# Chapter 17
# Classical (In)Determinism

Rather than giving a detailed account on the origin and varieties of classical determinism – which is a fascinating topic of its own [185, 186, 255, 282, 494, 495, 566] – a very brief sketch of some of its concepts will be given.

## 17.1 Principle of Sufficient Reason and the Law of Continuity

The *principle of sufficient reason* states that [495] *"a thing cannot come to existence without a cause which produces it ... that for everything that happens there must be a reason which determines why it is thus and not otherwise."*

This principle is related to another one which, in Diderot's *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers,* has been discussed as follows [218]: "The law of continuity *is a principle that we owe to Mr. Leibniz, which informs us that nothing jumps in nature and that one thing cannot pass from one state to another without passing through all the other states that can be conceived of between them. This law issues, according to Mr. Leibniz, from the axiom of sufficient reason. Here is the deduction. Every state in which a being finds itself must possess sufficient reason why this body finds itself in this state rather than in any other state; and this reason can only be found in its prior state. The prior state therefore contained something which gave birth to the actual state which it followed, and in such a way that these two states are so bound that it is impossible to place another in between them, for if there was a state between the actual state and that which immediately preceded it, nature would have left the first state even before it had been determined by the second to abandon the first; thus there would be no sufficient reason why it would sooner proceed to this state than to another."*



K. Svozil, *Physical (A)Causality*, Fundamental Theories of Physics 192,
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Note that irreversible many-to-one evolution is not excluded in this scheme, because in principle it is possible that many different states may evolve into a single one state; very much like the square function $f(x) = x^2$ maps both, say, $x = \pm 2$ into 4. If we interpret these concepts algorithmically and in terms of an evolution which amounts to a one-to-one permutation, then we arrive at a sort of hermetic and closed "clockwork universe" or virtual reality in which everything that happens is pre-determined by its past state, and ultimately by its initial state.

## 17.2 Possible Definition of Indeterminism by Negation

First of all it should be stated up-front that, as is always the case in formalizations, the following definitions and discussions merely apply to *models of physical systems,* and not to the physical systems themselves. Furthermore, indeterminism is just the *absence* or even *negation* of determinism.

## 17.3 Unique State Evolution

Determinism can be informally but very generally defined by the property that [382, Chapter on Indeterministic Physical Systems] *"the fixing of one aspect of the system fixes some other. ... In a (temporally) deterministic physical system, the present state of the system determines its future states"*. Alternatively one may say that the present determines both past and future [566]: *"determinism reigns when the state of the system at one time fixes the past and future evolution of the system."*. Here uniqueness plays a crucial role: deterministic systems evolve *uniquely.* If the past is also assumed to be determined by the present, then this amounts to an injective (one-to-one) state evolution; that is, essentially to a permutation of the state.

In classical continuum physics ordinary differential equations are a means to express the dynamics of a system. Thus determinism could formally be defined in terms of *unique solutions of differential equations*. In this approach determinism is essentially reduced to the purely mathematical question regarding the uniqueness of the solution of a differential equation.

According to the Picard–Lindelöf theorem an *initial value problem* (also called the *Cauchy problem*) defined by a first order ordinary differential equation of the form $y'(t) = f(t, y(t))$ and the initial value $y(t_0) = y_0$ has a unique solution if $f$ satisfies the Lipschitz condition and is continuous as a function of $t$.

A mapping $f$ satisfies (global/local) *Lipschitz continuity* (or, used synonymously, *Lipschitz condition*) with finite positive constant $0 < k < \infty$ if it increases the distance between any two points $y_1$ and $y_2$ (of its entire domain/some neighbourhood) by a factor at most $k$ [20, Sect. 4.3, p. 272]:

$$|f(t, y_2) - f(t, y_1)| \leq k|y_2 - y_1|. \tag{17.1}$$

That is, $f$ may be nonlinear as long as it does not separate different points "too much." $f$ must lie within the "outward cone" spanned by the two straight lines with slopes $\pm k$.

An initial value problem defined by a *second order* linear ordinary differential equation of the form $y''(t) + a_1(t)y'(t) + a_0(t)y = b(t)$ and the initial values $y(t_0) = y_0$ and $y'(t_0) = y_1$ has a unique solution if the functions $a_1$, $a_0$ and $b$ are continuous.

Systems of higher order ordinary differential equations which are normal are equivalent to first order normal systems of ordinary differential equations [61, Theorem 4, p. 180]. Therefore, uniqueness criteria of such higher order normal systems can be reduced to uniqueness criteria for first-order ordinary differential equations, which is essentially Lipschitz continuity.

## 17.4 Nonunique Evolution Without Lipschitz Continuity

There are other definitions of determinism *via* ordinary differential equations which (mostly implicitly) do not requiring Lipschitz continuity. Consequently (weak) solutions may exist, which may result in nonunique solutions.

The history of determinism abounds in proposals for indeterminism by nonunique solutions to ordinary differential equations. These proposals, if they are formalized, are mostly "exotic" in the sense that they do not satisfy the criteria for uniqueness of solutions mentioned earlier.

There are a plethora of such "examples of indeterminism in classical mechanics;" in particular, discussed by Poisson in 1806, Duhamel in 1845, Bertrand in 1878, and Boussinesq in 1879 [162, 494].

In 1873, Maxwell identified a certain kind of *instability* at *singular points* as rendering a gap in the natural laws [359, pp. 440]: "... *when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable. It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate, and not accurate.*"

Figure 17.1 (see also Frank's Fig. 1 in Chap. III, Sect. 13) depicts a one dimensional gap configuration envisioned by Maxwell [359, p. 443]: a *"rock loosed by frost and balanced on a singular point of the mountain-side*, ...." On top, the rock is in perfect balanced symmetry. A small perturbation or fluctuation causes this symmetry to be broken, thereby pushing the rock either to the left or to the right hand side of the potential divide. This dichotomic alternative can be coded by 0 and by 1, respectively.

One may object to this scenario of *spontaneous symmetry breaking* for physical reasons; that is, by maintaining that, if indeed the symmetry is perfect, there is no movement, and the particle or rock stays on top of the tip (potential).

However, any slightest movement – either through a microscopic asymmetry or imbalance of the particle, or from fluctuations of any form, either in the particle's position due to quantum zero point fluctuations, or by the surrounding environment of the particle – might topple the particle over the tip; thereby spoiling the original

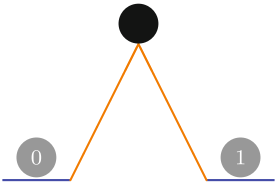


**Fig. 17.1** (Color online) A gap created by a black particle sitting on top of a potential well. The two final states are indicated by grey circles. Their positions can be coded by 0 and 1, respectively

symmetry. For instance, any collision of gas molecules with the rock may push the latter over the edge by thermal fluctuations.

Maxwell's scenario resembles Norton's dome [159, 216, 329, 379, 566], and a similar configuration studied already by Boussinesq in 1879 [494, pp. 176–178] which violates Lipschitz continuity: The ordinary differential equation of motion (for its derivation and motivation, we refer to the literature) associated with the Norton dome is given by

$$y'' = \sqrt{y}. \tag{17.2}$$

It can be readily verified by insertion that (17.2) has two solutions, namely (i) a trivial one $y_1(t) = 0$ for all times $t$, and (ii) a *weak solution* which can be interpreted as distribution: $y_2(t) = \frac{1}{144}(t-T)^4 H(t-T)$, where $H$ symbolizes the (Heaviside) unit step function. Note that the left hand side of (17.2) needs to be interpreted as a generalized function (or distribution); that is, as a linear functional integrated over a test function $\varphi$ which in this case could be 1):

$$\begin{aligned}
y_2(t)[\varphi] &= \left\{\frac{1}{144}(t-T)^4 H(t-T)\right\}[\varphi],\\
y_2'(t)[\varphi] &= \left\{\frac{1}{36}(t-T)^3 H(t-T)\right\}[\varphi] + \left\{\frac{1}{144}(t-T)^4 \delta(t-T)\right\}[\varphi]\\
&= \left\{\frac{1}{36}(t-T)^3 H(t-T)\right\}[\varphi], \qquad (17.3)\\
y_2''(t)[\varphi] &= \left\{\frac{1}{12}(t-T)^2 H(t-T)\right\}[\varphi] + \left\{\frac{1}{36}(t-T)^3 \delta(t-T)\right\}[\varphi]\\
&= \left\{\frac{1}{12}(t-T)^2 H(t-T)\right\}[\varphi].
\end{aligned}$$

The right hand side of Eq. (17.2) contains the square root of this distribution, in particular the square root of the unit step function. One way of interpretation would be in terms of theta-sequences such as $H(x) = \lim_{\epsilon\to 0} H_\epsilon(x) = \lim_{\epsilon\to 0}\left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{x}{\epsilon}\right]$. The square root of the unit step function could also be understood in terms of Colombeau theory [149]; or one might just define $\left\{\frac{1}{144}(t-T)^4 H(t-T)\right\}^{\frac{1}{2}}$ to be $\left\{\frac{1}{12}(t-T)^2 H(t-T)\right\}$.

Colombeau theory provides another rich source of pseudo-indeterminism [150] as it deals with situations in which "tiny micro-irregularities" are "mollified" and "blown up" to "macro-scales" [9, 28].

The Picard–Lindelöf theorem, which applies for first-order ordinary differential equations, cannot be directly applied to this second order ordinary differential equation. Therefore we have to use the aforementioned method of conversion of a higher order ordinary differential equation into systems of first order ordinary differential equations [140, Sect. II.D, pp. 94–96]. Suppose the initial value (or Cauchy) problem is

$$y''(t) = f(t, y(t), y'(t)) \text{ with } y(t_0) = a_0 \text{ and } y'(t_0) = a_1. \tag{17.4}$$

This equation can be rewritten into a coupled pair of equation, with $v = y'$:

$$y' = v, \quad v' = f(t, y, v) \text{ with } y(t_0) = a_0 \text{ and } v(t_0) = a_1. \tag{17.5}$$

The only modification for the Lipschitz condition is that instead of the absolute value of the numerical difference we have to use the difference in the plane

$$\|(y, v) - (z, w)\| = \left((y - z)^2 + (v - w)^2\right)^{\frac{1}{2}}. \tag{17.6}$$

For the rewritten Picard–Lindelöf theorem we have to assume that $f(t, y, v)$ is continuous as a function of $t$, and that the modified Lipschitz condition holds: for finite positive constant $0 < k < \infty$,

$$|f(t, y, v) - f(t, z, w)| \le k\|(y, v) - (z, w)\|. \tag{17.7}$$

A generalization to higher orders is straightforward.

In the Norton dome case, $f(t, y, v)$ is identified with $y^{\frac{1}{2}}$. This function does not satisfy the Lipschitz condition for $y = 0$, as its slope is $\frac{df}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$ which diverges for $y = 0$; hence no finite bound $k$ exists at that point: $f(t, y, v) = y^{\frac{1}{2}}$ grows "too fast" for $y$ approaching 0.

Similar considerations apply to other configurations violating Lipschitz continuity [216, 494].

There are other instances of classical determinism, all involving infinities of some sorts [382, Chapter on Indeterministic Physical Systems]. Neither shall be mentioned nor discuss here.