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Orly Buchbinder
Sebastian Kuntze *Editors*

Mathematics Teachers Engaging with Representations of Practice

A Dynamically Evolving Field

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Reflecting on Representations Which Reflect Practice—A Preface

Mathematics education is a science which has accumulated theoretical knowledge, and is related to fields of practice such as creating and implementing learning environments for students in mathematics classrooms. Representations of practice help to connect theory of mathematics education with practice of teaching mathematics and therefore are extremely important for teacher education and professional development.

The chapters of this monograph focus on reflecting on the role of representations of practice for pre-service teacher education and for in-service teacher professional development, while, at the same time, highlighting the potential for researching these areas. These reflections make visible the broad spectrum of possible uses of representations of practice, and this diversity underpins how fruitful it can be to enter in an exchange of ideas about practical, methodological, and theoretical issues underlying creation and use of different types of representations of practice.

Given the importance of this topic, we would like to thank the program committee of the 13th International Congress on Mathematical Education (ICME 13) held in Hamburg, Germany, in July 2016, for supporting a discussion group on representations of practice, which aimed to collect and thoroughly examine the role of representations of practice for pre-service and in-service teachers' professional development and for research into aspects of teacher expertise. The chapters of this book originate from that discussion group, and multiple international scientific contacts arose from the discourses which will hopefully enrich further collaboration within the mathematics education community in the future. We would like to thank all the participants and attendants of the discussion group, and especially the authors who contributed the chapters for this monograph. We also extend our gratitude to Rina Zazkis and Dan Chazan, who acted as discussants in our group, and who provided valuable insight throughout the discussion group meetings and their later comments and elaborations of the chapters in this volume.

Durham, USA — Orly Buchbinder
Ludwigsburg, Germany — Sebastian Kuntze

Contents

Representations of Practice in Teacher Education and Research—Spotlights on Different Approaches

Orly Buchbinder and Sebastian Kuntze

Abstract Representations of practice provide an opportunity to refer to teachers' professional environment both when designing tasks for teacher education or professional development, and when investigating aspects of teacher expertise. This volume amalgamates contributions by the members of the discussion group on representations of practice, which took place during ICME 13. The discussion group sought to collect experiences with different forms of representations of practice in pre-service and in-service teacher professional development settings, and of the use of representations of practice for researching into aspects of teacher expertise and its development. In this introductory chapter we provide an overview of different approaches to representing practice, and address key methodological issues that came up in the monograph's chapters and in the discussion group's meetings. We suggest four key questions along which such approaches can be discussed.

Keywords Representations of practice · Pre-service teacher education
Professional development for in-service teachers · Analyzing classroom situations

This monograph originated from a discussion group on the representations of practice at the 13th International Congress on Mathematical Education (ICME 13) held in Hamburg, Germany in July 2016. The discussion group aimed to collect and thoroughly examine the role of representations of practice for pre-service and in-service teachers' professional development and for research into teacher expertise. The attendants and the presenters of the discussion group shared an agreement

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that this topic is timely, especially when considering rapid technological developments that offer advanced tools for representing practice. In light of these developments and with the wide use of representations of practice, it is critical to attend to theoretical and methodological issues associated with their design and use.

The use of representations of practice for teacher education and research around the world has a long-standing tradition (Grossman et al. 2009; Herbst and Chazan 2011). Representations of practice can be different types of artefacts such as videos of classrooms or of individual students (e.g., Boaler and Humphreys 2005; Borko 2016), written cases (e.g., Smith et al. 2004), sample student work (Heid et al. 2015), scenarios (Zazkis et al. 2013), animations or story-boards (Herbst et al. 2016a), comic strips (e.g., Herbst et al. 2011), photographs (Carter et al. 1988), and combinations of several types of artefacts. A common, and sometimes implicit assumption underlying the use of representations of practice is that learning of the practice of teaching is a complex, multifaceted, context-specific process, situated in multiple social and cultural contexts. The resulting knowledge of the teaching practice is often held implicitly and is hard to access. Thus, representations serve as mediating tools between the world of professional practice and the educational or the research setting.

In their mediating role representations of practice provide teachers with specific learning opportunities which are close to action and reaction requirements of the classroom, yet they differ from the actual practice. Representations allow teachers to immerse themselves in a particular situation and to establish cognitive and emotional connection with it in ways that might not be possible at the rapid rate of actual classroom interaction. As such, representations of practice afford powerful learning opportunities for teachers to reflect on and analyze classroom situations and instances of individual student thinking (e.g., Santagata and Guarino 2011); envision potential responses to a situation (e.g., Webel and Conner 2015); contemplate various pedagogical moves and their consequences and examine teaching styles, which may be either close or removed from what teachers are familiar with (e.g., Seidel et al. 2011), including international and multicultural perspectives (Clarke et al. 2006; Stigler and Hiebert 1997).

These kinds of experiences provide meaningful support for teacher learning and create rich contexts for research into aspects of teacher expertise, views and convictions (e.g., Shulman 1986; Ball et al. 2008; Kersting et al. 2012; Kuntze 2012), competence facets such as professional vision (Sherin and van Es 2009), teacher noticing in the sense of selective attention (e.g., Seidel et al. 2013) or in the sense of knowledge-based reasoning (e.g., Sherin et al. 2011; Sherin 2007), competency of teachers' analysis of content-specific situations (e.g., Kuntze et al. 2015), mathematical content knowledge (Buchbinder in press; Zazkis et al. 2013), mathematical knowledge for teaching (e.g., Herbst and Kosko 2014), and rationality of teacher decision making (Herbst et al. 2016b).

Some constructs, such as noticing, describe aspects of expertise which are directly connected with classroom situations, while other aspects of teacher expertise, such as the degree of connectedness or coherence of pedagogical content knowledge (e.g., Doerr and Lerman 2009), can be harder to connect with observations of how teachers deal with representations of practice. The duality of theoretical constructs related to teacher expertise on the one hand and of the case-specific "mini-worlds" opened up by representations of practice on the other hand, raise questions of validity: e.g. "How is the construct reflected in an instrument which uses one or more representations of practice?", questions of relevance, e.g., "How meaningful is the construct for professional requirements in general and for situation-specific contexts, in particular?", and questions of generalizability: e.g. "To what extent does the case-based instrument design afford making inferences about a more general construct related to teacher expertise?" or "To what extent can inferences be made for a targeted group of teachers?"

Design of research instruments or professional development activities, addressing the constructs mentioned above, involves a vast range of considerations. There is a multi-faceted spectrum of decisions to make when choosing the format of the representation and its mode of use. Representations of practice may vary along multiple aspects, such as whether they are created by teacher educators/researchers or by the participating teachers themselves, whether they are staged or show an authentic classroom situation, whether situations are taken from teachers' own classrooms or the classroom of other teachers. They may vary by the amount of contextual information they bring, from video, which is considered as most context-rich, to static images realized with non-descript characters, to a written text vignette, with minimal context information. In addition, multiple methodological decisions must be made regarding the kinds of prompts to accompany the representations, and their formats: open or forced-choice; regarding the mode of interaction with the representation: individual or group; and regarding the nature of facilitation: open and exploratory or oriented toward a specific goal.

To guide the discussion of these complex issues in our discussion group we introduced a set of key questions:

- How can representations of practice encourage and afford pre-service and in-service teacher professional development, and by what means?
- How can representations of practice help to investigate aspects of teacher expertise, beliefs and conceptions?
- What kinds of methodological challenges emerge when designing opportunities for professional learning which make use of representations of practice? How can these challenges be addressed?
- What methodological challenges emerge when designing research settings based on representations of practice? How can these challenges be addressed?

Consequently, these questions also guided the writing of the chapters for this monograph, after the conference. The chapters in this volume were contributed by many of the presenters, panelists, and participants of the discussion group. The authors share insights from their own experiences with using representations of practice in their work as teacher educators and/or researchers, and offer their unique perspectives on some of the critical issues raised in the discussion group.

The first three chapters of this volume concentrate on representations of practice in video format. Karen Koellner, Nanette Seago, and Jennifer Jacobs report from their work with videotaped classroom situations in in-service teacher professional development with a specific empirical focus on the reported use of information from video cases by the participating teachers. Starting from the noticing concept as key framework, the chapter deals with the relationship between participating teachers' noticing of teacher actions in video cases and the development of their own classroom practice. The video cases in this project are framed by specific materials, which aim to foster mathematical content knowledge by demonstrating their significance for learning in the classroom. Based on a qualitative analysis of group interview data, the authors distinguish between different types of users according to the participating teachers' reports on the aspects of video and corresponding curriculum materials on transformational geometry they did or did not implement in their own practice. The findings suggest that teachers used information from the video cases in different ways, depending on the teachers' school context and their experiences in the professional development project. The authors conclude that even if representations of practice are a good way of underpinning the significance of specific content and of showing the enactment of acceding this content, additional research should be undertaken to further explore the relationship between noticing and the teachers' uptake of the stimuli provided in the professional development for their own classroom practice.

Jessica Hoth, Gabriele Kaiser, Martina Döhrmann, Johannes König, and Sigrid Blömeke present an analysis from the context of the Teacher Education and Development Study in Mathematics—Follow Up (TEDS-FU). In this study, three video vignettes were used to assess so-called situation-specific skills as components of noticing and teachers' professional competences. The staged video vignettes lasted three to five minutes and covered different topics. In-service teachers were asked to answer several questions about each video, related to both general pedagogical knowledge and to pedagogical content knowledge. The empirical part of the chapter concentrates on a qualitative coding of answers to one question related to one vignette, in which a primary student presented an incomplete and incorrect solution, which might be attributed to the way the teacher introduced the task. The results show that the in-service teachers' answers covered a relatively wide spectrum of views, which were condensed to thematic categories. Some of these categories appear to be related to scores in other variables measured by the TEDS-FU instruments. This suggests that in-service teachers who have noticed specific aspects of the classroom situation shown in the video representation tended to succeed on other test parts, e.g. the (shortened) TEDS-M pedagogical content knowledge test.

Sebastian Kuntze discusses a video-based in-service teacher professional development project foregrounded in the participants' learning related to aspects of instructional quality. The project focused on the participating teachers' criteria-based observation related to cognitive activation, intensity of argumentation, and learning from mistakes. Video representations of practice from authentic classrooms were used as learning opportunities in order to further develop the teachers' professional knowledge. Two more video cases were used in the video-based evaluation research on the professional development of in-service teachers participating in the project. The findings suggest that the participating teachers' situation-related views, addressed in the professional development, changed significantly with respect to more positive views of discourse in the classroom. Moreover, these findings were supported by participants' rating of the perceived similarity of the videotaped classrooms with their own classroom practices. Against these findings, the chapter discusses how representations of practice can encourage in-service teacher professional development and related evaluation research.

Libuše Samková explores the potential of concept cartoons for investigating pre-service teachers' professional knowledge. Whereas concept cartoons have first been suggested as learning opportunities for elementary students, they offer the possibility of gaining deep insight into content-related components of pre-service teachers' knowledge, and their analysis of hypothetical students' conceptions. The chapter refers to a theoretical background around components of professional knowledge and sketches how concept cartoons have been developed in earlier studies. On this basis, Samková developed the setting of a study with pre-service teachers, who were preparing to teach in primary schools. The author emphasizes the diagnostic potential of concept cartoons, and the results indicate that content knowledge and pedagogical content knowledge components are interwoven in the pre-service teachers' answers. The chapter argues that concept cartoons—as artificially designed representations of practice in connection with relevant theory—offer unique affordances for teacher education settings in order to diagnose and promote professional knowledge.

Students' thinking also plays a core role in the representations of practice presented by Corey Webel, Kimberly Conner, and Wenmin Zhao. The authors used the online tools provided by the Lesson*Sketch* platform in order to design what they call—teaching simulations—for pre-service elementary teachers. In addition to noticing students' thinking and reflecting on teaching, such teaching simulations offer opportunities for pedagogical action, for example, around teacher questioning techniques. In particular, the participating pre-service teachers are asked to make pedagogical choices within a simulated teaching situation and reflect on the consequences of these choices. This allows a learner (e.g., a pre-service teacher) to "test out" various decisions and draw conclusions by comparing outcomes. Some challenges with this approach are that learners bring their own criteria into their evaluations of outcomes, so learning depends on carefully crafted choices and outcomes.

Marita Friesen and Sebastian Kuntze focus on research about the teachers' competence of analyzing classroom situations pertaining to the use of mathematical representations. In this larger research context, the use of representations of practice appears as a core methodological feature in order to investigate the teachers' analysis. However, as the role of different vignette formats such as text, comic or video has hardly been explored in prior research, the chapter presents results from a format-aware research design including text, comic, and video vignettes. This research design allows comparisons of presentation format through a multi-matrix distribution of vignettes in several test booklets, and through a Rasch analysis of the participants' answers. In addition, the participants' reported engagement with the vignettes has been measured on four dimensions: authenticity, immersion, motivation, and resonance. The results indicate that for the teachers' competence of analyzing, the vignette format did neither produce significant differences nor impede the empirical unidimensionality of the competence construct as tested through Rasch modeling. Moreover the perceived immersion, motivation, and resonance did not differ for vignette formats either. The authors conclude that despite systematic design differences in the format of representing practice such as temporality, the different vignette formats are equally suitable for assessing the competence construct examined in this research.

Orly Buchbinder and Alice Cook analyzed scripts written by pre-service elementary and middle school teachers of mathematics to examine their mathematical and pedagogical knowledge pertaining to proving, with the particular focus on the roles of examples in proving. The participating pre-service teachers completed a multi-step instructional module. A critical element of the module was writing a one-page script—a continuation of a given classroom scenario—showing students presenting their arguments and challenging each other regarding which of their quadrilaterals constitute a counterexample to a certain geometrical statement. The analysis of the written scripts revealed three out of four theorized categories of mathematical knowledge for teaching of proving, and a category of general pedagogical knowledge. The results point to the importance of strengthening pre-service teachers' subject matter knowledge of geometry and of the logical aspects of proving, as prerequisite knowledge for implementing productive pedagogical practices. The chapter also highlights the potential of using representations of practice, produced by teachers, such as scripts, for enhancing professional knowledge and as a research tool.

The monograph concludes with two commentaries by Dan Chazan and Rina Zazkis, who joined the discussion group, during the ICME 13 conference, as discussants. These commentary chapters further examine the issues brought up in the chapters against the backdrop of the discussants' areas of expertise around the theory and practice of the use of representations.

This monograph does not intend to provide comprehensive answers to all the key questions—naturally, the chapters address only some of them. In all of the chapters and for all of the key questions presented above, the mediating role of representations of practice between teachers as individual learners on the one hand and professional development goals or research target constructs on the other hand is crucial. Each chapter proposes a different way of dealing with this mediating role, and each presents a different perspective on the advantages but also on the methodological challenges related to this mediating role. By providing spotlights in this sense, this collection of chapters and commentaries builds on, and contributes to the growing body of work on designing and using representations of practice for teacher education and research. We hope that it will inspire more research in this area to support teacher education and professional development of mathematics teachers.

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Representations of Practice to Support Teacher Instruction: Video Case Mathematics Professional Development

Karen Koellner, Nanette Seago and Jennifer Jacobs

Abstract What do teachers take up and use from mathematics professional development (PD) focused around video cases as representations of practice? In this chapter we explore what teachers took back to their classrooms based on a video case-based PD experience. Data gathered from focus group interviews and a set of reflection questions on teachers' learning and uptake from the PD form the basis of the analysis for this chapter. Teachers were classified into four different user categories—Generative, Transformative, Incremental, or Non Users—based on how they carried their PD experiences into their mathematics classrooms. These classifications contribute to our understanding of how, what and why teachers take up information from PD programs, and that they do that in unique ways and to varying degrees.

Keywords Video case · Mathematics professional development
Inservice teachers · Representations of practice · Teacher practice
Teacher learning

Introduction

Video-based professional development (PD) generally relies on selected video clips to serve as representations of practice that support teachers' collaborative discussion and analysis. Video is a tool that brings a slice of the classroom into the PD setting, helping to guide meaningful inquiry, reflection, and learning (Borko et al. 2011; Brophy 2004). Video can be used in a wide range of PD models, to guide teachers' attention and address particular learning goals.

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Previously we posited that PD models fall on a continuum from adaptive to specified, and we described the use of video representations in both types of models (Borko et al. 2011; Koellner and Jacobs 2015). On one end of the continuum are *adaptive* models, in which the learning goals and resources are derived from the local context and shared video is generally from the classrooms of the participating teachers. Examples of adaptive, video-based, mathematics PD models are video clubs (Sherin et al. 2009) and the Problem-Solving Cycle (Borko et al. 2015). In these models, video is selected and sequenced by the facilitator and/or the participating teachers, and video viewing and related activities are based on general guidelines that take into account the perceived needs and interests of the group.

On the other end of the continuum, *specified* models of PD typically incorporate published materials that specify in advance teacher learning goals. In video-based specified PD, the video clips are typically pre-selected and come from other teachers' classrooms. For example, Tom Carpenter and colleagues created a specified PD program using video clips from teachers' classrooms that were unknown to PD participants for the *Cognitively Guided Instruction* program. The clips were selected to elicit inquiry and discussion focused on students' mathematical thinking about arithmetic concepts (Carpenter et al. 2000). Another example is the *Learning and Teaching Linear Functions* (Seago et al. 2004), a set of specified PD materials that include video cases to help teachers deepen their understanding of ways to conceptualize and represent algebra content within their classroom practice. Across both adaptive and specified models of video-based PD, authentic video footage can be used as representations of practice that promote productive discussion around targeted content, pedagogical strategies, and/or student thinking (Borko et al. 2008).

Because adaptive models of video-based PD focus primarily on the analysis of the participating teachers' lessons, the experience of viewing video can serve as a reflective mirror into one's own practice (Lundeberg et al. 2008; Tunney and Van Es 2016). By contrast, analyzing video of unfamiliar teachers' practice, as is common in specified models of video-based PD, offers a window into alternative teaching practices (Zhang et al. 2011; van Es 2012; Calandra and Rich 2015; Givvin et al. 2005). Viewing someone else's classroom can prompt teachers to consider more wide-ranging instructional possibilities, while at the same time the experience can help teachers to see themselves in others and reflect on their own practice.

This chapter contributes to the literature focused on better understanding the use and impact of video cases as representations of authentic instructional practice in specified PD programs. The research questions that guided this work are: What do teachers report learning and using from a specified video case-based PD? What is the nature of the variation across teachers?

Noticing as a Conceptual Frame

Mathematics teachers come to PD workshops with varying levels of knowledge, much like the students who come to their math classrooms. One unique aspect of teachers' knowledge is their "professional vision," which refers to their ability to notice and analyze features of classroom interactions (Sherin 2007). Van Es and Sherin (2002) defined noticing as a process rather than a static category of knowledge and argued that it includes three components: (1) identifying important features of a classroom situation; (2) making connections between classroom interactions and the broader principles of teaching and learning; and (3) using what one knows about the context to reason about classroom events. Over the years, diverse conceptions of noticing have emerged in the literature, but in general most discussions of mathematics teacher noticing involve two main processes: (1) Attending to particular events in an instructional setting (i.e., teachers choose where to focus their attention and for how long) and (2) Making sense of events in an instructional setting (i.e., teachers draw on their existing knowledge to interpret what they notice in classrooms) (Sherin et al. 2011). Sherin et al. (2011) argue that these two aspects of noticing are not discrete, but rather interrelated. Teachers attend to events based on their sense-making, and how they interpret classroom interactions influences where they choose to focus their attention.

The conceptual frame of noticing is relevant to our consideration of what teachers take up from video-based PD for mathematics teachers and the impact on their classroom practice. It is well established that PD programs that incorporate video representations of practice foster the development of teachers' noticing skills (Roller 2016; Santagata and Yeh 2013). As they attend to and make sense of instructional events viewed during PD workshops, teachers are also likely to consider the implications for own practice (Koh 2015). In other words, what teachers notice appears directly relevant to how they elect to carry their learning into their classrooms (Sherin and van Es 2009). In addition, participants in video-based PD do not all make sense of the video clips or the classroom situations they depict in the same way; rather individuals bring differing knowledge and beliefs about teaching and learning, students, content, and curriculum to bear on what they notice (Erickson 2011; van Es 2011). Furthermore, these are important individual differences in terms of what teachers bring to and take from video-based mathematics PD experiences (Kazemi and Hubbard 2008; Kersting et al. 2010; Santagata and Yeh 2014). Teachers bring diverse perspectives on teaching and learning, experiences as classroom teachers, and content knowledge based on their own backgrounds and context. This individual diversity impacts what they notice in the videos, how they engage in the professional development and what they take and use in their own practice. More research is needed to understand and categorize these differences, and connect the use of video to both noticing and uptake.

LTG Video Case Materials and Design

The Learning and Teaching Geometry[1] (LTG) materials (Seago et al. 2017) use video as a centerpiece in the professional development designed to improve the teaching and learning of mathematical similarity based on geometric transformations. The authors of the materials conjectured that viewing and discussing video footage, on its own, would be insufficient to meet the LTG materials' learning goals. Therefore, the PD design incorporates pre and post-video viewing tasks, which together constitute a 'video case' and serve as a holistic basis for supporting learning from representations of practice (Seago et al., in press).

The LTG materials engage teachers in learning about similarity, congruence, and transformations and offer access to specific and increasingly complex mathematical concepts that are presented within the dynamics of classroom practice (Seago et al. 2010). The learning goals for the LTG PD were chosen because this content was a critical area of need for teachers—new U.S. Common Core Standards for Mathematics required them to teach transformations-based geometry, which had not previously been part of state standards (Seago et al. 2013). Therefore, it is likely that neither teachers nor their students have had many opportunities to engage with the specific mathematics content covered in the PD materials.

In addition to learning the content, a central goal of the materials was to support teachers' ability to provide classroom experiences to promote their students' learning. Sustained and in-depth engagement with video-cases, led by a knowledgeable facilitator, was hypothesized to be a powerful tool to promote teacher learning, instructional change, and student learning. The representations of practice form the backbone of what constitutes the LTG materials—a specified PD curriculum that is organized into 18, three-hour sessions, intentionally sequenced to follow a mathematical trajectory. In total, the program includes over 50 video clips, selected from real classroom footage of mathematics lessons across the United States. All of the video clips were examples of productive instruction, yet the clips vary in how similarity and congruence are taught. Only two teachers' instructional practice display expert content knowledge of transformations-based instruction, however the series of video clips are used purposefully in the sequenced mathematical trajectory. By focusing on classroom video from across multiple and varied contexts, the materials provide insight into what an emerging understanding of similarity looks like as well as a variety of instructional strategies that can foster this understanding.

The professional learning activity that most commonly comes before watching a given video clip in the LTG materials is working on the mathematical task that is in the clip. Solving the same task as the students in the video allows teachers to develop an adequate understanding of the mathematical demands faced by the

[1]The National Science Foundation supported both the Learning and Teaching Geometry Study (NSF Award#0732757) and the Learning and Teaching Geometry Efficacy Study (NSF Award#1503399).

students, and helps them to better engage with and interpret the student thinking and pedagogical moves captured by the video clip. In some cases, teachers are prompted to make predictions about how students will solve the problem or discuss the types of mistakes they think students might make. The assumption behind this type of pre-video activity is that teachers need a period of time to become sufficiently immersed in and familiar with the mathematics content they are about to see, so that they can readily follow the pertinent issues that arise in the video episodes.

Post-video viewing activities in the LTG materials include: careful unpacking of the ideas presented in the video clip, considering how those ideas apply in different mathematical contexts, discussing the pedagogical issues that were brought up by the video clip, and reflecting on how teachers can apply their emerging insights to make improvements in their own lessons. Certainly not all of these topics are discussed after each video, but they are generally part of each session. Facilitators draw on guiding questions provided by the materials, but they are also free to improvise based on their understanding of the teachers' needs and interests (Jacobs et al., in press).

The LTG Efficacy Study

The LTG Efficacy Study aims to explore the effectiveness of the LTG PD program using a randomized, experimental design. The sample is comprised of 111 mathematics teachers (serving grades 6–12) and their students from two contexts—one in the northeast United States and the other in the western mountain region. All teachers volunteered to participate in the study. Sometimes there were groups of teachers from one school but at times only one teacher represented a school site. Approximately half of the teachers were randomly assigned to take part in the LTG PD in the first intervention year (treatment group) and half will take part in the second intervention year (delayed treatment group/control group). The treatment group consisted of two sites and included 24 middle and high school teachers in the western mountain region and 25 middle and high school teachers in the northeast. The control group of teachers consisted of 31 teachers in both settings. Treatment teachers in both settings participated in the entire LTG PD program, including a one-week summer institute and four days of academic year follow-up sessions beginning in Summer 2016. Control teachers will participate in the same experiences beginning in Summer 2017. One facilitator led professional development workshops for all groups of teachers. She was one of the expert videotaped teachers found in the video-clips. She had a high level of transformations-based geometry content knowledge. During a pilot, she was found to facilitate with a high degree of fidelity to the LTG PD curriculum goals. The data gathered in this project include focus group interviews, reflection questions, videotape of PD sessions, videotape of teacher's classroom practice, teacher content assessments, teacher pre/post PD assessments, and student content assessments. Although a large amount of data was collected and used to quantitatively understand the efficacy and impact of the materials, the methodological challenge to understand qualitatively what exactly the teachers learned and took up from the PD and enacted in their classrooms still existed. This is the first step in addressing that challenge. The focus of this chapter is to understand what teachers learned from the PD and what they noticed from the video cases that appeared relevant to their own classrooms, and how they incorporated their learning into their current instructional practice.

Data Collection and Analysis

As an exploratory study, the analysis for this chapter is drawn from data gathered in focus group interviews and a set of reflection questions from the treatment teachers in both settings at the last two PD sessions. Focus group interviews allow teachers to collaboratively discuss their experiences and opinions on a selected topic (Vaughn et al. 1996), in this case their uptake of the LTG materials. Structured group conversations have the benefit of sparking memories and experiences a teacher may have forgotten, in contrast to the reflection questions which allowed only for independent thinking. The focus group interviews were conducted by a project staff member with groups of 3–4 participating teachers. A total of 20 teachers were interviewed in this manner. Teachers were encouraged to informally share their learning from the PD, what they had taken up in their practice as a result of participating in the PD, and why they did or did not use tasks, instructional strategies or tools that they experienced/viewed in the LTG PD in their classrooms. The interviewer prompted teachers to talk specifically about the video cases and any other influences from the PD they brought into their classrooms (including student materials and teaching practices), and to describe relevant information about their school or teaching context. Additionally, a set of reflection questions were given to 15 teachers during one of the last PD workshops held in December after the workshop was over. Teachers were asked to think back on the week-long summer PD workshops. They were asked to think about what they learned and what they actually used in their classroom. These questions prompted teachers to describe in writing how they had used content and pedagogical information from the PD in their classrooms. They were also asked what influenced them the most and if they were planning on incorporating any other aspects of the PD during the academic year.

We analyzed notes from the focus group interviews and teachers' written reflections using a modified grounded theory approach to look for patterns and themes that emerged from the data (Glaser and Strauss 2009). Specifically, we used the data to create and define categories based on teachers' reported use of the LTG materials in their classrooms. The themes identified allowed us to sort teachers into categories that highlighted the differences between them based on what they reported using in their practice as a result of their PD experience. In particular, we noticed that there seemed to be different 'levels of use' related to the content and pedagogy. For instance, we documented what participants reported using from the

LTG PD along with their explanations for using (or not using) specific components. In some cases, teachers reported only implementing some pedagogical practices covered in the PD (such as using tracing paper to teach transformations) whereas other teachers reported using almost all of the math problems and applets provided in the materials. We also discovered that some participants reported adapting the given problems and pedagogical strategies in ways that were aligned with but moved beyond the PD curriculum. From this inductive process, we generated four categories that appear to fit the patterns of use described by all of the interviewed teachers. For instance, when teachers were using the math problems, applets, and pedagogical strategies we originally put these teachers in a category and labeled it transformative because it appeared they changed their teaching of transformation-based geometry. However, then we realized that not only had some teachers used all of the materials but that they generated some new material. For instance, some teachers modified curriculum to make a static problem dynamic or another used the content of the PD and created applets that were dynamic to support student learning. These teachers became a new category. In the next section, we describe the four categories and provide case examples of teachers who are representative of each category, using illustrative quotes and other relevant information related to their experience of the PD.

Findings

Based on qualitative data analyses conducted to date, we found that participants report using information from the LTG video cases in very different ways depending on their experiences during the PD, their school context, and the mathematics courses they currently teach. We identified four categories of teachers that highlight the different ways they describe incorporating the mathematics content and pedagogical strategies learned from the PD in their practice: Generative Users, Transformative Users, Incremental Users, and Non Users.

Generative users are teachers who reported going beyond the scope of the LTG PD by using the knowledge and skills gained from the workshops to generate new and innovative instructional materials for their classrooms. Generative users described incorporating both their own newly developed instructional materials, along with materials and practices taken from the LTG PD program, in order to engage their students in the types of content and pedagogical experiences they noticed and considered beneficial during the PD. *Transformative users* intentionally brought what they learned about content and pedagogy from the LTG PD into their classrooms, using many of the given materials and observed practices in a substantive way to transform their mathematics instruction; however they did not generate any new instructional materials. *Incremental users* took up some of the materials and/or pedagogical strategies from the PD for use in their own classroom, but not to the degree of the transformative users. Lastly, *Non Users* are participants who did not use either the LTG content-based materials or pedagogical strategies in their classrooms. In the next section, we provide examples of each type of user, highlighting what they noticed and took up from the PD program and how particular elements of the PD appeared to influence their learning.

Generative User Example

Teachers were classified as generative users if they not only applied what they learned from the LTG PD, but used that learning to generate new instructional materials that expanded on critical mathematical and instructional components of the PD. For example, Peter, a high school geometry teacher with a strong math background, was classified as a generative user of the LTG materials because he developed new computer-based materials (applets created using Geogrebra software) for his students, building from his viewing, use, and discussions of similar materials during the PD. Peter was heavily influenced by the PD's emphasis on mathematical transformations in understanding geometric similarity, and he noticed that his own learning was deeply impacted by opportunities to explore technology on this topic (both through representations of practice and connected activities). Peter explained why he was driven to generate innovative classroom materials based on his PD experience:

> I am someone who has very strong visual-spatial reasoning. I regularly manipulate shapes and objects in my mind. I know that this is not something that everyone else has. So it was very beneficial to get to see something that would allow everyone to have a common dynamic vision of similarity. Using Geogebra applets during the workshops inspired me to develop my own Geogebra applets and also worksheets so my students can self-guide through some of our investigations.

The LTG PD highlights the importance of a visual, transformations-based approach to teaching and learning about congruence and similarity. As part of many of the post video-viewing experiences during the workshops, teachers had opportunities to explore Geogebra applets that supported their visualization of the dynamic relationships among similar figures. Peter was inspired by these experiences to develop his own Geogebra applets and accompanying classroom materials that went beyond the scope of the LTG PD materials.

Peter also shared that the video clips helped him to notice the range of student understanding around particular concepts, which then prompted a broad change in his teaching practice. Peter explained:

> The most significant thing about the video clips was the ability to analyze different "levels" of student understanding. I think understanding these different levels has helped me encourage more students to share their thinking. Understanding students' levels of thinking allows us as teachers to compare between partially correct and correct responses in class discussion. It actually would allow us to make rubrics that are explicitly focused on students thinking.

Nicole is an 8th grade teacher who was also classified as a generative user because she not only used the LTG materials in her classroom but she also generated new, related materials to help her students learn the focal geometric concepts. Nicole, reported,

> I used most of the materials from the PD. For example, I used all of the dilation problems, the rectangle problem, and I used the transparency paper with markers.

Over the course of the LTG PD workshops, Nicole had multiple opportunities to view a variety of representations of practice that highlighted and contrasted different student approaches to solving similarity problems—in particular, transformations-based student approaches and static-based student approaches. The majority of teachers came into the PD with a strong knowledge of static-based approaches and little knowledge of transformation-based approaches. In fact, most teachers were like Nicole in that they had no experience with using dilation as a tool for solving similarity problems. Nicole explained that she was inspired by her experiences in the PD to modify problems from her mathematics curriculum that encouraged static-based approaches so that they would also allow for transformations-based approaches. Nicole noted that she learned a great deal about both approaches during the workshops, and wanted to make sure she was providing her students with numerous opportunities to explore similarity problems using different methods. Therefore, after the PD, Nicole went carefully through her 8th grade curriculum, identified relevant tasks, and adapted them to be sure her students would become sufficiently proficient with transformations-based geometry.

Transformative User Example

Whereas Peter was particularly attentive to the impact that technology could have on teaching and learning similarity and Nicole was struck by the distinction between static and transformations-based approaches, Nancy was very interested in the use of tracing paper. Nancy found herself learning important content during the LTG workshops by watching videos of students using tracing paper to solve geometry problems. She then decided to transform her teaching by bringing this experience to her own classroom. However, unlike Peter and Nicole, Nancy did not report generating new ways to use tracing paper that were different from those she explored during the PD. Nevertheless, Nancy described the use of tracing paper as supporting a significant shift in her students' learning:

> I used patty [tracing] paper with transformations, which was helpful because students moved them around and we haven't ever done that before. This definitely helped them learn similarity in more conceptual ways.

Using tracing paper as a tool to solve problems and understand transformations-based geometry is an important focus of the LTG PD materials, and one that is highlighted in multiple representations of practice. During those clips, students use tracing paper in unique and (mostly) mathematically accurate ways, which commonly influences the participating teachers to begin exploring how they can bring tracing paper into their own classrooms. Nancy, like many other teachers, became cognizant of the learning opportunities afforded by this tool and encouraged her students to use it, closely following the examples of the videotaped classrooms and the pre- and post-video activities in which she herself used tracing paper. Nancy is considered a transformative user because she incorporated a new tool, spotlighted by the PD, into her classroom instruction focused on transformations-based geometry, in what appears to be a substantive and appropriate manner.

Incremental User Example

Carol, who is currently teaching Algebra II but no geometry classes, is an example of an incremental user. Although Carol did not mention bringing any of the content focused materials from the LTG PD into her classroom, she described changes in her pedagogy that she attributed to her PD experience. Carol explained that she has not yet had the opportunity to utilize her increased content knowledge due to the fact that she is not currently assigned to teach geometry, however she has intentionally incorporated newly learned instructional practices in her algebra classes. Carol told us,

> I am trying to incorporate some of the teaching methodologies that we observed in the videos from the workshops. For instance, I am having students present and explain their work to the others and making students defend their positions by further questioning them when they are not clear in their responses.

The video clips that Carol and her colleagues viewed, discussed and analyzed over the course of the LTG PD motivated her to reflect on her own practice and to consider aspects that she could improve on, such as student presentations and teacher questioning. In many of these clips, as Carol noticed, students presented their ideas to their classmates in whole and small groups, questioned each other, disagreed with each other's methods or solutions, or defended and clarified their mathematical arguments. These representations of practice helped Carol to recognize new pedagogical possibilities, and prompted her to incorporate them into all of her math classes regardless of the content focus.

George is another example of an incremental user because he has used some of the content pieces and tasks from the PD in his classes, but in a somewhat sporadic way. George explained,

> For me, I liked the emphasis on transformations to explain rigid motions. I liked the triangle proofs and how we used dilation strategies to determine similarity. I used these approaches in my classroom this fall and I can see the fruit of my labor. I didn't use all of the problems or content that I learned but I used some and it was great.

It appears George appreciated some of the key mathematical ideas from the summer institute that focused on transformation-based approaches. It is not clear why he only chose to use some specific problems and not others, but this type of response led him to be identified as an incremental user. These users up took portions of the mathematics content or pedagogical strategies that resonated well with them, but did not appear to use the content and/or pedagogy from the PD in a holistic or comprehensive way, nor did they generate new materials or practices based on their PD experiences.

Non User Example

Very few participants reported that they had not brought any of the content materials or pedagogical tools from the LTG PD into their classrooms. However, one high school teacher, Barb, who fell into this category explained her non-use by describing the school-imposed barriers she faced in this regard. Barb noted,

> I haven't used anything so far. We teach 2-hour daily block periods of math, covering one year of material each semester. It is hard to use stuff from this PD with the rapid pace of our math blocks. The pace is harder for me as a teacher than the students. I have so much to do and I can't change my teaching that quickly.

Barb teaches in a high achieving school, and she expressed concern that incorporating the materials and tools from the LTG PD program would cause her to slow down her instruction too much. Although she recognized the benefits of incorporating a transformations-based approach to the study of geometry, she could not see a way to utilize anything from the PD within her own classroom given her school's demands to cover a large amount of information in a short time frame.

Conclusions

The LTG PD materials, particularly through the use of representations of practice, provide extensive opportunities for teachers to notice and deeply consider the dynamic relationships among mathematics content, pedagogy, and student thinking. Many teachers in our study reported that seeing effective pedagogical strategies in the video clips helped them to envision how certain pedagogical strategies or mathematics content might play out in their own classroom. At the same time, it is clear that the teachers learned not only from the video, but from the activities that supported viewing and discussion of the clips. As we have noted, LTG video cases incorporate not only video clips but pre- and post-video viewing activities. As such, video cases provide teachers with multiple avenues to stimulate content learning and access pedagogical strategies in ways that are aligned with teachers' prior experiences and unique contexts.

Although video cases in specified PD models like LTG target carefully composed content and pedagogical learning goals, individual teachers may find particular components of the video representations of practice to be personally meaningful and relevant to their own classrooms. Individual differences in teachers' knowledge and beliefs impact what they learn from the LTG PD. Teachers will notice and attend to the events in the videos that they connect to, are puzzled by, or concern them. Taking part in a collaborative learning setting, they gain new insights from their colleagues as they notice and discuss a multitude of topics. Their individual and collective noticing impacts what they learn from their PD experience as well as what they choose to use in their own teaching practice.

We found that the purposefully designed LTG video cases anchored teachers' noticing and insights in particular ways, around a variety of issues related to teaching and learning mathematics. We conjecture that teachers' unique experiences in learning from the LTG PD and the specified representations of practice was likely due to differences in their noticing skills and/or their instructional context including grade level, courses taught, and curriculum requirements. We further hypothesize that this combination of differential noticing and variation in instructional context contributed to teachers' classification as different types of users of the PD materials in their classrooms.

It is clear that representations of practice in video based PD serve as a stimuli for reflection and noticing. However, additional research should be undertaken to explore and disentangle this connection between noticing and uptake from the PD as well as the representations of practice. For instance a detailed examination of what individual teachers attended to and brought up during the workshop discussions and whether those PD experiences are correlated with their classroom use categorizations is needed. In addition, objective analyses based on teachers' observed classroom practices is essential to validating data on their self-reported uptake of information from the PD.

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A Situated Approach to Assess Teachers' Professional Competencies Using Classroom Videos

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Abstract Teaching practice and its representation by videos are a central part of many empirical studies concerning the field of teaching and learning. In order to analyze how videos can help to investigate aspects of teachers' expertise, the data from 131 primary mathematics teachers who participated in the TEDS-Follow-Up study were evaluated. The teachers answered questions referring to scripted video-clips describing classroom situations. The questions were qualitatively analyzed covering the spectrum of aspects mentioned by the teachers and its relation to aspects of teachers' expertise. The analyses showed that teachers notice and mention a great number of aspects that were either directly observable in the video-clip shown, or could be identified using the given information. In addition, it is pointed out that teachers with high professional knowledge notice possible reasons for a student's error more accurately, while teachers with low professional knowledge focus on aspects that are not directly connected to the student's learning.

Keywords Teacher competence · Video based assessment · Noticing
Teacher knowledge

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Introduction

Research about teachers' expertise and teachers' competencies used a variety of approaches to gather information about the multifaceted abilities and skills that teachers require for their teaching profession. In addition to large-scale assessments that tested teachers' knowledge and beliefs by paper-and-pencil tests (e.g., Cognitively Activating Instruction and Development of Students' Mathematical Literacy (COACTIV), Kunter et al. 2011; Teacher Education and Development Study in Mathematics (TEDS-M), Blömeke et al. 2014; and Mathematics Teaching in the 21st century (MT21), Schmidt et al. 2011), research approaches used representations close to real classroom practice in order to assess teachers' professional competences using videos (e.g., Kersting 2008; Kersting et al. 2010, 2012; Star et al. 2011; Kaiser et al. 2015), text-vignettes (e.g., Dreher and Kuntze 2015) or comic scenes (e.g., Herbst et al. 2015). In connection with these approaches, the theoretical bases of these studies often also included more situation-specific facets of teachers' professional competencies. In order to analyze how video as a tool for representing teaching practice can help to investigate aspects of teachers' expertise, this paper presents a qualitative approach to analyze teachers' noticing of a video episode and links the results to their professional knowledge. In the following, the theoretical basis concerning teachers' expertise and teachers' professional competencies will be described. Subsequently, the methodological approach of a qualitative analysis of one selected question of the TEDS-FU video instrument will be presented as well as the results of this analysis.

Theoretical Background

Teachers' Expertise and Teacher Noticing

In order to identify characteristics of expert teachers, research in the field of teachers' expertise usually contrasted experts and novice teachers (Berliner 2001). In this regard, expert and novice teachers differed with regard to their situation-specific skills that become relevant in the course of teaching. The three situation-specific facets—the perception, interpretation and decision-making during class—were prominent facets concerning the concept of teachers' noticing (Sherin et al. 2011a; Jacobs et al. 2010) and also were central components of teachers' professional competencies as they were conceptualized and assessed in some studies as the TEDS-Follow-Up study (see Section "The TEDS-Follow-Up Study", Kaiser et al. 2015).

Research showed that expert and novice teachers' *perception* differ with regard to identifying relevant aspects for children's learning. Expert teachers distinguish important and less important information while novice teachers more often perceive surface characteristics (Berliner 2001). In addition, novice teachers may more often focus on the teacher and aspects of classroom management than on the subject and the classroom discourse (Star and Strickland 2008). "When issues of content were noticed, preservice teachers tended to comment only about whether the content was presented accurately and clearly and/or to provide a chronological description of what the teacher wrote on the board during the lesson" (ibid., p. 122).

These kinds of differences also became obvious when comparing expert and novice teachers' *interpretation* of classroom situations. In this regard, Sherin et al. (2011b, p. 5) pointed out that (teachers') perception and interpretation seem to be more "interrelated and cyclical". While novice teachers rather use descriptions of what happened in class, expert teachers interpret the situation deeply and precisely (Sabers et al. 1991; Carter et al. 1988). Problems in student learning processes, which are based on complex teaching situations, may be identified faster by expert teachers while novices rather identify the error itself and, again, describe the error but do not interpret it (Chi et al. 1981).

As another difference between expert and novice teachers could be identified that "as a group, experts are much more interested in analyzing why things are happening instead of critically commenting on the fact that events have happened" (Sabers et al. 1991, p. 81). In her learning to notice framework, Van Es (2011) described different phases of teacher noticing development. Here, teacher noticing evolved with regard to what is noticed and how teachers notice. While teachers with baseline noticing may more often attend to the teacher and his or her pedagogy as well as students' behavior, teachers' noticing shifts in the subsequent stages to students' learning and the relationship between students' mathematical thinking and teaching strategies (ibid.). With regard to how teachers notice, baseline noticing is characterized by mentioning general aspects of what occurred as well as descriptive and evaluative comments. Then again, extended noticing, as the most advanced stage of noticing, includes identifying noteworthy events and providing interpretative comments that refer to specific events and interactions as evidence for what was noticed (ibid.).

In addition, expert and novice teachers seemed also to differ with regard to their *decision-making*. As Jacobs et al. (2010) analyzed—in addition to teachers' attending to and interpreting children's strategies—teachers decide on how to respond on the basis of children's understanding. The authors pointed out that the development of this specific facet can be characterized by various aspects such as "a shift from general comments about teaching and learning to comments specifically addressing the children's understandings; a shift from overgeneralizing children's understandings to carefully linking interpretations to specific details of the situation" (ibid., p. 196).

Teachers' Professional Knowledge and Its Connection to Teachers' Noticing

The differences between expert and novice teachers' perception, interpretation and decision-making processes—as described in the previous section—may have resulted from their different knowledge bases (Livingston and Borko 1989). Teachers' perceptions and interpretations were both assumed to be knowledge-based because teachers' knowledge guides their perceptions and provides the basis for their interpretations of the perceived instances (e.g., Schäfer and Seidel 2015). Following the question of how teachers' noticing is linked to their professional knowledge, the definition of teacher noticing by Van Es and Sherin (2002, p. 573; Sherin 2010b) proposed an interrelation between perception, interpretation and knowledge. "We propose three key aspects of noticing:

(1) Identifying what is important or noteworthy about a classroom situation
(2) Making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent and
(3) Using what one knows about the context to reason about classroom interactions"

As Van Es and Sherin pointed out, teachers need to perceive important classroom situations and interpret them with regard to broader principles of teaching and learning using their professional knowledge. According to Shulman (1986, 1987), the main facets of teachers' professional knowledge are (1) content knowledge (in the case of mathematics teachers, this would be the mathematics content knowledge, short: MCK), their pedagogical content knowledge ([M]PCK) and their general pedagogical knowledge (GPK). "Thus, teachers must use their knowledge of the subject matter, knowledge of how students think of the subject matter, as well as knowledge of their local context to reason about events as they unfold" (Van Es and Sherin 2002, p. 574f.).

Linking teachers knowledge to their actual performance in the classroom, Blömeke et al. (2015, see Fig. 1) proposed a model of competence as a continuum, and integrated the situation-specific skills perception, interpretation and decision-making into their model of competence as a mediator between disposition and performance. Therefore, knowledge was hypothesized underlying performance but the relation may be mediated by the situation-specific skills, "cobbled together in response to task demands, somewhat differently for each person" (ibid., p. 6).

Referring to the comparison of novice and expert teachers, it was assumed that expert teachers use their knowledge more flexibly to interpret classroom incidences while this is more problematic for novice teachers (Berliner 2001). Kersting (2008) and Kersting et al. (2010, 2012) found that teachers' mathematical content knowledge for teaching positively related to their ability to interpret classroom videos, concluding that "teachers used their pedagogical and mathematical content knowledge for teaching when analyzing classroom situations" (Kersting 2008, p. 14). Accordingly, König et al. (2014) found that teachers' skill to interpret

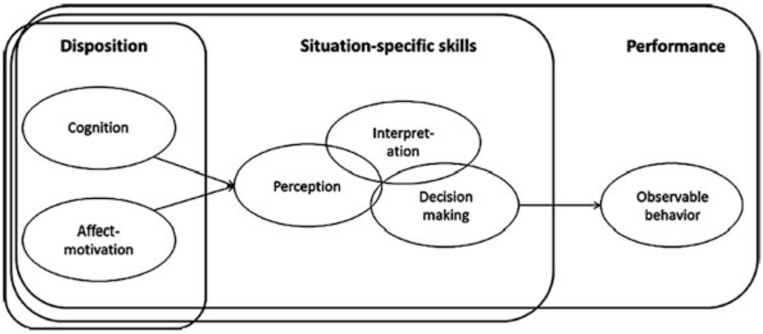

Fig. 1 Modeling competence as a continuum. Reproduced with permission from Blömeke et al. (2015), © 2015 Hogrefe Publishing

classroom incidents significantly correlates with their GPK while this connection was not found for teachers' skill to perceive specific classroom events and their GPK. With regard to comparing a video-based assessment of teachers' classroom management expertise and teachers' general pedagogical knowledge assessed by a paper-and-pencil test, König and Kramer (2016, p. 148) found that "teachers' general pedagogical knowledge and classroom management expertise are two different constructs, although they are substantially and positively inter-correlated."

However, Blomberg et al. (2011) proposed—based on their findings—that professional vision is a generic ability but not necessarily subject-related, while Dreher and Kuntze (2015, p. 110) found that "there is not a simple relationship between successful theme-specific noticing and a single component of professional knowledge. Instead, drawing on a variety of different components of professional knowledge and views can result in successful theme-specific noticing".

In this regard, Schäfer and Seidel (2015) pointed out that there is still only little research about the connection between teacher noticing and different knowledge facets. This was one of the main starting points for the following analyses that focused on mathematics teachers' noticing and its connection to their knowledge base.

Research Question

Many of the empirical studies presented in the previous section used video as a tool for representing practice to analyze teachers' expertise and teachers' noticing (cf. Schäfer and Seidel 2015; Kersting 2008; Kersting et al. 2010, 2012; Blomberg et al. 2011; Van Es and Sherin 2002). These videos had different functions in accordance with each of the respective study's research aim. For example, Van Es and Sherin (2002) used videos in teacher training (so-called video clubs) to discuss and reflect the teachers' practices with the group of participating teachers while Kersting (2008) used video to measure the quality of teachers' classroom analysis.

Following the aim to identify aspects that characterize how expert teachers notice classroom processes, the present study was based on data collected with a video test instrument and aimed at studying the following research question:

How can video, as a tool for representing teaching practice, support the investigation of aspects of teacher expertise, such as noticing (in this study the subdimensions of perception and interpretation)?

To answer this question, data collected with the video instrument of the TEDS-Follow-Up study was used, which will be described in the following section. Primary mathematics teachers' responses to a video sequence will be the focus of the following analyses and will be complemented by information about teachers' knowledge from another test part of the TEDS-Follow-Up study. The following section presents the methodological approach and describes the TEDS-Follow-Up study and its test instruments.

Methodological Approach

In order to utilize the potential that video offers to investigate aspects of teacher expertise, a qualitative approach was chosen using the data from the video-based test of the TEDS-Follow-Up study.

The TEDS-Follow-Up Study

TEDS-Follow-Up was the longitudinal Follow-Up to the international Teacher Education and Development Study in Mathematics (TEDS-M; e.g., Blömeke et al. 2014). TEDS-M was conducted under the auspices of the IEA (International Association for the Evaluation of Educational Achievement), and assessed the professional competence of future mathematics teachers at the end of their education in 16 participating countries. In Germany, about 2000 preservice teachers participated in the study in 2008. A subset of these teachers was reassessed in 2012 in the TEDS-Follow-Up study. In addition to 171 secondary school teachers, 131 primary school teachers who had about 4 years of work experience took part in the study which was realized as an online assessment. Following the model of competence as a continuum (see Fig. 1) and widening the theoretical framework, the TEDS-Follow-Up study closely referred to work in the field of teachers' expertise (Li and Kaiser 2011) and the concept of teacher noticing (Sherin et al. 2011a) and also included situation-specific skills as one main facet of teachers' competencies. More precisely, three situation-specific skills are considered:

Video-based test 1 (Questions on one video) Situation-specific skills in mathematics education (M_PID) and general pedagogy (P_PID)	Video-based test 2 (Questions on one video) Situation-specific skills in mathematics education (M_PID) and general pedagogy (P_PID)	Video-based test 3 (Questions on one video) Situation-specific skills in mathematics education (M_PID) and general pedagogy (P_PID)	Online test 1 (Shortened proficiency test from TEDS-M) Mathematics content knowledge, Mathematics pedagogical content knowledge	Online test 2 (Shortened proficiency test from TEDS-M) General pedagogical knowledge

Fig. 2 Shortened design of the TEDS-follow-up study

"(a) Perceiving particular events in an instructional setting,
(b) Interpreting the perceived activities in the classroom
(c) Decision making, either as anticipating a response to students' activities or as proposing alternative instructional strategies" (Kaiser et al. 2015, p. 373).[1]

The video test instrument of the TEDS-Follow-Up study consisted of three scripted video clips with corresponding questions. In order to identify expert teachers in the sample of TEDS-Follow-Up, the knowledge scores that resulted from the online tests 1 and 2 were used to provide additional information. As discussed in Section "Teachers' Professional Knowledge and Its Connection to Teachers' Noticing", teachers' noticing may be closely connected to their expertise. Figure 2 shows a short version of the design of the TEDS-Follow-Up study containing only the parts relevant for this paper.

The video-based tests were developed for the TEDS-Follow-Up study to assess the situation-specific skills perception, interpretation and decision-making. In the video-based tests, the study participants watched three short video clips (three to five minutes) showing excerpts of mathematics classes, and were asked to answer corresponding questions afterwards. The questions were presented in closed and open formats, and referred to didactical and pedagogical aspects of the teaching episode. The videos itself were scripted classroom scenes. Each of the videos had a different mathematical topic and showed classes in different instructional phases. Concerning the videos of the primary school study, all of the three video clips presented third-grade mathematics classes. One video dealt with geometry as mathematical content, while the instructional phases in this video were the introduction of the mathematical content by the teacher as well as part of the working phase of the children. Another video showed a mathematics class searching and discussing patterns and structures in Pascal's triangle. This video started during the

[1]For a detailed description of the theoretical base and conceptualizations in the TEDS-M study, see e.g., Blömeke et al. (2014); for the TEDS-Follow-Up study e.g., Hoth et al. (2016a), Kaiser et al. (2015).

working phase and included the beginning of the final phase of the lesson where the students presented their results in a whole class discussion. The third video dealt with third-grade learners working on a real-life mathematics task. The video showed the introductory phase of the lesson as well as the accompanying group work of the students subsequent to the introductory phase.

The participating teachers were provided with background information about the mathematical content of the lesson, additional information about the class and their learning conditions as well as information about what happened in former lessons. The teachers could always re-access this information when they answered the test questions subsequent to watching the video. In order to be as close to real teaching situations as possible, the teachers were only able to watch the videos once without the option to pause, rewind or fast-forward.

Data Sampling and Data Analysis

In order to illustrate how representations of classroom practice can help to investigate aspects of teacher expertise, we selected one question of the video vignette Geometry that required teachers to notice crucial aspects with regard to children's learning in the teaching episode presented in the video. More precisely, the question focuses on those aspects of the teaching sequence that basically led to a student's errors (as described below). Therefore, all information which is given within the complex and multifaceted teaching sequence became relevant and had to be interpreted with regard to the student's understanding as shown in the video. In the following, the video vignette Geometry is described in more detail as well as the selected question.

The video shows the beginning of a geometry lesson about Pentominoes[2] in a third-grade mathematics classroom. The students and the teacher sit in a circle of chairs while the teacher introduces these special geometric figures. She explains to the students how Pentominoes are built, presents their names and also mentions the concept of congruence to the children. Thereby, she shows one Pentomino example to the children (all squares are arranged in one row; see Fig. 3) by placing five squares in the middle of the circle. She also provides a poster that lists the building criteria of Pentominoes and shows some examples (see Fig. 4); she does not give an example for congruent Pentominoes but puts up a poster that verbally covers all aspects of congruence on an abstract level. The teacher then presents the task to the children who are asked to find all existing Pentominoes and justify the number of varying figures. As assisting materials, the teacher provides little squares that each student can use individually to build their Pentominoes. After building them, the

[2]Pentominoes are plane geometric figures that consist of five squares. Each of those squares must be connected to at least one other square with one side. Figures with four squares are called Tetrominos etc. For more information about Pentominoes see Golomb (1994).

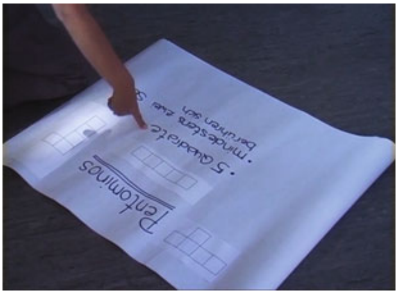

Fig. 3 Poster

Fig. 4 Teacher example

students are also asked to draw a representation of the Pentomino into their notebooks. Finally, the children get the chance to ask questions about the content presented as well as the given task.

In the following, the video shows one girl who presents her solution to the teacher. She explains that there must be 10 Pentominoes because she has 10 options to place the 5th square (see Fig. 5). The girl also provides an idea to prove her statement. However, she also makes two mistakes in her solution process. First, she finds Pentominoes only on the basis of one specific Tetromino and, second, she does not consider congruency.

The question that was selected for the analyses in this paper draws on these mistakes and asks the teachers to analyze the teaching sequence that they saw in the video-vignette with regard to the student's errors. The teachers were asked to identify three instances in the course of the teaching process shown that may have caused the student's errors (the errors itself were named in the question: not

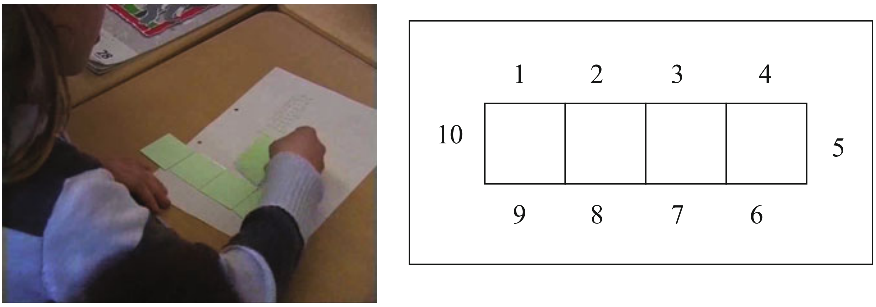

Fig. 5 Student's solution in the video-vignette "Geometry"

considering congruency and identifying Pentominoes only on the basis of one specific Tetromino). The teachers were given three open response fields.

This specific question was chosen for the following analyses because these aspects are typical for expert teachers' perception and interpretation and can be revealed through this complex question that requires the teachers to evaluate all the information they gathered in the course of watching the video. In the TEDS-Follow-Up study, teachers' answers were coded based on an extensive coding manual, which was based on in vivo-codes developed out of the test persons' answers and extensive expert ratings. For each question, 20% of the data was coded by two trained coders individually—inter-rater reliability resulting to $\kappa \approx 0.74$. In order to evaluate content-validity, several rounds of expert ratings were realized. Here, experts commented on the validity of the proposed questions (for details on the approach for analyzing of the rating-scale items see Hoth et al. 2016b).

Referring to the selected question, answers were coded as 'correct' if they presented instances that actually happened in the video. In addition, they had to refer to the student's errors and had to either explain her disregarding congruency or identifying Pentominoes only on the basis of one Tetromino. These requirements applied to five instances: (1) The teacher directly started with Pentominos, excluding preceding figures such as Tetrominoes, Triominoes etc., (2) the teacher presents one Pentomino example that also consists of that specific Tetromino, (3) the Pentomino examples on the poster were all build on the basis of that Tetromino, (4) the student did not understand the concept of congruency and (5) the didactic material which the teacher offered to find Pentominoes did not allow the students to flip and rotate their figures.

The data for the analyses were the answers of the 131 primary mathematics teachers who participated in the TEDS-Follow-Up study. As evaluation method, qualitative text analysis (cf. Mayring 2015; Kuckartz 2014) was used. Here, reducing evaluation procedures (Mayring 2015) were used to analyze the facets of the teachers' answers to the selected question. "The object of the analysis [using reducing processes] is to reduce the material such that the essential contents remain, in order to create through abstraction a comprehensive overview of the base material which is nevertheless still an image of it" (ibid., p. 373). Thus, the summarizing categories represent every aspect that the teachers mentioned with regard to the selected question and built the basis for analyzing aspects of teachers' expertise. Each category could be classified as more or less significant for explaining the student's errors and, therefore, provided information about the teachers' expertise. In this context, meaningful was related to the preciseness of the teachers' interpretations of the classroom incidents with regard to the student's errors. In addition, frequency analyses (cf. ibid.) gave insight into the distribution and emphasis of teachers' answers. The results of this inductive coding process as well as example of answers for each of the resulting categories are presented in Table 1.

The qualitative results were then related to the scores from the standardized MCK, MPCK and GPK assessments. As in the TEDS-M study, the international

Table 1 Category results of the inductive coding, teachers' example answers and frequency analysis

Category name and description	Example of an Answer[a]	Number of occurrences
Incomprehension of congruence: The student's mistake is based on her incomprehension of the concept of congruence	"The children did not understand the concept of congruency." (Teacher1164)	48
Ensuring students' comprehension: The teacher in the video did not ensure that the students understood the subject matter/the work assignment and so on	"The teacher did not check whether the children actually understood everything." (Teacher1050)	22
The teacher's example in the introduction: The student's mistake can be ascribed to the teacher presenting only one example of a Pentomino which was based on only one Tetromino in the introductory phase of the lesson	"The teacher placed that example which is based on that specific Tetromino in front of the children." (Teacher1090)	48
Missing student activities: Since the students did not have the opportunity to find and explore the Pentominoes/ their structure/congruent figures, they did not develop understanding	"The children should have worked practically on congruency." (Teacher1668)	20
More than five squares as working material: The student's mistakes may have happened due to the fact that the students had more than five squares available to work with during the working phase of the lesson	"She has more than five squares to work with." (Teacher1061)	4
The abstractness of teacher's description: The children did not understand the lesson's subject matter because the teacher's explanations were too abstract, involved too many technical terms and/or mathematical inaccurate facts	The teacher's explanation was too specialized. The students did not understand." (Teacher1294)	49
Missing motivation: The mistakes can be ascribed to missing motivation on the side of the student or the teacher failure to motivate her students	"Missing motivation" (Teacher1063)	3
Missing clarification of preceding figures: The student did not consider other Pentominoes because the teacher and her instruction did not consider the preceding figures such as Dominoes, Trominoes and Tetrominoes	"It would be possible to begin by searching all the Tetrominos (in the sitting circle)." (Teacher987)	10

(continued)

Table 1 (continued)

Category name and description	Example of an Answer[a]	Number of occurrences
Missing visual example and/or counterexample: The teacher failed to present a visual example and/or counterexample to the students that may have strengthened their understanding	"The teacher did not show an example of congruent Pentominoes." (Teacher 1217)	38
Pentomino examples on the poster: The poster that the teacher presents to the students about the building structure of Pentominoes also only shows Pentominoes that are based on the one Tetromino with all squares in one row	"The examples on the poster also show two Pentominoes made of that Tetromino." (Teacher1729)	11
Missing response to students' questions: The teacher did not respond (appropriately) to students' questions which may have negative influence on their understanding. In addition, she did not provide appropriate support such as strategic advice	"The teacher did not react on questions and suggestions of students in the beginning of the lesson." (Teacher1132)	16
Manageability of didactic material: The material that the teacher offers the students to work with does not enable the students to flip and rotate their found Pentominoes. Therefore, Karola is not able to consider congruency	"How is the girl supposed to test congruency if the Pentominoes that she finds only consist of individual components? This offers few opportunities to try out." (Teacher1278)	6
Color highlighting the Pentomino examples: The Pentominoes that the teacher provided on her poster are not optimally color highlighted	"All squares have the same color." (Teacher1665) "The alternatives on the poster are not colored clearly enough." (Teacher1209)	4
At least two sides are touching: The teacher explained the building structure of Pentominoes in that each of the five squares is touching at least one side of another square. However, the children did not understand the meaning of this statement	"Two sides of the squares have to be touching." (Teacher1691)	14
Pentominoes consist of five squares: The children did not know that the Pentominoes consist of five squares	"In Karola's solution, there are five squares in a row. She keeps positioning the sixth square to present her solution. She did not understand that a Pentomino consists of only five squares." (Teacher 1671)	4
Pentominoes' name: The children did not understand the term Pentomino as the name for the figures	"Karola does not know what Pentominos are and how they are build." (Teacher 1517)	3

(continued)

Table 1 (continued)

Category name and description	Example of an Answer[a]	Number of occurrences
Lack of student-to-student interaction: The mistakes may have occurred due to the lack of student-to-student interaction	"There was no group work or a possibility for the students to communicate with each other." (Teacher1433)	5
Answer does not refer to the question: This category subsumes all teachers' answers that do not correctly refer to the given question	"She uses the wrong equation." (Teacher979)	15

[a]A teacher's answer can include more than one aspect

average for these scores was set to 500 with a standard deviation of 100.[3] The subsample of German primary teachers who participated in the Follow-Up study had an average of 531 in their MCK and MPCK and an average of 644 in their GPK. More precisely, for each of the qualitatively found categories, the mean value was determined of those teachers' knowledge scores who named that category. The mean differences were then analyzed using *t*-tests in order to verify significant differences.

Results

In the selected task, 17 categories were constructed inductively. This high amount of categories already shows the complexity and variance of teachers' answers (see Table 1). In the following, the different categories are described and elaborated on with sample answers of the participating teachers. The third column reports the number of occurrences of each respective category within the teachers' answers. Since the teachers were asked to name three aspects[4] that may have caused the student's mistakes, teachers predominately named more than one aspect [min = 0; max = 6]. Therefore, one teacher may be assigned to more than one category and the number of occurrences does not equal the number of teachers who participated in the study. Furthermore, the answer a teacher provided in one open-response field could include more than one aspect. This evaluation procedure along categories with variable teacher groups seemed to be adequate for the evaluation of the richness of the categories identified and named by the teachers. For this qualitative coding 50% of the data was coded by two researchers independently, the interrater-reliability was satisfactory with $\kappa = 0.808$.

[3]For further details about the instruments and the scaling of the TEDS-M study see Tatto et al. (2012).

[4]There were three open response fields in the web-based test.

This analysis shows that the same video sequence and this one specific question provide the basis for different and multifaceted teacher answers. Obviously, teachers perceive different aspects in the same situation and with the same question as the starting point for their analyses, and their answers differ in regard to the aspects perceived. Some of the categories refer more closely to the student's errors and a possible explanation for the errors while others do not.

In order to understand the complexity of these categories as well as the approach that teachers used to perceive and interpret the information in the video, the categories are analyzed with regard to their appearance in the video in the following analysis. Some of the aspects mentioned by the teachers are directly observable in the video (have observable evidence in the video) while others are results of interpreting processes. This analysis, therefore, closely refers to the idea of evidence-based analysis (Van Es and Sherin 2002). For example, the category "Pentomino examples on the poster" directly refers to visual objects within the video—namely the Pentominoes which the teacher placed on her poster. However, in the video the teacher does not address these figures directly, they only appear on the poster but are not addressed by any actor in the video. Therefore, teachers who perceive these figures in the video have a rather holistic view on the teaching situation (meaning that they notice relevant aspects for student learning even if their attention is not directly alerted to it). Other categories such as "Manageability of didactic material" are an interpretative result of what was visually presented in the video but are not directly observable. In the video, one girl is shown presenting her solution to the teacher. She uses the material that the teacher offers to demonstrate her solution approach. However, she does not try to turn and rotate the Pentominoes which she already found and, therefore, the teachers whose answers belong to this category interpret the manageability of the material without seeing the student actually struggling to flip and rotate the figures.

Here, we can distinguish between categories that are perceived based on directly observable aspects in the teaching sequence and categories that result from interpreting processes. Then again, we can distinguish whether the aspects that are directly observable in the teaching scene are part of the action and, therefore, teachers' attention is directly drawn on them, or whether the aspects are observable in the background. As suggested by expertise research, novice teachers more often perceive surface characteristics in a teaching sequence while expert teachers distinguish important and less important information and interpret the situation deeply and profoundly (cf. Berliner 2001). In this regard, novice teachers may name categories that are directly observable and part of the main action while expert teachers more often name categories that result from interpreting processes.

In addition, the categories presented above can be classified with regard to their chosen perspective. While some teachers name more mathematical didactics aspects (such as the didactical material in the category "Manageability of didactical material" or the building of instruction of Pentominoes in the category "At least two sides are touching") other teachers focus on rather general pedagogical aspects (such as the mode of classroom interaction chosen by the teacher in the categories "Lack of student-to-student interaction" or the teacher's decision about the amount

Table 2 Classification of inductive categories and frequency analysis

Category name and description	Observability in the video	Number of occurrences	Perspective	Number of occurrences
The teacher's example in the introduction	Observable and part of the main action	66	Mathematical didactics aspects	235
At least two sides are touching				
More than five squares as working material				
Pentomino examples on the poster	Observable but not part of the main action	11		
Missing clarification of preceding figures	Not observable and not part of the main plot	158		
Missing visual example and/or counterexample				
Manageability of didactic material				
Pentominoes consist of five squares				
Pentominoes' name				
The abstract teacher's description				
Incomprehension of congruence				
Color highlighting the Pentomino examples	Observable but not part of the main plot	4	General pedagogical aspect	65
Missing response to students' questions	Not observable and not part of the main plot	61		
Lack of student-to-student interaction				
Missing student activities				
Ensuring students' comprehension				
Missing motivation				
Answer does not refer to the question		15		15

of student participation in the category "Missing students activities"). Here, the categories do not refer as much to the mathematical basis of the student's mistakes but conclude that specific (missing) aspects of the lesson's design may have caused the girl's errors. These classifications of categories are presented in Table 2.

The table shows that there are more categories about aspects that are not directly observable in the video, and this applies to mathematical didactics as well as to general pedagogical ones. These categories are results from interpreting the classroom events. This refers to 12 of the 18 categories. Moreover, 219 of the 300 mentioned aspects (235 mathematical didactics and 65 general pedagogical ones) belong to this classification (73%). 66 teachers' answers (22%) refer to aspects that are directly observable in the video and are also part of the main plot. Three categories belong to this classification. Finally, aspects that are observable in the video but not part of the main plot belong to the two categories "Color highlighting the Pentomino examples" and "Pentomino examples on the Poster". These categories are mentioned in only 15 of the teachers' answers (5%). In the following analyses, the observability of aspects will be taken into consideration as a numerical value. In this regard and considering the assumption that novice teachers predominately mention surface characteristics, aspects that are observable in the video and part of the main plot are coded as "0", aspects that are observable but not part of the main plot as "1" and aspects that result from interpreting process as "2".

In order to clarify whether these differences may be related to the teachers' different knowledge bases, Table 3 shows the mathematical didactics categories that resulted from the reducing process. For each category, the table shows the average estimate of the MCK (Mathematics Content Knowledge) and MPCK (Mathematics Pedagogical Content Knowledge) scores of all teachers who mentioned that category in their answers. Table 4 shows that connection between general pedagogical categories and teachers' average estimate of the MPCK and GPK (General Pedagogical Knowledge). The following analysis aims at identifying connections between teachers' noticing (in this case represented by their perception and interpretation) and their professional knowledge. Here, average scores of teachers' knowledge are presented for each of the categories in order to find out whether there are categories (resulting from expert teachers' noticing) that may be mentioned by teachers with higher professional knowledge and vice versa. In Tables 3 and 4, all average scores are colored light grey that significantly lie above the average score of the entire sample of German primary teachers, dark grey if it significantly lies below.

The results displayed in Tables 3 and 4 show that the relation between the categories and the teachers' professional knowledge is variable and not stable. Some of the categories that focus on mathematical didactics aspects of the teaching sequence (Table 3) were mentioned by teachers with over-average MPCK and MCK ("The teachers example in the introduction", "missing clarification of preceding figures" as well as "Incomprehension of congruency") while other categories were named by teachers with under-average MPCK and MCK ("Pentomino's name"). With regard to categories that focus on general pedagogical aspects of the teaching sequence (Table 4) there was only one category that was named by teachers with above-average knowledge (Color highlighting the Pentomino example) and one category that teachers mentioned who had under-average MPCK and GPK (Missing student-to-student interaction).

Analyzing the categories that were mentioned by teachers with rather low professional knowledge shows that those categories describe aspects of the teaching

Table 3 Contingency analysis between the mathematical didactics categories from the inductive codes and teachers' professional knowledge

Categories' names	Observability	N	Mean value MCK	Sign. (2-tailed)	Standard deviation	N	Mean value MPCK	Sign. (2-tailed)	Standard deviation
The teacher's example in the introduction	0	38	555	$p < .05$	115.1	38	552	$p < .05$	84.8
At least one side in common		12	516		105.6	12	518		79.1
More than five squares as working material		3	514	$p < .05$	46.7	3	561	$p < .005$	49.4
Pentomino examples on the poster	1	9	533		55.6	9	568	$p < .0005$	64.8
The abstract teacher's description	2	38	550		115.4	38	545		95.4
Missing clarification of preceding figures		10	572	$p < .00005$	79.2	10	555	$p < .05$	74.8
Missing visual example and/or counterexample		32	554	$p < .05$	107.3	32	538		84.2
Manageability of material		4	538		38.3	4	551	$p < .05$	40.8
Pentominos consist of five squares		4	513	$p < .05$	51	4	581	$p < .00005$	45.1
Incomprehension of congruence		38	567	$p < .0005$	92.3	38	559	$p < .005$	67.6
Pentominos' name		3	480	$p < .00005$	74.9	3	486	$p < .00005$	24.9
Total		214	545		99.7	214	544		82.4

sequence that do not directly relate to the student's errors. For example, the awareness of the figures' names ('Pentominoes') does not affect the solution process which is dominated by the missing consideration of congruency and using only one Tetromino as the basis. The girl could be able to find all existing figures if she knew the building requirements of the figures and understood the concept of congruency. However, she does not need to know the name of the figures, and this category does not relate to the girl's errors. However, it is true that the teacher introduces the figures and their name only very shortly and does not provide enough time for the student to learn the complex name "Pentomino" or to discover the meaning of it. Therefore, teachers who named this category obviously noticed a specific teaching decision that they did not agree with or they would have done differently. However, this category does not explain the girl's errors.

Analyzing the category "Missing student-to-student interaction" results in similar conclusions. It is true that the teacher in the video chose to let the students work for themselves and did not offer opportunities for student-to-student interaction. It is

Table 4 Contingency analysis between the general pedagogical categories from the inductive codes and teachers' professional knowledge

Categories' names	Observability	N	Mean value MPCK	Sign. (2-tailed)	Standard deviation	N	Mean value GPK	Sign. (2-tailed)	Standard deviation
Ensuring students' comprehension	1	14	531		42.1	15	651		79.1
Missing student activities	2	15	549		66.5	16	637		79.1
Missing motivation		2	523		43.6	2	637		47.6
Missing response to students' questions		12	519		113.4	13	657		75
Color highlighting the Pentomino examples		3	576	$p < .0005$	71.4	4	666	$p < .05$	40.2
Missing student-to-student interactions		5	462	$p < .0005$	126.4	4	620	$p < .005$	114.7
Total		51	529		82.3	54	647		75.7

possible that other students did not make the same mistakes as the girl, and student-to-student interaction could have resulted in the girl correcting her mistakes due to the interaction. However, this does not provide possible reasons for the girl's errors as inquired by the question. In addition, the other students followed the same introduction as the girl and might have made the same mistake. Again, teachers noticed an element in the teaching situation (the missing group work) that they would probably include into their own teaching. However, this was not linked to the student's understanding.

The categories named by teachers with average knowledge, have a close connection to the girl's errors. Indeed, the girl does not consider congruency (category "Incomprehension of congruency") and the teacher in the video shows only Pentomino examples that are based on the one specific Tetromino that the girl uses exclusively to construct her figures (categories "The teacher's example in the introduction" and "Examples on the Poster"). In addition, the teacher's organizational decisions that may have caused the error are addressed in the category "Missing clarification of preceding figures". If the teacher discussed the preceding figures in class, this specific error might not have happened.

Overall, it appears that some categories are not linked to the girl's errors while other categories refer to them closely. Tables 3 and 4 show that, for this specific situation, teachers who were able to notice relevant teaching instances and made connections between these specific instances and the girl's learning, often had above-average knowledge. Here, noticing teaching instances as problematic for the understanding and learning of students becomes obvious as one main aspect of teachers' expertise. In an additional analysis, the observability was correlated with the teachers' knowledge scores. However, this analysis did not show significant connections.

Table 5 Contingency analysis between the amount of mathematical didactics aspects noticed and the professional knowledge

Number of didactical aspects	N	Mean value MCK	Sign. (2-tailed)	Standard deviation	N	Mean value MPCK	Sign. (2-tailed)	Standard deviation
0	1	564	$p < .005$	.	1	441	$p < .00005$	.
1	16	509	$p < .05$	128.3	16	515	$p < .05$	120.9
2	33	543		73.8	33	548		65.4
3	21	547		99.3	21	533		92.3
4	9	518		37.2	9	536		27.5
5	3	437	$p < .00005$	62.2	3	534		48.3
Total	83	531		91.3	83	535		82.5

In addition to the accuracy of teacher noticing, its complexity and multi-facetted nature may be a characteristic of teachers' expertise. In this regard, the following analyses link the amount of aspects listed by the teachers (as an indication of teachers' wide-ranging perception) and their professional knowledge. Table 5 shows these relations for the amount of mathematical didactics aspects mentioned by the teachers, Table 6 provides that information for the general pedagogical aspects.

Table 5 shows the relation between the number of didactical aspects that the teachers mentioned and their mean value of MCK and MPCK. This analysis may be a first indication that novice teachers mention only few aspects because the teachers who mentioned only one or less didactical aspects have under-average MPCK. Regarding the contingency analysis between the number of pedagogical aspects mentioned by the teachers and their professional knowledge Table 6 shows that teachers addressing the most pedagogical aspects (in this specific case this amounts to two aspects) have above-average MPCK. This may be a first indication that expert teachers notice a variety of different aspects. However, no significant correlations were found between the number of the didactical aspects and the teachers' knowledge scores, nor between the number of pedagogical aspects and the teachers' knowledge scores.

Table 6 Contingency analysis between the amount of general pedagogical aspects noticed and the professional knowledge

Number of pedagogical aspects	N	Mean value MPCK	Sign. (2-tailed)	Standard deviation	N	Mean value GPK	Sign. (2-tailed)	Standard deviation
0	39	549		84.2	38	646		95.3
1	31	508	$p < .005$	87	31	652		82.2
2	31	556	$p < .05$	45.7	12	638		66.9
Total	83	535		82.5	81	647		85.9

Summary, Discussion and Conclusions

With the aim to analyze how video as a tool for representing classroom practice can help to investigate aspects of teachers' expertise, the video instrument of the TEDS-Follow-Up study served as an example of analyzing what teachers notice and how accurate they are in analyzing important classroom incidents, i.e. those classroom incidents that may have caused the student's errors. In the study described, 131 primary mathematics teachers analyzed possible causes for one student's errors that were presented in one video-vignette. This specific task was selected for the analyses because it required the teachers to analyze the entire teaching sequence and link specific aspects to the student's understanding. The qualitative analysis resulted in 17 categories that show the diversity of aspects mentioned by the teachers. The aspects that are central in the categories analyzed refer either to objects and incidents that were directly observable in the video and were also part of the main action, while other aspects were observable but not part of the main action and even other aspects were a result of interpretation processes and were not observable and not part of the main action in the video. Analyses showed that the most mentioned categories were the interpretative ones, second most common, were aspects that were directly observable in the video, and the least often mentioned were aspects that were observable but not part of the main action. Contingency analyses between the categories and teachers' professional knowledge showed that categories directly linked to the student's errors corresponded with above-average knowledge, while categories that were not the cause of the errors but represented weaknesses of the teaching sequence shown in the video were connected with below-average knowledge. Finally, a connection between high knowledge and noticing a variety of aspects in the teaching sequence was indicated but could not be confirmed quantitatively.

The analyses showed that video as a tool for representing practice can offer - to a specific extent - the multifaceted activities that also occur in real classes. The great amount of categories as well as the different perspectives that the teachers chose in their answers provided evidence that teachers noticed very different things in the same (3-minutes!) video while referring to the same question. With regard to teachers' expertise, it became clear that some teachers noticed crucial aspects for the learning of students while other teachers perceived rather surface characteristics such as the working arrangement in class (individual instead of group work) that are not directly relevant for the students' understanding (cf. Berliner 2001). In addition, and in accordance with the findings of Sabers et al. (1991), the results showed that expert teachers analyzed the reasons for the errors in greater depth—categories that were directly linked to the student's errors were named by teachers with comparatively high knowledge. As Van Es (2011) pointed out, teachers at the lower stages in the Learning to Notice Framework more often attended only to the teacher's pedagogy without linking it to subject-based reflections. Many of the categories that evolved in the present analyses, also focus on the teacher's pedagogy such as her decision on work arrangement ("Lack of student-to-student interaction"), her coloring of the Pentominoes ("Color highlighting the Pentomino examples"), her waiver of student activities and answering student questions ("Missing students' activity" and "Missing response to students' questions") or that she did not ensure the student's understanding of the work assignment ("Ensuring students' understanding"). Most of these categories that focused on the teacher's pedagogy were not directly linked to the student's errors because they were not subject-related and, therefore, did not refer to the student's mathematical thinking.

Focusing on the connections between teachers' noticing and their professional knowledge, the findings suggest that high professional knowledge is linked to noticing crucial aspects of student learning. However, and in accordance with findings from other studies (e.g. Dreher and Kuntze 2015), the results did not indicate that the categories that were more subject-related are linked to high subject-specific knowledge, while categories with a general pedagogical focus are linked to high general pedagogical knowledge. For this specific teaching incidence —analyzing instruction with regard to students' understanding—it showed that teachers with high professional knowledge (subject-specific *and* general pedagogical) identified crucial elements while teachers with low professional knowledge focused on elements that they observed and clarified as insufficient but they were not able to relate them to the student's understanding.

However, critically reflecting on the video instrument and its potential to assess teachers' professional competencies, involves reflecting on what the participating teachers analyzed, evaluated and judged in a teaching sequence with unknown students. The participating teachers did not have the same background knowledge about the students, their behavior and prior knowledge as they would have had from their own class. Since the videos were scripted and edited, teacher noticing was assessed in very condensed situations, where significant things happened in a very short time. Although our qualitative analysis yielded interesting results, interviewing teachers about this specific question, using the video as prompt, might result in even richer variety of categories.

In general the analyses pointed out that the relation between teachers' professional knowledge and noticing facets of professional competencies are not as clear-cut as our study assumed at the beginning. Further analyses with broader instruments are necessary in order to come to more secure results. However, these analyses showed that video as a tool for representing practice can help to investigate aspects of teacher expertise, such as noticing crucial aspects of students' learning.

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Representations of Practice in a Video-Based In-Service Teacher Professional Development Project and in Its Evaluation

Sebastian Kuntze

Abstract A key motivation behind using classroom videos in professional development activities is the way how videos can represent classroom situations. However, video offers a variety of possibilities of representing classrooms and the framing of classroom videos in contexts of both professional development and its evaluation plays a decisive role. To find out more about the role of video representations of practice, there is a need to analyze learning opportunities in professional development and possibilities of investigating the development of the participating teachers. This need of analysis is consequently addressed on the basis of experiences and empirical findings from a video-based in-service teacher professional development project. The analysis suggests that the way teachers perceive classroom situations can play a key role for their learning, not only in the process of professional development activities, but also in their evaluation.

Keywords Representations of practice · Video-based evaluation study
In-service teachers · Professional development

Introduction

The question of how representations of practice can promote in-service teacher professional development (PD)—a key question both for practice decisions and for research in the domain of PD—leads to several follow-up questions. Firstly, it targets the learning opportunities connected with representations of practice in professional development activities. These learning opportunities in turn depend on the aims and contents of the PD activity, as well as on the instructional framing of the representations of practice, such as for example specific stimuli for reflection, criteria-based analysis, or structured observation. Corresponding follow-up

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questions thus focus on an analysis of the learning opportunities around representations of practice which are provided by specific PD activities.

Secondly, the initial question leads to the follow-up question, how the teachers' actual development during the work with representations of practice can be described methodologically, yielding insight into the potential effectiveness of such PD activities. For answering this question, a theoretical framework providing relevant criteria is needed, as well as a methodology which affords carrying out related research into the teachers' development. If corresponding research instruments use representations of practice, we are confronted with the additional question: how can representations of practice help to investigate aspects of teacher expertise (such as e.g. criteria-based aspects of teachers' noticing or analyzing)—and their development?

These questions imply that methodological challenges have to be faced, both in designing and analyzing PD which uses representations of practice, and in the empirical investigation of teacher development in such PD.

Form these thoughts it follows that in particular when considering the initial question it becomes apparent that the key questions of the discussion group which are reflected in this volume (cf. Buchbinder and Kuntze, this volume) are interdependent and that they support themselves mutually, in the sense that answers should address these questions simultaneously. The questions of the discussion group can thus help to reflect on the status of the research presented here, that focuses on the role of representations of practice and their potential impact on the development of teacher expertise. Beyond the example considered in the following, empirical research about PD activities which use representations of practice should in particular consider these questions carefully, as continuous reflection helps to evaluate the contribution such studies can make for theory development and practical implications.

This chapter aims at reflecting on a specific video-based in-service teacher PD project—it will be called ViPD project in the following—and the related evaluation research, which were both based on the use of representations of practice, in this case on the use of videotaped classroom sequences (cf. Kuntze 2006). The discussion questions presented above will thus be answered for the example of the ViPD project and its evaluation. Implications for a broader context can be drawn for both theory elements related to the role of representations of practice in professional development and for practical decisions regarding video-based professional development activities and the design of their evaluation research. Consequently, the reflection in this chapter can contribute to the discussion of both empirical findings and the development of professional development programs.

This chapter discusses the ViPD project and related empirical findings against the background of the question 'how can representations of practice promote in-service teacher professional development?' and its follow-up questions. However, for the follow-up discussion, the chapter introduces a theoretical background which is not only specific for the ViPD Project. After an overview of this theoretical background, which focuses on video representations of practice and teachers' professional knowledge components, the specific theoretical scope of the

ViPD project is introduced and perspectives of evaluation research of PD in general and the ViPD project in particular are discussed. Then, both the research interest related to the discussion group questions and the corresponding research questions from the empirical evaluation research of the ViPD project are presented. The follow-up sections present a description and discussion of the design of the ViPD project, as well as design, methods and results of the associated evaluation research, which will be discussed in light of the leading question: how can representations of practice promote in-service teacher professional development?

Theoretical Background

Valid reference to teachers' professional practice contexts is a requirement both for professional learning, such as it is in the scope of professional development (PD) activities, and for research into aspects of teacher expertise and their development. Whereas real classroom situations only happen once in a certain way, representations of classroom practice allow repeated access to classroom practice. The notion of representation of (classroom) practice is understood analogously to Goldin and Shteingold's (2001) notion of representations of mathematical objects: a representation of (classroom) practice is something that stands for this classroom situation, such as, for example, a drawing showing the situation, a cartoon, a narrative, a transcript, or a video showing the situation. A representation of a classroom situation can mostly be viewed (or e.g. read) repeatedly, and hence it affords a relatively easy access to analysis by individuals, by groups of teachers, or by researchers. However, regardless of how a classroom situation is represented, the representation is never congruent with the situation itself: it would be barely impossible to fully represent all individual perspectives of the persons involved in a classroom situation, for instance. Even if a situation is videotaped, it is represented from a certain perspective, and even if video includes a large amount of context information (cf. Petko et al. 2003), it is impossible to capture all potentially meaningful context aspects. In Nanette Seago's words, "video is but a tool" (Seago 2004, p. 263)—so videotaped classrooms should not be confounded with the classroom situations they represent.

Representations of Practice in Video Format and Their Potential Role for Professional Development

As video technology affords showing classroom situations in a relatively information-rich format as far as the situation context is concerned (cf. Petko et al. 2003), representations of practice in video format have a great potential for teacher professional development. Compared with transcript-like text formats, for example, videotaped classroom situations can bring a rich spectrum of nonverbal components of interaction in the form of visual and acoustical information. It is a widely shared experience that in PD activities videotaped instructional situations are mostly very inviting for teachers to discuss instruction-related issues. Since situations may be viewed repeatedly, the interactions which are of interest can be thoroughly identified and examined very closely (cf. e.g. Beck et al. 2002).

When examining representations of practice in teacher professional development activities, the framing of the representations plays a key role for teachers' observations and learning: Teachers' situation-specific observation and analysis might, for example, be prompted with respect to specific criteria in explicit observation tasks. Even if a PD activity aims to avoid influencing the teachers' thinking by asking them very open questions only, the participants' implicit understanding of the goals of a PD activity might trigger their noticing (Sherin 2003; Borko et al. 2008; Krammer et al. 2008). Krammer et al. (2008) suggest an explicit meta-focus on different approaches to videotaped representations of practice in the work with PD activity participants, so that they are supported in distinguishing between observations on the one hand and judgments about a situation on the other.

It is also necessary to recall that any observation is based on prior knowledge (e.g. Ernest 1993), which, thus, has to be asserted also for the case of teachers' situation-related noticing (van Es and Sherin 2008; Sherin et al. 2011). In particular, noticing in the sense of knowledge-based reasoning (van Es and Sherin 2008) highlights this aspect: The design of PD activities should take into account that representations of practice can be perceived differently by different teachers. Studies such as Kuntze (2012) and Dreher and Kuntze (2015a, b) have investigated whether teachers' situation-specific views interdepend with their less situated views, and whether teachers' noticing is connected with professional knowledge aspects, for example. The findings suggest that professional knowledge and teachers' views may impact on their perception of classroom situations (cf. Lerman 1990).

Addressing Components of Professional Knowledge with Representations of Practice

Consequently, it seems possible to address professional knowledge (e.g. Shulman 1987; Leinhardt and Greeno 1986; Kuntze 2012) and teachers' views (e.g. Pajares 1992; Törner 2002; Kuntze 2012) through the use of representations of practice in PD. More specifically, teachers' situation-specific knowledge and views can be directly activated when watching and reflecting on classroom situations. Moreover, the empirical links to less situation-specific components of teachers' professional knowledge indicate that these less situation-specific components of teachers' professional knowledge can be developed together with the situation-specific components of the teachers' professional knowledge. This can be visualized particularly well in Fig. 1, which shows a model of professional knowledge (discussed in more detail in Kuntze 2012). This model shows components of professional knowledge according to three dimensions. Shulman's (1986a, b, 1987) domains of professional knowledge—pedagogical knowledge, pedagogical content knowledge, curricular knowledge, and subject matter/content knowledge—are included in the vertical rows. The model further includes teachers' views and beliefs as aspects of their professional knowledge in a corresponding spectrum, which are visualized by the front and rear cells. All the cells in the model are not considered as strictly separable (cf. Pajares 1992)—their purpose is mainly to support orientation. Most relevant for the following discussion is the distinction of different levels of situatedness of components of professional knowledge, which are shown as the horizontal layers. Teachers' knowledge and views can be very global, such as, for example, the epistemological "mathematical world views" described by Törner (2002), but views and knowledge can also be content domain-specific (e.g. knowledge or views specific for geometry), content-specific (e.g. knowledge about students' typical difficulties related to quadratic equations) or even situation-specific (e.g. knowledge or views how best to react in a specific learning situation). This distinction of levels of situatedness takes into account that elements of professional knowledge have shown to be structured episodically (Leinhard and Greeno 1986) and it can help to understand how different components of professional knowledge on different levels of situatedness can be consistent with each other or even be in conflict with each other (cf. e.g. Doerr and Lerman 2009). A teacher who holds very constructivist convictions on the global level (as far as learning mathematics is concerned in general), for example, might at the same time hold rather non-constructivist situation-specific views (e.g. views about teaching and learning by rote of a specific formula or views about how to react when a specific mistake appears in the classroom with respect to a particular content). In such cases being able to

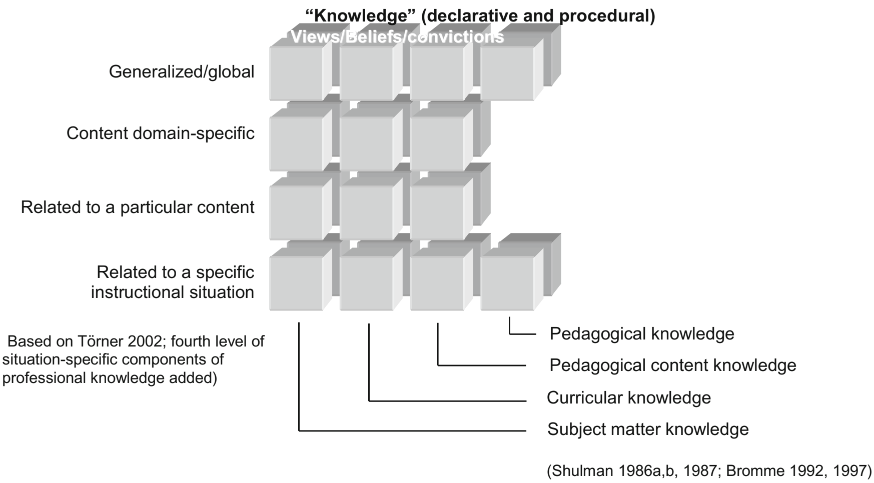

Fig. 1 Model for components of professional knowledge (Kuntze 2012, p. 275)

distinguish between components of professional knowledge on different levels of situatedness is helpful (e.g. for describing the teacher's knowledge-based decision-making).

Supporting Teachers' Knowledge and Views Related to Classroom Interaction with Representations of Practice

Reflecting on video representations of practice in PD activities can be expected to address the situation-specific level of professional knowledge of the PD participants. Through examining specific classroom situations, teachers are very likely to connect their thoughts and observations with corresponding situation-specific views and knowledge, e.g. by comparing with what they know about a similar situation they may have experienced in the past. They might also generalize from the situation towards more general components of professional knowledge or apply more global knowledge to the specific case of the classroom situation they are dealing with.

Against the background of the model in Fig. 1, it becomes visible that PD activities, which use representations of practice and aim at developing professional knowledge, primarily access professional knowledge and its development through the situation-specific level, as situation-specific professional knowledge is most close to what can be observed in representations of classroom situations. Accordingly, focusing on situation-specific views of the participating teachers over time can help to describe developments in these elements of professional knowledge of the participants particularly well. Moreover, possible changes on the situation-specific level might be a first indicator of possible changes also in other components of professional knowledge.

In the evaluation study presented and discussed from Section "Design and Methods of the Evaluation Research: Investigating Teachers' Situation-Specific Views with the Use of Representations of Practice" on, teachers' situation-specific views were thus in the foreground. The findings from this evaluation study will help to discuss how such views may develop during PD.

A Specific Video-Based PD Project

As an example of a PD project—the so-called ViPD project—which aimed at developing teachers' professional knowledge and views through representations of practice in video format, this chapter refers to a PD project which concentrated on classroom interaction with a particular focus on the quality aspects of (1) cognitive activation, (2) intensity of argumentation and (3) learning from mistakes (e.g. Clausen et al. 2003; Clausen 2002; Heinze 2005; cf. Klieme 2002). In general, teachers' professional knowledge components connected to these three quality criteria can be global (e.g. views about the importance of reasoning and argumentation for mathematics and mathematics instruction in general), content domain-specific (e.g. knowledge about approaches to proof methods in geometry), content-specific (e.g. knowledge about introducing specific proof problems) or situation-specific (e.g. views about a teacher's reaction in a specific classroom or knowledge about possibilities of reacting differently).

The quality aspects of cognitive activation, intensity of argumentation and learning from mistakes had been chosen for the PD project, as prior research had shown the prevalence of a teacher-centered small-step interaction pattern in Germany (e.g. Baumert et al. 1997; Stigler et al. 1999), associated with rather low cognitive activation. This type of interaction pattern had been observed even in topics which should be marked by argumentation such as geometrical proof (e.g. Kuntze and Reiss 2004). Also, opportunities of learning from mistakes had very rarely been used by teachers for creating productive learning opportunities. Since the time of these video studies, classroom practice might have changed—however fundamental changes in the way how interaction is orchestrated in German mathematics classrooms have not been reported since then.

In the ViPD project, the use of video representations of practice appeared as highly appropriate, as the three quality aspects of classroom interaction introduced above can be observed in videotaped classrooms, e.g. by external observers (Clausen et al. 2003; Clausen 2002). The work with representations of practice in the ViPD project targeted the teachers' situation-specific professional knowledge in the first place, with the additional aim that teachers connect this knowledge with other situations in their instructional practice as well.

Evaluation Research

According to an overview study by Lipowsky (2010, 2004) about characteristics of effective PD, effects of PD activities can be measured on four different levels: research can focus on impacts of the PD activity on (1) the level of teachers' perceptions, e.g. reports about the perceived usefulness of contents or reports about effects by the participants of a PD activity, (2) the level of professional knowledge or views of the teachers, (3) the level of the participants' instructional practice and (4) the level of learner variables such as development in achievement or in motivation.

For the evaluation research into possible effects of the reflection-oriented work on video representations of practice in the ViPD project, the first two levels by Lipowsky are particularly interesting. This decision was made as it allowed to avoid high complexity in the research design and the available testing time was used to deepen the research on the second level, with a specific emphasis on the teachers' situation-specific views. Also as part of the evaluation research methodology, representations of practice can be used so as to yield insight into participants' situation-specific views (Kuntze and Friesen 2016a; Friesen et al. 2015). As such research designs are still relatively scarce, considering the ViPD project's evaluation research allows local answers to the questions: 'How can representations of practice be used to investigate aspects of teachers' professional knowledge and views?' and 'How can corresponding methodological challenges be addressed when designing research settings based on representations of practice?'. In research designs which use representations of practice, solutions to these questions have to be found when empirical answers to the initial question "how can the use of representations of practice promote in-service teacher PD?" have to be generated—with this respect, discussing the ViPD project can make a significant contribution to the discussion related to this question.

Research Questions for the Review of the Professional Development Project and Its Evaluation

According to the thoughts presented in the previous section, the following questions guide a review and discussion of the ViPD project and of the evaluation research results related to that PD project. The key question is:

How can the use of representations of practice promote in-service teacher professional development?

As already pointed out in the introduction, finding answers to this question requires answering several follow-up questions. Figure 2 highlights this relationship. The research interest related to the key question (see light gray arrow

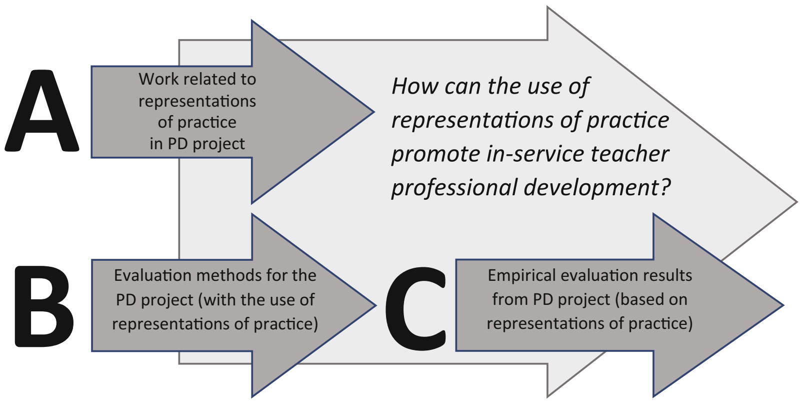

Fig. 2 Prerequisites (A and B) for answering the key discussion question—and need for reflection related to A and B

in Fig. 2) is coupled with questions associated with the arrows A and B, which allow empirical answers corresponding to arrow C.

The key question may thus be accentuated through the following more specific sub-questions, which connect the key question with the perspective of the ViPD project (cf. Fig. 2):

(A) What work related to representations of practice was carried out in the PD project and how were representations of practice used?
(B) How can teachers' situation-specific views relevant for the evaluation of the PD project be investigated with the use of representations of practice?
(C) How did the teachers' situation-specific views develop throughout the PD project?
(D) What implications beyond the ViPD project can be drawn from the findings with respect of the key question above?

This means that the key question can be answered in a combined way, by describing the work on representations of practice in the ViPD project (A), by describing the development of the teachers' views (C) which in turn requires reflecting about the design of the evaluation research (B), as the evaluation research was based on representations of practice as well. Questions (A) up to (D) will be addressed in the following sections.

The Intervention: Design of the PD Project's Video-Based Work with Representations of Classroom Situations

This section aims at providing answers to question (A) above. Corresponding to the aims described in the theoretical background section, criteria related to cognitive activation, intensity of argumentation, and learning from mistakes were in the center of the video-based work of the ViPD project. These criterion domains were already introduced above, the corresponding theoretical and empirical framework was the base for the aims of the work on classroom situations together with the participating teachers.

The aims of the ViPD project were to develop professional knowledge and views of secondary teachers. In the ViPD project, video-based work on classroom situations was in the foreground: the participating teachers reflected on the way how classroom interaction was orchestrated by the teachers in the videotaped classroom situations with respect to the three quality aspects: cognitive activation, intensity of argumentation and learning from mistakes. Authentic videos of relevant situations were used in the PD. The video sequences showed every-day classroom instruction, rather than 'good examples' of classrooms, to focus the discussion of participating teachers also on reflecting on possibilities of improvement. The use of every-day authentic classrooms has also additional pros: observing such representations of practice may make it easier for participating teachers to connect with their own every-day experience, and it is almost impossible to question the authenticity of the situations. Among the cons are (1) the potential non-availability of situations with specific characteristics and (2) the need for providing the participating teachers with very specific information about the situation context. Moreover, a further inconvenience of authentic videotaped classrooms consists (3) in the complexity of authentic classrooms, which often also show additional processes with potential relevance for criteria which might rather not be in the special focus of a PD activity. However in the case of the ViPD project, the video sample (and its size) allowed to solve these potential problems.

For the video-based evaluation research in the ViPD project, it was possible to select two videos which had a high relevance for all three focus quality aspects (i.e. for cognitive activation, intensity of argumentation and learning from mistakes). At the same time, the videos were selected so as to span a contrast between them with respect to the quality aspects. The two videos were comparable in length.

The videos selected for the learning opportunities in the ViPD project varied in content, so that in addition to cognitive activation, intensity of argumentation and learning from mistakes, also other quality aspects were discussed.

The ViPD project consisted of three working phases, conducted on weekends, with the whole group of teachers, and two intermediary phases, in which the participating teachers were asked to observe their own practice and to try to improve cognitive activation, intensity of argumentation and learning from mistakes in their own classrooms. The PD project, thus, aimed at bridging the gap between the work on representations of practice in the group, and the participants' individual classroom practice at their schools.

Group sessions around a classroom video started typically with context information about the video (grade, topic, school type, teaching prior to the sequence, etc.). The video was then shown, corresponding transcripts were provided. If required, it was possible to watch the video, or parts of it again. Then, any participants' observations they chose to mention were discussed in the group. As the videotaped classrooms showed every-day situations, the teachers had the opportunity to suggest ideas for improvement of the classroom interactions, which were discussed among the participants as well in terms of enhancing cognitive activation, intensity of argumentation, and learning from mistakes. Thus, the work was oriented towards reflection and the analysis of ideas for alternative teacher actions. The setting was different from lesson study approaches (cf. Dreher and Kuntze 2012), as the goal was not to develop an "ideal lesson", but rather to find starting points for improving every-day classrooms with respect of the three criterion domains.

The representations of practice, hence, were intended to play an activating role for evoking situations which the participating teachers might have experienced in similar ways, so that they could connect their experience with criterion knowledge about cognitive activation, intensity of argumentation, and learning from mistakes. Frequently, the discussion did not return to the video, but open up towards teachers' possibility to react to certain types of events in classrooms in a more general way. For instance, some general strategies of enhancing argumentation were derived both from the representations of practice and the teachers' experience.

If necessary, the facilitator focused the discussion back to one of the focus quality aspects, i.e. by inviting the group to draw conclusions related to the focus criteria, to assure that some observations were connected to these criteria. The discussions usually lasted from 40 up to 90 min per videotaped classroom situation.

To sum up, the work with the videotaped representations of practice in the ViPD project used methods of participant-centered, cooperative and open discussion around criteria associated with cognitive activation, intensity of argumentation, and learning from mistakes.

The video-based work was complemented with the encouragement of the participating teachers to observe their own classrooms (and their teaching) with respect of the three criterion domains and to experiment in their classroom in order to enhance cognitive activation, intensity of argumentation, and learning from mistakes during the two months-long intermediary phases between the three PD weekends. In this way, the work on representations of practice during the ViPD project meetings was combined with opportunities of connecting with the participants' "real" classroom practice.

The video-based work was preceded by a pre-test at the beginning of the ViPD project (details will be described below), and by an introductory presentation on the criteria of cognitive activation, intensity of argumentation and learning from mistakes. The three PD weekends also contained other activities, such as preparing the intermediary phases, discussing the results and experiences participants had collected in the intermediary phases, small-group work, etc. Towards the end of the PD project, the participants were asked to fill in the post-test (for details see next section).

Design and Methods of the Evaluation Research: Investigating Teachers' Situation-Specific Views with the Use of Representations of Practice

Research question (B) concentrates on how teachers' situation-specific views can be investigated with the use of representations of practice. As highlighted above, representations of practice have the potential of activating teachers' professional knowledge and beliefs, in particular those on the situation-specific and content-specific levels. However, designing specific research instruments requires facing methodological challenges (Kuntze and Friesen 2016a). One of these challenges is how "representative" a representation of practice is for the construct being assessed: For instance, if a study is interested in how teachers see the intensity of argumentation in a classroom situation, it is indispensable that the situation contains at least hypothetical opportunities for argumentation, which should be clear to the teachers who are being asked to comment on the classroom scenario.

In the case of the evaluation study considered here, two sequences ("video A" and "video B") from lessons on geometrical proof were used, so already the content domain suggests that intensive mathematical argumentation should be in the foreground of the situations. Moreover, both sequences show the development of a proof in a classroom conversation, so that validity and relevance of the representation of practice are also established from the task the students and the teachers in the classroom situation work on. As the videotaped representations of practice were selected from the video sample of a study with particular focus on cognitive activation, argumentation processes, and dealing with mistakes (Kuntze and Reiss 2004; Kuntze et al. 2004), it was possible to use the coding from this video study as a base for the design of the PD evaluation instruments.

Video A and video B were selected to represent contrasting types of classroom interaction. The framework conception of the evaluation study distinguished between an instructional style marked by intense argumentation, discourse in the classroom and cognitively activating reactions to mistakes on the one hand (represented by video A) and a teacher-centered small-step interaction with questions on a rather low level of complexity comparable to the dominant teaching script in Germany, as described in the TIMS Study (Baumert et al. 1997) on the other hand (represented by video B). The basis for intended contrast between video A and B consisted in the findings outlined in the theoretical background section, complemented by a framework related to proving in the mathematics classroom (e.g. Reiss 2002; Reiss et al. 2001) and in the above-mentioned results of our video studies on argumentation and geometrical proof (Kuntze and Reiss 2004; Kuntze et al. 2004). Both videos lasted about 10 min each. Further framework information on the context of the classroom situation was given to the participating teachers and transcripts of the classroom situations were at hand.

To control the validity of the question formats the participants were asked to comment also openly on the instructional quality of the situation, prior to being asked to rate items related to the given criteria of cognitive activation, intensity of argumentation, and learning from mistakes. Sample items for the corresponding scales and reliability values are displayed in Table 1. The scales had been adapted for use in this study so as to address the key focus areas of the PD. The open answers were used to check the validity of the scales. The reliability values presented in Table 1 are good.

The validity of the instrument in relationship with the goals of the ViPD project is supported even from a different perspective: A key aim of the PD project was to enhance the participating teachers' analysis of classroom situations in two ways: by analyzing classrooms as represented in video during the PD, and teachers' analyzing of their own classrooms during the intermediary phases. In a different study not reported here in detail, the participants' answers to the open questions were coded according to quality criteria of the teachers' analysis of representations of practice (Kuntze and Friesen 2016b). The results of that study suggest that teachers' higher quality of analysis corresponded to more developed situation-specific views about cognitive activation, intensity of argumentation, and learning from mistakes. These findings support the validity of the scales shown in Table 1, as the quality of analysis indicators and the situation-specific views showed connections (Kuntze and Friesen 2016b) which suggested that more expertise in the analysis

Table 1 Scales and sample items for standardized format questions related to videos A and B

Scale	Sample item	Number of items	Video A: Cronbach's α	Video B: Cronbach's α
Cognitive activation	"The students were encouraged to learn intensively."	3	0.87	0.77
Intensity of argumentation	"The classroom interaction was characterized by an argumentational interchange between the students and the teacher."	3	0.84	0.87
Opportunities for learning from mistakes	"By the manner, in which mistakes were treated in the classroom, the students were supported to build up meaningful knowledge, that is relevant for tasks and problem solving."	2	0.80	0.90

interdepended with more positive situation-specific views of video A even at the beginning of the PD project. As described above, video A was the classroom which showed more discourse and more intense learning from mistakes according to our prior video study (Kuntze and Reiss 2004; Kuntze et al. 2004).

Before viewing the two videos, teachers were given the opportunity to engage with the topic of geometrical proof in a multi-step approach: The teachers were first asked about criteria of instructional quality in general and about their non-situation-specific instruction-related views, then they were asked to conceive an overview of an introductory lesson on geometrical proof and to complete a questionnaire about proof-specific views. This way, the teachers had had the opportunity to activate their content-specific professional knowledge before being confronted with the videotaped representations of practice (Kuntze 2006, p. 415), related to that content domain. The data from these preliminary questionnaires indicates interdependencies between components of professional knowledge across levels of situatedness (Kuntze 2008)—these findings suggest that the work on representations of practice may also have an impact on less situation-specific components of professional knowledge.

Another component of the PD evaluation was a questionnaire at the end of each of the intermediary phases during which the teachers had been asked to observe and experiment in their own classrooms with respect of cognitive activation, intensity of argumentation, and learning from mistakes. The purpose of this questionnaire was to collect the teachers' perception of their efforts to "bridge the gap" between the PD and their own practice. As the participating teachers' observation and experimenting in their own classrooms can be considered as criteria-based work on classroom situations as well, the teachers' answers can provide insight whether the teachers saw themselves using the criteria-based analysis as emphasized in the PD

Table 2 Scales of teachers' perception of their efforts in the intermediary phases

Scale	Sample item	Number of items	Cronbach's α phase 1/2
Focused observation	"I have observed the cognitive activation of my mathematics instruction with more attention than before."	3	0.89/0.84
Experimenting/cognitive activation	"I have noticed changes in my mathematics instruction, which I attribute to my experimenting in the classroom."	5	0.83/0.88
Using opportunities for learning from mistakes and fostering intensity of argumentation	"I have observed that the intensity of argumentation in the classroom interaction was increased due to the measures I took."	4	0.73/0.78

also with respect of their own classroom practice. Table 2, shows sample questionnaire items and reliability values, according to three factors revealed through factor analysis.

The teachers were also asked to rate how close to their own instruction they perceived the classroom interaction in the two videos, with respect to both content and communication/classroom interaction. As general evidence for relationships between professional knowledge and instructional practice is still lacking (Tillema 2000; Lipowsky 2004), and as the findings are based on teachers' self-report, the participants' answers to these questions should be interpreted with care. However, these answers may complement the findings collected with other instruments.

The results reported in Lipowsky (2004) about teacher learning projects suggest that changes in professional knowledge play a role of a necessary but not sufficient condition for changes in the instructional practice of teachers. Seen from this point of view, the teachers' comparisons between videotaped instructional situations and their own instructional practice might be an interesting additional indicator. Against the background of the results of the TIMS Study (Baumert et al. 1997), it appears likely that at the beginning of the PD project, video B as an example following the prevalent German teaching script would be perceived as rather close to the participants' own classroom practice.

Summing up this section with respect of research question (B), the evaluation research of the ViPD project used representations of practice in an instrument which targeted the teachers' situation-specific views related to videos A and B. These views can be used as indicators of the teachers' professional knowledge and their potential learning. Additional insight can be gained from complementary questions about the teachers' reported activity of analyzing their own classrooms and about the perceived similarity to the videos. In the next section, selected empirical results will be presented to answer research question (C).

Results of the Evaluation Research: Evidence Indicating Development of the Teachers' Situation-Specific Views

The emphasis of this section is on research question (C), i.e., on the development of the teachers' situation-specific views throughout the PD project. Figure 3 shows the mean values of the scales measuring situation-specific views of the participating teachers before and after the PD. These results indicate a change towards more positive situation-specific views about the classroom situation represented in video A, the situation marked by rich discourse, whereas no significant changes could be detected for video B.

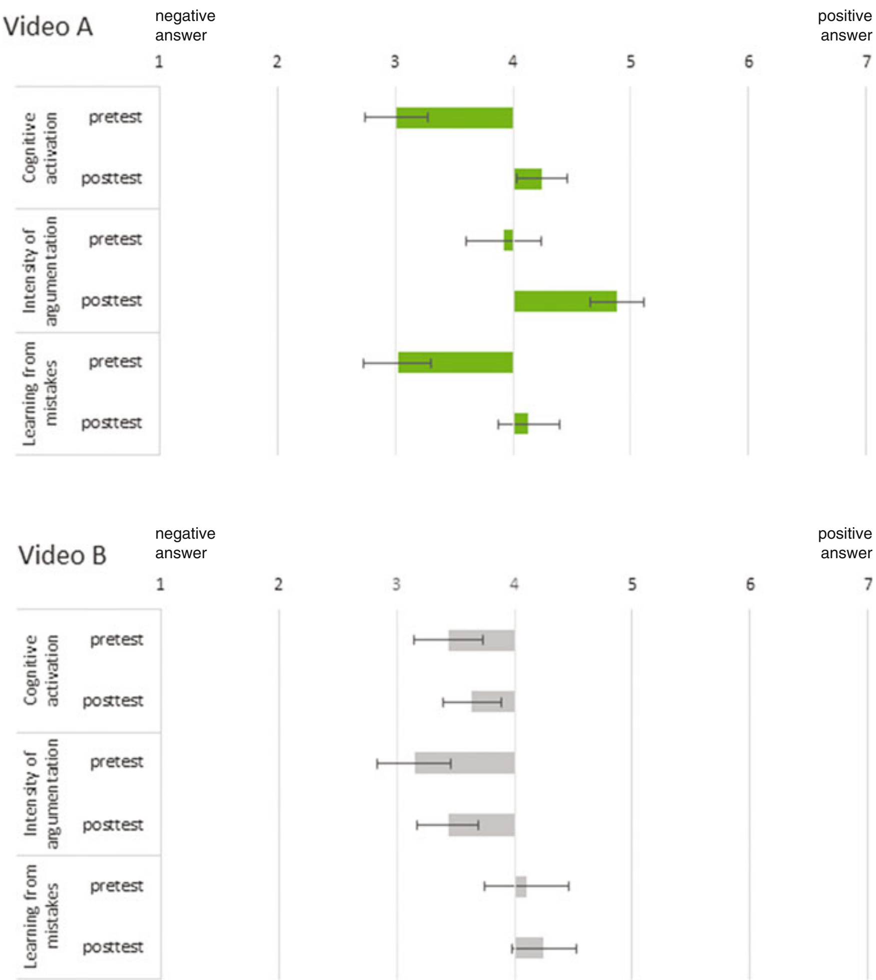

Fig. 3 Situation-specific views of participating teachers at the beginning and at the end of the PD project: mean values for the whole group and their standard errors

Examining the data more closely, a cluster analysis (Ward method) was carried out to make visible differences in the group which might be hidden behind the mean values. The cluster analysis yielded two sub-groups, which were labelled "traditionally oriented" teachers and teachers favoring discourse. The profiles of the mean situation-specific views in these sub-groups (at the beginning and at the end of the ViPD project, Kuntze 2006) are shown in Fig. 4. The findings indicate a relatively strong disagreement between the teacher sub-groups at the beginning of the PD, and somewhat more converging situation-specific views by the end of the PD. The teachers "favoring discourse" saw cognitive activation and intensity of argumentation in the situation shown in video A as positive, already at the beginning of the PD project. This teacher sub-group only changed significantly in their views related to learning from mistakes in video A, whereas the other sub-group improved their views related to video A in all criterion domains.

As shown in Fig. 5, this development coincided with reported efforts of most of the teachers in observing and experimenting related to the intensity of argumentation and learning from mistakes. According to the reports by the participating teachers, these efforts increased significantly from the first intermediary phase to the second one. Moreover, the teachers reported that their own classroom practice was closer to video A at the end of the project than at the beginning of the project: Fig. 6 shows how the teachers responded on average when being asked to rate the similarity of their own classroom practice to the two representations of practice, video A and video B. As expected, at the beginning of the PD both sub-groups of the participating teachers reported their classrooms to be closer to video B than to video A, both in

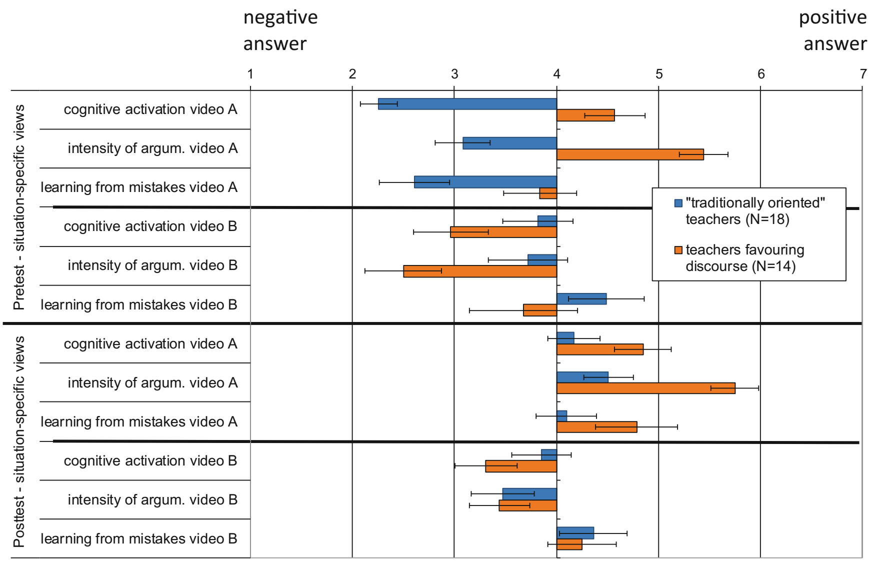

Fig. 4 Situation-specific views of participating teachers at the beginning and at the end of the ViPD project (for sub-groups according to a cluster analysis (Ward method): mean values and their standard errors)

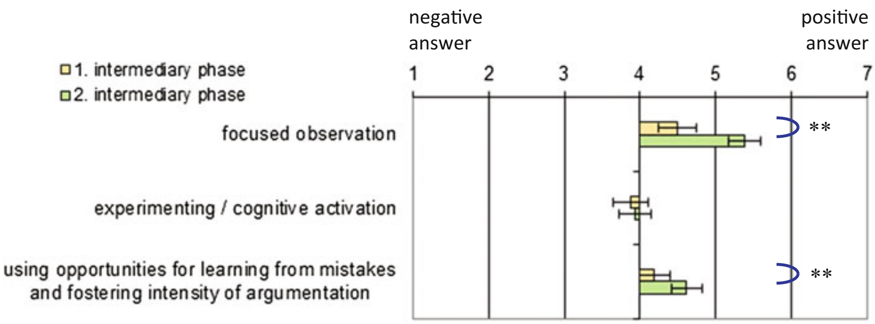

Fig. 5 Reported activities concerning classroom interaction during the two intermediary phases (mean values and their standard errors)

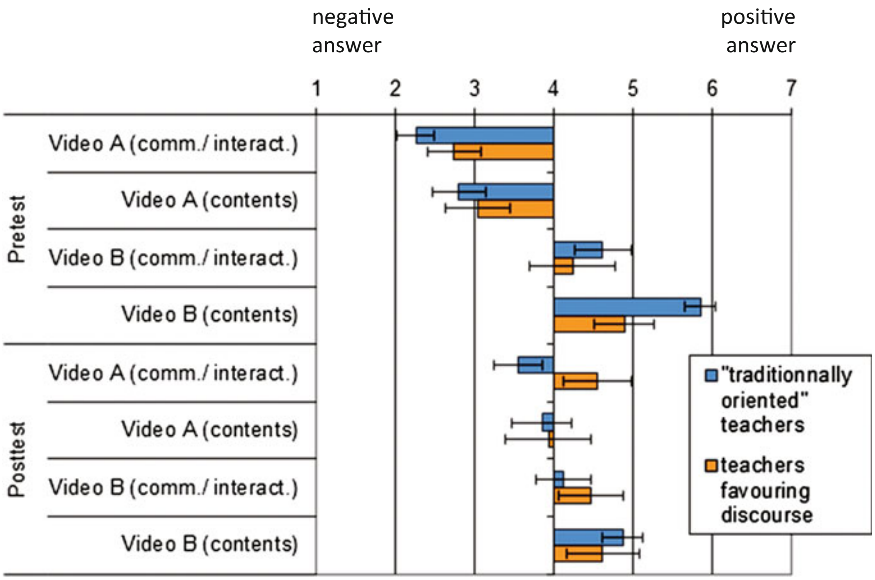

Fig. 6 Reported similarity of own instruction compared to the videotaped representations of practice (mean values and their standard errors)

terms of content and communication/interaction. At the end of the PD project, the answers suggest that the participants saw their classrooms less distant from video A. However, the teachers do not claim that their classrooms are really close to video A now, and still viewed video B to be not very different from their own instruction.

Of course, the data in Fig. 6 does not allow to infer directly about the classroom practice of the participating teachers—the results just reflect views of the teachers. These views are dependent on both the representations of practice and the teachers' perception of their own classrooms. This makes conclusions from the data in Fig. 6

difficult: the teachers' views of video A might have developed and/or the teachers' classroom practice might have undergone changes and/or the teachers' views of their own classroom practice might have evolved.

Summing up the empirical results relevant for research question (C), especially situation-specific views related to video A (the video showing more argumentation and discourse) have been subject to changes. According to the study of Kuntze and Friesen (2016b), more positive views related to cognitive activation, intensity of argumentation and learning from mistakes in video A are associated with a higher quality of analysis of the teachers' open answers. Seen against this background, i.e. combining the evidence of the two studies, the findings suggest an increase in teachers' expertise also with respect of analyzing classrooms in a broader sense.

Reflection, Discussion and Conclusions

The aim of this chapter was to reflect on how representations of practice can promote in-service teacher professional development, considering the example of the ViPD project while focusing on the participating teachers' views and reports of effects of the PD (cf. Lipowsky 2010, 2004). In short, the answer is: By using the kinds of intervention strategies described in Section "The Intervention: Design of the PD Project's Video-Based Work with Representations of Classroom Situations", and under the scope of the evaluation research methods (as laid out in Section "Design and Methods of the Evaluation Research: Investigating Teachers' Situation-Specific Views with the Use of Representations of Practice"), the situation-specific views of in-service teachers developed towards more positive views of classroom interaction marked by more argumentation and discourse (as reported in Section "Results of the Evaluation Research: Evidence Indicating Development of the Teachers' Situation-Specific Views"). Of course, this short answer should be deepened and completed by more detailed reflection and discussion.

Overall, the work on representations of practice in the PD project appears to have led to developments in the participants' instruction-related views. Even if the participating teachers have not completely lost their faith in the teacher-centered, small-step classroom interaction prevalent in Germany, they might have developed insight into the benefits of alternative ways of orchestrating classroom interaction, which are marked by more discourse and student-centered argumentation.

Still, the empirical results have to be interpreted with care: As there was no control group, we cannot be completely sure whether it was really the work with representations of practice in the PD project that has led to the observed developments. Moreover, the design of the evaluation research does not allow conclusions related to the role of specific aspects of the PD project, such as whether videotaped representations of practice are more effective than representations in text format, or whether cooperation among the participants has played a facilitating role. For such conclusions, it would have been necessary to compare different PD learning activities in a corresponding research design.

Further limitations of the study consist in the circumstance that the results somehow depend on the specific classroom situations in video A and B. Even if these situations had been selected to stand for two contrasting types of classroom interaction, including more situations and/or additional expert ratings could provide deeper insight in the future. With respect to the general aim of evaluating PD based on representations of practice, overcoming this limitation of the evaluation of the ViPD project can be a worthwhile methodological challenge.

Coming back to question (D) which focuses on potential implications of the findings to promoting in-service teacher professional development through the use of representations of practice, the following points should be noted:

- Reflecting on representations of practice in video format can have an impact on the teachers' situation-specific views. As these views can be asserted to be interdependent with the teachers' analysis of classroom situations (Kuntze and Friesen 2016b), the findings suggest that the teachers' competencies related to classroom situation-specific analysis have evolved.
- Examining representations of practice in connection with focused observation and experimenting in the teachers' own classrooms, might support the transfer of the PD contents into the participating teachers' classroom practice: As suggested by the data in Figs. 4 and 5, there were impacts at least in the teachers' perception. In Lipowsky's (2004, 2010) terms, these data are on the levels of teachers' self-reports (1) and their views and knowledge (2). Further studies should also include the teachers' actual practice, using student data or external observers (level 3 according to Lipowsky 2004) to examine the transfer into classroom practice.
- Representations of practice can also be used as a reference to describe one's own classroom or one's own teaching. Ratings of similarity as shown in Fig. 5 are only one possibility of such use. In-service teachers' professional development can be enhanced by reflecting on another teachers' classroom, and also by considering how close (or not) are the given representations of practice to one's own classroom practice, and by considering potential differences in teaching styles among the participating teachers. An alternative would be that teachers represent their classroom practice by themselves (e.g. in written texts or by designing cartoon representations with corresponding tools such as www.lessonsketch.org) and describe or compare their instructional practice through such representations.
- Representations of practice also allow the evaluation of PD projects. As it is possible to describe the learning outcomes of PD activities empirically, different PD settings and their quality may be assessed by using representations of practice. The study presented above shows only one out of a broad spectrum of possibilities. As shown in the case of the ViPD project, evaluation instruments should be adapted to the specific aims of PD activities, in order to provide valid insight. In more comprehensive designs of PD activity evaluation research, not only situation-specific developments in professional knowledge could be considered, but also developments on other levels of situatedness, and the

interrelatedness of different components of professional knowledge could be in the focus as well, especially across levels of situatedness.

- Finally, the findings presented above show a complexity of using representations of practice: different teachers can interpret representations of practice differently, even when they are prompted to focus on the same criteria. At the beginning of the ViPD project, there were very contrasting views related to video A, even though the questionnaire prompted the participants to focus on the three focus areas cognitive activation, intensity of argumentation and learning from mistakes. Criteria-based observation is thus not self-sufficient, but the use of criteria when working with representations of practice needs professional knowledge corresponding to these criteria. Although Shulman's (1986a, b, 1987) categories are very useful, describing the backing of teachers' criteria use in professional knowledge can benefit from further distinctions regarding levels of situatedness (Kuntze 2012). Indeed, it was possible to explain a part of the variance in the teachers' views prior to the PD by professional knowledge components on less situation-specific levels (Fig. 1). However, the unexplained variance calls for follow-up research into further variables which might impact the teachers' noticing (van Es and Sherin 2008; Sherin et al. 2011) when working with representations of practice.

The findings indicate that the teachers developed more convergent views by the end of the ViPD project—one possible reason is that they had elaborated a shared understanding of criteria. This would be a result of a learning process—the work on classroom situations in teacher groups (e.g. van Es and Sherin 2008) can help teachers to deepen their professional knowledge around such criteria, and to cross-link criteria-based professional knowledge on different levels of situatedness.

Dreher and Kuntze (2015a) have shown that, when asked to reflect on classroom situations, teachers used professional knowledge components on different levels of situatedness, regardless of any PD activity. Hence, the differences in initial teachers' views might also be a result of the teachers' drawing on professional knowledge from different, potentially disconnected, levels of situatedness, in addition to inter-individual differences in professional knowledge.

Looking at these phenomena from another perspective, such variance in professional knowledge and views can be a springboard for PD activities which use representations of practice: The question of elaborating observation criteria is an opportunity for facilitators to "unpack" the facilitator's knowledge-based reasoning and their criteria together with the participating teachers and to identify relevant professional knowledge on different levels of situatedness. Successful PD projects should consequently put an explicit focus on how a representation of practice can be interpreted, and how different perceptions might be linked with observable aspects of the representation of practice on the one hand and professional knowledge components on the other. Like this, the PD should explicitly develop a shared language for the analysis of representations of practice and reflect on the language use against the background of professional knowledge and instruction-related views. The development of such a language for the work on representations of practice might be a crystallization point for observing the teachers' professional development.

The use of representations of practice in the evaluation of PD activities does not only provide insight into the teachers' profession-related learning, but it can even be used as a further learning opportunity for the participating teachers: Offering them an opportunity to reflect on the answers they gave at the beginning and at the end of the PD activity may serve as a feedback on their own learning progress. Teachers may discover what is different in their noticing (Sherin et al. 2011) or analysis (Kuntze et al. 2015)—corresponding self-explanation processes may reinforce the consolidation of professional learning on a meta-level. This is a further facet of how representations of practice can promote in-service teacher professional development.

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Concept Cartoons as a Representation of Practice

Libuše Samková

Abstract The chapter focuses on using Concept Cartoons as a representation of practice in pre-service primary school teachers' education, especially on the possibility to employ them as a tool for investigating informal foundations of pedagogical content knowledge. The chapter introduces Concept Cartoons, and reports qualitative empirical research with a preparatory study. The preparatory study suggests the form of the Concept Cartoons environment suitable for investigating pedagogical content knowledge, and the main study analyzes displays of pedagogical content knowledge revealed in data collected from pre-service primary school teachers before their entering the course on didactics of mathematics. The results confirmed that Concept Cartoons were suitable for the studied purpose.

Keywords Concept Cartoons · Pedagogical content knowledge
Pre-service primary school teachers · Representation of practice
Teacher education

Introduction

In this chapter, I will present an educational tool called Concept Cartoons in a novel role—as a representation of practice and a diagnostic tool in pre-service primary school teachers' education. I will show how this new approach to Concept Cartoons can lead to successful investigation of informal foundations of pedagogical content knowledge of pre-service primary school teachers. From a general perspective, the chapter intends to contribute to the discussion about how representations of practice can help to investigate aspects of teacher expertise.

As an educator of pre-service primary school teachers I appreciate that data collected with the help of Concept Cartoons during mathematics content courses

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can give me valuable information on pre-service teachers' knowledge: data related to content knowledge provide me with continuous feedback on my own teaching, and data related to pedagogical content knowledge provide me with an introductory overview that allows to prepare didactics courses tailored to a particular sample of pre-service teachers.

The reported research focuses on two questions: "What form of the Concept Cartoons environment is suitable for investigating pedagogical content knowledge of pre-service teachers?", and "What are the informal foundations of pedagogical content knowledge with which pre-service primary school teachers enter the course on didactics of mathematics?" The research on the first question serves as a preparatory study for the research on the second question. During the preparatory study, more than 20 various Concept Cartoons with a set of questions were assigned to more than 100 pre-service teachers, in order to investigate how compositions of particular Concept Cartoons relate to quality and amount of displays of pedagogical content knowledge found in the collected data. A set of eight Concept Cartoons was selected on the basis of this preparatory study. During the main study, the selected Concept Cartoons were assigned to a group of 29 pre-service primary school teachers in the time before entering the course on didactics of mathematics, in order to analyze which particular displays of informal foundations of pedagogical content knowledge can be found in the collected data.

Background of the Research

Teachers and Their Knowledge

Teachers and their knowledge that influences the course of teaching are the focus of many educational frameworks. This contribution deals with *pedagogical content knowledge* in the sense of Shulman (1986) and Grossman (1990), i.e. with the construct that includes four central components: knowledge of teaching purposes, curricular knowledge, knowledge of pupils, and instructional knowledge.

As pedagogical content knowledge is a combination of miscellaneous components, also the range of the methods used to investigate pedagogical content knowledge is vast: tests, questionnaires, lesson observations, etc. (Depaepe et al. 2013). In mathematics education, an extensive study on pedagogical content knowledge was conducted under the research project COACTIV (Krauss et al. 2008). One of the studies that built on the COACTIV project investigated pedagogical content knowledge of lower-secondary mathematics teachers at different points in their teaching careers (Kleickmann et al. 2013). Its tests with open questions assessed three facets of pedagogical content knowledge: *knowledge of pupils* (of their strategies, conceptions and misconceptions, possible difficulties, sources of pupils' misunderstanding, etc.), *knowledge of tasks* (of multiple ways of solving, potential for pupils' learning), and *knowledge of instruction* (of different representations, models, modes of explanation, etc.). For instance, the following assignment belonged to a question on knowledge of tasks: “How does the surface area of a square change when the side length is tripled? Show your reasoning. Please note down as many different ways of solving this problem (and different reasoning) as possible.” (ibid., p. 102), and the other to a question on knowledge of pupils: “The area of a parallelogram can be calculated by multiplying the length of its base by its altitude… Please sketch an example of a parallelogram to which students might fail to apply this formula.” (ibid., p. 102).

Depaepe et al. (2015) conducted a research focusing on pedagogical content knowledge of pre-service primary and lower-secondary teachers. They also employed tests with open questions in their study but distinguished only two components of pedagogical content knowledge: *knowledge of pupils' misconceptions*, and *knowledge of instructional strategies and representations*. For instance, the following assignment belonged to a test question on knowledge of pupils' misconceptions: “Below are illustrations of elementary students' answers to the problem … For each student's answer write down the presumable student's reasoning and evaluate whether the answer is correct.” (ibid., p. 87).

Based on my previous experience that knowledge related to tasks might originate from different sources than knowledge related to pupils, I prefer to distinguish between knowledge of tasks and knowledge of pupils. Thus the reported study employs the classification of pedagogical content knowledge provided by Kleickmann et al. (2013).

Concept Cartoons

The name *Concept Cartoons* belongs to an educational tool that was developed in the 1990s by Keogh and Naylor (1993). They introduced Concept Cartoons as an instrument that might help support teaching and learning in primary school science classrooms by generating discussion, stimulating investigations, and promoting learners' involvement and motivation.

Each Concept Cartoon is a simple independent picture that shows a situation well known to pupils from school or everyday reality, and a group of several children discussing the situation through a bubble-dialog. The texts in bubbles are short and employ simple language. The discussion is composed in such a way that each of the children presents an alternative viewpoint on the situation or an alternative solution to a problem arising from the situation. Some alternatives may be correct, some incorrect, the correctness may also be unclear or conditional, depending on additional conditions not explicitly mentioned in the picture. One of the bubbles is blank, in order to indicate that there might exist other alternatives that have not been included in the dialog yet.

For a sample of one of the first Concept Cartoons created by Keogh and Naylor (1993) see Fig. 1. In that picture, the correctness of alternatives depends on two main factors: on the material of the coat, and on actual weather conditions

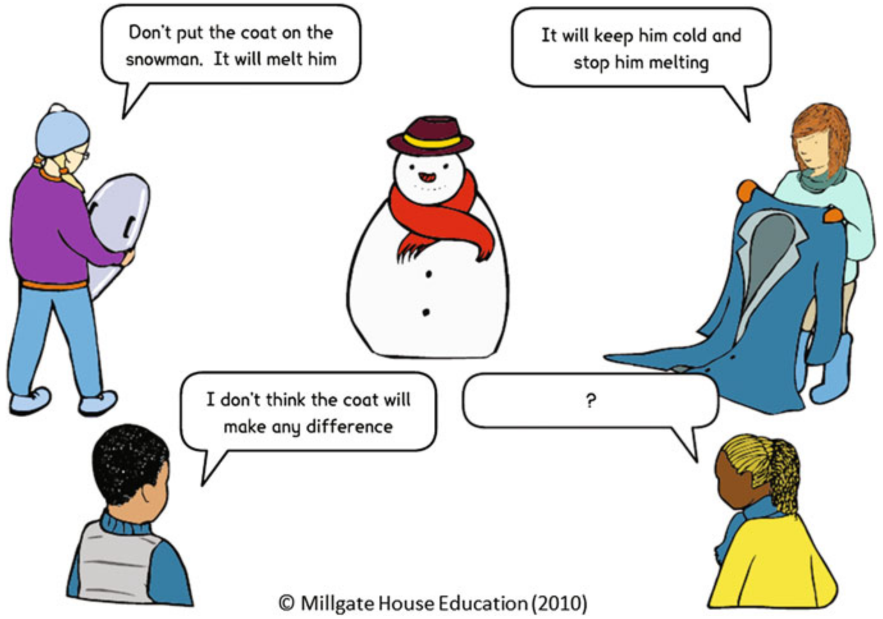

Fig. 1 Original Concept Cartoon; picture taken from (Naylor and Keogh 2010, no. 3.2)

(sun, temperature, etc.). This Concept Cartoon was composed in such a way that each of the alternatives can be true if you choose a suitable combination of the factors.

When using Concept Cartoons as an educational tool in the classroom, the teacher usually presents the picture to pupils with questions "What do you think about it?", "Which of the children are right?", "Why?", "What can we write into the blank bubble?", and the pupils discuss the answers.

Authors of Concept Cartoons performed several studies on the use of the tool in science classroom. The large-scale research investigated how pupils responded to the use of Concept Cartoons at primary and secondary school levels (Keogh and Naylor 1999). Among other results, the research confirmed that Concept Cartoons are able to support teaching and learning, and promote learners' motivation and engagement. The latter was confirmed even in case of usually less confident pupils—having pictured children speaking for them gives the pupils the confidence to discuss their ideas—from their point of view, the blame for a potential incorrect idea is not on the pupil but on the pictured children. As Keogh and Naylor pointed out in the study, also the evaluation looked differently with Concept Cartoons—pupils' ideas were not evaluated directly by the teacher as usual, instead the pupils themselves evaluated ideas of the pictured children, and this fact might have positively affected willingness to participate in the discussion as well. These attributes of Concept Cartoons relate to learners' motivation as a consequence of cognitive incongruity, a matter that was discussed by Hatano (1988). Hatano distinguishes three types of cognitive incongruity, and suitably composed Concept Cartoons may meet some or all of them: *surprise* (which is induced when a person encounters information that disconfirms a prediction based on prior knowledge), *perplexity* (which is induced when a person is aware of equally plausible but competing ideas), and *discoordination* (which is induced when one recognizes a lack of coordination among some of the pieces of knowledge involved).

Another research performed by authors of Concept Cartoons focused in detail on the form of the discussion and on the quality of arguments that appeared there (Naylor et al. 2007). Its results show that the lack of agreement amongst the pictured children encourages pupils to join the discourse with their own opinions, explanations and justifications, and that such discourse can take a form of sustainable and purposeful argumentation.

Since Concept Cartoons proved to be useful in science education, they naturally expanded to education of other school subjects, including mathematics (Dabell et al. 2008). In that case no large-scale research was conducted by the authors; the authors suppose that the results related to motivation, engagement and argumentation are of general character, and that they do not depend on the subject.

For a sample of one of the mathematical Concept Cartoons see Fig. 2. In that picture, one alternative is correct, and the others are incorrect. No conditionality appears there.

Fig. 2 Original Concept Cartoon; picture taken from (Dabell et al. 2008, no. 2.16)

Concept Cartoons as a Representation of Practice and a Diagnostic Tool

When I first encountered Concept Cartoons, I was attracted by the fact that each of the pictures shows various children's opinions in a certain situation related to topics that are taught in the classroom, so that the pictures can be considered as models of classroom discussions of pupils—as specific representations of practice.

The role of the teacher is not included in these representations, thus there is a big space for its integration and elaboration. For instance, I may assign the picture to a person, and ask the person to play the role of the teacher, i.e. to moderate the discussion, judge and evaluate the opinions, provoke further contributions to the debate, provide hints, explanation or advice that would be comprehensible for pupils, seek possible sources of misunderstanding or misconceptions, provoke or plan other activities that would clarify the situation, encourage looking for other alternatives that could be filled in the blank bubble.

Such use of Concept Cartoons resembles some of the test items intended for assessing teachers' pedagogical content knowledge presented in Section "Teachers and Their Knowledge":

- the setting consisting of various opinions (answers) of pupils and the requirement to judge and evaluate them, to provide advice or to present other possible answers appears in test items that focus on knowledge of pupils (e.g. on the ability to recognize pupils' misconceptions, difficulties and solving strategies—Kleickmann et al. 2013; Depaepe et al. 2015);
- the requirement to present other possible answers also appears in test items that focus on knowledge of tasks (e.g. on the knowledge of multiple ways to solve a problem—Kleickmann et al. 2013);
- the requirement to provide explanation also appears in some test items that focus on knowledge of instruction (e.g. on knowledge of different representations and explanations to standard problems—Kleickmann et al. 2013).

Following this resemblance, I decided to probe Concept Cartoons as a diagnostic tool for assessing teachers' pedagogical content knowledge. As the question of using Concept Cartoons for such a purpose is new, and the terrain is unknown, I narrowed the range of the question to an environment that is rather informal (similarly as Concept Cartoons are). So that I decided to address only pre-service primary school teachers and the informal foundations of pedagogical content knowledge that they might have gained from their own learning experiences (K–12, non-didactical teacher training courses). Thus, the research survey presented in this chapter focuses on pedagogical content knowledge of pre-service primary school teachers in the time before their entering the course on didactics of mathematics.

In comparison with the original use of Concept Cartoons, some modifications came about during the survey:

- instead of primary school pupils, pre-service primary school teachers worked with Concept Cartoons;
- instead of joint discussion in the classroom, the assessment was conducted individually and in written form;
- the original set of questions assigned with Concept Cartoons is not sufficient for such diagnostic purposes, so that questions on possible pupil's considerations and possible sources of pupil's misunderstanding or misconceptions, as well as requirements to provide explanations comprehensible to the pupil were added to the set;
- the original purpose of Concept Cartoons was educational, so that some particular pictures may not be suitable as diagnostic—the suitability of particular pictures had to be tested, some new pictures created.

To resolve and clarify all the above-mentioned differences, a preparatory study took place ahead of the main survey.

Preparatory Study

As indicated above, the main research required a preparatory study, and the research question for this preparatory study was "What form of the Concept Cartoons environment is suitable for investigating pedagogical content knowledge of pre-service teachers?" The preparatory study was conducted in two separate stages: the first stage explored Concept Cartoons from the original educational set, while the second stage dealt with Concept Cartoons that were newly created specifically for the purpose of this study. Participants of the preparatory study were 127 pre-service teachers, full time or distance university students from various years of the teacher training program.

The First Stage

For the first stage, four Concept Cartoons were selected from the original educational set created by Dabell et al. (2008); one of them is shown in Fig. 3.

The selected pictures differed in several composition factors: type of the pictured situation (classroom vs. everyday event), type of the text in bubbles (a proposal of a result vs. a proposal of a procedure and a result vs. an advice to a pupil who made a mistake), and/or number of alternatives that could be declared as correct (one vs. three). These Concept Cartoons were assigned to students on a worksheet with six common questions:

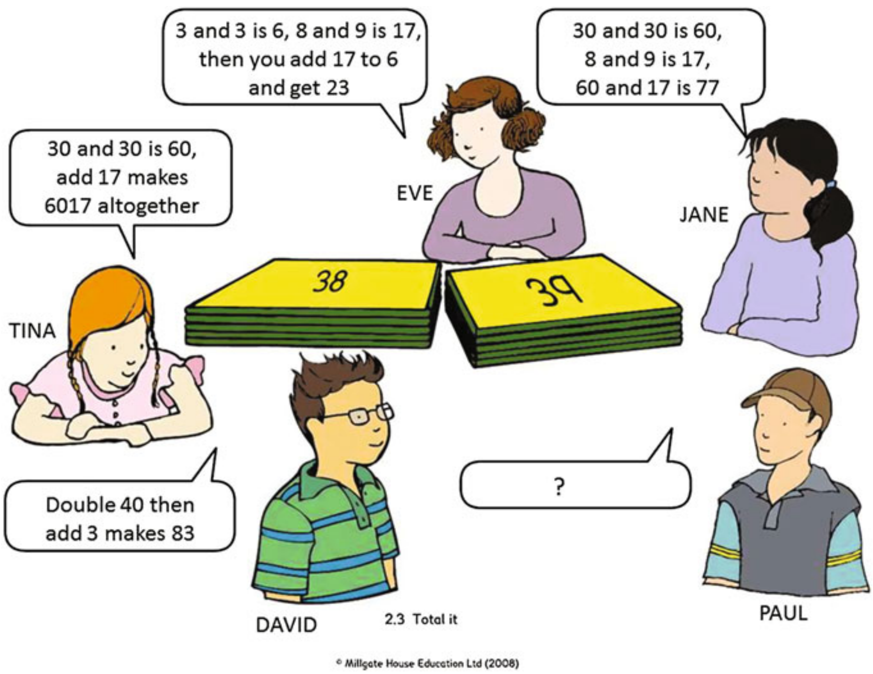

Fig. 3 Original Concept Cartoon; picture taken from (Dabell et al. 2008, no. 2.3), names added

(1) Which child do you strongly agree with?
(2) Which child do you strongly disagree with?
(3) Decide which ideas are right and which are wrong. Give reasons for your decision.
(4) Try to discover the cause of the mistakes.
(5) Advise the children who made the mistakes how to correct them.
(6) Propose a text that could be filled in the blank bubble—does not matter whether correct or incorrect. It might relate to another correct way of solving, or to another misconception.

Students worked on worksheets individually, during a lesson. The work took them approximately 80 min.

Original Concept Cartoons do not have the children in the picture named (see Figs. 1 and 2) which appeared uncomfortable for the respondents—many of them announced during the work that they did not know how to refer to particular pictured children. So that I let the respondents add letters (A, B, C, D …) to the children, and from that time I always label the pictured children. I prefer labeling by names (as in Fig. 3), to make the Concept Cartoons authentic—pupils in the classroom are also called by names, not by letters.

Data from worksheets were processed qualitatively, using the method of substantive coding and constant comparison (Bryant and Charmaz 2007). I focused on displays of pedagogical content knowledge, e.g. displays related to provision and recognition of right and wrong answers, to recognition of procedures used by pictured children, to identification of the causes of mistakes. Detailed description of analysis of data connected with two of the Concept Cartoons, and partial results belonging to 64 pre-service primary school teachers we already reported in Samková and Hošpesová (2015). We presented there two of the Concept Cartoons in detail, and showed how they allowed us to distinguish between subject matter knowledge and pedagogical content knowledge in the sense of Shulman (1986), as well as between procedural knowledge and conceptual knowledge in the sense of Baroody et al. (2007).

The other two Concept Cartoons that were not reported in Samková and Hošpesová (2015) appeared to be problematic from the perspective of the diagnostic purpose, because data collected with them offered almost no information on pedagogical content knowledge. Respondents' responses to indicative questions were very short, often giving only the opinion on the correctness, without attempts to comment on reasons or search for possible sources of misconceptions. Since the four tasks from the four Concept Cartoons were of similar difficulty, the obvious inequality in the responses turned my attention to the possible unsuitability of some Concept Cartoons for my purpose, and to the need to observe closely particular composition factors related to particular Concept Cartoons.

As turned out, both the pictures with enough collected data displayed an everyday event, the first of them used proposals of procedures and results in its bubbles, the second one used just results. Both the pictures with lack of collected data displayed a classroom event, the first of them used advices to a pupil in its bubbles, the second one used just results. From the perspective of alternatives in bubbles, the picture with advices to a pupil offered three alternatives that could be declared as correct, all other pictures just one. For further study I decided to focus closely on diverse combinations of the three composition factors, and explore the further potential of the factors. Since the original Concept Cartoons do not offer enough diversity for such a study (e.g. most of the pictures offer just one correct alternative, and bubbles within the pictures usually have the same type of the text), I had to prepare some new pictures.

The Second Stage

With focus on variability of combinations of the three composition factors from the first stage of the preparatory study, 22 new Concept Cartoons were created and used in the second stage. Among them, 11 Concept Cartoons were slight modifications of original Concept Cartoons. The modifications consisted for instance in changing the number of correct alternatives (e.g. by modifying numbers in the assignment of the task in such a way that the task gained more than one solution, or by replacing one of the incorrect alternatives by a correct one), in adjusting some incorrect alternatives to look more plausible (e.g. to take a form of a typical pupil's

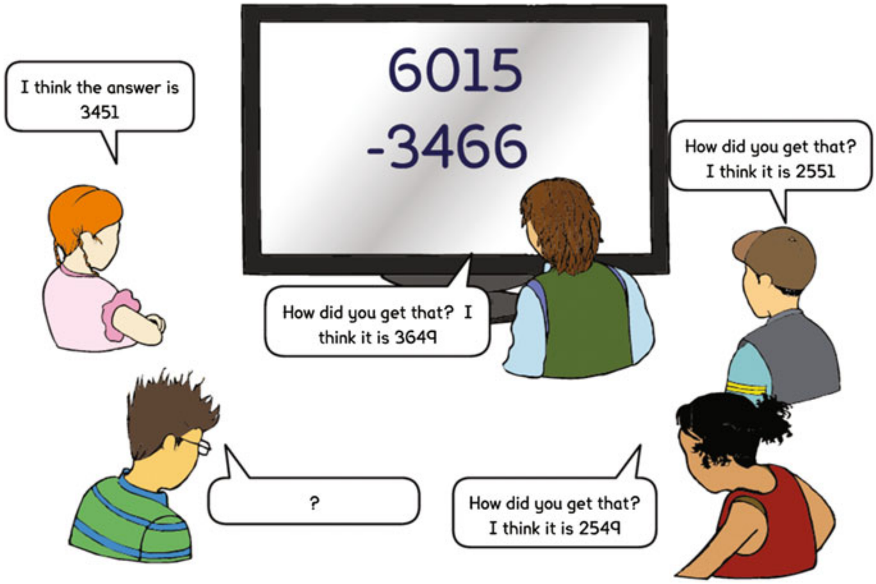

Fig. 4 Slightly modified Concept Cartoon (just decimal marks after the second digit deleted from all numbers, i.e. the range of the task changed from decimal to natural numbers); picture taken from (Dabell et al. 2008, no. 2.13)

misconception known from educational research), or in making small shifts in mathematical content (for such a shift see Fig. 4).

The other 11 Concept Cartoons were brand new. They presented a new pictured situation, a new perspective on the situation, and/or a quite new mathematical content (compare Figs. 2 and 5). I searched for inspiration in my own teaching experience and in the teaching experience of my colleagues (e.g. Tichá and Hošpesová 2010), in results of educational research (e.g. Ryan and Williams 2011; Bana et al. 1995), in books and textbooks (Ashlock 2002, 2010). Again, the newly created Concept Cartoons contained in their bubbles various more or less usual pupils' conceptions or misconceptions, descriptions of various correct ways of solving, or plausible incorrect ways of solving, and also some intentionally prepared unusual but authentically looking misconceptions (Samková and Tichá 2015). In comparison to original Concept Cartoons, some new types of the text in bubbles were established for the newly created pictures: a proposal of a procedure, a proposal of a statement, an opinion on the validity of a statement, an opinion on the number of solutions, and a reference to a drawing that was not displayed in the picture (i.e. a reference to a missing drawing). Some of the Concept Cartoons had the same mathematical content but different types of the text in bubbles (Samková et al. 2015), in order to allow monitoring the influence of the composition on collected data. All the 22 Concept Cartoons were assigned to various groups of participants, under the same conditions as in the first stage.

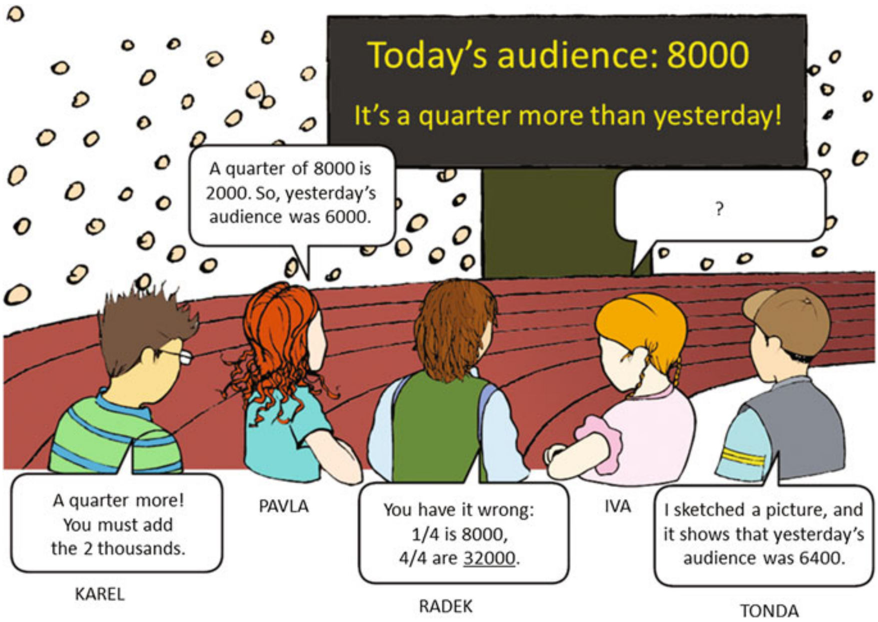

Fig. 5 Concept Cartoon re-designed for this study; template with empty bubbles and empty notice board taken from Fig. 2, new texts and names added

Data from the second stage were analyzed qualitatively. This time, the analysis focused on displays of pedagogical content knowledge in relation to composition factors of given Concept Cartoons. I also monitored the amount of relevant data obtained from participants in relation to various Concept Cartoons. According to quality and amount of relevant data collected with the Concept Cartoons, data analysis revealed three significant types of bubble content: bubbles with procedures and results, bubbles with references to a missing drawing, and bubbles with just results.

When using Concept Cartoons containing both procedures and results in their bubbles (as in Fig. 3), respondents could comment on described results and procedures, and look for errors in procedures leading to incorrect results as well as in procedures leading to correct results. This kind of Concept Cartoons offered the respondents a lot of concrete facts to judge and discuss, and the responses provided a lot of relevant data.

Another type of Concept Cartoons that appeared diagnostically valuable was the one that combined several bubbles containing procedures and results with a bubble introducing a result together with a reference to a missing drawing leading to this result (as in Fig. 5 where Tonda's bubble refers to such a drawing). These combinations of bubbles were often thought-provoking: respondents first commented on the procedures described in the bubbles, and then focused on the bubble without the procedure, attempting to find out what drawing the child was talking about. They often proposed their own drawings that could lead to the result. In this case, the Concept Cartoon played a similar diagnostic and thought-provoking role as problem posing, e.g. as posing problems corresponding to a given calculation (Tichá and Hošpesová 2010).

Some of the Concept Cartoons with just results proved to be problematic from the diagnostic perspective, especially when the task was in a form of a calculation (e.g. as in Fig. 4). The respondents often tended just to compare the correct result of the calculation with the numbers in bubbles, did not comment on reasons, and did not attempt to seek the procedures hidden behind the incorrect results or possible sources of misconceptions. This kind of Concept Cartoons offered the respondents few concrete facts to judge and discuss, and the responses provided few relevant data on pedagogical content knowledge. On the other hand, with an unusual composition, the Concept Cartoon with a calculation task and just results in bubbles attracted respondents' attention, and provoked them to respond widely. Such an unusual composition belonged to one of the original Concept Cartoons from the first stage of the study, which presented a task based on a calculation 5904 + 5106. What was unusual about the results proposed in bubbles is that all of them were composed only of digits 1 and 0: 1110, 11100, 11010, and 1010010. With this Concept Cartoon, I obtained a lot of relevant data; the respondents sought for the procedures hidden behind the proposed results, and suggested various sources of mistakes. Since some of the respondents' responses were not correct, this Concept Cartoon also helped to reveal weaknesses in pedagogical content knowledge: there were respondents who just compared the incorrect results in the bubbles with the correct result, and gave the children advices regardless of the possible source of the mistakes. For instance, one of the participants gave the children with the 11100 result the following advice: "Peter, recount it again, your result is 90 more than the correct result." (Samková and Hošpesová 2015).

Unlike the first stage of the preparatory study, some Concept Cartoons based on classroom events appeared suitable for assessing pedagogical content knowledge. All of them had a common characteristic: at least two correct alternatives in the picture that could be declared as correct, and procedures or procedures with results in bubbles. The correctness of the alternatives was either general, or conditional. With these Concept Cartoons, the respondents seemed to be surprised by the existence of multiple correct alternatives, and thus paid more attention to the reasoning related to the alternatives that looked correct. As a positive consequence, some of them paid more attention also to alternatives that looked incorrect, and offered detailed justifications on the incorrectness. As a negative consequence, some of the participants paid too much attention to the alternatives that looked correct, and improperly found mistakes in some completely correct formulations.

Generally, the results of the preparatory study showed that various pictured situations, various numbers of alternatives that can be considered as correct, and various types of the text in bubbles allowed to reach diverse components of pedagogical content knowledge, so that a set of Concept Cartoons with various combinations of composition factors is needed to get a comprehensible overview of participants' pedagogical content knowledge (Samková 2017).

On the basis of the results of the preparatory study I selected a set of eight Concept Cartoons for the main survey. This set contained Concept Cartoons from the preparatory study, four of them original Concept Cartoons (e.g. the one in Fig. 3), and four newly created (e.g. the one in Fig. 5). Tasks on the pictures were of diverse focus and difficulty, number of alternatives in a picture that could be declared as correct varied from one to all, and texts in bubbles were of diverse types: just results, just procedures, procedures and results, references to a missing drawing, opinions on validity, recommendations, and proposals of general statements or rules. Four of the Concept Cartoons were based on an everyday event, and four on a classroom event.

Main Survey

As already mentioned above, the research question for the main survey was "What are the informal foundations of pedagogical content knowledge with which pre-service primary school teachers enter the course on didactics of mathematics?"

Participants and Data Collection

Respondents of the research were 29 full time university students of master degree training for pre-service primary school teachers. In our country, pre-service primary school teachers' training covers all the primary school curriculum subjects and lasts 5 years, i.e. it serves as an equivalent of a 3-year bachelor program followed by a 2-year master program. Students come to the university directly from the secondary school, with no experience in teaching. During the first and second years of the training program, the students attend mandatory courses on mathematics, and during the third year they attend mandatory courses on didactics of mathematics. The survey took place in the second year of the training program, i.e. in the time before the students entered the courses on didactics of mathematics. All of the second year students participated in the research, none of them participated in the preparatory study.

The eight Concept Cartoons selected on the basis of the preparatory study were assigned to the respondents in two separate stages (due to time constraint reasons), four Concept Cartoons per stage. I placed them on a worksheet with six common questions that were the same as in the preparatory study (Section "The First Stage"). Students worked on worksheets individually, during a lesson. The work took them approximately 80 min each stage.

Data Analysis

First, all the materials were open-coded, and the codes sorted to 12 categories in such a way that, if applicable, the categories included both strengths and weaknesses related to the category label. In the following list, the categories including both strengths and weaknesses are provided with two examples of codes in brackets, the first example is considered as referring to some strength, and the second one to some weakness:

A. strong (dis)agreement (e.g. “strongly disagrees with Radek”, “strongly disagrees with Tonda”);
B. recognition of a correct/incorrect statement (e.g. “found all incorrect statements”, “thinks that a correct statement is incorrect”);
C. recognition of a procedure and its particular steps (e.g. “reveals a step that is incorrect”, “sees the procedure as one indivisible whole”);
D. explanation (e.g. “illustrative explanation”, “imprecise explanation”);
E. advice (e.g. “helpful advice”, “missing advice”);
F. identification of the cause of a mistake (e.g. “plausible cause of a mistake”, “just compares the result in a bubble with his own result”);
G. differentiation between identification of a mistake, its cause, and its remedy (e.g. “successful differentiation”, “one common answer for questions 3, 4, 5”);
H. own respondent errors and mistakes that appeared as a part of explanation or advice (e.g. “confuses part and whole”);
I. blank bubble (e.g. “alternative way of solving”, “unrealistic alternative”);
J. formal arrangement (e.g. “carefully follows the order of questions and answers”, “does not number answers”);
K. orientation in the picture (e.g. “did not link bubbles to names”);
L. not specified (e.g. “interesting”, “unclear”, “read again”).

Then the method of constant comparison was employed, data were repeatedly read over, labeled by new codes when needed, codes repeatedly compared with data and among themselves, rearranged, adjusted. Some codes were removed. For better clarity of the process, the codes were marked with plus or minus sign to denote aspects that were considered as positive or negative from the perspective of teachers’ pedagogical content knowledge. During the process, the list of categories was re-organized:

- codes from J, K and L categories were adjusted and replaced or removed, the three categories were canceled;
- codes from categories C, D, E, F appeared to be too tight together, so that these categories were unified under one common category labeled CF;
- when comparing codes within particular respondents, the heterogeneity of data related to Concept Cartoons with diverse composition was disruptive, so that the analysis was additionally enriched with codes related to the composition of particular Concept Cartoons (type of the text in bubbles, number of bubbles with

correct statement, number of solutions, etc.), and a new category M was established for them.

Thus, categories A, B, CF, G, H, I, M remained for the final analysis. The final analysis identified B, CF and I as the categories with the biggest density of data.

General Findings

In this section, I present general findings of the research, accompanied in brackets by references to particular responses that appear as transcripts in Section "Illustrative Data Excerpts". The references are in the following form: (respondent number: name of the child or children to which the response relates).

In spite of the fact that respondents of the research were students who had not attended a course on didactic of mathematics yet, data revealed 15 of the 29 students with good informal foundations of pedagogical content knowledge. These students were able to

- recognize various pupils' misconceptions (S30: Pavla, S31: Pavla, Radek);
- find a mistake in a procedure, clearly describe its possible cause, and advice how to remedy it (S4: Pavla, Karel, Radek);
- check/verify results of a task that they themselves did not solve, and explain to children with wrong results why their results cannot be correct (S6: Tonda, Pavla, Radek, S10: Pavla, Karel, Radek);
- usefully employ visualization (S2: Tonda, S4: Tonda, S22: Tonda, S24: Tonda);
- present various alternative ways of solving (S5: Paul, S10: Paul), also the elegant ones that advantageously utilize certain specific relations (S3: Paul, S4: Paul, S18: Paul, S26: Paul);
- present plausible potential pupils' misconceptions (S23: Paul).

On the other side, data revealed also 8 students with low level of knowledge related to pedagogical content. These students

- proposed possible pupils' alternative solutions that were unrealistic (S29: Iva);
- proposed explanations of pupils' procedures that were unrealistic (S1: Tonda, S21: Tonda) or with no relation to the task (S15: Tonda);
- tended to reject procedures which they themselves did not grasp (S12: Tonda);
- did not know common misconceptions (S16: Pavla).

The remaining 6 students showed unbalanced knowledge since some of their responses could be considered as displays of good knowledge, and some could not. For instance, the respondent S11 presented a proper alternative way of solving as a response to Paul's bubble (S11: Paul) but an unrealistic incorrect alternative as a response to Iva's bubble (S11: Iva).

Further, 10 of the 29 respondents in their worksheets made an effort to differentiate between identification of a mistake, its cause, and its remedy. All of them were successful in this endeavor. Without exception, they were all students who were successful also in other aspects of pedagogical content knowledge, e.g. in presenting various alternative solutions of a task (S4: Paul).

The remaining 19 respondents offered answers that did not differentiate between identification, cause and remedy. They often gave one common answer for questions 3, 4 and 5 (S12: Tonda, S15: Tonda). This fact is not surprising since the respondents had not attended any didactical courses yet.

Illustrative Data Excerpts

The findings reported generally in the previous section will now be illustrated by particular data excerpts. The transcripts will be presented in the following form:

respondent no.	question no.	respondent's answer to the question or its part (name of the pictured child included)

In cases when respondents gave one common answer for questions 3, 4 and 5, the question number in the transcript will be denoted by 3/4/5.

As an illustrative example, I have selected the Concept Cartoon from Fig. 5. The scene described in this Concept Cartoon is located outside the classroom, is based on a word problem with fractions, and the word problem has a unique solution. A similar task appeared in 2015 in state matriculation exam, where only 33% of the students solved the task correctly (Řídká 2015). I changed the context of the task, and enlarged the quantity given in the task from 800 to 8000. When creating the bubbles, I based three of them on three most frequent incorrect solutions (Pavla, Karel, Radek), and the fourth on the correct solution (Tonda). The incorrect solutions are offered in the form of procedures with results, the correct solution is in the form of a result with a reference to a missing drawing.

Like in the state exam, the task appeared to be rather difficult to solve: only 41% of the respondents properly choose Tonda's bubble as the correct one, the other 59% improperly choose Pavla's bubble.

Agreement with Pavla was often supported by the justification that is known as a common misconception:

S16	2)	Pavla—right answer, 8000: 4 = 2000 → yesterday 6000.

The respondents who choose Pavla, often strongly disagreed with Tonda. Some of them admitted that it is because they did not understand Tonda at all, that they did not grasp how could have he come to the result 6400:

S12	2)	I do not agree with Radek and Tonda.
	3/4/5)	Tonda's opinion is not right. I entirely did not understand his thinking.

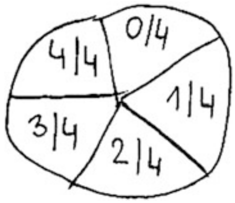

Fig. 6 Picture with a "pie" drawn by the respondent S21

Some respondents presented explanations of potential Tonda's procedure that were based on unrealistic misconceptions:

S1 3) Tonda—this opinion is the most wrong → first he subtracted one quarter (2000), and then added 100 from each quarter.
S21 4) Tonda—he probably drew a "pie", and included zero to his calculations (see Fig. 6).

Some of the explanations did not even relate to numbers from the assignment of the task:

S15 3/4/5) Tonda calculated $80 \cdot 80$.

Among the responses, I also found some explanations of Tonda's procedure that were plausible but incorrectly rejected by their authors as wrong:

S9 2) Tonda
4) Tonda—he probably drew a picture with four quarters, and added one quarter to it, so that he divided 8000 by 5.
5) Tonda—draw a picture, and the amount 8000 has to be divided to how many pieces if you want to find 1/4?

Respondents who agreed with Tonda supported their agreement diversely. Some of them offered a picture (see Fig. 7), others just solved the task without a picture, and compared their result with Tonda's.

These respondents were usually able to find mistakes in procedures presented by other children (Pavla, Karel and Radek), and gave reasons why the mistakes might occur. Some of them also advised a remedy:

S30 4) Pavla did not realize that the quarter must be calculated from the previous whole. That today's audience is
yesterday's whole + its quarter = 8000.
S31 4) Karel—he proceeded from 8000, i.e. from today's audience, not from the yesterday's
Pavla—the same as Karel
S4 4) Radek—he calculated that today is quarter of yesterday's, instead of quarter more than yesterday
5) Radek, read the task carefully: quarter of $\neq$ quarter more!

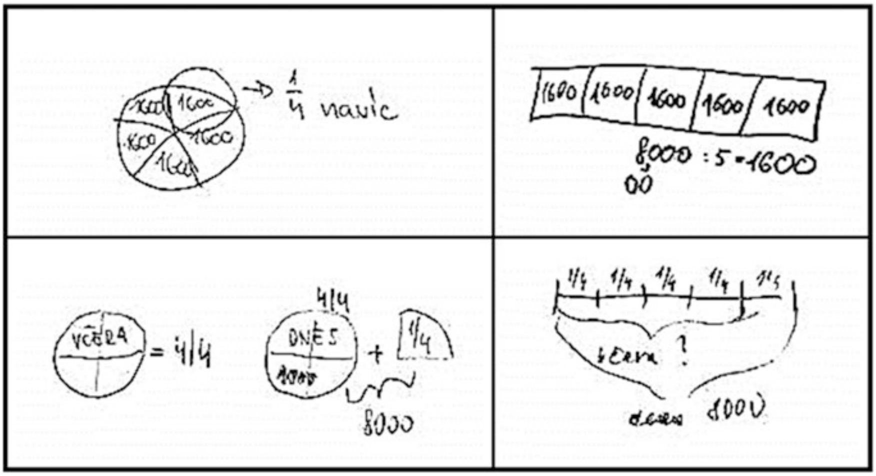

Fig. 7 Excerpts from four different worksheets where respondents agreed with Tonda (S24, S4, S22, S2). Translation: včera = yesterday, dnes = today, navíc = extra

Among the respondents there were also some that did not solve the task at all. They just verified all the results from the bubbles, one by one, and then gave explanations to children with wrong results which showed why these results could not be correct:

S10 3) Tonda—right, 6400 + quarter = 8000
Pavla—quarter of the yesterday's audience, not today's
Radek—wrong, when yesterday was some audience, and today is quarter more, then yesterday could not be more people than today
(bigger number—32,000—nonsense)

S6 3) Pavla—if yesterday came 6000 people, one quarter would be 1500, which does not result in 8000 but only in 7500.
Karel—if yesterday came 8000 people, one quarter would be 2000, and this number would be added to 8000 to get the people that have come today.
Radek—if his bubble were true, then the notice board would say:
Today's audience: 8000
It's a quarter of yesterday's.

Alternatives proposed to Iva's bubble depended on what result the respondent considered as correct. Students who agreed with Tonda often proposed hints to Tonda's result:

S10 6) Iva: 8000 is 5/4
S22 6) Iva: I think that yesterday's audience was 1600 less.
S4 6) Iva: It is 6400, because the picture shows me that it is 4/5 of 8000.

Students who agreed with Pavla often proposed pictures illustrating her procedure—a pie divided into quarters (S16, S18) or a hint about 8000 being 4/4 (S12). Some of the worksheets contained unrealistic alternative solutions to the task:

S29 6) Iva: A quarter of 8000 is 6000, so that yesterday came only 2000 people.
S11 6) Iva: Yesterday came 4000 people. Because 1/4 is 4000.

The Concept Cartoon from Fig. 5 was unique because students did not offer any alternative ways of solving to the blank bubble. However, students offered many alternative ways of solving to the Concept Cartoon from Fig. 3. This Concept Cartoon focuses on a numerical task 38 + 39 from the perspective of mental calculation, and its bubbles offer various pupil's solutions in the form of a procedure with a result. The worksheets contained many diverse correct alternative procedures proposed to Paul's bubble:

S5 6) Paul: First I add tens, 30 + 30 = 60, then I add units, 8 + 9 = 17. Then I add these numbers, 60 + 17, and get the number 77.
S11 6) Paul: 38 + 30 = 68, 68 + 9 = 77
S10 6) Paul: 2 · 30 = 60, and 9 is 69, and 8 is 77.

Besides the usual standard alternatives mentioned above, some other alternatives advantageously utilized the fact that both addends are close to 40 which is a multiple of ten, or that they are close to each other:

S26 6) Paul: I borrow one from 38, and add it to 39 → 40 + 37 = 77. It is better to count, because you don't need to carry any figures.
S3 6) Paul: 2 are missing in 38 to get 40, I borrow them from 39. So that I will have 40 + 37 = 77.
S18 6) Paul: Two times 40, and subtract 3.
S4 6) Paul: (38 · 2) + 1 = 77

Some of the worksheets also offered plausible potential pupils' misconceptions:

S23 6) Paul: (3 + 3) = 6
(8 + 9) = 17
38 + 39 = 617

Discussion

Results of the presented research are in accordance with findings of similar studies focusing on pedagogical content knowledge. Like in Kleickmann et al. (2013), Krauss et al. (2008), the research showed that some of the pre-service teachers were able to gain informal foundations of pedagogical content knowledge from their own learning experience prior to didactical courses and own teaching practice.

As an object in the first part of the research and a tool in the second part of the research I used the Concept Cartoons environment. Results of the research showed that Concept Cartoons provided with a set of indicative questions might become a suitable tool for exploring various aspects of pedagogical content knowledge in mathematics. In accordance with the earlier research on Concept Cartoons (Keogh and Naylor 1999; Naylor et al. 2007), Concept Cartoons appeared to be able to encourage presenting own opinions on texts in bubbles as well as answering the indicative questions about the picture. Although unlike the earlier research, I used Concept Cartoons in slightly different ways: with pre-service teachers instead of pupils, individually instead of collectively, in mathematics, and in written form.

As an important component of Concept Cartoons I see the blank bubble which allows to gain insight into knowledge of alternative ways of solving and knowledge of common pupils' misconceptions. The spectrum of responses to these bubbles that appeared in collected data is vast but the analysis of data shows that the wording of the sixth question which required to propose *one* alternative *or* misconception was unnecessary limiting the possible range of responses. For future use, it would be better to get an inspiration from the study of Kleickmann et al. (2013), and make some changes in the sixth question: divide the question into two parts, in the first part ask for *as many as possible* alternative ways of solving, and in the second part ask for *as many as possible* potential pupils' misconceptions.

The reported research was conducted at the Faculty of Education as a part of a three-year project focusing on opportunities to influence professional competences of pre-service primary school teachers. In some other studies under this project that I performed together with my colleagues, we investigated the possible use of Concept Cartoons in pre-service teachers' education from various perspectives and with various groups of pre-service primary school teachers. In mathematics courses, we used Concept Cartoons to support problem solving, reasoning and argumentation (Samková and Tichá 2016a, 2017b), and as a diagnostic tool for assessing pre-service teachers' mathematical content knowledge (Samková and Tichá 2015, 2016b, 2017a). In didactics courses, we used them in problem posing activities (Samková and Tichá 2016b, 2017b). Among others, Concept Cartoons helped us to show how diverse information could be provided by problem posing and problem solving, and thus confirmed the importance of linking problem solving and problem posing that was emphasized in a recently issued monograph on problem posing (Singer et al. 2015). We also used Concept Cartoons when preparing pre-service teachers for their own teaching practice: such a tool helped them to become aware of potential pupils' mistakes and misconceptions that might appear in the classroom. A similar but not the same tool called *discussion prompt sheets* was mentioned by Ryan and Williams (2011).

To place this current research on Concept Cartoons into the broader framework, it is useful to realize that I employed Concept Cartoons to ascertain responses of pre-service teachers to various alternative opinions of virtual (pictured) pupils. So, in a sense it can be said that I investigated informal foundations of pre-service teachers' ability to notice, namely of the aspects related to comments on pupils' talks (category *pupil commentary* in Vondrová and Žalská 2015).

Further, Concept Cartoons also address the concerns that were raised in the study of Depaepe et al. (2015) which focused on investigating mathematical content knowledge and pedagogical content knowledge of pre-service teachers. Depaepe et al. reproached some previous surveys on pedagogical content knowledge for not investigating pedagogical content knowledge independently of mathematical content knowledge, i.e. that respondents of those surveys first solved a task to demonstrate mathematical content knowledge, and then in the context of the *same* task commented on issues related to pedagogical content knowledge. By Depaepe et al., such circumstances negatively affect data related to pedagogical content knowledge. When using Concept Cartoons, no similar dependence occurs, because no solution of the task is explicitly required, and the respondents may demonstrate some aspects of their pedagogical content knowledge even in the case when they do not know how to solve the task or do not want to solve it. For instance, by proper checking and verifying all offered results, or by proper justification of why a result or a procedure offered in a bubble cannot be correct, the respondents can present their ability to evaluate pupils' answers—an ability that is an integral part of pedagogical content knowledge. In particular, the ability to evaluate pupils' answers without solving the task turns out to be important in teaching in the moment when, as a direct consequence of a discussion or as a prompt support to an explanation, a quite new task not previously prepared by the teacher appears in the classroom.

Conclusion

In this chapter I introduced a study focusing on the opportunity to use an educational tool called Concept Cartoons as a representation of practice in pre-service primary school teachers' education, especially on the possibility to employ Concept Cartoons as a tool for investigating informal foundations of pedagogical content knowledge of pre-service primary school teachers. The study suggests the form of the Concept Cartoons environment suitable for such a purpose, and analyzes displays of pedagogical content knowledge that were collected in the environment from pre-service primary school teachers before entering the course on didactics of mathematics. The results confirmed that Concept Cartoons were suitable for the studied purpose, namely for investigating informal foundations of knowledge of pupils (e.g. ability to recognize pupils' misconceptions, difficulties and solving strategies), knowledge of tasks (e.g. knowledge of multiple ways to solve a problem), and knowledge of instruction (e.g. knowledge of different representations and explanations to standard problems).

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Simulations as a Tool for Practicing Questioning

Corey Webel, Kimberly Conner and Wenmin Zhao

Abstract In this chapter we discuss some of the affordances and constraints of using online teaching simulations to support reflection on specific pedagogical actions. We share data from a research project in which we implemented multiple iterations of a set of simulated teaching experiences in an elementary mathematics methods course. In each experience, preservice teachers contrasted the consequences of different pedagogical choices in response to a particular example of student thinking. We share how their evaluations of their choices shifted within experiences at certain points, and their criteria for "good" questions began to evolve. We end with implications for how simulations can promote critical reflection on teaching practice.

Keywords Representations of practice · Teaching simulations · Questioning
Preservice teacher education · Elementary mathematics

Introduction

Representations of practice are being increasingly used to engage preservice teachers (PSTs) in problems of instruction (Amador et al. 2017; Bartell et al. 2013; de Araujo et al. 2015; Herbst et al. 2011; Sun and van Es 2015). These can include videos, animations, comic strips, vignettes, photos, and real or manufactured representations of student work. These representations have various affordances and limitations, but in general, they help PSTs and their instructors decompose instructional practice into manageable pieces that can be described, interpreted, analyzed, and practiced. Part of the purpose of this monograph is to explicate various ways that representations of practice can be used in teacher education to promote learning through, for example, stimuli for reflection, criteria-based analysis, or structured observation.

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In this chapter, we focus on a particular kind of representation of teaching which we describe as a *teaching simulation*, created with tools provided by the Lesson*Sketch* online platform (www.lessonsketch.org). A simulation, as we use the term, is a representation of practice that provides an opportunity for pedagogical action, in addition to opportunities for activities like noticing student thinking and reflecting on teaching. We designed the simulations to gain insights about the decision-making processes of novice teachers as they make pedagogical choices. They allow us to set up opportunities for learning directly from teaching (Hiebert et al. 2007), as PSTs try different pedagogical actions and then reflect on their consequences.

In this paper, we reflect on some specific design considerations that have resulted from the first two years of implementation. We specifically targeted the questioning practices of the PSTs in our program, documenting their pedagogical choices as well as their rationales and reflections. We analyzed what PSTs noticed about the consequences of the questions they selected in specific pedagogical situations and documented changes in their choices and explanations within and across multiple experiences with the simulations.

Background

Questioning Practices

International comparisons in mathematics teaching have shown that low-level questions, which require students to recall specific facts or carry out certain procedures, are especially prevalent in the United States (Givvin et al. 2005; Kawanaka and Stigler 1999; Stigler et al. 1996). Similarly, sequences of closed questions, intended to direct students through a series of procedural steps until they obtain the correct answer, have been referred to as *funneling* (Herbel-Eisenmann and Breyfogle 2005; Wood 1998). These types of questions position students as recipients of information rather than contributors to their own knowledge development (Boaler 2003; Webb et al. 2006), and are unlikely to spur correct and complete explanations on the part of students (Franke et al. 2009).

Recent research has described questioning practices that, in contrast to funneling or recall questions, are responsive to student thinking, drawing out and building on the specifics of students' ideas rather than imposing the teacher's idea (Jacobs and Empson 2016; Kazemi and Stipek 2001; Sherin 2002). Building on this research, The National Council of Teachers of Mathematics' *Principles to Actions* (2014) advocated teacher questions that "build on, but do not take over or funnel, student thinking," and those that "make mathematical thinking visible" (p. 41). In this study, we are looking specifically at follow-up questions that teachers might pose immediately after eliciting an initial explanation about a student's solution. Based on the literature described above, we defined types of questions as "low leverage"

Table 1 Classifications of question types

Type	Description	Example
Low leverage (directive)	Suggests a specific alternate strategy, does not refer to student work	Do you know how to do find common denominators?
Low leverage (invalidate)	Specific to student's work, but invalidates the student's strategy	Is it really two whole brownies?
Low leverage (funnel)	Responds to student work, but funnels to a correct answer. Often includes a binary choice (either/or, yes/no, etc.)	Are those pieces [in your diagram] sixths, or tenths?
High leverage (elicit)	Elicits student's thinking	Can you tell me more about the sixths in your diagram?
High leverage (build)	Help students build on their own thinking	Based on your diagram, who would you say gets the most amount of brownie?

or "high leverage" as shown in Table 1. For example, the low leverage (funnel) example refers to a student's work, but presents a binary choice that funnels the student toward a correct answer. In contrast, the high leverage questions reference specific aspects of student work ("your diagram"), but either elicit more information from the student about the work (i.e., they make mathematical thinking visible), or push students to consider the meaning of their work without conveying that it is correct or incorrect (i.e., they provide opportunities for students to build on their own thinking).

Learning to Ask Better Questions

Teacher educators have used various approaches to help PSTs and practicing teachers improve their questioning (Milewski and Strickland 2016; Moyer and Milewicz 2002; Nicol 1999; Spangler and Hallman-Thrasher 2014; Wagner 1973). For example, Moyer and Milewicz (2002) introduced a questioning framework to support PSTs in recognizing questions with different features. PSTs who worked with the framework began to ask more follow-up questions, but inconsistently (e.g., in some cases, they did so only when students had incorrect answers). Spangler and Hallman-Thrasher (2014) used imaginary task dialogues to support PSTs' ability to anticipate and respond to student thinking. When PSTs enacted the tasks with real students, the researchers found that while PSTs were able to develop and use a repertoire of "standard" responses, such as "How did you get that?" and "Can you tell me what you were thinking?", they struggled to respond to students in ways that were task-specific. Nicol (1999) found that PSTs struggled to reconcile different purposes for questions, such as learning more about student thinking but also helping them arrive at a correct solution. All of this work shows that supporting PSTs in developing high leverage questioning practices is challenging, and that learning about frameworks or categories of questions does not always translate into the ability to respond to students with high leverage questions. One explanation is that knowledge is situated; that is, "how a person learns a particular set of knowledge and skills, and the situation in which a person learns, become a fundamental part of what is learned" (Borko et al. 2000, p. 195). This view of knowledge as situated implies that that if PSTs are going to draw on the knowledge and skills that they gain in their education courses, the context of their learning experience needs to feel like teaching. Such experiences could include approximations that represent some authentic aspects of practice but also provide low-risk opportunities for novices to try, fail, and learn from their practice (Grossman et al. 2009).

Representations of Practice

To approximate practice, one must first represent it. There are many ways to represent teaching practice, including vignettes, depictions of student work, photos, animations, comic-strips, and videos. These have various affordances, but in general they aid in the decomposition of practice and support reflection on specific pedagogical situations (e.g., Kuntze et al. 2015). Also important are the ways that learners are asked to engage with representations (Beilstein et al. 2017). Videos in particular have been shown to help PSTs analyze and attend to details about the work of teaching (Star and Strickland 2008; Sun and van Es 2015). For example, Sun and Van Es (2015) found that PSTs who took a video-based course attended to and took up student ideas better than the students who took the previous course that did not utilize videos. They concluded that "learning to systematically analyze teaching with video can help PSTs learn to enact practices that afford opportunities to access and examine student thinking" (p. 210).

We define simulations as representations of practice that provide the possibility of pedagogical action. When a PST engages in a simulation, they can engage in activities similar to those associated with other representations (noticing, interpreting, describing, and reflecting), but in addition, they can make choices that actually *affect* the representation. They can see the results of those choices, and can make judgments about those choices on the basis of their effects. While we have designed our simulations within Lesson*Sketch*, other types of simulated experiences have been employed in mathematics teacher education, such as the use of trained actors or peers playing the role of students (Baldinger et al. 2016; Lampert et al. 2013; Shaughnessy et al. 2015). These similarly put novices in the position of making choices that have consequences within the simulation, though such "rehearsals" require decisions to be made quickly and may not afford as much time for reflecting on specific decisions.

In this project, we used the Lesson*Sketch* platform to design online storyboard teaching scenarios, which include some aspects of the teaching context such as a classroom, students, student work, dialogue (represented by text bubbles), etc. Lesson*Sketch* provides tools to aid in the reflection process; moments where the situation is paused and the user can be asked to make a choice, provide a comment, or ask a question. Finally, Lesson*Sketch* includes a "media chooser" tool, in which the user can be asked to select one out of a number of representations (in our case, these represented possible teacher actions). Each choice represents a unique path, and the designer can establish in advance how the situation will unfold in response to particular choices made by the user. The use of this feature is what distinguishes our Lesson*Sketch* experiences as simulations (see Kosko 2016 for a similar use of Lesson*Sketch*).

In contrast to interviews with real students (e.g., Moyer and Milewicz 2002; Nicol 1999), or with peers playing the role of students (Baldinger et al. 2016), the Lesson*Sketch* tool allows the designer a high degree of control over what the user can see, do, and notice within the representation (Herbst et al. 2011). Because Lesson*Sketch* experiences are standardized, the quality of the experience is not dependent upon the expertise of facilitators, actors, or peers playing the part of students. This is both a strength of situating the simulation with Lesson*Sketch* (we can compare how different participants respond in the same instructional situation) as well as a limitation (it cannot respond as flexibly to individual differences, and only includes a limited number of choices). In addition, because our simulations are online, they can be accessed easily by many participants and can generate substantial data in a short amount of time. Tweaks to the design can be made with little effort and new iterations can be subsequently tested with new populations.

One of the goals of the simulations was to not only see what could be revealed about PSTs' questioning practices, but to see if the simulations might impact the way they reflected on questions, including their purposes for questions and whether they believed their selected questions were "good." In this paper, we discuss our findings related to the research question, "How might cartoon-based teaching simulations be used to challenge novice teachers' mathematics questioning practices?"

Methods

Participants

In the first year of implementation, we engaged PSTs (n = 53) in three simulations during their first of two elementary methods courses at a four-year university in the Midwestern region of the United States. After analyzing and reporting on this data (Webel and Conner 2015, 2017), we revised these simulations and administered them with a new population (n = 86) the following year. In both administrations, PSTs were generally in their junior (third) year of university study, approximately 20–21 years old. The first course in the sequence targeted fraction concepts for the first eight weeks of the semester, focused specifically on helping PSTs appreciate the role of the unit in constructing and naming fractions (Chval et al. 2013), while the second course focused on measurement and geometry. Course assignments included explorations of mathematics with an emphasis on justification and reasoning, as well as analyzing and interpreting student work. As part of the program, each PST was assigned a field placement in an elementary classroom in which they spent at least 60 hours over the course of each semester during their junior year.

Data Collected from the Simulations

Each simulation involved a mathematical task, a classroom scenario, and a representation of student work. PSTs completed each experience as a homework assignment; the three experiences were spaced out, with about three weeks between them. A map of one experience titled Brandon is provided in Fig. 1.

First, PSTs solved a mathematical task (Step 1) and were asked to describe the mathematical ideas addressed in the task (Step 2). They watched a classroom episode that culminated in the teacher asking Brandon to explain his work. Then the PST was presented with several prompts, including requests to interpret the thinking represented by Brandon's work (Step 3), compose a question for Brandon (Step 4), and then select a question from a pre-established list (Step 5) and provide a rationale for why they believed the selected question would be the best to ask Brandon (Step 6). The choices included both high and low leverage questions. In the example shown in Fig. 1, the high leverage question (Step 5.3) aimed at eliciting Brandon's thinking by focusing on the critical misconception in his solution without directing him down a particular path. One of the low leverage question directed Brandon to a specific (procedural) strategy (Step 5.1), and the other funneled Brandon to a yes or no answer and conveyed that his solution strategy was incorrect (Step 5.2).

After selecting one of these questions, the PSTs viewed a predesigned response from Brandon (Step 7) and then were asked to evaluate their question once more (Step 8). For the high leverage question in Fig. 1, Brandon's response showed explicitly that he was now thinking of the previously established fourths and sixths as tenths, and in doing so had also changed the referent whole from one cup to two cups. This response has potential to help PSTs see these misconceptions more clearly than in Brandon's original response, and also opens up possibilities for Brandon to recognize, on his own, the inconsistencies in his solution (for example, he drew all of the sixths in the bottom cup to be the same size, but does not recognize that not all of the "tenths" are the same size). In response to the low leverage question, "Are fourths the same as sixths?" Brandon's response gave little information about his thinking; he merely responded with the expected answer of "no." Rather than providing an opportunity for Brandon to recognize and confront his misconception, the teacher's question allowed the misconception to remain unexamined while simultaneously conveying that his solution was incorrect.

After viewing Brandon's response to their selected question, PSTs were asked to imagine they could "go back in time" to see what would have happened had they asked the other question (Step 9). They viewed the Brandon's new response, and

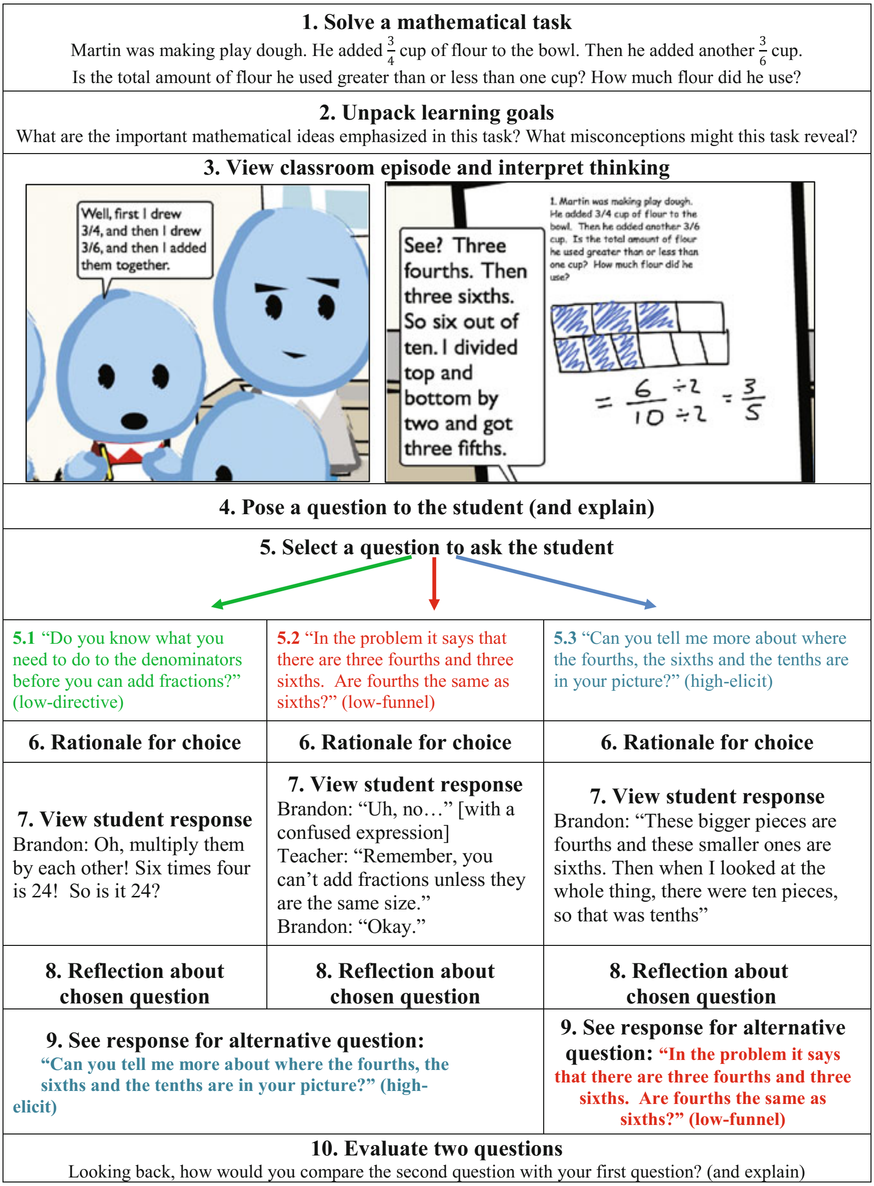

Fig. 1 A flowchart of the Brandon experience

then concluded the experience by determining which of the two questions they believed was "better" and explained why (Step 10). In some of the experiences, this was the final step. In the Brandon experience, we included another set of questions that could have been posed to Brandon, and repeated Steps 5–10 with this new set of questions (hereafter referred to as Brandon B. The set of questions shown above will be referred to as Brandon A).

Although we have data from six experiences in total (three from Year 1 and three from Year 2), only four of the experiences follow the format shown in Fig. 1, and these are the focus of this chapter. The two experiences that are excluded did not have Steps 9 and 10, in which PSTs compared the effects of different questions. Our previous analysis of data from Year 1 (as described in Webel and Conner 2017) suggested that Steps 9 and 10 were important for challenging PSTs' perspectives about effective questions. In this chapter, one of our main aims is to document the differences between how PSTs responded to the prompt in Step 8 versus the prompt in Step 10, and so we only include analysis from experiences that included both steps.

The experiences and question sets analyzed in this paper include the following:

- Matthew 2015: Matthew is depicted as questioning whether "3/4 of two brownies" can be an accurate way to describe a picture of two square brownies, with one whole brownie shaded and half of the second brownie shaded. He responds, "No, because they are cut in half, not in four squares." This is the only experience analyzed from the first iteration of the simulations in 2015.
- Matthew 2016: The same experience offered in 2015, but with a new group of PSTs.
- Brandon A: The experience described in Fig. 1.
- Brandon B: A second set of questions offered at the end of the Brandon experience. This set consisted of two questions: "What is the whole?" and "Where is the cup of flour in your picture?"
- Cedric: A new experience involving the task, "If I have four square yards, how many square feet is that?" Cedric draws a picture of a 4 by 3 rectangle, multiplies 4 by 3, and gives the answer of 12 ft^2.

In each experience, the student (Matthew, Brandon, or Cedric) produced work that revealed a significant mathematical misconception that had been previously discussed with the PSTs in the methods class. In none of the responses to questions did any of the simulated students completely resolve their misconception. This reflected our desire to represent student thinking authentically and challenge the naïve belief that misconceptions can be easily resolved in a short exchange (Spangler and Hallman-Thrasher 2014).

Data and Analysis

In this chapter, we describe what questions PSTs selected in each experience (in terms of high leverage or low leverage) and how they evaluated their questions at two time points (Step 8 and Step 10 in Fig. 1). In Step 8, PSTs selected one of three options:

- It was a good question; it accomplished what I wanted it to accomplish
- It was a good question, but [the student] didn't respond in the way I expected
- It was maybe not the best question; I should have asked something different.

After seeing the student's response to the second question (Step 10), they chose from the following options:

- The second question was better than the first
- My first question was better
- They were the same.

For each of these questions, PSTs were asked to type an explanation for their choice. We used a constant comparative process (Glaser and Strauss 1967) to place these explanations into 11 categories according to emerging themes, and then consolidated these themes into five larger conceptual categories. Once codes were agreed upon, we coded approximately 25% of the data individually between two researchers, reaching an agreement rate of 84% on the five large categories. We then coded the rest of the data, individually, and resolved all discrepancies through discussion. Table 2 shows the final codes, some of the most prevalent initial codes, and examples of explanations given by PSTs in each category.

Most of our findings will report numerical patterns in how PSTs answered the multiple choice questions (Steps 5, 8 and 10) across different experiences, focusing mostly on those who chose a low leverage question at Step 5 and whether they expressed doubt about the effectiveness of that question in either Step 8 or 10. However, we will also supplement these findings with summaries of explanation codes (from Table 2) and examples of explanations that PSTs provided to support their choices.

Results

Impact of the Simulated Experiences on PSTs' Question Preferences

The experiences did appear to have some influence on how PSTs thought about the questions they chose initially, particularly if they chose a low leverage question. Figure 2 shows all of the experiences in which PSTs had opportunities to compare the effects of a high and low leverage question.

For example, in the Matthew 2015 experience, 30% of PSTs initially selected a high leverage question in Step 5, and after the experience, 43% preferred that question over the alternative low leverage question that they had viewed in Step 9. Of the 70% who initially chose a low leverage question, only 13% still preferred that question after seeing the high leverage option (the remaining PSTs did not prefer one question over the other).

Table 2 Most prevalent codes for evaluations of questions after seeing student response

Final code	Initial code	Example explanation for choice
Directing: Question led to teacher take over of strategy/ thinking	The question provided an opportunity for the teacher to explain or tell.	"This is enough information for me as the teacher that I need to pull Brandon aside and have a mini lesson with him."
Addressing misconceptions: PST claims that the question helped the student understand, focused on a misconception, or failed to "fix" a misconception	The student understands now.	"Brandon understands that the denominator needs to be the same in order to add the fractions."
	The question directed the student to the misconception.	"I did give him a clue about what he should do next, he just didn't use it to find his answer."
	The question did not fix the student's misconception.	"This question was useless because Brandon has no idea how to find a common denominator."
Understanding student thinking: PST claims that the question helped the teacher to better understand Brandon's thinking or allowed the student to explain his thinking	The question helped the teacher understand the student's thinking.	"I think it was a really great question to ask Brandon because although he did not discover the correct answer I, the teacher got a much better understanding of his thinking."
	The question did not provide information about the students' thinking.	"I was hoping he would give some more explanation as to why he added them all up."
Building on student thinking: PST claims that the question provided an opportunity for Brandon to come to a new realization *on his own*	The question caused (or will cause) student to extend *his own thinking* to come to a new realization.	"I wanted Brandon to realize his confusion without me having to point it out to him. By asking this question, he reevaluated his answer and decided it may have not been the best solution."
	The question was too leading or gives away the answer.	"I should not have asked this question because the teacher gave away the answer and it did not probe Brandon to think on his own about the problem."
Other	Other (does not give a clear evaluation of the question).	"I expected Brandon to ask why the pieces must be the same size in order to add them."

Across all of the experiences, the chart shows inconsistent results for those who started by choosing a high leverage question (left hand side of Fig. 2)—sometimes, after viewing both types of questions, more PSTs expressed preference for the high

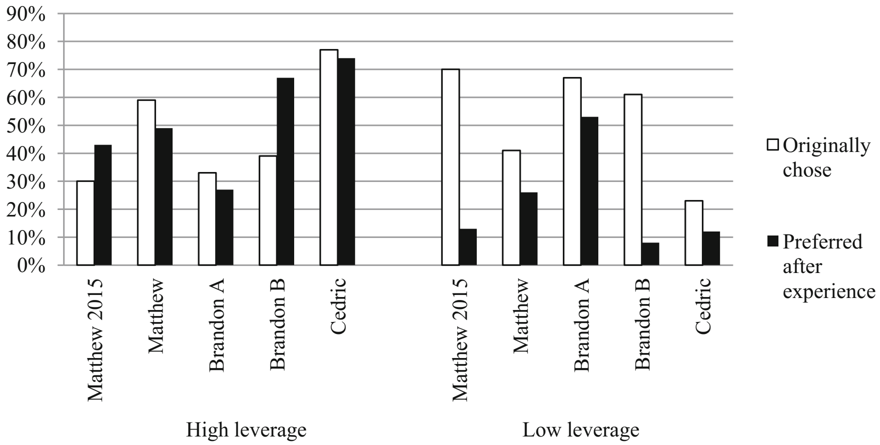

Fig. 2 Percentages of types of questions selected at the beginning of each experience and preferred at the end of each experience

leverage question they had initially selected, and sometimes they did not. In contrast, those who initially selected the low leverage questions (right hand side of Fig. 2) were more consistent in stating that they did not prefer their initial question after viewing both responses. This suggests that in general, the experiences supported PSTs in being more critical of their low leverage questions—but also did not necessarily increase their confidence in their high leverage questions.

The rationales PSTs wrote for their evaluations of these questions gave us indications about why some PSTs were impacted by the responses given by Brandon and some were not. For example, after initially selecting the low leverage question for Brandon A ("In the problem it says that there are three fourths and three sixths. Are fourths the same as sixths?") and seeing Brandon's reaction ("uh, no"), one PST explained why she thought the question was effective:

> I think this was a good question to ask Brandon because he realized that the sixths and fourths are different sized parts. He also realized that you cannot add fractions if the denominators are different numbers (and represent different sized parts). This is leading Brandon in the right direction of adding his fractions again, but the correct way. Because he knows that you cannot add fractions if the denominators are different, Brandon's next step would be to find common denominators.

This PST considered Brandon's response ("uh, no") to constitute evidence of understanding, and conveyed in her reflection that the question helped resolve his misconception. After seeing both questions, this PST still believed that the initial question was better:

> The first question prompted him into knowing that you cannot add fractions if the parts are different sizes and the denominators are different. The second question did not change Brandon's idea about the 6/10ths being incorrect. He was able to identify where the fourths were represented and where the sixths were represented, but he did not notice that the parts were different sizes. He still continued to count all of the parts together.

This was typical of the 39 PSTs who initially chose and then maintained their preference for the low leverage question; all but four of their explanations were coded as “addressing misconceptions.” On the other hand, of the 10 PSTs who changed their minds after going through the Brandon experience, six explanations were coded as “understanding student thinking.” For example, one PST wrote,

> I think the second question was a lot better because we actually get the chance to observe Brandon’s thinking and strategies. He is able to explain his thought process for us. The other question was more of the teacher telling Brandon what is right and what is wrong.

This suggests that when PSTs change their minds about the low leverage question they initially picked, they are doing so because they are attending to positive consequences of the question other than whether it supposedly resolves the student’s misconception; in this case, the PST values getting more information about Brandon’s thinking.

Important Features of the Experiences

The quotes in the previous paragraph suggest that *the opportunity to see the results of different pedagogical actions* was an important part of the experience. That is, the PST’s criteria for effective questioning began to shift when she compared the consequences of a low leverage question with the consequences of a high leverage question. Across all of the experiences, we saw that, indeed, PSTs became more critical of their initial low leverage questions only *after* comparing with a high leverage question (Fig. 3). For example, in the Brandon B experience, 36% of the PSTs who initially selected a low leverage question selected “It was maybe not the best question; I should have asked something different” after seeing Brandon’s response (Step 8). But after comparing with the high leverage question (Step 10), the percentage who selected “The second question was better than the first” was 62%. This shows that more PSTs had begun to doubt the effectiveness of the question they originally chose. In fact, in all of the experiences, more PSTs expressed doubt about their selection after seeing both high and low leverage questions.

This pattern was stronger in some experiences than others. In fact, the Brandon A experience was the least effective in terms of prompting PSTs to be more critical about their initial question choice. The Matthew experience, in contrast, revealed that some PSTs questioned their choice after seeing Matthew’s initial response, but substantially more PSTs doubted their original choice after seeing both questions. Explanations for these patterns are suggested by the PSTs’ evaluations of their questions. For example, after choosing a low leverage question and seeing Matthew’s response, one PST wrote,

> I still think the question was a good question, it shows the teacher that Matthew doesn’t understand what the partitioned pieces represent of the whole. She knows this is where she’ll have to work more with him and maybe the whole class.

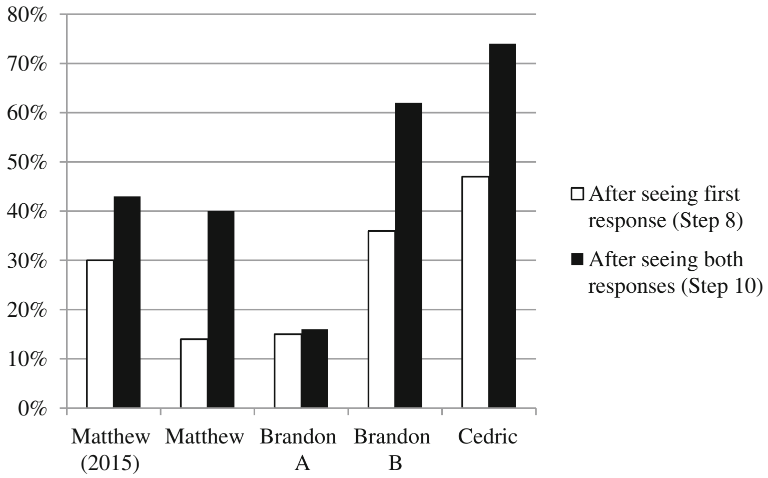

Fig. 3 When PSTs expressed doubt about their low leverage question choice. *Note* The total for each experience is the number of PSTs who initially selected a low leverage question, which was different for each experience

But after seeing Matthew's response to the high leverage question, the same PST wrote,

> The second one might have been better because it has him draw out what he is thinking. Therefore, you can see exactly what he is thinking because sometimes the explanation can get confusing and he might not be saying what he is thinking. But if he draws it out then you know for sure what he is thinking.

This PST, and several of her peers, only questioned her first choice when shown how Matthew responded to the high leverage question. Initially, she based her evaluation of the question on whether Matthew understood the mathematical idea, but after the second question, she based her evaluation on how clearly she could see Matthew's thinking. This suggests that providing a contrast between different pedagogical moves created a learning opportunity for several PSTs, and encouraged them to consider affordances of questions that they might not initially see as valuable (such as "knowing for sure" what a student is thinking).

In summary, the results show that, in general, individual experiences tended to support reductions in the number of PSTs who preferred low leverage questions. PSTs who changed their minds often shifted their criteria for evaluating questions from addressing misconceptions to drawing out or building on student thinking, while those who did not change their minds continued focusing on whether the question resulted in "fixing" a misconception. When we looked more closely at the features of the experiences, we saw that providing a student response to the initially chosen low leverage question (Step 8) resulted in some doubts about this question, but that providing a second response to a contrasting (high leverage) question (Step 10) increased the number of PSTs expressing doubts about that initial question.

Discussion

These findings suggest that the teaching simulations have some potential for challenging PSTs' initial questioning practices and provide some affordances that are not present in other representations. For example, some representations are primarily examples of student work (e.g., Bartell et al. 2013; Jacobs et al. 2010). Preservice teachers are able to analyze this work and even say what they would do in response, but they do not get to see the consequences of their decisions. In our simulations, the user engages in many of the same analyses, but then makes a choice that has an effect. This means that rather than getting feedback about their analysis and decisions from an instructor, feedback is contained *within* the simulation, in the form of the response from the student. This feedback can cause PSTs to reevaluate their interpretation of the students' thinking, their thoughts about the mathematics itself, and/or their pedagogical choice. In this sense, the PST is learning *from* teaching (Hiebert et al. 2007), rather than just learning *about* teaching.

Videos are another popular representations of work used in teacher education (e.g., Beilstein et al. 2017; Sun and van Es 2015; van Es and Sherin 2010). Videos have the benefit of realism—the students and teachers in the videos are real people doing the real work of teaching, in real time. However, this realism comes at a cost. First, the complexity of a video means that there are many things PSTs might pay attention to (what students are wearing, what students in the background are doing, how desks are arranged, etc.), which may or may not be the particular object of learning intended by the teacher educator. A simulated experience, while less realistic, allows the designer to reduce the complexity of an instructional situation to focus attention on specific objects of learning (Herbst et al. 2011). Secondly, as with analyzing student work, when watching a video PSTs do not have the possibility of making any choices. They can watch what happens, but they can only participate vicariously. In this sense, simulations provide the additional affordance of providing the opportunity for PSTs to engage in the scenario and "interact" with the student (albeit in a limited manner), allowing them to do some of the work of teaching rather than just observing and talking about it (Ball and Forzani 2009). Finally, when watching a video, PSTs can only see what actually happened in the recorded episode. In our simulations, PSTs see multiple versions of what *might* have happened, and then consider the affordances of different decisions that could have been made at a particular moment in time. Our data supports the conjecture that this weighing of different outcomes led to increased critique of practice, and indeed, this appears to be one of the main affordances of our simulations.

At the same time, there are certainly challenges involved with designing and using simulations within the Lesson*Sketch* environment. For example, since we created the teacher and student contributions, we cannot be sure that these represent realistic interactions or that PSTs will accept them as possible events that might occur in a real classroom. When designing the experiences, we drew on many of our own experiences working with teachers and students and sought to avoid simplistic or inauthentic interactions, but ultimately, we cannot say that the interactions we designed represent real teaching. On the other hand, the experiences we have designed are, in a sense, stories, and stories need not be true to be educative. What they need to do is feel authentic to the listener; the user of an experience must be able to imagine that real students and teachers could do and say the things depicted.

Another challenge with our simulations is that the interaction between student and teacher are necessarily much shorter than a real interaction. Jacobs and Empson (2016) argue that single talk turns (like posing a question) are sometimes inadequate for capturing the intent of a teaching move "because teachers often need to persist to support or extend children's thinking" (p. 188). Thus, judging PSTs' intentions based on a single question might be viewed as overly simplified. We would agree that real interactions with real students are messier and less structured than in our simulations, and that in such interactions, teachers have many more opportunities to either build on or take over student thinking. The nature of a designed simulation makes it difficult to create scenarios in which users make more than a few consecutive decisions, as possible outcomes increase exponentially with the addition of more decision points.

However, we would also argue that our simulations mitigate this in three ways. First, the PSTs not only select a question, but also explain why they chose the question and then evaluate it afterwards, which gives us more information about where they expect the conversation to go after asking the question. Second, in some experiences, we included indications (e.g., with thought bubbles) about the how the simulated teacher envisions the ensuing conversation, increasing our confidence that PSTs who choose a question are doing so with an understanding of the teachers' intent. Third, the question choices come after the teacher in the scenario has already elicited some initial information about the student's thinking. The question that comes next reveals what the PST plans to do with that thinking. In particular, if the teacher's initial move is to take over student thinking, it is not likely that later moves will start building on thinking. In this case, we assume that the teacher's goal is to direct students towards a particular approach, and indeed our analysis of rationale for PSTs' question critique supports this—PSTs who pick leading questions are much more likely to talk about "fixing" students' misconceptions by explaining or telling. The converse, however, is not assumed. If a teacher begins with a question that draws out or builds on thinking, they may or may not take over student thinking later in the interaction. We have some indications of this in the PSTs' evaluations of their selected questions; for example, in some cases they talked favorably about a high leverage question, explaining that it provided an opportunity for them to explain how to do the problem. These cases give some support to the idea that the PSTs' question choice only gives partial indication of the PSTs' overall intention for an interaction with a student.

A final challenge involves interpreting PSTs' pedagogical decisions across multiple simulation experiences. It has proven difficult to design different pedagogical situations in which we can be confident that the underlying features of questions are similar enough that we can tell whether PSTs are attending to them for consistent reasons. For example, in Brandon B, 33% of PSTs initially selected a high leverage question, but in the Cedric experience, this percentage was 77% (see Fig. 2). Is this because the PSTs' criteria for effective questioning was different, or because there is something about the Cedric experience (the mathematical task, the student work, the question choices) that is influencing their initial choice? There is simply too much variance across the experiences to know. Within an experience, this is less of an issue, because the only variation is the questions—the mathematical task and student work are the same. That is why, instead of comparing directly across simulations in terms of the numbers of PSTs who select each type of question, we have examined change *within* an experience, and documented whether PSTs who start by selecting a low leverage question become critical of that choice.

Conclusion and Future Considerations

Representations of teaching can clearly provide learning experiences for PSTs; simulations are a particular kind of representation that has certain benefits and limitations. Simulations slow down the action, reduce complexity, and allow decisions and reflection on their consequences. Also, because these are completed online, we generate data quickly in a form that is relatively easy to organize and analyze.

One feature that is important for teacher learning, especially in the context of trying out teaching within simulations or rehearsals, is feedback (Baldinger et al. 2016; Lampert et al. 2013); our feedback comes primarily from the simulation itself. In our second year of the project, we decided to include, at the end of each experience, explanations about what was advantageous about the high leverage questions (e.g., they did not do the mathematics for the student, they elicited additional thinking, they provided opportunities for students to build on their own ideas). We wondered whether these would start to be internalized in subsequent experiences. Our data do not allow us to address this question, but this raises the question of whether some additional in-class discussion might further support efforts to help PSTs more critically examine their pedagogical choices. Originally we had hoped that the simulations might become stand-alone modules that could be accessed more widely without requisite in-class activities. Additional testing is needed to see whether these response patterns can be strengthened and become consistent across contexts, and also to see whether they translate into changes in practice in real teaching situations, such as one-on-one tutoring sessions or small group tasks.

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Competence Assessment with Representations of Practice in Text, Comic and Video Format

Marita Friesen and Sebastian Kuntze

Abstract Representations of classroom practice are considered to be particularly suitable for assessing aspects of teacher competence. However, the role of representation formats in the design of test instruments has been investigated only scarcely so far. Consequently, the study presented in this chapter addresses the question whether N = 162 pre-service teachers' analyzing of six classroom situations is related to the format those situations are represented in (text, comic or video). Given the high relevance of dealing with multiple representations in the mathematics classroom, the study focuses on pre-service teachers' competence of analyzing how multiple representations of mathematical objects are used and connected to each other. The results indicate that representations of practice in the formats video, text and comic are comparably suitable for competence assessment in this context.

Keywords Representations of practice · Competence assessment
Video · Comic · Analyzing

Analyzing the Use of Multiple Representations

Due to the double role they play in the mathematics classroom, multiple representations can be described as "aid and obstacle for the learning of mathematics" (Dreher and Kuntze 2015b, p. 26): As mathematical objects are abstract and can only be accessed through representation, the use of multiple representations plays an indispensible role for problem solving and students' conceptual understanding (Duval 2006; Goldin and Shteingold 2001; Goldin 2008; Acevedo Nistal et al. 2009). Being able to use more than one representation of a mathematical object is essential as any representation will express some but not all information of the related mathematical object, stress some aspects and hide others (Dreyfus 2002).

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Many tasks involve at least two representation registers (Duval 2006; Lesh et al. 1987) and dealing with multiple representations becomes necessary whenever another representation appears to be more efficient in the process of problem solving (Dreyfus 2002; Ainsworth 2006).

At the same time, changing between multiple representation registers of a mathematical object (e.g., between algebraic and pictorial representations) is cognitively challenging as the learner must discriminate mathematically relevant features from those that are not relevant in a mathematical sense and transfer the information from one representation register to the other (Duval 2006). Therefore, changes between multiple representations of a mathematical object, so-called *conversions* (Duval 2006), are described as a source of problems in understanding in every domain of mathematics and at every level of teaching (Ainsworth 2006; Duval 2006; Lesh et al. 1987). This is in particular the case when learners are not sufficiently supported in connecting different representations of a mathematical object to each other when they carry out conversions (Duval 2006; Ainsworth 2006).

Consequently, an adequate support of students in making connections between multiple representations of a mathematical object requires that teachers are able to *analyze* how representations are used in the mathematics classroom: Teachers have to be able to identify and interpret situational aspects that are relevant for learning with multiple representations, such as unconnected conversions (Dreher and Kuntze 2015a; Friesen and Kuntze 2016; Friesen et al. 2015). Therefore, professional knowledge regarding the use of multiple representations is required to provide criteria as a basis on which relevant classroom observations can be interpreted (Kuntze et al. 2015; Friesen et al. 2015; Sherin et al. 2011).

As specific and context-dependent abilities to cope with profession-related demands can be described as competences (Weinert 1999; Baumert and Kunter 2013), *analyzing classroom situations regarding the use of multiple representations* can be regarded as an important profession-related competence for mathematics teachers (Friesen and Kuntze 2016). This is also supported by studies showing that such analyzing is an important characteristic of teacher expertise (Dreher and Kuntze 2015a) and can be learned in the context of professional teacher development (Friesen et al. 2015). Accordingly, we define the *competence of analyzing the use of multiple representations* as a teacher's ability to link relevant observations in a classroom situation to corresponding criterion knowledge so that unconnected changes of representations can be identified and interpreted with respect to their role as potential learning obstacle. Such competence can be seen as an important prerequisite for mathematics teachers in order to be able to provide students with adequate support in making connections between multiple representations of a mathematical object.

Competence Assessment with Representations of Practice

As profession-related competences are characterised by the range of situations and tasks which have to be mastered, competence assessment should be done by confronting test-takers with a sample of such (simulated) situations (Weinert 1999; Shavelson 2013). Accordingly, representations of practice can be implemented in corresponding test instruments in order to assess competence in close relation to professional requirements of teachers (e.g., Oser et al. 2009). In contrast to direct classroom observations, test instruments making use of representations of practice allow to assess competences under standardised conditions as the test-takers' responses to the same classroom situations become comparable (Kaiser et al. 2015; Oser et al. 2009). In addition, such instruments enable the systematic assessment of competences with larger samples of teachers (Borko 2016).

Many studies in the field of competence assessment argue for the use of video-based representations of practice as video is supposed to allow the perception of meaningful real-life job situations (Blömeke et al. 2015). Furthermore, representations of practice in video format appear to enhance teachers' engagement with classroom situations in terms of perceived authenticity and resonance with own classroom experience (Seidel et al. 2011; Kleinknecht and Schneider 2013). Teachers have also found to be motivated when working with video-based representations of practice and reported high immersion into the presented classroom situations (ibid.). However, recent studies have drawn attention to representations of practice in other formats than video: A comparison between pre-service teachers' analysis of the same classroom situation in the formats video and animation showed that the participants rated the genuineness of the representation significantly higher in the case of video (Herbst et al. 2013). The pre-service teachers' analyzing, however, appeared not to be related to the format the classroom situations were represented in, as corresponding analyzing results did not show any significant differences (ibid.). Herbst et al. (2013) concluded that representations of practice in the animation format might be comparably effective to video in order to elicit pre-service teachers' analyzing.

Another possible format to assess teachers' analyzing of mathematics classroom situations are text-based representations of practice as used by Dreher and Kuntze (e.g. 2015a, b) in order to assess teachers' theme-specific noticing: They applied four short transcript-like texts with fictitious classroom situations to elicit teachers' ability to notice potentially obstructing demands of unconnected changes of representations for students' understanding (ibid.). Similar to text-based representations of practice, comics allow to sketch numerous and systematic variations of a classroom situation that can hardly be found and recorded in reality (Herbst and Kosko 2014). Comic-based representations of practice were, for example, used by Herbst et al. (2016) who implemented cartoon storyboards in order to assess teachers' instructional decision-making.

There are, however, so far only very few studies in the field of competence assessment that are format-aware in the sense that they investigate how teachers engage with different formats, e.g., in terms of perceived authenticity or if teachers' analyzing of a classroom situation is related to the format it is represented in. In the following, texts, comics and video-based representations of practice will be compared with respect to the assessment of pre-service teachers' analyzing of classroom situations as investigated in this study.

Representations of Practice in Text, Comic and Video Format

Individual characteristics of different formats of representations of practice might play a role for the test-takers' engagement with and analysis of the implemented classroom situations (see Fig. 1). By engagement we mean perceptions of authenticity, immersion, motivation and resonance. In particular, we use the term immersion to describe the effect that representations of practice can provide the test-takers with enough information to be "inside" the presented classroom situation (Seidel et al. 2011). The term resonance describes the effect that representations of practice can facilitate the test-takers ability to relate to their own teaching experiences (ibid.).

To describe how various types of representations of practice can differ, Herbst et al. (2011, cf. Herbst and Kosko 2014) propose the categories of temporality and individuality. Accordingly, videos often reproduce the passing of time and preserve the individual features of people and places in the presented classroom events. Texts, however, neutralise individuality and temporality to a high degree by using

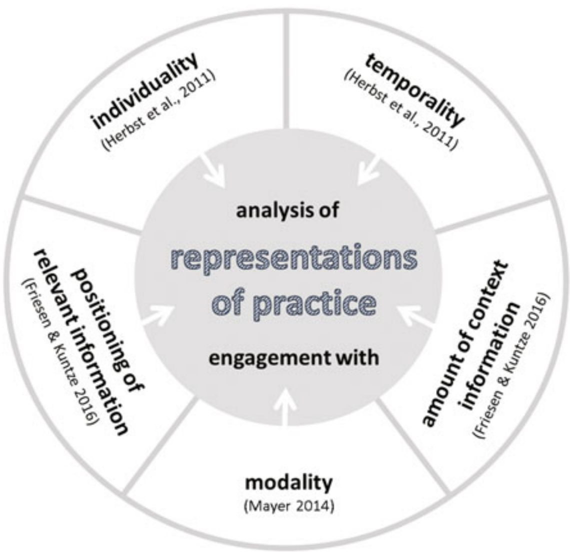


Fig. 1 Key categories for comparing different representation formats (e.g., text, comic, video) and their possible role for the engagement with and analysis of representations of practice

expressions such as "the students" and expand or collapse the duration of the presented classroom situations. The position of comics might be somewhere in between: Regarding temporality, a comic strip with speech bubbles can be compared to a text, whereas regarding individuality, it is closer to video (Herbst et al. 2011).

Other categories describing the characteristics of different representation formats are the positioning of relevant information and the amount of context information in these formats (Friesen and Kuntze 2016) as well as the modality in which such information is provided (cf. Mayer 2014). Whereas text-based representations of practice can present classroom situations in a clear structure, rather lengthy descriptions might be necessary to picture what students and teachers are doing. In videos, the information relevant for the analysis of a classroom situation might be somewhere hidden in the vast amount of context information as visual and acoustic information as well as moving pictures have to be processed. However, the larger amount of context information provided in video-based representations of practice could also support the understanding of a classroom situation and help to perceive it as more authentic and more motivating (Friesen and Kuntze 2016; Seidel et al. 2011). In comics, individual characteristics that might be important to fully comprehend a situation can be added without leading to lengthy descriptions that would be necessary in text-based representations of practice. At the same time, unnecessary context information that might be hindering for the engagement with and analysis of a classroom situation can be left out (Friesen and Kuntze 2016). Low individuality as provided by nondescript characters in comic-based representations of practice might also help to project an observer's individual teaching experience on a classroom situation and could thus facilitate the engagement with a classroom situation in terms of immersion, the perceived authenticity, motivation and resonance (cf. Herbst and Kosko 2014; Seidel et al. 2011).

To our knowledge, there are hardly any empirical studies which systematically investigate the possible role of different formats such as text, comic and video when representations of practice are used in competence assessment. As the individual characteristics of different representation formats might, however, be related to the test-takers engagement and analysis as described above, corresponding research questions are addressed in this study.

Research Interest and Research Questions

The research interest of this study is to explore whether format (text, comic, video) plays a role in assessing pre-service teachers' competence of analyzing classroom situations regarding the use of multiple representations. In particular, the research questions are the following:

- Is there a relationship between the pre-service teachers' analyzing regarding the use of multiple representations and the format of the presented classroom situations (text, comic, video)?
  In particular: Does the format of the represented mathematics classroom situations play a role for the pre-service teachers' ability to identify unconnected changes of representations and to interpret them with respect to their role as potential learning obstacles?
- Is there a difference in the pre-service teachers' engagement with representations of practice regarding the format of the representation (text, comic, video)?
  In particular: Do the pre-service teachers perceive texts, comics and video-based representations of practice differently regarding authenticity or with respect to the pre-service teachers' immersion, motivation and resonance?

Development of the Implemented Representations of Practice

In order to assess the pre-service teachers' competence of analyzing regarding the use of multiple representations, we developed a test instrument involving classroom scenarios situated in grade 6. All representations of practice have a similar structural design and show classroom situations with group work in the context of fraction learning. Each classroom situation starts with the teacher being asked for help by a group of students who have already started to solve a given problem using a certain representation (e.g., algebraic or pictorial). The situations were designed on purpose in such a way that the teachers' support of the students is not in line with the theory regarding the use of multiple representations as outlined above. In attempt to support the students' understanding, the teacher shifts away from the representation the students have already been using and changes to an additional representation. However, this change of representations remains unexplained as the teacher fails to connect it to the representation the students have already been using. Due to the lack of connections between the different representations the students and the teacher make use of, the teacher's reaction could potentially lead to further problems in the students' understanding rather than supporting it (Friesen and Kuntze 2016).

With the aim to explore validity of the designed representations of practice described above, the classroom situations were presented to $N = 5$ expert teachers who are not only experienced practitioners but also hold positions as teacher educators for pre-service teachers who are in their induction phase at secondary schools. Therefore, these expert teachers can be expected to be well experienced in observing and analyzing classroom situations. They were separately asked to evaluate the teacher's reaction to the students' question regarding the use of multiple representations in each classroom situation. In addition, the expert teachers judged the authenticity of the designed classroom situations, for example regarding the questions the students asked and the representations that were used by students and teachers. According to these expert ratings, six classroom situations were chosen for the test instrument in which the support given by the teachers was identified as potentially impeding for the students' understanding due to the unexplained and unconnected change of representations as outlined above. These classroom situations were also rated as highly authentic and representative for mathematics classrooms in grade 6 by the experts.

In order to investigate the pre-service teachers' responses to different formats, we implemented each of the six classroom situations as text, comic and video (see Fig. 2 for an example). The texts were used as blueprints to design the comics and the comics provided the storyboards for the video recordings. In order to avoid dependencies between the video clips, each video was recorded in another classroom showing six different teachers and learning groups. After editing the video recordings, we adapted the comics and the texts, so that the conversations in the classroom situations would have the same wording in each format and the representations used by students and teachers (e.g., fraction circles) would look the same (Friesen and Kuntze 2016).

In order to provide more insight into the content and plot of the representations of practice implemented in the test instrument, one of the classroom situations will be described in more detail in the following (see Fig. 3).

In this classroom situation, the students struggle with converting an improper fraction into a mixed number. They have already started to solve the problem by changing registers, namely from the given register of representation (fraction number $\frac{13}{5}$) to a division (13:5). As they do not know how to continue, the teacher explains that they can write the remainder of the division as a fraction. As this idea involves a conversion from the division register (13:5 = 2R3) back to the fraction register $\left(2\frac{3}{5}\right)$, which the students obviously are not able to carry out, the teacher introduces two further registers of representation: The problem is now represented in a real-world situation where thirteen pizza slices are put together in a way that they form two whole pizzas and three slices. While telling the pizza story, the

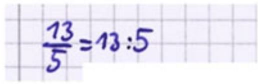
Fig. 2 Representations of practice in text, comic and video format; comic drawn by Juliana Egete

PRACTICE LESSON GRADE 6: Rewriting improper fractions as mixed numbers

S1: Can you please help us here? We have a question…

(Teacher comes to the students' desk.)

S2: We want to rewrite this fraction as a mixed number. And because the fraction bar means the same as dividing, we have started to write it like this *(shows entry in their notes).* But we don't know how to go on now…

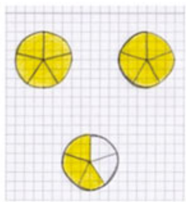


T: Ok, we call this a division with a remainder and, as you know, you can write the remainder as a fraction.

S2: Well, honestly, I still don't get it…

T: Let's explain it with a pizza then, ok? I`ll make a quick drawing for you.

(Teacher draws and explains.)



T: Here you have 13 pizza slices. And now we put them together: five slices make one whole pizza. Then you get two pizzas and three slices are left: Two wholes and three fifths!

Fig. 3 Representation of practice in text format

teacher uses another register of representation by drawing fraction circles that stand for the pizzas. The situation finishes with the teacher verbally providing a mixed number as the solution to the initial problem: "Two wholes and three fifths". On the surface, it seems that the teacher's support has finally led to the correct solution of the given problem. Moreover, the teacher's idea to move away from symbolic representations to the potentially motivating pizza story and colored fraction circles seems to be a student-oriented approach.

However, analyzing this classroom situation against the theoretical background of dealing with multiple representations in the mathematics classroom leads to a different result. Throughout the situation, the students are hardly supported in relating different registers of representations to each other when changes of representations occur: The first conversion from the fraction to the division register is initiated by the students themselves, but they cannot complete it. Instead of supporting the students in doing so, the teacher changes registers again by introducing the pizza story and the circular pies, however, without making any connections to the registers used before. It remains, for example, unexplained why there are thirteen pizza slices and why always five slices make one whole. The problem is finally solved by the teacher in the “pizza” register, again without making any connections back to the registers the students were struggling with at the beginning of the situation. The teacher does not explain, for example, how the solution “two wholes and three fifth”, which is only verbally expressed, is related to the solution in the pizza register (two whole pizzas and three slices are left) or in the division register (2R3), where fifths do not appear at all. For these reasons, the teachers’ reaction can hardly be regarded as a support for students’ understanding, but might rather be seen as a potential obstacle for the successful integration of multiple registers of representation in the process of students’ learning of fractions.

Although the corresponding text, comic and video represent the same classroom situation (see Fig. 2), they differ from each other regarding aspects of individuality, temporality (Herbst et al. 2011), modality (Mayer 2014), the positioning of relevant information regarding multiple representations and the amount of context information in general (Friesen and Kuntze 2016). Text-based representations require to get engaged with the classroom situations while only providing basic information and little individuality. As such, text-based representations might help pre-service teachers to focus their analysis on the use of multiple representations. The reduced amount of context information might, on the other hand, make it difficult to immerse into the situation and could make a situation look less authentic.

In contrast, a video-based representation of practice provides high individuality showing concrete students, teachers and classrooms which might contribute to the perceived authenticity of a situation and might be particularly motivating for the pre-service teachers. Analyzing a video-based representation of practice could, however, be more difficult as the shown teacher explains and draws at the same time so that visual and acoustic information has to be perceived simultaneously with a temporality close to a real classroom situation.

The analysis of comic-based representations of practice requires from the pre-service teachers to connect graphical elements (comic storyboard, depicted representations such as the fraction circle) to the text in the speech bubbles in order to make sense of the classroom situation. The reduced amount of context information, in contrast to the video-based representation of practice, might help to focus the analysis on the use of multiple representations.

Sample, Design and Administration of the Test Instrument

The sample of this study consists of $N = 162$ mathematics pre-service teachers (66.9% female; $M_{age} = 21.55$, $SD_{age} = 2.38$) in the first three semesters of their professional teacher education ($M_{semester} = 1.80$; $SD_{semester} = 1.40$). All student teachers were enrolled in courses for teaching mathematics at secondary school level and came from different Universities of Education in the State of Baden-Wuerttemberg, Germany. They completed the test instrument described above in a course at their home university.

In order to assess the pre-service teachers' competence of analyzing the use of multiple representations, they were asked to evaluate the teachers' support in each of the six classroom situations by responding to the following open-ended item: *How appropriate is the teacher's response in order to help the students? Please evaluate the use of representations and give reasons for your answer*. With the aim to investigate the role of the different formats for the pre-service teachers' engagement with the classroom situations and for their analyzing the use of multiple representations, a multiple matrix design comprising of six test booklets was applied (see Table 1). Each booklet included the six classroom situations while always two situations were implemented in the same format. The links amongst the booklets can be seen in Table 1: Always two booklets were linked to each other by sharing the same cluster of three representations of practice. Thus, a balanced distribution of the six classroom situations in the three formats could be achieved (Friesen and Kuntze 2016). The test booklets were randomly assigned to the pre-service teachers. The videos lasted about 1.5 min each and could be paused or watched several times.

With the aim to investigate how authentic the pre-service teachers found a given classroom situation and how they perceived their motivation, immersion and resonance when dealing with it, they were asked to evaluate their engagement with

Table 1 Multiple matrix booklet design (T ≙ text, C ≙ comic, V ≙ video)

Classroom situation	Booklet 1	Booklet 2	Booklet 3	Booklet 4	Booklet 5	Booklet 6
1	T	T	C	C	V	V
2	C	C	V	V	T	T
3	V	V	T	T	C	C
4	T	V	V	C	C	T
5	C	T	T	V	V	C
6	V	C	C	T	T	V

Table 2 Rating scale statements related to the pre-service teachers' engagement (cf. Seidel et al. 2011)

Engagement (in terms of)	Sample item
Authenticity	*The classroom situation appeared as authentic to me*
Immersion	*I felt part of the situation, as if I had been there in the classroom*
Motivation	*I found it motivating to deal with the classroom situation*
Resonance	*Dealing with the situation made me think of my own classroom experience*

each of the six situations (cf. Seidel et al. 2011). Therefore, the pre-service teachers evaluated four statements (see Table 2) according to a six-point Likert scale (1 = *I strongly disagree*; 6 = *I strongly agree*) after analyzing a classroom situation regarding the use of multiple representations.

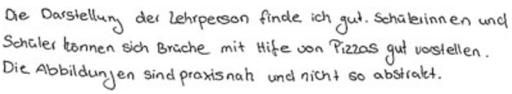
Data Analysis and Selected Results

Addressing the first research question regarding the pre-service teachers' analyzing of the six classroom situations, their answers were coded by two independent raters reaching a good inter-rater reliability with $\kappa = 0.85$ (Cohen's kappa). The top-down coding scheme was derived from how we defined the competence of analyzing the use of multiple representations, namely as the ability to identify unconnected and unexplained conversions in classroom situations and interpret them as potential learning obstacles. Accordingly, code 0 was assigned to answers that referred only to representations used by the teacher without making any connections to the students' question or the representation used by the students, thus indicating that the unconnected change of representations has not been identified (see Fig. 4 for a corresponding coding sample).

Code 1 was assigned to answers indicating that a pre-service teacher has identified the change of representations, however, without mentioning that it remains unconnected and might consequently be problematic for students' understanding (see Fig. 5 for a corresponding coding sample).

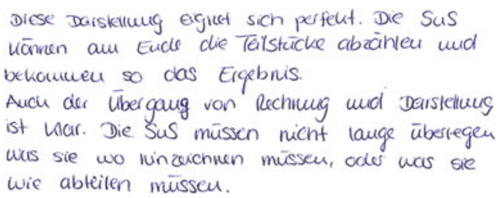
Code 2 was assigned to pre-service teachers' answers indicating that the unconnected change of representations has been identified and interpreted with respect to its role as potential learning obstacle (see Fig. 6 for a corresponding coding sample). Code 2 was thus taken as indicator for the competence of analyzing the use of multiple representations in a classroom situation. All coding samples (see Figs. 4, 5 and 6) refer to the classroom situation shown in Fig. 3.

The distribution of the three codes (see Fig. 7) shows that only 25.1% of the pre-service teachers' answers indicated that the unconnected change of representations has been identified and interpreted with respect to its role as potential learning obstacle.


Die Darstellung der Lehrperson finde ich gut. Schülerinnen und Schüler können sich Brüche mit Hife von Pizzas gut vorstellen. Die Abbildungen sind praxisnah und nicht so abstrakt.

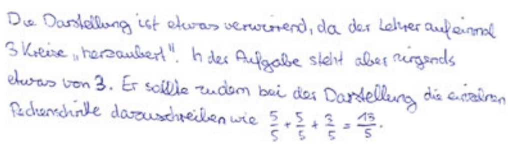
I think the teacher's representation is good. Pizzas can help students to get a clear idea of fractions. The drawings are close to everyday life and not too abstract.

Fig. 4 Coding sample for code 0 (pre-service teacher A)


Diese Darstellung eignet sich perfekt. Die SuS können am Ende die Teilstücke abzählen und bekommen so das Ergebnis.
Auch der Übergang von Rechnung und Darstellung ist klar. Die SuS müssen nicht lange überlegen was sie wo hinzeichnen müssen, oder was sie wie abteilen müssen.

This representation is ideal. The students can count the slices at the end and will get the solution. The shift from the calculation to the (graphical) representation is also clear. The students don't have to think long about what goes where or what should be divided how.

Fig. 5 Coding sample for code 1 (pre-service teacher B)


Die Darstellung ist etwas verwirrend, da der Lehrer aufeinmal 3 Kreise „herzaubert". In der Aufgabe steht aber nirgends etwas von 3. Er sollte zudem bei der Darstellung die einzelnen Rechenschritte dazuschreiben wie $\frac{5}{5}+\frac{5}{5}+\frac{3}{5}=\frac{13}{5}$.

The (graphical) representation is somewhat confusing because the teacher comes up with 3 circles "as if by magic". There is, however, no "3" in the task. He (the teacher) should also write down each step next to the (graphical) representation:

$$\frac{5}{5}+\frac{5}{5}+\frac{3}{5}=\frac{13}{5}$$

Fig. 6 Coding sample for code 2 (pre-service teacher C)

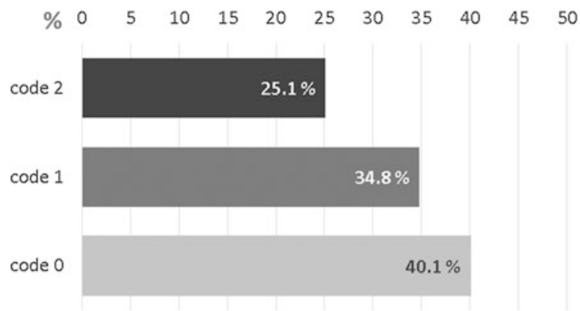


Fig. 7 Pre-service teachers' answers: distribution of codes (cf. Friesen and Kuntze 2016)

In order to address the question of format in a first step, a chi-square test was computed to explore if the pre-service teachers' analyzing results as reported above (see Fig. 7) were related to the format of the classroom situations (text, comic, video). The results of the chi-square test revealed no significant association between the codes and the format in which the classroom situations were presented to the pre-service teachers ($\chi^2(4) = 7.09$, $p > 0.05$).

With the aim to address the question of format in further analyses, a Rasch model was applied to the data of this study for two important reasons. First, the format of a classroom situation (text, comic or video) can increase or decrease the demands that are required to analyze it (e.g., Hartig 2008). Hence, the empirical item difficulties calculated in the Rasch model are particularly useful to investigate possible relations between the pre-service teachers' analyzing and the format of the classroom situations. The second reason is that the Rasch model provides a mathematical framework against which data can be compared with respect to unidimensionality (Bond and Fox 2015). For this purpose, residual-based fit statistics can be used to determine how well each item fits within the underlying test construct and whether the requirement for unidimensionality holds up empirically (ibid.). This can not only be regarded as a control of the quality of the measures (Bond and Fox 2015) but can also be seen as an important indicator regarding the question of format raised in this study, since different demands involved in analyzing representations of practice in different formats (text, comic, video) might not only lead to significant differences in item difficulties but could also cause items to measure different latent traits or dimensions (e.g., Rauch and Hartig 2010).

In order to conduct the Rasch analysis and estimate the empirical item difficulties of the representations of practice in the different formats, the six classroom situations in the three formats were taken as 18 items. In order to reflect the coding of the pre-service teachers' answers (see Fig. 7), a partial credit model was applied to the data. The Rasch analysis revealed good fit values for all 18 items ($0.91 \leq wMNSQ \leq 1.16$; $-0.6 \leq t \leq 1.0$) indicating that they sufficiently fit the Rasch model (Bond and Fox 2015). The EAP/PV-reliability was obtained by dividing the variance of the individual expected a posteriori ability estimates by the estimated total variance of the latent ability (Wu et al. 2007). It appeared to be rather low (0.45) which might be due to the comparatively small number of items (Bond and Fox 2015). However, it can also be due to the fact that analyzing the classroom situations implemented in the test instrument was quite difficult for the pre-service teachers at the beginning of their university studies, as has already been reflected in the distribution of codes as described above (see Fig. 7).

As the difficulty estimates of the items can be interpreted as interval data (Bond and Fox 2015), an analysis of variance (ANOVA) could be conducted in order to investigate the association of the estimated item difficulties and the different formats of the six classroom situations (text, comic and video). The results of the ANOVA showed no significant effect of format on the item difficulties ($F = 0.047$, $df = 4$; $p = 0.996$), indicating that the pre-service teachers' analyzing of the use of multiple representations was not systematically related to the format of the representation. These findings are hence in line with the results of the chi-square test reported above.

Addressing research question two, the pre-service teachers' evaluations regarding their engagement with the representations of practice were explored. Mean values between $M = 4.0$ ($SD = 1.3$, resonance to videos) and $M = 4.9$ ($SD = 0.9$, authenticity of comics) indicate on average positive ratings with respect to the authenticity of the representations of practice and the pre-service teachers' perceived immersion, motivation and resonance while dealing with the classroom situations (see Fig. 8).

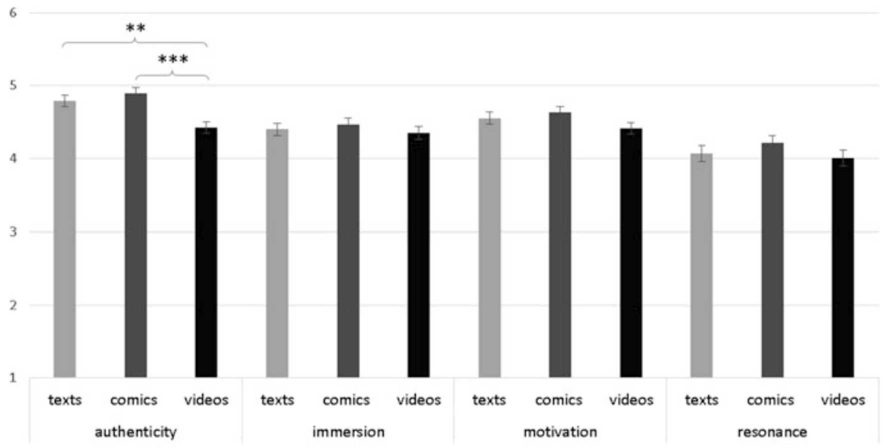


Fig. 8 Pre-service teachers' evaluations regarding their perceived engagement with the classroom situations (means and standard errors, 1 ≙ I strongly disagree/6 ≙ I strongly agree)

In order to investigate the possible role of format for the pre-service teachers' perceived engagement, an analysis of variance (ANOVA) was conducted. It revealed a small but significant effect ($F = 9.897$, $df_1 = 2$, $df_2 = 12$, $p < 0.001$; $r = 0.20$) of the format on the perceived authenticity, indicating that video-based representations of practice were on average rated as less authentic ($M_{video} = 4.4$, $SD_{video} = 1.1$) than texts ($M_{texts} = 4.8$, $SD_{texts} = 0.9$) and comics ($M_{comics} = 4.9$, $SD_{comics} = 0.9$). No significant differences were found between the ratings of texts, comics and videos with respect to the pre-service teachers' perceived immersion, motivation and resonance (see Fig. 8).

Discussion

The aim of this study was to contribute to the methodological question of format when representations of practice are used in research into aspects of teacher competence. Focusing on teachers' competence of analyzing the use of multiple representations in mathematics classroom situations, the question was raised if pre-service teachers' engagement with classroom situations and their analyzing of it are related to the format in which those classrooms are represented in a test instrument. The multiple matrix design of the study made is possible to compare the results of pre-service teachers' analyzing for the same six classroom situations in the three formats text, comic and video, each playing an important role in recent studies assessing aspects of teacher competence. Although the limitations of the study have to be taken into consideration when interpreting the results (e.g., the sample is not representative and is restricted to pre-service teachers, the classroom situations focus on learning of fractions in grade 6, the investigated formats are restricted to text, comic and video), the research questions could be answered and some implications for further research settings in the field of competence assessment with representations of practice can be derived.

The results show that pre-service teachers engage comparably well with representations of practice in the formats text, comic and video with regard to the perceived motivation, immersion and resonance. In the case of authenticity, the video-based representations of practice were rated significantly lower than the texts and comics (see Fig. 8). These findings contrast, for example, with findings by Herbst et al. (2013) who found that a video-based representation of practice was perceived significantly more genuine by pre-service teachers than the same representation of practice in animation format. The lower ratings of the authenticity revealed in the case of the video-based representations of practice might be due to specific characteristics of the video clips: The high individuality in the videos might, for example, decrease the perceived authenticity when the classroom surroundings differ widely from those familiar to a participant. Furthermore, individual characteristics of the students and teachers in the video clips (e.g., complexion, way of speaking) might diminish the perceived authenticity. Another reason for the lower ratings regarding the authenticity of the video-based representations of practice might be that the clips implemented in the test instrument were staged videos whereas the participants might have expected to see recordings of real classrooms. Further research in this context should also consider the reverse way of designing the different formats text, comic and video by generating texts and cartoons on the basis of recordings from real classrooms.

However, the overall positive mean values of the perceived immersion, motivation, resonance indicate that the participants were sufficiently engaged with the representations of practice implemented in the test instrument, regardless of format. The findings of this study are thus in line with the results found by Seidel et al. (2011) in the case of video-based representations of practice. They add to these findings by showing that with regard to pre-service teachers' engagement with representations of practice, texts and comics can be comparably effective to tap into the competence of analyzing mathematics classroom situations.

The results regarding the pre-service teachers' analyzing of classroom situations presented in the three different formats show that there are no significant differences between the item difficulties related to texts, comics and videos. Furthermore, no significant association between the distribution of the codes for the pre-service teachers' analyzing results and the different formats of the classroom situations (text, comic, video) could be found. In line with these findings, the Rasch model applied to the data showed good fit values indicating that the requirement for unidimensionality holds up empirically and that all items contribute in a meaningful way to the competence of analyzing, regardless of format. It can thus be concluded that the pre-service teachers' analyzing of the use of multiple representations was not systematically related to the specific characteristics of the different formats (text, comic, video) as have been described above. These findings are in line with results reported by Herbst et al. (2013), showing that pre-service teachers' analyzing of a classroom situation was not associated with the implemented formats video and animation. The findings of the study presented here can add to these results, as the multiple matrix design made it possible to compare six classroom situations in three formats (text, comic, video). It can be concluded that videos, texts and comic-based representations of practice were comparably suitable to elicit pre-service teachers' analyzing regarding the use of multiple representations in mathematics classroom situations.

Bearing in mind the high expense involved in the production of video-based representations of practice, the findings of this study encourage further research into the development of alternative formats in order to assess profession-related competences of teachers. They add to findings in the field of video-based measurement which were, for example, made by Santagata et al. (2007) and Kaiser et al. (2015) and encourage the use of representation formats other than video when aspects of teacher competence are assessed. When test instruments involving representations of practice are developed, it should particularly be taken into account whether specific characteristics inherent to a certain format are suitable to facilitate the analysis of a presented classroom situation or whether such characteristics could also impede analyzing. However, certain characteristics of formats, such as a high degree of temporality or high amount of context information, might also be implemented by purpose when they form a part of the professional competence under investigation. In future research, pre-service teachers at an advanced level and in-service teachers should be taken into account as they might perceive texts, comics and video-based representations of practice in a different way, due to their different professional knowledge and teaching experience.

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Examining the Mathematical Knowledge for Teaching of Proving in Scenarios Written by Pre-service Teachers

Orly Buchbinder and Alice Cook

Abstract In this chapter, we examine what aspects of Mathematical Knowledge for Teaching of Proving (MKT-P) can be observed in written scenarios of classroom interactions, produced by pre-service teachers (PSTs) of mathematics. A group of 27 elementary and middle school PSTs completed an online interactive module, intended to trigger reflection on, and crystallization of their knowledge of the roles of examples in proving. To ground these processes in the context of teaching, the module engaged PSTs in analysis of several representations of practice such as a questionnaire about quadrilaterals with sample student work imbedded in it, and a classroom scenario in a storyboard format realized with cartoon characters. In addition, PSTs wrote a one-page continuation of that scenario describing how they would handle the situation if they were teaching the class. These scenarios proved to be a rich source of data on several aspects of MKT-P as well as general pedagogical knowledge.

Keywords Pre-service teachers · Written scenarios · Conceptions of proving
The role of examples in proving · Geometry

Introduction

Preparing mathematics teachers who can facilitate learning of all students through sense making, reasoning, justification, and argumentation is a complex endeavor. It requires providing rich opportunities for pre-service teachers (PSTs) of mathematics to develop a broad knowledge base, i.e., mathematical knowledge for teaching (Shulman 1986; Hill et al. 2008). Designing learning opportunities for PSTs that bridge the university teacher preparation setting and the actual practice of teaching

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can be quite challenging (Ponte and Chapman 2008). To achieve this important goal, researchers have been using various representations of practice, such as video, written vignettes, and sample student work, in a variety of ways: to promote teacher noticing, to engage PSTs' in analysing student thinking, to reflect on certain teaching moves, to learn about standards, and many others (e.g., Herbst et al. 2016). By engaging PSTs with representations of practice, teacher educators create a simplified context resembling the actual teaching practice, but without the same level of complexity (Grossman et al. 2009).

Yet another way for bringing practice into the teacher preparation setting is to engage PSTs in creating their own representations of practice, such as lesson plans, depictions of classroom scenarios or written scripts (Herbst et al. 2016; Zazkis et al. 2013). This process requires PSTs to imagine themselves in a teaching situation, attend to its multiple aspects and details, and practice implementing both mathematical and pedagogical knowledge related to it. As a result, these PSTs' created artifacts provide a unique window into PSTs' knowledge, and beliefs.

In this chapter, we used PSTs' written scripts to examine their mathematical knowledge for teaching of reasoning and proving (MKT-P), which we describe in detail below. We chose to focus on this content because reasoning and proving are fundamental to doing and learning mathematics meaningfully, at all levels. In line with Ellis et al. (2012) we view *proving* as a broad collection of activities such as recognizing patterns, conjecturing, generalizing, justifying, making and evaluating arguments, and reasoning deductively. To engage students in these processes and help bringing their conceptions closer to conventional mathematical knowledge, PSTs themselves need to develop a strong knowledge base and teaching practices specific to proving.

Hence, we designed a multi-step instructional module, with the dual purpose of diagnosing PSTs' MKT-P and enhancing this knowledge. The mathematical content of the module was geometry, specifically, quadrilaterals, because this topic appears in curricula throughout multiple grades, from elementary through high school and beyond, at different levels of complexity. As such, this topic would be equally relevant to PSTs in an elementary, middle or high school levels. In addition, geometry is considered as a natural place for students to encounter argumentation and proving. The module particularly focused on the role of examples in proving or disproving a statement because of the critical role examples play in these processes (Mason et al. 2010). The literature suggests that many PSTs lack strong understanding of the roles of examples in proving (Reid and Knipping 2010), making them an important topic to address with PSTs. Another key aspect of the module was reflecting on a scenario of a whole-class discussion in which students were confused about the role of examples in proving. PSTs wrote a detailed script of a teaching situation imagining how they would help students resolve this mathematical conflict if it was their own classroom.

In this chapter we report on the analysis of 27 scripts written by elementary and middle school PSTs enrolled in an undergraduate course on reasoning and proving. For PSTs writing these scripts served as an opportunity to envision themselves as teachers, and contemplate both the mathematical content and the pedagogical enactment of classroom interaction. As researchers, we aimed to understand what kinds of knowledge resources PSTs drew upon when writing the scripts. Hence, within our mixed sample of elementary and middle-school pre-service teachers of mathematics, and with a particular geometrical statement at hand, our overarching research question was: *What aspects of MKT-P become visible in written scenarios of classroom interactions produced by pre-service mathematics teachers?*

In the following sections, we present the theoretical grounds of our work, followed by description of the setting and the methods of the study. We devote the results section to presenting categories of MKT-P that came up in our data, using excerpts of PSTs' scripts. We conclude by discussing the advantages of using PSTs' produced representations of practice as an educational and research tool.

Theoretical Background

Mathematical Knowledge for Teaching Proving

Building on the construct of Mathematical Knowledge for Teaching (MKT) (e.g., Shulman 1986; Hill et al. 2008) researchers have proposed that engaging students in proving activities, such as exploration, conjecturing and justification, requires a special type of knowledge—the Mathematical Knowledge for Teaching Proving (MKT-P) (e.g., Corleis et al. 2008; Steele and Rogers 2012). Stylianides (2011) introduced a "comprehensive knowledge package for teaching proof", consisting of three types of knowledge: knowledge of mathematical content, knowledge of students' proof-related conceptions, and knowledge of pedagogies for supporting students' learning. Lesseig (2016) suggested a framework with four types of knowledge: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching—all specifically related to proving. In the similar vein, we view MKT-P as comprised of four interrelated types of knowledge: two specific to Subject Matter Knowledge (SMK) and two specific to Pedagogical Content Knowledge (PCK) (Fig. 1).

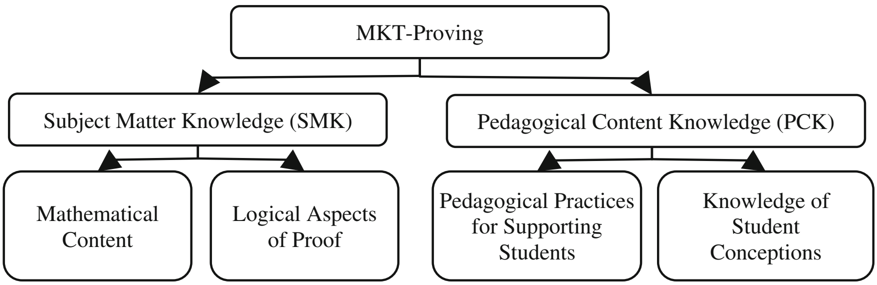


Fig. 1 Mathematical knowledge for teaching of proving

The first category of the SMK for proving is Mathematical Content Knowledge, which requires understanding of concepts, procedures and relationships among them, along with their underlying mathematical principles. The second category is the Knowledge of the Logical Aspects of Proof, such as knowledge of different types of proofs, proof components, modes of argumentation, understanding of the roles of examples in proving, and meta-knowledge about proving, such as knowledge of functions of proof (Hanna and deVillers 2012). The two categories of PCK for proving include the knowledge of Pedagogical Practices for Supporting Students in developing conceptions of proof and proving, which are in line with conventional mathematical knowledge; and the Knowledge of Students Conceptions (and misconceptions) related to proof. These two types of PCK supplement general pedagogical knowledge, which include knowledge of planning and carrying out instruction, questioning techniques, and facilitating discussions (Shulman 1986).

Teaching and Learning About Reasoning and Proving

The vast body of research on reasoning has identified various types of difficulties related to proving. Among them are over-reliance on examples for proving a general statement, rejection of counterexamples as exceptions, reliance on authority as a source of validation and justification (Harel and Sowder 2007; Healy and Hoyles 2000), misunderstanding of the generality of proof, and/or assuming that a proven theorem can be refuted by a counterexample (Chazan 1993), difficulty in reasoning with conditional statements (e.g., Durand-Guerrier 2003), and many others. These findings seem to be persistent over time and appear in students of all ages (Reid and Knipping 2010), pre-service teachers, and even in-service teachers (see Ko 2010 for an overview).

Many of these types of difficulties are related to the interplay between empirical and deductive reasoning, or in other words, between examples and proving. When it comes to making inferences about mathematical statements, examples play multiple roles. Supporting examples are used in exploring patterns and creating conjectures. Although they do not prove a statement, they increase one's confidence in it and even may suggest a proving strategy, as in case of generic examples (Leron and Zaslavsky 2013). Counterexamples can help to refine conjectures by disproving false ones (Mason et al. 2010). Counterexamples can also suggest a way to prove the statement, by virtue of describing the types of objects, i.e., counterexamples, whose existence must be disproved (Yoop 2017). Buchbinder and Zaslavsky (2009) proposed a mathematical framework for describing the relationships between examples and proving. Next, we illustrate the part of the framework pertaining to universal statements, with examples and the statement from our module.

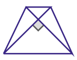
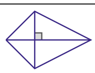
The Mathematical Framework for the Role of Examples in Proving—An Illustration

A mathematical statement can be characterized by a set of objects to which it refers—the domain of the statement D(x) and a proposition P(x), which describes a property of the objects in the domain, or makes a claim about them. For example, a statement: *A quadrilateral with congruent and perpendicular diagonals is a kite,* refers to the set of all quadrilaterals with congruent and perpendicular diagonals—D(x), asserting that such quadrilaterals are kites—P(x). This statement is false because it omits a necessary condition for a kite—at least one diagonal bisects another—but instead includes the unnecessary condition of congruent diagonals.

With this domain and proposition, four types of examples can be defined, depending on whether an example belongs to a domain or not and whether it has the property or not. The four types of examples and their roles are summarized in Table 1, and explained below.

A *supportive* example, is the object in the domain for which the proposition is true: $x \in D, P(x)$. In our case, it is a quadrilateral with congruent and perpendicular diagonals, which is also a kite, e.g., a square. Supportive examples are insufficient for proving a universal statement, since such statements require a general proof showing that the proposition is true for all the objects in the domain. A *counterexample* is an object in the domain, for which the proposition is false: $x \in D, \neg P(x)$. For instance, an isosceles trapezoid with perpendicular diagonals is not a kite, and therefore disproves the statement at hand. A convex kite, whose diagonals are not congruent, is irrelevant to either proving or disproving the given statement—since it is not in the statements' domain, it cannot be used to infer whether a property holds for the objects in the domain. This type of example: $x \notin D, P(x)$, is *irrelevant* to the statement (we term it Irrelevant Type 1), but since it

Table 1 A framework for describing the status of (a single) example in determining the truth-value of a universal mathematical statement

Type of example	To prove a universal statement $\forall x \in D,\ P(x)$	To disprove a universal statement $\forall x \in D, P(x)$
Supporting $x \in D, P(x)$ 	**Insufficient**	Non-applicable
Counter example $x \in D, \neg P(x)$ 	Non-applicable	**Sufficient**
Irrelevant Type 1 $x \notin D, P(x)$ 	Non-applicable	Non-applicable
Irrelevant Type 2 $x \notin D, \neg P(x)$ 	Non-applicable	Non-applicable

satisfies the property (it is a kite), such an example can trick students into thinking that it can be of use for determining the truth-value of the statement. What might increase confusion even more is that the example of a kite constitutes a valid counterexample to the converse statement: "All kites have congruent and perpendicular diagonals." Finally, an object which is not in the domain and does not have the property: $x \notin D, \neg P(x)$, is another type of irrelevant example (Irrelevant Type 2). For instance, a general rectangle does not have perpendicular diagonals and is not a kite. We maintain that knowledge of the kinds of inferences that can or cannot be drawn based on various types of examples is critical for exploring, generalizing and modifying conjectures.

Using Representations of Practice in Teacher Education—The Focus on Scripts

Learning to teach occurs across multiple settings, both informal and structured, including personal experience as students, to university-based courses, school observations, internship, own practice, and in-service professional development. In order to provide PSTs with opportunities to experience teaching while in the university setting, teacher educators use representations of practice in the forms of video, scenarios, animations, comics, storyboards, and sample student work (e.g., Santagata and Yeh 2014; Steinet al. 2000). By examining and analyzing representations of practice, PSTs can experience particular aspects of teaching in a safe environment of reduced complexity (Grossman et al. 2009). Another way of bridging the university and classroom setting is through *approximations* of practice, which are opportunities for PSTs to actively engage in practices resembling teaching. This can be done through simulations, rehearsals, or writing detailed scripts of hypothetical lessons (e.g., Herbst et al. 2016).

Script writing, in particular, has been increasingly used in teacher education as a learning and as an assessment tool (Crespo et al. 2011; Zazkis et al. 2013; Zazkis and Zazkis 2014). The scripts are produced by the PSTs, and constitute written accounts of events in a classroom in which a PST envisions him or herself as a teacher. The scripts are written in the form of screenplays, using first person voices of both the teacher and students, giving rise to the name *lesson-plays* (Zazkis et al. 2013). When writing the actual dialog turns between teacher and students, or among the students, PSTs need to envision a situation in great detail and to attend to both mathematical and pedagogical aspects of classroom communication. Zazkis and colleagues describe this as following:

> At a mathematical level, the imagined verbal exchanges necessarily bring into focus both the actual use of mathematical language in communicating and the forms in which the ideas are explained or justified. At the pedagogical level, the imagined exchange articulates assumptions about how students are thinking and how their thinking might be changed; it also articulates possible teaching trajectories (p. 13).

As such, scripts provide a unique window into both PSTs' MKT and their conceptions about teaching mathematics. For example, one can learn about PSTs' attention to precision in the use of mathematical language, about their knowledge of mathematics as reflected in the imagined teacher explanations, or about PSTs' pedagogical moves and questioning strategies. In addition, PSTs' representation of student dialog may shed light on their knowledge of student conceptions and difficulties, and of the ways to address potential students' misconceptions.

Method

The Setting and the Participants

This study draws on data collected in an undergraduate content course, *Reasoning and Proving for Elementary and Middle-School Teachers*, at a state university in the Mid-Atlantic area, USA. The course aimed to strengthen PSTs' content knowledge on logic and reasoning within topics from middle school mathematics such as geometry, the number system, and proportional reasoning. We report on the data from 27 pre-service teachers: 8 elementary, 18 middle, and 1 undeclared. There were 23 female and 4 male participants; 11 PSTs were in their 4th year of university studies, 14 in their 3rd year, 1 was in their 2nd year, and 1 had missing data.

The Module

Our multi-step module was constructed around an interactive online experience, *What can you infer from this example?*, which centered on content knowledge needed to determine the truth value of the following statement: *A quadrilateral with congruent and perpendicular diagonals is a kite*. Note: for ease of communication, throughout the rest of the chapter we refer to this as the *main* statement. We focused specifically on understanding the relationships between examples and proving, such as understanding that supportive examples are insufficient for proving a universal statement, while a counterexample disproves a false statement. The robust knowledge of this topic is crucial for teachers to guide students through such cornerstone activities as exploration, conjecturing and justifying (Ellis et al. 2012).

The module was designed to address both pedagogical and content knowledge of PSTs by embedding content-related prompts in pedagogical settings using cartoon-based representations of classroom situations (see Buchbinder et al. 2016 for discussion of design principles and theoretical underpinnings of the instructional module). The module was comprised of nine parts, listed below; the data for this chapter came from parts 6 and 7 (in bold):

1. Review definitions of special quadrilaterals and their properties.
2. Determine the truth-value of six statements about quadrilaterals.
3. Identify the domain, the proposition and the truth-value of the main statement.
4. Given six students' examples, determine whether each example: proves the statement, only confirms the statement, contradicts the statement, or is irrelevant to the statement.
5. Re-evaluate and revise answer to Part 3.
6. **Examine a cartoon-based scenario of a classroom discussion of the main statement. Determine which of the two students' examples (or both or neither) is a counterexample to the main statement.**
7. **Write a 1–2 page continuation of this scenario from the teacher perspective.**
8. Whole class discussion of the online module and sharing the teaching scenarios.
9. Optional: Re-write and re-submit the script.

Parts 1–6 were created and administered online, through Lesson*Sketch*, a web-based platform for teacher education (Herbst et al. 2016). The teaching scenario that PSTs analyzed in part 6 aimed to support PSTs' understanding of the role of counterexamples in proving. The scenario prompted PSTs to clarify the difference between the two student examples: an isosceles trapezoid and a general kite, which represent a counterexample and an irrelevant example of type 1, respectively (Table 1). The scenario shows a class discussing the main statement. Two students disagree with each other on whose quadrilateral is a counterexample to the statement. An isosceles trapezoid is proposed by student Red, and a kite is proposed by student Blue (named by the color of their shirt). Other students join the discussion, offering their opinion or expressing confusion. One potential source of mathematical confusion is that a general kite satisfies the proposition of the main statement. The scenario leaves the issue unresolved, with the teacher placing the two quadrilaterals on the board side by side (Fig. 2).

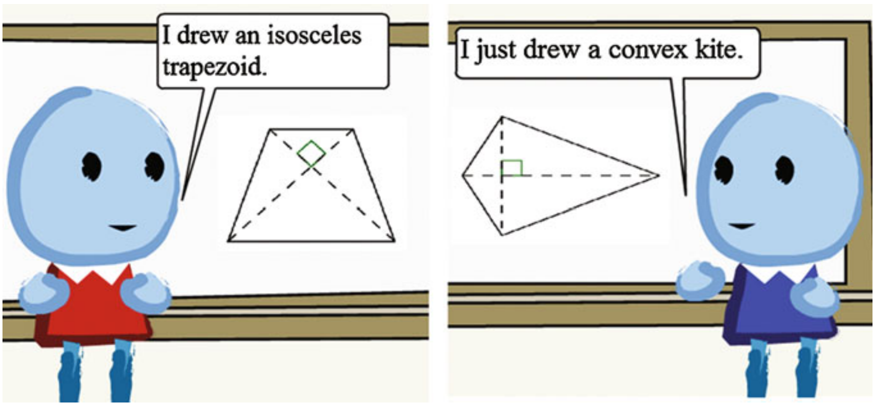


Fig. 2 The two students in the scenario present their work on the board. Red (on the left) drew an isosceles trapezoid, Blue (on the right) drew a kite (Graphics are © 2017, The Regents of the University of Michigan, used with permission)

The instructions for writing the continuation of this scenario were: *If you were the teacher in this class, how would you lead the discussion to support students' understanding? Write a scenario that describes in detail how you would see the classroom events unfold from here, if this was your classroom. The description should be in the form of a scenario (like a screenplay) that will lay out the discussion in the exact words of teachers and students. Include text and figures you would place on the board.*

Data Collection and Analysis

The module and the script writing task were assigned in the first part of the course, after PSTs learned to identify a domain and proposition of statements, and after they learned about conditional statements and negation. PSTs had two weeks to complete the module and the script writing individually. This assignment comprised 10% of the overall course grade, and was scored by completion.

To analyze the PSTs' scripts, we used open coding, partially informed by our theoretical framework (Miles et al. 2013). The two authors of this chapter read all the scripts individually, and took analytic notes on the types of instances they would categorize as each of the four types of MKT-P (Fig. 1). We also took note of whether these instances conveyed strengths or weaknesses in mathematical or pedagogical knowledge. We then compared the notes, discussed and resolved any discrepancies, and created a coding scheme by clustering PSTs' mathematical and pedagogical ideas into sub-categories of MKT-P. In this process, we identified categories related to three out of four types of MKT-P: knowledge of mathematical content, knowledge of the logical aspects of proving, and knowledge of pedagogical practices for supporting student learning. We also identified two categories of general pedagogical moves, unrelated to proving.

The fourth type of MKT-P, knowledge of students' conceptions of proving, was virtually indistinguishable from the knowledge of pedagogical practices for supporting students' learning; hence we do not report on it. This can be attributed to the design of the prompt and the particular scenario depicting students disagreeing about which quadrilateral constitutes a counterexample to the main statement. Extending this scenario requires analysis of the mathematical ideas in the students' arguments and devising a pedagogical strategy to resolve the students' mathematical confusion. This required PSTs to draw on their knowledge of mathematics and of pedagogical practices, respectively. Thus, the nature of the scripting task was less conducive to revealing PSTs' knowledge of students' conceptions of proving, than, for example, a task in which PSTs analyze students' common proof-related mistakes.

Results

We organize the presentation of results by the types of MKT-P (Fig. 1), excluding the knowledge of students' conceptions of proving, but including the general pedagogical knowledge. For each type of knowledge, we describe their related thematic categories and specific codes which came up in data analysis, and illustrate some of them with excerpts of PSTs' scripts. These excerpts are brought here verbatim, using PSTs' chosen names for the students or teacher characters, and descriptions of classroom actions. Square brackets are used for clarifying comments and to indicate the number of omitted dialog turns, in cases when the scripts had to be shortened to highlight a particular aspect.

SMK: The Knowledge of Mathematical Content

Two main categories were identified as related to the knowledge of mathematical content. The first contains instances indicative of PSTs' knowledge of quadrilaterals. The second category is related to precision in the use of mathematical vocabulary.

Category 1: Knowledge of quadrilaterals, specifically:

(a) Knowledge of properties of quadrilaterals.
(b) Knowledge of hierarchy between types of quadrilaterals.

Overall, PSTs' scripts illustrated good command of definitions of quadrilaterals, their properties and hierarchy among them. Although this content was reviewed in class during the module, we detected a few gaps in this area. For instance, some PSTs did not consider a square as being a kite, and thus interpreted it as a counterexample to the main statement. This is inconsistent with the inclusive definition of a kite, which was adopted and reviewed in the course, however, it is possible that some PSTs operated with a definition of a kite that excludes a square, following some existing textbooks (Usiskin and Dougherty 2007). Alternatively, PSTs' might be relying not on a particular definition, but on their personal concept image of a kite (Tall and Vinner 1981), which could have been limited by a prototypical image of a kite as having two pairs of adjacent congruent sides, not all congruent to each other.

In addition to this expected issue, we detected instances of overgeneralization of certain properties, or limitation of the scope of quadrilaterals to special types only, as shown in the excerpt by PST-8:

Teacher: "Student 4 please list the properties of a kite. Student 5 please list the properties of a rectangle." Students come up to the board to list the various properties.
[7 turns later]

Student 1: "If a quadrilateral has congruent diagonals, then it is a rectangle. If a quadrilateral has perpendicular diagonals, then it is a kite or a rhombus."
Student 2: "No kite is a rectangle."
Teacher: "Therefore, the statement must be…?"
Class: "False!"

In this script the teacher asked students to compare properties of rectangles and kites, attempting to show that it is not possible for a quadrilateral with congruent and perpendicular diagonals to be kite. The logical sequence can be represented like this: congruent diagonals ⇒ rectangle; perpendicular diagonals ⇒ kite or rhombus; intersection of kites and rectangles is an empty set ⇒ congruent and perpendicular diagonals cannot imply kite. This argument fails because all of its three assumptions are mathematically incorrect: there are many quadrilaterals with congruent diagonals that are not rectangles, there are quadrilaterals with perpendicular diagonals that are neither kites nor rhombi, and a square is both a kite and a rectangle.

Category 2: Attention to precision in mathematical vocabulary

Appropriate vocabulary is one of the attributes of mathematical knowledge. While some PSTs used correct and precise mathematical language, others implemented incorrect or ambiguous mathematical vocabulary, especially when describing the relationships between different types of examples and the statement. Following are several examples of the latter.

(a) Limited vocabulary for describing *counterexamples or irrelevant* examples.

This included such phrases as "[the] example does not follow the statement", "does not fit the statement", "does not apply to the drawing", "inconsistent with the statement", "does not agree with the statement", "examples that break the rules of the statement". The ambiguity stems from the fact that these expressions were used by some PSTs to describe both counterexamples and irrelevant examples, reflecting PSTs' difficulty to distinguish between these two types of examples.

(b) Limited vocabulary to describe *supportive* examples,

We documented multiple instances of PSTs using expressions such as: "[the] example proves the statement", or "[the] example makes the statement is true". This might simply indicate PSTs' limited mathematical-logical vocabulary to signify an example that satisfies both the domain in the property of the statement. For instance, although PST-17 used some ambiguous language in her script, she concluded it with a comment showing that she understands the distinction between supportive examples and proving: "I would then do a follow up lesson with new examples reinforcing the difference between confirming a statement and proving a statement is true".

However, incorrect or ambiguous vocabulary could also indicate a wrong conception that supportive examples are sufficient to prove a statement. As we analyzed the scenarios, we saw a strong connection between using precise mathematical language and robust knowledge of the logical aspects of proof. All PSTs who used ambiguous mathematical language also showed gaps in the knowledge of the logical aspects of proof, as will be illustrated below.

SMK: Knowledge of the Logical Aspects of Proof

The logical aspects of proof addressed by the module were: the recognition of different types of examples and understanding of their roles in proving or disproving a statement. The instances, pertaining to this type of knowledge were grouped into two main categories.

Category 1: The role of supportive examples

Since most PSTs correctly indicated that supportive examples are insufficient for proving a universal statement, some scripts contained evidence that PSTs hold a wrong conception of supportive examples. In her script, PST-10 led the imagined class to correctly identify the domain and the property of the statement, and to explain why an isosceles trapezoid is a counterexample. Next, she asked students to find an example that *proves* the statement, and wrote this dialog:

Green student: I think I found an example that proves the statement. The definition of a kite is a quadrilateral with two non-overlapping pairs of congruent adjacent sides. So, I decided to draw a kite with congruent sides. Since, the sides are congruent I realized that the diagonals would also be congruent and perpendicular to each other. [Shows a picture of a square]

Mr. Gray: Raise your hand if you agree. *Everyone raises his or her hand*

Although we could not determine from the script whether or not the word "prove" was used just as a figure of speech, we triangulated this data with PST-10's response to Part 5 of the module, in which PTSs could revise their initial stance on the truth-value of the main statement. PST-10 wrote: "Based on the definition of a kite this statement is true. Kite is a quadrilateral with two non-overlapping pairs of congruent adjacent sides. I stand by my answer." This scenario, confirmed by additional data, reveals a problematic conception: considering a statement true on account of a supporting example, even after acknowledging the existence of a counterexample.

Category 2: Recognition of and inference from counterexamples

Some PSTs' scripts correctly described what kind of mathematical object constitutes a supportive example, a counterexample or an irrelevant example (often using the language of domain and claim/property), and explained what can, or cannot be inferred about the truth-value of the statement based on these examples.

However, the majority of the scripts showed evidence of confusion and revealed several problematic issues. We identified three sub-categories:

(a) Determining the truth-value of the statement when both supportive examples and counterexamples exist.

Difficulties in this area were often manifested in diverting the discussion in the script from dealing with counterexamples to finding supportive examples, or to modifying the statement to be true.

(b) Recognition of the hierarchical structure of the statement.

The following excerpt illustrates a difficulty in this area:

Purple: I don't think that it matters which part of the statement is false or which example proves that it is false. I think it only matters that we have examples that prove it to be false.
[2 turns later]

Teacher: It sounds like we agree that the statement is false. When something is false, can't it be false for one or more reasons? It is still false.

This script, by PST-7, shows that although she understands that a false statement can have multiple counterexamples, she does not understand what kind of object constitutes a counterexample, or, more precisely, she considers an object that does not satisfy *any* part of a statement as a counterexample.

(c) Distinction between counterexamples and irrelevant examples.

This distinction posed the most difficulties to PSTs in our sample, with two-thirds of the scripts demonstrating some sort of difficulty in this area. In the next excerpt, PST-24 correctly explains that the example of a kite has the required property, but is not in the domain, and therefore cannot be used to disprove the statement.

Student C: Well the kite doesn't fit our criteria because it doesn't have congruent diagonals, the trapezoid fits both criteria and since it isn't a kite that shows that our statement is false.

Me: Okay so that is a critical point, because the kite here doesn't have congruent diagonals, it is outside the groups of quadrilaterals that we are looking at, so we can't use it to disprove our statement. We are trying to prove that a kite is the only type of shape that could fit this category, so by proving that our trapezoid here fits in the categories, it shows us that there is an exception to the statement given, making our statement false.

Interestingly, being able to clearly describe the domain and the property, and even explain why an isosceles trapezoid is a counterexample, was not sufficient for some PSTs to infer that a kite is irrelevant, as the next excerpt from PST-25's script shows:

Teacher: Great! Does the example of the kite whose diagonals are not equal in length confirm or contradict the statement?
Student: That example contradicts the statement.
Teacher: How?
Student: Because it shows that some quadrilaterals can be kites based on the fact that their adjacent sides are equal even though their diagonals are not equal in length.
Teacher: But the diagonals of that example are perpendicular to each other.
Student: Yes, but in order to fully meet the criteria for the statement, the diagonals must be congruent and perpendicular to each other. Because that kite is still a kite even though it does not have both of those properties it proves that both of the examples contradict the statement.
Teacher: Well argued!

In this script, the student explains that an example that does not "meet the criteria for the statement" disproves it. The fact that the teacher praised this explanation is an evidence of PST-25's own difficulty to recognize an irrelevant example as such.

PCK: Knowledge of Pedagogical Practices for Supporting Students

In this section we include the different types of pedagogical moves, suggested by PSTs, to support students understanding of the logical aspects of proving. We organized the moves into five categories by their intended purpose, or by the use of a particular tool.

Category 1: Analyze the logical structure of the statement.

We identified multiple moves whose goal was to support students' understanding of the logical structure of the statement. These could be carried through the teacher's explanation, or through a whole class discussion with teacher questioning. The examples of such pedagogical moves are:

(a) Break the statement into domain and property (verbally or in writing, possibly with color, possibly done by students); explain/discuss the meaning of each part of the statement.
(b) Practice identifying domain and claim in easier statements suggested by the teacher, or by the students.
(c) Re-write the given quantified statement as a conditional statement. Discuss the meaning of "for all" statements.
(d) Explain/discuss the difference between a conditional statement and its converse ($P \rightarrow Q \neq Q \rightarrow P$), using a specific example or in general terms.

The excerpt from PST-15's script below shows how he intended to compare the structure of the given statement and its converse side by side, by having students highlight the domain and the property of each statement.

Teacher: Let's take a closer look at the statement. "A quadrilateral which diagonals have the same length and are perpendicular to each other is a kite." I'm going to write another statement on the board. *Teacher writes, "Kites are quadrilaterals which diagonals have the same length and are perpendicular to each other." Teacher labels the first statement "1" and the second statement "2".*
[9 turns later]

Teacher: We call the thing we are saying something about the domain, and what we are saying about that thing is our claim. Can I get a volunteer to come underline the domain of each statement in red? *Orange comes to the front of the class and correctly underlines the domain of each statement in red marker.*

Teacher: Thank you Orange. Now can someone come underline the claim of each statement in Blue? *Blue comes to the front of the class and correctly underlines the domain of each statement in blue marker.*

Teacher: Thank you Blue. So now we can clearly see that statement 1 and 2 are very different; in fact, they are opposites. So let's look back at example A, the convex kite. In statement 1 we are claiming something about all the quadrilaterals with equal and perpendicular diagonals. Is this convex kite a quadrilateral with equal and perpendicular sides?

The pedagogical moves described in this script are specifically oriented towards enhancing students' understanding of the logical structure of the statement.

Category 2: Consider different types of examples and their role in proving

This category includes pedagogical moves whose goal is to convey the role of different types of examples in proving or disproving a universal statement.

(a) Explain/discuss that only examples that are in the statements' domain can be used to make inferences about the statement.
(b) Explain/discuss how to recognize a counterexample—an object in the domain, which does not have the property; that a single counterexample disproves a universal statement; that a false statement can have multiple counterexamples.
(c) Ask students to come up with examples that support, disprove or are irrelevant to a certain statement (suggested by the teacher or the students).

The following excerpt from PST-1's script shows implementation of some of these pedagogical moves in a mathematically correct way.

Student B: So, in order to contradict it [the statement], we didn't need to draw a kite at all. We actually needed to draw a quadrilateral that isn't a kite, but still has congruent and perpendicular sides, like that trapezoid.

Me: Yes! That is why contradicting a mathematical statement can be so tricky. You have to figure out exactly what you are targeting in your statement before you can move forward. We can think of a statement in terms of P and Q. When a statement is universal, meaning it makes a claim about ALL elements in a certain domain, you must find an example within that domain that does not fit the claim.

Similar to PST-1, the majority of PSTs addressed the roles of examples in proving to some extent in their scripts. Unfortunately, the quality of pedagogical moves was often significantly impeded by the gaps in PSTs' own subject matter knowledge. Consider the excerpt from a script by PST-26:

Teacher: Well we have 2 different quadrilaterals on the board that Red and Blue have provided us. Both of these shapes are examples of quadrilaterals that help disprove the claim or in other words what is being stated. What is the claim or what is being stated about quadrilaterals that have congruent and perpendicular diagonals?

Student A: That they are kites!

Teacher: So from our class discussion, do you guys think that this statement is true or false?

Student B: False, because we found 2 shapes that help prove the claim wrong.

Teacher: Very good! The two shapes that we have drawn on the board are called counterexamples. Counterexamples are examples that help disprove a claim. Although there might be examples that help prove a claim, if there is even one counterexample then the statement/claim is false.

The pedagogical content knowledge specific to proving is reflected in PST-26's move to explain/discuss the structure of the statement, and that a single counterexample disproves the statement, thus outweighing supportive examples. PST-26 does this part correctly, but considers both an isosceles trapezoid and a kite as counterexamples, without any explanation. thus, revealing a gap in her subject matter knowledge. The rest of PST-26's scenario is devoted to discussing that a false statement can have multiple counterexamples, using other statements about quadrilaterals.

Category 3: Use of real life analogies

This move involved using an analogy from a real life situation to model the mathematical statement and to explain the role of different types of examples through a non-mathematical context. There were only two scripts in this category, but they stood out in their creativity. PST-27 wrote:

Teacher: So right now we think that the kite and trapezoid both contradict the statement. But is this allowed? Let's try a different example, how about a funny one. Let's say any human that has eyes legs and has long hair is a girl.

In the rest of the scenario, PST-27 used pictures of female and male celebrities with long hair to illustrate supporting examples and counterexamples (respectively). She also used an example of a horse, a non-human with long hair, legs and eyes, to illustrate an irrelevant example. Next, PST-27 went back to the statement and explained how each of these examples corresponds to the mathematical content. The next excerpt shows this move.

Teacher: So back to our example, Student A and Kesha [both females] are like the square. A square is a kite but has congruent and perpendicular diagonals. Neither Student A, Kesha, or the square disprove the statement. What is Ashton Kutcher [male with long hair] like?

Student O: Ashton Kutcher is the contradicting kite! Wait, no, he's the trapezoid…I don't know.

Teacher: Well, he has the hair like the trapezoid has the congruent diagonals, but he is not a girl the way the trapezoid is not a kite.

PST-2 also used a real life analogy, with a statement: "An animal which is white in color and furry is a dog". In her script, the teacher correctly led the students to recognize that white and furry cat is analogous to an isosceles trapezoid in the main statement. However, she also explained that a brown retriever or a hairless terrier disprove the statement about dogs, and, by analogy, concluded that a kite disproves the mathematical statement, because it "fails to recognize one part of the statement".

This script illustrated the complexity of using a real life analogy to model logical reasoning, and the intertwined nature of pedagogical content knowledge and subject matter knowledge. The script may also suggest an alternative explanation of the difficulty to dismiss the example of a kite as irrelevant. Although the real-life statement "An animal which is white in color and furry is a dog" and the mathematical statement: "A quadrilateral with congruent and perpendicular diagonals is a kite" have similar logical structure, the former can be interpreted as an attempt to define a dog. A definition implies a bi-conditional statement, even if it is not worded as such, and should be understood as "If an animal is white and furry—it is a dog; and if an animal is a dog—it is white and furry". Therefore, examples of non-furry or non-white dogs disprove the statement about dogs. It is possible that some PSTs, including PST-2, interpreted the main statement as an alternative definition of a kite; in other words, they implicitly interpreted the given statement as a bi-conditional. Under this interpretation, a kite can be seen as a counterexample to the bi-conditional statement. This could be one plausible explanation of the data, although other explanations are possible.

Category 4: Use a Venn diagram

Three PSTs proposed using a Venn diagram to visualize the relationships between different groups of quadrilaterals in the statement, but only two PST succeeded to create mathematically accurate Venn diagrams.

Figure 3a shows a Venn diagram that is somewhat problematic. The two upper circles represent quadrilaterals with congruent and perpendicular diagonals, which are kites (the middle circle) and not kites (the right circle). The empty intersection, supposedly, indicates that kites and isosceles trapezoids do not have common membership, which is true. However, the diagram gives a wrong impression that

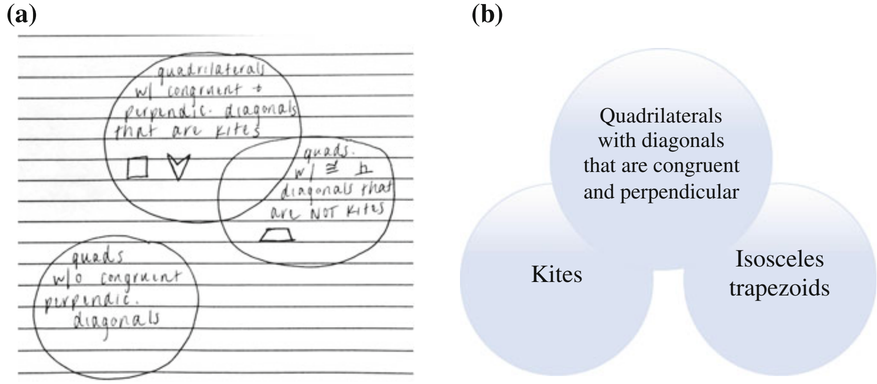


Fig. 3 Two examples of uses of Venn diagrams

quadrilaterals with congruent and perpendicular diagonals must be either kites or isosceles trapezoids. More can be said about this figure, but what we found striking is the conclusion PST-20 drew based on this diagram:

Teacher: As you guys can see, this statement can either be proved true or false, depending on the quadrilateral. Both Red and Blue were correct in claiming their examples contradicted the statement, they just picked examples that fell within different circles in the Venn diagram.

This script is an example of a creative pedagogical move, executed in a way that has some potential. However, it is accompanied by a mathematically wrong explanation. Figure 3b shows a correct Venn diagram by PST-12. It depicts three circles, with the two lower circles representing kites and isosceles trapezoids, which partially intersect a circle representing quadrilaterals with congruent and perpendicular diagonals, but not each other. Yet again, the conclusion from this diagram was surprising:

Teacher: So can someone summarize why the statement is not true and how we disproved it?

Blue: Just because a quadrilateral has diagonals that are congruent and perpendicular it does not imply that it is a kite. We provided two counterexamples: a kite without congruent diagonals and an isosceles trapezoid with both congruent and perpendicular diagonals.

Teacher: Thank you Blue. That was a great summary. So we can disprove a mathematical statement by showing that P does not always imply Q.

It seems that PST-12 has everything needed to make a logically correct inference—she constructed a correct Venn diagram, and explained with logical notation how to disprove a conditional statement. In PST-12's notation, a kite without congruent diagonals corresponds to $(\neg P)$, making it not applicable to disproving the statement.

Her interpretation of a kite as a counterexample signals a deeper conceptual issue at play in her reasoning.

Category 5: Support background knowledge

It was noteworthy that PSTs noticed the need to have stronger knowledge of relevant geometrical content in order to advance to proving. These PSTs introduced pedagogical moves into their scenarios with the intention to strengthen student' background knowledge of geometry, to support argumentation.

(a) Explain/discuss the relationship between special quadrilaterals. E.g., can a square be a kite?
(b) Remind/discuss definitions of concepts (e.g., congruent, perpendicular) and properties of quadrilaterals (kite, isosceles trapezoid, square, rectangle).

At the beginning of her script PST-23 wrote: "I would start the class with a list of known polygons and quadrilaterals on a poster with definitions that the class has agreed upon in a prior class period". Then, PST-23 provided a table with correct definitions of the concepts: congruent, perpendicular, quadrilateral, kite, square and rectangle. Another example of this sub-category is the excerpt from PST-17:

Teacher: How do we know that it [an isosceles trapezoid] is not a kite? Let's look back at our defining properties

1. Two disjoint pairs of consecutive sides are congruent by definition.
2. The diagonals are perpendicular.
3. One diagonal is the perpendicular bisector of the other.
4. One of the diagonals bisects a pair of opposite angles.
5. One pair of opposite angles are congruent.
   Which properties does the picture on the board NOT have?

Student: It does not have properties one, three, or five. So, it is not a kite.

Teacher: Good, so know we know that although the diagonals are congruent and perpendicular, the shape is not a kite.

(c) Use of manipulatives and multiple representations.

Pedagogical techniques in this sub-category involved using visualization aids to support student understanding of quadrilaterals, such as graphic organizers, technology resources to demonstrate properties of quadrilaterals, and manipulatives such as geo-boards and rubber bands.

General Pedagogical Knowledge

In addition to PCK specific to proving we identified two categories of general pedagogical knowledge emerging from the PSTs' scripts. Within each category, there were general pedagogical moves that can be described as productive—the ones that allow access to mathematical ideas and thinking, and unproductive moves, which limit students' learning opportunities. Although wrong mathematical content renders any type of pedagogical move as unproductive, here, we tried to highlight pedagogical techniques, without referring to their mathematical content.

Category 1: Setting up and managing student work

(a) Strategies to support individual student thinking: structuring time for individual thought, giving clear guidelines for work, e.g., draw, measure, or write a response to a given prompt.
(b) Strategies for group work: partner talk, think-pair-share, and groups reporting on their ideas.

Most scripts included some combination of individual and group work strategies, reflecting that PSTs valued the importance of varied ways to support students as creators and owners of knowledge. Productive use of individual and group work included providing students with a specific mathematical prompt on what they should be doing or thinking about, with clear instructions on how they should work (individually, with a partner, with a group), and for how long. For example, PST-24 wrote: "Okay, I want you guys to take a minute to turn and talk to the person next to you, look at our statement and tell me what information is given, and what information is the thing we are trying to prove". In an unproductive use of student group work PST-21 suggested "encourage[ing] students to work out their frustration by talking about it in separate groups", without a particular mathematical prompt or materials to guide discussion or help capturing ideas.

(c) Strategies for monitoring student work

Some scripts allowed for student-led work, and even described teacher actions during this time. For example, PST-21 wrote: "I will walk around and observe their work to determine who understands and who is struggling", indicating clear understanding of the specific actions that a teacher can take to improve student learning during group work.

Category 2: Managing whole-class discussion

Whole class discussion was a key aspect in nearly all of the PSTs' scripts, and included a rich variety of moves. We identified the following sub-categories:

(a) Assessing agreement with ideas emerging from the discussion. E.g., "close your eyes and raise your hands if you agree", "does everyone understands this?".
(b) Pressing students to explain their thinking or others' thinking more deeply. E.g., "what does it tell us?", or "in your own words explain what Blue said".
(c) Encouraging discussion and strategically providing (or withholding) affirmation. E.g., "let's hear from someone who has not spoken yet", "Let's hold on to this idea".

Productive applications of these general pedagogical techniques included asking rich and thought-provoking questions, engaging multiple students and having students listen to and build on each other's argumentation. Non-productive moves involved praising students for any type of contribution, even an incorrect one, turning a whole-class discussion into a single student-teacher dialog, and resolving mathematical confusion "democratically" on account of general agreement or disagreement.

(d) Teacher-led explanations.

Almost all PSTs' scripts had some portion of teacher led explanation—an appropriate pedagogical move, considering the scenario provided in the module. While there were scripts which specified the content, the purpose, and student involvement in this process, some PSTs merely mentioned that they would "give a short lecture on proving", without accompanying details. Other unproductive moves included completely taking over the lesson by the teacher, and/or diverting the focus of explanation towards a different topic.

Discussion

Our study adds to the growing literature on using PSTs' produced scripts in university courses to foster professional knowledge and to evaluate it (Zazkis et al. 2013; Crespo et al. 2011). Among all parts of the module, the scripts of classroom interactions, written by PSTs, generated the richest corpus of data. The scripts were much longer and more elaborate than open responses to the questionnaire; they were even more detailed than oral responses captured on audio during the whole-class discussion (Part 8 of the module). They were also the most coherent account of PSTs' thinking. Since PSTs attended to the roles of various types of examples in proving within a single scenario, it allowed for knowledge gaps or inconsistent mathematical ideas to surface. This was harder to detect within other parts of the module, such as Part 4, in which PSTs evaluated each type of examples separately. As opposed to methodologies relying on self-reported descriptions of what teachers might do in certain situations, the PSTs' produced scripts also revealed *how* they envision enactment of that situation, in the way that bring to forth their subject matter and pedagogical knowledge.

When designing this module, we aimed to create a sequence of tasks that would evoke PSTs' reflection on their content knowledge of geometry and of the logical aspects of proving, specifically—the roles of examples in proving. We also aimed to foster PSTs' thinking about pedagogical practices for supporting students' understanding of proving. Throughout the module we designed multiple opportunities for PSTs to engage with geometry of quadrilaterals, recall and review relevant concepts, analyze sample student work and classroom scenario, write a script, and brainstorm mathematical and pedagogical ideas with their peers. As researchers, we sought to learn from PSTs' responses about their mathematical knowledge for teaching of proving (MKT-P).

Indeed, the imagined classroom interactions, their mathematical content, the choice of mathematical language, and the pedagogical approaches imbedded in the scripts revealed a lot about PSTs' MKT-P, as well as their general pedagogical knowledge. The following three types of MKT-P were most prominent in the data: knowledge of mathematical content, knowledge of the logical aspects of proving and knowledge of the pedagogical practices for supporting students' conceptions of proving. We were particularly encouraged by the range and the richness of the pedagogical practices, which came up in PSTs' data. Among them, were such techniques as the use of the real-life or mathematical analogies, Venn diagrams, review of background knowledge, attending to the logical structure of statements, leading discussions through questioning, and having students share mathematical arguments and justify them. This is especially impressive, since these pedagogical strategies have not been discussed with PSTs in the class prior to the module.

Unfortunately, in the majority of the scripts these productive pedagogical practices were hindered by mathematically incorrect or imprecise content and/or ambiguous language. We see this as a critical point, especially considering the setting of the task. PSTs were encouraged by the teacher educators to use web-resources and textbooks to review geometrical definitions. They were given ample time to complete the online portion of the module and write the scripts. They could have potentially consulted each other or the instructor on issues they were confused about. Nevertheless, the scripts reveal an array of conceptual difficulties, some related to quadrilaterals, but mainly, to the logical aspects of proving, and the relationships between examples and proving. In the results section, we suggested potential explanations to some of these difficulties, but their detailed discussion is beyond the scope of this chapter.

Our results highlight the need for greater attention to mathematical content knowledge of PSTs in the area of geometry and especially, of the logical aspects of proving. We echo the recommendations of the Conference Board of the Mathematical Sciences (CBMS 2012) and the recent Standards for Preparing Teachers of Mathematics by the Association of Mathematics Teacher Educators (AMTE 2017) to strengthen the mathematical preparation for teachers at all levels. However, we also agree that mathematical experiences of pre-service teachers should be grounded in the practice of teaching (Grossman et al 2009; Lampert 2010). Our results also point to the importance of developing PSTs' pedagogical knowledge within content-specific situations. Such pedagogical moves as group work or questioning should be embedded within the context of mathematics. Without this, there is a possibility that PSTs' attempts to apply these pedagogical moves will result in either detracting from mathematical learning, or reinforcing ideas not fully aligned to mathematics knowledge.

Engaging PSTs in creating their own representations of practice, in the form of written and detailed scripts, grounded in specific mathematical content can be a useful tool in addressing both issues raised in our study. The script writing can help PSTs to link mathematical content and pedagogy in meaningful ways. The creative activity of script writing can help to enhance PSTs' repertoire of content specific pedagogical practices, with clear mathematical focus.

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Ceci n'est pas une Pratique: A Commentary

Rina Zazkis

Abstract I highlight the main issues discussed in the chapters and wonder about the effect of engaging with representations of practice on actual teaching practice. I offer avenues for future studies in which representations of practice are designed by teachers, rather than researchers and teacher educators.

Keywords Teaching practice · Lesson play · Scripting

Teacher Education via Representations of Practice

What is the meaning of 'representation'? What is the meaning of 'practice'? Both constructs have been intensively discussed and defined by researchers (e.g., Grossman et al. 2009; Hall 1997; Herbst et al. 2011; Lampert 2010) in reference to preparation for professional practice in general, and to teacher education in particular. I do not attempt to summarize or declare a preference towards one perspective or another. I refer an interested reader to a concise and informative summary by Herbst (2018), who elaborates on representations of practice and points to similarities among and nuances within various perspectives. However, for my commentary a rather simplistic view suffices: Practice is the practice of teaching and it is represented by a variety of artifacts, such as videos, animations, comic strips, vignettes, scripted interactions, or excerpts of student work. Some of the artifacts are carefully chosen excerpts of actual teaching practice, while others are imagined, designed and simulated.

Considering these artifacts as representations of practice described and analyzed in this volume brings to mind René Magritte's famous picture, see Fig. 1.

While initially perceived as a contradiction, "Ceci n'est pas une pipe" (This is not a pipe) directs the viewer's attention that this is an image of an object, rather than an object itself. When asked about the picture, Magritte noted, "Of course it

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Fig. 1 A copy of a famous picture by René Magritte

was not a pipe, just try to fill it with tobacco.[1]" His response points to the difference between an object (or a concept) and its representation, a theme that I attend to in these notes.

While not always explicitly stated as such, the chapters in the volume have a dual goal: (1) to investigate various components of teachers' knowledge or aspects of teachers' competence/expertise, and (2) to contribute to the preparation of teachers for instructional practice or to teachers' professional development. While chapters by Buchbinder and Cook, by Samkova, by Hoth et al., and by Friesen and Kuntze focus mainly on (1), chapters by Kuntze, by Koellner et al., and by Webel et al. study the effect on (2).

The authors offer thoughtful and informative elaboration on particular features of the representations of practice used in their research, pointing to advantages and limitations of various choices. However, when comparing the suitability of different representations, Friesen and Kuntze found video, text and comic format to be "comparably suitable," as teachers engage with each format "comparably well". This reinforces prior research findings of Herbst et al. (2013), by providing stronger evidence via rigorous methodological design.

As a collective, the chapters offer a wide variety of learning experiences for teachers and describe the benefits of continuous professional development as a

[1]http://www.mattesonart.com/biography.aspx.

result of engagement with representations of practice. They describe how pedagogical choices shifted or enhanced and how critical reflection evolved. For example, teachers participating in the study by Webel et al. became more skillful in posing questions, prospective teachers participating in Samkova's study became more knowledgeable in predicting and handling students' errors, Kuntze's participants became more thoughtful in their critique of lessons.

Acknowledging the explicit and often profound effect on teachers' knowledge when engaged with representations of practice, I echo René Magritte, saying "Ceci n'est pas une pratique" (This is not a practice). A great ballet critic may have never danced. An expert wine taster may have never brewed. A famous sports commentator may not play ball. That is, extended ability to critique a practice does not necessarily correspond to the ability to carry out the practice.

From Representations to Practice

Kuntze refers to Lipowsky (2004), who noted that changes in professional knowledge play the role of a necessary but insufficient condition for changes in the instructional practice of teachers. While participating teachers show evidence of improvement when attending to particular aspects of knowledge studied via representations of practice, how did their personal practice evolve? The authors appear in agreement that the effect of experience in critique and analysis of representation of practice on the "real practice" of teaching has yet to be examined. For example, Kuntze explicitly suggests that further studies should include actual practice, and study a transfer of professional development content to classroom practice. Koellner et al. claim that "objective analyses based on teachers' observed classroom practices is essential to validating data on their self-reported uptake of information from the PD."

While the need to draw an explicit connection between experiencing representations of practice and "real" practice is clearly established, how this need can be addressed remains unclear. It will be necessary not only to overcome the logistics of following teachers who participated in research and professional development, but also to establish the validity of the potential correlation when attributing particular instructional choices to teachers' prior experiences with representations of practice. This is an extremely complicated and challenging task. Avoiding this challenge, I offer an alternative.

On Representations of Practice Designed by Teachers

Note that some of the representations of practice discussed in this volume are carefully chosen excerpts of practice (e.g., video clips in Kuntze and in Hoth et al.), while others are designed (e.g., concept cartoons in Samkova's study), or imagined and simulated (e.g., comics in Webel et al.). However, the choices of excerpts are made by researchers, and the representations and simulations are created by researchers. But what if we turn the task around and ask teachers to create representations of teaching, rather than respond to what is created by others? Buchbinder and Cook have done just that, asking prospective teachers to continue a conversation between students and teacher in a form of a screenplay.

Acknowledging the enormous difficulty in examining ‘real teaching’, I have been working for a while on representations of practice composed by prospective teachers, rather than those designed by experts. This route started as a ‘lesson play’—presenting part of a lesson in a form of a dialogue between a teacher and students (Zazkis et al. 2009). With colleagues, I analyzed lesson plays composed by prospective teachers and argued that they provide a lens into how teachers imagine practice (Zazkis et al. 2013). In a more recent work, the method of involving prospective teachers in composing dialogues was extended and described as a “scripting approach.” Analyzing teachers’ scripts provided insights into various aspects of their mathematical and pedagogical knowledge (e.g., Zazkis and Zazkis 2014; Zazkis and Kontorovich 2016). In what follows, I offer possible extensions of the studies in this volume, capitalizing upon the scripting approach.

Consider for example a teacher from the Hoth et al. study who, after watching the video, is asked to imagine her/his conversation with Karola and present it in a format of a scripted interaction between a student and a teacher. Will s/he point to the student’s mistake or will s/he design an approach that would lead the student to discover her mistake and possibly reconsider her answer? A scripting task can be implemented either instead of, or in addition to, providing an open response analysis of the teaching sequence that led to Karola’s mistake. Teachers in the Hoth et al. study provided multifaceted and occasionally constructive critiques to the teaching episode in video. However, how would they themselves carry out the lesson? How would they ensure students’ comprehension? A scripted dialogue may provide some answers.

I point out that there is a big difference in describing what one would do and actually doing it, or at least pretending/imagining doing it. In my experience, teachers describe more fluently what they would or could ask, than actually formulating particular questions. In fact, the difficulty of prospective teachers in role-playing a particular interaction led to the development of lesson play tasks, in which the role-play is imagined, without the necessity to “think on your feet”. Webel et al. make an important step towards teachers’ productions when asking teachers to pose their own question to a student following a student’s idea presented in a comic simulation. However, rather than presenting teachers with pre-programmed students’ responses to the chosen questions, how would teachers themselves imagine the response? How will they choose follow up questions, if necessary? A scripted dialogue composed by a teacher may shed light on these questions.

In Kuntze’s chapter teachers commented in open format on two videos selected from authentic classrooms on a geometric proof. Suppose these (or other) teachers were asked to imagine, and present in a form of a script, how their classroom may look like. I wonder, how will the scripts attend to particular issues identified in the teachers' responses to the videos. Kuntze commented on how teachers may perceive their own practice before and after their engagement with the videos in the professional development project. I suggest that teacher designed scripts, rather than self-reports, may provide an additional and potentially closer look at their practice, via imagined practice. Similarly, the Koellner et al. chapter focuses on what teachers take away from a video based project related to teaching and learning geometry. Their classification of participants is based preliminary on the participants' self-reports. Acknowledging redundancy in my suggestions, I wonder what if the participants were asked to present a scripted dialogue on how they foresee a classroom interaction on a particular topic. Will the script correspond to the self-report? Will particular issues learned from the video be evident? The researchers indicate that validation with classroom practice is needed to further substantiate their findings. Scripts of imagined classroom interactions will provide an intermediate stepping stone for comparison, given the difficulty in following up all of the participants' teaching of the same topic.

Teachers' created representations of practice should not be limited to text-based scripts, which I suggested above. Samkova's chapter provides an interesting analysis of teachers' responses to concept cartoons. I wonder, how a concept cartoon designed by a prospective teacher may look like? I believe it will provide insight about the cartoon-designer's particular aspects of pedagogical content knowledge.

Friesen and Kuntze concluded that different formats of representation were comparably suitable to assess teachers' competence. I wonder, what if teachers were asked to create their own representations in different formats? Will the aspects they chose to address in text be comparable to those addressed via video or via comics? Of interest here is a study of Rougée and Herbst (2018), who compared representations of practice composed by teachers in storyboards and text formats. They found unexpected and nuanced differences and concluded that "medium matters". Obviously, this conclusion depends on the particular aspects of representations that were studied and compared.

Continuing a consideration of the medium, I note that prospective teachers participating in Buchbinder and Cook's study completed their scripted interactions between a teacher and students in the text format, while the prompt was presented as a cartoon-based scenario. Given that these teachers were exposed to Lesson*Sketch*, as their instructional module was administered in this platform, the setting provides a suitable venue for varying the format of scripts and exploring further the affordances and relative advantages of text and storyboard media. Such exploration can be especially applicable in the context of geometry, where it is reasonable to expect that visual artifacts accompany the dialogue.

I hope the authors will consider these suggestions as avenues for future research, which are a natural extension and follow up from their studies. I note, considering the suggestion to extend the presented studies using scripting or other teacher-designed representations of practice, that "Ceci n'est pas une pratique," either. But I assert that scripting practice brings teachers a step closer to the 'real practice' of teaching and brings researchers a step closer to evaluating how engagement with representations of practice may influence practice.

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Considering What We Want to Represent

Daniel Chazan

Abstract Both the fields of mathematics teacher education and research on mathematics teachers have been making extensive use of representations of teaching, whether through written cases, video clips of actual practice, or a range of designed representations (like storyboards or animations with cartoon characters, e.g., Chazan and Herbst in Teachers Coll Rec 114(3):1–34, 2012). This reflection on the contributions to this monograph suggests that as mathematics educators continue to grapple with what representations of teaching are and might be, we give greater attention to the objects to which these representations, as signs, refer.

Keywords Representations of teaching · Mathematics teacher education
Research on teaching

This monograph is a welcome contribution to the growing scholarly attention to representations of teaching and their use in teacher education and research on teaching (e.g., Zazkis and Herbst 2018). In responding to this collection, I suggest that as we continue to grapple with what representations of teaching are, we give greater attention to the objects to which these representations, as signs, refer. In doing so—though it may be challenging—in our publications when we describe uses of representations of teaching, I suggest that we try to specify more clearly the representing that is being done with these representations; said another way, we should try to specify what it is that the representations offered by researchers or teacher educators are intended to represent.

The importance of representing teaching is one of the shared features of research on teaching and the practice of teacher education. This importance of representing practice is not limited to teacher education as a field of professional preparation; indeed Grossman and colleagues document the importance of representing practice for those preparing clergy and clinical psychologists for practice (Grossman et al. 2009). Stimulated in part by ways in which mathematical activity makes use of

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representations of mathematical objects, this monograph focuses on the artifacts, representations, that are used in the process of representing teaching to teachers or teacher candidates. Contributions to the volume use representations of practice to do research on teachers' mathematical content knowledge (Buchbinder and Cook, this volume), to establish the impacts of professional development (Koellner et al., this volume), to seek to understand how particular representations—like Concept Cartoons (Samková, this volume) or video (Hoth et al., this volume)—can be a resource for such research, or to even compare teacher candidates' reactions to different formats of representations to explore the affordances and constraints of these formats for a variety of purposes (Friesen and Kuntze, this volume). Similar to the work in this monograph, recently, other researchers have explored the degree to which teacher candidates treat representations as authentic (Herbst et al. 2013) and the degree to which representations of teaching are effective tools for eliciting knowledge of practice (Herbst and Kosko 2013).

With the advent of technologies that have eased the capturing and sharing of video and those that have supported the creation of graphic arts representations of classroom interaction, self-consciousness about the use of representations of teaching in research on teaching and in teacher education has grown substantially, perhaps explaining the existence of the sort of research presented here. Much of this work has focused on characteristics of the representations themselves. For example, as a way to understand the proliferation of representations of practice used in teacher education and research on teaching, Herbst and colleagues have suggested dimensions for distinguishing representations of practice. First, they suggest that: "Representations [of practice] can be characterized and distinguished according to their origin, from found to transformed to designed" (Herbst et al. 2016, p. 82). One might think of found representations as ones like unedited video clips where it is hard to see the specific decisions that have gone into the creation of the representation (though Hall 2000, reminds us that many such decisions have been made). Transformed representations, like the edited video clip, have undergone an explicit and evident process of editing. Designed representations, like storyboards and animations that use two-dimensional cartoon characters, by contrast, are much more clearly created. Then, Herbst and colleagues offer two other dimensions for characterizing representations: Temporality and Individuality (p. 84). While these dimensions are useful for representations in a range of what Friesen and Kuntze (this volume) label formats, these last two are especially useful for distinguishing designed representations, like storyboards and animations, that use semiotic resources for the creation of representations of teaching (Herbst et al. 2011). Temporality helps distinguish how such representations, as opposed to unedited clips of video, do not seek to represent the ways in which time elapses in classroom interaction. Individuality as a dimension helps a viewer understand decisions the creator of a designed representation has made in selecting what aspects of characters to represent.

The work reviewed so far, focuses on dimensions of the representation itself and how those dimensions make certain qualities of classroom interaction available or not available to the end user. Returning to the analogy to representation of mathematical objects, and considering more particularly multiple representations of functions in mathematics education, the work reviewed so far helps us understand different formats of representations of teaching as analogous to the tables, graphs, and expressions that provide different insights about the functions they represent. Yet, in much of this work on multiple representations of functions, the learner represents the same function in multiple ways and coordinates what is learned from the variety of representations into a deeper understanding of the mathematical object itself. The situation when it comes to representations of teaching feels quite different. We tend not to have different representations of the same interaction (though Friesen and Kuntze, this volume, explore such a possibility). And, it is unclear whether the object whose representation is intended is indeed the same across the use of different formats of representation, let alone within each format (in this sense storyboards and video are not as different from one another as they might seem on first blush). For example, sometimes a video is intended to represent what happened on a particular day with particular students in a particular teacher's class and thus represents this teacher's practice. But, that same video can also represent a kind of teaching that teacher candidates are meant to emulate. Or, the video can represent a dilemma that is common in teaching. More generally, in the hands of mathematics teacher educators, I suggest that representations of teaching are often not intended as a representation of a particular classroom interaction.

Thus, another way to seek to understand ways in which practice is represented focuses less on the artifacts themselves—their characteristics and the media in which they are created—and more on the nature of the representing activity, on what representational artifacts are meant to represent. In a recent review of the work of one dozen teacher educators using the Lesson*Sketch* platform (Chazan et al., accepted), we note teacher educators' uses of representations of teaching to capture the complexity of teaching practice by articulating dilemmas experienced by teachers, or a particular aspect of practice, or teacher candidates' initial efforts to carry out some aspect of practice. In this reflection on the contributions to this monograph, I would similarly like to close by focusing not on the utility of particular representational formats, but instead on what it is that contributors to this volume seek to represent, though in some cases I find it challenging to identify exactly what is intended.

A number of the contributions to this volume seek to represent actual classroom interaction as it occurred in some particular place at some particular time. For example, the videos of classroom practice that Koellner et al. (this volume) share with teachers are meant to represent practice that teachers should emulate in teaching similarity from a transformational approach and illustrate what they call specified professional development. By contrast, Kuntze (this volume) shares everyday examples of classroom practice, that are not viewed as exemplary, to have teacher candidates review what they see in these representations when they focus on cognitive activation, intensity of argumentation, and learning from mistakes. This focus may actually show teacher candidates that exemplary practice is relatively rare, even as they work to attempting to enact such practice themselves.

By contrast, some of the other contributions to the monograph seem to be the result of a process of what Grossman et al. (2009) might describe as a decomposition of practice into constituent parts. The representing of practice that seems to occur around these representations seems focused on particular aspects of teaching. For example, Webel et al. (this volume) focus on teachers questioning techniques as one aspect of practice that teacher candidates can work on improving. Similarly, Samková (this volume) focuses on how students might respond to a question and how to create discussion around ideas elicited from students.

Looking forward, it seems to me that a continued focus on representations of teaching both in research on teaching and in the context of teacher education is warranted and is quite likely to continue. Perhaps future work will help us learn more about relationships between the dimensions of representations of practice and the nature of the representing of teaching being done both in the context of teacher education and research on teaching.

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