The Absence of Discrete Mathematics in Primary and Secondary Education in the United States… and Why that Is Counterproductive

Joseph G. Rosenstein

Abstract This chapter describes the opportunity that discrete mathematics provides for supporting reasoning, problem solving, and systematic thinking in the school mathematics curriculum and illustrates this opportunity by providing a set of discrete mathematics problems that begin "Find all… ." It also provides a year-by-year model for how discrete mathematics can be included in the primary and secondary curriculum. Finally, the article describes some of the possible reasons why discrete mathematics was not included in the new national mathematics standards in the U.S., and why we consider these reasons misguided, in light of the opportunities provided when discrete mathematics is part of the curriculum.

Keywords Counting · Combinatorics · Graphs · Systematic listing
Divide and conquer · Reasoning · Problem solving · Standards
Common Core State Standards in Mathematics

The title of this article speaks of the absence of discrete mathematics in primary and secondary education in the United States. Why has this happened?

The *Common Core State Standards for Mathematics* (NGA Center and CCSSO 2010) that were developed in 2009 and adopted soon afterwards by almost all of the states in the United States essentially excludes discrete mathematics. More specifically, no mention is made of graphs (those with vertices and edges), no mention is made of systematic listing and counting (combinatorics) except as an adjunct to the probability standard in the 8th grade, no mention is made of modern issues involving fairness (including fair division, apportionment, and elections), recursion (including Fibonacci numbers), or codes and cryptography, and the word *pattern* barely occurs in the standards even though mathematics is often referred to

In the video at https://www.youtube.com/watch?v=ff29i_yPoZ0 the author discusses many of the issues raised in this article.

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as the science of patterns. Since these topics are not in the Common Core, they are not addressed in the assessment tools that are based on the Common Core. Since most teachers, schools, and districts are being judged on the performance of their students on assessment tools based on the Common Core, teachers, schools, and districts feel they cannot devote time or effort to topics that are extraneous to the Common Core, including topics in discrete mathematics. Thus discrete mathematics is now absent from primary and secondary education in the United States.

Was discrete mathematics present in U.S. education before the advent of the Common Core? While not universally present, nevertheless discrete mathematics topics were taught in many schools by many teachers, teachers were introduced to discrete mathematics topics in professional development activities, and these topics were the focus of several textbooks (e.g., DeBellis and Rosenstein 2005, Rosenstein 2014) and appeared in many standard textbook series (e.g., COMAP 2013; Hirsch et al. 2015). This was in large part as a consequence of the recommendation of discrete mathematics in the 1989 publication of *Curriculum and Evaluation Standards* by the National Council of Teachers of Mathematics, and recommendation for the integration of "the main topics of discrete mathematics" in the K-12 curriculum in its 2000 publication, *Principles and Standards for School Mathematics*, which notes that: "As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum."

Unfortunately, in the Common Core era discrete mathematics is hardly present in the U.S. school curriculum. The situation in the United States prior to the Common Core standards is described more thoroughly in an article entitled *Discrete mathematics in primary and secondary schools in the United States* by DeBellis and Rosenstein (2004).

1 Why Should Discrete Mathematics Be Included?—Reasoning and Problem Solving

We argue that school mathematics curricula should take seriously the idea that an important reason for studying mathematics is to understand reasoning and to learn how to solve problems: Discrete mathematics is an excellent vehicle to help students at all grade levels become the problem solvers and reasoners that we desire. Although the Common Core speaks positively about reasoning and problem solving, it ignores this important arena that is both accessible to all students at all grade levels and that can foster the desired reasoning and problem solving.

In the examples that follow, keep in mind that the content in these examples may not be significant in itself, but that the experience that students have with learning how to use systematic reasoning can be very significant in shaping their thinking in mathematics, and in other areas of human endeavor.

One class of problems for which systematic reasoning is critical consists of those problems that begin with the phrase "Find all … ." For example, if you give young children cut-out shirts and trousers of several different colors and ask them to “find all” outfits that can be made, they will typically proceed in a random fashion. When they show you all the outfits they have made, you will find that some outfits are omitted and that some are duplicated.

Discrete mathematics provides a way of learning to answer such questions systematically. Students should construct a chart (see Table 1) where the shirts are listed at the top and the trousers at the left, and the outfits can be located in the cells of the chart. Actually, before constructing a chart with words in the cells, they should put the outfits themselves in the cells of a “chart.”

Subsequently, they should use a tree diagram (see Fig. 1), where the first branching is for the shirts and the second branching is for the trousers.

Ultimately, students should use the Multiplication Principle of Counting, which in this context says that “if there are four ways of selecting a shirt and, in each case, there are three ways of selecting trousers (independently of which shirt is selected) then there are 4×3 or 12 ways of selecting both—that is, there are 12 outfits altogether.

Note that the tree diagram also provides a way of systematically listing the 12 outfits; if you follow each of the branches of the tree, you end up with the vertical list of all 12 outfits on the right (see Fig. 2).

Similarly, if you ask secondary students to “find all” factors of 200, they will most likely proceed randomly. The same strategies presented above can also be used in this situation. Consider using a chart, as in Table 2. Since $200 = 8 \times 25 = 2^3 \times 5^2$, any factor of 200 is a product of a power of 2 from 0 to 3—that’s four possibilities—and a power of 5 from 0 to 2—that’s three possibilities, so 200 has 4×3 or 12 factors. Thus students can arrive at a systematic way of finding and of listing all the factors of 200, and can see that this problem is essentially the same as the previous problem of finding all the outfits.

In learning how to solve both problems above, the focus is moving from random behavior to systematic behavior.

Here is another “find all” example. Find all graphs that have exactly four vertices. Rather than proceed randomly, students can learn to break a difficult problem into cases, a problem-solving strategy that might be called *divide and conquer*. This is not an obvious strategy; students have to learn when and how it can be used. In this example, the cases might involve the number of edges that the graph has, and the students might solve the problem by constructing all four-vertex graphs with no

Table 1 Using a chart to solve a “Find all …” outfits problem

	Red shirt	Striped shirt	Yellow shirt	Green shirt
Black trousers	Red shirt & black trousers	Striped shirt & black trousers	Yellow shirt & black trousers	Green shirt & black trousers
White trousers	Red shirt & white trousers	Striped shirt & white trousers	Yellow shirt & white trousers	Green shirt & white trousers
Grey trousers	Red shirt & grey trousers	Striped shirt & grey trousers	Yellow shirt & grey trousers	Green shirt & grey trousers

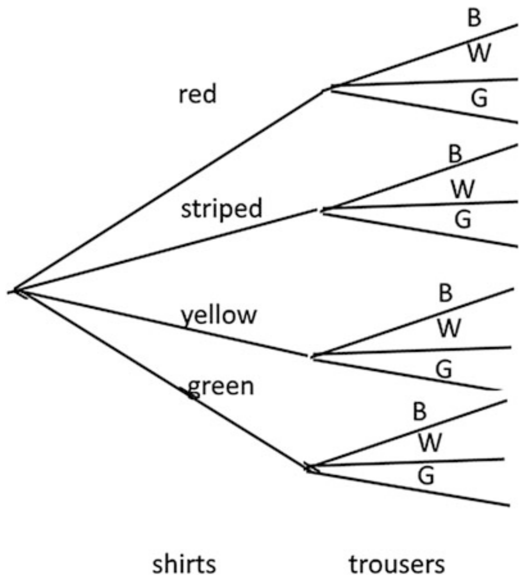


Fig. 1 Using a tree diagram to solve a "Find all …" outfits problem

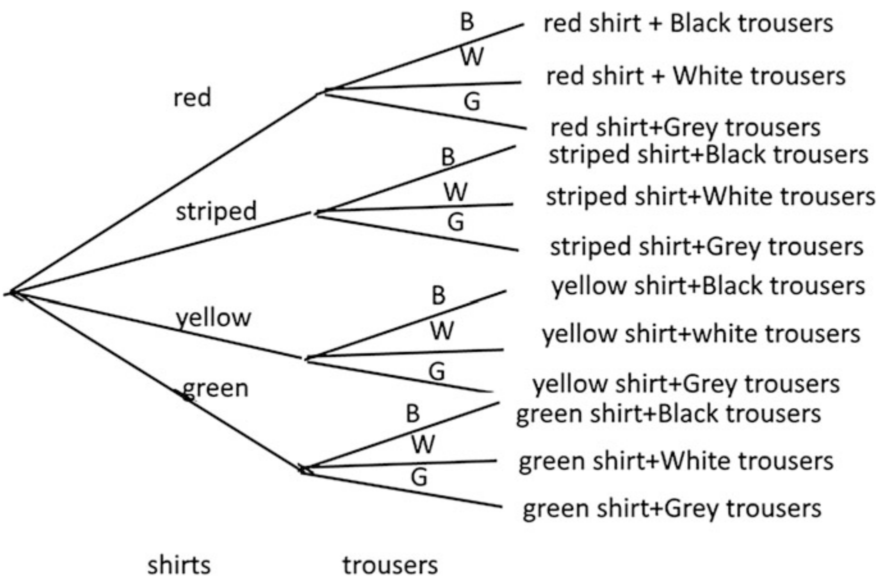


Fig. 2 A tree diagram provides a way of systematically listing all possibilities

Table 2 Using a chart to systematically find all factors of 200

	2^0	2^1	2^2	2^3
$\mathbf{5^0}$	$2^0 \times 5^0 = 1$	$2^1 \times 5^0 = 2$	$2^2 \times 5^0 = 4$	$2^3 \times 5^0 = 8$
$\mathbf{5^1}$	$2^0 \times 5^1 = 5$	$2^1 \times 5^1 = 10$	$2^2 \times 5^1 = 20$	$2^3 \times 5^1 = 40$
$\mathbf{5^2}$	$2^0 \times 5^2 = 25$	$2^1 \times 5^2 = 50$	$2^2 \times 5^2 = 100$	$2^3 \times 5^2 = 200$

edges, then all four-vertex graphs with one edge, then all four-vertex graphs with two edges, and so on. The solution to the initial problem is then obtained using the Addition Principle of Counting, which says that if you split the items to be counted into groups that have no elements in common, then the total number of items is the sum of the numbers of items in each group.

So, how many different graphs are there with four vertices? The Addition Principle of Counting says that the answer is the number of four-vertex graphs with 0 edges plus the number of four-vertex graphs with 1 edge, and so on, up to the number of four-vertex graphs with 6 edges (see Fig. 3). Thus the total number of different graphs with four vertices is 11.

"Wait a minute!" you or your students might exclaim. Aren't there six different four-vertex graphs with one edge? (See Fig. 4). Doesn't the fact that each edge connects two different vertices make these six graphs different? This is an important question: When are two graphs the same and when are they different?

This question is an example of a common and fundamental question in mathematics: When are two objects the same and when are they different?

For example, when are two numbers the same? Although 2 and 2/1 look very different from one another, as do the pair 12/16 and 21/28, and as do the pair 1/2

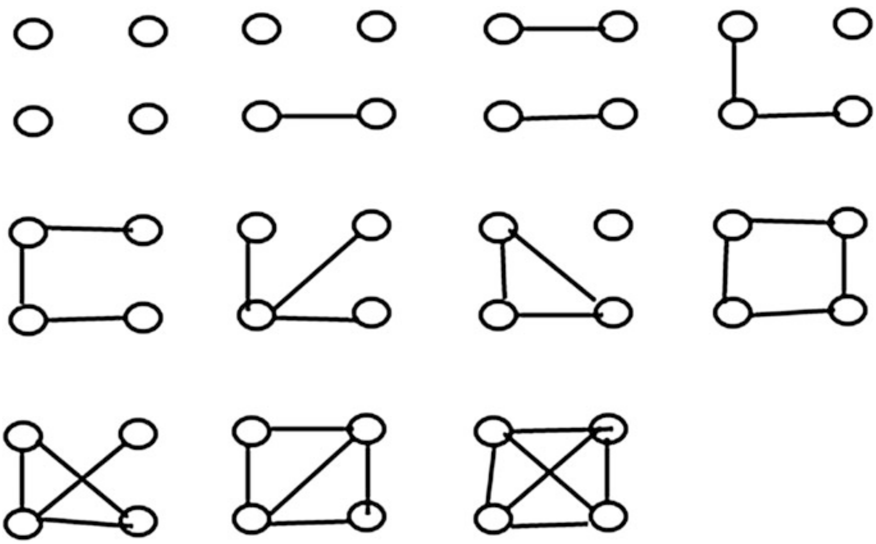


Fig. 3 Using a *divide and conquer* strategy to find how many graphs there are with four vertices

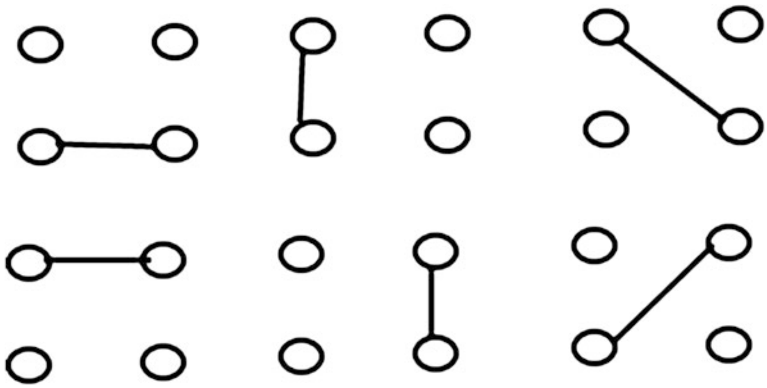


Fig. 4 Are there six different four-vertex graphs with one edge?

and 0.5, students come to learn that they are actually the same—that is, equal. Many are never able to reconcile themselves to the fact that 0.999… is equal to 1.

When are two triangles the same? The first two in Fig. 5 are recognizably the same, even at a young age, but it takes a while before children recognize that the third triangle (resulting from flipping the second triangle about a vertical axis) is the same as the second, and probably longer to recognize that the fourth triangle (resulting from rotating the third triangle counterclockwise by 90°) is the same as the third, and even longer to recognize that the fifth is the same as the fourth. Eventually, they understand that if they can reposition a shape so that it matches another shape, then the two shapes are the "same"—in which case they are referred to as *congruent*.

The same principle is used in dealing with graphs. If one graph can be repositioned so that it matches another graph, then the two graphs are the same—they are referred to as *isomorphic*. Thus the six four-vertex graphs with one edge in Fig. 4 are all isomorphic, since each graph can be repositioned so that it matches each of the other graphs.

"Wait a minute!" you or your students might exclaim. Can't you get another four-vertex graph with five edges beside the one we had before? The one we had before is on the left in Fig. 6; a new one is on the right. These two graphs certainly look different. But you can transform the new one into the old one! Just reposition

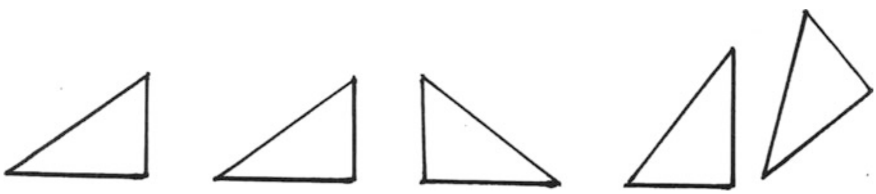


Fig. 5 Which triangles are the same?

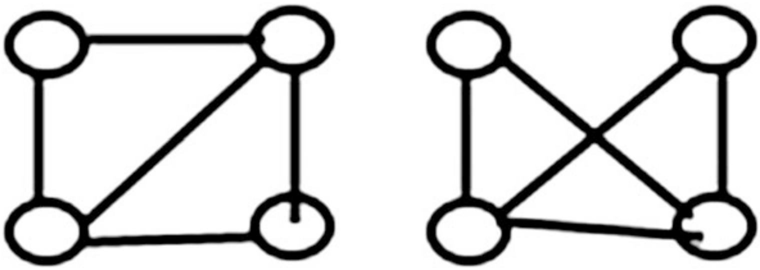


Fig. 6 Two isomorphic four-vertex graphs with five edges

its top right vertex, and the edges adjacent to it, so that it is located below its bottom right vertex … and you'll see the old graph at the left.

As an exercise, systematically determine how many different 2-regular graphs there are with 15 vertices. What is a 2-regular graph? That's a graph where each vertex has degree 2. What's the degree of a vertex? The number of edges that meet at the vertex. First, convince yourself that any 2-regular graph must be a collection of cycles.

The problem of finding all different graphs with four vertices is one example of systematic construction. Another kind of example is to find all different graphs with five vertices of which two have degree 3 and three have degree 2.

How do you even begin to solve this problem? Again, you have to think systematically.

You know that there is a vertex of degree 3 so you draw one vertex—call it A, and link it to three other vertices, B, C, and D. (See Fig. 7) Drawing this part of a graph is not an obvious first step; students have to learn to convert verbal information into graphic form.

You know that there is another vertex of degree 3. Where is it?

It could be E, in which case, (see Fig. 8) since E has degree 3, it must connect to B, C, and D, the only vertices that have room for another link. Look! We have found a graph with the desired properties, two vertices of degree 3 and three of degree 2.

But the second vertex of degree 3 could also be one of the vertices B, C, or D in the graph at the top left of Fig. 9. Let's suppose that it is C. Then we need to connect C to two other vertices. If we connect C to B and D, as in the graph at the top right of Fig. 9, then there are no vertices to which E can be connected, although it must have degree 2. So connecting C to both B and D doesn't work. We have to connect C to E and one of B and D; let's connect it to B, as in the graph at the bottom left of Fig. 9. Finally, we connect D to E, since those are the only two vertices which have empty slots for an edge. When we do that, we find that we have a second graph with the desired properties, two vertices of degree 3 and three of degree 2; this graph is at the bottom right of Fig. 9.

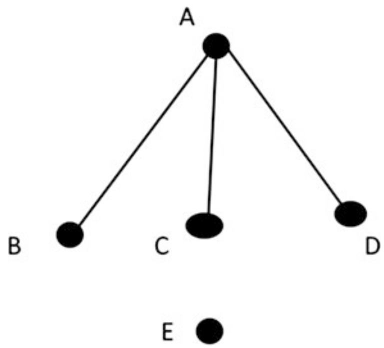


Fig. 7 Draw a vertex of degree 3, call it A, link it to three other vertices, B, C, and D

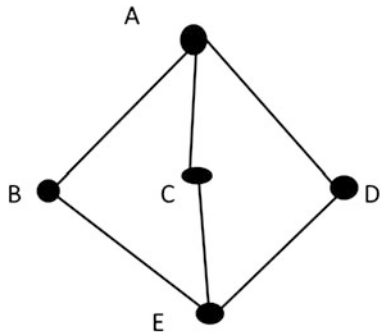


Fig. 8 Case 1: E is the other vertex of degree 3

Are there any other graphs with these properties? For example, if instead of selecting C to have degree 3, we had selected B to have degree 3, perhaps we would have found a third graph with these properties. In fact, it can be proved that any graph that has two vertices of degree 3 and three vertices of degree 2 must be isomorphic to one of the two graphs we have constructed.

Wait a minute! Isn't it possible that these two graphs are also isomorphic? After all, they both have the same number of vertices of each degree.

However, when we look at these two graphs side-by-side (see Fig. 10), we see that they are different. One way that they are different is that the graph on the right has three vertices that form a triangle, but there is no triangle with three vertices in the graph on the left. That makes it impossible for the two graphs to be isomorphic—no matter how we move the vertices of the second graph, those three vertices will still form a triangle.

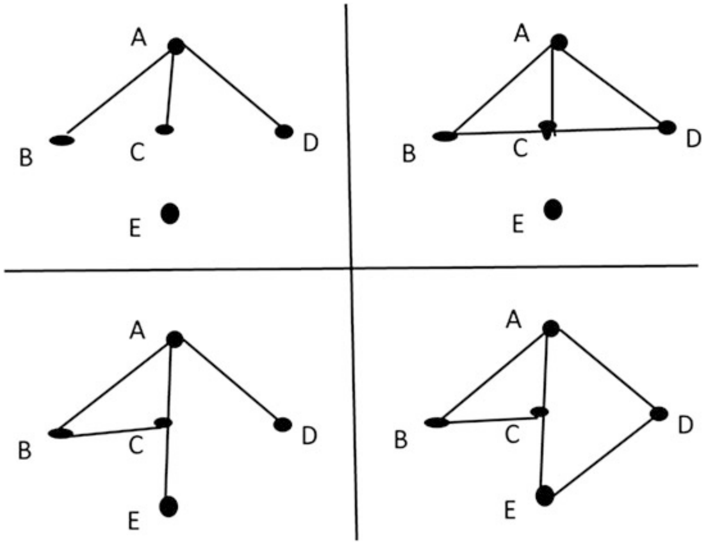
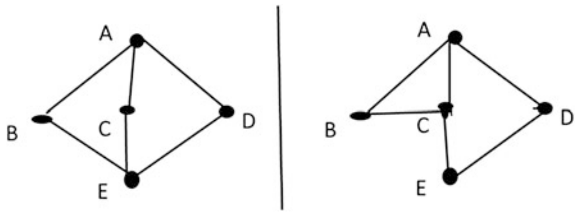


Fig. 9 Case 2: B, C, or D could be the other vertex of degree 3. Consider C

Fig. 10 Two non-isomorphic graphs with the desired properties



The net result is that we have systematically constructed all graphs with two vertices of degree 3 and three vertices of degree 2. There are exactly two of them.

As an exercise, you might try to construct systematically all different graphs with two vertices of degree 3 and four vertices of degree 2, or all different graphs with two vertices of degree 3 and five vertices of degree 2, etc. If you want a real challenge, try constructing systematically all graphs with eight vertices, all of degree 3.

As a final "Find all …" example of systematic reasoning, let us consider the question of how many different dishes containing three scoops of ice cream you can make, if eight flavors are available.

Your immediate response might be "8 choose 3" since there are that many ways of selecting three out of the eight flavors. But that would only be correct if the problem stated that the three scoops have different flavors. It doesn't.

Or your immediate response might be "$8 \times 8 \times 8$," which would be correct if we were counting the number of ice cream cones, since a cone that has chocolate, then vanilla, then strawberry is certainly different than a cone that has strawberry, then chocolate, then vanilla—so there are 8 choices for each scoop. But with a dish of three scoops, the order of the scoops doesn't matter.

How do you solve this counting problem, where order doesn't matter and repetition is allowed? You have to consider several cases, solve each using the Multiplication Principle of Counting, and then add the results together, using the Addition Principle of Counting. A student who has learned to solve this problem has mastered several different techniques of mathematical reasoning and problem solving.

In this problem, there are three possibilities:

- all three scoops could be different, in which case there actually are "8 choose 3" or 56 possibilities;
- all three scoops could be the same, in which case there are 8 possibilities, since there are 8 flavors; or
- there could be two of one flavor (8 choices there) and one of another flavor (7 choices there), another 56 possibilities.

So adding these all together, we get a total of 120 possible ice cream dishes.

As an exercise, determine how many different dishes there are with four scoops of ice cream, where there are eight flavors and where two dishes are the same if they have the same number of scoops of each flavor.

As noted earlier, although the content information in these examples may not be very significant, the experience that students have with learning how to use systematic reasoning to solve problems like this—which have no prerequisites beyond elementary algebra—can be very significant in shaping their thinking in mathematics, and in other areas of human endeavor.

2 Reasoning and Problem Solving at All Grade Levels

When should students start having these experiences? Students should start having reasoning and problem solving experiences in the early grades, so that they can build on those experiences in the later grades, and so they can avoid developing the erroneous conclusion that mathematics equals computation. As can be seen from the examples above, discrete mathematics is a useful arena for introducing reasoning and problem solving.

A systematic approach to incorporating reasoning and problem solving experiences through discrete mathematics appeared in the New Jersey Mathematics Standards, originally adopted in 1996, and modified in 2002, which listed appropriate topics for each grade level. The New Jersey Mathematics and Science Coalition, which I served as Director for many years, reviewed the mathematics standards in 2007 and, taking into consideration the feedback from many New

Jersey teachers and the emerging drafts of the national standards, proposed a new set of mathematics standards for New Jersey, building on both the earlier successful versions of the standards and the recommendations in the national standards. These standards were not seriously considered in the rush to adopt the national standards.

The discrete mathematics portion of those 2007 recommendations is presented in Table 3. The proposed New Jersey standards include two topics of discrete mathematics that were recommended by the National Council of Teachers of Mathematics (NCTM) in *Principles and Standards for School Mathematics* (2000) as appropriate for all grades—systematic listing and counting and vertex-edge graphs—and that were developed more fully in NCTM's two books *Navigating Through Discrete Mathematics in Grades K–5* and *Grades 6–12* (DeBellis et al. 2009; Hart et al. 2008).

Systematic listing and counting is essential preparation for probability—it involves sorting and classifying in the early grades, organizing information in grades 3–5, and using the Multiplication Principle of Counting in the middle grades. Essentially all of this is missing from the Common Core, which evidently assumes that all students will be able to absorb these topics as they learn probability in high school, not a good assumption since probability is the trickiest of all mathematical topics. We hope that systematic listing and counting will be added to our national standards.

The study of vertex-edge graphs enables students to discuss and solve a variety of modern applied problems involving networks—such as efficient routes for snow plows or delivery trucks. This topic is also valuable, as we have seen, because it provides an accessible arena for students to focus on problem solving and reasoning in an interesting context.

The study of vertex-edge graphs also introduces the important modern topic of algorithms, for example, one that given a street map will help generate an efficient route from A to B. At early grade levels, the study of algorithms involves following directions, and later devising instructions and developing strategic thinking skills. Following various kinds of directions in the early grades helps children understand and follow the arithmetical algorithms (like the algorithm for adding two two-digit numbers) that they are later expected to learn with fluency. The study of algorithms, like systematic counting, had been included in the New Jersey Mathematics Standards since 1996, but is entirely absent from the Common Core. We hope that vertex-edge graphs and algorithms will also be added to our national standards.

These topics and reasoning and problem solving experiences with these topics are valuable to *all* students in *all* countries[1] and, given the dozen years of experience with these topics in New Jersey, we believe it is possible to adjust the current mathematics curriculum so that these topics can be added, with a decreased emphasis on some other topics.

[1]A video I prepared that discusses the discrete math topics that all students should be exposed to by the time they complete secondary school is also posted on YouTube.

Table 3 Standards for discrete mathematics proposed for New Jersey in 2007

Standards at each grade level	Topic
Pre-Kindergarten	
1. Determine whether or not an object has a particular attribute	Sorting
2. Sort objects into groups (e.g., sort basket of collected items into piles of pinecones, acorns, and twigs.)	Sorting
Kindergarten	
1. Sort and classify objects according to one attribute (e.g., color, size, shape, kind, or student-generated attribute), and order the resulting groups by the number of objects (each smaller than 10)	Sorting
Grade 1	
1. Sort and classify objects according to one or two attributes (e.g., color, size, shape, kind), noting that a single object can belong to more than one class	Sorting
2. Follow simple sets of directions (e.g., from one location to another, or from a recipe)	Algorithms
Grade 2	
1. Use Venn diagrams to sort and classify objects according to two attributes (e.g., color, size, shape, kind)	Sorting
2. Generate and list all possibilities in simple counting situations (e.g., all outfits involving two shirts and three pants)	Systematic counting
3. Follow the directions for simple two-person games (e.g., tic-tac-toe) and recognize that some strategies work better than others	Algorithms
Grade 3	
1. Represent all possibilities for a simple counting situation in an organized way (e.g., lists, charts) and draw conclusions from this representation	Systematic counting
2. Follow, devise, and describe practical and logical sets of directions for a simple sequence of events (e.g., to add two 2-digit numbers)	Algorithms
3. Find paths in concrete examples of vertex-edge graphs	Vertex-edge graphs
4. Color maps (e.g., NJ counties) using as few colors as possible	Vertex-edge graphs
Grade 4	
1. Use Venn diagrams to represent and classify data according to three attributes, such as shape, color, and size	Sorting
2. Represent all possibilities for a simple counting situation in an organized way (e.g., tree diagrams) and draw conclusions from this representation	Systematic counting
3. Devise strategies for winning two-person games (e.g., "make 5" where players alternately add 1 or 2 and the person who reaches 5, or another designated number, is the winner)	Algorithms
Grade 5	
1. Solve counting problems, including those where multiplication can be used (e.g., you can make $3 \times 4 = 12$ outfits using 3 shirts and 4 skirts)	Systematic counting
2. Justify that all possibilities in a counting problem have been enumerated and that there is no duplication	Systematic counting

(continued)

Table 3 (continued)

Standards at each grade level	Topic
3. Follow, devise, and describe practical and logical sets of directions for a complex sequence of events (e.g., to multiply two 2-digit numbers)	Algorithms
4. Represent problem solving situations using vertex-edge graphs and determine the degree of any vertex (i.e., the number of adjacent vertices), whether or not a graph is connected (i.e., can you get from any vertex to any other vertex?), and how many paths there are from one vertex to another vertex	Vertex-edge graphs
Grade 6	
1. Solve counting problems involving Venn diagrams with two attributes (e.g., there are 15 students in the chess club and 20 students on the debating team. Eight students are in both clubs. How many different students are there participating in one or more of these two activities?)	Sorting
2. Apply the multiplication principle of counting in various situations (e.g., find the number of possible outcomes when three coins are tossed or when three officers are selected from a six-person club)	Systematic counting
3. Devise strategies for winning simple games and express those strategies as sets of directions	Algorithms
Grade 7	
1. Apply the multiplication principle of counting to situations involving permutations (where order is important), including situations with and without replacement, and use factorial notation to condense the results	Systematic counting
2. List the possible combinations of two or three elements chosen from a given set (e.g., the handshake problem and the number of 2- or 3-person committees selected from a group of 12 people)	Systematic counting
3. Use vertex-edge graphs to represent and find reasonable solutions to practical problems • Travel route from one site on a map to another • Delivery route that stops at specified places • Drawing a picture with a single line without repeating an edge • Scheduling project meetings (to avoid conflicts) using graph coloring	Vertex-edge graphs
Grade 8	
1. Solve counting problems involving Venn diagrams with three attributes	Systematic counting
2. Distinguish between permutations and combinations and solve counting problems of both types using the multiplication principle of counting (no formulas)	Systematic counting
3. List and count the number of paths from the top cell of Pascal's Triangle to another one and describe the connection between such problems and problems involving combinations.	Systematic counting
4. Use vertex-edge graphs and algorithmic thinking to find solutions to practical problems • Finding the shortest network connecting specific sites • Finding the shortest travel route from one site on a map to another • Finding a low-cost circuit that visits each vertex exactly once	Algorithms Vertex-edge graphs

(continued)

Table 3 (continued)

Standards at each grade level	Topic
High School	
1. Apply the multiplication principle of counting in complex situations, distinguish between situations with and without replacement, distinguish between ordered and unordered counting situations, and justify solutions to counting problems	Systematic counting
2. Use Pascal's Triangle to solve problems involving combinations and probability	Systematic counting
3. Use vertex-edge graphs and algorithmic thinking to represent and solve practical problems • Is there a circuit that includes every edge in a graph just once (snow-plow or delivery routes)? • Is there a circuit that includes every vertex in a graph just once? • Critical path analysis	Algorithms Vertex-edge graphs
4. Develop and use strategies for making fair decisions, including combining individual preferences into a group decision (voting methods) and determining how many representatives each constituency gets (apportionment)	Algorithms

We have focused in this section on the value of discrete mathematics as a vehicle for improving the reasoning and problem solving skills of our students. However, there are other reasons why topics in discrete mathematics should be included in the school mathematics curriculum. Here are a few reasons for including discrete mathematics at all grade levels … and in programs for prospective and practicing teachers:

- Discrete mathematics includes valuable concepts and tools that show students the usefulness of mathematics, and that respond to the question, "How is math useful in the real world?"
- Discrete mathematics facilitates focus on modeling, problem solving, and reasoning at all grade levels.
- Discrete mathematics offers students who have been unsuccessful in traditional school mathematics a new start in mathematics.
- Discrete mathematics offers an opportunity to generate in primary and secondary teachers a new enthusiasm for teaching mathematics in new ways.

These and other rationales for discrete mathematics are discussed in detail in previous articles (e.g., Rosenstein 1997, 2007; Rosenstein et al. 1997), as is their value to all students, so I will not provide further explanations here.

3 Why Was Discrete Mathematics Excluded from the Standards in the United States?

Why would a state like my own, New Jersey, which had included discrete mathematics in its state mathematics standards since 1996 and which has consistently been among the top states in the independently conducted National Assessment of Educational Progress (NAEP), abandon its standards in favor of the Common Core? That's easy to answer: Each state's eligibility to receive federal funds for education was made contingent on its adopting Common Core.

Given these positive affordances of discrete mathematics, we must ask, why did the mathematics community go along with standards that essentially excluded discrete mathematics?

First, a major concern for many years is the number of students who come to college with an inadequate background in mathematics. In practice, that means that many students who have taken courses at the secondary level that presumably prepared them for college level mathematics courses are not actually prepared for those courses. As a result, colleges and universities have increasingly provided remedial courses in mathematics (and, similarly, in language arts—reading and writing).

This is not a new problem. Indeed, 35 years ago the Rutgers mathematics department first instituted a placement examination for incoming students and we discovered—that is, we now have data—about how many of our students were not prepared for calculus or even precalculus. As director of the undergraduate math program, I became co-chair of a university-wide committee on precollege preparation for the university, which strengthened Rutgers' entrance requirements, and subsequently became a member of a similar state-level committee, which in part led to the New Jersey state standards adopted in 1996. This was my first involvement in mathematics education.

Although this is not a new problem, it has been an ever-increasing problem since colleges and universities have expanded and ever-higher percentages of secondary school graduates are going to college. One of the negative effects of this democratization of education is that more students are enrolling in college who are unprepared for college-level work in mathematics.

One of the principal motivations for the movement to create state standards in the 1990s was to ensure that secondary school graduates were prepared for college, careers, and citizenship. In the past 20 years, as a higher percentage of secondary students went to college, the focus in primary and secondary schools became more on preparation for college.

The mistake that has been made is that the focus has shifted from college-readiness to calculus-readiness, and the driver of the entire mathematics curriculum in the Common Core has become preparing students for calculus. That requires what one of my colleagues described as "a fanatical focus on fractions" and an early emphasis on algebra. In such a curriculum, there is no time for the *frills* of discrete mathematics.

Shifting from college-readiness to calculus-readiness assumes that all college students, real or potential, and therefore all secondary students, need to prepare themselves for calculus. That is simply not so. Most secondary students will have no need for calculus and therefore no need for all the topics whose main role is to prepare them for calculus, including division of fractions when they are 13 years old. For these students, elementary algebra (what is referred to in the U.S. as Algebra I), elements of geometry, and exposure to the applications of trigonometric and exponential functions are the appropriate and sufficient topics from the calculus track.

Instead of writing standards that would prepare students for college, careers, and citizenship, as originally intended, the goal of the writing group, it seems to me, was shifted to prepare students for calculus.

At this point it is appropriate for me to note that I am not opposed to having standards in mathematics. Indeed, I played a leading role in the organization, creation, and adoption of *high* mathematics standards for New Jersey in 1996 (20 years ago!) and again in 2002, standards which certainly played a role in New Jersey's consistently high performance on the National Assessment of Educational Progress. Accompanying the standards was a 600+ page volume *New Jersey Mathematics Curriculum Framework,* (Rosenstein et al. 1997) that provided assistance to school personnel in implementing the standards.

Standards are, I believe, very important. But the standards should be applied to, and be appropriate for, *all* students. Not *all* students need to take calculus. Not *all* students need to be able to find 64 to the 2/3 power, and not *all* students need to manipulate algebraic fractions—a skill that is primarily useful in calculus.

Many topics in discrete mathematics are more valuable for *all* students than some of the topics needed to succeed in calculus, and are more accessible to *all* students than the *intense algebra* needed for calculus. Many topics in discrete mathematics can demonstrate to *all* students how mathematics is applied in today's data-driven world. And many topics in discrete mathematics provide opportunities for *all* students to learn about mathematical reasoning and problem solving.

That is just as true for students who are planning to take calculus. Although the Common Core may prepare them for calculus courses, they are not prepared for later courses involving problem solving, reasoning, and proof, skills which they could have developed through discrete mathematics topics in the school curriculum. This inadequate preparation may be partially responsible for the prevalence of courses in mathematical reasoning for college juniors who seek to become math majors. Including discrete mathematics in the school curriculum could provide prospective math majors with the reasoning skills needed to succeed in college.

Thus, the Common Core has left non-college intending students and non-calculus intending students without the problem solving and reasoning skills necessary for jobs in the current economy, and has left Science, Technology, Engineering, and Mathematics- (STEM-) intending students without the requisite problem-solving and reasoning abilities to succeed in math and science courses beyond calculus. Thus, the new standards are not serving students who are non-calculus intending nor are they serving students who plan to go on in STEM disciplines.

A second motivation behind the calculus focus of the core curriculum was the belief that the United States needs more students who are preparing themselves for STEM careers, and the belief that the way to increase the STEM pipeline is to ensure that more students take calculus in secondary school.

Leaving aside the question of whether there is a shortage of STEM personnel and STEM-prepared personnel, our research on Rutgers students who have taken Advanced Placement (AP) Calculus—the prime candidates for the STEM pipeline—reveals that a substantial percentage of these students do not continue in the STEM pipeline (Ahluwalia and Rosenstein 2017).

This suggests that we are not doing what is needed to convince these students to stay in the STEM pipeline and, just maybe, are acting counterproductively, exposing them to topics for which they are not adequately prepared and in ways that do not encourage them to be interested in pursuing mathematical or scientific careers. Others were only virtually in the STEM pipeline, that is, they took advanced math and science courses only because doing so enhanced their ability to get into the colleges of their choice, not because of their interest in these subjects.

Thus, the problem of the STEM pipeline is not that of recruitment, but of retention. (The 2007 report, *Rising Above the Gathering Storm*, seems unaware of this issue and simplistically recommends increasing the number of students taking AP Calculus in high school.) Thus, *it is not that the STEM pipeline is too small, but rather that it is too leaky*. Too many students exit the STEM pipeline. The solution to this problem should focus more on retaining students who are already in the STEM pipeline than on recruiting more students into the pipeline.

What I have said above applies primarily to students who live in relatively high socio-economic areas, where all students have the opportunity to take high level mathematics classes. However, in many low socio-economic areas, students do not have such opportunities and the talents of many students remain undeveloped, in mathematics and in other areas. It is a tragedy that thousands of students in America's urban and rural areas never are provided the resources and support that will enable them to be successful. For those students, STEM-based efforts must be escalated. This is a real challenge, a challenge that can be helped by the addition of discrete mathematics topics, which are often more engaging to students, in part because they have fewer prerequisites, and can help encourage students from low-resource schools to stay in the STEM pipeline.

A third reason for omitting discrete mathematics from the curriculum is concern about the scores of United States students on international assessments. If our students are not performing well in comparison with other countries, then it must be our curriculum that is defective, so they argue, and improving our standards will improve our curriculum, which in turn will improve the performance of our students.

There are a number of problems with this expressed concern. What do the assessments actually show? Are there other explanations for the apparent gap in performance? Is the gap indeed substantial? Is it really important that there is a gap? Is changing the curriculum the appropriate response? Will it have the intended effect?

International assessments are based on the curriculum that is *common* to all of the participating countries. Countries that have a broader curriculum are at a disadvantage on international tests because if their students spend only 90% of the time on the common topics they will not do as well as countries whose students spend 100% of time on those topics. Therefore, if increasing scores is important and the obstacle is curriculum, then it follows that we should narrow the curriculum so that our curriculum is in line with the international tests. Thus, from this perspective, discrete mathematics must be deleted from the curriculum. This reasoning is presumably an issue in other countries besides the United States; any country that wants to broaden its mathematics curriculum risks putting its students at a disadvantage and lowering its scores compared to other countries.

Because of this, the Common Core decides that the focus of mathematics education in the early grades should be primarily on fractions, which leaves little time for inclusion of discrete topics suggested for the early grades in Table 3. Those topics are considered *frills* and are not considered mathematics but *play*, and including them in our curriculum will not help us *catch up* to the students of other countries. But it is precisely these kinds of mathematical explorations that will enable our students to achieve the reasoning and problem-solving skills that we want them to have; they should not be excluded from our curriculum.

Perhaps we should rather promote the adoption of a different perspective, namely, that all countries should be encouraged to adopt a broader curriculum. How to do this is a problem that we all need to address.

To summarize, all three areas of concern—the focus on preparation for calculus, the desire to expand the STEM pipeline, and the concerns about international assessments—all seem to support the conclusion that discrete mathematics is not important. And, in each of those three areas of concern, I believe that this conclusion is not justified.

My hope and belief is that the US will eventually recognize that taking discrete mathematics will have a positive impact on students' mathematical preparation, their interest in STEM careers, and their performance on international assessments—by improving their reasoning and problem-solving skills and by introducing them to the many ways in which discrete mathematics is applied in today's world.

When will this recognition take place? I don't know. But it can only happen if we come to a recognition that we do not need to prepare all students for calculus, that not all students need *intense algebra*, and that we should not accelerate all students into Algebra courses and then into AP Calculus courses before they are ready for them.

I believe that this recognition *will* happen and I anticipate that in the coming years these topics will have the prominence in the school curriculum that they deserve, and that our students deserve. To help this come about, I have written a high school text, *Problem Solving and Reasoning with Discrete Mathematics* (Rosenstein 2014), a much expanded, revised, and refocused version of a book that was developed by myself and Valerie DeBellis (DeBellis and Rosenstein 2005) over ten years ago. This text presents one vision of how discrete mathematics can be incorporated into the high school curriculum.

My hope is that other mathematicians will be motivated to actively inform the mathematical education community, and the broader community, about the importance and value of introducing discrete mathematics into the curriculum of their countries' schools, by developing their own curriculum materials, and that together we will promote the importance of a broader curriculum.

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