Discrete Mathematics Is Essential Mathematics in a 21st Century School Curriculum

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Abstract In this chapter we discuss discrete mathematics in the school curriculum. We make the case, based on many years of curriculum research and design, that discrete mathematics is essential in a modern, robust school mathematics curriculum, and that five broad problem types emerge as ways to organize the diversity of discrete mathematics contexts that are important and appropriate for the curriculum —enumeration, sequential change, relationships among a finite number of elements, information processing, and fair decision-making. In this chapter, these five problem types are briefly described and three classroom examples are provided. Subsequent chapters in this volume provide additional analysis, research, and more classroom examples.

Discrete mathematics is a robust field of mathematics with many modern applications. Yet it has no succinct definition, so we begin this chapter, and indeed this volume, by considering the nature and relevance of discrete mathematics.

1 The Rise of Discrete Mathematics

Discrete mathematics is described by *Topic Study Group 17: Teaching and Learning of Discrete Mathematics* at the 13th International Congress on Mathematical Education (Hart et al. 2017) as follows:

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Discrete mathematics is a comparatively young branch of mathematics with no agreed-upon definition but with old roots and emblematic problems. It is a robust field with applications to a variety of real world situations, and as such takes on growing importance in contemporary society. We take discrete mathematics to include a wide range of topics, including logic, game theory, algorithms, graph theory (networks), discrete geometry, number theory, discrete dynamical systems, fair decision making, cryptography, coding theory, and counting. Cross-cutting themes include discrete mathematical modeling, algorithmic problem solving, optimization, combinatorial reasoning, and recursive thinking.

A recent publication that looks to the future of mathematics, *The Mathematical Sciences in 2025* (Committee on the Mathematical Sciences 2025 2013), states that "over the years there have been important shifts in the level of activity in certain subjects—for example, the growing significance of probabilistic methods, the rise of discrete mathematics, and the growing use of Bayesian statistics" (p. 72). The book identifies two new drivers of mathematics–computation and big data, and for both of these drivers it describes how discrete mathematics plays an important role–for example, discrete mathematics algorithms for information processing, dynamical systems in ecology, networks in industry and the humanities, and discrete optimization (p. 77).

Discrete mathematics is particularly suited to applications involving technology and computers. In fact, discrete mathematics is sometimes considered the mathematics of computer science. The *discrete* aspect of discrete mathematics is often contrasted with the *continuous* mathematics of calculus. It is appropriate to connect discrete mathematics to computers and contrast it with calculus, but neither characterization is complete.

We may take the following as a definition: *Discrete mathematics is a collection of mathematical concepts and methods that help us solve problems that involve a countable (often finite) number of elements or processes and connections among them.*

It is more helpful in coming to understand discrete mathematics to consider the wide variety of important and interesting problems that can be solved. Examples of questions that can be naturally investigated with discrete mathematics include:

How can you avoid conflicts when scheduling meetings, shipping hazardous chemicals, or assigning frequencies to radio stations? How can you schedule a project for shortest completion time when it consists of numerous interconnected sub-projects? How can you fairly decide among competing alternatives, like candidates standing for election? How can you fairly divide or apportion objects, such as seats of congress or property in an inheritance? How can you ensure accuracy, security, and efficiency in digital transactions such as transferring files, making online purchases, or posting to social networks? How many different Personal Identification Numbers (PINs), IP addresses, or pizzas with different toppings are possible? How can you model and analyse processes of sequential change, such as year-to-year growth in population, month-to-month change in credit card debt, or periodic medicine dosage?

The breadth and diversity of the questions above reflect the power and applicability of discrete mathematics. Discrete mathematics is indeed essential to understanding our modern technological world, and as such it is essential to include discrete mathematics in the school curriculum. But which parts of discrete mathematics should be included?

2   Incorporating Discrete Mathematics into the School Curriculum: Five Essential Discrete Mathematics Problem Types

The power of discrete mathematics lies in mathematical modeling and solving problems, so instead of focusing on which topics to include, we consider broad problem types. Based on curriculum research, development, and implementation in classrooms and textbooks over many years, five broad problem types emerge as the most potent types of discrete mathematical problems that are relevant for the school curriculum—*enumeration, sequential step-by-step change, relationships among a finite number of elements, information processing, and fair decision-making*.

These five problem types, along with the respective discrete mathematics domains of combinatorics, recursion, graph theory, informatics, and the mathematics of voting and fair division (which can be viewed as part of game theory), have been identified as fundamental for school mathematics based on the following work over the past 30 years:

- Initial momentum provided by the collegiate discrete mathematics movement in the 1980s (e.g., Hart 1985; Ralston 1989; Dossey 1991);
- Federally-funded professional development programs in the 1990s in the United States to implement the National Council of Teachers of Mathematics (NCTM 1989) standard on discrete mathematics (Hart and Schoen 1989–1993; Rosenstein 1989–1996; Sandefur 1990–1994; Kenney 1992–1996);
- Articles and books to support implementation of the discrete mathematics recommendations in NCTM's (1989, 2000) standards (Hart et al. 1990, 2008; Hart 1991; Hirsch and Kenney 1991; Debellis et al. 2009);
- Articles recommending more discrete mathematics in the *Common Core State Standards for Mathematics* (NGA Center and CCSSO 2010) in the United States (Hart and Martin 2008, 2016; Martin and Hart 2012);
- Curriculum research and design to develop high school textbooks that integrate discrete mathematics (Hart 1997, 1998, 2008, 2010; Hirsch et al. 2015, 2016).

Each of the five problem types is briefly discussed below in this chapter. Subsequent chapters in this volume provide further analysis:

Rosenstein takes a broad look at several of the problem types in elementary and secondary school in the United States. Gaio and Di Paola consider discrete mathematics in the lower grades in Italy, focusing on graph theory (*relationship-among-elements*) and cryptography (an aspect of *information processing*). The *enumeration* problem type is the focus of the chapters by Coenen et al., Hoveler, Lockwood and Reed, and Vancso et al., as they consider combinatorics and combinatorial reasoning. The *sequential change* problem type is the focus of the chapters by Devaney and by Sandefur et al., as they discuss recursion and recursive thinking. Cozzens and Koirala and Ferrarello and Mammana focus on the *relationship-among-elements* problem type in their chapters on networks and graphs. Regarding *fair decision-making*, this can be considered part of the broad area of game theory,

and three chapters are in this broad category—Garfunkel discusses fairness models for fair division and bankruptcy problems, while Colipan and Rougetet analyze combinatorial games. Finally, Goldin considers the affective dimension of studying discrete mathematics generally, and Igoshin considers logic, which can be considered part of discrete mathematics but is of course fundamental to all parts of mathematics, discrete or not.

The following sections provide a brief description of the five problem types. Classroom examples follow.

2.1 Enumeration

Enumeration problems involve counting. Perhaps to solve a problem or part of a problem it would be helpful to count something. What are you counting? What is the structure of the counting situation? Are you counting choices from a collection of objects, outcomes from a sequence of tasks, or counting from "this or that" distinct situation? General consideration of these questions leads to the following three common and useful types of enumeration problems, which are particularly relevant in the school curriculum.

- Count the number of choices from a collection of objects—Are you choosing from a collection of objects? If so, consider the issues of order and repetition in your choice, yielding four possible problem types, including permutations and combinations.
- Count the number of possible combined outcomes from a sequence of tasks—Is there a sequence of tasks? Can the situation be represented by a tree diagram, where the outcomes from each task are represented in each level of branching in the tree? If so, try applying the Multiplication [Fundamental] Principle of Counting. But be careful! This seemingly simple principle requires that there is a *sequence* of tasks, that the number of outcomes at each stage is *independent* of the choices in the previous stages, and that the *combined* outcomes at the end, which is what you are counting, are all *distinct*.
- Count this or that—Are there two sets involved, the members of which need to be counted? If so, be careful of any overlap and try the Addition Principle of Counting or the inclusion/exclusion principle.

2.2 Sequential Change

Sequential, step-by-step change is a natural part of our world. Think about all the situations where some quantity is changing yearly, monthly, or daily; from an amount in one period or at one state to a different amount in the next period or state. A recursive model is often used to analyze such situations. You can build a recursive model in school mathematics courses by considering questions such as below:

- Can you write an equation using the words NOW and NEXT to describe how the quantity in the NEXT period compares to the quantity NOW? This provides an intuitive and accessible way to model the situation with a recursive equation. A recursive equation using subscripts or function notation can be helpful later for further analysis, but such notation is infamously difficult for students to understand and need not be a barrier for earlier use of recursive models to analyze sequential change situations.
- How many steps of recursion are there? That is, does just the current step determine the next step, or are two (or more) previous steps needed to determine the next step? If more than one step is needed, then a simple NOW-NEXT equation will not work and something more complicated, perhaps with subscripts, will be required.
- How does the process start? What is the initial amount? To build a recursive model, you need to know where to start as well as how to get from one step to the next.

2.3 Relationship Among a Finite Number of Elements

This broad class of problems arises in situations where there are many objects or elements and a relationship between pairs of those elements. These are problems about networks, such as communication networks, transportation networks, or social networks. Such problems can be modeled using vertex-edge graphs.

- What are the elements? Represent those as vertices. What is the relationship between pairs of elements? Draw an edge between vertices (elements) that are related, thus creating a vertex-edge graph model.
- Is it a conflict relationship? Try a vertex coloring model.
- Is it a prerequisite relationship? Try a critical path analysis.
- Does the context suggest visiting *each vertex* or using *each edge* of the vertex-edge graph? Try a Hamilton path or an Euler path, respectively.
- Does the context suggest reaching all the vertices of the graph without any redundancy (no loops)? Try a spanning tree. Are the edges weighted? Perhaps a minimal spanning tree will be useful.

Networks are everywhere in modern life and thus vertex-edge graphs are an important and relevant topic. Arguably, fluent use of vertex-edge graphs may be a mathematical skill rivaling many of the traditional skills currently included in the curriculum. We need to think hard about whether traditional topics squeeze out important discrete mathematics topics. And if so, should they?

2.4   Information Processing

Consider problems about information processing, that is, problems about searching, securing, sending, or receiving information, especially digital information, particularly in contexts related to the Internet. There are four fundamental issues of information processing that could be involved, each of which can be addressed using fundamental school mathematics.

- *Access*—For information to be useful it must be accessible. How does this relate to school mathematics? Well, for example, elementary set theory and logic are part of Internet search engines and often can be used explicitly to specify advanced searches. Or, in the theme of algorithmic problem solving, specifically related to coding, searching and sorting algorithms are fundamental aspects of computer science that help make information accessible and can be productively studied in school mathematics.
- *Accuracy*—It is important to ensure the accuracy of information as it is sent or received. For example, when sending a photo from deep space or scanning a UPC product code at a grocery store checkout station, error-detecting and -correcting codes are used. Such codes often use elementary number theory, like modular arithmetic, or, in more advanced settings, linear algebra.
- *Security*—In many situations it is essential that information is kept secure and private. For example, private email and secure credit card numbers are often sent using public-key cryptography, which is based on number theory. This fascinating topic is relevant to almost all students in their daily digital lives. The mathematics of secret codes begins simply and extends to be as challenging as students and teachers desire.
- *Efficiency*—In our modern digital information age we consume more and more data. With streaming video, online mapping, photos, and constant social media updates, everyone wants faster Internet service and more memory with a larger data plan for their phones. All this requires efficient data transfer and storage. Students can learn about data compression using variable-length codes and Huffman trees. These accessible mathematical topics are highly relevant, for example they are used in most compressed file formats today, like .jpg images and .mpg videos. We also want the algorithms we use to solve problems to be efficient. Basic consideration of algorithm complexity is also relevant and appropriate for school mathematics, such as the qualitative differences among linear, exponential, and factorial growth. For example, the famous *Traveling Salesman Problem* in graph theory is easy to state and begin analyzing, but a simple *brute-for*ce solution algorithm that checks all possible routes grows factorially and thus is impractical, even with the world's fastest supercomputer.

2.5 Fair Decision-Making

Many problems require making a fair decision. In particular, consider problems about fair voting or fair division.

Must one alternative among several be fairly chosen by a group of people? Consider a voting model. Is it a *one-person-one-vote* situation, as in elections for government office, or a *one-person-many-votes* situation, as in stockholder voting where each stockholder has as many votes as shares owned?

- In a one-person-one-vote situation in which there are more than two alternatives or candidates, *ranked-choice voting* is often the best option, whereby people vote by ranking the candidates rather than just designating their favorite candidate. The data gathered from ranked-choice voting provides rich information about voter preferences, which can be analyzed using a number of efficient vote-analysis methods to choose a good winner.
- In a one-person-many-votes situation, a *weighted voting* model can be used, in which both *weight* (the number of votes an individual has) and *power* (a measure of how critical an individual's vote is) are analyzed.

Does something need to be fairly divided or apportioned? Consider a fair division model. The choice of an effective fair division method depends on what is being divided.

- Is it divisible (like land or cake), or indivisible (like seats of congress or antiques)?
- If divisible, is it homogeneous (like a flat tract of land) or heterogeneous (like land that is hilly and forested)?
- If indivisible, are the objects identical (like seats of congress) or non-identical (like antiques)?

Depending on answers to these questions you can use different models and methods of fair division, many of which are accessible, engaging, and relevant for school mathematics.

We conclude this chapter with three classroom examples, related to three of the five problem types—sequential change, relationships among elements, and fair decision-making. These examples are appropriate for middle or high school students. Other chapters in this book provide additional classroom examples, including several at the elementary school level and with respect to all five problem types.

3   Classroom Examples

3.1   Example 1: Proper Medicine Dosage

This example is modified from Hart and Martin (2016). It illustrates the *sequential change* problem type.

Consider the common situation of taking repeated daily doses of a medication. Suppose a hospital patient is given an antibiotic to treat an infection. He is initially given a 30 mg dose and then receives another 10 mg at the end of every six-hour period thereafter. Through natural body metabolism, about 20% of the antibiotic is eliminated from his system every six hours. This situation raises many interesting questions, such as:

> What is the long-term amount of antibiotic in the patient's system?
>
> How should this prescription be modified if the doctor decides that a long-term amount of 25 mg is desired?

This problem is about a process of sequential change, namely, the change in the amount of antibiotic in the patient's system, which changes every six hours. Thus, a recursive model may be useful. Note that a student does not need to know the precise definition of recursion to continue, he or she can build a model as follows.

Is it possible to describe this process of sequential change with an equation using the words NOW and NEXT? In this case, if NOW is the amount of antibiotic in the patient's system now, and NEXT represents the amount after the next six-hour dose, then NEXT = $0.8 \cdot$ NOW + 10. (This model assumes that the amount is measured after the regular dose is taken.) Is there an initial amount? Yes, 30 mg.

Thus, we have a model: Start with 30, then represent the step-by-step change based on 20% elimination and a regular 10 mg dose with NEXT = $0.8 \cdot$ NOW + 10.

Now a spreadsheet or calculator can be used to easily compute the amounts over time. Initially, the amount is 30 mg. Then, six hours later, $0.8 \cdot 30 + 10 = 34$ mg, then, another six hours later, $0.8 \cdot 34 + 10 = 37.2$ mg, and so on.

What about the long-term amount? Is there an *equilibrium value*? With technology, we can quickly see that the long-term amount stabilizes at about 50 mg, as shown in the first two columns of Fig. 1.

How can we change the prescription to get a long-term amount of 25 mg? Using our recursive model and a spreadsheet, we can easily try different adjustments. Maybe we start by cutting the *initial dose* of 30 mg in half, since the goal is to cut the long-term amount in half. (Try it; it doesn't work! Surprisingly, the long-term amount stays at 50. See the third and fourth columns of Fig. 1.) How about cutting the *regular dosage* in half? (Try it; it works. See the fifth and sixth columns of Fig. 1.)

	A	B	C	D	E	F
1	*Recurring dose: 10 mg*		*Recurring dose: 10 mg*		*Recurring dose: 5 mg*	
2	Dose #	Mg in system	Dose #	Mg in system	Dose #	Mg in system
3	1	30	1	15	1	30
4	2	34	2	22	2	29
5	3	37.2	3	27.6	3	28.2
6	4	39.76	4	32.08	4	27.56
7	5	41.808	5	35.664	5	27.048
8	6	43.4464	6	38.5312	6	26.6384
9	7	44.75712	7	40.82496	7	26.31072
10	8	45.805696	8	42.659968	8	26.048576
11	9	46.6445568	9	44.1279744	9	25.8388608
12	10	47.31564544	10	45.30237952	10	25.67108864
13	11	47.85251635	11	46.24190362	11	25.53687091
14	12	48.28201308	12	46.99352289	12	25.42949673
15	13	48.62561047	13	47.59481831	13	25.34359738
16	14	48.90048837	14	48.07585465	14	25.27487791
17	15	49.1203907	15	48.46068372	15	25.21990233
18	16	49.29631256	16	48.76854698	16	25.17592186
19	17	49.43705005	17	49.01483758	17	25.14073749
20	18	49.54964004	18	49.21187007	18	25.11258999
21	19	49.63971203	19	49.36949605	19	25.09007199
22	20	49.71176962	20	49.49559684	20	25.07205759
23	21	49.7694157	21	49.59647747	21	25.05764608
24	22	49.81553256	22	49.67718198	22	25.04611686
25	23	49.85242605	23	49.74174558	23	25.03689349
26	24	49.88194084	24	49.79339647	24	25.02951479
27	25	49.90555267	25	49.83471717	25	25.02361183
28						

Fig. 1 Spreadsheet showing the sequential change in the amount of antibiotic in a patient's system for different initial and recurring doses

The recursive model is very accessible and useful, especially when using technological tools such as spreadsheets, graphing calculators, or computer coding. Further investigation can be carried out as needed and as deemed appropriate by the curriculum and teacher. For example, students can create more formal recursive equations using subscript or function notation; they can devise and analyze *closed-form* equations; they can generate graphs showing the relationship between dose number and amount of antibiotic in the system or use so-called *cobweb graphs* that show the relationship between successive amounts of antibiotic.

3.2 Example 2: Optimally Assigning Frequencies to Radio Stations

This example is adapted from *Focus in High School Mathematics* (NCTM 2009, pp. 70–72, as adapted there from Hirsch et al. 2015). It illustrates the use of vertex-edge graphs to model and solve a problem about a *relationship among elements*.

Task: The Federal Communications Commission (FCC) needs to assign radio frequencies to seven new radio stations located on the grid in Fig. 2. Such assignments are based on several considerations, including the possibility of creating interference by assigning the same frequency to stations that are too close together. In this simplified situation, we assume that broadcasts from two stations located within 200 miles of each other will create interference if they broadcast on the same frequency, whereas stations more than 200 miles apart can use the same frequency to broadcast without causing interference with each other.

Consider these questions:

> How can a vertex-edge graph be used to assign frequencies so that the fewest number of frequencies is used and no stations interfere with each other?
>
> What would each vertex represent? What would an edge represent?
>
> What is the fewest number of frequencies needed?

1 unit = 50 miles

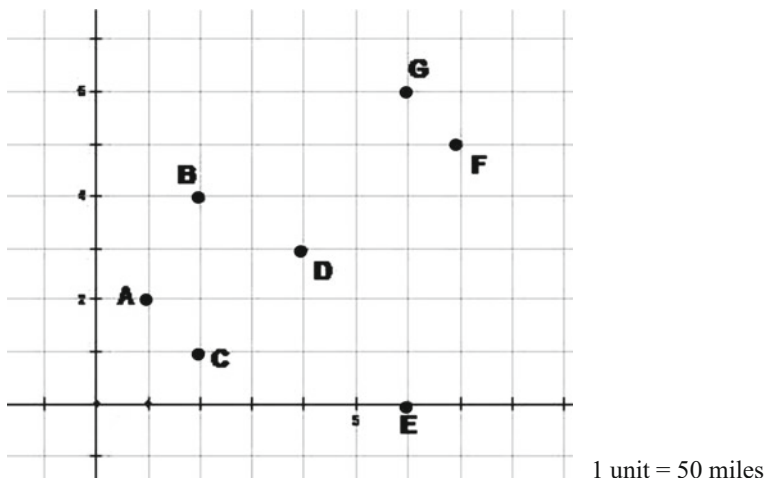**Fig. 2** Grid showing placement of seven radio stations

In the Classroom: Students work on the task in groups. Each group agrees that a vertex in the graph represents a radio station. So the graph will have seven vertices. What about edges? Some groups decide to make a graph model in which two vertices will be joined by an edge if the stations they represent are within 200 miles of each other. Other groups suggest that two vertices will be joined with an edge if the stations are *more* than 200 miles apart. After some discussion, realizing that both options can lead to a solution but one may be easier, the choice is made to use the first suggestion, that of joining vertices with an edge if the distance between them is less than (or equal to) 200 miles, because that will show stations that cannot share the same frequency and thus when different frequencies are needed.

The next step in building the model requires determination of the distance between each pair of stations. Some groups compute the distances using the distance formula or Pythagorean theorem and a calculator. Other groups use a length of string or paper strip marked off using the given scale to show 200 miles. In any case, groups produce models similar to the one in Fig. 3.

Now students work at assigning frequencies to vertices (which can be thought of as coloring the vertices) so that no two vertices joined by a single edge have the same frequency (color). They look for a method that will generate the fewest frequencies for this particular graph. Once they find a solution, they are asked to explain.

One group explained their solution this way:

> We found that the fewest number of frequencies that could be used was four. We reasoned this way: First, look at the collection of vertices *A* through *D*. Each of these vertices is joined by an edge to each of the other three in the collection. So no two vertices in this collection of four vertices can have the same frequency. That means you need *at least* four frequencies. But can you actually get by with four frequencies for *all* the vertices? Well,

Fig. 3 Vertex-edge graph model of the radio station problem—vertices are radio stations, an edge between two vertices indicates that the two radio stations can interfere with each other and thus must have different assigned frequencies
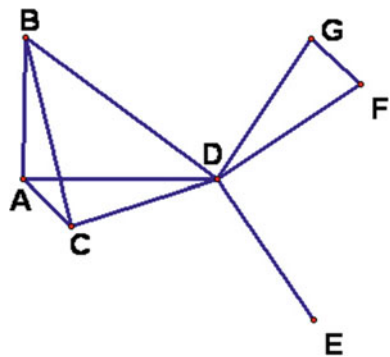
suppose you assign frequency 1 to vertex A, frequency 2 to vertex B, frequency 3 to vertex C, and frequency 4 to vertex D. You can finish the assignment by assigning frequency 1 to vertex G (because G doesn't interfere with A), frequency 2 to vertex F (because F doesn't interfere with B), and frequency 3 to vertex E (because E doesn't interfere with C). So four frequencies will do the job! So you need at least four and four actually works, so that proves it—four is the fewest!"

Brief Analysis: Many aspects of mathematics, modeling, and mathematical practices are evident in this example. For example: (a) choosing and using appropriate mathematics—vertex-edge graphs, specifically vertex coloring, as well as the Pythagorean theorem and the distance formula, (b) explicitly building a model—carefully considering what the vertices and edges will represent, in particular notice the students' discussion about whether to connect vertices with an edge if they are more than 200 miles apart or less than 200 miles apart, which is in fact an important decision (technically between a conflict graph or a compatibility graph) that significantly affects the model and the solution method, (c) analyzing the model in order to understand the situation and make better decisions—students analyze the graph model to figure out how to assign frequencies in a way that results in no interference and uses the fewest number of frequencies, (d) reasoning and constructing arguments—the students essentially prove that a *complete graph* on four vertices requires four colors, and (e) algorithmic problem solving—they devise a systematic procedure for assigning frequencies (coloring vertices) and explain why the procedure works. Finally, note an interesting connection and contrast to geometry. While in this problem distance is needed to decide when two radio stations are *related*, that is, close enough that their signals will interfere with each other, distance is not an intrinsic feature of a vertex-edge graph. What matters in a vertex-edge graph are connections, not the actual placement of the vertices or lengths of the edges. Thus, while vertex-edge graphs can be seen as geometric objects, and studied, for example, in a high school geometry course, the conventional geometric notions of location, size, and shape are not essential features of a vertex-edge graph model.

3.3 Example 3: Ranked-Choice Voting

In this student investigation, adapted from Hirsch et al. (2016), a fair choice is sought from among more than two alternatives. Thus, this is an example of the *fair decision-making* problem type. In the investigation below, answers to some of the posed questions are provided in brackets.

Suppose your class is taking a trip to a nearby park. The class must decide what to do for lunch. The options are to buy food at the park (P), bring a sack lunch (S), or eat at a nearby restaurant (R). Everyone must do the same thing for lunch.

What is a fair way to decide what your class will do for lunch?

Let's vote!

Instead of just voting for your favorite, you can get more information about everyone's opinion by *ranking* the three options. You will rank your favorite with a 1, second-favorite with a 2, and your least-favorite with a 3.

Suppose the results of your class voting are summarized in Table 1. A table like this is called a *preference table*.

Examine this table and each of the opinions in Fig. 4 about which lunch option is the winner. Answer and discuss the questions that follow.

- With which of these students do you agree? Why?
- Give a reasonable explanation for Taylin's thinking. [Taylin's opinion that Park should be the winner is reasonable since Park gets the most 1st and 2nd choice votes, and also the fewest 3rd choice votes.]
- How could Isaure explain to Andreas that Restaurant should not win? [Restaurant would be a poor choice for winner since most voters rank Restaurant as their least preferred option.]

Table 1 Preference table showing the results of voting by ranking the lunch options

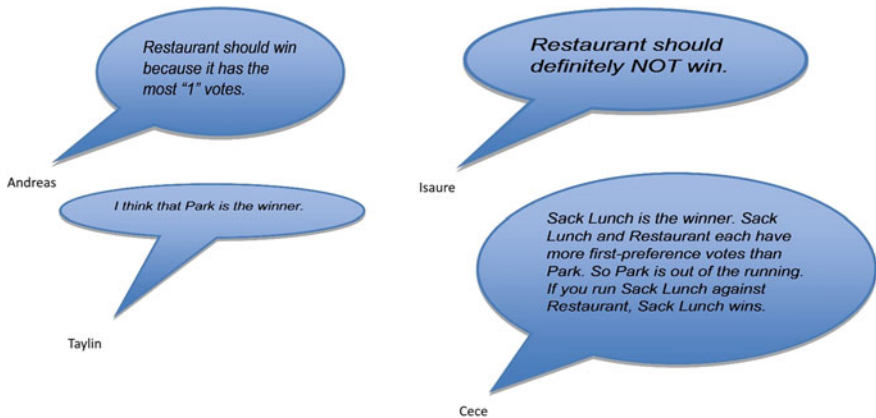|	Rankings					
Park (P)	1	1	2	2	3	3
Sack lunch (S)	2	3	1	3	1	2
Restaurant (R)	3	2	3	1	2	1
Number of voters	6 voters	4 voters	6 voters	7 voters	5 voters	5 voters

Fig. 4 Four students' opinions and reasoning about which lunch option is the winner

- Verify Cece's claim that Sack Lunch and Restaurant each have more first-preference votes than Park. Explain why Sack Lunch is the winner using Cece's method. [If you reallocate the 1st choice votes for Park to the other two options, since Park is eliminated in Cece's method, then the recomputed 1st choice totals for Sack Lunch and Restaurant are 17 to 16, respectively, so Sack Lunch wins.]
- Suppose everyone only voted for their favorite lunch option, and they did not rank the options by preference. In this case, which lunch option is the winner? Do you see any drawbacks to the voting method where you only vote for your favorite? [The winner based on voting only for your favorite is Park, since Park has the most 1st choice votes. But Park is the least preferred option for the most students! That's just not fair! This is a common drawback to the method where the winner is whoever gets the most 1st choice, or *favorite*, votes.]

The students' ideas above show that there are many ways to analyze the data from ranked-choice voting. In the full lesson, you will analyze several of the most common vote-analysis methods.

For now, consider Cece's method more carefully, which can be called the *top-two runoff method*. The top-two runoff method works by finding the top two candidates based on 1st-choice votes, and then running those two against each other to find the winner. Here is how it works in detail.

Step 1. Count the 1st choice votes to find the top two candidates.
Step 2. Eliminate all the other candidates. So now you have just two candidates.
Step 3. Some voters have had their 1st choice candidate eliminated. So reassign their votes and recompute.

Table 2 summarizes the results of voting for class president of the sophomore class at Northern High School. Use the *top-two runoff method* to find the winner.
Hints:

- Who are the top two candidates?
- Who gets eliminated? Cross out the row for that candidate.
- How many voters voted for the eliminated candidate as their 1st choice? Who will they now vote for as 1st choice?
- Now who has the most 1st choice votes?

Table 2 Preference table showing the results of ranked-choice voting for class president

	Rankings			
Jamal	1	1	3	3
Shirley	3	2	1	2
Ken	2	3	2	1
Number of voters	4 voters	6 voters	7 voters	8 voters

The top-two runoff method is just one of many common vote-analysis methods. As you have already seen, different methods can yield different winners. So the question arises: *Is there a perfect method that we should always use?* Surprisingly, the answer is, no! There is a famous theorem in mathematics, called *Arrow's Impossibility Theorem*, which states that no voting or analysis method is ideal in all situations. Thus, you must examine each particular voting situation, consider any rules or laws that already exist, take into account all the factors that you can, and make a decision about the best method to use for that situation.

While there is no perfect voting method, some are better than others. The commonly-used plurality method—where you vote for your favorite and whoever gets the most votes wins—is arguably, and unfortunately, the worst (when there are three or more candidates). As stated in a summary of an experts workshop on voting at the Centre for Voting Power and Procedures at the London School of Economics (2010):

> Plurality Voting is the worst of any known system to elect fairly a single winner from three or more candidates. The most serious problem, they [the co-directors of the Centre for Voting Power and Procedures] said, is that Plurality Voting often elects the candidate least preferred by an absolute majority of voters.

Experts often recommend the *points-for-preferences method* (also called the Borda method), *Instant Runoff Voting* (IRV), *approval voting*, or, if a winner is produced, the *all-pairs runoff method* (also called the Condorcet method).

4   Conclusion

Discrete mathematics includes core mathematical content and practices that are essential for a robust school mathematics curriculum. It naturally extends mathematical analysis into additional contexts that are interesting and relevant, such as fairness, networks, sequential change, and the Internet. It also introduces students to mathematics that may become increasingly useful to them as fields based on computation and coding expand.

Discrete mathematics stretches students to think about mathematics in different ways that may help them to see mathematics in a new light, as being about more than solving an equation or evaluating a formula. It provides an opportunity for developing students' reasoning ability, communication skills, problem solving ability, and modeling skills, as well as mathematical habits of mind that are specifically cultivated through studying discrete mathematics, such as algorithmic problem solving, combinatorial reasoning, and recursive thinking.

In short, discrete mathematics is *empirically powerful*, as a tool for modeling and solving fundamental contemporary problems, and it is *pedagogically powerful* in

that it can be used in the curriculum to simultaneously address content, process, and affect goals of mathematics education. As such, discrete mathematics is indeed essential mathematics for a 21st century school curriculum. Read on in this volume to find out more about the teaching and learning of discrete mathematics worldwide.

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