9 Time-Dependent Problems

9.1 Introduction

We now give a brief introduction to time-dependent problems through the equations of elastodynamics for *infinitesimal deformations*

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \rho_o \frac{d^2\boldsymbol{u}}{dt^2} = \rho_o \frac{d\boldsymbol{v}}{dt}, \tag{9.1}$$

where $\nabla = \nabla_X$ and $\frac{d}{dt} = \frac{\partial}{\partial t}$ (see Appendix B).

9.2 Generic Time Stepping

In order to motivate the time-stepping process, we first start with the dynamics of single point mass under the action of a force $\boldsymbol{\Psi}$. The equation of motion is given by (Newton's Law)

$$m\dot{\boldsymbol{v}} = \boldsymbol{\Psi}, \tag{9.2}$$

where $\boldsymbol{\Psi}$ is the total force applied to the particle. Expanding the velocity in a Taylor series about $t + \theta \Delta t$, where $0 \leq \theta \leq 1$, for $\boldsymbol{v}(t + \Delta t)$, we obtain

$$\boldsymbol{v}(t + \Delta t) = \boldsymbol{v}(t + \theta \Delta t) + \frac{d\boldsymbol{v}}{dt}|_{t+\theta \Delta t}(1-\theta)\Delta t + \frac{1}{2}\frac{d^2\boldsymbol{v}}{dt^2}|_{t+\theta \Delta t}(1-\theta)^2(\Delta t)^2 + \mathcal{O}(\Delta t)^3 \tag{9.3}$$

and for $\boldsymbol{v}(t)$, we obtain

$$\boldsymbol{v}(t) = \boldsymbol{v}(t + \theta \Delta t) - \frac{d\boldsymbol{v}}{dt}|_{t+\theta \Delta t}\theta \Delta t + \frac{1}{2}\frac{d^2\boldsymbol{v}}{dt^2}|_{t+\theta \Delta t}\theta^2(\Delta t)^2 + \mathcal{O}(\Delta t)^3. \tag{9.4}$$

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Subtracting the two expressions yields

$$\frac{d\boldsymbol{v}}{dt}|_{t+\theta\Delta t} = \frac{\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t)}{\Delta t} + \hat{\mathcal{O}}(\Delta t), \tag{9.5}$$

where $\hat{\mathcal{O}}(\Delta t) = \mathcal{O}(\Delta t)^2$, when $\theta = \frac{1}{2}$, otherwise $\hat{\mathcal{O}}(\Delta t) = \mathcal{O}(\Delta t)$. Thus, inserting this into Eq. 9.2 yields

$$\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}(t) + \frac{\Delta t}{m}\boldsymbol{\Psi}(t+\theta\Delta t) + \hat{\mathcal{O}}(\Delta t)^2. \tag{9.6}$$

Note that a weighted sum of Eqs. 9.3 and 9.4 yields

$$\boldsymbol{v}(t+\theta\Delta t) = \theta\boldsymbol{v}(t+\Delta t) + (1-\theta)\boldsymbol{v}(t) + \mathcal{O}(\Delta t)^2, \tag{9.7}$$

which will be useful shortly. Now expanding the position of the mass in a Taylor series about $t+\theta\Delta t$ we obtain

$$\boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t+\theta\Delta t) + \frac{d\boldsymbol{u}}{dt}|_{t+\theta\Delta t}(1-\theta)\Delta t + \frac{1}{2}\frac{d^2\boldsymbol{u}}{dt^2}|_{t+\theta\Delta t}(1-\theta)^2(\Delta t)^2 + \mathcal{O}(\Delta t)^3 \tag{9.8}$$

and

$$\boldsymbol{u}(t) = \boldsymbol{u}(t+\theta\Delta t) - \frac{d\boldsymbol{u}}{dt}|_{t+\theta\Delta t}\theta\Delta t + \frac{1}{2}\frac{d^2\boldsymbol{u}}{dt^2}|_{t+\theta\Delta t}\theta^2(\Delta t)^2 + \mathcal{O}(\Delta t)^3. \tag{9.9}$$

Subtracting the two expressions yields

$$\frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t} = \boldsymbol{v}(t+\theta\Delta t) + \hat{\mathcal{O}}(\Delta t). \tag{9.10}$$

Inserting Eq. 9.7 yields

$$\boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + (\theta\boldsymbol{v}(t+\Delta t) + (1-\theta)\boldsymbol{v}(t))\,\Delta t + \hat{\mathcal{O}}(\Delta t)^2, \tag{9.11}$$

and using Eq. 9.6 yields

$$\boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + \boldsymbol{v}(t)\Delta t + \frac{\theta(\Delta t)^2}{m}\boldsymbol{\Psi}(t+\theta\Delta t) + \hat{\mathcal{O}}(\Delta t)^2. \tag{9.12}$$

The term $\boldsymbol{\Psi}(t+\theta\Delta t)$ can be handled in a simple way:

$$\boldsymbol{\Psi}(t+\theta\Delta t) \approx \theta\boldsymbol{\Psi}(t+\Delta t) + (1-\theta)\boldsymbol{\Psi}(t). \tag{9.13}$$

We note that

- When $\theta = 1$, then this is the (implicit) Backward Euler scheme, which is very stable (very dissipative) and $\hat{\mathcal{O}}(\Delta t)^2 = \mathcal{O}(\Delta t)^2$ locally in time,
- When $\theta = 0$, then this is the (explicit) Forward Euler scheme, which is conditionally stable and $\hat{\mathcal{O}}(\Delta t)^2 = \mathcal{O}(\Delta t)^2$ locally in time,
- When $\theta = 0.5$, then this is the (implicit) "Midpoint" scheme, which is stable and $\hat{\mathcal{O}}(\Delta t)^2 = \mathcal{O}(\Delta t)^3$ locally in time.

In summary, we have for the velocity[1]

$$\boldsymbol{v}(t + \Delta t) = \boldsymbol{v}(t) + \frac{\Delta t}{m} (\theta \boldsymbol{\Psi}(t + \Delta t) + (1 - \theta)\boldsymbol{\Psi}(t)) \tag{9.14}$$

and for the position

$$\begin{aligned} \boldsymbol{u}(t + \Delta t) &= \boldsymbol{u}(t) + \boldsymbol{v}(t + \theta \Delta t)\Delta t \\ &= \boldsymbol{u}(t) + (\theta \boldsymbol{v}(t + \Delta t) + (1 - \theta) bf v(t)) \Delta t, \end{aligned} \tag{9.15}$$

or in terms of $\boldsymbol{\Psi}$

$$\boldsymbol{u}(t + \Delta t) = \boldsymbol{u}(t) + \boldsymbol{v}(t)\Delta t + \frac{\theta(\Delta t)^2}{m} (\theta \boldsymbol{\Psi}(t + \Delta t) + (1 - \theta)\boldsymbol{\Psi}(t)) . \tag{9.16}$$

9.3 Application to the Continuum Formulation

Now consider the continuum analogue to "$m\dot{\boldsymbol{v}}$"

$$\rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho_o \frac{\partial \boldsymbol{v}}{\partial t} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} \stackrel{\text{def}}{=} \boldsymbol{\Psi} \tag{9.17}$$

and thus

$$\rho_o \boldsymbol{v}(t + \Delta t) = \rho_o \boldsymbol{v}(t) + \Delta t \,(\theta \boldsymbol{\Psi}(t + \Delta t) + (1 - \theta)\boldsymbol{\Psi}(t)) . \tag{9.18}$$

Multiplying Eq. 9.18 by a test function and integrating yields

$$\begin{aligned} \int_\Omega \boldsymbol{\nu} \cdot \rho_o \boldsymbol{v}(t + \Delta t)\, d\Omega &= \int_\Omega \boldsymbol{\nu} \cdot \rho_o \boldsymbol{v}(t)\, d\Omega \\ &\quad + \Delta t \int_\Omega \boldsymbol{\nu} \cdot (\theta \boldsymbol{\Psi}(t + \Delta t) + (1 - \theta)\boldsymbol{\Psi}(t)) \; d\Omega, \end{aligned} \tag{9.19}$$

[1] In order to streamline the notation, we drop the cumbersome $\mathcal{O}(\Delta t)$-type terms.

and using Gauss's divergence theorem and enforcing $\boldsymbol{v} = \boldsymbol{0}$ on Γ_u yields (using a streamlined time-step superscript counter notation of L, where $t = L\Delta t$ and $t + \Delta t = (L+1)\Delta t$)

$$\int_{\Omega} \boldsymbol{v} \cdot \rho_o \boldsymbol{v}^{L+1}\, d\Omega = \int_{\Omega} \boldsymbol{v} \cdot \rho_o \boldsymbol{v}^{L}\, d\Omega + \Delta t\theta \left(-\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma}\, d\Omega + \int_{\Gamma_t} \boldsymbol{v} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, dA + \int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega \right)^{L+1} + \Delta t(1-\theta) \left(-\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma}\, d\Omega + \int_{\Gamma_t} \boldsymbol{v} \cdot \boldsymbol{t}^*\, dA + \int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega \right)^{L}. \tag{9.20}$$

As in the previous chapter on linearized three-dimensional elasticity, we assume

$$\{\boldsymbol{u}^h\} = [\boldsymbol{\Phi}]\{\boldsymbol{a}\} \quad and \quad \{\boldsymbol{v}^h\} = [\boldsymbol{\Phi}]\{\boldsymbol{b}\} \quad and \quad \{\boldsymbol{v}^h\} = [\boldsymbol{\Phi}]\{\dot{\boldsymbol{a}}\}, \tag{9.21}$$

which yields, in terms of matrices and vectors

$$\{\boldsymbol{b}\}^T[M]\{\dot{\boldsymbol{a}}\}^{L+1} = \{\boldsymbol{b}\}^T[M]\{\dot{\boldsymbol{a}}\}^{L} - \Delta t\theta \{\boldsymbol{b}\}^T \left(-[K]\{\boldsymbol{a}\}^{L+1} + \{\boldsymbol{R}_f\}^{L+1} + \{\boldsymbol{R}_t\}^{L+1} \right) - \{\boldsymbol{b}\}^T \Delta t(1-\theta) \left(-[K]\{\boldsymbol{a}\}^{L} + \{\boldsymbol{R}_f\}^{L} + \{\boldsymbol{R}_t\}^{L} \right). \tag{9.22}$$

where $[M] = \int_{\Omega} \rho_o [\Phi]^T [\Phi]\, d\Omega$, and $[K]$,$\{\boldsymbol{R}_f\}$, and $\{\boldsymbol{R}_t\}$ are as defined in the previous chapters on elastostatics. Note that $\{\boldsymbol{R}_f\}^L$ and $\{\boldsymbol{R}_t\}^L$ are known values from the previous time-step. Since $\{\boldsymbol{b}\}^T$ is arbitrary

$$[M]\{\dot{\boldsymbol{a}}\}^{L+1} = [M]\{\dot{\boldsymbol{a}}\}^{L} + (\Delta t\theta) \left(-[K]\{\boldsymbol{a}\}^{L+1} + \{\boldsymbol{R}_f\}^{L+1} + \{\boldsymbol{R}_t\}^{L+1} \right) + \Delta t(1-\theta) \left(-[K]\{\boldsymbol{a}\}^{L} + \{\boldsymbol{R}_f\}^{L} + \{\boldsymbol{R}_t\}^{L} \right). \tag{9.23}$$

One should augment this with the approximation for the discrete displacement:

$$\{\boldsymbol{a}\}^{L+1} = \{\boldsymbol{a}\}^{L} + \Delta t \left(\theta \{\dot{\boldsymbol{a}}\}^{L+1} + (1-\theta)\{\dot{\boldsymbol{a}}\}^{L} \right). \tag{9.24}$$

For a purely implicit (Backward Euler) method $\theta = 1$

$$\left([M]\{\dot{\boldsymbol{a}}\}^{L+1} + \Delta t[K]\{\boldsymbol{a}\}^{L+1} \right) = [M]\{\dot{\boldsymbol{a}}\}^{L} + \Delta t \left(\{\boldsymbol{R}_t\}^{L+1} + \{\boldsymbol{R}_f\}^{L+1} \right), \tag{9.25}$$

augmented with

$$\{\boldsymbol{a}\}^{L+1} = \{\boldsymbol{a}\}^{L} + \Delta t\{\dot{\boldsymbol{a}}\}^{L+1}, \tag{9.26}$$

which requires one to solve a system of algebraic equations, while for an explicit (Forward Euler) method $\theta = 0$ with usually $[M]$ is approximated by an easy-to-invert matrix, such as a diagonal matrix, $[M] \approx M[\mathbf{1}]$, to make the matrix inversion easy, yielding:

$$\{\dot{\boldsymbol{a}}\}^{L+1} = \{\dot{\boldsymbol{a}}\}^{L} + \Delta t[M]^{-1} \left(-[K]\{\boldsymbol{a}\}^{L} + \{\boldsymbol{R}_f\}^{L} + \{\boldsymbol{R}_t\}^{L} \right), \tag{9.27}$$

augmented with

$$\{\boldsymbol{a}\}^{L+1} = \{\boldsymbol{a}\}^{L} + \Delta t\{\dot{\boldsymbol{a}}\}^{L}. \tag{9.28}$$

There is an enormous number of time-stepping schemes. For general time-stepping, we refer the reader to the seminal texts of Hairer et al. [1,2]. In the finite element context, we refer the reader to Bathe [3], Becker et al. [4], Hughes [5], and Zienkiewicz and Taylor [6].

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