rangle_X = \int_0^T \langle g(t), f\rangle_X \, dt\,;\ G \text{ is denoted by } \int_0^T g(t)\, dt\,.$$

Proposition 2.3.5 *Let $A \in \mathcal{L}(X, Y)$.*

- *The mapping $f \mapsto Af$ is continuous from $C^0([0, T]; X)$ to $C^0([0, T]; Y)$;*
- *For all $f \in L^1(0, T; X)$, $\displaystyle\int_0^T A(f(t))\, dt = A\left(\int_0^T f(t)\, dt\right)$.*

Proposition 2.3.6 *A bound and differentiation of integrals:*

- *For all $f \in L^1(0, T; X)$, $\displaystyle\left\|\int_0^T f(t)\, dt\right\|_X \leq \int_0^T \|f(t)\|_X\, dt$;*
- *For all $f \in C^0([0, T]; X)$,*

$$\forall t \in]0, T[,\ \lim_{h\to 0}\left(\frac{1}{h}\int_t^{t+h} f(s)\, ds\right) = f(t) \text{ and}$$

$$\lim_{h\to 0^+}\left(\frac{1}{h}\int_0^h f(s)\, ds\right) = f(0)\,;$$

- *For all $f \in C^1([0, T]; X)$, $\displaystyle\int_0^T \frac{df}{ds}(s)\, ds = f(T) - f(0)$.*

More generally, it is necessary to introduce the distributions valued into function spaces, that is, vector-valued distributions. According to [93], one can proceed as follows.

Definition 2.3.7 The space of X-valued distributions in $]0, T[$ is denoted by $\mathcal{D}'(]0, T[; X)$. It is the set of linear and continuous mappings defined on $\mathcal{D}(]0, T[)$ with a value in X, where continuity is considered with respect to uniform convergence on the bounded sets of $\mathcal{D}(]0, T[)$.

Now, as in Definition 2.1.6, for f in $\mathcal{D}'(]0, T[; X)$ and for ϕ in $\mathcal{D}(]0, T[)$, the action of f on ϕ is written with the help of duality brackets, with an index $_t$ to emphasize the fact that we are considering the time variable:

$$\langle f, \phi\rangle_t\,.$$

By definition, the result of these duality brackets belongs to X.

Remark 2.3.8 Note that the spaces $L^2(0, T; X)$ and $C^m([0, T]; X)$ can be identified with subspaces of $\mathcal{D}'(]0, T[; X)$.

Now, similarly to the case of standard distributions, i.e., the ones that depend on the space variable $\boldsymbol{x}$ alone, one can introduce the notion of differentiation.

Definition 2.3.9 Let f be an element of $\mathcal{D}'(]0,T[;X)$. Its time derivative is defined by

$$\forall\phi \in \mathcal{D}(]0,T[), \quad \langle \frac{df}{dt}, \phi\rangle_t = -\langle f, \frac{d\phi}{dt}\rangle_t.$$

Moreover, the time differentiation in the sense of distributions is internal, in other words…

Proposition 2.3.10 *Let $f \in \mathcal{D}'(]0,T[;X)$, then $\frac{df}{dt}$ belongs to $\mathcal{D}'(]0,T[;X)$.*

Definition 2.3.11 Let $A \in \mathcal{L}(X,Y)$ and $f \in \mathcal{D}'(]0,T[;X)$: Af, defined by

$$\forall\phi \in \mathcal{D}(]0,T[), \quad \langle Af, \phi\rangle_t := A\left(\langle f, \phi\rangle_t\right),$$

belongs to $\mathcal{D}'(]0,T[;Y)$.

Thus, one has…

Proposition 2.3.12 *Consider the setting of the previous Definition. Then, the mapping $f \mapsto Af$ is linear and continuous from $\mathcal{D}'(]0,T[;X)$ to $\mathcal{D}'(]0,T[;Y)$.*

From these last two definitions and related propositions, one can deduce the (expected but) *fundamental* result concerning the distributions in the $(t,\boldsymbol{x})$ variable, which basically claims that one can *invert* the time and space differentiations

Theorem 2.3.13 *For all $(f,A) \in \mathcal{D}'(]0,T[;X) \times \mathcal{L}(X,Y)$, we have the following identity:*

$$\frac{d}{dt}(Af) = A\left(\frac{df}{dt}\right).$$

From a practical point of view, this theorem allows us to perform the computations in a "natural" and expected way. This will be crucial for deriving the variational formulations of the time-dependent problems. For instance, if $\boldsymbol{u} \in \mathcal{D}'(]0,T[;\boldsymbol{H}(\mathbf{curl},\Omega))$, one knows that $\mathbf{curl}\,\boldsymbol{u} \in \mathcal{D}'(]0,T[;\boldsymbol{L}^2(\Omega))$. According to the above theorem,

$$\frac{d}{dt}(\mathbf{curl}\,\boldsymbol{u}) = \mathbf{curl}\left(\frac{d\boldsymbol{u}}{dt}\right) \text{ in } \mathcal{D}'(]0,T[;\boldsymbol{L}^2(\Omega)).$$

These considerations will be sufficient to give a meaning to the variational formulations of the subsequent chapters. For more details, we refer the reader to [157, 177] or [93] chap. XVIII.

In the remainder of the book, we will keep the notation $\boldsymbol{u}(t) : \boldsymbol{x} \mapsto \boldsymbol{u}(t,\boldsymbol{x})$ to denote the value of $\boldsymbol{u}$ at a given time t. We will also use primes to denote differentiation with respect to time of $\boldsymbol{u}$ (when it has a meaning), e.g., $\boldsymbol{u}'$, $\boldsymbol{u}''$, etc.. When $\boldsymbol{u}$ belongs to $C^m([0,T];X)$, for a Banach space X, this notation is justified.

If $\boldsymbol{u}$ belongs to $L^2(0, T; X)$, $\boldsymbol{u}(t)$ is known for almost all t. In the most general case, that is, if $\boldsymbol{u}$ belongs to $\mathcal{D}'(]0, T[; X)$, this is an *improper notation*. Nevertheless, this "generalized" notation allows us to give a more unified presentation of the results. Note also that it fits well into the physical perception, i.e., the knowledge of the electromagnetic fields at a given time. Moreover, from a mathematical point of view, this is an admissible notation, since one can invert the time derivative and the differentiation in space (see Theorem 2.3.13).