Weak Formulations in Three Dimensions

6

6.1 Introduction

Albeit a bit repetitive, we follow similar constructions as done in the one-dimensional analysis of the preceding chapters. This allows readers a chance to contrast and compare the differences between one-dimensional and three-dimensional formulations. To derive a direct weak form for a body, we take the balance of linear momentum $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}$ (denoting the strong form) and form a scalar product with an arbitrary smooth vector-valued function $\boldsymbol{\nu}$, and integrate over the body (Fig. 6.1),

$$\int_{\Omega} (\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}) \cdot \boldsymbol{\nu} \, d\Omega = \int_{\Omega} \boldsymbol{r} \cdot \boldsymbol{\nu} \, d\Omega = 0, \tag{6.1}$$

where $\boldsymbol{r}$ is the residual and $\boldsymbol{\nu}$ is a test function. If we were to add a condition that we do this for all possible test functions ($\forall \boldsymbol{\nu}$), Eq. 6.1 implies $\boldsymbol{r} = \mathbf{0}$. Therefore, if every possible test function was considered, then

$$\boldsymbol{r} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \tag{6.2}$$

on any finite region in Ω. Consequently, the weak and strong statements would be equivalent provided the true solution is smooth enough to have a strong solution. Clearly, $\boldsymbol{r}$ can never be nonzero over any finite region in the body, because the test function will locate them. Using the product rule of differentiation,

$$\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) = (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu} + \nabla \boldsymbol{\nu} : \boldsymbol{\sigma} \tag{6.3}$$

leads to, $\forall \boldsymbol{\nu}$

$$\int_{\Omega} (\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) - \nabla \boldsymbol{\nu} : \boldsymbol{\sigma}) \, d\Omega + \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\nu} \, d\Omega = 0, \tag{6.4}$$

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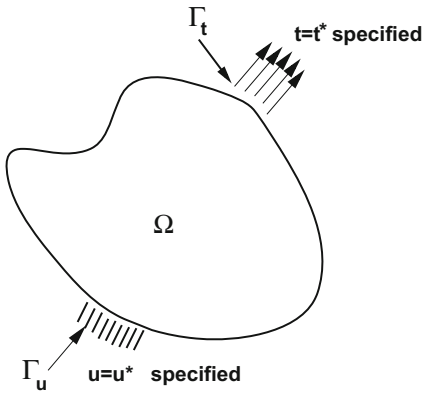

Fig. 6.1 A three-dimensional body

where we choose the $\boldsymbol{\nu}$ from an admissible set, to be discussed momentarily. Using the divergence theorem leads to, $\forall \boldsymbol{\nu}$,

$$\int_{\Omega} \nabla \boldsymbol{\nu} : \boldsymbol{\sigma} \, d\Omega = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\nu} \, d\Omega + \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{\nu} \, dA, \tag{6.5}$$

which, since the traction $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$, leads to

$$\int_{\Omega} \nabla \boldsymbol{\nu} : \boldsymbol{\sigma} \, d\Omega = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\nu} \, d\Omega + \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{\nu} \, dA. \tag{6.6}$$

If we decide to restrict our choices of $\boldsymbol{\nu}$'s to those such that $\boldsymbol{\nu}|_{\Gamma_u} = \mathbf{0}$, we have, where $\boldsymbol{u}^*$ is the applied boundary displacement on Γ_u, for infinitesimal strain linear elasticity

$$\boxed{\begin{array}{l} \text{Find } \boldsymbol{u}, \boldsymbol{u}|_{\Gamma_u} = \boldsymbol{u}^*, \quad \text{such that } \forall \boldsymbol{\nu}, \boldsymbol{\nu}|_{\Gamma_u} = \mathbf{0} \\ \underbrace{\displaystyle\int_{\Omega} \nabla \boldsymbol{\nu} : \mathbb{E} : \nabla \boldsymbol{u} \, d\Omega}_{\stackrel{\text{def}}{=} \mathcal{B}(\boldsymbol{u}, \boldsymbol{\nu})} = \underbrace{\displaystyle\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\nu} \, d\Omega + \int_{\Gamma_t} \boldsymbol{t}^* \cdot \boldsymbol{\nu} \, dA}_{\stackrel{\text{def}}{=} \mathcal{F}(\boldsymbol{\nu})}, \end{array}} \tag{6.7}$$

where $\boldsymbol{t} = \boldsymbol{t}^*$ on Γ_t. As in the one-dimensional formulation, this is called a "weak" form because it does not require the differentiability of the stress $\boldsymbol{\sigma}$. In other words, the differentiability requirements have been *weakened*. It is clear that we are able to consider problems with quite irregular solutions. We emphasize that if we test the solution with all possible test functions of sufficient smoothness, then the weak solution is equivalent to the strong solution. *Futhermore, that provided the true solution is smooth enough, the weak and strong forms are equivalent, which can be seen by the above constructive derivation.*

6.2 Hilbertian Sobolev Spaces

As in one dimension, a key question is the selection of the sets of functions in the weak form. Somewhat naively, the answer is simple, the integrals must remain finite. Therefore the following restrictions hold ($\forall \boldsymbol{\nu}$), $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\nu}\, d\Omega < \infty$, $\int_{\partial\Omega} \boldsymbol{t}^* \cdot \boldsymbol{\nu}\, dA < \infty$ and $\int_\Omega \nabla\boldsymbol{\nu} : \boldsymbol{\sigma}\, d\Omega < \infty$, and govern the selection of the approximation spaces. These relations simply mean that the functions must be square integrable. In order to make precise statements one must have a method of "book keeping." Such a system is to employ so-called Hilbertian Sobolev spaces. We recall that a norm has three main characteristics for any functions $\boldsymbol{u}$ and $\boldsymbol{\nu}$ such that $||\boldsymbol{u}|| < \infty$ and $||\boldsymbol{\nu}|| < \infty$ are

- (1) $||\boldsymbol{u}|| > 0$, $||\boldsymbol{u}|| = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$,
- (2) $||\boldsymbol{u} + \boldsymbol{\nu}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{\nu}||$ and
- (3) $||\alpha\boldsymbol{u}|| = |\alpha|||\boldsymbol{u}||$,

where α is a scalar constant. Certain types of norms, so-called Hilbert space norms, are frequently used in solid mechanics. Following standard notation, we denote $H^1(\Omega)$ as the usual space of scalar functions with generalized partial derivatives of order ≤ 1 in $L^2(\Omega)$, i.e., square integrable, in other words $u \in H^1(\Omega)$ if

$$||u||^2_{H^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_j}\, d\Omega + \int_\Omega u u\, d\Omega < \infty. \tag{6.8}$$

Similarly, we define $\boldsymbol{H}^1(\Omega) \stackrel{\text{def}}{=} [H^1(\Omega)]^3$ as the space of vector-valued functions whose components are in $H^1(\Omega)$, i.e.,

$$\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \text{ if } ||\boldsymbol{u}||^2_{\boldsymbol{H}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}\, d\Omega + \int_\Omega \sum_{i=1}^{3} u_i u_i\, d\Omega < \infty, \tag{6.9}$$

and we denote $\boldsymbol{L}^2(\Omega) \stackrel{\text{def}}{=} [L^2(\Omega)]^3$. Using these definitions, a complete boundary value problem can be written as follows. The data (loads) are assumed to be such that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{t}^* \in \boldsymbol{L}^2(\Gamma_t)$, but less smooth data can be considered without complications. Implicitly we require that $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega)$ without continually making such references. Therefore, in summary we assume that our solutions obey these restrictions, leading to the following infinitesimal strain linear elasticity weak form:

$$\boxed{\begin{array}{l} \text{Find } \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \boldsymbol{u}|_{\Gamma_u} = \boldsymbol{u}^*, \quad \text{such that } \forall \boldsymbol{\nu} \in \boldsymbol{H}^1(\Omega), \boldsymbol{\nu}|_{\Gamma_u} = \boldsymbol{0} \\ \\ \displaystyle\int_\Omega \nabla\boldsymbol{\nu} : \mathbb{E} : \nabla\boldsymbol{u}\, d\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\nu}\, d\Omega + \int_{\Gamma_t} \boldsymbol{t}^* \cdot \boldsymbol{\nu}\, dA. \end{array}} \tag{6.10}$$

We note that if the data in (6.10) are smooth and if (6.10) possesses a solution $\boldsymbol{u}$ that is sufficiently regular, then $\boldsymbol{u}$ is the solution of the classical linear elastostatics problem in strong form:

$$\boxed{\begin{aligned} &\nabla \cdot (\mathbb{E} : \nabla \boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0}, \qquad \forall \boldsymbol{x} \in \Omega, \\ &\boldsymbol{u} = \boldsymbol{u}^*, \qquad \forall \boldsymbol{x} \in \Gamma_u, \\ &\boldsymbol{\sigma} \cdot \boldsymbol{n} = (\mathbb{E} : \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t} = \boldsymbol{t}^*, \qquad \forall \boldsymbol{x} \in \Gamma_t. \end{aligned}} \tag{6.11}$$

6.3 The Principle of Minimum Potential Energy

Repeating the procedure that we performed for one-dimensional formulations earlier in the monograph, we have

$$\begin{aligned} ||\boldsymbol{u} - \boldsymbol{w}||^2_{E(\Omega)} &= \mathcal{B}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u} - \boldsymbol{w}) \\ &= \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - 2\mathcal{B}(\boldsymbol{u}, \boldsymbol{w}) \\ &= \mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) - 2\mathcal{B}(\boldsymbol{u}, \boldsymbol{w}) + 2\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) \\ &= \mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) - 2\mathcal{B}(\boldsymbol{u}, \boldsymbol{w} - \boldsymbol{u}) \\ &= \mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) - 2\mathcal{F}(\boldsymbol{w} - \boldsymbol{u}) \\ &= \mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - 2\mathcal{F}(\boldsymbol{w}) - (\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) - 2\mathcal{F}(\boldsymbol{u})) \\ &= 2\mathcal{J}(\boldsymbol{w}) - 2\mathcal{J}(\boldsymbol{u}), \end{aligned} \tag{6.12}$$

where similar to the one-dimensional case, we define the *elastic potential* as

$$\mathcal{J}(\boldsymbol{w}) \stackrel{\text{def}}{=} \frac{1}{2}\mathcal{B}(\boldsymbol{w}, \boldsymbol{w}) - \mathcal{F}(\boldsymbol{w}) = \frac{1}{2}\int_\Omega \nabla \boldsymbol{w} : \mathbb{E} : \nabla \boldsymbol{w}\, d\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t}^* \cdot \boldsymbol{w}\, dA. \tag{6.13}$$

This implies

$$0 \leq ||\boldsymbol{u} - \boldsymbol{w}||^2_{E(\Omega)} = 2(\mathcal{J}(\boldsymbol{w}) - \mathcal{J}(\boldsymbol{u})) \;\; or \;\; \mathcal{J}(\boldsymbol{u}) \leq \mathcal{J}(\boldsymbol{w}), \tag{6.14}$$

where Eq. 6.14 is known as the Principle of Minimum Potential Energy (PMPE). In other words, the true solution possesses the minimum potential. As in one dimension, the minimum property of the exact solution can be proven by an alternative technique. Let us construct a potential function, for a deviation away from the exact solution $\boldsymbol{u}$, denoted $\boldsymbol{u} + \lambda\boldsymbol{\nu}$, where λ is a scalar and $\boldsymbol{\nu}$ is any admissible variation (test function)

$$\mathcal{J}(\boldsymbol{u} + \lambda\boldsymbol{\nu}) = \frac{1}{2}\int_\Omega \nabla(\boldsymbol{u} + \lambda\boldsymbol{\nu}) : \mathbb{E} : \nabla(\boldsymbol{u} + \lambda\boldsymbol{\nu})\, d\Omega - \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} + \lambda\boldsymbol{\nu})\, d\Omega - \int_{\Gamma_t} \boldsymbol{t}^* \cdot (\boldsymbol{u} + \lambda\boldsymbol{\nu})\, dA. \tag{6.15}$$

If we differentiate with respect to λ,

$$\frac{\partial \mathcal{J}(\boldsymbol{u}+\lambda\boldsymbol{\nu})}{\partial\lambda} = \int_{\Omega} \nabla\boldsymbol{\nu} : \mathbb{E} : \nabla(\boldsymbol{u}+\lambda\boldsymbol{\nu})\, d\Omega - \int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{\nu}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t}^*\cdot\boldsymbol{\nu}\, dA, \tag{6.16}$$

and set $\lambda = 0$ (because we know that the exact solution is for $\lambda = 0$), we have

$$\frac{\partial \mathcal{J}(\boldsymbol{u}+\lambda\boldsymbol{\nu})}{\partial\lambda}|_{\lambda=0} = \int_{\Omega} \nabla\boldsymbol{\nu} : \mathbb{E} : \nabla\boldsymbol{u}\, d\Omega - \int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{\nu}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t}^*\cdot\boldsymbol{\nu}\, dA = 0. \tag{6.17}$$

Clearly, the minimizer of the potential is the solution to the field equations, since it produces the weak form as a result. This is a minimum since

$$\frac{\partial^2 \mathcal{J}(\boldsymbol{u}+\lambda\boldsymbol{\nu})}{\partial\lambda^2}|_{\lambda=0} = \int_{\Omega} \nabla\boldsymbol{\nu} : \mathbb{E} : \nabla\boldsymbol{\nu}\, d\Omega \geq 0. \tag{6.18}$$

It is important to note that the weak form, derived earlier, requires no such potential, and thus is a more general approach than a minimum principle. Thus, in the hyperelastic case, the weak formulation can be considered as a minimization of a potential energy function. This is sometimes referred to as the Rayleigh–Ritz method.

6.4 Complementary Principles

There exist another set of weak forms and minimum principles called complementary principles. Starting with $\nabla\cdot\boldsymbol{\tau} = \boldsymbol{0}$, $\boldsymbol{\tau}\cdot\boldsymbol{n}|_{\Gamma_t} = \boldsymbol{0}$, multiplying by the solution $\boldsymbol{u}$ leads to

$$\int_{\Omega} \nabla\cdot\boldsymbol{\tau}\cdot\boldsymbol{u}\, d\Omega = 0 = \int_{\Omega} \nabla\cdot(\boldsymbol{\tau}\cdot\boldsymbol{u})\, d\Omega - \int_{\Omega} \boldsymbol{\tau} : \nabla\boldsymbol{u}\, d\Omega. \tag{6.19}$$

Using the divergence theorem yields

$$\boxed{\begin{array}{l}\text{Find } \boldsymbol{\sigma},\ \nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\boldsymbol{0},\ \boldsymbol{\sigma}\cdot\boldsymbol{n}|_{\Gamma_t}=\boldsymbol{t} \ \text{ such that} \\ \underbrace{\int_{\Omega} \boldsymbol{\tau} : \mathbb{E}^{-1} : \boldsymbol{\sigma}\, d\Omega}_{\stackrel{\text{def}}{=}\mathcal{A}(\boldsymbol{\sigma},\boldsymbol{\tau})} = \underbrace{\int_{\Gamma_u} \boldsymbol{\tau}\cdot\boldsymbol{n}\cdot\boldsymbol{u}^*\, dA}_{\stackrel{\text{def}}{=}\mathcal{G}(\boldsymbol{\tau})} \qquad \forall\boldsymbol{\tau},\ \nabla\cdot\boldsymbol{\tau}=\boldsymbol{0},\ \boldsymbol{\tau}\cdot\boldsymbol{n}|_{\Gamma_t}=\boldsymbol{0}.\end{array}} \tag{6.20}$$

This is called the complementary form of Eq. 6.7. Similar restrictions are placed on the trial and test fields to force the integrals to make sense, i.e., to be finite. Similar boundedness restrictions control the choice of admissible complementary functions. In other words we assume that the solutions produce finite energy. *Despite*

the apparent simplicity of such principles they are rarely used in practical computations, directly in this form, because of the fact that it is somewhat difficult to find approximate functions, $\boldsymbol{\sigma}$, that satisfy $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$. However, in closing, we provide some theoretical results. As in the primal case, a similar process is repeated using the complementary weak form. We define a complementary norm

$$0 \leq ||\boldsymbol{\sigma} - \boldsymbol{\gamma}||^2_{E^{-1}(\Omega)} \stackrel{\text{def}}{=} \int_{\Omega} (\boldsymbol{\sigma} - \boldsymbol{\gamma}) : \boldsymbol{I\!E}^{-1} : (\boldsymbol{\sigma} - \boldsymbol{\gamma})\, d\Omega = \mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\gamma}, \boldsymbol{\sigma} - \boldsymbol{\gamma}). \tag{6.21}$$

Again, by direct manipulation, we have

$$\begin{aligned} ||\boldsymbol{\sigma} - \boldsymbol{\gamma}||^2_{E^{-1}(\Omega)} &= \mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\gamma}, \boldsymbol{\sigma} - \boldsymbol{\gamma}) \\ &= \mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) + \mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - 2\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\gamma}) \\ &= \mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - 2\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\gamma}) + 2\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \\ &= \mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - 2\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\gamma} - \boldsymbol{\sigma}) \\ &= \mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - 2\mathcal{G}(\boldsymbol{\gamma} - \boldsymbol{\sigma}) \\ &= \mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - 2\mathcal{G}(\boldsymbol{\gamma}) - (\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - 2\mathcal{G}(\boldsymbol{\sigma})) \\ &= 2\mathcal{K}(\boldsymbol{\gamma}) - 2\mathcal{K}(\boldsymbol{\sigma}), \end{aligned} \tag{6.22}$$

where we define $\mathcal{K}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \frac{1}{2}\mathcal{A}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) - \mathcal{G}(\boldsymbol{\gamma}) = \frac{1}{2}\int_{\Omega} \boldsymbol{\gamma} : \boldsymbol{I\!E}^{-1} : \boldsymbol{\gamma}\, d\Omega - \int_{\Gamma_u} \boldsymbol{\gamma} \cdot \boldsymbol{n} \cdot \boldsymbol{u}^*\, dA$. Therefore,

$$||\boldsymbol{\sigma} - \boldsymbol{\gamma}||^2_{E^{-1}(\Omega)} = 2(\mathcal{K}(\boldsymbol{\gamma}) - \mathcal{K}(\boldsymbol{\sigma})) \;\; or \;\; \mathcal{K}(\boldsymbol{\sigma}) \leq \mathcal{K}(\boldsymbol{\gamma}), \tag{6.23}$$

which is the Principle of Minimum Complementary Potential Energy (PMCPE). By directly adding together the potential energy and the complementary energy we obtain an equation of balance:

$$\begin{aligned} \mathcal{J}(\boldsymbol{u}) + \mathcal{K}(\boldsymbol{\sigma}) &= \frac{1}{2}\int_{\Omega} \nabla \boldsymbol{u} : \boldsymbol{I\!E} : \nabla \boldsymbol{u}\, d\Omega - \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{u}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t}^* \cdot \boldsymbol{u}\, dA \\ &\quad + \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{I\!E}^{-1} : \boldsymbol{\sigma}\, d\Omega - \int_{\Gamma_u} \underbrace{\boldsymbol{t} \cdot \boldsymbol{u}}_{(\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{u}^*}\, dA \\ &= 0. \end{aligned} \tag{6.24}$$

Remark: Basically, the three-dimensional and one-dimensional formulations are, formally speaking, virtually identical in structure.