

# The Differential Dimension Polynomial for Characterizable Differential Ideals



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**Abstract** We generalize the notion of a differential dimension polynomial of a prime differential ideal to that of a characterizable differential ideal. Its computation is algorithmic, its degree and leading coefficient remain differential birational invariants, and it decides equality of characterizable differential ideals contained in each other.

**Keywords** Dimension polynomial • Characterizable ideal

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## 1 Introduction

Many systems of differential equations do not admit closed form solutions or any other finite representation of all solutions. Hence, such systems cannot be solved symbolically. Despite this, increasingly good and efficient heuristics to find solutions symbolically have been developed and are implemented in computer algebra systems [4, 5]. Of course, such algorithms can at best produce the subset that admits a closed form of the full set of solutions. Given such a set of closed form solutions returned by a computer algebra system, the natural question remains whether this set is a complete solution set (cf. Example 5.4).

Classical measures, e.g. the Cartan characters [3] and Einstein's strength [7], describe the size of such solution sets. However, they have a drawback: one can easily find two systems  $S_1$  and  $S_2$  of differential equations such that the solution set of  $S_1$  is a proper subset of the solution set of  $S_2$ , but these two solution sets have identical measures (cf. Example 5.3). In particular, if  $S_1$  is given by a solver of

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differential equations, these measures cannot detect whether this is the full set  $S_2$  of solutions.

Kolchin introduced the differential dimension polynomial to solve this problem for solution sets of systems of differential equations corresponding to prime differential ideals [12–15]. This polynomial generalizes the Cartan characters and strength by counting the number of freely choosable power series coefficients of an analytical solution. Recently, Levin generalized the differential dimension polynomial to describe certain subsets of the full solution set of a prime differential ideal [18].

Even though decomposing the radical differential ideal generated by a set of differential equations into prime differential ideals is theoretically possible, it is expensive in practice (cf. [2, §6.2]). Thus, there is a lack of practical methods which decide whether a subset of the solution set of a system of differential equations is a proper subset. This paper solves this problem for greater generality than solution sets of prime differential ideals. It generalizes the differential dimension polynomial to characterizable differential ideals and thereby gives a necessary condition for completeness of solution sets. Such ideals can be described by differential regular chains, and there exist reasonably fast algorithms that decompose a differential ideal into such ideals [1, 2].

To formulate the main theorem, we give some preliminary definitions; the missing definitions are given in Sect. 2. Denote by  $F\{U\}$  a differential polynomial ring in  $m$  differential indeterminates for  $n$  commuting derivations over a differential field  $F$  of characteristic zero. For a differential ideal  $I$  in  $F\{U\}$  let  $I_{\leq \ell} := I \cap F\{U\}_{\leq \ell}$ , where  $F\{U\}_{\leq \ell}$  is the subring of  $F\{U\}$  of elements of order at most  $\ell$ . We define the differential dimension function using the Krull dimension as

$$\Omega_I : \mathbb{Z}_{\geq 0} \mapsto \mathbb{Z}_{\geq 0} : \ell \mapsto \dim(F\{U\}_{\leq \ell} / I_{\leq \ell}) .$$

By the following theorem, this function is eventually polynomial for large  $\ell$  if  $I$  is characterizable. Such polynomials mapping  $\mathbb{Z}$  to  $\mathbb{Z}$  are called numerical polynomials, and there exists a natural total order  $\leq$  on them.

**Theorem 1.1** *Let  $I \subset F\{U\}$  be a characterizable differential ideal.*

1. *There exists a unique numerical polynomial  $\omega_I(\ell) \in \mathbb{Q}[\ell]$ , called differential dimension polynomial, with  $\omega_I(\ell) = \Omega_I(\ell)$  for sufficiently big  $\ell \in \mathbb{Z}_{\geq 0}$ .*
2.  *$0 \leq \omega_I(\ell) \leq m \binom{\ell+n}{n}$  for all  $\ell \in \mathbb{Z}_{\geq 0}$ . In particular,  $d_I := \deg_\ell(\omega_I) \leq n$ .*
3. *When writing  $\omega_I(\ell) = \sum_{i=0}^n a_i \binom{\ell+i}{i}$  with  $a_i \in \mathbb{Z}$  for all  $i \in \{0, \dots, n\}$ , the degree  $d_I$  and the coefficients  $a_i$  for  $i \geq d_I$  are differential birational invariants, i.e., they are well-defined on the isomorphism class of total quotient ring of  $F\{U\}/I$ .*
4. *The coefficient  $a_n$  is the differential dimension of  $F\{U\}/I$ , as defined below.*

*Let  $I \subseteq J \subset F\{U\}$  be another characterizable differential ideal.*

5. *Then  $\omega_J \leq \omega_I$ .*

Assume  $\omega_I = \omega_J$ , and let  $S$  respectively  $S'$  be differential regular chains with respect to an orderly differential ranking  $<$  that describe  $I$  respectively  $J$ .

- 6. The sets of leaders of  $S$  and  $S'$  coincide, and
- 7.  $I = J$  if and only if  $\deg_x(S_x) = \deg_x(S'_x)$  for all leaders  $x$  of  $S$ , where  $S_x$  is the unique element in  $S$  of leader  $x$ .

This theorem can be slightly strengthened, as  $I \subseteq J$  and  $\omega_I = \omega_J$  already imply  $\deg_x(S_x) \leq \deg_x(S'_x)$  for all leader  $x$  of  $S$  (cf. Lemma 3.5). Thus  $I = J$  if and only if  $\prod_x \deg_x(S_x) = \prod_x \deg_x(S'_x)$ . It would be interesting to have a version of Theorem 1.1, where this product is an intrinsic value, similar to the leading differential degree [9].

The importance of characterisable differential ideals and their connection to differential dimension polynomials appear in [6, §3.2], building on Lazard’s lemma [2]. In particular, the invariance conditions were implicitly observed. To the best of the author’s knowledge, testing equality by means of invariants does not appear in the literature. Testing equality of differential ideals is connected to Ritt’s problem of finding a minimal prime decomposition of differential ideals.<sup>1</sup>

Recently, the author introduced the differential counting polynomial [16, 17]. It gives a more detailed description of the set of solutions than the differential dimension polynomial, in fact so detailed that it seems not to be computable algorithmically. In particular, it provides a necessary criterion of completeness of solution sets, whereas the differential counting polynomial only provides a sufficient criterion. The intention of this paper is a compromise of giving a description of the size of the set of solutions that is detailed enough to be applicable to many problems, but that is still algorithmically computable.

A more detailed description of the content of this paper in the language of simple systems is a part of the author’s thesis [17].

Section 3 proves Theorem 1.1, Sect. 4 discusses the computation of the differential dimension polynomial, and Sect. 5 gives examples.

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<sup>1</sup>It is easy to test equality of two prime differential ideals given by a characteristic set (cf. exercise 1 in [14, §IV.10]). However, the unsolved Ritt problem states that there is no algorithm known to find a *minimal* decomposition of a differential ideal given by a set of generators into prime ideals given by characteristic sets [20], [14, §IV.9]. Under mild conditions, Ritt’s problem is equivalent to several other problems, among them (1) deciding whether a differential ideal given by a set of generators is prime, (2) finding a set of generators of a prime differential ideal given by a characteristic set, and (3) given the characteristic sets of two prime differential ideals  $I_1$  and  $I_2$  determine whether  $I_1 \subseteq I_2$  [10].

## 2 Preliminaries

### 2.1 Squarefree Regular Chains

Let  $F$  be a field of characteristic zero,  $\overline{F}$  its algebraic closure, and  $R := F[y_1, \dots, y_n]$  a polynomial ring. We fix the total order, called ranking,  $y_1 < y_2 < \dots < y_n$  on  $\{y_1, \dots, y_n\}$ . The  $<$ -greatest variable  $\text{ld}(p)$  occurring in  $p \in R \setminus F$  is called the leader of  $p$ . The coefficient  $\text{ini}(p)$  of the highest power of  $\text{ld}(p)$  in  $p$  is called the initial of  $p$ . We denote the separant  $\frac{\partial p}{\partial \text{ld}(p)}$  of  $p$  by  $\text{sep}(p)$ .

Let  $S \subset R \setminus F$  be finite. Define  $\text{ld}(S) := \{\text{ld}(p) \mid p \in S\}$  and similarly  $\text{ini}(S)$  and  $\text{sep}(S)$ . The set  $S$  is called triangular if  $|\text{ld}(S)| = |S|$ ; in this case denote by  $S_x \in S$  the unique polynomial with  $\text{ld}(S_x) = x$  for  $x \in \text{ld}(S)$ . We call the ideal  $\mathcal{I}(S) := \langle S \rangle : \text{ini}(S)^\infty \subseteq R$  the ideal associated to  $S$ . Let  $S_{<x} := \{p \in S \mid \text{ld}(p) < x\}$  for each  $x \in \{y_1, \dots, y_n\}$ . The set  $S$  is called a squarefree regular chain if it is triangular and neither  $\text{ini}(S_x)$  is a zero divisor modulo  $\mathcal{I}(S_{<x})$  nor  $\text{sep}(S_x)$  is a zero divisor modulo  $\mathcal{I}(S)$  for each  $x \in \text{ld}(S)$ .

**Proposition 2.1 ([11, Prop. 5.8])** *Let  $S$  be a squarefree regular chain in  $R$  and  $1 \leq i \leq n$ . Then  $\mathcal{I}(S_{<y_i}) \cap F[y_1, \dots, y_{i-1}] = \mathcal{I}(S) \cap F[y_1, \dots, y_{i-1}]$ . Furthermore, if  $p \in F[y_1, \dots, y_{i-1}]$  is not a zero-divisor modulo  $\mathcal{I}(S_{<y_i})$ , then  $p$  is not a zero-divisor modulo  $\mathcal{I}(S)$ .*

Note that the last sentence follows easily using that the zero divisors (and zero) are the union of the associated primed, cf. [8, Thm. 3.1].

**Theorem 2.2 (Lazard’s lemma, [11, Thm. 4.4, Coro. 7.3, Thm. 7.5], [2, Thm. 1])** *Let  $S$  be a squarefree regular chain in  $R$ . Then  $\mathcal{I}(S)$  is a radical ideal in  $R$ , and the set  $\{y_1, \dots, y_n\} \setminus \text{ld}(S)$  forms a transcendence basis for every associated prime of  $\mathcal{I}(S)$ . Let such an associated prime  $\mathcal{I}(S')$  be given by a squarefree regular chain  $S'$ . Then  $\text{ld}(S) = \text{ld}(S')$  and, in particular,  $R/\mathcal{I}(S)$  is equidimensional of dimension  $n - |\text{ld}(S)|$ .*

### 2.2 Differential Algebra

Let  $F$  be a differential field of characteristic zero with pairwise commuting derivations  $\Delta = \{\partial_1, \dots, \partial_n\}$ . Let  $U := \{u^{(1)}, \dots, u^{(m)}\}$  be a set of differential indeterminates and define  $u_\mu^{(j)} := \partial^\mu u^{(j)}$  for  $\partial^\mu := \partial_1^{\mu_1} \dots \partial_n^{\mu_n}$ ,  $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{Z}_{\geq 0})^n$ . For any set  $S$  let  $\{S\}_\Delta := \{\partial^\mu s \mid s \in S, \mu \in (\mathbb{Z}_{\geq 0})^n\}$ . The differential polynomial ring  $F\{U\}$  is the infinitely generated polynomial ring in the indeterminates  $\{U\}_\Delta$ . The derivations  $\partial_i : F \rightarrow F$  extend to  $\partial_i : F\{U\} \rightarrow F\{U\}$  by setting  $\partial_i \partial_1^{\mu_1} \dots \partial_n^{\mu_n} u^{(j)} = \partial_1^{\mu_1} \dots \partial_i^{\mu_i+1} \dots \partial_n^{\mu_n} u^{(j)}$  ( $1 \leq i \leq n$ ,  $1 \leq j \leq m$ ) via additivity and Leibniz rule. We denote the differential ideal generated by  $p_1, \dots, p_t \in F\{U\}$  by  $\langle p_1, \dots, p_t \rangle_\Delta$ .

A ranking of the differential polynomial ring  $F\{U\}$  is a total ordering  $<$  on the set  $\{U\}_\Delta$  satisfying additional properties (cf. e.g. [14, p. 75]). A ranking  $<$  is called orderly if  $|\mu| < |\mu'|$  implies  $u_\mu^{(i)} < u_{\mu'}^{(j)}$ , where  $|\mu| := \mu_1 + \dots + \mu_n$ . In what follows, we fix an orderly ranking  $<$  on  $F\{U\}$ . The concepts of leader, initial and separant carry over to elements in the polynomial ring  $F\{U\}$ .

Let  $R$  be a residue class ring of a differential polynomial ring by a differential ideal. A differential transcendence basis  $\{p_1, \dots, p_d\} \subset R$  is a maximal set such that  $\bigcup_{i=1}^d \{p_i\}_\Delta$  is algebraically independent over  $F$ . The differential dimension of  $R$  is the corresponding cardinality  $d$ .

A finite set  $S \subset F\{U\} \setminus F$  is called (weakly) triangular if  $\text{ld}(p)$  is not a derivative of  $\text{ld}(q)$  for all  $p, q \in S, p \neq q$ . Define  $S_{<x}$  and  $S_x$  as in the algebraic case. We call  $\mathcal{S}(S) := \langle S \rangle_\Delta : (\text{ini}(S) \cup \text{sep}(S))^\infty \subseteq F\{U\}$  the differential ideal associated to  $S$ . The set  $S$  is called coherent if the  $\Delta$ -polynomials of  $S$  are reduced to zero with respect to  $S$  [21], and it is called a differential regular chain if it is triangular, coherent, and if neither  $\text{ini}(S_x)$  is a zero divisor modulo  $\mathcal{S}(S_{<x})$  nor  $\text{sep}(S_x)$  is a zero-divisor module  $\mathcal{S}(S)$  for each  $x \in \text{ld}(S)$ . An ideal  $\mathcal{S}(S)$  is called characterizable if  $S$  is a differential regular chain.

Let  $S$  be a differential regular chain in  $F\{U\}$ ,  $\ell \in \mathbb{Z}_{\geq 0}$ , and  $L := \{\partial^\mu y \mid y \in \text{ld}(S)\} \cap F\{U\}_{\leq \ell}$  be the set of derivatives of leaders of elements in  $S$  of order at most  $\ell$ . For each  $x \in L$  there exists a  $\mu_{[x]} \in \mathbb{Z}_{\geq 0}^n$  and a  $p_{[x]} \in S$  such that  $\text{ld}(\partial^{\mu_{[x]}} p_{[x]}) = x$ . Define an algebraic triangular set associated to  $S$  as  $S_{\leq \ell} := \{\partial^{\mu_{[x]}} p_{[x]} \mid x \in L\}$ . Although  $S_{\leq \ell}$  depends on the choice of  $\mu_{[x]}$  and  $p_{[x]}$ , it has properties independent of the choice.

**Lemma 2.3 (Rosenfeld’s Lemma)** *Let  $S$  be a differential regular chain in  $F\{U\}$ ,  $\ell \in \mathbb{Z}_{\geq 0}$ , and  $<$  orderly. Then  $S_{\leq \ell}$  is a squarefree regular chain and  $\mathcal{S}_{F\{U\}_{\leq \ell}}(S_{\leq \ell}) = \mathcal{S}(S)_{\leq \ell}$ .*

The idea is due to [21]. For a detailed proof cf. [17, Lemma 1.93].

### 2.3 Numerical Polynomials

Numerical polynomials are elements in the free  $\mathbb{Z}$ -module  $\left\{ \binom{\ell+k}{k} \in \mathbb{Q}[\ell] \mid 0 \leq k \leq n \right\}$ , i.e., rational polynomials that map an integer to an integer. They are totally ordered by  $p \leq q$  if  $p(\ell) \leq q(\ell)$  for all  $\ell$  sufficiently large. Then  $p \leq q$  if and only if either  $p = q$  or there is a  $j \in \{0, \dots, d\}$  such that  $a_k = b_k$  for all  $k > j$  and  $a_j < b_j$ , where  $p = \sum_{k=0}^d a_k \binom{\ell+k}{k}$  and  $q = \sum_{k=0}^d b_k \binom{\ell+k}{k}$ .

### 3 Proofs

#### 3.1 Proof of Existence and Elementary Properties

We prove Theorem 1.1.(1), (2), (4), and (5). Therefore, let  $I \subseteq J \subset F\{U\}$  be characterizable differential ideals,  $S$  be a differential regular chain with respect to an orderly differential ranking  $<$  with  $\mathcal{S}(S) = I$ , and  $\ell \in \mathbb{Z}_{\geq 0}$  be sufficiently big.

Lemma 2.3 implies  $I_{\leq \ell} = \mathcal{S}(S_{\leq \ell})$  and Theorem 2.2 states that the dimension  $\dim(F\{U\}_{\leq \ell}/I_{\leq \ell})$  can be read off from the number of polynomials in  $S_{\leq \ell}$ , which only depends on  $\text{ld}(S)$ . Thus, to prove Theorem 1.1.(1) and 1.1.(2) we may assume  $S = \text{ld}(S)$ . In this case  $\mathcal{S}(S)$  is a prime differential ideal, and hence the statements follow from Kolchin’s original theorem [14, §II.12].

For the proof of Theorem 1.1.(4) note that the transcendence bases of all associated primes of  $\mathcal{S}(S)$  are equal by Theorem 2.2, and for each of these associated prime the claim follows from Kolchin’s original theorem.

To prove Theorem 1.1.(5) note that  $I \subseteq J$  implies  $I_{\leq \ell} \subseteq J_{\leq \ell}$  for all  $\ell \geq 0$ . In particular, the map from  $F\{U\}_{\leq \ell}/I_{\leq \ell}$  to  $F\{U\}_{\leq \ell}/J_{\leq \ell}$  is surjective and, thus,  $\dim(F\{U\}_{\leq \ell}/I_{\leq \ell}) \geq \dim(F\{U\}_{\leq \ell}/J_{\leq \ell})$ . ■

#### 3.2 Invariance Proof

The differential polynomial ring  $F\{U\}$  is filtered by the finitely generated  $F$ -algebras  $F\{U\}_{\leq \ell}$ . This filtration induces a filtration on  $F\{U\}/I$  for a differential ideal  $I$ . To prove the invariance statement in Theorem 1.1.(3) we show that this filtration extends to  $\text{K}(F\{U\}/I)$  if  $I$  is characterizable, where  $\text{K}$  denotes the total quotient ring. Thereby, standard techniques of filtrations can be adapted from Kolchin’s proof.

*Example 3.1* Consider  $\Delta = \{\partial_r\}$ ,  $U = \{u, v\}$ , and  $I := \langle u_0 \cdot v_1 \rangle_{\Delta}$ . Then  $u_0$  is not a zero-divisor in  $F\{U\}_{\leq 0}/I_{\leq 0} \cong F[u_0, v_0]$ , but  $u_0 \cdot v_1 = 0$  in  $F\{U\}/I$ . So, even though the inclusion  $\alpha : F\{U\}_{\leq 0}/I_{\leq 0} \hookrightarrow F\{U\}_{\leq 1}/I_{\leq 1}$  is injective, the image of this map under the total quotient ring functor  $\text{K}$  is no longer injective, as  $\text{K}(\alpha) : \text{K}(F\{U\}_{\leq 0}/I_{\leq 0}) \rightarrow \text{K}(F\{U\}_{\leq 1}/I_{\leq 1}) = \text{K}(F[u_0, v_0, u_1, v_1]/\langle u_0 \cdot v_1 \rangle)$  maps  $u_0$  to zero, as zero divisors become zero in the total quotient ring, cf. e.g. [8, Prop. 2.1].

**Lemma 3.2** *Let  $I \subseteq F\{U\}$  be a characterizable differential ideal and  $\ell \in \mathbb{Z}_{\geq 0}$ . Then,  $F\{U\}_{\leq \ell}/I_{\leq \ell} \hookrightarrow F\{U\}_{\leq \ell+1}/I_{\leq \ell+1}$  induces an inclusion*

$$\text{K}(F\{U\}_{\leq \ell}/I_{\leq \ell}) \hookrightarrow \text{K}(F\{U\}_{\leq \ell+1}/I_{\leq \ell+1}) .$$

*Proof* Any non-zero-divisor in  $F\{U\}_{\leq \ell}/I_{\leq \ell}$  is a non-zero-divisor when considered in  $F\{U\}_{\leq \ell+1}/I_{\leq \ell+1}$  (cf. Proposition 2.1), and thus a unit in  $\text{K}(F\{U\}_{\leq \ell+1}/I_{\leq \ell+1})$ . Hence,  $F\{U\}_{\leq \ell}/I_{\leq \ell} \rightarrow \text{K}(F\{U\}_{\leq \ell+1}/I_{\leq \ell+1})$  factors over  $\text{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})$  by the universal property of localizations. This induces a map  $\iota : \text{K}(F\{U\}_{\leq \ell}/I_{\leq \ell}) \rightarrow$

$\mathbb{K}(F\{U\}_{\leq \ell+1}/I_{\leq \ell+1})$ . Now,  $\ker \iota \cap (F\{U\}_{\leq \ell}/I_{\leq \ell})$  is zero, since it is the kernel of the composition  $F\{U\}_{\leq \ell}/I_{\leq \ell} \hookrightarrow F\{U\}_{\leq \ell+1}/I_{\leq \ell+1} \hookrightarrow \mathbb{K}(F\{U\}_{\leq \ell+1}/I_{\leq \ell+1})$  of monomorphisms. By [8, Prop. 2.2] there is an injection<sup>2</sup> from the set of ideals in  $\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})$  into the set of ideals in  $F\{U\}_{\leq \ell}/I_{\leq \ell}$ . This implies  $\ker \iota = 0$ . ■

This filtration is well-behaved under differential isomorphisms.

**Lemma 3.3** *Let  $I \subseteq F\{U\}$  and  $J \subseteq F\{V\}$  be characterizable differential ideals. Let  $\varphi : \mathbb{K}(F\{U\}/I) \rightarrow \mathbb{K}(F\{V\}/J)$  be a differential isomorphism. Then there exists an  $\ell_0 \in \mathbb{Z}_{\geq 0}$  such that*

$$\varphi(\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})) \subseteq \mathbb{K}(F\{V\}_{\leq \ell+\ell_0}/J_{\leq \ell+\ell_0}).$$

*Proof*  $F\{U\}/I$  is a (left)  $F[\Delta]$ -module for every differential ideal  $I \subset F\{U\}$ , where  $F[\Delta]$  is the ring of linear differential operators with coefficients in  $F$ . The filtration of  $F[\Delta]$  by the linear differential operators  $F[\Delta]_{\leq k}$  of order  $\leq k$  is compatible with the filtration of  $F\{U\}$  in the sense that  $F[\Delta]_{\leq k}(F\{U\}_{\leq \ell}/I_{\leq \ell}) \subseteq F\{U\}_{\leq \ell+k}/I_{\leq \ell+k}$ . Note that the canonical image of  $F[\Delta]_{\leq \ell}(F\{U\}_{\leq 0}/I_{\leq 0})$  in  $F\{U\}_{\leq \ell}/I_{\leq \ell}$  generates the latter as an  $F$ -algebra. Abusing notation, given any  $F$ -module  $M$  of an  $F$ -algebra, denote by  $\mathbb{K}(M)$  the total quotient ring of the  $F$ -algebra generated by  $M$ . In particular,  $\mathbb{K}(F[\Delta]_{\leq \ell}(F\{U\}_{\leq 0}/I_{\leq 0})) = \mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})$ .

There exists an  $\ell_0 \in \mathbb{Z}_{\geq 0}$  with  $\varphi(F\{U\}_{\leq 0}/I_{\leq 0}) \subseteq \mathbb{K}(F\{V\}_{\leq \ell_0}/J_{\leq \ell_0})$ , as  $F\{V\}/J = \bigcup_{\ell \in \mathbb{Z}_{\geq 0}} F\{V\}_{\leq \ell}/J_{\leq \ell}$ . Now

$$\begin{aligned} \varphi(\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})) &= \varphi(\mathbb{K}(F[\Delta]_{\leq \ell}(F\{U\}_{\leq 0}/I_{\leq 0}))) \\ &= \mathbb{K}(F[\Delta]_{\leq \ell} \varphi(F\{U\}_{\leq 0}/I_{\leq 0})) \\ &\subseteq \mathbb{K}(F[\Delta]_{\leq \ell} \mathbb{K}(F\{V\}_{\leq \ell_0}/J_{\leq \ell_0})) \\ &\subseteq \mathbb{K}(F\{V\}_{\leq \ell+\ell_0}/J_{\leq \ell+\ell_0}) \end{aligned}$$

■

The Krull-dimension changes when passing to total quotient rings. Instead, we use  $\dim_F(R) := \max_{P \in \text{Ass}(R)} \text{trdeg}_F(\mathbb{K}(R/P))$  as notion of dimension for  $F$ -algebras  $R$ . Then,  $\dim(R) = \dim_F(R) = \dim_F(\mathbb{K}(R))$  allows to prove the invariance condition.

*Proof of Theorem 1.1.(3)* Let  $\varphi$  be as in Lemma 3.3. Then,

$$\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell}) \cong \varphi(\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})) \subseteq \mathbb{K}(F\{V\}_{\leq \ell+\ell_0}/J_{\leq \ell+\ell_0})$$

with the  $\ell_0 \in \mathbb{Z}_{\geq 0}$  from Lemma 3.3, and thus

$$\begin{aligned} \dim(F\{U\}_{\leq \ell}/I_{\leq \ell}) &= \dim_F(\mathbb{K}(F\{U\}_{\leq \ell}/I_{\leq \ell})) \\ &\leq \dim_F(\mathbb{K}(F\{V\}_{\leq \ell+\ell_0}/J_{\leq \ell+\ell_0})) \\ &= \dim(F\{V\}_{\leq \ell+\ell_0}/J_{\leq \ell+\ell_0}). \end{aligned}$$

<sup>2</sup>The image consists of those ideals which do not contain any zero divisors.

Thus  $\omega_I(\ell) \leq \omega_I(\ell + \ell_0)$  and by symmetry  $\omega_I(\ell) \leq \omega_I(\ell + \ell_0)$ . Now, an elementary argument implies that the degrees and leading coefficients of  $\omega_I$  and  $\omega_J$  are the same. ■

### 3.3 Comparison Proof

The proof of Theorem 1.1.(6) and (7) uses two propositions, which relate ideals and squarefree regular chains. The first proposition is a direct corollary to Lazard’s Lemma (Theorem 2.2).

**Proposition 3.4** *Let  $S, S'$  be squarefree regular chains in  $F[y_1, \dots, y_n]$  with  $\mathcal{I}(S) \subseteq \mathcal{I}(S')$  and  $|S| = |S'|$ . Then  $\text{ld}(S) = \text{ld}(S')$ .*

The following lemma is used to prove the second proposition. It captures an obvious property of the pseudo reduction with respect to a squarefree regular chain  $S$ : if a polynomial  $p$  can be reduced to zero by  $S$ , but  $\text{ini}(p)$  cannot be reduced to zero, then there must be a suitable element in  $S$  to reduce the highest power of  $\text{ld}(p)$ .

**Lemma 3.5** *Let  $S$  be a squarefree regular chain and  $p \in F[y_1, \dots, y_n]$  with  $\text{ld}(p) = x$ ,  $p \in \mathcal{I}(S)$ , and  $\text{ini}(p) \notin \mathcal{I}(S)$ . Then  $S$  has an element of leader  $x$  and  $\text{deg}_x(S_x) \leq \text{deg}_x(p)$ .*

**Proposition 3.6** *Let  $S$  and  $S'$  be squarefree regular chains in  $R = F[y_1, \dots, y_n]$  with  $\mathcal{I}(S) \subseteq \mathcal{I}(S')$  and  $|S| = |S'|$ . Then,  $\mathcal{I}(S) = \mathcal{I}(S')$  if and only if  $\text{deg}_x(S_x) = \text{deg}_x(S'_x)$  for all  $x \in \text{ld}(S) = \text{ld}(S')$ .*

*Proof* Let  $\text{deg}_x(S_x) = \text{deg}_x(S'_x)$  for all  $x \in \text{ld}(S)$ . We show  $\mathcal{I}(S) \supseteq \mathcal{I}(S')$  by a Noetherian induction. The statement is clear for the principle ideals  $\mathcal{I}(S_{<y_2})$  and  $\mathcal{I}(S'_{<y_2})$ . Let  $p \in \mathcal{I}(S')$  with  $\text{ld}(p) = y_i$  and  $\text{deg}_{y_i}(p) = j$ . Assume by induction that  $q \in \mathcal{I}(S')$  implies  $q \in \mathcal{I}(S)$  for all  $q$  with  $\text{ld}(q) < y_i$  or  $\text{ld}(q) = y_i$  and  $\text{deg}_{y_i}(q) < j$ . Without loss of generality  $\text{ini}(p) \notin \mathcal{I}(S'_{<y_i}) = \mathcal{I}(S)_{<y_i}$ , as otherwise  $p$  has a lower degree in  $y_i$  or a lower ranking leader when substituting  $\text{ini}(p)$  by zero. Now, Lemma 3.5 implies  $y_i \in \text{ld}(S)$  and  $\text{deg}_{y_i}(p) \geq \text{deg}_{y_i}(S_{y_i})$ . Then,

$$r := \text{ini}(S_{y_i}) \cdot p - \text{ini}(p) \cdot y_i^{\text{deg}_{y_i}(p) - \text{deg}_{y_i}(S_{y_i})} \cdot S_{y_i}$$

is in  $\mathcal{I}(S)$  if and only if  $p \in \mathcal{I}(S)$  is, but  $r$  is of lower degree or of lower ranking leader than  $p$ . The claim follows by induction.

Let  $\mathcal{I}(S) = \mathcal{I}(S')$  and  $x \in \text{ld}(S)$ . This implies  $\text{ini}(S_x) \notin \mathcal{I}(S')$ , and thus  $\text{deg}_x(S'_x) \leq \text{deg}_x(S_x)$  by Lemma 3.5. By symmetry  $\text{deg}_x(S'_x) \geq \text{deg}_x(S_x)$ , and thus  $\text{deg}_x(S_x) = \text{deg}_x(S'_x)$ . ■

*Proof of Theorem 1.1.(6) and (7)* Lemma 2.3 reduces the statements to the algebraic case. In this case, Proposition 3.4 implies Theorems 1.1, and 1.1 follows from Proposition 3.6, because all polynomials in  $S_{\leq \ell}$  ( $\ell \in \mathbb{Z}_{\geq 0}$ ) of degree greater than one in their respective leader already lie in  $S$ . ■



### 4 Computation of the Differential Dimension Polynomial

To compute the differential dimension polynomial  $\omega_{\mathcal{S}(S)}$  of a characterizable differential ideal  $\mathcal{S}(S) \subseteq F\{U\}$  for a differential regular chain  $S$  we may assume  $S = \text{ld}(S)$  (cf. Sect. 3.1). This assumption implies that  $\mathcal{S}(S)$  is a prime differential ideal, and for this case there exist well-known combinatorial algorithms for  $\omega_{\mathcal{S}(S)}$  [15].

Alternatively, the differential dimension polynomial  $\omega_{\mathcal{S}(S)}$  can be read off the set of equations  $S$  of a simple differential system [1]. Such a set  $S$  is almost a differential regular chain, except that weak triangularity is replaced by the Janet decomposition, which associates a subset of  $\Delta$  of cardinality  $\zeta_p$  to each  $p \in S$ . Then, the differential dimension polynomial is given by the closed formula

$$\omega_{\mathcal{S}(S)}(l) = m \binom{n + l}{n} - \sum_{p \in S} \binom{\zeta_p + l - \text{ord}(\text{ld}(p))}{\zeta_p},$$

involving only the cardinalities  $\zeta_p$  and the orders  $\text{ord}(\text{ld}(p))$ .

### 5 Examples

For each prime differential ideal  $I$  there exists a differential regular chain  $S$  with  $I = \mathcal{S}(S)$ . Thus, the differential dimension polynomial defined in Theorem 1.1 includes the version of Kolchin. However, the following example shows that Theorem 1.1 is more general.

*Example 5.1* Consider  $U = \{u, v\}$ ,  $\Delta = \{\partial_t\}$ ,  $p = u_1^2 - v$ , and  $q = v_1^2 - v$ . The characterizable differential ideal  $I := \mathcal{S}(\{p, q\})$  is not prime, as  $p - q = (u_1 - v_1)(u_1 + v_1)$ .

Prime differential ideals  $I \subseteq J$  are equal if and only if  $\omega_I = \omega_J$  by Kolchin’s theorem. By the following example, this is wrong for characterizable ideals and any generalization to such ideals needs to consider the degrees of polynomials in a differential regular chain.

*Example 5.2* Consider  $\langle u_0^2 - u_0 \rangle_{\Delta} = \mathcal{S}(\{u_0^2 - u_0\}) \subsetneq \langle u_0 \rangle_{\Delta} = \mathcal{S}(\{u_0\})$  in  $F\{u\}$  for  $|\Delta| = 1$ . Both differential ideals are characterizable and have the differential dimension polynomial 0. However, they are not equal.

The next example shows that the Cartan characters and other invariants do not suffice to prove that two solution sets are unequal.

*Example 5.3* For  $\Delta = \{\partial_x, \partial_y\}$  consider the regular chains  $S_1 = \{u_{1,0}\}$  and  $S_2 = \{u_{2,0}, u_{1,1}\}$  in  $\mathbb{C}\{u\}$ . Then  $\mathcal{S}(S_2) \subseteq \mathcal{S}(S_1)$ . The strength and first Cartan character are one and the second Cartan character and differential dimension are zero for both

ideals (in any order high enough), i.e., these values are the same for both ideals. However,  $\mathcal{J}(S_2) \neq \mathcal{J}(S_1)$ , as  $\omega_{\mathcal{J}(S_1)}(\ell) = \ell + 1 \neq \ell + 2 = \omega_{\mathcal{J}(S_2)}(\ell)$ .

In the last example, the differential dimension polynomial proves that a symbolic differential equation solver does not find all solutions.

*Example 5.4* Let  $U = \{u\}$  and  $\Delta = \{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\}$ . The viscous BURGERS' equation  $b = u_{0,2} - u_{1,0} - 2u_{0,1} \cdot u_{0,0}$  has the differential dimension polynomial  $2\ell + 1$ . MAPLE's `pdsolve` [19] finds the set

$$T := \left\{ c_1 \tanh(c_1 x + c_2 t + c_3) - \frac{c_2}{2c_1} \mid c_1, c_2, c_3 \in \mathbb{C}, c_1 \neq 0 \right\}$$

of solutions, which only depends on three parameters. The differential dimension polynomial shows that the set of solutions is infinite dimensional, and hence  $T$  is only a small subset of all solutions.

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