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Eric W. Hart
James Sandefur *Editors*

Teaching and Learning Discrete Mathematics Worldwide: Curriculum and Research

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Preface

"Discrete mathematics is the math of our time." So declared the immediate past president of the National Council of Teachers of Mathematics, John Dossey, in 1991 (as referenced in the first chapter, "Discrete Mathematics is Essential Mathematics in a 21st Century School Curriculum," of this volume). Nearly 30 years later that statement is still true, although the news has not yet fully reached school mathematics curricula. Nevertheless, much valuable work has been done, and continues to be done. This volume reports on some of that work. It provides a glimpse of the state of the art in learning and teaching discrete mathematics around the world, and it makes the case once again that discrete mathematics is indeed mathematics for our time, even more so today in our digital age, and it should be included in the core curricula of all countries for all students.

The chapters in this book are extended versions of papers presented at the thirteenth International Congress on Mathematical Education (ICME-13), held in Hamburg, Germany, in July 2016, as part of Topic Study Group 17 on discrete mathematics. The chapters are organized into six sections. The first section examines current thinking about discrete mathematics in the school curriculum. The remaining sections focus on core discrete mathematics content and practices for school mathematics—combinatorics and combinatorial reasoning, recursion and recursive thinking, networks and graphs, fair decision-making and game theory, and, finally, logic and proof. A brief description of the chapters in each section follows.

Part I: The Landscape of Discrete Mathematics in the School Curriculum

Hart and Martin's chapter launches the book. It identifies five essential discrete mathematics problem types that should be included in robust twenty-first century school curricula. Besides discussing these problem types in some detail—including what they are, what mathematics is involved, and why these five, classroom examples are given that make the discussion immediately practical and implementable. This chapter also serves as an introduction to the entire volume.

Rosenstein looks backward and ahead at the school discrete mathematics movement in the United States. He describes the opportunity that discrete mathematics provides for supporting reasoning, problem solving, and systematic thinking in the school mathematics curriculum and illustrates this opportunity with a set of problems that begin "Find all …." He analyzes the current lack of discrete mathematics in primary and secondary education in the U.S., despite a strong beginning in the 1980s and 1990s. He provides some rationale for why this has happened, why it is a counterproductive development, and then looks optimistically to a future in which discrete mathematics will play a more prevalent role in school curricula. He includes specific recommendations and examples.

In *Discrete Mathematics in Lower School Grades? Situation and Possibilities in Italy*, Gaio and Di Paola first describe results of a survey they have taken over 150 Italian mathematics teachers at all school levels. These results indicate that, especially at the lower grades, teachers have minimal knowledge of discrete mathematics topics such as cryptography, algorithms, and graph theory. On the other hand, teachers indicate a willingness and desire to learn about these topics and to teach them in their classes. Much of this desire is based on wanting students to engage actively in mathematical problem solving. Later, the authors describe an ongoing project resulting from their earlier survey in which they are working with teachers to design and implement discrete mathematics activities involving binary numbers, algorithms, and cryptography in grades 3 through 8.

In *Discrete Mathematics and the Affective Dimension of Mathematical Learning and Engagement*, Goldin addresses the importance of student attitudes, beliefs, emotions, and motivations when learning mathematics. The author argues that for many students, negative attitudes and emotions toward the traditionally taught curriculum can hinder their learning of mathematics. He goes on to describe some possible positive affordances arising by teaching discrete mathematics topics through problem solving contexts set in familiar experience. This can lead to exploration, enhanced engagement, and personal satisfaction. The chapter ends with a call for more research on the affective and conative dimensions of the learning and teaching of discrete mathematics.

Part II: Combinatorics and Combinatorial Reasoning

Combinatorics might be considered the mathematical art of counting. Combinatorial reasoning is the skill of reasoning about the size of sets, the process of counting, or the combinatorial setting to answer the question, How many? Combinatorics is a key part of discrete mathematics and combinatorial reasoning is a powerful mode of reasoning, a mathematical habit of mind, that is specifically developed in the learning and teaching of discrete mathematics.

This section of the book begins with the chapter by Coenen, Hof, and Verhoef, *Combinatorial Reasoning to Solve Problems*, which studies the mathematical thinking of students, aged 14–16, as they try to solve combinatorial problems involving combinations and permutations, such as how many dishes of ice cream can we have using three flavors, and how many ice cream towers on a cone can we have. One result of their research is observing that students often start problem solving at the highest level of attention, which often leads to more mistakes and incorrect answers then for students who start at a lower level. From the research, the authors observe that guidance from teachers is important. In fact, the chapter suggests how students, guided correctly, can develop relational understanding using combinatorial reasoning, which can lead to a better understanding of the problems and more confidence in their solutions.

In Höveler's chapter, *Children's Combinatorial Counting Strategies and their Relationship to Mathematical Counting Principles*, the results of a qualitative study are presented, the main goals of which were to identify how children solve combinatorial counting problems and to gain insights into the relationship between their strategies and conventional mathematical counting principles. Counting strategies identified and discussed include multiplicative, additive, and compensation strategies, in addition to recursive strategies and comparing structural features of problems. These strategies are examined with respect to their conceptual and operational differences and similarities to combinatorial counting principles, including the multiplication principle, the principle of inclusion/exclusion, and the so-called shepherd's principle.

The chapter by Lockwood and Reed, *Reinforcing Mathematical Concepts and Developing Mathematical Practices through Combinatorial Activity*, focuses on a rich combinatorial task involving counting passwords. The authors provide examples of affordances that undergraduate students gained by engaging with the task. They highlight two kinds of affordances—those that strengthened understanding about fundamental combinatorial ideas, and those that fostered meaningful mathematical practices. They consider pedagogical implications and, in particular, maintain that combinatorics is an area of mathematics that offers students chances to engage with accessible yet complex mathematical ideas and to develop important mathematical practices. They present examples of sophisticated student work that they hope will contribute to an overall goal of elevating the status of combinatorics specifically, and discrete mathematics more broadly, in the school and undergraduate curriculum.

The chapter by Vancso, et al. summarizes the ideas and background of a combinatorics research and teaching project, including historical reforms in the school curriculum in 1978 in Hungary and T. Varga's work. Thereafter they discuss the main elements of their current project: a pretest and developed teaching materials, including student worksheets with rich problems and some tools for teaching combinatorics such as *Poliuniversum*. In choosing the problems for the worksheets they were led by two research questions: (a) how students handle open tasks (which are presented in many of the combinatorial problems), and (b) how they use various manipulatives at different ages.

Part III: Recursion and Recursive Thinking

Recursion involves describing a given step in a sequential process in terms of the previous step(s). Such a description is often captured in a recursive formula (also called a difference equation), which might be informal, such as NEXT = NOW + 2, or formal, such as $a_n = a_{n-1}+ 2$, and is in contrast to an explicit (or closed-form) formula, such as $a_n = 2n$. Recursion and recursive thinking are powerful modeling and problem-solving strategies used throughout mathematics. They are particularly developed in the teaching and learning of discrete mathematics. The two chapters in this section discuss the benefits of recursion and recursive thinking in the classroom, as seen particularly in the study of difference equations and discrete dynamical systems.

In *Discrete Dynamical Systems: A Pathway for Students to Become Enchanted with Mathematics*, Devaney points out that the traditional mathematics curriculum consists primarily of fourth century, BC, geometry, eleventh century algebra, and possibly some seventeenth century calculus. He goes on to argue that to attract students to mathematics, they should have experiences in which they engage with some of the exciting areas of contemporary mathematics. Discrete mathematics offers a number of opportunities for engaging students in contemporary mathematics, as described throughout this monograph. In this chapter, the author describes a number of activities involving the modern field of discrete dynamical systems, particularly chaos and fractals. These activities, which have been successfully used with students for years, lead to the construction of some strange and beautiful shapes. Combining recursion and iteration with traditional geometric topics, such as the geometry of transformations, leads to students developing an understanding of why these shapes arise. The activities involve exploration and creativity on the part of the students as they learn important mathematics.

In *How Recursion Supports Algebraic Understanding*, Sandefur, Somers, and Dance propose the integration of recursive thinking with algebraic thinking. The chapter first gives a number of simple models, appropriate at a variety of school levels, that can be approached using both standard algebra and recursion. These models, building on the ideas that repeated addition is multiplication and repeated multiplication is exponentiation, lead to a more complete understanding of linear, quadratic, and exponential functions. The focus is on covariational thinking, particularly the differences between constant and variable change, an understanding that is at the core of learning calculus. The chapter gives a vision of how this integration can be achieved from early middle school through secondary school, within minimal change in the curriculum.

Part IV: Networks and Graphs

The two chapters in this section show how teaching networks and graphs (also called vertex-edge graphs) can not only help students learn important mathematical content, but also foster mathematical thinking and give students the experience of approaching problems similarly to how a research mathematician might approach a problem.

In *Food Webs, Competition Graphs, and a 60-Year Old Unsolved Problem*, Cozzens and Koirala define how a food web can be constructed by knowing the predator-prey relationships in a particular habitat. They then relate food webs to other types of graphs, most importantly competition graphs and interval graphs. After some discussion of these different types of graphs and their interrelationships, the authors proceed to discuss how, historically, these graphs have been used to try to understand relationships between species competing for the same resources. This gives a nice example of how the use of contemporary mathematics, which is accessible and relevant for high school students, can lead to a better understanding of our world, ecological relationships in this case.

In *Graph Theory in Primary, Middle and High School*, Ferrarello and Mammana report on research they conducted on the introduction of graph theory in grades 3 through 10 in Sicily. The activities center around the Königsberg bridges problem, and more generally, the idea of determining when a graph has an Euler cycle. While the focus is similar, the level of the activities is adjusted depending on grade level. The activities are described in some detail for the different grade levels and summaries of students' responses to these activities are given.

Part V: Fair Decision-Making and Game Theory

Game theory is an area of mathematics dealing with situations of cooperation and conflict involving players, moves, strategies, and outcomes. Broadly viewed, it includes the mathematics of fair decision-making as well as combinatorial games. Fair decision-making is the focus of Garfunkel's chapter; Colipan and Rougetet consider combinatorial games.

Garfunkel discusses a number of fairness models related to fair division and bankruptcy. Several models have a very long and colorful history. He emphasizes the role of mathematical modeling in solving such fairness and equity problems. In addition to showing how accessible these discrete models can be, he attempts to show their intrinsic interest and the fact that they can and should be introduced in high school and even middle school mathematics curricula.

In *Mathematical Research in the Classroom via Combinatorial Games*, Colipan describes the Chocolate Game, one of several Nim-type combinatorial games that can be used to give students an authentic research experience, similar to those of research mathematicians. In particular, the students consider questions that are mathematically easy to access, have a variety of strategies for going forward, and solutions to one question bring out new questions. While these games could be used for a wide variety of student levels, the chapter includes research results when this approach was used with a group of 50 first-year college students.

Rougetet's chapter provides a rich history of the Nim game, a prototypical combinatorial game, and she considers the role of this game in the latest reform of the French high school education system. This reform has led to changes in the content of the mathematics curricula including a new theme, *algorithmic and programming*, which aims at initiating pupils (7th–9th grades) to "write, develop and run a simple program". To achieve this, the curriculum offers several class activities centered on "games in a maze, …, Nim game and Tic-Tac-Toe". As the mathematical solution of Nim relies on the binary system, easily characterized by bistable circuits, the first electromechanical Nim playing machines were built in the 1940s, followed later by smaller and purely mechanical machines. This chapter presents these inventions—which claimed pedagogical purposes—and considers their use in classrooms as a recreational application to tackle the algorithmic and programming theme of curricula.

Part VI: Logic and Proof

Logic and proof are of course fundamental to all mathematics, discrete or not. Discrete mathematics provides an opportunity for students to develop their logical thinking and proof abilities in possibly new and more accessible settings.

In *Mathematics and Logic: Their Relationship in the Teaching of Mathematics*, Igoshin argues that we must give our future teachers a deeper understanding of mathematical logic for them to be totally effective in teaching their students the fundamentals of mathematical thinking. He breaks down what future teachers should know into four principles: (1) learning the structure of mathematical statements, (2) understanding the concept of proof of a mathematical statement, (3) training methods for proving, and (4) learning the structure of mathematical theories. He goes into each of these principles in some detail, including numerous examples, and shows how discrete mathematics supports the learning of logic.

Conclusion

Taken together, these authors provide a contemporary view of the teaching and learning of discrete mathematics worldwide. We hope that this volume prompts new work, future collaborations, and further progress in improving mathematics education for students and teachers everywhere, particularly through incorporation of the deep ideas, powerful methods, and modern applications of discrete mathematics.

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Part I
The Landscape of Discrete Mathematics in the School Curriculum

Discrete Mathematics Is Essential Mathematics in a 21st Century School Curriculum

Eric W. Hart and W. Gary Martin

Abstract In this chapter we discuss discrete mathematics in the school curriculum. We make the case, based on many years of curriculum research and design, that discrete mathematics is essential in a modern, robust school mathematics curriculum, and that five broad problem types emerge as ways to organize the diversity of discrete mathematics contexts that are important and appropriate for the curriculum—enumeration, sequential change, relationships among a finite number of elements, information processing, and fair decision-making. In this chapter, these five problem types are briefly described and three classroom examples are provided. Subsequent chapters in this volume provide additional analysis, research, and more classroom examples.

Keywords Discrete mathematics · Curriculum · Sequential change Information processing · Fair decision-making · Enumeration · Vertex-edge graphs

Discrete mathematics is a robust field of mathematics with many modern applications. Yet it has no succinct definition, so we begin this chapter, and indeed this volume, by considering the nature and relevance of discrete mathematics.

1 The Rise of Discrete Mathematics

Discrete mathematics is described by *Topic Study Group 17: Teaching and Learning of Discrete Mathematics* at the 13th International Congress on Mathematical Education (Hart et al. 2017) as follows:

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> Discrete mathematics is a comparatively young branch of mathematics with no agreed-upon definition but with old roots and emblematic problems. It is a robust field with applications to a variety of real world situations, and as such takes on growing importance in contemporary society. We take discrete mathematics to include a wide range of topics, including logic, game theory, algorithms, graph theory (networks), discrete geometry, number theory, discrete dynamical systems, fair decision making, cryptography, coding theory, and counting. Cross-cutting themes include discrete mathematical modeling, algorithmic problem solving, optimization, combinatorial reasoning, and recursive thinking.

A recent publication that looks to the future of mathematics, *The Mathematical Sciences in 2025* (Committee on the Mathematical Sciences 2025 2013), states that "over the years there have been important shifts in the level of activity in certain subjects—for example, the growing significance of probabilistic methods, the rise of discrete mathematics, and the growing use of Bayesian statistics" (p. 72). The book identifies two new drivers of mathematics–computation and big data, and for both of these drivers it describes how discrete mathematics plays an important role–for example, discrete mathematics algorithms for information processing, dynamical systems in ecology, networks in industry and the humanities, and discrete optimization (p. 77).

Discrete mathematics is particularly suited to applications involving technology and computers. In fact, discrete mathematics is sometimes considered the mathematics of computer science. The *discrete* aspect of discrete mathematics is often contrasted with the *continuous* mathematics of calculus. It is appropriate to connect discrete mathematics to computers and contrast it with calculus, but neither characterization is complete.

We may take the following as a definition: *Discrete mathematics is a collection of mathematical concepts and methods that help us solve problems that involve a countable* (*often finite*) *number of elements or processes and connections among them.*

It is more helpful in coming to understand discrete mathematics to consider the wide variety of important and interesting problems that can be solved. Examples of questions that can be naturally investigated with discrete mathematics include:

> How can you avoid conflicts when scheduling meetings, shipping hazardous chemicals, or assigning frequencies to radio stations? How can you schedule a project for shortest completion time when it consists of numerous interconnected sub-projects? How can you fairly decide among competing alternatives, like candidates standing for election? How can you fairly divide or apportion objects, such as seats of congress or property in an inheritance? How can you ensure accuracy, security, and efficiency in digital transactions such as transferring files, making online purchases, or posting to social networks? How many different Personal Identification Numbers (PINs), IP addresses, or pizzas with different toppings are possible? How can you model and analyse processes of sequential change, such as year-to-year growth in population, month-to-month change in credit card debt, or periodic medicine dosage?

The breadth and diversity of the questions above reflect the power and applicability of discrete mathematics. Discrete mathematics is indeed essential to understanding our modern technological world, and as such it is essential to include discrete mathematics in the school curriculum. But which parts of discrete mathematics should be included?

2 Incorporating Discrete Mathematics into the School Curriculum: Five Essential Discrete Mathematics Problem Types

The power of discrete mathematics lies in mathematical modeling and solving problems, so instead of focusing on which topics to include, we consider broad problem types. Based on curriculum research, development, and implementation in classrooms and textbooks over many years, five broad problem types emerge as the most potent types of discrete mathematical problems that are relevant for the school curriculum—*enumeration, sequential step-by-step change, relationships among a finite number of elements, information processing, and fair decision-making*.

These five problem types, along with the respective discrete mathematics domains of combinatorics, recursion, graph theory, informatics, and the mathematics of voting and fair division (which can be viewed as part of game theory), have been identified as fundamental for school mathematics based on the following work over the past 30 years:

- Initial momentum provided by the collegiate discrete mathematics movement in the 1980s (e.g., Hart 1985; Ralston 1989; Dossey 1991);
- Federally-funded professional development programs in the 1990s in the United States to implement the National Council of Teachers of Mathematics (NCTM 1989) standard on discrete mathematics (Hart and Schoen 1989–1993; Rosenstein 1989–1996; Sandefur 1990–1994; Kenney 1992–1996);
- Articles and books to support implementation of the discrete mathematics recommendations in NCTM's (1989, 2000) standards (Hart et al. 1990, 2008; Hart 1991; Hirsch and Kenney 1991; Debellis et al. 2009);
- Articles recommending more discrete mathematics in the *Common Core State Standards for Mathematics* (NGA Center and CCSSO 2010) in the United States (Hart and Martin 2008, 2016; Martin and Hart 2012);
- Curriculum research and design to develop high school textbooks that integrate discrete mathematics (Hart 1997, 1998, 2008, 2010; Hirsch et al. 2015, 2016).

Each of the five problem types is briefly discussed below in this chapter. Subsequent chapters in this volume provide further analysis:

Rosenstein takes a broad look at several of the problem types in elementary and secondary school in the United States. Gaio and Di Paola consider discrete mathematics in the lower grades in Italy, focusing on graph theory (*relationship-among-elements*) and cryptography (an aspect of *information processing*). The *enumeration* problem type is the focus of the chapters by Coenen et al., Hoveler, Lockwood and Reed, and Vancso et al., as they consider combinatorics and combinatorial reasoning. The *sequential change* problem type is the focus of the chapters by Devaney and by Sandefur et al., as they discuss recursion and recursive thinking. Cozzens and Koirala and Ferrarello and Mammana focus on the *relationship-among-elements* problem type in their chapters on networks and graphs. Regarding *fair decision-making*, this can be considered part of the broad area of game theory, and three chapters are in this broad category—Garfunkel discusses fairness models for fair division and bankruptcy problems, while Colipan and Rougetet analyze combinatorial games. Finally, Goldin considers the affective dimension of studying discrete mathematics generally, and Igoshin considers logic, which can be considered part of discrete mathematics but is of course fundamental to all parts of mathematics, discrete or not.

The following sections provide a brief description of the five problem types. Classroom examples follow.

2.1 Enumeration

Enumeration problems involve counting. Perhaps to solve a problem or part of a problem it would be helpful to count something. What are you counting? What is the structure of the counting situation? Are you counting choices from a collection of objects, outcomes from a sequence of tasks, or counting from "this or that" distinct situation? General consideration of these questions leads to the following three common and useful types of enumeration problems, which are particularly relevant in the school curriculum.

- Count the number of choices from a collection of objects—Are you choosing from a collection of objects? If so, consider the issues of order and repetition in your choice, yielding four possible problem types, including permutations and combinations.
- Count the number of possible combined outcomes from a sequence of tasks—Is there a sequence of tasks? Can the situation be represented by a tree diagram, where the outcomes from each task are represented in each level of branching in the tree? If so, try applying the Multiplication [Fundamental] Principle of Counting. But be careful! This seemingly simple principle requires that there is a *sequence* of tasks, that the number of outcomes at each stage is *independent* of the choices in the previous stages, and that the *combined* outcomes at the end, which is what you are counting, are all *distinct*.
- Count this or that—Are there two sets involved, the members of which need to be counted? If so, be careful of any overlap and try the Addition Principle of Counting or the inclusion/exclusion principle.

2.2 Sequential Change

Sequential, step-by-step change is a natural part of our world. Think about all the situations where some quantity is changing yearly, monthly, or daily; from an amount in one period or at one state to a different amount in the next period or state. A recursive model is often used to analyze such situations. You can build a recursive model in school mathematics courses by considering questions such as below:

- Can you write an equation using the words NOW and NEXT to describe how the quantity in the NEXT period compares to the quantity NOW? This provides an intuitive and accessible way to model the situation with a recursive equation. A recursive equation using subscripts or function notation can be helpful later for further analysis, but such notation is infamously difficult for students to understand and need not be a barrier for earlier use of recursive models to analyze sequential change situations.
- How many steps of recursion are there? That is, does just the current step determine the next step, or are two (or more) previous steps needed to determine the next step? If more than one step is needed, then a simple NOW-NEXT equation will not work and something more complicated, perhaps with subscripts, will be required.
- How does the process start? What is the initial amount? To build a recursive model, you need to know where to start as well as how to get from one step to the next.

2.3 Relationship Among a Finite Number of Elements

This broad class of problems arises in situations where there are many objects or elements and a relationship between pairs of those elements. These are problems about networks, such as communication networks, transportation networks, or social networks. Such problems can be modeled using vertex-edge graphs.

- What are the elements? Represent those as vertices. What is the relationship between pairs of elements? Draw an edge between vertices (elements) that are related, thus creating a vertex-edge graph model.
- Is it a conflict relationship? Try a vertex coloring model.
- Is it a prerequisite relationship? Try a critical path analysis.
- Does the context suggest visiting *each vertex* or using *each edge* of the vertex-edge graph? Try a Hamilton path or an Euler path, respectively.
- Does the context suggest reaching all the vertices of the graph without any redundancy (no loops)? Try a spanning tree. Are the edges weighted? Perhaps a minimal spanning tree will be useful.

Networks are everywhere in modern life and thus vertex-edge graphs are an important and relevant topic. Arguably, fluent use of vertex-edge graphs may be a mathematical skill rivaling many of the traditional skills currently included in the curriculum. We need to think hard about whether traditional topics squeeze out important discrete mathematics topics. And if so, should they?

2.4 Information Processing

Consider problems about information processing, that is, problems about searching, securing, sending, or receiving information, especially digital information, particularly in contexts related to the Internet. There are four fundamental issues of information processing that could be involved, each of which can be addressed using fundamental school mathematics.

- *Access*—For information to be useful it must be accessible. How does this relate to school mathematics? Well, for example, elementary set theory and logic are part of Internet search engines and often can be used explicitly to specify advanced searches. Or, in the theme of algorithmic problem solving, specifically related to coding, searching and sorting algorithms are fundamental aspects of computer science that help make information accessible and can be productively studied in school mathematics.
- *Accuracy*—It is important to ensure the accuracy of information as it is sent or received. For example, when sending a photo from deep space or scanning a UPC product code at a grocery store checkout station, error-detecting and -correcting codes are used. Such codes often use elementary number theory, like modular arithmetic, or, in more advanced settings, linear algebra.
- *Security*—In many situations it is essential that information is kept secure and private. For example, private email and secure credit card numbers are often sent using public-key cryptography, which is based on number theory. This fascinating topic is relevant to almost all students in their daily digital lives. The mathematics of secret codes begins simply and extends to be as challenging as students and teachers desire.
- *Efficiency*—In our modern digital information age we consume more and more data. With streaming video, online mapping, photos, and constant social media updates, everyone wants faster Internet service and more memory with a larger data plan for their phones. All this requires efficient data transfer and storage. Students can learn about data compression using variable-length codes and Huffman trees. These accessible mathematical topics are highly relevant, for example they are used in most compressed file formats today, like .jpg images and .mpg videos. We also want the algorithms we use to solve problems to be efficient. Basic consideration of algorithm complexity is also relevant and appropriate for school mathematics, such as the qualitative differences among linear, exponential, and factorial growth. For example, the famous *Traveling Salesman Problem* in graph theory is easy to state and begin analyzing, but a simple *brute-for*ce solution algorithm that checks all possible routes grows factorially and thus is impractical, even with the world's fastest supercomputer.

2.5 Fair Decision-Making

Many problems require making a fair decision. In particular, consider problems about fair voting or fair division.

Must one alternative among several be fairly chosen by a group of people? Consider a voting model. Is it a *one-person-one-vote* situation, as in elections for government office, or a *one-person-many-votes* situation, as in stockholder voting where each stockholder has as many votes as shares owned?

- In a one-person-one-vote situation in which there are more than two alternatives or candidates, *ranked-choice voting* is often the best option, whereby people vote by ranking the candidates rather than just designating their favorite candidate. The data gathered from ranked-choice voting provides rich information about voter preferences, which can be analyzed using a number of efficient vote-analysis methods to choose a good winner.
- In a one-person-many-votes situation, a *weighted voting* model can be used, in which both *weight* (the number of votes an individual has) and *power* (a measure of how critical an individual's vote is) are analyzed.

Does something need to be fairly divided or apportioned? Consider a fair division model. The choice of an effective fair division method depends on what is being divided.

- Is it divisible (like land or cake), or indivisible (like seats of congress or antiques)?
- If divisible, is it homogeneous (like a flat tract of land) or heterogeneous (like land that is hilly and forested)?
- If indivisible, are the objects identical (like seats of congress) or non-identical (like antiques)?

Depending on answers to these questions you can use different models and methods of fair division, many of which are accessible, engaging, and relevant for school mathematics.

We conclude this chapter with three classroom examples, related to three of the five problem types—sequential change, relationships among elements, and fair decision-making. These examples are appropriate for middle or high school students. Other chapters in this book provide additional classroom examples, including several at the elementary school level and with respect to all five problem types.

3 Classroom Examples

3.1 Example 1: Proper Medicine Dosage

This example is modified from Hart and Martin (2016). It illustrates the *sequential change* problem type.

Consider the common situation of taking repeated daily doses of a medication. Suppose a hospital patient is given an antibiotic to treat an infection. He is initially given a 30 mg dose and then receives another 10 mg at the end of every six-hour period thereafter. Through natural body metabolism, about 20% of the antibiotic is eliminated from his system every six hours. This situation raises many interesting questions, such as:

> What is the long-term amount of antibiotic in the patient's system?
>
> How should this prescription be modified if the doctor decides that a long-term amount of 25 mg is desired?

This problem is about a process of sequential change, namely, the change in the amount of antibiotic in the patient's system, which changes every six hours. Thus, a recursive model may be useful. Note that a student does not need to know the precise definition of recursion to continue, he or she can build a model as follows.

Is it possible to describe this process of sequential change with an equation using the words NOW and NEXT? In this case, if NOW is the amount of antibiotic in the patient's system now, and NEXT represents the amount after the next six-hour dose, then NEXT = 0.8 · NOW + 10. (This model assumes that the amount is measured after the regular dose is taken.) Is there an initial amount? Yes, 30 mg.

Thus, we have a model: Start with 30, then represent the step-by-step change based on 20% elimination and a regular 10 mg dose with NEXT = 0.8 · NOW + 10.

Now a spreadsheet or calculator can be used to easily compute the amounts over time. Initially, the amount is 30 mg. Then, six hours later, 0.8 · 30 + 10 = 34 mg, then, another six hours later, 0.8 · 34 + 10 = 37.2 mg, and so on.

What about the long-term amount? Is there an *equilibrium value*? With technology, we can quickly see that the long-term amount stabilizes at about 50 mg, as shown in the first two columns of Fig. 1.

How can we change the prescription to get a long-term amount of 25 mg? Using our recursive model and a spreadsheet, we can easily try different adjustments. Maybe we start by cutting the *initial dose* of 30 mg in half, since the goal is to cut the long-term amount in half. (Try it; it doesn't work! Surprisingly, the long-term amount stays at 50. See the third and fourth columns of Fig. 1.) How about cutting the *regular dosage* in half? (Try it; it works. See the fifth and sixth columns of Fig. 1.)

	A	B	C	D	E	F
1	***Recurring dose: 10 mg***		***Recurring dose: 10 mg***		***Recurring dose: 5 mg***	
2	Dose #	Mg in system	Dose #	Mg in system	Dose #	Mg in system
3	1	30	1	15	1	30
4	2	34	2	22	2	29
5	3	37.2	3	27.6	3	28.2
6	4	39.76	4	32.08	4	27.56
7	5	41.808	5	35.664	5	27.048
8	6	43.4464	6	38.5312	6	26.6384
9	7	44.75712	7	40.82496	7	26.31072
10	8	45.805696	8	42.659968	8	26.048576
11	9	46.6445568	9	44.1279744	9	25.8388608
12	10	47.31564544	10	45.30237952	10	25.67108864
13	11	47.85251635	11	46.24190362	11	25.53687091
14	12	48.28201308	12	46.99352289	12	25.42949673
15	13	48.62561047	13	47.59481831	13	25.34359738
16	14	48.90048837	14	48.07585465	14	25.27487791
17	15	49.1203907	15	48.46068372	15	25.21990233
18	16	49.29631256	16	48.76854698	16	25.17592186
19	17	49.43705005	17	49.01483758	17	25.14073749
20	18	49.54964004	18	49.21187007	18	25.11258999
21	19	49.63971203	19	49.36949605	19	25.09007199
22	20	49.71176962	20	49.49559684	20	25.07205759
23	21	49.7694157	21	49.59647747	21	25.05764608
24	22	49.81553256	22	49.67718198	22	25.04611686
25	23	49.85242605	23	49.74174558	23	25.03689349
26	24	49.88194084	24	49.79339647	24	25.02951479
27	25	49.90555267	25	49.83471717	25	25.02361183

Fig. 1 Spreadsheet showing the sequential change in the amount of antibiotic in a patient's system for different initial and recurring doses

The recursive model is very accessible and useful, especially when using technological tools such as spreadsheets, graphing calculators, or computer coding. Further investigation can be carried out as needed and as deemed appropriate by the curriculum and teacher. For example, students can create more formal recursive equations using subscript or function notation; they can devise and analyze *closed-form* equations; they can generate graphs showing the relationship between dose number and amount of antibiotic in the system or use so-called *cobweb graphs* that show the relationship between successive amounts of antibiotic.

3.2 *Example 2: Optimally Assigning Frequencies to Radio Stations*

This example is adapted from *Focus in High School Mathematics* (NCTM 2009, pp. 70–72, as adapted there from Hirsch et al. 2015). It illustrates the use of vertex-edge graphs to model and solve a problem about a *relationship among elements*.

Task: The Federal Communications Commission (FCC) needs to assign radio frequencies to seven new radio stations located on the grid in Fig. 2. Such assignments are based on several considerations, including the possibility of creating interference by assigning the same frequency to stations that are too close together. In this simplified situation, we assume that broadcasts from two stations located within 200 miles of each other will create interference if they broadcast on the same frequency, whereas stations more than 200 miles apart can use the same frequency to broadcast without causing interference with each other.

Consider these questions:

> How can a vertex-edge graph be used to assign frequencies so that the fewest number of frequencies is used and no stations interfere with each other?
>
> What would each vertex represent? What would an edge represent?
>
> What is the fewest number of frequencies needed?

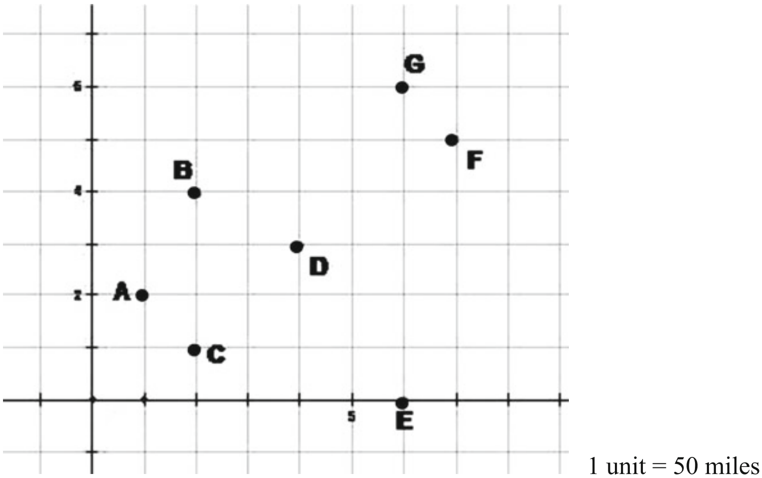

Fig. 2 Grid showing placement of seven radio stations

In the Classroom: Students work on the task in groups. Each group agrees that a vertex in the graph represents a radio station. So the graph will have seven vertices. What about edges? Some groups decide to make a graph model in which two vertices will be joined by an edge if the stations they represent are within 200 miles of each other. Other groups suggest that two vertices will be joined with an edge if the stations are *more* than 200 miles apart. After some discussion, realizing that both options can lead to a solution but one may be easier, the choice is made to use the first suggestion, that of joining vertices with an edge if the distance between them is less than (or equal to) 200 miles, because that will show stations that cannot share the same frequency and thus when different frequencies are needed.

The next step in building the model requires determination of the distance between each pair of stations. Some groups compute the distances using the distance formula or Pythagorean theorem and a calculator. Other groups use a length of string or paper strip marked off using the given scale to show 200 miles. In any case, groups produce models similar to the one in Fig. 3.

Now students work at assigning frequencies to vertices (which can be thought of as coloring the vertices) so that no two vertices joined by a single edge have the same frequency (color). They look for a method that will generate the fewest frequencies for this particular graph. Once they find a solution, they are asked to explain.

One group explained their solution this way:

> We found that the fewest number of frequencies that could be used was four. We reasoned this way: First, look at the collection of vertices *A* through *D*. Each of these vertices is joined by an edge to each of the other three in the collection. So no two vertices in this collection of four vertices can have the same frequency. That means you need *at least* four frequencies. But can you actually get by with four frequencies for *all* the vertices? Well,

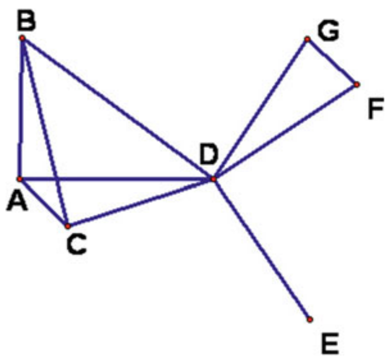

Fig. 3 Vertex-edge graph model of the radio station problem—vertices are radio stations, an edge between two vertices indicates that the two radio stations can interfere with each other and thus must have different assigned frequencies

suppose you assign frequency 1 to vertex *A*, frequency 2 to vertex *B*, frequency 3 to vertex *C,* and frequency 4 to vertex *D*. You can finish the assignment by assigning frequency 1 to vertex *G* (because *G* doesn't interfere with *A*), frequency 2 to vertex *F* (because *F* doesn't interfere with *B*), and frequency 3 to vertex *E* (because *E* doesn't interfere with *C*). So four frequencies will do the job! So you need at least four and four actually works, so that proves it—four is the fewest!"

Brief Analysis: Many aspects of mathematics, modeling, and mathematical practices are evident in this example. For example: (a) choosing and using appropriate mathematics—vertex-edge graphs, specifically vertex coloring, as well as the Pythagorean theorem and the distance formula, (b) explicitly building a model—carefully considering what the vertices and edges will represent, in particular notice the students' discussion about whether to connect vertices with an edge if they are more than 200 miles apart or less than 200 miles apart, which is in fact an important decision (technically between a conflict graph or a compatibility graph) that significantly affects the model and the solution method, (c) analyzing the model in order to understand the situation and make better decisions—students analyze the graph model to figure out how to assign frequencies in a way that results in no interference and uses the fewest number of frequencies, (d) reasoning and constructing arguments—the students essentially prove that a *complete graph* on four vertices requires four colors, and (e) algorithmic problem solving—they devise a systematic procedure for assigning frequencies (coloring vertices) and explain why the procedure works. Finally, note an interesting connection and contrast to geometry. While in this problem distance is needed to decide when two radio stations are *related*, that is, close enough that their signals will interfere with each other, distance is not an intrinsic feature of a vertex-edge graph. What matters in a vertex-edge graph are connections, not the actual placement of the vertices or lengths of the edges. Thus, while vertex-edge graphs can be seen as geometric objects, and studied, for example, in a high school geometry course, the conventional geometric notions of location, size, and shape are not essential features of a vertex-edge graph model.

3.3 Example 3: Ranked-Choice Voting

In this student investigation, adapted from Hirsch et al. (2016), a fair choice is sought from among more than two alternatives. Thus, this is an example of the *fair decision-making* problem type. In the investigation below, answers to some of the posed questions are provided in brackets.

Suppose your class is taking a trip to a nearby park. The class must decide what to do for lunch. The options are to buy food at the park (P), bring a sack lunch (S), or eat at a nearby restaurant (R). Everyone must do the same thing for lunch.

What is a fair way to decide what your class will do for lunch?

Let's vote!

Instead of just voting for your favorite, you can get more information about everyone's opinion by *ranking* the three options. You will rank your favorite with a 1, second-favorite with a 2, and your least-favorite with a 3.

Suppose the results of your class voting are summarized in Table 1. A table like this is called a *preference table*.

Examine this table and each of the opinions in Fig. 4 about which lunch option is the winner. Answer and discuss the questions that follow.

- With which of these students do you agree? Why?
- Give a reasonable explanation for Taylin's thinking. [Taylin's opinion that Park should be the winner is reasonable since Park gets the most 1st and 2nd choice votes, and also the fewest 3rd choice votes.]
- How could Isaure explain to Andreas that Restaurant should not win? [Restaurant would be a poor choice for winner since most voters rank Restaurant as their least preferred option.]

Table 1 Preference table showing the results of voting by ranking the lunch options

	Rankings					
Park (P)	1	1	2	2	3	3
Sack lunch (S)	2	3	1	3	1	2
Restaurant (R)	3	2	3	1	2	1
Number of voters	6 voters	4 voters	6 voters	7 voters	5 voters	5 voters

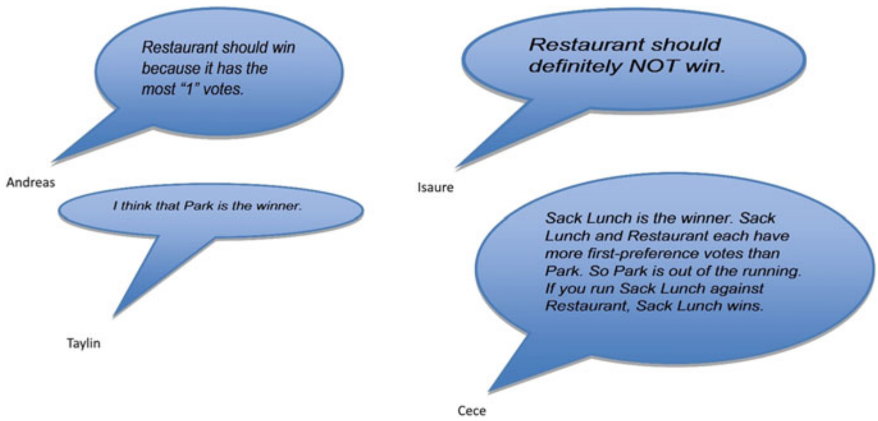

Fig. 4 Four students' opinions and reasoning about which lunch option is the winner

- Verify Cece's claim that Sack Lunch and Restaurant each have more first-preference votes than Park. Explain why Sack Lunch is the winner using Cece's method. [If you reallocate the 1st choice votes for Park to the other two options, since Park is eliminated in Cece's method, then the recomputed 1st choice totals for Sack Lunch and Restaurant are 17 to 16, respectively, so Sack Lunch wins.]
- Suppose everyone only voted for their favorite lunch option, and they did not rank the options by preference. In this case, which lunch option is the winner? Do you see any drawbacks to the voting method where you only vote for your favorite? [The winner based on voting only for your favorite is Park, since Park has the most 1st choice votes. But Park is the least preferred option for the most students! That's just not fair! This is a common drawback to the method where the winner is whoever gets the most 1st choice, or *favorite*, votes.]

The students' ideas above show that there are many ways to analyze the data from ranked-choice voting. In the full lesson, you will analyze several of the most common vote-analysis methods.

For now, consider Cece's method more carefully, which can be called the *top-two runoff method*. The top-two runoff method works by finding the top two candidates based on 1st-choice votes, and then running those two against each other to find the winner. Here is how it works in detail.

Step 1. Count the 1st choice votes to find the top two candidates.
Step 2. Eliminate all the other candidates. So now you have just two candidates.
Step 3. Some voters have had their 1st choice candidate eliminated. So reassign their votes and recompute.

Table 2 summarizes the results of voting for class president of the sophomore class at Northern High School. Use the *top-two runoff method* to find the winner.

Hints:

- Who are the top two candidates?
- Who gets eliminated? Cross out the row for that candidate.
- How many voters voted for the eliminated candidate as their 1st choice? Who will they now vote for as 1st choice?
- Now who has the most 1st choice votes?

Table 2 Preference table showing the results of ranked-choice voting for class president

	Rankings			
Jamal	1	1	3	3
Shirley	3	2	1	2
Ken	2	3	2	1
Number of voters	4 voters	6 voters	7 voters	8 voters

The top-two runoff method is just one of many common vote-analysis methods. As you have already seen, different methods can yield different winners. So the question arises: *Is there a perfect method that we should always use?* Surprisingly, the answer is, no! There is a famous theorem in mathematics, called *Arrow's Impossibility Theorem*, which states that no voting or analysis method is ideal in all situations. Thus, you must examine each particular voting situation, consider any rules or laws that already exist, take into account all the factors that you can, and make a decision about the best method to use for that situation.

While there is no perfect voting method, some are better than others. The commonly-used plurality method—where you vote for your favorite and whoever gets the most votes wins—is arguably, and unfortunately, the worst (when there are three or more candidates). As stated in a summary of an experts workshop on voting at the Centre for Voting Power and Procedures at the London School of Economics (2010):

> Plurality Voting is the worst of any known system to elect fairly a single winner from three or more candidates. The most serious problem, they [the co-directors of the Centre for Voting Power and Procedures] said, is that Plurality Voting often elects the candidate least preferred by an absolute majority of voters.

Experts often recommend the *points-for-preferences method* (also called the Borda method), *Instant Runoff Voting* (IRV), *approval voting*, or, if a winner is produced, the *all-pairs runoff method* (also called the Condorcet method).

4 Conclusion

Discrete mathematics includes core mathematical content and practices that are essential for a robust school mathematics curriculum. It naturally extends mathematical analysis into additional contexts that are interesting and relevant, such as fairness, networks, sequential change, and the Internet. It also introduces students to mathematics that may become increasingly useful to them as fields based on computation and coding expand.

Discrete mathematics stretches students to think about mathematics in different ways that may help them to see mathematics in a new light, as being about more than solving an equation or evaluating a formula. It provides an opportunity for developing students' reasoning ability, communication skills, problem solving ability, and modeling skills, as well as mathematical habits of mind that are specifically cultivated through studying discrete mathematics, such as algorithmic problem solving, combinatorial reasoning, and recursive thinking.

In short, discrete mathematics is *empirically powerful*, as a tool for modeling and solving fundamental contemporary problems, and it is *pedagogically powerful* in that it can be used in the curriculum to simultaneously address content, process, and affect goals of mathematics education. As such, discrete mathematics is indeed essential mathematics for a 21st century school curriculum. Read on in this volume to find out more about the teaching and learning of discrete mathematics worldwide.

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The Absence of Discrete Mathematics in Primary and Secondary Education in the United States… and Why that Is Counterproductive

Joseph G. Rosenstein

Abstract This chapter describes the opportunity that discrete mathematics provides for supporting reasoning, problem solving, and systematic thinking in the school mathematics curriculum and illustrates this opportunity by providing a set of discrete mathematics problems that begin "Find all… ." It also provides a year-by-year model for how discrete mathematics can be included in the primary and secondary curriculum. Finally, the article describes some of the possible reasons why discrete mathematics was not included in the new national mathematics standards in the U.S., and why we consider these reasons misguided, in light of the opportunities provided when discrete mathematics is part of the curriculum.

Keywords Counting · Combinatorics · Graphs · Systematic listing
Divide and conquer · Reasoning · Problem solving · Standards
Common Core State Standards in Mathematics

The title of this article speaks of the absence of discrete mathematics in primary and secondary education in the United States. Why has this happened?

The *Common Core State Standards for Mathematics* (NGA Center and CCSSO 2010) that were developed in 2009 and adopted soon afterwards by almost all of the states in the United States essentially excludes discrete mathematics. More specifically, no mention is made of graphs (those with vertices and edges), no mention is made of systematic listing and counting (combinatorics) except as an adjunct to the probability standard in the 8th grade, no mention is made of modern issues involving fairness (including fair division, apportionment, and elections), recursion (including Fibonacci numbers), or codes and cryptography, and the word *pattern* barely occurs in the standards even though mathematics is often referred to

In the video at https://www.youtube.com/watch?v=ff29i_yPoZ0 the author discusses many of the issues raised in this article.

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as the science of patterns. Since these topics are not in the Common Core, they are not addressed in the assessment tools that are based on the Common Core. Since most teachers, schools, and districts are being judged on the performance of their students on assessment tools based on the Common Core, teachers, schools, and districts feel they cannot devote time or effort to topics that are extraneous to the Common Core, including topics in discrete mathematics. Thus discrete mathematics is now absent from primary and secondary education in the United States.

Was discrete mathematics present in U.S. education before the advent of the Common Core? While not universally present, nevertheless discrete mathematics topics were taught in many schools by many teachers, teachers were introduced to discrete mathematics topics in professional development activities, and these topics were the focus of several textbooks (e.g., DeBellis and Rosenstein 2005, Rosenstein 2014) and appeared in many standard textbook series (e.g., COMAP 2013; Hirsch et al. 2015). This was in large part as a consequence of the recommendation of discrete mathematics in the 1989 publication of *Curriculum and Evaluation Standards* by the National Council of Teachers of Mathematics, and recommendation for the integration of "the main topics of discrete mathematics" in the K-12 curriculum in its 2000 publication, *Principles and Standards for School Mathematics*, which notes that: "As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum."

Unfortunately, in the Common Core era discrete mathematics is hardly present in the U.S. school curriculum. The situation in the United States prior to the Common Core standards is described more thoroughly in an article entitled *Discrete mathematics in primary and secondary schools in the United States* by DeBellis and Rosenstein (2004).

1 Why Should Discrete Mathematics Be Included?—Reasoning and Problem Solving

We argue that school mathematics curricula should take seriously the idea that an important reason for studying mathematics is to understand reasoning and to learn how to solve problems: Discrete mathematics is an excellent vehicle to help students at all grade levels become the problem solvers and reasoners that we desire. Although the Common Core speaks positively about reasoning and problem solving, it ignores this important arena that is both accessible to all students at all grade levels and that can foster the desired reasoning and problem solving.

In the examples that follow, keep in mind that the content in these examples may not be significant in itself, but that the experience that students have with learning how to use systematic reasoning can be very significant in shaping their thinking in mathematics, and in other areas of human endeavor.

One class of problems for which systematic reasoning is critical consists of those problems that begin with the phrase "Find all … ." For example, if you give young children cut-out shirts and trousers of several different colors and ask them to “find all” outfits that can be made, they will typically proceed in a random fashion. When they show you all the outfits they have made, you will find that some outfits are omitted and that some are duplicated.

Discrete mathematics provides a way of learning to answer such questions systematically. Students should construct a chart (see Table 1) where the shirts are listed at the top and the trousers at the left, and the outfits can be located in the cells of the chart. Actually, before constructing a chart with words in the cells, they should put the outfits themselves in the cells of a “chart.”

Subsequently, they should use a tree diagram (see Fig. 1), where the first branching is for the shirts and the second branching is for the trousers.

Ultimately, students should use the Multiplication Principle of Counting, which in this context says that “if there are four ways of selecting a shirt and, in each case, there are three ways of selecting trousers (independently of which shirt is selected) then there are 4×3 or 12 ways of selecting both—that is, there are 12 outfits altogether.

Note that the tree diagram also provides a way of systematically listing the 12 outfits; if you follow each of the branches of the tree, you end up with the vertical list of all 12 outfits on the right (see Fig. 2).

Similarly, if you ask secondary students to “find all” factors of 200, they will most likely proceed randomly. The same strategies presented above can also be used in this situation. Consider using a chart, as in Table 2. Since $200 = 8 \times 25 = 2^3 \times 5^2$, any factor of 200 is a product of a power of 2 from 0 to 3 —that’s four possibilities—and a power of 5 from 0 to 2—that’s three possibilities, so 200 has 4×3 or 12 factors. Thus students can arrive at a systematic way of finding and of listing all the factors of 200, and can see that this problem is essentially the same as the previous problem of finding all the outfits.

In learning how to solve both problems above, the focus is moving from random behavior to systematic behavior.

Here is another “find all” example. Find all graphs that have exactly four vertices. Rather than proceed randomly, students can learn to break a difficult problem into cases, a problem-solving strategy that might be called *divide and conquer*. This is not an obvious strategy; students have to learn when and how it can be used. In this example, the cases might involve the number of edges that the graph has, and the students might solve the problem by constructing all four-vertex graphs with no

Table 1 Using a chart to solve a “Find all …” outfits problem

	Red shirt	Striped shirt	Yellow shirt	Green shirt
Black trousers	Red shirt & black trousers	Striped shirt & black trousers	Yellow shirt & black trousers	Green shirt & black trousers
White trousers	Red shirt & white trousers	Striped shirt & white trousers	Yellow shirt & white trousers	Green shirt & white trousers
Grey trousers	Red shirt & grey trousers	Striped shirt & grey trousers	Yellow shirt & grey trousers	Green shirt & grey trousers

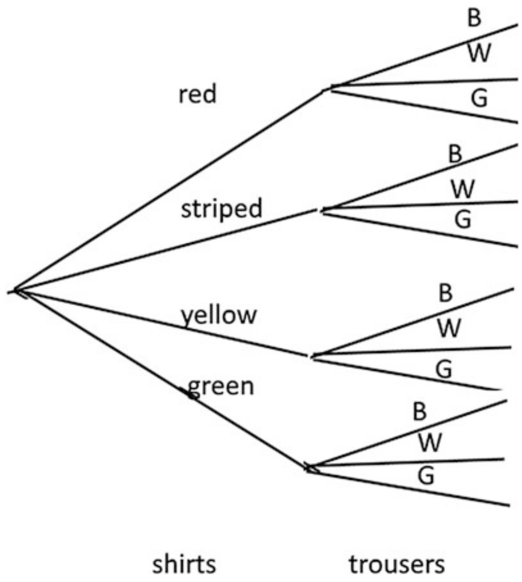


Fig. 1 Using a tree diagram to solve a "Find all …" outfits problem

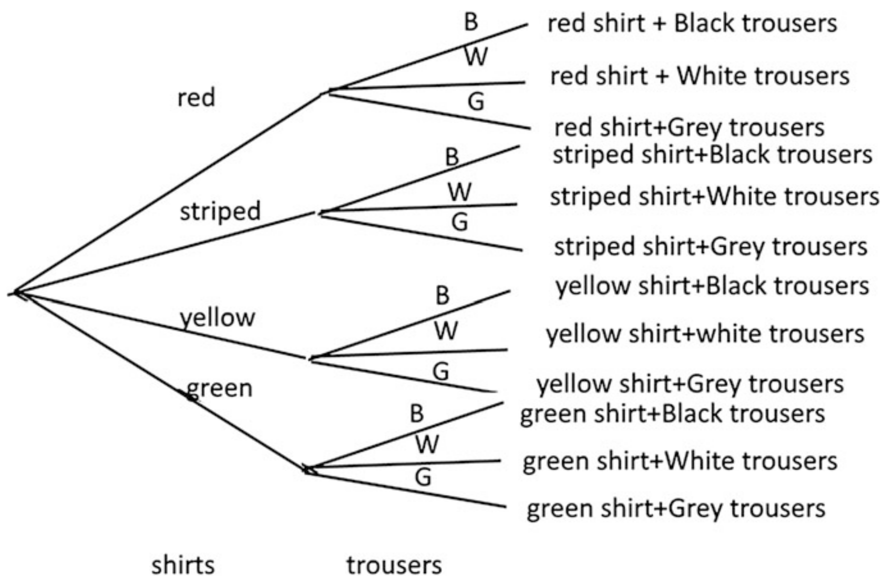


Fig. 2 A tree diagram provides a way of systematically listing all possibilities

Table 2 Using a chart to systematically find all factors of 200

	2^0	2^1	2^2	2^3
$\mathbf{5^0}$	$2^0 \times 5^0 = 1$	$2^1 \times 5^0 = 2$	$2^2 \times 5^0 = 4$	$2^3 \times 5^0 = 8$
$\mathbf{5^1}$	$2^0 \times 5^1 = 5$	$2^1 \times 5^1 = 10$	$2^2 \times 5^1 = 20$	$2^3 \times 5^1 = 40$
$\mathbf{5^2}$	$2^0 \times 5^2 = 25$	$2^1 \times 5^2 = 50$	$2^2 \times 5^2 = 100$	$2^3 \times 5^2 = 200$

edges, then all four-vertex graphs with one edge, then all four-vertex graphs with two edges, and so on. The solution to the initial problem is then obtained using the Addition Principle of Counting, which says that if you split the items to be counted into groups that have no elements in common, then the total number of items is the sum of the numbers of items in each group.

So, how many different graphs are there with four vertices? The Addition Principle of Counting says that the answer is the number of four-vertex graphs with 0 edges plus the number of four-vertex graphs with 1 edge, and so on, up to the number of four-vertex graphs with 6 edges (see Fig. 3). Thus the total number of different graphs with four vertices is 11.

"Wait a minute!" you or your students might exclaim. Aren't there six different four-vertex graphs with one edge? (See Fig. 4). Doesn't the fact that each edge connects two different vertices make these six graphs different? This is an important question: When are two graphs the same and when are they different?

This question is an example of a common and fundamental question in mathematics: When are two objects the same and when are they different?

For example, when are two numbers the same? Although 2 and 2/1 look very different from one another, as do the pair 12/16 and 21/28, and as do the pair 1/2

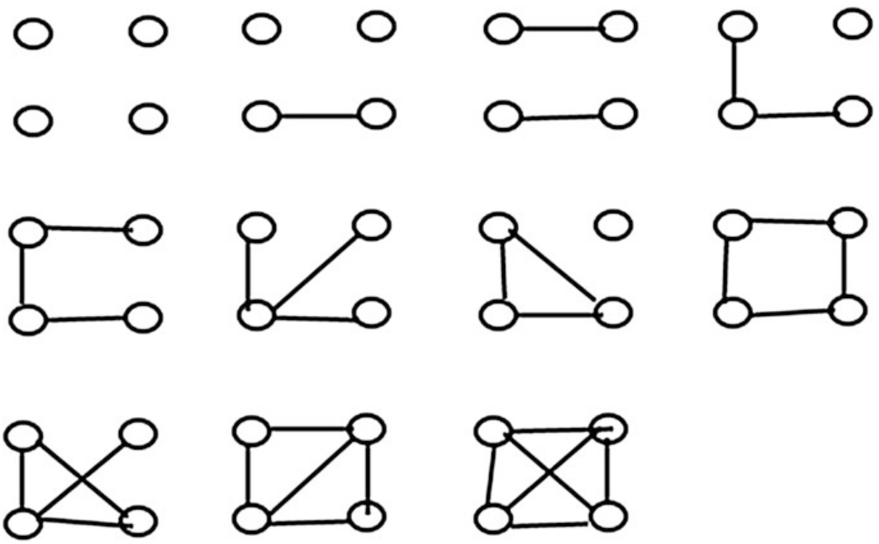


Fig. 3 Using a *divide and conquer* strategy to find how many graphs there are with four vertices

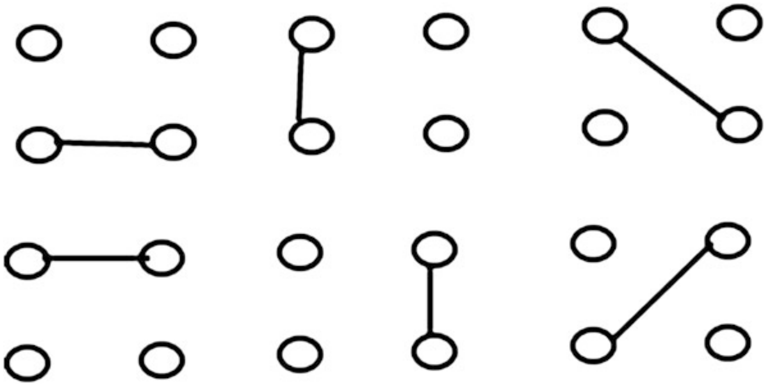


Fig. 4 Are there six different four-vertex graphs with one edge?

and 0.5, students come to learn that they are actually the same—that is, equal. Many are never able to reconcile themselves to the fact that 0.999… is equal to 1.

When are two triangles the same? The first two in Fig. 5 are recognizably the same, even at a young age, but it takes a while before children recognize that the third triangle (resulting from flipping the second triangle about a vertical axis) is the same as the second, and probably longer to recognize that the fourth triangle (resulting from rotating the third triangle counterclockwise by 90°) is the same as the third, and even longer to recognize that the fifth is the same as the fourth. Eventually, they understand that if they can reposition a shape so that it matches another shape, then the two shapes are the "same"—in which case they are referred to as *congruent*.

The same principle is used in dealing with graphs. If one graph can be repositioned so that it matches another graph, then the two graphs are the same—they are referred to as *isomorphic*. Thus the six four-vertex graphs with one edge in Fig. 4 are all isomorphic, since each graph can be repositioned so that it matches each of the other graphs.

"Wait a minute!" you or your students might exclaim. Can't you get another four-vertex graph with five edges beside the one we had before? The one we had before is on the left in Fig. 6; a new one is on the right. These two graphs certainly look different. But you can transform the new one into the old one! Just reposition

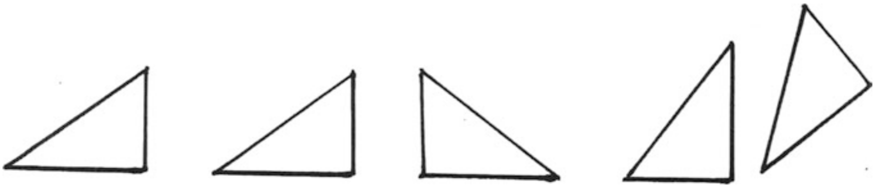


Fig. 5 Which triangles are the same?

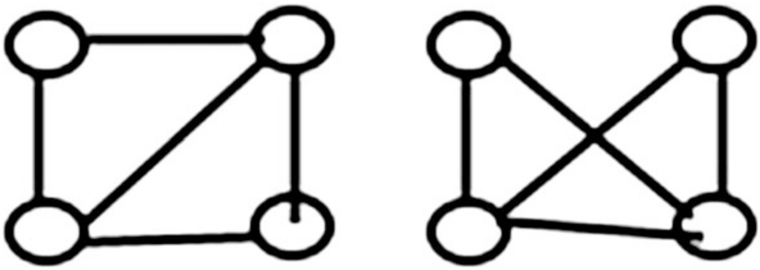


Fig. 6 Two isomorphic four-vertex graphs with five edges

its top right vertex, and the edges adjacent to it, so that it is located below its bottom right vertex … and you'll see the old graph at the left.

As an exercise, systematically determine how many different 2-regular graphs there are with 15 vertices. What is a 2-regular graph? That's a graph where each vertex has degree 2. What's the degree of a vertex? The number of edges that meet at the vertex. First, convince yourself that any 2-regular graph must be a collection of cycles.

The problem of finding all different graphs with four vertices is one example of systematic construction. Another kind of example is to find all different graphs with five vertices of which two have degree 3 and three have degree 2.

How do you even begin to solve this problem? Again, you have to think systematically.

You know that there is a vertex of degree 3 so you draw one vertex—call it A, and link it to three other vertices, B, C, and D. (See Fig. 7) Drawing this part of a graph is not an obvious first step; students have to learn to convert verbal information into graphic form.

You know that there is another vertex of degree 3. Where is it?

It could be E, in which case, (see Fig. 8) since E has degree 3, it must connect to B, C, and D, the only vertices that have room for another link. Look! We have found a graph with the desired properties, two vertices of degree 3 and three of degree 2.

But the second vertex of degree 3 could also be one of the vertices B, C, or D in the graph at the top left of Fig. 9. Let's suppose that it is C. Then we need to connect C to two other vertices. If we connect C to B and D, as in the graph at the top right of Fig. 9, then there are no vertices to which E can be connected, although it must have degree 2. So connecting C to both B and D doesn't work. We have to connect C to E and one of B and D; let's connect it to B, as in the graph at the bottom left of Fig. 9. Finally, we connect D to E, since those are the only two vertices which have empty slots for an edge. When we do that, we find that we have a second graph with the desired properties, two vertices of degree 3 and three of degree 2; this graph is at the bottom right of Fig. 9.

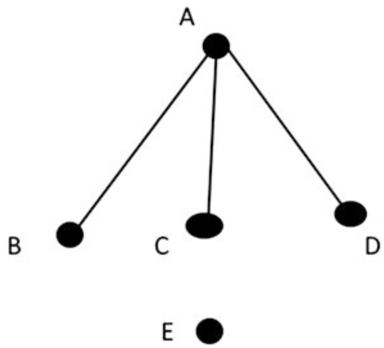


Fig. 7 Draw a vertex of degree 3, call it A, link it to three other vertices, B, C, and D

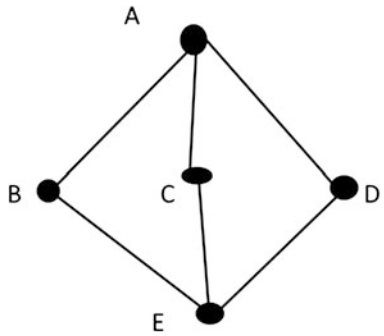


Fig. 8 Case 1: E is the other vertex of degree 3

Are there any other graphs with these properties? For example, if instead of selecting C to have degree 3, we had selected B to have degree 3, perhaps we would have found a third graph with these properties. In fact, it can be proved that any graph that has two vertices of degree 3 and three vertices of degree 2 must be isomorphic to one of the two graphs we have constructed.

Wait a minute! Isn't it possible that these two graphs are also isomorphic? After all, they both have the same number of vertices of each degree.

However, when we look at these two graphs side-by-side (see Fig. 10), we see that they are different. One way that they are different is that the graph on the right has three vertices that form a triangle, but there is no triangle with three vertices in the graph on the left. That makes it impossible for the two graphs to be isomorphic—no matter how we move the vertices of the second graph, those three vertices will still form a triangle.

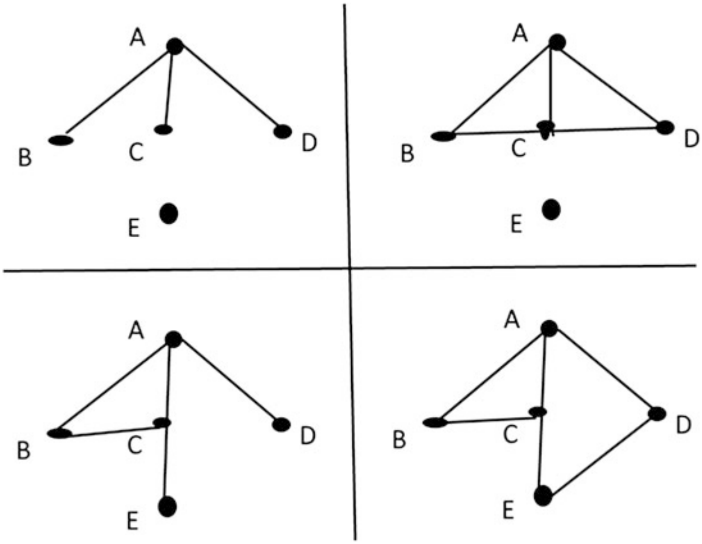
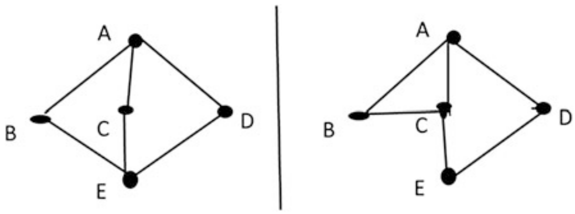


Fig. 9 Case 2: B, C, or D could be the other vertex of degree 3. Consider C

Fig. 10 Two non-isomorphic graphs with the desired properties



The net result is that we have systematically constructed all graphs with two vertices of degree 3 and three vertices of degree 2. There are exactly two of them.

As an exercise, you might try to construct systematically all different graphs with two vertices of degree 3 and four vertices of degree 2, or all different graphs with two vertices of degree 3 and five vertices of degree 2, etc. If you want a real challenge, try constructing systematically all graphs with eight vertices, all of degree 3.

As a final "Find all …" example of systematic reasoning, let us consider the question of how many different dishes containing three scoops of ice cream you can make, if eight flavors are available.

Your immediate response might be "8 choose 3" since there are that many ways of selecting three out of the eight flavors. But that would only be correct if the problem stated that the three scoops have different flavors. It doesn't.

Or your immediate response might be “$8 \times 8 \times 8$,” which would be correct if we were counting the number of ice cream cones, since a cone that has chocolate, then vanilla, then strawberry is certainly different than a cone that has strawberry, then chocolate, then vanilla—so there are 8 choices for each scoop. But with a dish of three scoops, the order of the scoops doesn’t matter.

How do you solve this counting problem, where order doesn’t matter and repetition is allowed? You have to consider several cases, solve each using the Multiplication Principle of Counting, and then add the results together, using the Addition Principle of Counting. A student who has learned to solve this problem has mastered several different techniques of mathematical reasoning and problem solving.

In this problem, there are three possibilities:

- all three scoops could be different, in which case there actually are “8 choose 3” or 56 possibilities;
- all three scoops could be the same, in which case there are 8 possibilities, since there are 8 flavors; or
- there could be two of one flavor (8 choices there) and one of another flavor (7 choices there), another 56 possibilities.

So adding these all together, we get a total of 120 possible ice cream dishes.

As an exercise, determine how many different dishes there are with four scoops of ice cream, where there are eight flavors and where two dishes are the same if they have the same number of scoops of each flavor.

As noted earlier, although the content information in these examples may not be very significant, the experience that students have with learning how to use systematic reasoning to solve problems like this—which have no prerequisites beyond elementary algebra—can be very significant in shaping their thinking in mathematics, and in other areas of human endeavor.

2 Reasoning and Problem Solving at All Grade Levels

When should students start having these experiences? Students should start having reasoning and problem solving experiences in the early grades, so that they can build on those experiences in the later grades, and so they can avoid developing the erroneous conclusion that mathematics equals computation. As can be seen from the examples above, discrete mathematics is a useful arena for introducing reasoning and problem solving.

A systematic approach to incorporating reasoning and problem solving experiences through discrete mathematics appeared in the New Jersey Mathematics Standards, originally adopted in 1996, and modified in 2002, which listed appropriate topics for each grade level. The New Jersey Mathematics and Science Coalition, which I served as Director for many years, reviewed the mathematics standards in 2007 and, taking into consideration the feedback from many New

Jersey teachers and the emerging drafts of the national standards, proposed a new set of mathematics standards for New Jersey, building on both the earlier successful versions of the standards and the recommendations in the national standards. These standards were not seriously considered in the rush to adopt the national standards.

The discrete mathematics portion of those 2007 recommendations is presented in Table 3. The proposed New Jersey standards include two topics of discrete mathematics that were recommended by the National Council of Teachers of Mathematics (NCTM) in *Principles and Standards for School Mathematics* (2000) as appropriate for all grades—systematic listing and counting and vertex-edge graphs—and that were developed more fully in NCTM's two books *Navigating Through Discrete Mathematics in Grades K–5* and *Grades 6–12* (DeBellis et al. 2009; Hart et al. 2008).

Systematic listing and counting is essential preparation for probability—it involves sorting and classifying in the early grades, organizing information in grades 3–5, and using the Multiplication Principle of Counting in the middle grades. Essentially all of this is missing from the Common Core, which evidently assumes that all students will be able to absorb these topics as they learn probability in high school, not a good assumption since probability is the trickiest of all mathematical topics. We hope that systematic listing and counting will be added to our national standards.

The study of vertex-edge graphs enables students to discuss and solve a variety of modern applied problems involving networks—such as efficient routes for snow plows or delivery trucks. This topic is also valuable, as we have seen, because it provides an accessible arena for students to focus on problem solving and reasoning in an interesting context.

The study of vertex-edge graphs also introduces the important modern topic of algorithms, for example, one that given a street map will help generate an efficient route from A to B. At early grade levels, the study of algorithms involves following directions, and later devising instructions and developing strategic thinking skills. Following various kinds of directions in the early grades helps children understand and follow the arithmetical algorithms (like the algorithm for adding two two-digit numbers) that they are later expected to learn with fluency. The study of algorithms, like systematic counting, had been included in the New Jersey Mathematics Standards since 1996, but is entirely absent from the Common Core. We hope that vertex-edge graphs and algorithms will also be added to our national standards.

These topics and reasoning and problem solving experiences with these topics are valuable to *all* students in *all* countries[1] and, given the dozen years of experience with these topics in New Jersey, we believe it is possible to adjust the current mathematics curriculum so that these topics can be added, with a decreased emphasis on some other topics.

[1]A video I prepared that discusses the discrete math topics that all students should be exposed to by the time they complete secondary school is also posted on YouTube.

Table 3 Standards for discrete mathematics proposed for New Jersey in 2007

Standards at each grade level	Topic
Pre-Kindergarten	
1. Determine whether or not an object has a particular attribute	Sorting
2. Sort objects into groups (e.g., sort basket of collected items into piles of pinecones, acorns, and twigs.)	Sorting
Kindergarten	
1. Sort and classify objects according to one attribute (e.g., color, size, shape, kind, or student-generated attribute), and order the resulting groups by the number of objects (each smaller than 10)	Sorting
Grade 1	
1. Sort and classify objects according to one or two attributes (e.g., color, size, shape, kind), noting that a single object can belong to more than one class	Sorting
2. Follow simple sets of directions (e.g., from one location to another, or from a recipe)	Algorithms
Grade 2	
1. Use Venn diagrams to sort and classify objects according to two attributes (e.g., color, size, shape, kind)	Sorting
2. Generate and list all possibilities in simple counting situations (e.g., all outfits involving two shirts and three pants)	Systematic counting
3. Follow the directions for simple two-person games (e.g., tic-tac-toe) and recognize that some strategies work better than others	Algorithms
Grade 3	
1. Represent all possibilities for a simple counting situation in an organized way (e.g., lists, charts) and draw conclusions from this representation	Systematic counting
2. Follow, devise, and describe practical and logical sets of directions for a simple sequence of events (e.g., to add two 2-digit numbers)	Algorithms
3. Find paths in concrete examples of vertex-edge graphs	Vertex-edge graphs
4. Color maps (e.g., NJ counties) using as few colors as possible	Vertex-edge graphs
Grade 4	
1. Use Venn diagrams to represent and classify data according to three attributes, such as shape, color, and size	Sorting
2. Represent all possibilities for a simple counting situation in an organized way (e.g., tree diagrams) and draw conclusions from this representation	Systematic counting
3. Devise strategies for winning two-person games (e.g., "make 5" where players alternately add 1 or 2 and the person who reaches 5, or another designated number, is the winner)	Algorithms
Grade 5	
1. Solve counting problems, including those where multiplication can be used (e.g., you can make $3 \times 4 = 12$ outfits using 3 shirts and 4 skirts)	Systematic counting
2. Justify that all possibilities in a counting problem have been enumerated and that there is no duplication	Systematic counting

(continued)

Table 3 (continued)

Standards at each grade level	Topic
3. Follow, devise, and describe practical and logical sets of directions for a complex sequence of events (e.g., to multiply two 2-digit numbers)	Algorithms
4. Represent problem solving situations using vertex-edge graphs and determine the degree of any vertex (i.e., the number of adjacent vertices), whether or not a graph is connected (i.e., can you get from any vertex to any other vertex?), and how many paths there are from one vertex to another vertex	Vertex-edge graphs
Grade 6	
1. Solve counting problems involving Venn diagrams with two attributes (e.g., there are 15 students in the chess club and 20 students on the debating team. Eight students are in both clubs. How many different students are there participating in one or more of these two activities?)	Sorting
2. Apply the multiplication principle of counting in various situations (e.g., find the number of possible outcomes when three coins are tossed or when three officers are selected from a six-person club)	Systematic counting
3. Devise strategies for winning simple games and express those strategies as sets of directions	Algorithms
Grade 7	
1. Apply the multiplication principle of counting to situations involving permutations (where order is important), including situations with and without replacement, and use factorial notation to condense the results	Systematic counting
2. List the possible combinations of two or three elements chosen from a given set (e.g., the handshake problem and the number of 2- or 3-person committees selected from a group of 12 people)	Systematic counting
3. Use vertex-edge graphs to represent and find reasonable solutions to practical problems • Travel route from one site on a map to another • Delivery route that stops at specified places • Drawing a picture with a single line without repeating an edge • Scheduling project meetings (to avoid conflicts) using graph coloring	Vertex-edge graphs
Grade 8	
1. Solve counting problems involving Venn diagrams with three attributes	Systematic counting
2. Distinguish between permutations and combinations and solve counting problems of both types using the multiplication principle of counting (no formulas)	Systematic counting
3. List and count the number of paths from the top cell of Pascal's Triangle to another one and describe the connection between such problems and problems involving combinations.	Systematic counting
4. Use vertex-edge graphs and algorithmic thinking to find solutions to practical problems • Finding the shortest network connecting specific sites • Finding the shortest travel route from one site on a map to another • Finding a low-cost circuit that visits each vertex exactly once	Algorithms Vertex-edge graphs

(continued)

Table 3 (continued)

Standards at each grade level	Topic
High School	
1. Apply the multiplication principle of counting in complex situations, distinguish between situations with and without replacement, distinguish between ordered and unordered counting situations, and justify solutions to counting problems	Systematic counting
2. Use Pascal's Triangle to solve problems involving combinations and probability	Systematic counting
3. Use vertex-edge graphs and algorithmic thinking to represent and solve practical problems • Is there a circuit that includes every edge in a graph just once (snow-plow or delivery routes)? • Is there a circuit that includes every vertex in a graph just once? • Critical path analysis	Algorithms Vertex-edge graphs
4. Develop and use strategies for making fair decisions, including combining individual preferences into a group decision (voting methods) and determining how many representatives each constituency gets (apportionment)	Algorithms

We have focused in this section on the value of discrete mathematics as a vehicle for improving the reasoning and problem solving skills of our students. However, there are other reasons why topics in discrete mathematics should be included in the school mathematics curriculum. Here are a few reasons for including discrete mathematics at all grade levels … and in programs for prospective and practicing teachers:

- Discrete mathematics includes valuable concepts and tools that show students the usefulness of mathematics, and that respond to the question, "How is math useful in the real world?"
- Discrete mathematics facilitates focus on modeling, problem solving, and reasoning at all grade levels.
- Discrete mathematics offers students who have been unsuccessful in traditional school mathematics a new start in mathematics.
- Discrete mathematics offers an opportunity to generate in primary and secondary teachers a new enthusiasm for teaching mathematics in new ways.

These and other rationales for discrete mathematics are discussed in detail in previous articles (e.g., Rosenstein 1997, 2007; Rosenstein et al. 1997), as is their value to all students, so I will not provide further explanations here.

3 Why Was Discrete Mathematics Excluded from the Standards in the United States?

Why would a state like my own, New Jersey, which had included discrete mathematics in its state mathematics standards since 1996 and which has consistently been among the top states in the independently conducted National Assessment of Educational Progress (NAEP), abandon its standards in favor of the Common Core? That's easy to answer: Each state's eligibility to receive federal funds for education was made contingent on its adopting Common Core.

Given these positive affordances of discrete mathematics, we must ask, why did the mathematics community go along with standards that essentially excluded discrete mathematics?

First, a major concern for many years is the number of students who come to college with an inadequate background in mathematics. In practice, that means that many students who have taken courses at the secondary level that presumably prepared them for college level mathematics courses are not actually prepared for those courses. As a result, colleges and universities have increasingly provided remedial courses in mathematics (and, similarly, in language arts—reading and writing).

This is not a new problem. Indeed, 35 years ago the Rutgers mathematics department first instituted a placement examination for incoming students and we discovered—that is, we now have data—about how many of our students were not prepared for calculus or even precalculus. As director of the undergraduate math program, I became co-chair of a university-wide committee on precollege preparation for the university, which strengthened Rutgers' entrance requirements, and subsequently became a member of a similar state-level committee, which in part led to the New Jersey state standards adopted in 1996. This was my first involvement in mathematics education.

Although this is not a new problem, it has been an ever-increasing problem since colleges and universities have expanded and ever-higher percentages of secondary school graduates are going to college. One of the negative effects of this democratization of education is that more students are enrolling in college who are unprepared for college-level work in mathematics.

One of the principal motivations for the movement to create state standards in the 1990s was to ensure that secondary school graduates were prepared for college, careers, and citizenship. In the past 20 years, as a higher percentage of secondary students went to college, the focus in primary and secondary schools became more on preparation for college.

The mistake that has been made is that the focus has shifted from college-readiness to calculus-readiness, and the driver of the entire mathematics curriculum in the Common Core has become preparing students for calculus. That requires what one of my colleagues described as "a fanatical focus on fractions" and an early emphasis on algebra. In such a curriculum, there is no time for the *frills* of discrete mathematics.

Shifting from college-readiness to calculus-readiness assumes that all college students, real or potential, and therefore all secondary students, need to prepare themselves for calculus. That is simply not so. Most secondary students will have no need for calculus and therefore no need for all the topics whose main role is to prepare them for calculus, including division of fractions when they are 13 years old. For these students, elementary algebra (what is referred to in the U.S. as Algebra I), elements of geometry, and exposure to the applications of trigonometric and exponential functions are the appropriate and sufficient topics from the calculus track.

Instead of writing standards that would prepare students for college, careers, and citizenship, as originally intended, the goal of the writing group, it seems to me, was shifted to prepare students for calculus.

At this point it is appropriate for me to note that I am not opposed to having standards in mathematics. Indeed, I played a leading role in the organization, creation, and adoption of *high* mathematics standards for New Jersey in 1996 (20 years ago!) and again in 2002, standards which certainly played a role in New Jersey's consistently high performance on the National Assessment of Educational Progress. Accompanying the standards was a 600+ page volume *New Jersey Mathematics Curriculum Framework,* (Rosenstein et al. 1997) that provided assistance to school personnel in implementing the standards.

Standards are, I believe, very important. But the standards should be applied to, and be appropriate for, *all* students. Not *all* students need to take calculus. Not *all* students need to be able to find 64 to the 2/3 power, and not *all* students need to manipulate algebraic fractions—a skill that is primarily useful in calculus.

Many topics in discrete mathematics are more valuable for *all* students than some of the topics needed to succeed in calculus, and are more accessible to *all* students than the *intense algebra* needed for calculus. Many topics in discrete mathematics can demonstrate to *all* students how mathematics is applied in today's data-driven world. And many topics in discrete mathematics provide opportunities for *all* students to learn about mathematical reasoning and problem solving.

That is just as true for students who are planning to take calculus. Although the Common Core may prepare them for calculus courses, they are not prepared for later courses involving problem solving, reasoning, and proof, skills which they could have developed through discrete mathematics topics in the school curriculum. This inadequate preparation may be partially responsible for the prevalence of courses in mathematical reasoning for college juniors who seek to become math majors. Including discrete mathematics in the school curriculum could provide prospective math majors with the reasoning skills needed to succeed in college.

Thus, the Common Core has left non-college intending students and non-calculus intending students without the problem solving and reasoning skills necessary for jobs in the current economy, and has left Science, Technology, Engineering, and Mathematics- (STEM-) intending students without the requisite problem-solving and reasoning abilities to succeed in math and science courses beyond calculus. Thus, the new standards are not serving students who are non-calculus intending nor are they serving students who plan to go on in STEM disciplines.

A second motivation behind the calculus focus of the core curriculum was the belief that the United States needs more students who are preparing themselves for STEM careers, and the belief that the way to increase the STEM pipeline is to ensure that more students take calculus in secondary school.

Leaving aside the question of whether there is a shortage of STEM personnel and STEM-prepared personnel, our research on Rutgers students who have taken Advanced Placement (AP) Calculus—the prime candidates for the STEM pipeline—reveals that a substantial percentage of these students do not continue in the STEM pipeline (Ahluwalia and Rosenstein 2017).

This suggests that we are not doing what is needed to convince these students to stay in the STEM pipeline and, just maybe, are acting counterproductively, exposing them to topics for which they are not adequately prepared and in ways that do not encourage them to be interested in pursuing mathematical or scientific careers. Others were only virtually in the STEM pipeline, that is, they took advanced math and science courses only because doing so enhanced their ability to get into the colleges of their choice, not because of their interest in these subjects.

Thus, the problem of the STEM pipeline is not that of recruitment, but of retention. (The 2007 report, *Rising Above the Gathering Storm*, seems unaware of this issue and simplistically recommends increasing the number of students taking AP Calculus in high school.) Thus, *it is not that the STEM pipeline is too small, but rather that it is too leaky*. Too many students exit the STEM pipeline. The solution to this problem should focus more on retaining students who are already in the STEM pipeline than on recruiting more students into the pipeline.

What I have said above applies primarily to students who live in relatively high socio-economic areas, where all students have the opportunity to take high level mathematics classes. However, in many low socio-economic areas, students do not have such opportunities and the talents of many students remain undeveloped, in mathematics and in other areas. It is a tragedy that thousands of students in America's urban and rural areas never are provided the resources and support that will enable them to be successful. For those students, STEM-based efforts must be escalated. This is a real challenge, a challenge that can be helped by the addition of discrete mathematics topics, which are often more engaging to students, in part because they have fewer prerequisites, and can help encourage students from low-resource schools to stay in the STEM pipeline.

A third reason for omitting discrete mathematics from the curriculum is concern about the scores of United States students on international assessments. If our students are not performing well in comparison with other countries, then it must be our curriculum that is defective, so they argue, and improving our standards will improve our curriculum, which in turn will improve the performance of our students.

There are a number of problems with this expressed concern. What do the assessments actually show? Are there other explanations for the apparent gap in performance? Is the gap indeed substantial? Is it really important that there is a gap? Is changing the curriculum the appropriate response? Will it have the intended effect?

International assessments are based on the curriculum that is *common* to all of the participating countries. Countries that have a broader curriculum are at a disadvantage on international tests because if their students spend only 90% of the time on the common topics they will not do as well as countries whose students spend 100% of time on those topics. Therefore, if increasing scores is important and the obstacle is curriculum, then it follows that we should narrow the curriculum so that our curriculum is in line with the international tests. Thus, from this perspective, discrete mathematics must be deleted from the curriculum. This reasoning is presumably an issue in other countries besides the United States; any country that wants to broaden its mathematics curriculum risks putting its students at a disadvantage and lowering its scores compared to other countries.

Because of this, the Common Core decides that the focus of mathematics education in the early grades should be primarily on fractions, which leaves little time for inclusion of discrete topics suggested for the early grades in Table 3. Those topics are considered *frills* and are not considered mathematics but *play*, and including them in our curriculum will not help us *catch up* to the students of other countries. But it is precisely these kinds of mathematical explorations that will enable our students to achieve the reasoning and problem-solving skills that we want them to have; they should not be excluded from our curriculum.

Perhaps we should rather promote the adoption of a different perspective, namely, that all countries should be encouraged to adopt a broader curriculum. How to do this is a problem that we all need to address.

To summarize, all three areas of concern—the focus on preparation for calculus, the desire to expand the STEM pipeline, and the concerns about international assessments—all seem to support the conclusion that discrete mathematics is not important. And, in each of those three areas of concern, I believe that this conclusion is not justified.

My hope and belief is that the US will eventually recognize that taking discrete mathematics will have a positive impact on students' mathematical preparation, their interest in STEM careers, and their performance on international assessments—by improving their reasoning and problem-solving skills and by introducing them to the many ways in which discrete mathematics is applied in today's world.

When will this recognition take place? I don't know. But it can only happen if we come to a recognition that we do not need to prepare all students for calculus, that not all students need *intense algebra*, and that we should not accelerate all students into Algebra courses and then into AP Calculus courses before they are ready for them.

I believe that this recognition *will* happen and I anticipate that in the coming years these topics will have the prominence in the school curriculum that they deserve, and that our students deserve. To help this come about, I have written a high school text, *Problem Solving and Reasoning with Discrete Mathematics* (Rosenstein 2014), a much expanded, revised, and refocused version of a book that was developed by myself and Valerie DeBellis (DeBellis and Rosenstein 2005) over ten years ago. This text presents one vision of how discrete mathematics can be incorporated into the high school curriculum.

My hope is that other mathematicians will be motivated to actively inform the mathematical education community, and the broader community, about the importance and value of introducing discrete mathematics into the curriculum of their countries' schools, by developing their own curriculum materials, and that together we will promote the importance of a broader curriculum.

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Discrete Mathematics in Lower School Grades? Situation and Possibilities in Italy

Aaron Gaio and Benedetto Di Paola

Abstract This paper presents an overview of the Italian situation in teaching discrete mathematics in primary and middle school, taking into account the national teaching guidelines and their connection with the subject. We describe research conducted with about 150 teachers, interviewed in a preliminary questionnaire. The data collected shows, for all teaching grades, interest in having more discrete mathematics in the school curriculum even if there are some difficulties in teaching it and in inserting it in the usual mathematical activities at school, mostly related to teachers' knowledge and self-confidence about the subject. We also discuss results and future plans for a continuing research project in the field. We describe in the conclusion a design research project involving teachers in the activity-designing process, aimed at bringing new mathematical knowledge and competences to students.

Keywords Design research · Computational thinking · Algorithms
Programming · Unplugged

1 Introduction and Context

Discrete mathematics, graph theory and cryptography, together with various algorithms found in computer science, can be a great teaching topic in lower school grades. Some projects about this have been tried around the World (Hart 1990; Kenney 1991; Rosenstein 1997; Casey et al. 1992; Bell et al. 1998–2015), but we feel that not much reached the Italian education system. Discrete mathematics is not clearly delimited in our curriculum and teachers are usually not aware that it actually could be. The goal of our work is to study the *learning of mathematical*

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skills through the teaching of discrete mathematics. We are studying both the teaching of general skills, such as reasoning and modeling, and skills particular to discrete mathematics, such as algorithmic and recursive thinking.

Our main research problem is to therefore support a proposal to alleviate the substantial lack of (sequences of) tasks in the Italian school curriculum about discrete mathematics, computer algorithms and cryptography, especially for primary and middle school. Both in the school programs and in textbooks, activities of this kind are missing almost entirely, despite much agreement that they can be really useful in improving the skills mentioned above.

1.1 In the Italian National Guidelines for Education

The Italian Ministry for Education, University and Research published in 2012 the current *Indicazioni Nazionali per il curricolo della scuola* dell'infanzia *e del primo ciclo di istruzione*, National Guidelines for the first cycle (kindergarten to 8th grade) of education (Ministero della Pubblica Istruzione 2012). These guidelines are not a detailed description of school curriculum to follow, but provide a guide from which single schools and institutes, and teachers, can draw the basic goals and competences to be reached. Some general standards are set with objectives for the educational achievements and learning goals. It is interesting to underscore how this first cycle is given its due relevance in the learning process of each person:

> La storia della scuola italiana […] assegna alla scuola dell'infanzia e del primo ciclo d'istruzione un ruolo preminente in considerazione del rilievo che tale periodo assume nella biografia di ogni alunno. Entro tale ispirazione la scuola attribuisce grande importanza alla relazione educativa e ai metodi didattici capaci di attivare pienamente le energie e le potenzialità di ogni bambino e ragazzo.
>
> Italian school history […] gives kindergarten and the first education cycle a prominent role considering the importance of this time in every student's life. Within this, the school attributes great relevance to the education and teaching methods that can fully activate energies and potentialities of every kid.

The importance of active student involvement is to be considered a key point in the modern school world. Multidisciplinary features play a leading role in a school environment where the boundaries between the different subjects should fade more and more. Moreover, other important aspects related to our research topic, underlined in the national guidelines, are connected to problem solving abilities:

> Favorire l'esplorazione e la scoperta, al fine di promuovere il gusto per la ricerca di nuove conoscenze. In questa prospettiva, la problematizzazione svolge una funzione insostituibile: sollecita gli alunni a individuare problemi, a sollevare domande, a mettere in discussione le conoscenze già elaborate, a trovare appropriate piste d'indagine, a cercare soluzioni originali.
>
> Support exploration and discovery as a means to promote curiosity towards new knowledge. In this perspective, the irreplaceable role of problems: students are stimulated to identify problems, pose questions, discuss these with knowledge which has already been developed and to finally look for original solutions. [pp. 26–27, translated by authors]

and the use of didactical activities as laboratories:

> Realizzare attività didattiche in forma di laboratorio, per favorire l'operatività e allo stesso tempo il dialogo e la riflessione su quello che si fa. Il laboratorio, se ben organizzato, è la modalità di lavoro che meglio incoraggia la ricerca e la progettualità, coinvolge gli alunni nel pensare, realizzare, valutare attività vissute in modo condiviso e partecipato con altri […]
>
> Didactical activities as laboratories, to promote practicality and dialog at the same time, also making the students reflect on what they are doing. A laboratory, if well administered, is the best way to encourage planning and search abilities, involve the students in thinking, realizing and evaluating processes, in a participative and shared with others. […] [p. 27, translated by authors]

The national guidelines include details of the mathematical competences. In all grades, the importance of the laboratory activities, interpreted as not only the physical location but above all as an opportunity to have activities and experimentation opportunities in first person, is reaffirmed. More specifically, there is a division between primary (in Italy, grade 1–5) and middle school (in Italy, grade 6–8. For primary school, the focus is:

- reading and understanding texts with logical content
- building lines of reasoning, having own ideas, defending and comparing them with others
- having positive attitude towards mathematics, realizing how mathematical tools are useful in the real world

While for middle school we have more specific topics:

- understanding logical aspects, as a result of correct argumentations and the possible changes of mind they can cause
- grasping the relationship between mathematics and reality
- having positive attitude towards mathematics, realizing how mathematical tools are useful in the real world

For primary school, the teaching objectives are not expected to go beyond basic arithmetic capacities and some geometry. Lots of freedom is left to the single school/ teacher to plan the teaching program. Algorithmic and logical thinking is also referred to as important in the technology chapter of the guidelines, for all school grades.

1.2 Key Competences for Lifelong Learning—European Union

In this way, the Italian school system is compatible with the European Union guidelines for learning, (2006), which explicitly gives necessary competencies for students to reach. Among the eight key points by the EU, we are strongly interested in key point 3, *Mathematical competence and basic competences in science and technology*, from which:

> Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge. […] An individual should have the skills to apply basic mathematical principles and processes in everyday contexts at home and work, and to follow and assess chains of arguments. An individual should be able to reason mathematically, understand mathematical proof and communicate in mathematical language, and to use appropriate aids.

and key point 4, *Digital competence*, from which:

> Digital competence requires a sound understanding and knowledge of the nature, role and opportunities of IST (Information Society Technology) in everyday contexts: in personal and social life as well as at work. This includes main computer applications such as word processing, spreadsheets, databases, information storage and management, and an understanding of the opportunities and potential risks of the Internet and communication via electronic media (e-mail, network tools) for work, leisure, information sharing and collaborative networking, learning and research.

2 Why Discrete Mathematics?

By discrete mathematics, we mean all those topics that can relate to academic discrete mathematics in the early school years. For example, many concepts are related to basic graph theory such as paths problems and coloring problems, binary numbers and arithmetics, algorithms (e.g. search algorithms, sorting algorithms) and also some sort of pre-coding activities useful in enhancing computational thinking in the students. Why should we teach these topics?

There are plenty of reasons to try to include these tasks into the curriculum.

- First of all, affect. The subject is quite **engaging**, as it is seen as an innovative topic, with the chances of increasing awareness of mathematics in everyday life and generating interest in all the subjects involved. It can be fun for everyone to try some new kind of teaching strategies; further arguments in this favor can be found in Goldin (2017, in publication, pp. 6–7)
- Students are directly involved in using *problem solving capacities* and *logical abilities*
- The approach and the teaching methodology help in enhancing *communication and creativity*
- It can further improve the *computational thinking* ability of students
- It supports *computer science competencies* which can be useful in later school grades
- There are *stand-alone activities*: a teacher can adapt a program to the time available and his/her willingness to make one rather than another task
- "Education should prepare young people for jobs that do not yet exist, using technologies that have not yet been invented, to solve problems of which we are not yet aware" (cit. Richard Riley)

3 A Preliminary Survey: Teachers' Competences and Difficulties

What do teachers think? We had this question in mind when beginning our research. The goal was to have a clear starting point about the situation in our Country and, at least some, teachers' thoughts about the topics described.

We had a first survey, with results collected from about 150 Italian teachers, mostly in-service and quite evenly divided between primary, middle and secondary school. The survey was done with an online platform; single answers to the questionnaire were given by the involved subjects and automatically registered in an online database. We analysed the results in a .csv file, collecting them in a table for the multiple-choice answers and dividing the different answers using keywords for the open questions. The resulting analysis is mainly a quantitative approach (Di Paola et al. 2016): the qualitative analysis was the codification of some particular key words used by teachers.

In our preliminary survey, teachers, especially at lower levels, admit that they do not have the necessary knowledge to teach discrete mathematics topics in school. We look at some details of some of the questions asked.

Questions 6-7-8-9-10-11 were about their previous experience in learning cryptography and graph theory, asking about school levels and what they were taught at which level.

The results are shown in the Tables 1 and 2. We have a vast majority of primary and middle school teachers not having had any prior school knowledge about the topics, while secondary school teachers had. This is confirmed by their answers to the question "write down some words that you think of when hearing the word

Table 1 Knowledge about cryptography, in their previous school/universities studies

Teachers in:	Knowledge about cryptography	No knowledge
Primary school (1–5)	3 (5.4%)	52 (94.6%)
Middle school (6–8)	5 (13.5%)	32 (86.5%)
Secondary school (9–13)	36 (63.1%)	21 (36.9%)
Total (%)	29.53	70.47

Table 2 Keywords to connect to cryptography/algorithm

Teachers in:	Related to computer science	Secret codes mystery/ games	Both
Primary school (1–5)	14 (27.4%)	31 (60.8%)	6 (11.8%)
Middle school (6–8)	15 (44.1%)	16 (47.1%)	3 (8.8%)
Secondary school (9–13)	37 (67.2%)	8 (14.6%)	10 (18.2%)
Total (%)	47.1	39.3	13.6

Table 3 Interest in teaching these topics

Teachers in:	Interested	Don't know	Not interested
Primary school (1–5)	32 (58.2%)	17 (30.9%)	6 (10.9%)
Middle school (6–8)	30 (83.3%)	2 (5.6%)	4 (11.1%)
Secondary school (9–13)	41 (71.9%)	12 (21.1%)	4 (7.0%)
Total (%)	69.59	20.95	9.46

cryptography/algorithm." Key-words were collected and grouped according to the kind of concepts they referred to. Generally, the primary teachers think about secret codes and hidden things, while teachers who had a mathematics background and answered yes in the knowledge question used words connected with computer science, data and information security. More specifically, "hidden" and "mystery" were more frequent in a *secret codes* thinking. Words such as "calculator" and "procedures" are more related to computer science, or, at least, give us some confidence that the compiler has something in mind about computers and concepts of informatics.

Also, teachers were asked if they had any previous experience in teaching the topics or if "they would be interested in teaching some algorithm, cryptography and other discrete mathematics topic to students", and their answers are quite promising, as seen in Table 3:

From a qualitative point of view, we had answers which were quite encouraging:

> I'm not an expert in this field, but I really think that some innovative teaching methods could be appreciated in our school; contextualizing mathematical topics to make them more appealing, and teach things that are both useful and feel real but at the same time make kids learn important mathematics is the way to go. (Math & Science teacher, 8th grade)

> I have some basic knowledge from my personal interests, but a serious project which can provide some guidelines for teaching it is missing. (Math teacher, 10th grade)

At the end, the data collected were giving us results as summarized above. Some qualitative reading of the meaningful answers is compatible with the quantitative results and confirm that we are on a path that can be appreciated by teachers, educators, schools and researchers.

4 The Research Project

As we said previously, a literature review on the subject of discrete mathematics, in text books, school curriculum and online resources, suggests that it is not a topic that we often deal with in the classroom. Pedagogical content available on this is, as a consequence, also quite poor. There are some documents about cryptography in secondary and high schools, but the problem we are facing is to bring some meaningful knowledge (or better, process of learning) into grades as low as the primary grades 3–5 and to the middle school.

From the questionnaire results and other contacts with teachers and educators in various Italian schools, we had encouragement to continue developing a project on discrete mathematics for various school levels in our country.

4.1 Methodology

The methodology we used is that of design research or design experiments (Brown 1992; Cobb et al. 2003; Barab and Squire 2004; Cobb and Whitenack 1996). For the purpose of this thesis, the developmental approach is taken into consideration (Plomp 2007); development studies function to design and develop a research based intervention, and to construct design principles in the process of developing it. The goal is to explore new learning and teaching environments, and to verify their effectiveness. Also, to develop new methods, instruments and teaching actions to further improve problem solving and logical thinking, using unusual topics such as algorithms and cryptography for primary school students. We therefore intend to contribute to the development of new teaching and learning theories with clearly defined content and goals, taking into consideration learning processes in specific situations (Battaglia et al. 2016).

Design research is quite appropriate in this situation, as we are facing a brand new experience in an environment that we need to analyze carefully on a local scale, considering all the different elements in the learning environment.

The intended design experiment is a classroom experiment in which the researcher (or researchers) cooperates with the teachers in assuming teaching responsibilities. On the one hand, the teacher is a part of the design team and will play a key role in the development and reviewing of the activities; on the other hand, they need a guide to experiment with this new experience and to present this new content (Gelderblom and Kotzé 2009).

The plan is therefore to design and develop some activities on discrete mathematics topics and to evaluate them in a dynamic process of design research. A starting point was the activities presented in the well-known *This is Mega-Mathematics* workbook (Fellows and Casey 1992) and in the *Computer Science Unplugged* project (Bell et al. 1998–2015).

4.2 Tasks and Implementation, The Plan

We are in the process of reviewing some activities, thinking about possible changes and new implementation, in cooperation with a group of teachers at two Primary and Middle Schools, in Trento, Italy; teachers from different grades are involved directly in the project planning, and their, and other students will be part of the project. A total number of about 370 students, from 3rd grade to the 8th, will be involved. The final goal is to work in a *vertical curriculum* approach, i.e. by competences and not by single notions. The idea is to get students from primary to middle school to acquire some competences, not only about discrete mathematics, but above all in the processes of problem solving and mathematical reasoning. Having these skills before entering secondary school could be of great use for supporting the current curriculum. *Teaching Discrete Mathematics in Grades 7–12* (Hart 1990) gives some ideas of a possible continuation of the path.

The teaching is done together with the teachers in the classroom setting and a continuous review and design process is planned to take place after every lesson or lesson cycle.

The first implementation step was a teaching activity (Steffe 1983), as a pilot evaluation, in 6th grade classrooms, chosen as a first preliminary tryout for the activity we plan to use. 6th grade is the central step in our project, lying between primary school and middle school and was therefore suitable for this preliminary experiment.

During the past school year many teachers showed interest in the projects that were made and were willing to have us go through these sequences of tasks with their classes. We developed many different tasks and activities which were meant to suit a particular age group. Some more detail regarding the topics chosen:

- 3rd and 4th grade: Binary numbers and algorithms, as for example games on sequence algorithms and selection, as well as basic search algorithms, not only in an unplugged approach but also with a longer view towards a future *scratch-like* activity.
- 4th and 5th grade: Binary numbers and algorithms as above plus sorting algorithms; computational complexity to see and grasp the difference between a *fast* algorithm and a *slow* one, even without being formal in a P versus NP definition.
- 6th grade: In addition to the above, some basic cryptography, such as substitution ciphers, both with symbols and drawing and some more algebra-related ones, e.g. Caesar's cipher;
- 7th and 8th grade: A mix of the previous with some higher cryptographic content added, as for example, Vigenere's cipher and a first example of public-key cryptography using graph theory concepts.

As this is a work in progress project, some of the activities have already been made and some are still in our intended plan.

As an example, we describe a part of an activity for 3rd and 4th grade in some detail. Our first *unplugged* sequence of tasks occupies about 3 lesson slots of about one and a half to two hours and follows a brief introduction on how computers work and binary numbers, in the form of games. Task 1 was an activity on paper, about binary image representation. Students had to color a grid which was provided with 0 s and 1 s and produce a drawing following the numbers. This task goal was about following instructions and beginning to understand how a computer transmits information.

Task 2 was about giving and receiving instructions. Students were divided in pairs and given a series of shapes and objects they could move on their table. One student for each pair was to create a composition on his table; without looking at each other (physical barrier between the two), student 1 had to explain to the other how to reproduce the same composition with the objects and shapes. Children were required to be as precise as possible while the game went on, and to try to find compositions that were harder to form. Only oral communication was allowed, not to make them "correct" the other mistakes or look at the other composition.

The following tasks had the goal of making programming even more tangible for children (Resnick et al. 2009; Hill et al. 2015). We wanted them to learn to give instruction as a calculator, through a path to walk on. Finally, with some of the classes we went on to construct some more complex sequence of instructions, posing different games to strengthen the concepts, but always with similar goals.

5 Conclusion

We noticed that the subject chosen was quite engaging for both students and teachers, as it is seen as an innovative topic with the chances of increasing awareness of mathematics in everyday life (Gravemeijer 1994; Freudenthal 1973) and generating interest in all the subjects involved.

In a design research paradigm, we plan on having another cycle of refinement of the activities we have already developed and the development of those we have planned earlier.

We are focusing our didactical research activity on some of the points mentioned above, as computational thinking (Kramer 2007) and problem solving general characteristics, or communication and creativity (Brousseau 2006), applied to problem solving, among students; but also, we did notice some implications coming from some tasks done in the classroom, in terms of group work and important consequences in cooperation or selfishness between students. For example, (Gaio and Di Paola 2016) focuses on one single activity which is presented to students as a group task, giving them the rules, but with the goal that they find themselves the algorithm for the solution. A qualitative analysis of the results, through some videos recorded in the classroom, shows, according to Vygotsky's perspective on the zone of proximal development (Vygotsky 1981), how children, playing together, realize that some greedy algorithms might never work, if we want to achieve group success. Some dynamics in which the game cannot finish if they seek to optimize their own result over the group result are shown.

In the last part of this project, we will focus on the collection of more focused results and a video-based analysis of the results, qualitative and fine-grained; in which both group activities and classroom discussions are recorded. We also have many of the transcripts, together with field notes, student's sheets, and interviews as other sources of evidence. As already said, the focus is put on students' learning and thinking, in reaction to the different tasks proposed.

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Discrete Mathematics and the Affective Dimension of Mathematical Learning and Engagement

Gerald A. Goldin

Abstract This paper describes certain characteristics of discrete mathematics that can enable teachers to evoke student interest and engagement, and develop students' powerful affect in relation to math—emotions, attitudes, beliefs, and values. Special affordances of discrete mathematics include interesting topics arising in familiar settings, special cases that are easy to set up and explore, a variety of natural representations embodying mathematical structures, and few prerequisites needed for in-depth inquiry. We also list several possible domain-specific sources of commonly-occurring *math anxiety* (long-term negative affect) which can be ameliorated through effective teaching making use of these features of discrete mathematics. An example from game theory illustrates our suggested approach. Pitfalls are also identified, including the too-early introduction of formal definitions, theorems, and problem-solving algorithms, or (alternatively) the relegation of discrete mathematics to a "slow track" in the curriculum.

Keywords Affect · Math anxiety · Game theory · Affective · Cognitive
Conative · Meta-affective · Metacognative

1 Introductory Discussion

Mathematical reasoning and problem-solving heuristics are often neglected when the teaching focus is on rules, procedures, memorization, and domain-specific methods. The latter tend to predominate in traditional mathematical topics—arithmetic, algebra, geometry, and analysis—which are the content of most of the curriculum and of most standardized testing in schools. DeBellis and Rosenstein (2004) expressed the idea that discrete mathematics could provide new ways for teachers to think about mathematics, and innovative strategies for them to engage their students.

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Discrete mathematics was then a relatively non-standard domain of the school curriculum in the United States, as it still is today. DeBellis and Rosenstein valued discrete mathematics as a set of topics whose teaching could place central emphasis on reasoning, strategy use, decision-making, and open-ended problem solving—i.e., on *cognitive* and *meta-cognitive* capabilities of unquestionable importance to understanding mathematical concepts and applying mathematics effectively. Writing in (2004), I adopted a stance similar to theirs, elaborating on some of the cognitions most relevant to discrete mathematics and addressing the development of students' internal systems of cognitive representation. The widely-applicable heuristic process, *modeling the general on the particular* served as an example for discussion (Goldin 2004).

A kind of groundswell of interest in discrete mathematics as a curriculum topic seems to have occurred from 1990 to around 2010. In 2002 Gila Hanna (Canada), Kristina Reiss (Germany), Jürgen Richter-Gebert (Germany), and Jacobus H. van Lint (Netherlands) organized an international mini-workshop at the Oberwolfach Mathematical Institute, *Discrete Mathematics and Proof in the High School*, with additional participants from Austria, England, Russia, Scotland, and the USA (MFO 2002). This workshop led to publications in special issues of *ZDM* (Heinze et al. 2004). In response to the *Principles and Standards for School Mathematics* (NCTM 2000), two subsequent publications in the *Navigations Series* published by the National Council of Teachers of Mathematics were devoted to discrete mathematics (DeBellis et al. 2009; Hart et al. 2008): these provide a wealth of teaching ideas and resources, at grades levels pre-K–5 and 6–12 respectively. Although there has been some decline in interest recently, at least in the United States, the recent high school level textbook by Rosenstein (2014), *Problem Solving and Reasoning with Discrete Mathematics*, offers a quite detailed presentation of how higher cognitive capabilities can be developed through discrete mathematics. This work, with an accompanying activity book, devotes much explicit attention to processes of reasoning and problem solving as the student explores coloring maps, graphs and graph theory in considerable depth, counting and combinatorics, and fair division problems.

The college level textbook by Rosen (2012, 7th edition) likewise highlights mathematical reasoning as a major theme, but takes a more traditional approach to its interpretation as establishing a logical foundation for methods of proof. Post-secondary level teaching often assumes a traditional, lecture-centered style, but discrete mathematics offers opportunities for changing this, too (e.g., Paterson and Sneddon 2011).

Despite the high level of topical interest and the abundance of teaching resources, discrete mathematics was not included as a *domain* of mathematics in the Common Core State Standards (CCSS 2010), adopted (as of August 2015) by 42 of the 50 states of the United States, by Washington, D.C., and by other jurisdictions. Here and there, a few topics of discrete mathematics occur in these standards: for example, permutations and combinations may be found in high school statistics and probability, and Pascal's triangle finds a home in the context of expanding binomial expressions in high school algebra. Most of the other areas discussed by DeBellis et al. (2009) and Hart et al. (2008) are, however, omitted or not deeply addressed.

Is this a bad thing? As high-stakes testing has come to be mandated by U.S. government policy, the issue of mathematics education's goals has become especially salient. The CCSS form the basis of standardized tests used widely (and controversially) in the USA to assess school success and teacher efficacy. While the absence of most topics in discrete mathematics from the CCSS might be seen as unfortunate by some advocates, because it removes a major incentive for schools to devote time and attention to the subject, there is an important redeeming aspect. When mathematical content is *not* included in high-stakes testing, a much freer approach to its teaching is possible, with very different purposes and very different attitudes surrounding those purposes. The *main* objectives can become not only mathematical exploration, discovery, and reasoning processes, but also motivation, engagement, and mathematically powerful structures of affect.

On the other hand, it is usually perceived that there is little or no incentive for teachers to allocate time to topics that are not to be included in standardized tests. When short-term gain scores on such tests form a component of teacher evaluations, the pressure to focus exclusively on traditional topics becomes still greater—and the level of anxiety associated with school mathematics increases further. This perception requires an answer. My advocacy of activity that enhances students' mathematical *motivation*, *engagement*, and *powerful affect* is based on the belief that achieving these objectives is essential to genuine, life-long mathematics achievement.

Achievement is not synonymous with test performance. Yes, one can envision students *turned off* by mathematics who manage nevertheless to acquire a level of proficiency in routine arithmetic and algebraic methods sufficient to answer many test questions correctly. But it is far more difficult to envision such turned off students pursuing mathematical problem solving with enthusiasm, embarking on STEM careers, or making meaningful use of mathematics in their planning and decision-making. It would be my contention that over the period of a school year, some significant time devoted to exploratory activity whose main objectives are *affective* in nature is likely to pay off in increased levels of student attention, greater persistence, and the kind of *growth mindset* conducive to meaningful learning (Dweck 2006).

Here I would like to explore opportunities afforded by discrete mathematics associated principally with the affective domain, which were mentioned only briefly in my 2004 article. Affect with regard to mathematics is taken to include students' emotions, attitudes, beliefs, and values (McLeod 1992; DeBellis and Goldin 2006; Hannula et al. 2016). I also draw in the related *conative* domain. Conation is a psychological term taken to include students' needs, desires, goals, and motivations—i.e., the *why* of their engagement (or disengagement) with mathematics (Snow 1996; Snow et al. 1996; Hannula 2006; Jansen and Middleton 2011; Middleton et al. 2017).

Why should we think that discrete mathematics offers something special in these domains? The phrase is a kind of catch-all for some diverse mathematical topics: logic and algebra, set theory, combinatorics, number theory, graph theory, game theory, algorithms, iteration and recursion, decision theory, fair division problems, and more. These find application in contexts ranging from computer science and statistics to international policy. What they have in common is far more than mere *discreteness*, in the sense of being mathematical topics where variables take (mostly) discrete rather than continuous values. They share most or all of the following characteristics:

1. They are topics that can be motivated directly by posing problems set in familiar, potentially intriguing situations: elections, children's games, art and coloring books, sharing, or counting combinations
2. There are often mathematically easy special cases that can be thought up and explored
3. Various natural representations can be constructed, and accessible, interesting questions can be asked about the mathematical structures implicit in the results of exploration and representation
4. The problem explorations typically involve few mathematical prerequisites—they do not require much, if any, algebra, formal or analytic geometry, trigonometry, or calculus, or even the arithmetic of fractions.

All these features—widely noted, but I believe underexploited—suggest opportunities to influence students' affective and conative orientations.

Next let us review the importance of the affective domain in mathematics learning, surveying some constructs pertaining to mathematical engagement. Some possible reasons specific to school mathematics for students' becoming turned off or disaffected, and for the prevalence of *math anxiety* in the population, are suggested in the literature. I shall then consider how features of discrete mathematics favor different affective experiences and more productive outcomes for students' motivation and engagement. I shall suggest some techniques, and discuss a few potential pitfalls.

2 Affective Issues in the Learning of Mathematics

2.1 General Perspectives and Construct

A growing literature addresses the importance of affect in mathematical learning, teaching, and problem solving. Attention has been devoted to the widespread aversion to mathematics in students and adults attributable to the trait termed *math anxiety* (e.g., Baloğlu and Koçak 2006; Tobias 1993), a trait that bears complex relationships to gender (e.g., Devine et al. 2012), to general test anxiety, and to mathematical performance. Much of the research on math anxiety consists of large-scale, questionnaire-based studies, and almost all is situated in or refers to the traditional topics of school mathematics—not to discrete mathematics. This in itself may highlight the need to take a different approach to mathematics teaching, in a domain that offers new possibilities.

To address the affordances provided by discrete mathematics requires attention to theoretical ideas emerging from more fine-grained, qualitative, descriptive research on affect, motivation, and engagement (e.g., DeBellis and Goldin 2006; Goldin 2000, 2002, 2014; Hannula 2006; Hidi et al. 2004; Leder et al. 2002; Maasz and Schlöglmann 2009; Op't Eynde et al. 2007; Philipp 2007; and extensive references within these sources). Let us mention briefly a few of these important ideas.

Affect may be seen as an internal *system of representation*, encoding information and exchanging information with internal cognitive representational systems during thinking and problem solving. It enables interaction and communication with others. Specifically, *emotions have meanings* which are context-dependent, and of which the person is not always consciously aware. They may carry information about a mathematical problem, the student's perceived status in understanding the problem or finding its solution, how the student stands in relation to other people in the immediate social context, how accessible are the near-term or longer-term goals important to the student, and so forth.

Affective pathways are sequences of in-the-moment emotions (local affect) that may occur and recur in mathematical situations, and interact dynamically with the student's strategic choices. For example, initial *curiosity* may lead to *puzzlement* and then *bewilderment*, followed by *frustration*, evoking a change of strategy or a request for help that leads to *encouragement*, *pleasure* as progress is made, *elation* if and when insight is achieved, and *satisfaction* with having understood the mathematics or solved the problem. Alternatively, frustration may generate *anxiety* in the student, leading to *fear*, *shame* and/or *despair*, evoking defense mechanisms and avoidance strategies.

Meta-affect refers to affect about affect, affect about cognition about affect, and the regulation of affect. Meta-affect is far more than emotional self-regulation: it includes the emotions one has *about* one's emotions, which can wholly transform the experience of affect. For example, one student may *hate* being frustrated, and become *angry* with his own frustration—thinking, in effect, "This is what always happens when I try to do math." Another student may *enjoy* frustration—she feels *intrigued* and *excited*, responding, "this is a good problem, I'm stuck but don't tell me how to do it, I want to solve it myself."

Recurring pathways of local affect and accompanying meta-affect in mathematical contexts result in the construction of longer-term *affective structures*—comprised of mutually reinforcing emotions, attitudes, goals, beliefs, and values. Such structures are evidenced in mathematical situations through characteristic affective pathways, interacting dynamically with thoughts, behaviors, social situations, traits and orientations, beliefs and values, and so forth. Examples of such affective structures include students' mathematical self-identity, their self-efficacy, their mathematical integrity, and the intimacy of their relationship to mathematics.

Engagement structures are conative/affective/cognitive structures descriptive of patterns in students' in-the-moment engagement during group mathematical activity (Goldin et al. 2011). Such a structure is activated by a student's immediate *motivating desire*, which evokes characteristic emotions, thoughts (self-talk), behaviors, social interactions, meta-affect, and so forth. Examples (among a larger set) include:

Get The Job Done, Look How Smart I Am, Let Me Teach You, Don't Disrespect Me, Pseudo-Engagement, and I'm Really Into This.

The last is based on the pattern of *flow* discussed in depth by Csikszentmihalyi (1990) and many subsequent researchers.

2.2 Domain-Specific Issues

The preceding constructs, while emerging from the mathematics education literature, are not limited to the domain of mathematics. However, some features specific to mathematics have been identified as having important affective implications, and are pertinent to the *domain-specificity* of math anxiety (Goldin 2014). These will help us understand the affordances in discrete mathematics.

1. School mathematics, unlike other subjects, places much emphasis on problem solving; and all but the most routine problem solving entails the experience of *impasse*—which thus plays a central role. This is likely to evoke frustration, at least to some degree, with accompanying consequences
2. In mathematics, knowledge of rules, procedures, and algorithms is frequently disconnected from understanding underlying reasoning, logic, or mathematical concepts. Skemp (1976) famously characterizes the distinction as being between *instrumental* and *relational* understanding. This disconnection can create unease, discomfort, and anxiety, as students follow the steps and are rewarded for obtaining correct answers by doing so, without understanding the *why* behind what they are doing
3. As their study of mathematics proceeds, students encounter *embedded conceptual challenges*—e.g., the first introduction of rational numbers (fractions), the transition to algebraic thinking, the introduction of formal proofs, and expectations for higher levels of abstraction from concrete situations. Students slow to meet these challenges may lose confidence in themselves and their mathematical abilities
4. The school curriculum in mathematics is *hierarchical*, with later concepts and skills depending heavily on prior ones. Any interruption of the sequence may result in frustration, discouragement, and even failure
5. Correct answers in mathematics are highly valued and centrally important, but by their nature, achieving them is *unreliable*—even if the student understands well the necessary concepts and processes. A single sign or character, misplaced or misread, can change the meaning of most mathematical expressions due to the non-redundancy of our system of formal notation. There is always the possibility of oversight or clerical error, leading potentially to disappointment and frustration
6. Certain *beliefs* about mathematics, prevalent in the wider culture, may fulfill some students' emotional needs but impede their mathematical development (Leder et al. 2002; Maasz and Schlöglmann 2009; Sheffield 2017). For example,

success in mathematical ability is often regarded as the result of an innate, special genius inaccessible to most people—a belief that may enable a sense of pride in being *gifted*, or alternatively insulate the failing student from feelings of guilt or shame.

3 Discrete Mathematics: Affordances and Pitfalls

Now let us consider some features of discrete mathematics in relation to the preceding discussion.

3.1 Cognitive and Metacognitive Features of Discrete Mathematics

Discrete mathematics offers manifold opportunities for taking a problem solving approach. Introductory questions may include:

> If each person in a group of 5 shakes hands once with everyone else, how many handshakes are there? How many different colors are needed to color the states in a map of the United States if different colors must be used for states sharing a common border? What is the shortest path by rail from A to B (on a presented map showing distances and rail connections)? Which player has the advantage in a two-person, zero-sum game (where a certain payoff matrix is presented)?

The contexts for such questions are not complex, but set in familiar experience. Approaching each problem requires few mathematical prerequisites, and solution processes do not need to make use of previously-taught mathematical procedures. The questions are easy enough to seem accessible from the start, creating a favorable metacognitive context, and the way they are posed suggests they may yield to trial and error exploration.

Such problems also suggest natural directions for generalization. For example, suppose there are 6 people in the group instead of 5, or 10 people, or n people. A handshake involves just 2 people at a time; but we may ask how many different ways a subset of 3 can be selected from the original 5 people, or a subset of 4, or a subset of m from a group of n.

Then we have the opportunity to *create* representations. For example, it is easy to represent a map by a planar (vertex-edge) graph, which leads naturally into a variety of new problems—not just coloring problems—in the broader context of graph theory. Or one can explore systematic methods for counting the possible subsets of a set, keeping track pictorially in different ways of those included in and those excluded from the subset—creating a variety of combinatorial situations for which Pascal's triangle provides a description.

In each such activity one can explore not only the underlying mathematical structure, but also the *process* of exploration itself—a kind of meta-exploration! How does someone actually invent a new representation, or how *might* one think it up if it weren't already known? How does someone find interesting questions to ask? How does one choose cases to explore in gaining an understanding of generally-posed problem situations, identifying simple but generic examples? While some questions may eventually be answered with a formula (for example, $n!/m!(n-m)!$ for the number of ways to choose subsets of size m from a set of size n), the presumption from the start that formulaic knowledge is the goal need not infuse the exploration. That is, students can experience the *doing* of mathematics, making use of the same cognitive and metacognitive processes used by researchers. They thus acquire skills of great value in science, technology, engineering, statistics, and other fields where mathematics finds application, as well as in mathematics itself.

Nevertheless, as with more traditional school mathematics, it is also possible to define ahead of time a body of mathematical knowledge and skills as learning objectives, introduce definitions, prove theorems, present formulas, and test for the student's acquisition of the desired competencies—and, in the process, miss out on the opportunities for exploration and discovery.

3.2 Affective, Meta-Affective, and Conative Affordances

The cognitive dimension is but part of the story, creating a context for the affective and conative opportunities offered by these features of discrete mathematics.

Initially in the introduction of a new topic, we have the chance to foster *curiosity-driven* engagement, based on a familiar situation and associated good feelings. The feeling of curiosity is both emotional and conative. It is a generally pleasant feeling, while the possibility of satisfying one's curiosity fulfills, in a very immediate way, a fundamental human need expressed in the natural propensity of children toward curiosity—to enhance one's knowledge or understanding.

Let us envision for concreteness a game theory activity for students at the middle school level. As children, most people have played some version of *odds and evens*, where each of two players (Alice and Bob) independently extends one or two fingers. Alice has called "odds." If the outcome is odd (3 fingers total), Alice wins; if the outcome is even (2 or 4 fingers total), Bob wins. The very context of a game suggests *fun*—pleasure in the activity itself.

Moreover, this is an out of school game, with its own rituals: sometimes the players call out, "One, two, three, *shoot*," both extending their fingers on the word *shoot* to ensure simultaneity. Bringing such a schoolyard practice into the classroom suggests something *unusual*, an activity *transgressive* of conventional boundaries (Pieronkiewicz 2015). This can create a *meta-affective* context of safety for curiosity and anticipatory pleasure.

The educational objectives of the activity are not mainly the concepts of formal game theory; rather, they are affective, meta-affective, metacognitive, and conative.

"Is the game fair to both players? How do we know?" Most students will conclude it is fair and be able to give coherent reasons, without any direct instruction. Then we may introduce the idea of changing the payoffs, through questions that encourage exploration. What if we change them so that evens is worth more than odds? … or so that the payoff for 2 fingers is worth more or less than the payoff for 4 fingers? Could we change them so that one player always wins?

We will need a way to keep track of the different payoffs. The students can *invent* representations to display them. The idea of a "payoff matrix" emerges from discussion, and the students who propose it experience satisfaction in the recognition by the class and the teacher of their discovery as an important mathematical idea.

The game context offers opportunities for productive social interactions, for experimentation through playing the game, for flow—activation of the engagement structure, *I'm really into this*. Frustration may occur when puzzling questions arise that lack obvious answers, but this occurs with positive meta-affect. There may be a sense of mathematical success achieved through exploration—the *aha* moments of insight, the thrills of discovery. For some students, these experiences may be entirely unusual or transgressive of their prior self-concepts with regard to mathematics.

The teacher guides the exploration. Concepts of probability and of expected value may come into play. Game-theoretic ideas such as a *zero-sum* game, games with a saddle point, and equalizing strategy may (or may not) emerge during the exploration. Of course the teacher knows these mathematical ideas, and gently tilts the discussion accordingly; but the teacher's affect is also different—since the learning objectives are not *standard*, arriving at formulas and routinized procedures need not be of the highest priority. The opportunities for building students' confidence and sense of self-efficacy are increased, as well as high-level, transferable problem solving strategies and heuristic processes.

Many of the domain-specific issues around mathematical affect, listed in the preceding section, are fundamentally shifted. Problem solving has been made safer, so that frustration can be experienced as intriguing. Mathematical rules, procedures, and formulas have nearly disappeared, or are subsidiary to the main ideas. More challenging ideas become accessible through student-generated representations of a fully-understood situation. The topic is mostly *outside* the usual hierarchical sequence of prerequisites. Correctness matters in a way that is comfortably less high-stakes.

And discrete mathematics embraces topics not even considered to be 'mathematical' in the wider culture. Thus the activity offers a good chance to provide *contradictory experience* to prevailing beliefs about the inaccessibility of mathematics to all but the most talented.

But none of the above is straightforward. It is important to point out ways that discrete mathematics instruction can easily, inadvertently, miss the opportunities available.

3.3 Possible Pitfalls

One possible pitfall is to fail to attend *explicitly* to affect, relying tacitly on the idea that since discrete mathematics is fun and its contexts intrinsically motivating, students will necessarily stay engaged when difficulties arise. Initial interest and curiosity may not be sufficient to carry the day. Students can easily fall back into well-worn affective pathways associated with more traditional school mathematics.

A second pitfall is to yield to the temptation of routinization—moving directly into presenting conventional terms, definitions and theorems, illustrative examples, or specific algorithms for solving classes of problems. This is more efficient, probably, in reaching more advanced mathematical results in a shorter time, but it shifts us away from the affective and meta-affective objectives we have highlighted.

For instance, in introducing game theory to older students, one may want to begin with an abstract and beautiful characterization of maximum generality, such as the following:

Definition 1 The strategic form, or normal form, of a two-person zero-sum game is given by a triplet (X, Y, A), where

(1) X is a nonempty set, the set of strategies of Player I
(2) Y is a nonempty set, the set of strategies of Player II
(3) A is a real-valued function defined on $X \times Y$. (Thus, $A(x, y)$ is a real number for every $x \in X$ and every $y \in Y$.)

The interpretation is as follows. Simultaneously, Player I chooses $x \in X$ and Player II chooses $y \in Y$, each unaware of the choice of the other. Then their choices are made known and I wins the amount $A(x,y)$ from II (Ferguson 2014, p. II-4)

The mathematical idea here is complete, and it is clearly communicated to readers fluent in formal language and notation. Those with some mathematical sophistication will appreciate the definition's succinctness and its elegant generality. But affective affordances in discrete mathematics may be lost if we begin—as we do routinely in other domains of mathematics—with already-polished results, expecting students to learn mainly by interpreting them, studying worked-out examples, and then imitating the methods in the examples.

My point here is *not* to remove such lovely descriptions from mathematics teaching. Rather, it is to regard them as the end point, rather than the beginning, of a set of meaningful student experiences.

On the other hand, a different pitfall (and very real one) would be to define discrete mathematics as part of a *slow track*, suitable for students unable to keep up with the pace of traditional topics in the curriculum. This in itself creates a negative affective, meta-affective, and conative context for the learners. Powerful affect and meta-affect, strategic and metacognitive capabilities, and long-term, productive motivation, should be developed in students at *all* ability levels in mathematics. Indeed, developing these may come to substantially *influence* what we call a student's *mathematical ability*, which is less fixed than is commonly believed (Sheffield 2017).

4 Conclusion

In earlier work I suggested some affectively-oriented principles for fostering inventiveness in mathematics education (Goldin 2009), in brief:

- Make *powerful affect* a key goal, as it is essential to mathematical success
- Create an *emotionally safe environment* for engagement with conceptually challenging ideas
- Foster *intimate engagement*
- Develop *positive meta-affect* around the emotions that occur naturally during problem solving
- Cultivate *personal satisfaction* in success and learning
- *Value mathematically inventive ideas*, and follow them up
- And *respect the individuality* of each student.

I believe such principles can and should be implemented in all of school mathematics. But as we have discussed here, discrete mathematics provides some exceptional affordances for them.

There is a great need for research on the affective and conative dimensions of discrete mathematics—very little now exists. Can the possibilities suggested here be realized systematically? How best can this be done? What are the longer-term affective and motivational consequences of current discrete mathematics offerings? If we are successful in enhancing students' affect and motivation through discrete mathematics, how can the transfer of new attitudes and orientations to more standard mathematical subjects be encouraged? I hope the discussion here contributes to focusing greater attention on these issues.

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Part II
Combinatorics and Combinatorial Reasoning

Combinatorial Reasoning to Solve Problems

Tom Coenen, Frits Hof and Nellie Verhoef

Abstract This study reports on combinatorial reasoning to solve problems. We observed the mathematical thinking of students aged 14–16 and study the variation of the students' combinatorial reasoning in terms of activity levels in a process of emergent modelling. We interpret student reasoning with the focus on stages of attention and describe the results in a framework of long-term mathematical thinking. The results show that the students are tempted to begin the problem solving process on the highest level and otherwise have difficulties transitioning from a lower to a higher level of activities. Qualitative analysis revealed some students' preference for the use of formulas, while at the same time other students showed more insight by their systematic approach of the problems, leading to better results. We advocate matching emergent modelling with teaching of combinatorial reasoning, stimulating students to create a relational network of knowledge.

Keywords Combinatorial reasoning · Tree diagram · Drawing with replacement Drawing without replacement · Referential activity

1 Introduction

Combinatorial analysis is an appropriate topic in the mathematics curriculum, because it has problems suitable for all grades. Many applications in different fields can be presented in teaching combinatorial analysis (Kapur 1970). Combinatorial problems stimulate students' construction of meaningful representations, mathematical

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reasoning, and the abstraction and generalization of mathematical concepts (Sriraman and English 2004). In this study we use emergent modelling indicating students' strategies to solve combinatorial reasoning problems (Gravemeijer 1999).

2 Theoretical Framework: Mathematical Thinking

Most combinatorial problems do not have readily available solution methods, and create students' uncertainty regarding how to approach them and what method to employ (Batanero et al. 1997). Mason (2004) distinguished four attention levels: (1) being aware of the whole situation; (2) focus on details and awareness of relations or similarities; (3) focus on properties as attributes that objects might satisfy; and (4) focus on reasoning solely on the basis of properties. Tall (2013) described a general framework for long-term development of mathematical thinking: (1) conceptual-embodied knowledge based on perceptions of and reflections on properties of objects; (2) operational-symbolic knowledge that grows out of the embodied form through *physical* action of the learner into mathematical procedures; and (3) axiomatic-formal knowledge based on formal definitions and proof. Gravemeijer (1999) emphasized students' thinking—related to Mason's attention levels and Tall's framework of development—in terms of levels of activity in a process of emergent modelling:

1. Activity in the task setting, in which interpretations and solutions depend on understanding how to act in the problem setting (often out-of-school settings)
2. Referential activity, in which models-of refer to activity in the setting described in instructional activities
3. General activity, in which models-for derive their meaning from a framework of mathematical relations
4. Formal mathematical reasoning, which is no longer dependent on the support of models-for mathematical activity.

We use Gravemeijer's levels to analyze student activities in the context of combinatorial reasoning problems. An example of emergent modeling can be found in the research of Batanero et al. (1997), exercise 6. In this exercise, students are given the question: "How many different ways are there for a grandmother to place four children in two different bedrooms, both with enough room for four children?". The addition "*the grandmother can place all the four children in one room, or she can have Alice, Bert and Carol on the first floor and Diana in the upstairs room*" reveals a clue to the solution strategy. It implies to distribute the children and this implicit model suggests considering all decompositions of the number 4. For example, when you place two children on the ground floor and two children upstairs, then there are 6 possibilities to distribute the 4 children—which is a combinatorial problem in itself to solve. If students systematically elaborate all the possible decompositions they can find the correct answer by adding

$1 + 4 + 6 + 4 + 1 = 16$. Batanero observed some students solving the problem correctly this way. However, Batanero suggests that the problem could have been solved in an easier fashion by a multiplication based on a selection model. Indeed, if we shift our attention to the fact that for each child one room out of two needs to be selected, then the problem is solved quickly: $2 \cdot 2 \cdot 2 \cdot 2 = 16$ possibilities in total. However, the first strategy is based on the model *of* the situation and can be deduced from the context. The second strategy is just a model *for* the mathematical solution procedure and can only be applied after a major shift of attention.

For the problems we posed to our students, imagine that students are working on the problem 'ice-cream top 3':

> Problem 1 "The ice-cream top 3 problem": Out of six different flavors, how many different ranked top threes can be made?

At the first level students may call some triples of flavors like banana-strawberry-chocolate, vanilla-banana-cherry, banana-chocolate-banana, and so on. At the second level, they may evaluate some of these triples as incorrect, because they notice from the context that flavors are not to be repeated, so banana-chocolate-banana is not a possible top 3, or they may wonder whether strawberry-banana-chocolate is different from banana-chocolate-strawberry or not. At the third level students could use a systematic enumeration like the odometer strategy (English 1991), or use tree-diagrams to represent all possibilities. At the fourth level they may deduce from the enumeration or the tree diagram the formal calculation $6 \times 5 \times 4$ as the solution to the problem.

3 Method

3.1 Participants

Three student groups participated: five boys and nine girls (aged 15–16) with a basic knowledge on tree diagrams; seven boys and eight girls (aged 14–15) with no prior education in combinatorial reasoning problems, tree diagrams or probability; seven boys and fourteen girls (aged 14–15) with knowledge how to draw a tree diagram and how to calculate basic probabilities.

3.2 Research Instrument: Field Notes of Live Observations

The field notes of live observation were in a Lesson Study context deepened by videotapes in order to validate the field notes (Verhoef and Tall 2011).

Table 1 Coding of student remarks at levels and quality

	Level	Correctness
		1 = right/0 = wrong
J: $10 \times 9 \times 8 \times 7$?	4	0
L: reads exercise out loud and draws ten books in a row	1	1
S: If you choose this one, there are only nine left	2	0
J: so $10 \times 9 \times 8 \times 7$?	4	0
S: would it be right?		

3.3 Analysis

The field notes and the transcribed videotapes were ordered in the levels of activity.

An example of the categorization of student's remarks is given in Table 1. The students are solving the "bookstore question": A bookseller sells top-ten books in order. He lists—from the first four customers buying a single book from the top ten—which book they decide to buy. How many lists can arise? (the answer is $10 \times 10 \times 10 \times 10$). Each student remark is noted, together with the level of emergent modelling. The final column shows if the student's remark is correct or not.

We characterize J's answer on level 4 because of J's direct symbolic answer followed by an action which confirms J's inner thoughts. S's remark is wrong on level 2—an activity in the setting described in instructional activities. J's next formal calculation is a remark on level 4 again. J confirms his previous answer again. The calculation, even though correct looking at S's remark, is not appropriate for the given problem, so it's labelled wrong. In summary J's answers are characterized as formal level 4 based on his reactions which confirm his own formal inner deepened knowledge.

4 Results

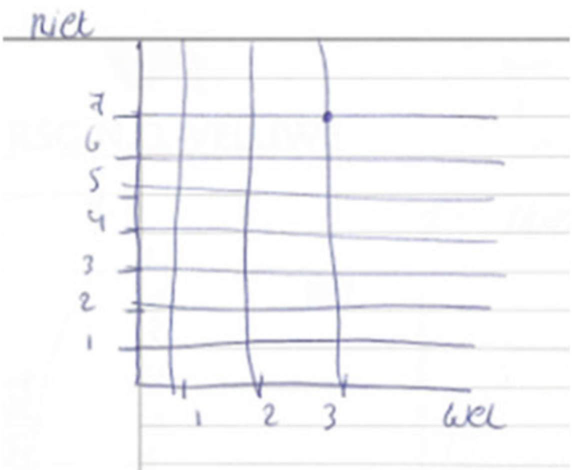
4.1 Procedures and Formulas

Lesson 1, one group started with the problem of t-shirts:

> Problem 2 The t-shirt problem": Going on a holiday, you want to take three of your ten shirts. How many options are there to take three?

Student B had been taught to calculate numbers of combinations with the calculator and found the right answer 120. Student A mentioned that the problem could be solved by using a tree diagram, which would be an action at level 3. Student A was convinced of the correctness of $10 \times 9 \times 8$, which she deduced

Fig. 1 An x–y grid about choosing 3 out of 10



from the tree diagram, and challenged student B to explain how it is possible that the formula for combinations gives the answer of 120 possibilities and 10×9 $8 = 720$ (an action at level 4). Student B couldn't explain why both answers are not the same: maybe counting lattice paths works? (B's activities go to level 3) Student A drew an x–y grid (Fig. 1), tried to calculate the numbers without writing them down, made a mistake calculating and didn't find the correct answer of 120 (A's activity on level 4). Now student A was convinced of her solution of 720 possibilities.

Student A and B's discussion limits to Mason's fourth level of the focus on reasoning solely on the basis of properties. In terms of Tall's framework of long-term development of mathematical thinking we see that both students A and B are not able to use conceptual-embodied knowledge. They only use operational-symbolic knowledge.

After solving the other problems, the students returned to the t-shirt problem. They acknowledged that taking the three t-shirts white, blue and red is the same as taking blue, white and red (at level 2), but they could not translate this notion to an appropriate solution strategy (advancing to level 3 and 4). Student B mentioned that probably there are double sets of shirts (at level 2). According to student A all the problems can be solved with tree diagrams (at level 3). But student B drew an x–y grid and mentioned, without any explanation, that you only have the choice between yes and no (at level 3). Not being able to agree on a solution, they decided to write down both answers (Fig. 2).

Student B argues in terms of Mason's level 2 with the focus on details and awareness of relations or similarities, while student A argues in terms of Mason's level 3 with the focus on properties as attributes that objects might satisfy. Student B starts with conceptual-embodied knowledge, but without deepening this knowledge he will end up using not understood operational-symbolic knowledge, as is illustrated by the unexplained use of an x–y grid.

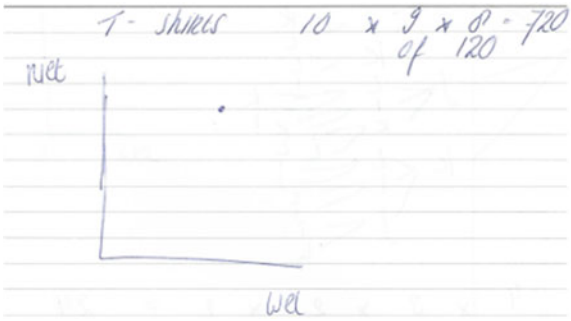


Fig. 2 The solution to choosing 3 out of 10

A different group in lesson 1 is working on the ice-cream tower problem.

Problem 3 “The ice-cream tower problem”: How many different ice-creams can one get if you place three scoops on top of each other. You may choose out of ten flavors and choose a flavor more than once.

Immediately, student C says correctly 10 times 10 times 10, so 1000 (10 * 10 * 10 = 1000 at level 4). The group writes down this answer. After a while, having solved a problem about a multiple choice test, they start all over to solve the problem of the ice-cream tower because they couldn’t decide about similarity or not with the multiple choice test problem. They discussed for about 10 min about it. Student D mentioned that it doesn’t matter whether you choose a flavor twice (at level 3). Student C agrees and repeats his answer by stressing ‘you can first choose this flavor, and after that, the same again and again. So 10 times 10 times 10’ (at level 4). Student E then states that ‘thus’ the order is not important, so he proposes to make a lattice-grid (at level 3). Student D asks whether they should take the order into account (at level 2). Student E states that the order is not important because you can choose a flavor more than once. She points out that you also may choose three different flavors (at level 2). Student C now agrees to draw a lattice-grid (at level 3). The next step in the discussion is about how big this grid has to be. Finally they agree that you have to say three times ‘yes’ to a flavor, but they hesitate on the number of ‘no’s’. After a while they decide that it is 1 time yes and 9 times no per scoop, so for a tower of three scoops you have to say 3 times yes and 27 times no. Student E draws a grid and starts to calculate. She makes mistakes and ends up with the answer 3276 (at level 4). Meanwhile student C and D are discussing the possible outcome when drawing a tree-diagram of the problem. They conclude correctly that this would yield a number of 10 times 10 times 10, which corresponds to the original answer (at level 4), but they never assessed the correctness of the tree-diagram for this problem (at level 3). Student C and D could not achieve the highest level 4. Now the students together discuss this ‘huge’ number 3276 and are tending to go back to the original answer 1000 (from level 3 to 4).

They can't make up their minds. At that moment the teacher ends the lesson. The first—correct—answer stays their solution, but they are not certain of its correctness (at level 4). In this case we see that the students are aware of the problem according to Mason's first level, and the whole situation using Mason's second level with the focus on details and awareness of relations or similarities. This relationship stimulates and deepens their arguments. In terms of Tall's framework of long-term mathematical thinking the students don't take time to interpret the context in conceptual-embodied knowledge. This omission interferes with the use of operational-symbolic knowledge and hence with finding and comprehending the solution to the problem.

4.2 *Construction of a Systematic Method*

Lesson 2, student F had the lead. He tried to solve the ice-cream top three problem (Problem 1). He wrote down: ice-cream top three. The problem involved an ordered selection without repetition out of six different ice-cream flavors (numbered 1, 2, 3, 4, 5 and 6). Student F started with writing down a 1. He then wrote down 5 and 4, because for scoop 2 and 3, there are $5 \times 4 = 20$ possibilities left. Next, student F wrote down 2. Again he thought that there were 20 possibilities left for scoop 2 and 3. He repeated this for 3. Then student F hesitated. He thought that if 2 is on top, there are only $4 \times 3 = 12$ possibilities left. He continued with 3 on top ($3 \times 2 = 6$ possibilities), 4 on top ($2 \times 1 = 2$ possibilities), etc. Again student F hesitated. The reason for this hesitation is not clear, but he didn't trust the solution. Student F decided to systematically write down all the possibilities. Student F worked in a structured way on level 2, starting with a column containing 123 till 126, followed by 134 till 136, leading up to 156. The next column started with 234 and so on. He noticed a pattern in the number of possibilities he wrote down: 4,3,2,1—3,2,1—2,1—1, making a total of 10,6,3 and 1, so a total of 20, giving him a (false) sense of security (Fig. 3).

The other students challenged him why for the example, 132 is not in the columns. Suddenly, student F realized on level 3 that the structure and combinations belong to the solution of the "ice-cream cup" problem.

> Problem 4 "the ice-cream cup problem": In how many ways can you pick three different flavors out of six to put into a cup?

He realizes the difference with the "ice-cream top three" question. He crossed the top of the page and replaced this with "ice-cream cup". It is unclear how he concluded this, as he made no remarks about it.

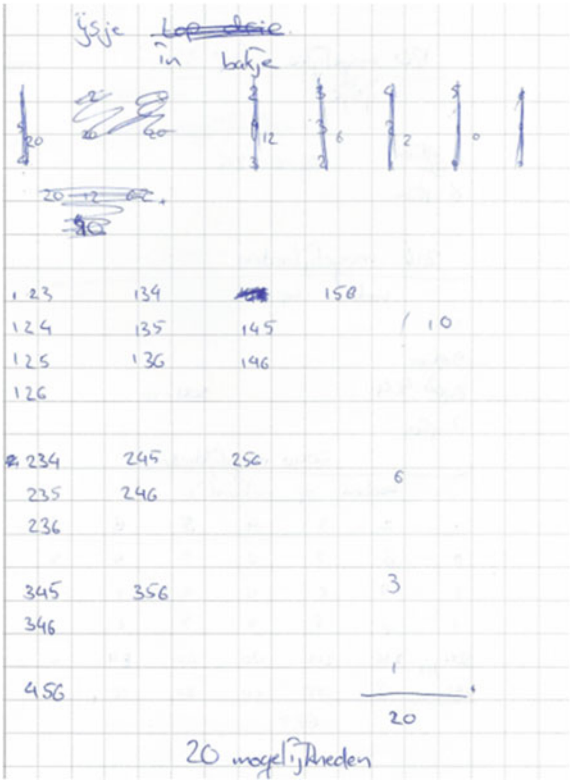
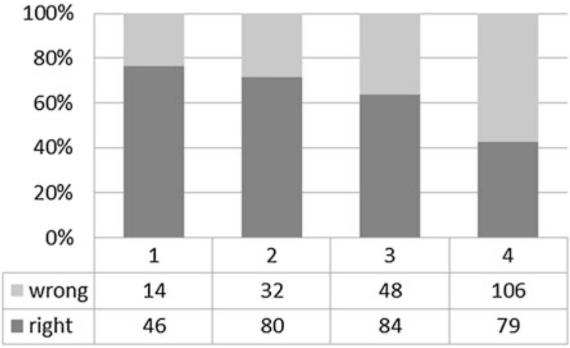


Fig. 3 The solution to ice-cream in cup

4.3 *Errors Within the Levels*

As students often start in higher levels, it is interesting to check if this doesn't cause more errors. Table 1 depicted the approach to classify all remarks into the levels and check whether the remark was correct. The results are presented in Fig. 4. As Fig. 4 shows, the number of remarks increases as the level increases, supporting the observation that students often start at a high level. Figure 4 also shows that more errors indeed occur when higher level remarks are made. A mistake on a lower level can lead to a wrong conclusion for the following levels, which may partly account for the increasing errors reported per level. Remarkable is that at the highest level, more mistakes are made than correct remarks, which again stresses the danger of students immediately starting at the highest level. It will need more research to find out whether transitions between the levels are causing mistakes, or more mistakes are made when levels are skipped completely.

Fig. 4 Errors made per level



	1	2	3	4
wrong	14	32	48	106
right	46	80	84	79

5 Discussion

The first example shows that students can start in the highest level intuitively and are then unable to check (interpret) their answers at level 3. It is hard to distinguish between students using a level 4 reasoning and students using rote operations on numbers. Often, students will go through a number of levels without communicating this. If a wrong assumption is made in one of the first steps, following steps can be correct, whereas the answer to the problem will be wrong. This emphasizes the need to teach students to write out each step of the process they go through to reach the answer. This will also provide more certainty for the students regarding their answers. In the second example we see that student F first uses a mostly formal approach. After choosing the first flavor of ice-cream on top, he calculates the number of possibilities for that one flavor with a multiplication. This multiplication is based on recognition of ‘one less left’ and can be characterized as an action on the highest level. He repeats this for flavor 2 and 3. The calculation can be continued in a correct way, but he changes his mind. He decreases both factors in the next multiplications—for flavor 2-by 1. Student F doesn’t express his thoughts about this, but probably again a sort of ‘one less each time’-idea makes him do it this way. The decrease is built on a wrong interpretation of the situation that, after flavor 1 is put in the first place, this flavor is not to be chosen in any other top 3. So, although the combinatorial characteristics of order and repetition seem to be considered by the student, which is an action on level 2, the calculation on the fourth level is wrong. Student F doubts himself, and after systematically writing down all possibilities (level 3) based on the combinatorial characteristics (level 2) the student reaches insight in what he was doing. His insight is so deep that he can interpret his formal calculation as wrong for the problem, and even better, he is able to match the solution to another—the right—question.

The number of errors made increases as the level increases, and at the highest level, more errors than correct remarks were made. This stresses the need for students to use an emergent approach, so they can go through the levels of Tall. The approach where formulas are studied through rote learning poses a danger of jumping to false conclusions, although the path through the different levels also does not guarantee a correct solution. We do believe that by going through the different levels, the students are capable of checking their results and gaining confidence in themselves concerning combinatorial problems.

The topic of combinatorics is very suitable to develop several skills of students. DeBellis and Rosenstein (2004) confirm that discrete mathematics should be viewed not only as a collection of new and interesting mathematical topics, but, more importantly, as a vehicle for providing teachers with a new way to think about traditional mathematical topics and new strategies for engaging their students in the study of mathematics. Based on research, Lockwood (2011) argued that these specific kinds of combinatorial reasoning problems are well suited for students to be able to determine similarity of problem types, situations and techniques.

6 Conclusion

In mathematical reasoning, students don't automatically develop in line with Tall's (2013) framework and the distinguished levels by Mason do not guarantee a correct solution process. The students don't focus on the whole situation but focus on a detail, in line with the findings of Mason (2004). Teachers should be aware of the fact that students often begin on the highest level without relational understanding and otherwise easily make mistakes going quickly to a higher level in their solution process. Guidance by the teacher seems important. We believe that education focused on relational understanding is of much more value than instrumental instruction (Skemp 1976). Students are more capable of verifying their strategies and justifying their reasoning when education is built on their individual informal approach (Eizenberg and Zaslavsky 2004). We believe that emergent modelling can play an important role in this type of education. Exploration at lower levels can help students to develop a relational network of knowledge and maybe prevent that they start automatically, without considering the problem setting, at a high level. There seems to be an important role for the teacher: students need guidance to develop a model-of, individually reaching a higher level. This will strengthen the confidence of the students in their approach and in their answers. This effect could even extend to other fields of mathematics. We believe future research should investigate how teaching combinatorial reasoning could be matched with emergent modelling and what type of guidance is most effective.

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Children's Combinatorial Counting Strategies and their Relationship to Conventional Mathematical Counting Principles

Karina Höveler

Abstract In this chapter results of a qualitative study are presented, the main goals of which were to identify how children solve combinatorial counting problems and to gain insights into the relationship between their strategies and conventional mathematical counting principles. Counting strategies identified and discussed include multiplicative, additive, and compensation strategies. These strategies are examined with respect to their conceptual and operational differences and similarities to combinatorial counting principles, including the multiplication principle, the principle of inclusion/exclusion, and the so-called shepherd's principle.

Keywords Combinatorics · Counting strategies · Problem solving
Primary school · Undergraduate mathematics education

1 Theoretical Background

Being able to determine cardinalities is an important competency, not only in probability contexts. Even though it is already a central issue in the first grade, a range of studies shows that high school students and even university students often struggle to determine cardinalities when they face combinatorial problems (e.g., Cadwallader et al. 2012; Eizenberg and Zaslavsky 2003; Godino et al. 1992; Hadar and Hadass 1981; Kavousian 2008; Lockwood 2010, 2012). Errors such as overcounting, which involves determining a quantity that is bigger than the wanted one, and difficulties in choosing the right operation occur (e.g., Kavousian 2008). It is predicted by several authors (e.g. English 2007; Halani 2012; Hefendehl-Hebeker and Törner 1984) that these errors and difficulties are based on a lack of understanding of the underlying combinatorial ideas. Maher et al. (2011) investigated in a long-term study over a period of twelve years how students developed central

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combinatorial ideas (such as formulas for basic combinatorial operations) based on their own notations, strategies, and ideas. The authors (Maher et al. 2011) suggest that the constant revision and the elaboration of key ideas in the sense of Bruner's idea of the spiral principle are of great importance for the acquisition of understanding combinatorial strategies and concepts. From this perspective, it is essential to address combinatorial problems early with primary school pupils. To support their learning processes in a socio-constructivist perspective, knowledge about students' individual strategies and underlying ideas is required. This points to the need to specify the relationship between children's strategies and conventional mathematical principles (Duit et al. 2012).

1.1 Analysis of Subject Matter: Conventional Combinatorial Counting Approaches

The term *combinatorics* is derived from the Latin word *combination* which originally describes the conclusion of two things (Knobloch 1973). Combining elements into a new object or, in Bernoullis words: "the mixing, grouping, and co-ordination of things to objects" forms the central content of the mathematical field of combinatorics (Bernoulli 1713, according to the German translation of Haussner 1899, p. 76f.; English translation by the author). The aim of combinatorial questions is to determine all permissible combinations ("Which ones are there?") and their quantity ("How many possibilities are there?") (Kütting and Sauer 2008). School-related combinatorics is described by Törner (1987, p. 121) as kind of a higher multiplication, for which "new" numbers and "new" counting methods are provided. Since these mathematical counting methods depend on the characteristics of combinatorial counting problems—more precisely on the characteristics of the set of objects to be counted—it is necessary not only to consider possible solution strategies more closely, but also the characteristics of these problems.

Characteristics of the Set of Objects to be Counted

In mathematics, a combinatorial object is understood as combining things or elements to new objects on the basis of given rules (Jeger 1973). Depending on the particular problem, the combinatorial objects to be carried out are based on different compositional laws. Considering that the order is or is not of importance, a selection of only some of the elements might or might not be necessary, and elements may occur simply or repeatedly, a total of six combinatorial configurations are derived for elements of a single set: (1) combinations of m elements, taken n at a time, (2) arrangements, in order, of m elements, taken n at a time, and (3) permutations, arranging m elements, taken m at a time), each with or without repetition (Batanero et al. 1997). For each of these combinatorial configurations, there are operations available to determine the total amount of outcomes. Since the same solution strategies can be used for the same sought-after configuration, these terms often also describe the categorization of combinatorial problems and different combinatorial operations (Jeger 1973).[1]

Approaches to Solve Combinatorial Counting Problems

From a mathematical point of view there are at least three approaches to solve combinatorial counting problems: systematic listing, the application of counting principles, and combinatorial operations, like combinations, variations or permutations. In addition, representations like tables, graphs, and tree diagrams are helpful tools. For primary school the former two approaches are of great interest. Both can already be applied with knowledge and skills developed so far in children's education. Counting principles, such as the multiplication principle or the principle of inclusion and exclusion, are of particular importance as almost any counting problem can be solved by their skillful application (Schrage 1996). Regardless which approach is used, the characteristics of the sought-after combinatorial configuration must be taken into account.

1.2 Current State of Research on Individual Combinatorial Counting Strategies and Ideas

Previous studies have investigated the long-term development of combinatorial ideas (Maher and Martino 1992, 2000; Maher et al. 2011) and the difficulties of learners of different ages (Batanero et al. 1997; Fischbein and Gazit 1988; Fischbein and Grossman 1997; Hadar and Hadass 1981; Piaget and Inhelder 1975). Other studies provide important findings about and ways to describe students' combinatorial thinking with regard to the set of elements to be counted (Lockwood 2012, 2013) and information about individual combinatorial problem-solving strategies (e.g., English 1991, 1993, 1996; Larivée and Normandeau 1985; Martino 1992; Maher and Martino 1992). The empirical studies on children's problem-solving strategies provide some information about their listing strategies. In particular, the stages in their strategy development and the use of different types of strategies have been examined. In a later study the great influence of task variables (e.g. operation, nature of elements, or value of parameters m and n) on the problem-solving strategies and success rates of learners has been demonstrated by a study of Batanero et al. (1997). This influence of different task variables was not considered in the afore mentioned investigations dealing with young children's problem-solving strategies. Furthermore, these studies did not explore the underlying ideas of children's strategies.

Early investigations of Piaget and Inhelder (1975) give hints that children at elementary school age already use additive or multiplicative calculations as well as

[1]The terminology to describe the different types of combinatorial objects and operations is applied inconsistently in different articles and books (compare for example Batanero et al. 1997 and Kütting and Sauer 2008). In this article, the terminology of Batanero et al. (1997) is used.

the idea of recurrence instead of counting all units particularly. But so far little is known about these counting strategies, as later studies (e.g. English 1991, 1993; Maher and Martino 1996; Hoffmann 2003) generally focused on solving existential problems (“Which outcomes are possible?”) instead of counting problems (“How many outcomes are possible?”). This lack of attention is contrary to their importance: Insights into students’ strategies and students’ underlying concepts are the basis for helping students understand and apply basic combinatorial counting principles. This is of vital importance since these combinatorial counting principles are the basis to understand and justify counting formulas, as the results of a teaching experiment with undergraduate students indicate (Lockwood et al. 2015). A conceptual understanding of principles may furthermore help to overcome some of the student’s known errors.

To assist learners in understanding and reinventing for themselves the ideas of counting principles, at least the following aspects are required: Firstly, more precise knowledge about learners’ counting strategies and especially their underlying concepts and secondly, information about similarities and differences between primary children’s strategies and mathematical principles.

2 The Study: Investigating Relationships between Children’s Counting Strategies and Combinatorial Counting Principles

Aim of the study. To address the previously described lack of knowledge about the combinatorial problem-solving strategies and the underlying concepts of primary children, a study with the following guiding questions was conceived:

> *How do primary children solve combinatorial counting problems without prior teaching at school?*
>
> *What is the relationship between primary children’s strategies and the conventional mathematical approaches?*

Due to the importance of counting principles one main focus of the study was to shed light on the questions above by focusing on the relationship between children’s counting strategies and mathematical counting principles:

> *What counting strategies do third graders use to solve combinatorial counting problems?*
>
> *What is the relationship between these strategies, including their underlying concepts, and mathematical counting principles?*

Data collection and tasks. Information was gathered from individual, clinical interviews (30–45 min) in two episodes with $n = 18$ and $n = 45$ third graders from different schools. Both were divided in three groups. Every group of children got one set of combinatorial problems. Unlike the studies of English (1991, 1993) and Hoffmann (2003), who focused on learners' combinatorial strategies solving combinatorial listing problems, in the present study the question "How many outcomes are possible?" was posed, instead of asking "Which outcomes are possible?". Task selection was based on two frame conditions: Offering the use of a broad range of counting principles, and considering the influence of different task variables (operation, nature of elements, value of parameters m and n). On the basis of these frame conditions three different types of problems were used: (1) combinations with repetition, (2) combinations without repetition, (3) arrangements without repetition. For the terminology in this study, "combination" describes combinatorial problems in which order does not matter, while "arrangement" describes combinatorial problems in which order does matter. Each problem set consisted of two isomorphic combinatorial problems, differing in context and elements to be combined (for further details see Höveler 2014).

Data analysis. The video-recorded and transcribed interviews were analyzed in two steps by central elements of the Grounded Theory (Glaser and Strauss 1967). First, classes of children's strategies were built. Afterwards relationships between their strategies, including the underlying concepts, and mathematical principles were identified by constant comparison.

3 Findings: Disparities and Similarities between Children's Counting Strategies and Combinatorial Counting Principles

Data analysis shows that children determined the cardinality of the sets of outcomes by *additive*, *multiplicative*, and *compensation strategies*. Also, they used *recursive strategies* and *structural correspondences* to solve the problems. Some counting strategies were developed without prior listing, whereas most were inferred from a systematic listing. The comparison of children's strategies and mathematical principles reveals disparities and similarities in their underlying concepts.

3.1 Disparities: Children's Multiplicative Strategy and the Multiplication Principle

Remarkably, the use of multiplicative calculations has been observed not only in arrangement problems but also in combination problems. In either case the determined amount was bigger than the wanted: the error of overcounting occurred. Analysis shows that all multiplicative approaches are based on the same underlying idea: Deducing the number of a set of outcomes on the basis of the number of a single element in the set of outcomes. This so-called 'from single to set

Table 1 Sets of combinatorial tasks used in the study

Set 1: Combination without repetition (C.wo.r.)	Set 2: Combination with repetition (C.w.r.)	Set 3: Arrangement without repetition (A.wo.r.)
Soccer: Four teams want to play a soccer tournament. Each team plays once against each other team. How many games are there in total?	Ice-cream: Here are four different flavors of ice cream. How many different sundaes with two scoops are possible, if the order of scoops does not matter?	Blocktowers: Here are four blocks of different colors. How many towers two blocks high can you build?
Lottery: Here are four different numbers in a bowl. How many different pairs of numbers are possible?	Domino: Suppose you create a set of dominoes: Here are empty dominoes. On each half of the domino there can be one to four dots. How many dominoes are there in the complete set?	Two digit numbers: Here are four cards, each with a different digit. How many two digit numbers can you build?

multiplication' and its underlying concept 'from single to set' are illustrated below with reference to an interview sequence with Phil who solved the block tower-problem (see Table 1):

Interviewer: Here are four blocks in different colors. How many two blocks high towers can you build?
Phil: [constructs all possibilities with a blue block, six altogether]. Hum, six blue towers. Means six towers with every color. Twenty-four.
Interviewer: Why twenty-four?
Phil: All in all there are four times six, twenty-four towers.

Phil first considers how many solutions with a fixed element, namely a blue block, are included in the set of outcomes. He finds out that there must be six towers with blue blocks in total. Based on the determined amount (six towers with blue blocks), he considers the amount of solutions with a fixed other element (six towers with every color). To determine the total amount of outcomes, he multiplies the cardinality of elements of the initial amount (four blocks of different colors) and the cardinality of solutions with a fixed element (six towers with every color). This approach is used by many children.

Both basic considerations are entirely correct and reasonable. The subsequent conclusion to multiply is presumably based on the known concept of multiplication as repeated addition: "The product $m \cdot n$ is the cardinality of the union of m pairwise disjoint sets, which all have the same cardinal number n" (Kirsch 2004, p. 23, translated by the author). However, since the composed quantities of towers with a special color are not disjoint (e.g., the six towers with a blue block contain already two towers of each other color) the essential condition of disjoint sets is not fulfilled and the overcounting occurs.

The comparison of this approach with the *multiplication principle* shows an essential difference. According to Tucker (2002, p. 170) the latter can be defined as follows:

> Suppose a procedure can be broken into m successive (ordered) stages with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the m^{th} stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and *if the composite outcomes are all distinct*, the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes (emphasis in original).

The 'from single to set multiplication' is based on the idea of building groups of solutions (for example all towers with a blue block, all towers with a green block and so on). It produces in many cases groups which are not disjoint. The above principle is often formulated as splitting objects into single elements or stages which are necessarily independent. The main difference is obvious: The 'from single to set multiplication' produces groups which are not disjoint, whereas the above principle is based on disjoint sets.

The misuse of the underlying concept of the multiplication principle is of further interest: none of the children's multiplication strategies were based on this underlying concept. Instead, all multiplication strategies were based on the 'from single to set' idea. There are several indications not to consider this underlying idea as a random error, but rather as a typical concept: Firstly, as mentioned before, the 'from single to set multiplication' appeared not only while solving the block tower-problem, but also in problems with different contexts and different underlying figures. Secondly, data analysis showed cognitive conflicts, which arose in many cases when students, for example Paul who solved the soccer-problem (see Table 1), noticed the emergence of double outcomes when using the 'from single to set' strategy:

> Paul: Hum, some games are twice. But actually, that does not make sense. Every team plays three times. Means there should be four times three games. But, still there are some double games, for example green plays against blue and blue against green? Hum, every team plays three times but there are 6 games?!

Apparently, learners assume that the 'from single to set' strategy leads to the right number of outcomes. The challenge seems to be the non-empty intersection of the subsets. Thirdly, some learners who determined the cardinality by systematic listing or addition, also used the 'from single to set' idea and determined an amount which was bigger than the wanted.

Further investigation shows the similarity between the 'from single to set multiplication' and the idea of multiplication as repeated addition, a concept, children know from the second year of primary school. It only differs in one important aspect: the sets are not disjoint. This similarity leads to the assumption that the occurrence of this error is due to learners' incomplete prior knowledge. For most of the typical counting problems, which children solve during the early grades, there is no need to reflect whether sets are disjoint or not, because the information is just given. In contrast, this reflection is essential while solving combinatorial problems, as the 'from single to set multiplication' shows.

3.2 Similarities: Children's Compensation Strategies and Mathematical Compensation Principles

Students who used strategies based on the mentioned 'from single to set' concept realized the overcounting in most instances. To compensate the overcounting, two systematic strategies have been observed: the 'take-away' and the 'classification strategy'. Both strategies show strong similarities to mathematical counting principles and are described below.

Take-away strategy: This strategy is characterized by the elimination of double units. Students, such as Phil, compared systematically the classes of outcomes with each other. In each step those outcomes of the 2nd, 3rd, …, *n*th group, which already appeared in the 1st group, were removed.

> Phil: Actually, that's easy, because, one… I have to…I have to pull out these *[pulls towers "blue-green" and "green-blue" aside]* …. And actually, everything looks as if – oh these ones can be put away *[puts towers "red-green" and "green-red" from the red group away*] because of these two *[points his finger on the equal objects in the green group]*. But now it looks like everything is ok with the green towers.

The comparison of this approach with counting principles shows that it includes basic ideas of the principle of inclusion and exclusion. This principle generalizes the addition principle of two sets whose subsets are not disjoint (Schrage 1996). Children's counting can be understood as forming the union of the quantities, whereas the removal of the duplicate objects can be understood as excluding elements in the intersection.

Classification strategy: Instead of removing double outcomes, some children, for example Jasmin, determined the total number by grouping those objects, which can be seen as equal under the given task conditions. Afterwards they counted the number of groups to determine the cardinality of the set of outcomes:

> Situation: Jasmin solved the lottery problem by systematic listing with the underlying idea 'from single to set' and arranged her solutions in four groups on the table: 1st group: 1-2; 1-3; 1-4; 2nd group: 2-1; 2-3; 2-4; 3rd group: 3-1; 3-2; 3-4; 4th group: 4-1; 4-2; 4-3. She determined a total number of twelve instead of six solutions. After she realized that some of her solutions were equal under the given task conditions, the interviewer asked her if there was a way to find out the right amount of possibilities:

Interviewer: What can you do to find out the right number of solutions without double outcomes?

Jasmin: I'll pool the equal ones together. [J. rearranges her set of outcomes in groups: 1st group: 1-2; 2-1; 2nd group: 1-3; 3-1; 3rd group: 1-4; 4-1; 4th group: 2-3; 3-2; 5th group: 2-4; 4-2; 6th group: 3-4; 4-3].

Interviewer: How many are there in total, now?

Jasmin: Hum, lets count. Well, one, two, three, four, five and six [While counting she points her finger on each group of outcomes].

A systematic comparison between this approach and mathematical counting principles points out the similarity between the classification strategy and the so-called shepherd principle, a second way of indirect counting. Its idea is illustrated by an anecdote of a shepherd described by Bourbaki (1963): To the question: how to determine the number of sheep, the answer was "by counting the legs and dividing the result by four." The principle is based on the rule: "Let L and S be two sets with finite cardinalities l and s. If there is a mapping f from L onto S (f: $L \rightarrow S$, surjective) such that $\left|f^{-1}(y)\right| = c$ for all $y \in S$ then $l = c \cdot s$." (Schrage 1996, p. 193). The number of s may then be determined by division of l by c, if l and c are known. Jasmin's strategy is based on the same idea: Firstly, she groups those outcomes, which are equal under the given task conditions. This equals the idea of dividing l by c. Secondly, she determines the wanted quantity of s by counting the number of groups.

During the study only two children used this strategy. This shows, however, that some learners already autonomously develop the underlying mathematical idea of the shepherd principle on the level of action (for detailed analysis and further explanations and strategies see Höveler 2014).

4 Discussion and Conclusion

This study indicates that third graders already use different counting strategies to solve combinatorial problems. Besides the multiplicative and compensation strategies, which were presented in this chapter, they also use additive strategies as well as recursive strategies and structural relationships when they are faced with isomorphic problems. As has been discussed, these strategies are cogently connected to mathematical counting principles. Some of the strategies are connected to combinatorial counting principles but with substantive disparities based on sustainable different concepts. For example, the 'from single to set multiplication' seems to be based on the idea of multiplication as repeated addition of equal addends without recognition that the groups need to be disjoint, whereas the multiplication principle requires disjoint sets. Other strategies, however, show a remarkable degree of convergence with associated counting principles, for example the 'take-away' and the 'classification' strategies, as connected to the principle of inclusion and exclusion and the so-called shepherd's principle, respectively.

Which conclusions can be drawn from these results and which further investigations are necessary?

The results provide one possible explanation for a frequently occurring type of error: the error of overcounting (e.g. Kavousian 2008; Lockwood 2011), whereby learners solve various counting problems by multiplication and determine a bigger amount than the wanted. The systematic comparison of their strategies with combinatorial principles leads to the conclusion that their strategies are not equivalent to the idea of the multiplication principle. Instead they are based on the ‘from single to set’ concept. As pointed out, the ‘from single to set multiplication’ may be based on prior knowledge, namely the notion of multiplication as continued addition of equal addends, but this does not take into account the requirement of disjoint quantities. As a consequence, the lack of awareness of disjoint sets should be perceived as the cause of the overcounting. Since this lack of the idea of disjoint quantities also appears in contexts of additive strategies and systematic listing, it seems to be a typical misconception while solving counting problems.

The results lead to the assumption that the ‘idea of disjoint quantities’ is a key concept in developing combinatorial understanding and a crucial point to overcome some overcounting errors. Further investigation is needed to clarify whether the ‘from single to set’ concept and the missing awareness of disjoint quantities are a study-specific phenomenon or whether they do typically occur when children are solving combinatorial counting problems.

The results lead to another consideration: In this study, none of the children used a multiplicative strategy which was based correctly on the idea of the multiplication principle. From a mathematical perspective this principle is of major significance while solving counting problems. Therefore, it is of special interest to find out whether students develop an entire concept of the multiplication—including the idea of disjoint sets—when classroom reflections focus on the awareness of disjoint quantities and the relationship between the amount of outcomes with a specific element and the total amount of outcomes.

Furthermore, the results show that approaches for viable mathematical counting strategies arise from initially faulty appearing strategies: Thus based on the ‘from single to set’ idea, the ‘take-away’ strategy and the ‘classification strategy’ were used by students to compensate for perceived overcounting. These strategies represent at a very basic level the ideas of counting principles, in this case the idea of the principle of inclusion and exclusion and the idea of the shepherd principle. This observation leads to pedagogical advice for handling the ‘from single to set’ idea in class: This strategy should not be discarded and considered as incorrect access. Rather, it is essential to perceive it as a starting point to discuss compensation strategies and to address the central idea of disjoint quantities.

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Reinforcing Mathematical Concepts and Developing Mathematical Practices Through Combinatorial Activity

Elise Lockwood and Zackery Reed

Abstract As a branch of discrete mathematics, combinatorics is an area of mathematics that offers students chances to engage with accessible yet complex mathematical ideas and to develop important mathematical practices. In this chapter, we focus on a combinatorial task involving counting passwords, and we provide examples of affordances that undergraduate students gained by engaging with the task. We highlight two kinds of affordances—those that strengthened understanding about fundamental combinatorial ideas, and those that fostered meaningful mathematical practices. We hope that these examples of rich and sophisticated student work will contribute to an overall goal of elevating the status of combinatorics specifically, and discrete mathematics more broadly, in the K–16 curriculum. We conclude with a handful of pedagogical implications.

Keywords Counting · Combinatorics · Mathematical practices Postsecondary students

1 Introduction and Motivation

Combinatorial tasks offer students opportunities to think deeply about accessible yet complex mathematical ideas. In his book *Applied Combinatorics*, Tucker (2002) says of his counting chapter, "We discuss counting problems for which no specific theory exists" and emphasizes that the problems require "logical reasoning, clever insights, and mathematical modeling" (p. 169). He goes on to say that, "for many students, this is the most challenging and most valuable chapter in this book" (p. 169). Martin (2001) shares a similar sentiment in his introduction to his chapter on counting, saying "One of the things that make elementary counting difficult is

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that we will encounter very few algorithms. You will have to think. There are few formulas and each problem seems to be different" (p. 1). Kapur (1970) points out that "the mathematics of the continuous" (p. 113) has powerful methods that allow some applications to be completed without requiring ingenuity. In contrast, in "the mathematics of the discrete, not many such powerful methods are available and ingenuity is always required" (p. 114). He goes on to list several reasons why combinatorial mathematics should be made a higher priority in school mathematics, and the same could be argued for its importance in the undergraduate curriculum.

We share these passages to point out that these mathematicians and researchers see value in the kinds of thinking that combinatorial tasks elicit. This contributes to an overall argument that combinatorial tasks, and discrete mathematics tasks more broadly, can offer many opportunities not only for reinforcing mathematical concepts but also for developing important mathematical practices.

In this chapter, we present a set of combinatorial tasks (which we call the Passwords Activity) that we found to be useful for work with undergraduate students. Although we focus on one particular task in this chapter, we hope that the benefits we share related to this particular task can serve as a representative example of benefits of combinatorial tasks more generally. We outline the task, and we present excerpts from student work to demonstrate three ways in which a combinatorial task both strengthened understanding about fundamental combinatorial ideas and also fostered meaningful mathematical practices. In terms of combinatorial ideas, we will show how this particular task helped students strengthen their understandings of the multiplication principle, the notion of 'choosing,' and combinatorial identities. In terms of practices, we will demonstrate how the task provided opportunities for students to develop their skills of justifying, generalizing, and proving (specifically, engaging in combinatorial proof).

2 Background Literature and Theoretical Perspective

One reason for us to develop and study tasks like the Passwords Activity is because there is ample evidence that students struggle to solve counting problems correctly (e.g., Batanero et al. 1997; Eizenberg and Zaslavsky 2004; Godino et al. 2005; Lockwood and Gibson 2016). Although strides have been made in recent years to learn more about how to help students solve counting problems more successfully, instructors who have taught counting may agree with the sentiment shared by Annin and Lai: "Mathematics teachers are often asked, 'What is the most difficult topic for you to teach?' Our answer is teaching students to count" (2010, p. 403). To this end, then, we hope to develop particular instructional techniques and activities to help students (and teachers) be more successful when solving counting problems. This is why we highlight some of the domain-specific combinatorial ideas that can be reinforced through this set of tasks. In particular, we seek to demonstrate connections to three important elements of combinatorics: the multiplication principle, combinations (binomial coefficients), and combinatorial identities. We elaborate each of these concepts in the Results section.

In addition, as noted in the introduction, there is a consensus among some researchers and educators that combinatorics and discrete mathematics afford key mathematical practices, such as those defined by the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010). For example, the 1991 book *Discrete Mathematics Across the Curriculum* made a case for the value of discrete mathematics topics in K–12, arguing that discrete mathematics tasks can foster practices such as making mathematical connections, problem solving, critical thinking, and mathematical reasoning (Kenney and Hirsch 1991, p. vii). Other mathematics education researchers have highlighted the utility of combinatorics as a context in which to help students make connections among representations (Maher et al. 2011) and engage in practices like generalization (Lockwood 2011), proof (Maher et al. 2011) and problem solving (Lockwood 2015). In this chapter, we want to contribute to this overall narrative by demonstrating some examples of how a rich combinatorial task facilitated combinatorial insight and reinforced mathematical practices.

3 Methods

In the study described in this chapter, we had 10 undergraduate students solve counting problems in individual, hour-long interviews. Nine of the students were calculus students who had not taken a university level course that covered counting, and one student was a senior math major who had taken an upper-division discrete mathematics course. We gave students the Passwords Activity, which we explicate below. This activity was part of a larger study and was designed to target students' generalizing activity in combinatorial tasks specifically. We sought both to learn about students' combinatorial reasoning and about their generalization in combinatorial contexts. For data analysis, we transcribed the interviews and used qualitative data software to review the interviews. We used the constant comparative method (Strauss and Corbin 1998) to identify relevant phenomena across the interviews, and to discover particularly salient episodes that could explain such phenomena. In the next section, we highlight details of the Passwords Activity.

4 The Passwords Activity

The Passwords Activity is designed to scaffold students' learning as they develop a statement of the binomial theorem. A general statement of the theorem is given as follows: If x and y are real numbers and n is a nonnegative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

We acknowledge that there are many ways to prove the binomial theorem and that there are a variety of contexts to which the binomial theorem might be naturally connected (block walking, Pascal's Triangle, and committee selection are common settings). We also recognize prior work that has examined students' reasoning about binomial coefficients and the binomial theorem in these different settings. For example, Maher and Speiser (2002) connect block walking with binomial coefficients, and Maher et al. (2011) demonstrate powerful student reasoning about the binomial theorem and Pascal's Triangle. Our activity offers but one alternative approach to an important combinatorial idea, but there are certainly opportunities for other meaningful connections that we do not explicitly make here. We focus on counting a set of passwords in two different ways, each of which is reflected by an expression on one side of the binomial theorem. In the next section, we present the progression of the Passwords Activity, which unfolds in three stages. We describe the progression of tasks and what the students were instructed to do, and in so doing we provide details about the mathematics of the task.

4.1 Details of the Task Progression

Stage 1: Passwords consisting of the letters A and B. We first had students consider the total number of possible length 3 passwords involving A and B (we will refer to these as 3-character A,B passwords), of which there are 8. To answer this, students tended to list all 8 passwords (and in some cases they justified this result by reasoning that for each entry in the password there are 2 possible options, and so by the multiplication principle there are $2^3 = 8$ possible passwords). Most students listed on their own, but if they seemed stuck on the problem we asked them to try listing some passwords. We then prompted students to make Table 1, which organizes the total number of passwords according to the number of As in the password (we instructed students to make such a table, but a template was not provided for them).

Students filled out the table either by simply listing the passwords for each row (or reading the respective numbers of passwords from a previously generated list that they may have made), or by recognizing that the entries in the rows of the table are binomial coefficients ($\binom{3}{k}$ for $k = 0, 1, 2, 3$). Some students recognized that binomial coefficients make sense because for a given number of As in a password, they may choose the positions in which the As will go. The placement of the As determines the password since there are only Bs remaining to fill the empty slots. We then asked students to create tables for passwords of length 4 and 5, and they generated tables like Tables 2 and 3.

Next, we had students explore the relationship between the values of the table and the total number of passwords in each case. Many students identified a pattern of 8, then 16, then 32 total passwords for 3, 4, and 5-character passwords, and they

Table 1 The 3-character A,B table

Number of As	Number of passwords
0	1
1	3
2	3
3	1

Table 2 The 4-character A,B table

Number of As	Number of passwords
0	1
1	4
2	6
3	4
4	1

Table 3 The 5-character A,B table

Number of As	Number of passwords
0	1
1	5
2	10
3	10
4	5
5	1

could generalize that the total number of n-character A,B passwords is 2^n. To justify this, some students applied the multiplication principle to an arbitrary n-stage counting process, in which there are two choices at each stage.

It is important to note that even if some students had not previously been exposed to binomial coefficients or were not able to come up with a general formula, they could still engage with subsequent stages in the activity and could potentially engage in generalization. Students made note of patterns of the respective tables (1, 3, 3, 1 for 3-character passwords, 1, 4, 6, 4, 1 for 4-character passwords, 1, 5, 10, 10, 5, 1 for 5-character passwords, etc.[1]), and some students used previously created tables in subsequent work to engage in both combinatorial reasoning and generalization, even without referring to binomial coefficients.

Stage 2: Passwords consisting of the letters A and B and the number 1. Next, we had students consider passwords that consist of the characters A, B, and 1. They engaged in the same kinds of activity as they had previously, exploring the two ways of generating the numbers of 3, 4, and 5-character 1, A,B passwords (i.e., using the multiplication principle or using binomial coefficients, whether explicitly

[1]Again, we recognize that these numbers are rows in Pascal's triangle, but pursuing the relationship with Pascal's triangle is not our goal in this set of tasks.

or not). We prompted them to generate similar expressions and tables for each case, where the tables were organized in terms of the number of 1s in each password (instead of the number of As). Importantly, students could use the tables created in the prior stage in order to complete this stage. As an example, we consider 4-character 1, A,B passwords.

First, we note that there are 3^4 total 4-character 1, A,B passwords, because there are three choices for each of the four positions in the password. Table 4 organizes the passwords according to the number of 1s in the password. In order to fill out an entry in the table, we had students first place the 1s (or select positions to place the 1s) and then fill in the rest of the positions with As or Bs. Note that this allowed the students to leverage their previous work in a couple of ways. First, they could draw on specific entries in a previous table, and second, they could use the fact that there are 2^k total k-character A,B passwords.

Again, in this case if students did not yet have a formula for binomial coefficients, they could still interact with the task and engage in generalization. In particular, they could look back at tables created for 3, 4, or 5-character A,B passwords and use those numerical results for the first stage in the counting process. They could also recognize that for any of the positions that were not 1s, they were simply creating A,B passwords, and previous results could be leveraged to complete the more current tables.

Stage 3: Passwords consisting of numbers and letters. Once the students had counted A,B passwords and 1,A,B passwords each in two different ways, we had them consider a situation in which we have multiple letters and multiple numbers. In order to discuss this, we describe one particular example of 5-character passwords consisting of the numbers 1 or 2 and the letters A, B, or C, which we call 5-character 1, 2, A, B, C passwords. As before, a student could count the total number of such passwords in two ways—first by simply computing the total by arguing about the number of choices for each position, and second by making a table, this time according to the number of numbers in a password. There are 5^5 total passwords, because there are five choices (3 numbers and 2 letters) for each of the five positions. The table can be filled out as in Table 5.

To justify the table entries, a student can consider one of the rows—for example, the fourth row counts the 5-character 1, 2, A, B, C passwords that have exactly 3 numbers. One can first select places that will be numbers (there are $\binom{5}{3}$ ways to

Table 4 The 4-character 1, A,B table

Number of 1s	Number of passwords
0	$1 \cdot 2^4$
1	$4 \cdot 2^3$
2	$6 \cdot 2^2$
3	$4 \cdot 2^1$
4	$1 \cdot 2^0$

Table 5 A 5-character 1, 2, A, B, C table

Number of numbers	Number of passwords
0	$1 \cdot 2^0 \cdot 3^5$
1	$5 \cdot 2^1 \cdot 3^4$
2	$10 \cdot 2^2 \cdot 3^3$
3	$10 \cdot 2^3 \cdot 3^2$
4	$5 \cdot 2^4 \cdot 3^1$
5	$1 \cdot 2^5 \cdot 3^0$

do this, which is 10), and each of those number places can be filled in with either 1 or 2, giving 2^3. Then the remaining two positions must be letters, and there are 3^2 ways to filling those positions with A, B, or C. The same line of reasoning holds for any of the rows, and summing the rows (which count disjoint cases of how many numbers are in the passwords) yields the total number of passwords.

If a student has previously seen binomial coefficients, this line of reasoning can be extended to a general case of counting n-length passwords consisting of x numbers and y letters. There is similarly a 3-stage process that determines how many length n passwords have exactly k numbers. First, one can select k of the n positions in which to place the numbers, and this leaves $(n - k)$ positions that will be letters. Then any of the k number positions can be any of the x numbers, and any of the $(n - k)$ letter positions can be any of the y letters. In this way, a general statement of the binomial theorem is achieved, which is the mathematical culmination of this activity:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The teaching and learning goal of this activity was not necessarily to generate a statement of the binomial theorem (specifically in the case where x and y are nonnegative integers), and most of the students we worked with did not get all the way to a full statement of the binomial theorem. However, as we will see in the following results, the activity provided opportunities for students to engage meaningfully in combinatorial thinking and mathematical practices in a number of ways.

5 Affordances of the Passwords Activity: Examples from Student Work

Having described the progression of the tasks, we now present examples of students engaging in the Passwords Activity. The aim here is both to demonstrate ways in which the tasks have been effectively implemented with students and to highlight affordances of the activity that we observed among students. We emphasize two kinds of affordances—those that facilitated learning of combinatorial content, and those related to the development of mathematical practices. We provide examples of three students that demonstrate affordances.[2] First, the task reinforced meaningful connections to the multiplication principle while also providing an opportunity for justification. Second, the task allowed students to strengthen their understanding of "choosing" while facilitating meaningful generalizing activity. Third, the task encouraged student reflection on combinatorial identities while supporting the broader practice of proving (in the context of combinatorial proof). The point is to emphasize how combinatorial tasks like the Passwords activity can serve both to help undergraduate students learn important combinatorial ideas and also help students develop important practices.

5.1 Reinforcing the Multiplication Principle and Providing Opportunities for Justification

In this case, we emphasize the content of the multiplication principle. The multiplication principle is one of the most foundational ideas in counting [some call it the "Fundamental Principle of Counting" (Richmond and Richmond 2009)]. Broadly, it is the idea that if a given counting process can be broken down into independent successive stages, then the total number of outcomes of that process is the product of the number outcomes of each stage, provided that the composite outcomes are all distinct (Tucker 2002). It underlies and provides justification for many of the counting formulas students learn. However, while it is an intuitive idea, and while it uses the familiar operation of multiplication, there are a number of subtle mathematical issues in the multiplication principle (see Lockwood et al. 2016, for more detailed discussion of the multiplication principle; see also Höveler 2017). In light of its importance, we feel that it is worthwhile for students to have opportunities to use and reason about the multiplication principle. Thus, one mathematical affordance of the Passwords Activity is that it can reinforce students' fluency with the multiplication principle.

We present an episode with Desmond,[3] a vector calculus student with a background in computer science. When asked for the total number of possible n-character A,B passwords, the following exchange occurred:

[2]Note, in organizing the results, in each example we pair an affordance related to combinatorial content with one related to mathematical practice. We do so to be efficient in our presentation of three student examples, but we do not claim that the affordances must be paired in this way. Indeed, in investigating some combinatorial idea (like the multiplication principle) students may engage in a variety of practices.

[3]All names are pseudonyms.

Int.	So if you had an n-length password, again using A and B, how many total passwords do you think there would be?
Desmond	2 to the n.
Int.	Okay. And can you just explain again your thinking on that?
Desmond	I mean … it's basically, okay two states for each. So every time you add another digit you're multiplying it by 2 …. So, okay, let's just abbreviate that into powers.

When Desmond is referring to these "states," we infer that he is considering that within the password, each entry can either be a "state" of A or a B, and he spoke of these states as binary choices (this language is not surprising given his experience in computer science). He was able to articulate why multiplication occurred in this binary case by arguing that there are two possible states at each position.

We then further prompted Desmond to discuss why the number of passwords double each time a new entry in the password is added, and he gave the following reasoning:

Desmond	…you just tack on an extra digit to every single one, which is either an A or a B.
Int.	Okay.
Desmond	So just based on—okay, we've had this many states for 4 digits [refers to the 4-character A,B password case, which yields 2^4 total passwords].
Int.	Uh-huh.
Desmond	We throw in there another digit, that's like, okay, wait, now, instead we're taking this [points to the 4-character A,B table] with A and this with B, so it's twice that.

In his response, Desmond seemed to understand that doubling occurred each time a new password entry was introduced because for each of the previous password arrangements there were two new passwords created, one that ends in A and one that ends in B. Because of this doubling, it made sense to him that we would multiply the previous number of outcomes by 2. Desmond's work demonstrates the kind of meaningful reasoning about multiplication that the task may elicit. It is worth noting that not all students were able to make meaning of the doubling or to connect the process for generating the total number of passwords to the multiplication principle. However, Desmond's work suggests the potential ways in which students might engage meaningfully with multiplication in counting, perhaps developing stronger foundations for understanding the multiplication principle.

In this example, we also see how a student could develop meaningful justifications. Here the multiplication principle highlighted a particular relationship, which Desmond recognized as involving the doubling of the total number of passwords from one stage to the next. He was able to justify this doubling in a couple of ways. First he could justify the multiplication by 2 because he envisioned having options for states, and this allowed him to explain and justify why the successive multiplication by 2 might make sense. Desmond also made the observation that the total set of outcomes was doubling (as seen in the second excerpt), and he revealed a correct justification for why the outcomes are doubling, one that is rooted in combinatorial reasoning and not simply a numerical pattern.

In this example from Desmond's interview, we see an instance in which the Passwords Activity afforded a student with the opportunity to learn more about the multiplication principle, both in considering multiple stages in a counting process and in determining the total number of passwords. Even more, though, we see an example of a situation in which a combinatorial task facilitated rich justification. Through this exploration Desmond was able to notice and make use of the combinatorial structure of the outcomes to formulate a sophisticated argument. We would argue that this task, like many combinatorial tasks, was accessible for Desmond in that he could explore the situation without invoking powerful theorems or requiring sophisticated prior mathematical knowledge.

5.2 *Strengthening Understanding of "Choosing" and Facilitating Meaningful Generalizations*

The Passwords Activity also allowed for students to strengthen their reasoning about choosing and binomial coefficients. We use the term *choosing* to refer to the act of selecting a subset of objects from a set of distinct objects, and this can be solved using binomial coefficients. Choosing is a fundamental aspect of understanding counting, both because it helps us solve a number of counting problems, but more importantly because choosing often serves as a stage in the counting process. In prior work, we have found that in some contexts, choosing can be particularly difficult for students to grasp. In particular, in one study, Lockwood et al. (2015a) found that although students demonstrated success on many problems that positioned choosing as selecting objects (such as choosing people to participate in a contest), they came to an impasse on a problem that would have required them to select positions in a binary string that would be occupied by zeros. Lockwood et al. (2015b) suspect that this was an issue of the students not properly encoding what they were trying to count as something they already knew how to count (in particular, they did not perceive of the positions as a set of distinct objects from which they could choose a subset). These findings were corroborated in other work Lockwood et al. (2016), in which the authors found quantitative evidence that students saw some combination problems as fundamentally different than others. Specifically, they had difficulty recognizing some problems (involving encoding of positions) as being solvable using binomial coefficients. These studies offer evidence that selection problems, especially those involving choosing something like positions or locations in a password, can be quite difficult for students.

In this context, we feel that the Passwords Activity could serve as an appropriate introduction to the notion of choosing and selecting positions. To see this, we demonstrate how a student could use choosing in order to complete the activity. James was a student who had taken discrete mathematics, and yet he was not familiar with the binomial theorem (or at least he did not seem to recognize or recall it). Initially he filled out the AB tables by counting outcomes, but he did not recognize that he could use the notion of choosing, which was a topic with which he had been familiar. Figure 1 shows his work on the table for a 4-character AB password—Fig. 1a shows the six outcomes he found for having exactly two A's in the password, and Fig. 1b shows his 4-character AB table (which we had prompted him to make).

Later, as he considered A, B, 1 passwords, he recognized that the numbers he was generating for how many ways to place the 1s in a password were familiar. When he was solving the 4-character A, B, 1 password case, he was trying to figure out how many such passwords have exactly two 1s. He found that there were 6 ways to place the two 1s in a 4-character password. We explicitly asked him why he got 6. He said "that 6 is counting how many ways…" and then he paused. Then he said "It's 4 choose 2 … 6 is 4 choose 2." When asked what he meant, he recognized how to interpret the problem as a way of choosing. "I'm counting how many ways you can choose to place two 1s in a sequence of 4." It is noteworthy to us that recognizing the 6 as 4 choose 2 was not an immediate connection for James. He did not seem to make the connection to choosing right away, but after some thought he could see why 4 choose 2 could make sense in this context. Given students' difficulties with realizing that they can choose positions, James' work suggests that perhaps the Passwords Activity could facilitate students' understanding of choosing in such a context.

After James made this connection, he seemed able to continue to use and understand the relationship through the remainder of his work on the task. For example, he went on to consider the question of how many passwords of length 8 could be made from the characters 1, 2, A, B, C, and D. He was able to make the table for this problem, as seen in Fig. 2.

James Yeah. So 8 choose 1 ways to choose where one digit goes. Two possible digits, so you have to account for if it's one or the other and then for the seven remaining slots there are four possible letters.

Int. Okay. Cool. Does that make sense? Were you looking at the table? Were you thinking about it both or what?

James Well, for me once I can convince myself that like this is true and this is true and this is true [points to 3 of the entries in the table], and I can like clearly see the pattern—I just like—yeah, if I like stop to think about it, like 8 choose 5 ways to place the numbers.

Int Uh-huh.

James And then 2 to the 5th ways to arrange those 5 numbers, and then 4 to the 3rd ways to arrange the last 3 slots.

Here we see that James seemed to be able to explain his work on this problem, and he could articulate what each term meant. He could use this notion of choosing effectively to pick spots in which the numbers could go. Again, this is a subtle

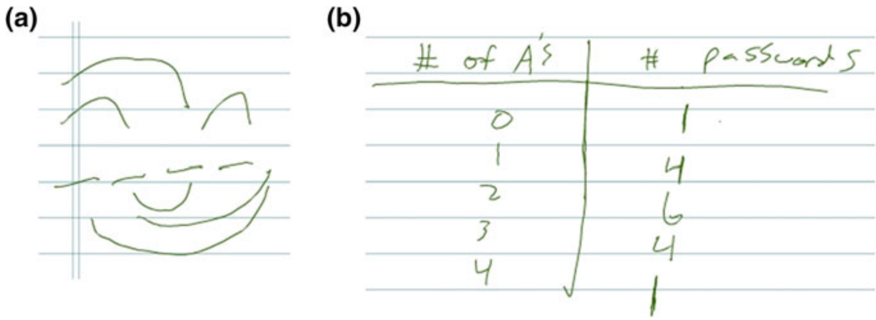

Fig. 1 Work on table for 4-character AB password

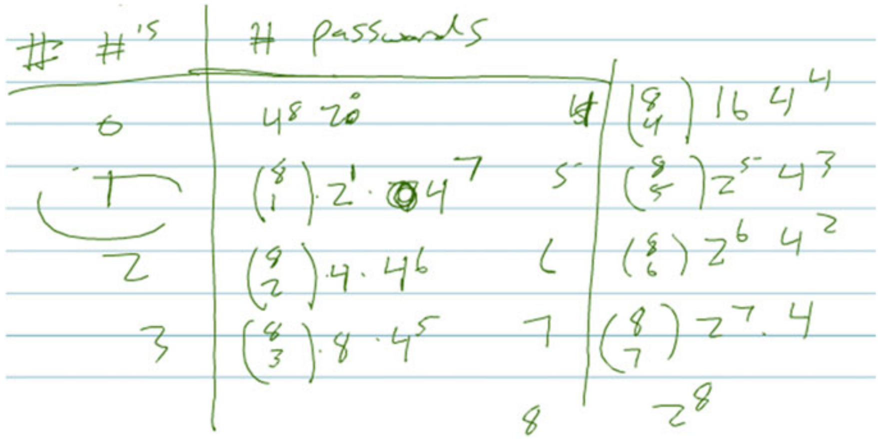

Fig. 2 James' work on the 8-character 1, 2, A, B, C, D password

aspect of counting combinations (Lockwood et al. 2015b, 2016a, b), and it is noteworthy that he was able to do this.

We also see that James was engaging in the mathematical practice of generalization. After James had just finished the work described above, we asked him to find multiple representations for the number of ways to count the n-length passwords consisting of x numbers and y letters. After attempting to separate the terms in the table to create two sums that he could multiply, James turned his attention to representing the entries in the table generally. He then wrote $\binom{n}{a}x^a y^{n-a}$ as a representation for the entries in the table (see Fig. 3).

When prompted to explain his generalization, James said the following, which indicated that he was drawing on his prior activity in order to generalize:

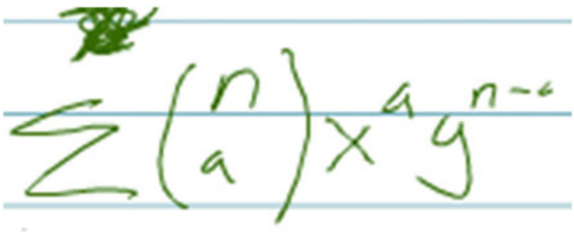

Fig. 3 James' expression for entries in a general table

James All right. Yeah. So it's like if this is *a*, it's like *n* choose *a* times *x* to the *a*, and then *y* to the *n* minus *a*. I think.

Int. So, yeah, you're choosing *a* to represent something. What does a represent? Or, I mean –

James The number of numbers in the password.

Int. Okay.

James Yeah, *a* is, number of numbers in the password.

Int. Okay. So once you—why are you doing *n* choose *a*?

James Why am I doing *n* choose *a*? Because that's choosing—or, I mean, that's telling you all the available—all the possible ways you could put a numbers into *n* slots.

Here we see James' extension of this idea of selecting positions for where the 1 (s) could go to selecting the number of ways to put numbers in the password. This is a nice generalization of his work. We wish to demonstrate here that James' generalization was more than symbolic abstraction, but rather it was rooted in his understanding of the combinatorial process that generated the outcomes. After negotiation of the intricacies of symbolic representation of his summation, James created the statement of the binomial theorem in Fig. 4. We argue that by engaging in this activity, James came to develop a statement of the binomial theorem that was meaningful for him, and he created and used the practice of generalization to do so.

The design of the Passwords Activity (particularly the fact that the number and types of characters increases through the tasks) is intended to promote mathematical generalization both within and across stages. In this activity, not only do students have the chance to generalize, but they also get the opportunity to make generalizations that are grounded in reflection on the combinatorial setting, and not just on formulaic patterning. This episodes highlights the ways in which combinatorial tasks can be designed to develop meaningful practices like generalization.

5.3 *Providing Insight into Combinatorial Identities and Offering Occasion for Combinatorial Proof*

Combinatorial identities are equality relations, and combinatorial proofs involve arguing that an identity is true by finding two counting processes that reflect the

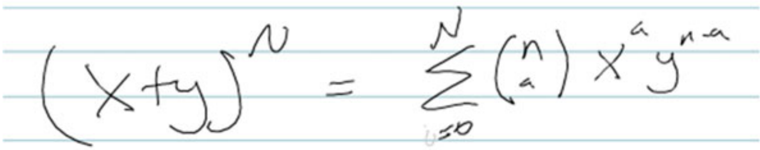

Fig. 4 James' final statement of the binomial theorem

respective expressions and that count the same set of outcomes. Benjamin and Quinn (2003) note that to prove an identity, they

> pose a counting question, and then answer it in two different ways. One answer is the left side of the identity; the other answer is the right side. Since both answers solve the same counting question, they must be equal. Thus the identity can be viewed as a counting problem to be tackled from two different angles (pp. ix–x).

Identities, particularly those involving binomial coefficients, are often introduced in discrete mathematics or combinatorics courses, and the art of combinatorial proof is powerful but difficult to master. Another affordance of the Passwords Activity is that it provides a natural introduction to combinatorial identities, which are closely related to combinatorial proof and to the practice of proving more broadly.

As an example of this phenomenon we look at the work of Sam, an integral calculus student. Upon initial generation of the table involving 4-character A,B passwords, we asked Sam whether and why it makes sense that there would be the same number of passwords with one A as there would be with three As. He responded by saying:

> Yes. Because for only one A your other three would have to be Bs out of the total. Where if you add three As only one of them could be a B, so it would just be like a mirror image of the two of them.

A number of other students also made observations about this symmetric relationship in the tables. Attention to this sort of symmetry in this context could be leveraged in a more general setting to justify the symmetry of the binomial coefficients. In particular, a student could be encouraged to reason about why it must be true that $\binom{n}{k} = \binom{n}{n-k}$. In the context of n-character A,B passwords, he or she could argue, similar to Sam's reasoning above, that the number of ways of selecting positions for some k As is the same as selecting the $n - k$ positions for the Bs. Arguing that both sides of the equation count the same set of outcomes (n-character A,B passwords with exactly k As) could be a productive introduction to a combinatorial proof.

There are multiple opportunities during the completion of this activity for students to generate and reflect on combinatorial identities, and, possibly, to be introduced to combinatorial proof. Each stage in the task is structured so that students count the number of passwords in a given context in two different ways, both having distinct mathematical expressions. By asking the students to come up with formal expressions for both representations of the total number of passwords and then asserting their equality, students can consider what it means for two expressions to be equal. Equality is not merely a matter of two expressions being computationally equivalent, but it can also entail identifying that two distinct combinatorial processes count the same set of outcomes. This is the basic principle of combinatorial proof, that, for some given combinatorial context, there is an isomorphism between the set of outcomes that the respective expressions count. In this way, the activity serves to underscore the general mathematical practice of proving and the specific practice of proving a combinatorial identity.

6 Conclusion and Implications

In this chapter, we have presented the Passwords Activity, and we have offered examples of student engagement with this sequence of tasks. We hope that this chapter contributes to the overall narrative that combinatorics, and even more broadly discrete mathematics, is an important and worthwhile topic for students at all levels. By focusing on undergraduate students and their work on the Passwords Activity, we hope to demonstrate some valuable affordances for engagement with combinatorial activities. We conclude with four closing points of discussion related to pedagogical implications.

First, we feel that the Passwords Activity may be useful for students in a couple of different situations. It would be an appropriate exercise for students who are just learning counting, perhaps who are familiar with solving basic problems involving arrangement and selection but who have not yet seen the binomial theorem or combinatorial identities. For students to arrive at a final statement of the binomial theorem, it would be most effective if they were familiar with a formula for binomial coefficients (that there are $\binom{n}{k}$ k-element subsets from a set of n elements). A sophomore or junior level discrete mathematics course would be an appropriate setting for the activity. However, it is also possible to have students engage with the activity without previously having seen binomial coefficients. Such students will be limited in how far they can generalize (their final general statement will perhaps be incomplete), but they can still engage in important mathematical practices. Because of this, the first stages of the task would be appropriate even for students with very little counting experience, and the activity might be appropriate during students' initial forays into counting. This might occur in a lower-division university course, or even in a high school classroom. Anecdotally, the authors used the activity in both a junior-level discrete mathematics class and a sophomore-level finite mathematics class. In both cases, the students were actively engaged and successfully completed the activity, suggesting that the materials could be appropriate for use in a variety of classroom settings. The activity is designed to help students engage with counting in an accessible context, and from this perspective, students in pre-calculus or calculus could meaningfully engage with these tasks. Whatever the audience, the activity could be employed during an hour-long class period, or it would be effective as a recitation activity or homework assignment.

Second, we offer a cautionary note about over-reliance on numerical patterning in combinatorial tasks. In creating the tables and the general expressions in the Passwords Activity, students were susceptible to focusing primarily on numerical relationships and not on deeper structural relationships that are grounded in the combinatorial context. When filling out the tables, we had some students observe and then use a numerical pattern that they could not explain, or, worse, guess and check with numbers that made little sense within the passwords context. This phenomenon is common in combinatorial activity, and it occurs more generally in mathematics learning and teaching. Thus, although some numerical patterning can be useful in developing conjectures and formulating generalizations, a general challenge for instructors is to make sure the students can connect numerical patterns back to the appropriate combinatorial context. We suggest that instructors consistently remind students of the problem context and ask them questions that draw their attention back to the outcomes that they are counting.

Third, in combinatorial tasks there are often opportunities for students to deal with formal mathematical notation. Students can struggle with coordinating the different variables in complicated expressions involving summation (e.g., Strand and Larsen 2013), and we conjecture that it may be beneficial to give students experience with generating and formalizing their own notation for complicated sums. When students generate expressions using their own notation, they can develop an inherent ownership of the mathematics that they adopt. In our task, writing the formal statements of the various expressions is not the only (or even the primary) goal of the activity, but some students were able to connect their intuitive combinatorial reasoning with more formal notation. In this way, activities like these have a potential benefit of giving students opportunities to engage in formalizing mathematical ideas.

Finally, as we alluded to before, the Passwords Activity is but one context through which to explore the binomial theorem and combinatorial identities. There are many other interesting contexts to which these tasks could be explicitly connected: Pascal's triangle, counting committees, block walking, and more. Once students have developed a robust understanding of the Passwords context, there may be rich opportunities for exploring connections to other contexts. Such connections could propagate further discussions about topics such as isomorphisms and bijections, as well as deeper explorations into combinatorial ideas and combinatorial proof.

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Complex Mathematics Education in the 21st Century: Improving Combinatorial Thinking Based on Tamás Varga's Heritage and Recent Research Results

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Abstract This paper summarizes the ideas and background of a combinatorics research and teaching project, including historical reforms in school curriculum in 1978 in Hungary and T. Varga's work. Thereafter we discuss the main elements of our current project: pretest and developed teaching materials, including worksheets with some examples and some tools for teaching combinatorics such as *Poliuniversum*. In choosing the problems of the worksheets we were led by two research questions: (1) how students handle open tasks (which are presented in many combinatorial problems) and (2) how they use various manipulatives at different ages.

Keywords Complex mathematics education · Combinatorics · Worksheet
Problem solving · Poliuniversum

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1 The Main Ideas of the Project Following T. Varga

Through a grant from the Hungarian Academy of Sciences, our research group investigated how Tamás Varga's heritage, recent research results, and new technology can improve mathematics education in Hungary, and how Varga's ideas can be modified for the 21st century. For the first year of our project, 2015, we chose to analyze combinatorics education. The reasons for this choice were that combinatorics played a major role in Varga's work, and that combinatorics and discrete mathematics are considered a Hungarian specialty, a Hungarian branch of mathematics based on the great contributions Hungarian mathematicians have made in this area.

As combinatorics developed as a branch of mathematics, several math educators started to examine the possibility of teaching combinatorics. And by combinatorics they meant not only permutations, variations, and combinations. One possible—not even the broadest—description of combinatorics is that it is the collection of counting problems with finite sets. This is an incredibly rich collection of problems that can already be started in primary school. But in teaching combinatorics—just like in teaching other branches of mathematics—the primary goal is not to teach the solutions of certain type of problems, but to teach problem solving methods, and how to approach unknown problems.

In Hungary Tamás Varga (1919–1987) introduced combinatorics into the school curriculum in his Grades 1–8 "complex math teaching experiment", which started in 1963, and lead to the new Hungarian school curriculum in 1978 (Halmos and Varga 1978). He started his work in Grades 1 and 2, replacing arithmetic and geometry by an integrated mathematics, containing many deep ideas of mathematics in their infancy (Varga 1967). He wrote about the possibility of starting math education early this way:

> Once Zoltán Kodály was asked, "When should music education of children start?" "In the uterus" - he answered. Mathematics education cannot be started too early either. Once we give up force feeding, children will learn mathematics joyfully and successfully. (Varga 1969).

T. Varga was the first Hungarian researcher who considered the entire primary and middle school curricula and methods as a whole. We would like to realize this idea in our project too, which is one reason why we made parallel experiments for Grades 6 and 10. He was in continuous contact with several research institutions of other countries, adopted the good ideas and practices, but rejected the practices of too much formalism (Freudenthal 1973; Halmos and Varga 1978). He called his concept "post-New Math" (Varga 1988), distancing himself from the too much formalism of the New Math, giving its "corrected" version. He put emphasis on problem solving, especially open problems, playful tasks, and he gradually built concepts through more individual activity done by both the students and teachers.

At the secondary level combinatorics was introduced to the official curriculum in the 1978 curriculum reform. The new mathematics curriculum was written by Lóránt Pálmay and János Urbán. Tamás Varga did not work directly at the high school level, but his ideas certainly had influence on it, and they were partially included in this new curriculum. There was an extensive educational experiment in the years after 1978 using Varga's ideas, especially worksheets and discovery style teaching at the secondary level. A Grade 9–12 textbook series was also published by Andrea Bartal using the worksheets and discovery style teaching. It did not become the number one textbook in Hungary, and after the start of liberalization of the educational system in 1989 and the appearance of many new textbooks, it disappeared.

Tamás Varga considered the use of tools extremely important. In his textbooks and instructor's manuals logical sets and colored rods are used in almost every chapter. In addition to these, in the combinatorics chapters the elements of the construction game Gabi, colored pearls, matches, boxes of matches, and elements of the construction game Babylon are also mentioned in some problems. Varga considered the various kinds of gaming experiences, different activities, such as paper folding, drawing, studying models, etc. very important in concept building and practice, too. "The theory of combinatorics is ideally taught to young children as it can be linked to manipulative activity. We can lead children to the abstract concept of this theory through a variety of personal experiences" (Varga 1967).

Varga and his colleagues defined their own levels of teaching combinatorics, which have been built into the methodology of Hungarian mathematics education. According to this theory, the levels of combinatorial problem solving are the following (Szitányi and Csíkos 2015, p. 226):

- Differentiating the cases
- Listing all possible cases as brainstorming
- Regular listing
  - Two types of representations: objects—images (drawings, letters, tables, graphs)
  - Strategies: change, fixing, cyclicality
- Applying formal methods
  - Two types of representations: objects—images
  - Strategies: multiplication, addition, one-to-one mapping, recursion
- Recognition of structures.

One motive of our project was our interest in analyzing the 1978 reform, and trying to find connections between the 1978 and 2016 situation. We would like to follow the ideas and concepts of T. Varga, but in an extended Grade range of 1–12 instead of 1–8, and we would also like to try using new tools and take advantage of new possibilities to teach combinatorics in school. Our research group consists of researchers in mathematics didactics and teachers, who were involved in planning and developing the experiment and who also taught the classes in which the experiment was being undertaken. Moreover, they documented the teaching process and their opinions of it.

We planned a pretest as the first step, focusing on grades 6 and 10, to see the current situation in Hungary. The pretest can be found at the end of this chapter. The pretest was completed by 446 pupils (230 in Grades 6–7, 216 in Grades 10–11), and statistics were compiled about every question. The main conclusions that we have drawn from the test are that younger pupils use manipulatives easier, and their thinking is more flexible, while the older students prefer formulas. Surprisingly, in problems with more conditions, and hence with no general formula, the results of the younger pupils are not worse than the results of the olders. The analysis of these data also gave us some ideas on how to develop our worksheets. For more detailed results, look at the research paper of Kosztolányi (2016) PME-40 in Szeged.

Our teaching experiment was undertaken in four middle schools with 134 students in five classes, and in two high schools with 104 students in seven classes.

We collected tasks and chains of exercises for worksheets ("Munkalapok" at Varga), which the pupils had to complete during the experimental teaching of combinatorics in six to eight lessons.

We would like to illustrate the style of problems and tasks, and also show how we were taking advantage of the tools that were used during the experiment, which included *Poliuniversum* [for description of Poliuniversum see Sect. 3 below and Stettner and Emese (2016)], and logical sets, among others. Teaching took place in 2015 in October and November. Leaders of the project visited the groups to collect some personal experiences, and we also made videos of the lessons and conducted interviews with teachers and students.

In the following sections we will show two of the worksheets we used and one separate question that gave surprising result to us. The choice of these worksheets presented here is based on special focuses of our research: one of them is the use of open tasks in teaching and the other is the use of tools. We chose an additional problem because we found it particularly interesting. We were wondering a lot about why this problem was so difficult for students.

2 Third Worksheet of Open Tasks (The Third Worksheet Both of the Seven Worksheets Comprising the Grade 6 Teaching Materials and of the Six Worksheets for the Grade 10 Teaching Materials)

The third worksheet, which contained two question papers is particularly fascinating, as it presents some tasks that can be interpreted in more than one way and we tried as well to investigate how pupils react to open questions, which very often occurs in the case of combinatorial problems. One teacher, our co-author H. Burian, summarized her experiences.

Overall 60 students worked on the 3rd worksheet, which contained two sets of questions. The 60 students were from two different high schools and were from six different classes. Twenty two of them were grade 11, eleven were grade 9, and twenty seven were in grade 8. None of them managed to solve all of the problems. Most of the questions were *open type* questions, and there were a lot of questions that required the student to give his or her opinion on the matter.

2.1 "Ice Cream": 1st Question Paper of the Third Worksheet

Write your solutions to the questions below on a separate piece of paper, based on this situation:

Dóri often goes to an ice cream shop near her home in the summer when she and her family are home. Because she loves ice cream, she gets some pocket money, from which she can afford at most 3 scoops of ice cream every day.

She told her friends that last summer she ate a different combination of scoops every day, even though she could choose from only 5 flavors.

Panka was unconvinced if that statement was true, because she knew that Dóri and her family spent only two weeks away from home during the summer.

Laci believes that this is only possible if she ate 4 scoops of ice cream sometimes.

Questions, solutions, and analysis

1. *Figure out and write down the circumstances that could influence the truthfulness of Dóri's statement!*
2. *Dóri ate 3 scoops of ice cream every day, and she could choose from 3 flavors every day. This means that she could eat different combinations of scoops for 10 days*
   (a) *Check whether the statement above is true or false! Write down your way of thinking.*
   (b) *Make up different "ice cream eating" conditions, and calculate the number of days on which Dóri could eat different combinations of scoops under these conditions! Write down the conditions and your calculations.*
   (c) *Based on your results, what is your opinion on the truthfulness of Dóri's and the others' statements?*

The first question was already one where the students had to give their opinion, and it allowed us to make one of the most interesting observations. The time spent on discussing this question was about 15 min in almost all groups. The reasons for this could be the following:

- The students usually meet with *closed type* questions only, so they do not have a lot of experience in solving *open type* questions.
- The meaning of the word “different” in the question is not obvious.

The most popular answers to the first question written above were the following. (The numbers in parentheses are the numbers of students giving that particular answer or using that particular method.)

- How long was the summer holiday? (25 students)
- Does the order of the purchased scoops matter? (23)
- Is it possible to buy more than one scoop from a flavor? (11)
- How many scoops did she eat a day? (8)
- Did she ask for them in a cone or a cup? (8)
- Was the ice cream shop open every day? (5)

The following questions were mentioned less often:

- Did Dóri have enough pocket money? (3)
- How much did one scoop cost? (3)

It can be concluded from the answers, that after the students came to understand the question, they gave mostly relevant answers. Most of them saw the length of the holiday and the order of the scoops as the most important aspect. It was fascinating to see that fewer students thought of “How many scoops did she eat a day?”, even though the text said “maximum”, which allows for 1, 2 or 3 scoops a day. Most of them did some calculations in the first question and/or in questions 2a or 2b, and then deduced their conclusions from these calculations. The following answers were the most popular based on these calculations:

- More than one scoop per flavor is allowed, the order matters, and she ate 3 scoops a day: $5^3 = 125$. (16 students)
- Only one scoop per flavor is allowed, and the order matters, and she ate 3 scoops a day: $5 \cdot 4 \cdot 3 = 60$. (13)
- Only one scoop per flavor is allowed, the order does not matter, and she ate 3 scoops a day: $(5 \cdot 4 \cdot 3)/(3 \cdot 2 \cdot 1) = 10$. (5)
- More than one scoop per flavor is allowed, the order does not matter, and she ate 3 scoops a day: $5 \cdot 5 \cdot 5/3!$, and they mentioned that this is not a whole number. (3)
- Only one scoop per flavor is allowed, the order matters, but she can eat 1, 2 or 3 scoops: $5 + 5 \cdot 4 + 5 \cdot 4 \cdot 3 = 85$. (8)
- More than one scoop per flavor is allowed, the order matters, but she can eat 1, 2 or 3 scoops: $5 + 5 \cdot 5 + 5 \cdot 5 \cdot 5 = 155$. (3)
- Only one scoop per flavor is allowed, the order matters, but she can only choose from 3 flavors, and she ate 3 scoops a day: $3! = 6$. (11)
- Only one scoop per flavor is allowed, the order does not matter, but she can only choose from 3 flavors, and she ate 3 scoops a day: 1. (3)
- More than one scoop per flavor is allowed, the order matters, she can choose from 3 flavors, and eats 3 scoops: $3^3 = 27$. (13)

- More than one scoop per flavor is allowed, the order matters, she can choose from 3 flavors, but she can eat 1, 2 or 3 scoops: $3 + 3 \cdot 2 + 1 = 10$. (6)

It is clear from the results that questions 1 and 2 could be interpreted in many ways. Those students who took the length of the summer holiday into account calculated with 9 weeks and 7 days/week, so 63 days overall. In their case, it depended on the calculation method they used whether Dóri's statement was true or false.

In question 2a most people calculated without taking the order and the possibility to have more than one scoop per flavor into account. An interesting aspect of the solutions is that roughly the same number of people interpreted 2a and 2b in a way that Dóri can choose from five flavors (as in the original question) and in a way that she can choose from only three flavors. In both cases the answer to whether Dóri's statement was true or false depended once again on the calculation method they used.

In 2b a few people mentioned, without making any calculations, that Dóri could eat 1 or 2 scoops but could only choose from two flavors, or that she could eat maximum 4 scoops but only choose from two flavors again (someone calculated the second case, $2 + 4 + 8 + 16 = 30$).

Very few people answered 2c, and even those who did only wrote a very short opinion in a few words or a sentence at most. The answers mostly commented on the unobvious interpretation of the statements. The most probable reason for the short opinions is that the students very rarely have to do declarations like this.

3. *Discuss in a group your "ice cream eating" conditions and the results you obtained from them, as well as your way of thinking and calculation methods. Try and come up with even more cases with different conditions, and do the appropriate calculations for these cases as well! Write down these conditions and the calculations as well.*
4. *Dóri's friend thinks that the order of the scoops in the cone should be taken into account as well. She believes that the results would change significantly if this aspect of the problem would be considered as well. Have you taken this option into account so far? If you have not, make extensions to your chart where you take this into account as well!*
   What is your opinion on the statements of Dóri and the others based on these new calculations?

About three students answered questions 3 and 4, which is not surprising due to the fact that they were not relevant after they discussed every possibility in questions 1 and 2.

5. *Ice cream eating "habits" of Dóri can change anytime, possibly even after one day, and they can change more than once. Come up with some new situations again based on this new piece of information, do and write down the appropriate calculations for these situations, then reassess your opinion on the statements of Dóri and the others.*

About three students answered this question as well.

2.2 *"Ice Cream": 2nd Question Paper of the Third Worksheet*

Write your solutions on a separate piece of paper, but do not forget to indicate which solution belongs to which question.

Questions, solutions, and analysis

1. *Zita frequently goes to another, bigger ice cream shop, where she can choose from 12 flavors. 6 of these flavors are her favorites, so she always chooses from these ones. If she eats 3 scoops of ice cream every day, calculate the number of days on which she can eat different combinations of ice cream scoops!*

   Define the conditions, do the necessary calculations, and make a chart from your results!

The first question did not cause any problems to the students this time, as they analyzed similar cases in the first set of questions. The only difference was that there were 6 flavors to choose from this time. Most of the students calculated with 3 scoops, and some of them calculated with more than one scoop being allowed per flavor, while some of them did not. They only took the order into account if they calculated with only one scoop per flavor. A few students attempted to calculate the case where more than one scoop per flavor is allowed and the order matters, but most of them failed with their answer being $\frac{6^3}{3!}$. As the result they got is a whole number, they didn't notice that the solution was wrong, unlike in the 1st set of questions were they realized that $\frac{5^3}{3!}$ is wrong due to it being a fraction.

2. *Create a task (tasks if possible), where the solution can be reached by using the same methods as in the ice cream eating questions above! They should be based on different everyday-life situations. Write them down together with their solutions, and comment on which "ice cream eating cases" have similar solutions.*

The second question was answered by almost everybody, and most of them defined similar problems as the one with the ice cream, but with different edibles or with some permanent qualities of a phone, like its color, memory capacity, or phone case.

3. *Make a question paper that consists of 4 tasks created by people in your group, and attach a separate piece of paper to it with the solutions.*

No one answered the 3rd question because the students ran out of time.

4. *Generalization*

   I. *We can choose from n flavors in an ice cream shop, and we eat 3 scoops.*

      (a) *If all of our scoops are of different flavors, and their order matters, then we can have n(n − 1)(n − 2) different possibilities.*
      (b) *Find cases that are different from the one above by changing the initial conditions, and try to come up with a formula for the number of different ice creams that consist of 3 scoops. Take notes of your results.*

   II. *We can choose from n flavors in an ice cream shop, and we eat g scoops.*

      (a) *If all of our scoops are of different flavors, and their order matters, then we can have n(n − 1)…(*n − g + 1*) different possibilities.*
      (b) *Find cases that are different from the one above by changing the initial conditions, and try to come up with a formula for the number of different ice creams that consist of g scoops. Take notes of your results.*

The fourth question, which was the generalization part, was dealt with by the students in grade 8 either with the help of the teachers, or they did not look into it at all. Some of those in grade 11 and grade 9 tried to generalize the formulae, mostly with success, but they failed in the cases where more than one scoop per flavor was allowed and the order mattered. The following answers were the most popular:

- Only one scoop per flavor is allowed, the order does not matter: $\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-g+1)}{g!}$. (12)
- More than one scoop per flavor is allowed, the order matters: n^g. (5)
- More than one scoop per flavor is allowed, the order does not matter: $\frac{n^g}{g!}$. (5)

For these questions, we provided an optional help table, which students could or could not use. See Table 1.

The most interesting observation made in the second set of the questions was related to Table 1. Thirty-two students tried to fill in this table, which is insufficient to draw well-backed conclusions, however it is still interesting to take a look at the results:

- Maximum of 2 scoops per flavor, the order does not matter: $\frac{n\cdot n\cdot(n-1)}{3!}$ or $\frac{6\cdot 6\cdot 5}{3!}$ (4); $\frac{6\cdot 5\cdot 4}{3!}+6\cdot 5=50$ (6); $\frac{6\cdot 6\cdot 5}{3}=60$ (4); $20+\frac{6\cdot 5}{3}=30$ (2)
- Maximum of 2 scoops per flavor, the order matters: $n\cdot n\cdot(n-1)$ or $6\cdot 6\cdot 5$ (6); $120+6\cdot 5=150$ (2); $120+6\cdot 5\cdot 2=180$ (2); $6\cdot 5\cdot 5$ (1); $120+6\cdot 5\cdot 3=210$ (3)
- Maximum of 3 scoops per flavor, the order does not matter: $\frac{n^3}{3!}$ or $\frac{6^3}{3!}=36$ (6); 30 + 6 = 36 (4); 50 + 6 = 56 (6); 30 + 10 = 40 (1); $6+6\cdot 5+6\cdot 5\cdot 4=156$ (1)
- Maximum of 3 scoops per flavor, the order matters: n^3 *or* $6\cdot 6\cdot 6=216$ (8); 150 + 6 = 156 (2); 180 + 6 = 186 (2); 210 + 6 = 216 (2)

Table 1 Optional help table for “ice cream” 2nd question paper

	The order does not matter	The order matters
Different scoops		
2 scoops per flavor at most		
3 scoops per flavor at most		

Most of the students filled in the first row correctly, but they either filled the second row and the first column of the third row incorrectly or they left it empty. It is impossible to determine that from the people who filled these in correctly, how many worked alone and how many copied the results from the board. As well we observed that there were students who filled in these problematic cells incorrectly, but at least attempted to break down the problem into sub-cases (most of them were grade 11, though).

There were no problems with the 2nd column of the 3rd row due to both n^3 and 6^3 being correct solutions, and even those students who filled in the previous row incorrectly got these correct.

As already mentioned, these are the results of only 32 students so the conclusions we made have to be checked within a wider population. Nevertheless in our opinion in these three problematic cases (the second row and the first column of the third row) the students failed to get the right results because they would have had to either break down the problem into sub-cases, or list all the possible combinations, or use a tree graph. They had only encountered the break-down method in the first set of questions of the 1st lesson before, and it seems that this was not enough for them to try and use it to solve this problem, and to throw away the ingrained method of deriving a formula with some multiplication in it and division by factorial due to repetitions. These results seem to back the hypothesis that “teachers insist on the students solving the problems with the method they are learning at that time”, and for students in higher year classes searching for a formula is seen most often, so this gets ingrained.

The failure to achieve the desired goal seemed to be caused by the questions and tables as well. Namely, the questions and tables might have discouraged the search for different methods to solve the problems.

Firstly, the first question in the first set of questions was solved by the students by taking the order into account and either with or without more than one scoop per flavor being allowed, which means the solution was mostly $5 \cdot 5 \cdot 5 = 125$ or $5 \cdot 4 \cdot 3 = 60$, both of which are too great a number for the students to list all the possible combinations. Therefore, it is unsurprising that they did not use this method in the question where more than one scoop per flavor was allowed, but the order did not matter (even though there were only 35 possible combinations). There were even more scoops to be calculated with in the second set of questions, which obviously means more combinations, and does not encourage to find an alternative method.

The table, which has limited cell size as per usual, prevents to find an alternative to the formula method. Thus it would have been a better option to ask questions which result in smaller numbers so students can list all possibilities and they can see that the sum of the different cases gives the result when more scoops per flavor is allowed.

In our point of view, a third factor also exists that contributed to the lack of success. That is the fact that in combinatorics the hardest cases to calculate are the ones where repetition is allowed, but the order does not matter (combination with repetition). "Selection problems, especially those involving choosing something like positions or locations in a password, can be quite difficult for students" (Lockwood et al. 2015). Only a very limited number of students can figure out the solutions to these by themselves, the others are either unable to solve the problem, or they find an incorrect solution, as it happened in this case as well.

2.3 Summary

The open-type questions confused the students at the beginning, because most of the questions they encounter in class are closed-type questions, and the previous question papers in this project were the latter type as well. They are even less used to having to communicate their opinions, therefore questions that required this mostly remained unanswered, or were answered in only a few words.

In the case of calculations, most answers were incorrect when more scoops per flavor were allowed and the order did not matter. Most of the students either did not give an answer, or used the usual strategy of finding a formula with which they can get the result by using simple multiplication and division. The only exceptions from this were a few students from the higher year classes.

3 Teaching Tools of the Experiment

Teaching mathematics is done too often at the symbolic level. Using well-chosen tools can improve creative thinking, according to Bruner et al. (1966) theory. Following Tamás Varga's spirit, we also considered the use of tools extremely important. In choosing the appropriate tools we took advantage of the experiences of the Experience Workshop lead by Kristóf Fenyvesi (http://www.elmenymuhely.hu/?lang=en) and we reviewed their publications (Fenyvesi et al. 2014; Fenyvesi and Stettner 2011). We found most tools in the publication *Adventures On Paper Math-Art Activities for Experience-centred Education of Mathematics*. We considered the ZomeTool (http://www.zometool.com/) and the 4D Frame (http://4dframe.com/eng/) modelling sets. For financial and other practical reasons, we finally chose the Poliuniversum set, the Logical set, and some hand-made tools/toys: pentominos, cat cards, and drawings of animals of Noah's Ark.

In the 5th worksheet of our experiment, pupils used the Poliuniversum set, which is intended to improve combinatorial and geometric skills. Boxes contain one of 3 plastic shapes: triangle, square and (almost) circle, made in red, yellow, green and blue. The two sides are of the same color. There are three semicircles attached to the boundary of the circle in directions making 120° angles with one another, the largest has radius that is half of the original circle, for the medium it is one fourth, and for the smallest it is one eighth. The diameters of the three semicircles cut off three segments from the original circle so the basic form is not exactly a circle. So this shape is bordered by 3 arcs and 3 line segments, so they can stand on their line segment parts in a stable way, as kids tried it. The forms having the same shape and colored in all possible ways are packed in a box. We bought boxes containing the circle figures. The mathematics of Poliuniversum can be explored at the following website: http://poly-universe.com/dimensions/mathematics.

3.1 Fifth Worksheet (Using the Poliuniversum Tool)[1]

The worksheet is unusual for an average pupil because they are only familiar with easier combinatorial tasks. These problems involve geometrical figures and transformation, which seem to be more complex and harder for pupils. From the pupils' solution it can be seen that this worksheet was the least successful for secondary students. There were many creative and "tricky" solutions, but the teacher had to help more than with other worksheets, and nobody answered all of the questions correctly. At the end of the experiment teachers used a questionnaire made by a researcher of the project in order to get information about the popularity of worksheets on a scale of 1–5. The "Poliuniversum worksheet" was ranged at the top. They liked to use the set of Poliuniversum, but they thought that this worksheet was the hardest of the six worksheets according to teachers' reports and student questionnaires.

> Let us get acquainted with Poliuniversum. It contains shapes of almost-circles of red, yellow, green, or blue color. Each almost-circular shape contains all four colors. One such shape is shown in Fig. 1. The shape in Fig. 1 is primarily green, with different-size half circles of blue, red, and yellow around the edge.

Questions, solutions, and analysis

1. *How many of the circular shapes are there in a box if there is one of each possible coloring?*

[1]Sections 3.1 and 4 are from Eliza Beregszászi's dissertation.

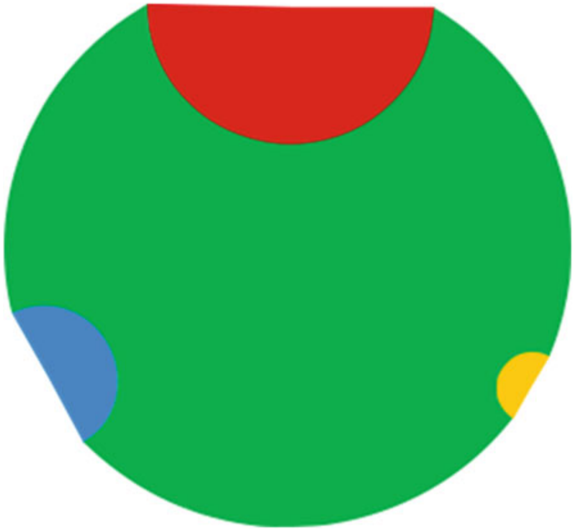


Fig. 1 A Poliuniversum circular shape with base color green

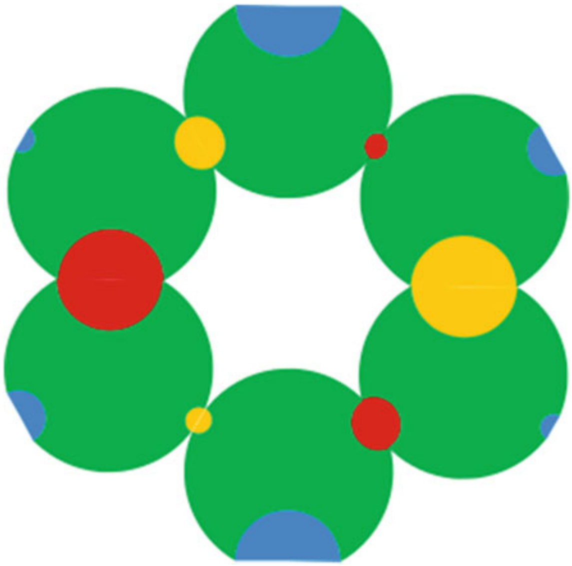


Fig. 2 A ring made of all six different green Poliuniversum circular shapes

2. *Let us choose all the different possible green circular shapes and make a ring as shown. Joining semicircles should be of the same size and color* (Fig. 2).

(a) *How many rings can we make this way?*

We would like to show some of the students' solutions now. The correct answer of 2a can be seen in Fig. 3. We chose green as base color. Let's look at only the circles with the same base color, which in our case are the green ones. There are six of these, because there are three semicircles on each piece, which can be three different colors, but every color has to be used once. The problem can be solved based on just one piece of the Poliuniversum, as the placement of that determines the order and position of the others. Let us say we draw a solution on a piece of paper, then we choose a random Poliuniversum circle, and place it in such a way that a semicircle is facing outside the greater circular ring. Then we can get all the solutions from this by rotating this small circle. Because there are three semicircles on a piece, rotating it results in two more solutions. Furthermore, if we flip the small circle, we can place three different semicircles facing outside again, which results in three more solutions. So overall there are 6 different solutions, as shown in Fig. 3.

This problem was extraordinarily troublesome and unusual for the students. Teachers rarely use such tasks in combinatorics lesson, due to them requiring highly complex geometrical thinking. Students have to decide which cases are different and which ones are the same, and these depend on specific geometrical transformations. This could explain why so many students got only three solutions instead of six.

Now consider parts b, c, and d of question 2:

(b) *Can we get rings that are symmetric about a line?*
(c) *How about ones with rotational symmetry?*
(d) *What would be your answer to (b) and (c) if we do not care about the colours, only about sizes?*

The teachers' guideline explains that if we do not take the colors into account, then form the six rings in Fig. 3, the ones in the first row are rotationally symmetric at a 180° angle. The ones in the second row have three axes of symmetry, and are rotationally symmetric at the angles of 120° and 240°.

We believe that the creators of this problem expected lengthy answers like the one above from the students, but they worded their questions as "yes or no" type questions, which resulted in all the students giving brief answers like "yes, no, there is, there is not"… Even though every student answered the questions right, we cannot be sure whether their thought process was right as well, due to their short answers. The teachers usually expect justifications to the answers but students can do it at different levels depending on the grade level and how strong the class and the school are mathematically.

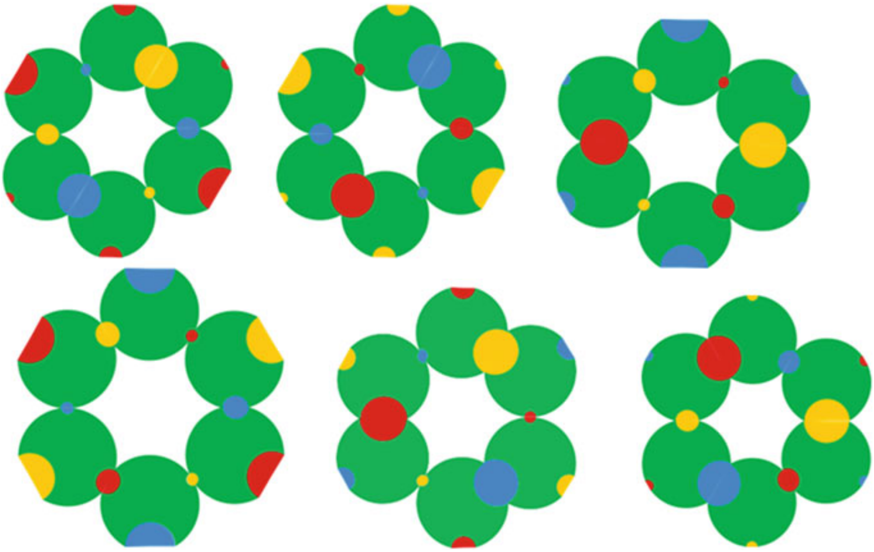


Fig. 3 The six different rings made with green Poliuniversum circles

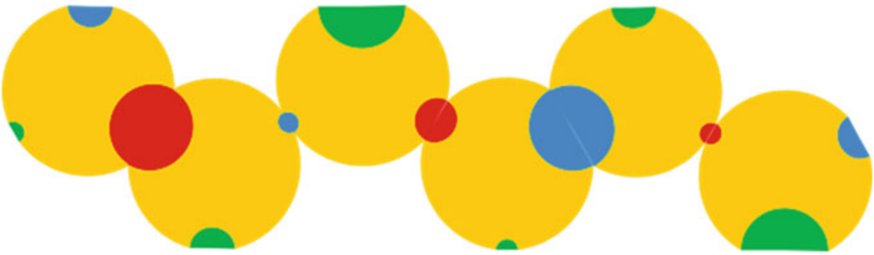


Fig. 4 A wavy chain of yellow Poliuniversum circles

3. *Make a "wavy" chain using 6 circles of the same base color. Only same size and same color semicircles can join* (see Fig. 4).

(a) *How many such chains of length 6 can be made?*
(b) *Can you see a connection to the previous problem?*

We believe that this question is much easier for the students than the previous one, but only if they managed to solve the previous question well. If they had to solve the worksheet alone, without any help, then those who could not figure out the second question would have been in a great disadvantage. Therefore in our opinion it was very useful that they discussed the solution of the previous problem before starting this one.

To the first question, which was "How many such chains of length 6 can be made?", the students gave some diverse answers. Based on the teachers' guideline,

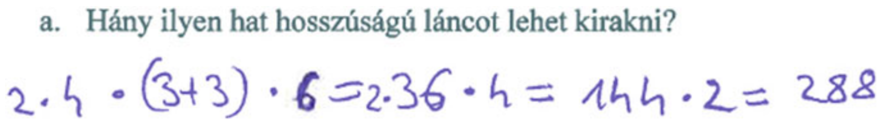


Fig. 5 One student's solution to question 3a

the correct answer is 144, which can be calculated by transforming each ring solution of the previous question into chains with 6 different starting pieces, then realizing that both end pieces can be attached to the chain in two different ways, because they can be flipped. Hence the number of solutions is $6 \cdot 6 \cdot 2 \cdot 2 = 144$.

It turned out from the remarks of the teachers of the Grade 10 students that some pupils think the correct answer is the one above multiplied by 2, due to their considering solutions rotated by 180° as different solutions. See Fig. 5. We agree with the solution in the guideline, that 144 is the better answer.

The majority of the students got 288 as the answer, which is what the teacher expected from them, but a few of them got 144. We noticed that those who wrote 144 as the answer, solved the question alone. Some of them even made comments on their choice numbers while multiplying. In Fig. 5 we can see the solution of a student who tried to solve the problem based on the previous question. He found 3 solutions in the previous question, then the teacher helped him, and he realized that there are 3 more solutions, so he corrected his solution for 2a by writing $3 + 3$. That is why he wrote $3 + 3$ in his solution to this question as well. The 4 stands for the two end pieces and their two possible positions, which he condensed into a 4 instead of $2 \cdot 2$. The 6 at the end stands for the six possible starting elements of the chain. It can be seen though that he added the $\cdot 2$ (for the rotating) subsequently, so in the end his answer was 288, even though his answer was 144 without the teachers' influence.

We believe that the reason for almost everyone obtaining 288 as the answer was that firstly the students discussed the solutions with the teacher, who expected an answer of 288, and only then did they put it down on the worksheet. Evidence for this is seen in Fig. 6, since the answer was written down first, and the calculation was only written down afterwards.

Another possible explanation of this is that they heard the correct solution from someone, and then they tried to backtrack it. In many combinatorial problems like this the correct answer depends on which configurations are considered different. In

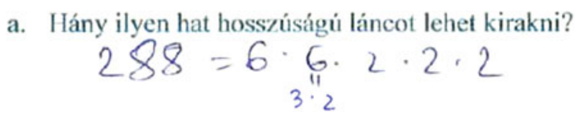


Fig. 6 Another pupil's solution

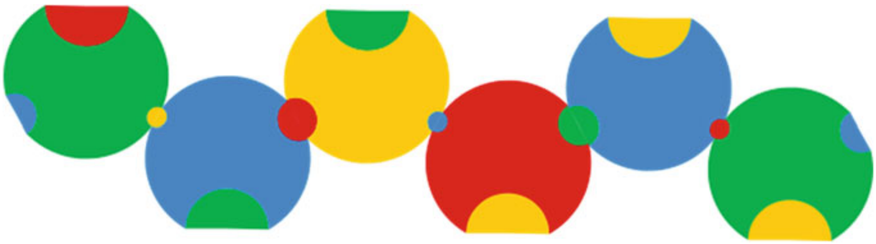


Fig. 7 A wavy chain with different base colors

our worksheets these were not defined beforehand deliberately. These issues were discussed in class when the problem came up.

Most of the students answered “yes” to question 3b, or they left the answer space empty.

4. *Solve the previous problem if the base colors can be different* (Fig. 7).

(a) *How many starting elements can you choose?*
(b) *How many second elements can you choose for a given starting element?*
(c) *In how many ways can you choose the third element? Does it depend on the choice of the first two elements?*

Everyone attempted the first three questions, but there was not enough time left to solve this question as well, therefore only a limited number of students tried to solve it. Those who were intrigued by the questions, and managed to obtain answers to the previous questions faster, started thinking about this problem, and they had some clever ideas and solutions. They did not discuss these on a class level, though.

Six students answered part a, four of them correctly said the answer was 24, while two of them got 144 as an answer, which they calculated as $(6 \cdot 4) \cdot 6$. Those who answered 144 did not continue solving the question.

Only three students answered part b, and they all answered correctly. All of them answered part a correctly as well.

There were two questions in part c, and two students tried to deal with these. One of them answered “it does not depend on the choice of the first two elements”, but did not answer the other question about the number of ways in which you can choose the third element.

The other student answered $2 \cdot 5$, which is the solution of one of the cases. He did not figure out the other case.

If there was more time to work on the worksheet, and the students got to this question, many of them probably would have made the same mistake of calculating with only this one case. It is a brilliant idea from the creators of the worksheet that they ask about the dependency of the third element on the first two elements, as this provokes the students to think about this, which would probably result in more of them answering correctly.

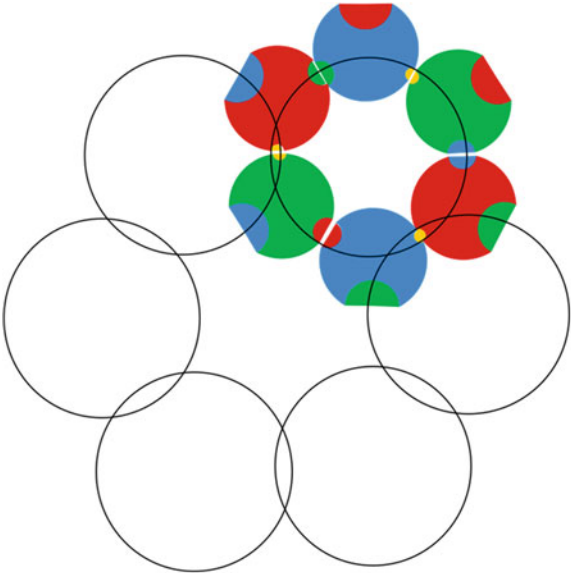


Fig. 8 A super ring of Poliuniversum circles

5. *Solve the problem of "full joining of circles"* (see Fig. 8), *using all elements to make six rings joining in a super ring (a seventh ring will be created in the middle).*

No one could solve Questions 5, as there was not enough time for it.

4 A Particularly Interesting Problem (In Worksheet 2 for Grade 10)

Finally let us see a very interesting task from the second worksheet in grade 10.

> A bug is walking on the edges of a cube from a vertex to the opposite vertex. Moving backwards is not allowed. How many different routes can the bug take?

This was the question in this worksheet that the students found the hardest to solve. The correct answer is six different routes, but there were eight cubes drawn for them to help them try out possibilities. They draw correct routes on the first few cubes, but then they created many different wrong solutions as well. Quite a few of them drew routes where the bug did not even arrive at the opposite vertex. There were some who were just scribbling, while some of them repeated a route they had drawn before, then realized their mistake, and crossed it out. It seems that the eight cubes confused them, and many of them were not confident enough to believe themselves when they had all six solutions, and they tried to find even more solutions.

There were barely any students who obtained the correct solutions by following some kind of system. Most of those who got all six routes obtained them in a completely random order. It is rather surprising that they found it so hard, because they could get as many physical cubes (in addition to the drawings) as they wanted from the teacher as a help, and they could draw the routes on these cubes with markers. The teachers' evaluations make it clear that this was a time-consuming activity. The students were enthusiastic about drawing, nevertheless they still found it hard to solve the question. One explanation for why they found it hard could be because it required stereopsis. One of the teachers mentioned that he put a great emphasis on telling the students that the bug is hurrying to the opposite vertex, so it does not turn back, but the eight cubes confused them, and some of them wanted to draw a route on all of them, which resulted in them drawing longer routes as well.

In our opinion, the problem was not unambiguously worded, since it only mentioned that the bug is hurrying to the opposite vertex, and many students did not conclude from this that only the shortest routes should be taken into account. We think it would have been a better choice to write down the number of ways in which the bug can get to the opposite vertex on the quickest route. We do not know whether only one, or every teacher drew the attention of the students on the fact that only the shortest routes should be considered as a solution, but the students found the problem hard to solve regardless of the school they were in.

In one of the schools many students did not know which one is the vertex opposite to their chosen starting vertex. Some of the students thought it was not the vertex opposite the whole cube, but only a side of the cube. A few of the students drew routes of length five on the last two cubes, which did end in the opposite vertex, but they were not the shortest routes, so they did not fulfil this condition.

In Fig. 9 we can see an edifying solution of a student, which illustrates the phenomenon that occurs with most students very often. Firstly, this student tried to draw in the possible routes of the bug as well, but he could not find all the solutions. Afterwards he started to calculate and write down the number of ways in which the bug can get to a vertex. Many students tried this in every school, and in our experience they wrote down the correct numbers, but they did not write down the final sum of six at the last vertex, which would be the correct solution. They got stuck at this last step, which was probably caused by a lack of stereopsis and their unfamiliarity with questions of such type.

At the beginning they can easily figure out that the bug can only get to the adjacent vertices in one way. At this point, the problem resembles the number line, which makes it easy to solve. At the next step they have to think in two dimensions already, and see a square where they have to count the routes from one vertex to the opposite one, and they have to take it into account that the bug can get there from two different vertices as well. They managed to think through the problem well up to this point. At the last step they have to look at the problem in three dimensions though, and consider that the bug can get to the opposite vertex from three different

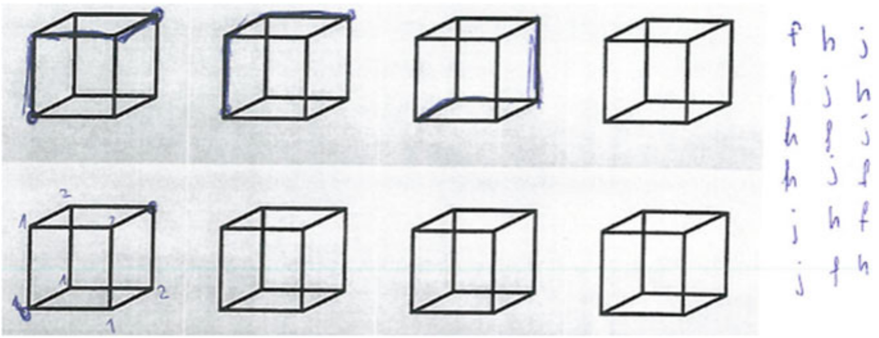


Fig. 9 A student's solution to the bug-walking problem

vertices. This proved troublesome for most of the students, which resulted in them not obtaining the final solution, even though they were very close. Afterwards some of the students, including the one whose solution can be seen on Fig. 9, denoted the directions with letters, and this way they found all six solutions.

5 Conclusion

We found that many of Tamás Varga's ideas can be used successfully today (e.g. worksheets with carefully constructed problems), his heritage is a very rich one. Both students and teachers liked these worksheets, many of them being discovery type worksheets: leading students to discover the solutions to certain type of problems using a chain of problems that students solve individually or in pairs and small groups, with the teacher walking around and helping the students.

A very positive consequence of implementing and testing the worksheets is that it revealed flaws (for example the ones described in the next three paragraphs) in the teaching methods for combinatorics that can be taken into account in the upcoming phases of the project or generally in teaching combinatorics.

It was clear that the students often struggled with the open-type questions, which meant a big cognitive load for them, because most of the questions they have solved are closed-type. We believe that this is the case with most of the students in the Hungarian teaching system. Teaching open-type questions should be started as early as possible, maybe even in lower year classes, and then continue for the entire duration of their schooling, so they would be continuously exposed to questions of this type.

It also became clear that students find it hard to solve questions where repetition is allowed and the order does not matter. One of the aims of the project is to make teaching guidelines for teachers. One possible recommendation is to have this type of question included in these guidelines with a great weight at the earliest possible stage.

It seems that in given situations students could not find the model best suited to solve certain questions even when they had tools to help them. One reason can be that the question was not unambiguously worded. At the same time, we have to recognize that students had more problems where they needed geometrical thinking as well to count the options. Another probable reason is that they are only familiar with a limited number of combinatoric models. It also needs to be included in the guidelines that students should be exposed, during the entire time they are taught combinatorics, to questions of different type and difficulty levels, which need to be solved with different methods, so just using a formula does not become too significant.

Analysis of student work on some worksheets was included in this paper. We plan to analyze other worksheets, too, and publish the results.

We have included the pretest (see the next section) and several problems from the worksheets in this paper. We plan to translate the other worksheets to English also, and make them available for interested teachers. It would be interesting to see how students in different countries like these worksheets and how successful they are on them. We already have some data on how students in different countries perform on the pretest. It would be nice to extend this database.

We hope the problems of our worksheets illustrate our desire that in teaching combinatorics—just like in teaching other branches of mathematics—the primary goal is not to teach the solutions of certain type of problems, but to teach problem solving methods, and how to approach unknown problems. We wish to encourage other teachers and educators to use these types of problems and this approach.

Acknowledgements This study was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences.

Appendix: Pretest

Sixty minutes are allocated to answer the following questions. Give reasons to your answers in detail. Make sure that you indicate your final answer. Have fun.

1. Robert, John, Kate and Elizabeth are sitting on a bench next to each other.
   (a) How many sitting arrangements are possible?
   (b) Another boy, called Michael joined them. How many different ways can they be seated on the bench if a girl can sit next to a boy and a boy can sit next to a girl only (if girls and boys alternate)?
   (c) Robert, Michael, John, Kate, Maria, Elizabeth and Susanne line up (in a row) in the schoolyard. How many ways is it possible?
   (d) How many different ways could this be done if boys and girls alternate? (A girl can stand next to a boy and a boy can stand next to a girl only.)

2. Consider clowns.

   (a) A clown has 3 buttons on each leg of his trousers, 3 on each arm of his coat and 3 in the middle of his coat. There were three colors used on each place: one red, one yellow and one green button. Is it possible that the order of the colors of the five places is different?

   (b) A class of the Riverside Clown School has students with a special uniform. Their uniform has 5 buttons on each leg of their trousers, 5 on each arm of their coats and 5 in the middle of their coats. There were 5 colors used on each place: one red, one yellow, one green, one blue and one purple. (See diagram.) We know that each student has a different dress (uniform). It means that the order of the colors on each of the 5 places is different. What is the maximum number of students in a class under the given conditions?

3. Mary made towers by using colored cubes of the same size.

   (a) How many four-story towers can be made by using 4 cubes if each cube has different color?

   (b) How many five-story towers can be made by using 5 cubes if 3 of them are red and 2 cubes are blue?

   (c) How many eight-story towers can be made by using 8 cubes if 3 of them are red and 5 of them are green?

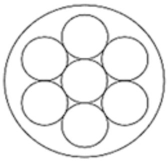
4. Consider the numbered cards below.

   (a) How many five digit numbers can be formed by using all these cards if each of them is used exactly once?

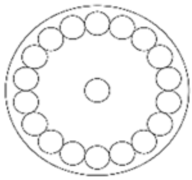
   (b) How many five-digit numbers can be formed by using all these cards if each of them is used exactly once?

   (c) How many seven-digit numbers can be formed by using all these cards if each of them is used exactly once?

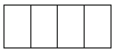
5. Consider a magician.
   (a) A magician put 2 red and 5 blue balls of the same size onto a table where the table can be turned around. (See diagram.) Two arrangements of the balls are identical if they match when the table turns. How many different arrangements of the balls are possible?



   (b) This time the magician put 2 red and 17 blue balls of the same size onto a table where the table can be turned around. (See diagram.) Two arrangements of the balls are identical if they match when the table turns. How many different arrangements of the balls are possible this way?

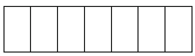


6. We put 1, 2, 3 and 4 into the boxes by using the following rules:



- Firstly 1 is placed in a box, followed by 2. Then 3 comes. Finally, 4 is put into the leftover place.
- We put a single number into each square.
- The first number can be put anywhere.
- Then you can put each leftover number into a box which has already a number NEXT to it.

   (a) How many ways can you fill the boxes in? (Two arrangements are different if you find at least one box with two different numbers.)
   (b) How will your answer change if you must use 1, 2, 3, 4, 5, 6 and 7 under the same conditions (see list above)?



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Part III
Recursion and Recursive Thinking

Discrete Dynamical Systems: A Pathway for Students to Become Enchanted with Mathematics

Robert L. Devaney

Abstract Discrete dynamical systems and fractal geometry are two of the most interesting fields of research in contemporary mathematics. One reason for this is the absolutely beautiful images that often arise in these fields. A second reason is that many topics in these fields are completely accessible to all, including high school students. One of the aims of this paper is to describe one such topic, namely, the chaos game. Not only do students get quite excited when they first encounter this topic, but they also see how the fractal geometry they use to understand the chaos game relates directly to what they are currently studying in their geometry classes.

Keywords Chaos game · Sierpinski triangle · Sierpinski hexagon
Fractalina · Fractanimate

1 Introduction

One of the things that we in the US do not do well is to expose our K–12 students to what is new, exciting, and beautiful in contemporary mathematics. We have these students in our math classes for twelve years, during which we show them fourth century BC geometry, eleventh century algebra, and, if they really work hard and do well, some seventeenth century calculus. No wonder many students think that there is nothing interesting or important going on in mathematics. Just imagine physicists restricting attention to eleventh century physics or biologists to fourth century BC biology! No way that would happen!

In an effort to change this in our area of the country, my University has organized Math Field Days for the past twenty years or so. These are held two or three times a year, and, each time, around five hundred high school students and their teachers show up for a day where we expose them to some exciting areas of contemporary mathematics. There are no competitions; students who participate in math contests

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are already “hooked” on mathematics and will very likely end up in STEM careers. But there are plenty of other students out there who are very talented and quite creative, but who have no idea of what is going on in today’s mathematics.

In an effort to reach out to these students, at the Field Days we focus primarily on discrete dynamical systems. This is a topic that is incredibly accessible to younger students. Indeed, one of the major areas of interest in the field is what happens when you iterate the simple quadratic expression $x^2 + c$. This iterative process often leads to extremely chaotic behavior, and viewing the corresponding chaotic regime in the complex plane produces incredibly beautiful fractal objects like the Julia and Mandelbrot sets. When students hear that we finally understood what happens when the real expression $x^2 + c$ is iterated in the 1990s, and that we still don’t understand what happens when the complex expression $z^2 + c$ is iterated, they become quite intrigued. I cannot count the number of letters and emails I have received from teachers and students over the years raving about how great it was to see how beautiful and exciting mathematics can be.

As an illustration of how these topics can be used to excite students, we shall restrict attention in this paper to just one of the many topics in discrete dynamics that we delve into at the Field Days, namely, the *chaos game*, or, as dynamicists call it, an iterated function system (See Barnsley 1988; Choate et al. 1998; Peitgen 1991). One of the beauties of this topic, besides the exquisite and quite surprising fractal images that arise, is the fact that it brings together many of the topics that high school students are currently studying, like the geometry of transformations, geometric measurement, and probability, in a very different and appealing way.

2 The “Classical” Chaos Game

The easiest chaos game to explain is played as follows. Start with three points at the vertices of an equilateral triangle. Color one vertex red, one green, and one blue. Take a die and color two sides red, two sides green, and two sides blue. Then pick any point whatsoever in the triangle; this is the *seed*. Now roll the die. Depending upon which color comes up, move the seed half the distance to the similarly colored vertex. Then repeat this procedure, each time moving the previous point half the distance to the vertex whose color turns up when the die is rolled. After a dozen rolls, start marking where these points land.

When you repeat this process many thousands of times, the pattern that emerges is a surprise: it is not a *random mess*, as most first-time players would expect. Rather, the image that unfolds is one of the most famous fractals of all, the Sierpinski triangle shown in Fig. 1. Notice that there are no points in the “missing” triangles in this set. This is why we did not plot the first few points when we rolled the die.

To enable students to understand what is going on here, it is helpful to provide them with a copy of the Sierpinski triangle. Then, given a particular point in the original triangle, have them plot the three possible points to which this point is

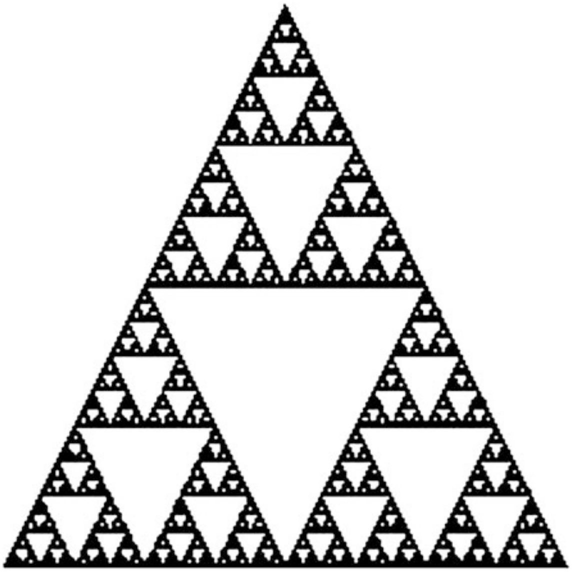

Fig. 1 The Sierpinski triangle. The original red, green and blue vertices are located at the vertices of this image

moved when the die is rolled. Then, have them plot the nine points at the next level, and the 27 points at the next level, and so on. It is probably easiest to start this process with a point in the middle of the largest empty triangle. This explains why, after just a very few rolls of the die, the corresponding point is in an empty triangle that is too hard to see because the size of this triangle has become miniscule. So this shows students why the Sierpinski triangle emerges when this game is played, and it also helps their geometric visualization as well as their measurement skills. Starting this process at a point that lies on the Sierpinski triangle leads to a more complicated process, and also helps the student to understand the algorithm for the chaos "game" described in Sect. 5.

It is nice at this point to show students an interesting connection between the Sierpinski triangle and Pascal's triangle. Have them list the numbers in Pascal's triangle down to some level. Then have them erase all of the even numbers and block out each odd number with a black disk. As this process continues down Pascal's triangle, they should begin to see the Sierpinski triangle emerging.

Now here is an observation that fosters other geometric skills: the Sierpinski triangle consists of three self-similar pieces, each of which is exactly one half the size of the original triangle in terms of the lengths of the sides. And these are precisely the numbers that we used to play the game: three vertices and move half the distance to the vertex after each roll. So we can go backwards: just by looking at the Sierpinski triangle, and with a keen eye for its geometry, we can read off the rules of the game we played to produce it.

3 Other Chaos Games

For a different example of a chaos game, put six points at the vertices of a regular hexagon. Number them one through six and erase the colors on the die. We change the rules a bit here: instead of moving the point half the distance to the appropriate vertex after each roll, we now compress the distance *by a factor of three*. By this we mean we move the point so that the resulting distance from the moved point to the chosen vertex is one-third the original distance. We say that the *compression ratio* for this game is three.

Again we get a surprise: after rolling the die thousands of times the resulting image is a *Sierpinski hexagon* depicted in Fig. 2. And again we can go backwards: this image consists of six self-similar pieces, each of which is exactly one-third the size of the full Sierpinski hexagon—the same numbers we used to design the game. By the way, there is much more to this picture than meets the eye at first: notice that the interior white regions of this figure are all bounded by the well-known Koch snowflake fractal!

This is where the geometry of transformations arises: given a fractal that results when a certain chaos game is played, can you determine the rules that were used to produce this image? In Fig. 2, the second fractal is the Sierpinski carpet. How many vertices were used to produce this image, and what was the compression ratio? You need to determine a collection of different geometric transformations that take the entire carpet onto a certain number of distinct, self-similar pieces. An applet called Fractalina that can be used to create a variety of chaos game images is available at the Boston University Dynamical Systems and Technology website (http://math.bu.edu/DYSYS/applets).

One quick question: which object emerges when we play the chaos game with four vertices at the corners of a square and a compression ratio of two?

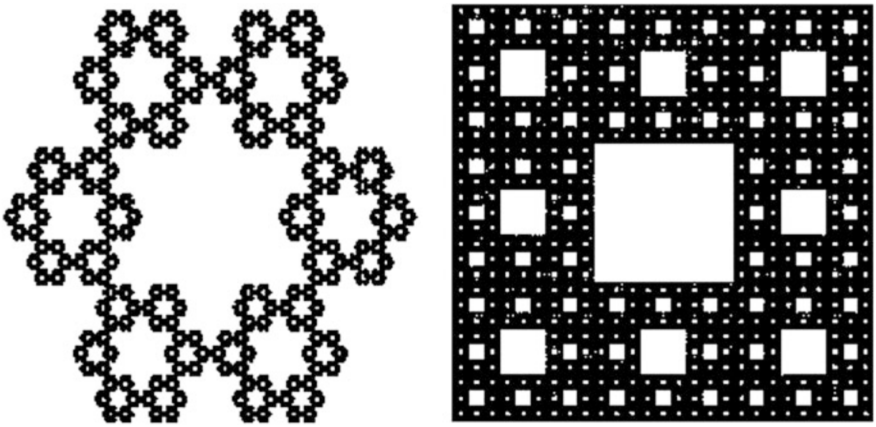

Fig. 2 The Sierpinski hexagon and carpet

(Answer is given in Sect. 6.) As you see, fractals do not always emerge when the chaos game is played.

4 Rotations

Now let's add rotations to the mix. This is where the geometry of transformations becomes even more important. Start with the vertices of a triangle as in the case of the Sierpinski triangle. For the bottom two vertices, the rules are as before: just move half the distance to that vertex when that vertex is called. For the top vertex, the rule is: first move the point half the distance to that vertex, and then rotate the point 90° about the vertex in the clockwise direction. The result of this chaos game is shown in Fig. 3a: note that there are basically three self-similar pieces in this fractal, each of which is half the size of the original, but the top one is rotated by 90° in the clockwise direction. Again, as before, we can use geometric transformations to go backwards and determine the rules of the chaos game that produced the image. In addition, plotting the possible images of a given point in the fractal now involves both contractions and rotations and hence more and different geometric skills.

Changing the rotation at this top vertex to 180° yields the image in Fig. 3b. This time, the top self-similar piece is rotated 180°. For the fractal in Fig. 3c, we rotated 20° in the clockwise direction around the lower left vertex, 20° in the counterclockwise direction around the lower right vertex, and there was no rotation around the top vertex.

In the math classes that most students take, usually the geometry of transformations involves rotations, expansions, or contractions of simple geometric objects, like squares or circles. Here the objects are much more interesting to look at, and determining these transformations can be difficult at times. We often challenge students at the Field Days to figure out the rules of a chaos game that produced a certain image. For example, in Fig. 4, we give you the opportunity to try your hand

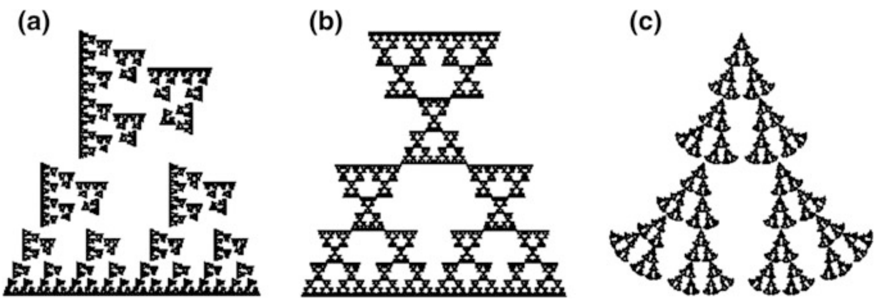

Fig. 3 Sierpinski with rotations

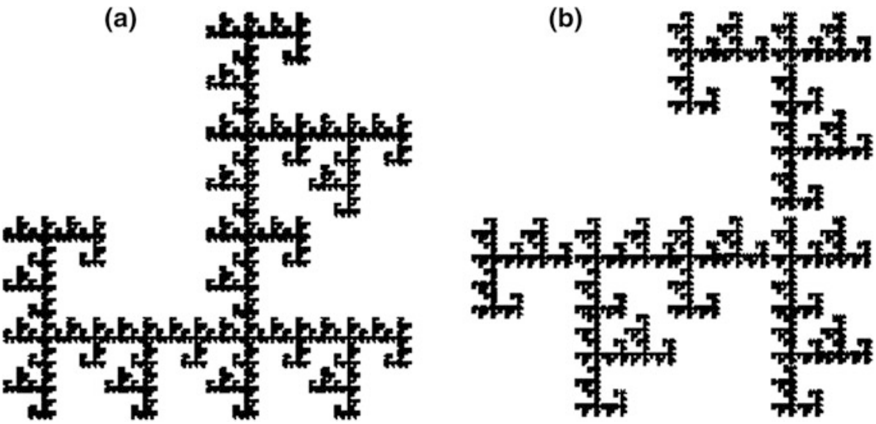

Fig. 4 Challenging chaos game images

at this. You must determine the number of vertices, the compression ratio, and the rotations involved in each case. Not so easy! (Answer in Sect. 6.)

Another activity that greatly motivates students is fractal movie-making (See Devaney 2004b). Once you know how to create a single fractal pattern via the chaos game, you can slowly vary some of the rotations, compression ratios, or locations of the vertices to create a fractal movie. I often challenge students to make a movie that is "beautiful" and that I cannot figure out how they made it. The students often work for hours to make these animations. Of course, beautiful here means "with a lot of symmetry," so there really is a lot of geometry in this activity. Another applet called Fractanimate is available to make these movies at the Boston University DS & T website. A number of fractal movies created by students are also posted at this site.

5 The Chaos "Game"

One final topic that is always a big hit at the Field Days arises when I challenge the students present to beat me at the Chaos Game (See Peitgen et al. 1991). To play this game, we begin with the outline of the Sierpinski triangle down to some level. That is, we begin with the original triangle and successively remove groups of sub-triangles at each level. The first level is defined to be the case where the *middle* triangle has been removed from the original triangle, leaving behind three equal-sized triangles. At the second level, the three smaller middle triangles are removed from these three, leaving behind nine equal-sized triangles. At the level n stage, there are then 3^n triangles. Then highlight one of the remaining small triangles at this given level. This triangle is the *target*. Now place a point at the lower right vertex of the original triangle. This is the *starting point*. The goal of the game is to move the starting point into the *interior* of the target. The moves are just the moves of our original chaos game: At each stage the point is moved half the

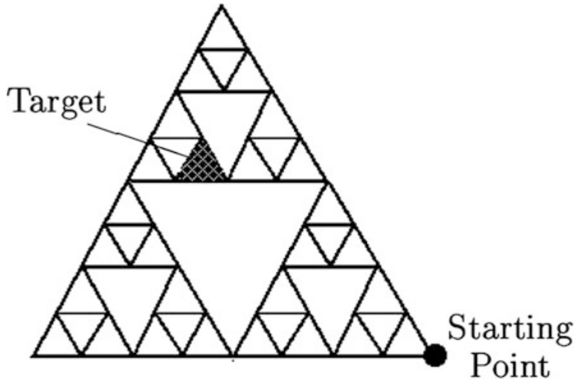

Fig. 5 Level three of the chaos game

distance to one of the three original vertices. The chaos game setup for a level three game is displayed in Fig. 5.

At a given level, it is always possible to move the starting point into the interior of the target in the same number of moves, no matter where the target is located. For example, for the three targets available at level one, it is possible to hit any target in exactly three moves. (Recall that you must end up in the interior of the target, not the boundary.) At level two, four moves suffice, and at level n exactly $n+2$ moves can be found to hit any target. The challenge to students is to figure out the algorithm for hitting any possible target. Students can usually come up fairly quickly with a way to hit a specific target, but the algorithm necessary to hit any target is much more difficult both to formulate and to explain. But that, as I always tell the students, is what mathematics is all about—being able to figure out a solution, and then being able to explain it in a coherent fashion.

For example, in Fig. 5, the moves to hit the prescribed target are, in order: top, left, right, left, and top. There is only one other way to hit this target in five moves: left, top, right, left, top. This in general is the case: there are exactly two sequences of moves that allow you to hit the target in the minimum number of moves. An interactive version of this game is also available at the DS & T website. At this website, there are also several other variants of this game that include rotations in the mix. These are even more challenging!

6 Some Solutions

In this section we briefly describe the answers to some of the questions posed earlier in this paper. First, what happens when you play the chaos game with four vertices at the corners of a square and a compression ratio of two? Well, the points generated by this process eventually fill up the entire square densely. Indeed, the square is a self-similar object: it can be broken into four equal-sized sub-squares, each with sides exactly half the length of the original square.

In Sect. 4, the two fractals displayed in Fig. 4 were obtained as follows. Each was generated using three vertices, a compression ratio of two, and a rotation of 90°. In Fig. 4a, the rotation was in the clockwise direction, and, in Fig. 4b, in the counterclockwise direction. The three vertices were placed at the corners of an isosceles right triangle. In particular, using the applet Fractalina, in the first case the vertices were placed at (100, 50), (−50, 0), and (50, −50) and, in the second, the vertices were placed at (0, 50), (−50, −100), and (50, −50).

Finally, winning the chaos game in the previous section is a two-step process. The first step involves moving the starting point into the interior of original triangle. This can be accomplished by either moving left then top, or by moving top then left, since the starting point is located at the lower right vertex of the triangle. The second step involves determining the *address* of the target triangle. For example, at stage one, there are three possible target triangles, one on the top, one at the left, and one at the right. We denote these targets by T, L, and R respectively. At stage two, each of these level one triangles is sub-divided into three smaller triangles. For example, the upper triangle T now contains three smaller target triangles, which we denote by TT, TL, and TR. Then each of these targets contains three smaller targets. So, for example, TL can be divided into TLT, TLL, and TLR. Continuing, each possible target at level n has a unique address consisting of a sequence of n letters T, L, and R. Then, to reach the given target at phase two of the process, we just reverse the letters in the address and follow that pattern to move the point into the interior of the target. That is why, in the example in Sect. 5, the winning strategy to reach the given target was either LTRLT or TLRLT.

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How Recursion Supports Algebraic Understanding

James Sandefur, Kay Somers and Rosalie Dance

Abstract This article discusses the advantages of integrating recursion and difference equations into the middle school and high school algebra curriculum as a means to promote a deeper mathematical understanding of algebraic topics, particularly the covariation of variables. This integration of difference equations builds on earlier mathematical concepts and prepares students for studying the mathematics of change: calculus. In addition, recursive problems can reinforce a student's ability to communicate mathematically through the use of contextual situations.

Keywords Difference equations · Recursion · Covariation · Constant change
Quadratic functions

1 Introduction

In recent years, there has been a push to include more discrete mathematics in the school curriculum. Advocates argue that it presents the students with different useful ways of thinking and that this mathematics is becoming more important in our data-heavy age. This movement was met with resistance. Some argued that the currently taught mathematics, including pre-algebra, algebra, geometry and analysis, is important for preparing students for calculus, which can be a gateway into STEM professions, and that limiting students' exposure to the traditional skills might limit their future options. What both groups have missed is that this need not

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be an either/or situation. Other chapters in this monograph describe how discrete topics such as graph theory and combinatorics can support the goals of the current curriculum taught in many countries. We describe in this chapter how the discrete topic of recursion can be integrated into the algebra curriculum in a way that develops alternative thinking skills among the students while simultaneously deepening and broadening their understanding of traditional algebra. We propose that this integration of recursion can better prepare students for calculus than the traditional algebra curriculum.

In early grades, students are frequently encouraged to think in terms of patterns and change, such as the recursive nature of many sequences. This thinking process is exploited when students learn that repeated addition is multiplication, and later, that repeated multiplication is exponentiation. Unfortunately, in most cases, recursive thinking is not further developed. For example, students know that $m + m + \cdots + m$ (n times) equals nm, After they are introduced to linear functions and their graphs, do they make the connection that, when x is an integer, the right-hand-side of the linear equation $y = mx + b$ is just $b + m + m + \cdots + m$, where we are adding m, x times? In some sense, the linear equation is just extending the idea of repeated addition to cases where we are adding m a non-integer number of times. Thinking about lines in this fashion reinforces one of the most important aspects of lines, constant rate of change; as x changes by 1, y changes by m. Further, recursive models with constant change resulting in linear solutions develop students' understanding of rate of change within contextual situations, as we will elaborate later.

In many situations, the variable x in the linear equation represents time. We consider these situations as *dynamic* with the slope describing how y changes per unit change in time. This early use of repeated addition resulting in multiplication, and more complex dynamic situations, which are described in later sections, can help support students' ability to use covariational reasoning, a skill that some research indicates is crucial for developing a robust understanding of calculus (Carlson et al. 2002). By covariational reasoning, we mean

> the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other (Carlson et al. 2002, p. 354).

For example, $y = -4.9t^2$ describes how distance fallen (in meters) relates to time passing (in seconds). We will discuss covariational thinking in more detail in examples given in later sections.

Mathematics at the secondary level and the first or second year of college is, in our opinion, too often focused on geometric facts and algebraic manipulation. Recently, there has been an increased effort to provide applications, but these applications are often exercises that involve little more than substitution into a formula followed by algebraic manipulation. Students often have little or no idea how the formula was derived nor do they understand if it really models the problem situation. Some of these problems can be worked while ignoring the context. It is our common experience that this approach fails to show students the power of mathematics to solve real problems; it also does not support the deeper understanding of algebra that can result from contextual thinking.

In our view, doing mathematics within a contextual situation should serve the dual purposes of

1. engaging the students and
2. having the context reinforce the logic of the mathematics they are doing.

We are strong believers that a *physical* context can lead to better conceptual understanding. Just as lower level students can use simple contexts, such as cutting a cake, to come to understand why

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

makes sense, contexts, such as finding the area of a triangular stack of squares, should help secondary students understand why an algebraic formula such as

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

makes sense. These two purposes can be accomplished through the use of difference equations and recursion, as our later examples illuminate.

Harel (2001) compared a traditional approach to teaching the recursive proof technique, induction, to an experimental approach used in his study. Instead of the mechanical approach of giving students a closed formula solution to a problem and being asked to prove it is true using induction, Harel introduced induction by having students solve problems in contexts that are recursive in nature. The students solved these problems by observing and using the recursive pattern. From here, it is natural to move into full proof by induction. His study indicates that through this contextual introduction to induction, students better understand what induction is about, how to apply it, and why the original statement is true. We describe in the following how a similar approach of introducing naturally recursive situations within the algebra curriculum can be used to promote algebraic understanding. In addition, we discuss specifics of how this integration of standard algebra, difference equations, and engaging contextual problem solving can promote deeper and more robust student learning.

2 Constant Change and Linear Functions

Situations resulting in constant change arise in numerous contexts and can be represented in a variety of manners. For example, consider the following situation:

Situation 1: Let T_n represent the number of 1 m by 1 m tiles needed to form a border around the outside of a square garden measuring n meters by n meters.

For example, $T_4 = 20$ as can be seen in Fig. 1. One way to think about the function T_n is that the border is made up of $4n$-meter long strips and 4 corners, that is,

$$T_n = 4n + 4. \tag{1}$$

Linear function (1) is considered a closed form representation of our function.

If our garden is increased to a 5-by-5 m garden, that is, each dimension is increased by 1 m, then one way to think about the border is that each of the strips will have to be lengthened by 1 m (keeping the tiles in the corners); that is

$$T_{n+1} = T_n + 4. \tag{2}$$

Equation (2) is called a recursive equation or a difference equation. This form emphasizes the constant rate of change, $T_{n+1} - T_n = 4$, telling us we need 4 additional tiles per additional meter increase in dimensions of the garden. We must note that for a recursive equation to be useful, we have to be given the value of the function for some value of n, such as $T_0 = 4$. If we do not consider a 0-by-0 garden as making sense, we could give $T_1 = 8$. Students generally agree a 0-by-0 garden is no garden at all, although we can certainly surround a point where we have no garden at all with 4 corner tiles, thus illustrating T_0.

We note that Eq. (2) can be depicted by a flow diagram, such as seen in Fig. 2, which describes pictorially how T_n is changing over time. A flow diagram is useful when we describe how a process is changing, and is of the form

$$T_{n+1} - T_n = \text{things added} - \text{things taken away}.$$

The circle in Fig. 2 represents the quantity T. An arrow is drawn going into the circle for each quantity added during a phase and an arrow is drawn going out for each quantity that is removed from T during that same phase. We have found that

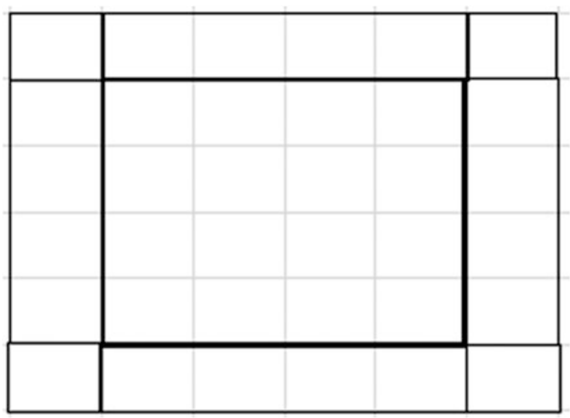

Fig. 1 4-by-4 garden with 1 meter border

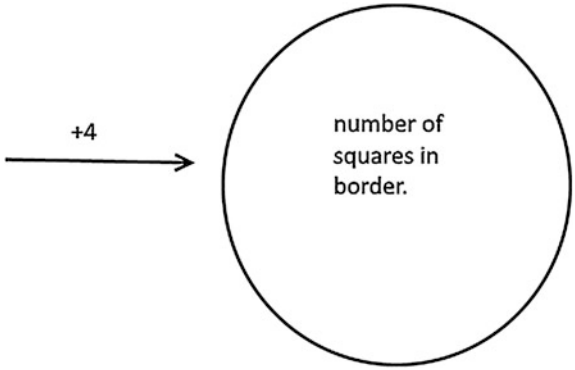

Fig. 2 Flow diagram for change in tiles in border per unit change in dimensions of garden

even with college students, both upper level math students and liberal arts students fulfilling a general education requirement, being able to visually depict a dynamic situation using a flow diagram aids in their ability to understand and model real situations. Different students have different learning styles, and while flow diagrams add to the understanding of most students, they are particularly valuable for visual learners. As the situation being modeled becomes more complex, as seen in later examples, flow diagrams become more valuable. It is helpful for students to first learn how to use flow diagrams in simple situations such as our garden problem.

A classroom presentation of this problem could ask students to generate an expression that describes this situation. Both the recursive and closed form solutions would be correct answers, and if different students generate different solutions, they should explain their reasoning. For example, some students may generate the closed formula $T_n = 4n + 4$, and explain that this formula arises because the border is comprised of the sum of the four sides ($4n$) and the four corners (4). Other students might explain that each side of the border consists of $n + 1$ tiles, n tiles against the garden plus one additional tile for a corner. When the four ($n + 1$) pieces of edging are fitted together, they complete the border, leading to $T_n = 4(n + 1)$. (One college student in a Discrete Mathematics class who provided this description added a comment that his experience as a carpenter led him to view the border in that way.)

When middle or secondary students are introduced to recursion, they may write a recursive equation such as

$$\text{'Next } T' = \text{'Current } T' \text{ plus } 4.$$

Secondary and college students should eventually learn the subscripted version of this recursive equation, either $T_n = T_{n-1} + 4$ or $T_{n+1} = T_n + 4$. If no student generates the recursive relation, the teacher can present one of these versions and ask the students to describe what this equation means to them. This gives an opportunity to focus student-thinking on the slope, or constant change aspect of linear equations. Discussion of a graph of the points (n, T_n) where $\text{n} = 0, 1, 2, 3, \ldots$ connected by the line they define can help students develop an understanding of the slope concept. A table of values with three columns, such as Table 1, can further

Table 1 Description of changes in sides of garden to size of boarder

n	T_n	Verbal description
0	4	No garden at all
1	8	1 tile on each side and 1 at each corner
2	12	2 on each side and 1 at each corner
3	16	3 on each side and 1 at each corner
⋮	⋮	⋮
n	$4n + 4$	n on each side and 1 at each corner

help students focus on how a change of 1 in one variable effects a change of 4 in the other, and thus can deepen their understanding of slope.

The point is that students need to understand that a linear function often describes a dynamic situation, something that is changing, not just a static object that answers a particular question for a specific value of the independent value. Thinking of this covariational relationship between the independent and dependent variables is vital for students who will eventually study calculus.

To help students understand the relationship between the recursive equation and the initial value, we could change the problem, stating we want the border as described previously, plus a 1-m^2 tile in the interior of the garden, on which we can place a statue or fountain. The closed form solution would become $T_n = 4n + 5$ but the recursive formula would remain $T_{n+1} = T_n + 4$. Now, though, we would have $T_1 = 9$, or if we do not view a 1-by-1 garden with a tile filling the interior as a garden, $T_2 = 13$ (12 tiles in the border of the 2 by 2 garden, plus a tile in the interior of the garden) as our initial value. For problems in which T_0 makes sense, students will discover that T_0 corresponds to the vertical intercept, 4 for the garden with no interior tile as the equation is $T_n = 4n + 4$, and 5 for the garden with an interior tile since the equation is $T_n = 4n + 5$. As this problem shows, the vertical intercept may not have physical meaning in some contextual situations. As discussed above, students will gain understanding from constructing both a table and a graph as well as the different symbolic representations of the new situation.

In the garden example, because of the tiles, the function only makes good sense for integer values for the independent variable. It is relatively easy to come up with contexts in which this is not the case. For example, suppose a car has a gas tank that holds 12 gallons of gas and it is full. Suppose we also know that the car gets 20 miles per gallon of gas. If we let $f(m)$ represent the amount of gas left after driving m miles, we could use the linear function $f(m) = -0.05m + 12$ or the recursive model, $f(m+1) = f(m) - 0.05$, with $f(0) = 12$, which would have been derived from the flow diagram in Fig. 3. (We use the convention that functions whose domain is inherently restricted to a subset of the integers will be given using subscripts, as T_n, but functions which can reasonably be extended to an interval of real numbers will be given in function notation, as in $f(m)$.)

We can present this car problem to students and ask them to find both a closed formula for $f(m)$ (in terms of m) and a recursive formula for $f(m)$. In each case, they should describe how they found their formulas. We note that the recursive

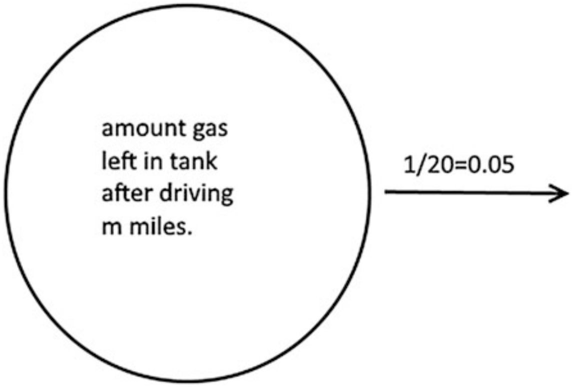

Fig. 3 Flow diagram for amount of gas left in tank after driving one additional mile

formula only works for an integer number of miles driven, but it can be used to help students extrapolate to the linear function for any real number of miles driven, up to 240. The recursive formula still helps to make the connection between the slope of the linear function and the constant change of −0.05 per mile driven.

Initially, students seem to be drawn to one particular way of thinking; requiring them to think in multiple domains about a single problem can broaden their thinking. It is productive to ask students to compare and contrast the advantages and disadvantages of recursive solutions versus closed form solutions, especially after they have worked on several examples of both, such as the previous two problems. From their writing, we have seen that most students initially consider the closed form solution more useful: To find the number of tiles needed for a garden that is 20-by-20, they know that a simple substitution into the closed formula is sufficient, while numerous calculations or access to a spreadsheet would be needed to answer the question with the recursive formula. They also note that the closed form solution can be used in situations where non-integer values of the independent variable are reasonable, such as m in the car model.

On the other hand, there are several strengths of the recursive formula that students should understand. One advantage of the recursive approach is that it emphasizes that the amount of gas in the tank is not static, but changes as the car is driven. Furthermore, the rate of change is clearly visible with the recursive approach. This visibility helps to focus students' attention on the change as they answer questions such as: How many more tiles would we need to enlarge an existing 20 m by 20 m garden by a meter (or two meters) in each direction?

Both approaches give students an opportunity to reason coherently about mathematics and learn to clearly explain their reasoning orally and in writing, a skill that needs to be developed in mathematics classrooms. For example, in situations where a geometric model is lacking, such as the gas-tank problem, the flow diagram gives a pictorial way of thinking about this problem; this allows a recursive development that can be translated into the closed form solution. In fact, after some exposure to flow diagrams, students will be able to quickly identify situations modeled by a linear equation and translate their flow diagram into closed formulas.

To further challenge students, they can be asked to prove that their formulas are correct. This question provides an opportunity to discuss the difference between explaining why they think their formula is correct and providing a proof that it is correct. It can lead to interesting discussions concerning what constitutes a mathematical proof. Teachers will need to make explicit what they consider a reasonable explanation versus what they consider a proof.

3 Linear Change and Quadratic Functions

Situations resulting in variable change arise in numerous contexts and can be represented in a variety of ways. In this section, we consider situations in which there is linearly increasing or decreasing *change*. To clarify, we now consider the following well-known situation, but describe a different twist on this problem which promotes its use in developing students' deeper conceptual understanding of quadratic functions and how they contrast with linear functions. Finally, it gives a foreshadowing of calculus.

> **Situation 2**: If the n people in a room each shake hands exactly one time with everyone else in the room, how many handshakes take place?

This question can be introduced by asking all the students in a class to introduce themselves to each other and shake hands. This works well in the first few days of a course as an ice-breaker. Afterwards, we can ask if everyone shook hands with everyone else. If the class is reasonably large, the answer is usually 'no.' We then ask how we can ensure that everyone meets everyone else. Some students may suggest that everyone line up. The first student goes around and shakes hands with all the other students. Then the second student does the same thing. At some point, someone notices that with this method, everyone shakes hands with everyone else twice. If there are n people in the room, this results in $n(n-1)$ handshakes with each counted twice, so the total number of handshakes required is only half of that,

$$h_n = \frac{n(n-1)}{2}.$$

To avoid the double-shaking, often someone suggests a modification of this approach by having each student shake hands only with those behind him or her in the line. This gives the total of

$$h_n = (n-1) + (n-2) + \cdots + 2 + 1$$

handshakes. If no one suggests it, the teacher can propose a third, similar approach: The teacher comes early to class. After the first student arrives, he or she shakes hands with the teacher. When the second student arrives, this student shakes hands with the two people already in the room. As each person arrives, that person shakes hands with everyone already in the room. This gives rise naturally to the recursion equation

$$h_{n+1} = h_n + n;$$

that is, the number of handshakes h_{n+1} equals the number of handshakes that have already taken place, h_n, plus the number of handshakes this (n + 1)th person engages in with the n people already in the room.

This handshake example, which many teachers already use, connects the closed formula and recursive approach, and gives a nice justification of the formula for the sum of the first n integers given in the introduction. Many students still prefer solving problems using a closed formula, but there is less enthusiasm for the direct approach in this problem because, in some sense, it was not direct: the students solved a different problem (the number of permutations of 2 people chosen from n people) and then had to divide by 2 after recognizing that this was twice as many handshakes as needed. The recursive approach gives a natural way to approach this problem.

In the handshake problem, the recursive and closed formulas both arise naturally in an engaging classroom activity. A similar problem can be introduced using the triangle numbers, that is, constructing 'triangles' by starting with a 1-by-1 square, then adding a 2-by-1 rectangle beside it, then a 3-by-1 and so forth (or by placing bowling pins, or balls, in a triangular configuration with rows containing 1 pin, 2 pins, 3 pins, …). Then, with the function t_n representing the number of squares (or pins) when there are n columns (or rows), we get $t_0 = 0,\ t_1 = 1,\ t_2 = 3,\ t_3 = 6,\ \ldots$, whereas for the handshake problem, $h_1 = 0,\ h_2 = 1,\ h_3 = 3,$ and so forth. Students will see from a table that the dependent values resulting from the triangle numbers are the same as those for the handshake problem. On the other hand, the function, $t_n = n(n+1)/2$, and the difference equation, $t_{n+1} = t_n + n + 1$, are different. It is valuable to have students discuss how the geometric (or physical) construction of the triangle numbers compares and contrasts to the handshaking situation. Comparison of their graphs will visually illustrate their similarity and their difference.

Let's refer back to the approach to the handshake problem in which you are in a room and people keep entering. As someone enters, you shake hands with them as does everyone else in the room. Students can be asked to develop a formula describing how the number of handshakes they make changes as the nth person enters the room and a formula for how the total number of handshakes changes as the nth person enters the room. Letting s_n represent the number of handshakes you make and h_n, as before, represent the total number of handshakes, the corresponding formulas would be

$$s_{n+1} = s_n + 1 \text{ and } h_{n+1} = h_n + n$$

The students should notice that one formula results in constant change, $s_{n+1} - s_n = 1$, while the other formula results in linearly increasing change, $h_{n+1} - h_n =$ n. The corresponding closed form expressions are

$$s_n = n - 1 \text{ and } h_n = \frac{n(n-1)}{2}$$

Students have already experienced that constant change produces a linear function. They can now extend that understanding: linearly increasing (or decreasing) change produces a quadratic function. The students can be asked to explain why

$$f_{n+1} - f_n = an$$

should result in a closed formula of the form

$$f_n = \frac{an\,(n-1)}{2} + f_0,$$

and more generally, why

$$f_{n+1} - f_n = an + b$$

should result in a closed formula

$$f_n = \frac{an(n-1)}{2} + bn + f_0,$$

They can discover this by computing one value at a time and substituting, using the formula

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}.$$

Students can be given similar problems to reinforce their understanding. One such example is the number of unit-squares needed to construct an n-by-n square. While students at a certain level should know that $r(n) = n^2$, we have seen middle school students working on this problem express the answer as, ‘each time you add two more unit-squares than the last time.’ In particular, they notice that $r(1) = r(0) + 1$, $r(2) = r(1) + 3$, $r(3) = r(2) + 5$, …, or more generally

$$r(n+1) = r(n) + 2n + 1.$$

Again, linear change results in a quadratic function. (Since a square does not have to have integer sides, this function can be extended to $r(x) = x^2$.)

Further investigations can be done on the relationships discussed above to support covariational reasoning. For example, students can be asked to graph the function h_n and then find the slope of the secant line connecting (n, h_n) to $(n+1, h_{n+1})$. Note that the slope of the secant line, which is actually n, increases as the value of n increases. Students can now observe that whereas linear functions have constant slope, the slope of a quadratic function is not constant, something that can be seen both in the graph and by looking at the slopes of the secant lines. Now when students reach calculus and find that the derivative of a quadratic function is a linear function, they can be asked why this makes intuitive sense. Hopefully they will refer back to their understanding that linear change (derivative) corresponds to a quadratic function.

In summary, we have seen that:

- constant change, $f(x+1) - f(x) = m$, corresponds to a linear function, $f(x) = mx + f_0$ and
- linear change, $f(x+1) - f(x) = ax + b$ corresponds to a quadratic function, $f(x) = \frac{ax(x-1)}{2} + bx + f_0$.

This should not be confused with the difference equation $T_{n+1} = aT_n + b$ which is a linear difference equation but is not linear change if $a \neq 1$. In this case $T_{n+1} - T_n = (a-1)T_n + b$ so the change is proportional to the amount. This difference equation corresponds to a much more complicated formula which will be discussed in the next section.

4 Proportional Change and Exponential Functions

In this section, we discuss situations where there is constant proportional change, such as a bank account that is increasing by a constant percent, say i percent. Flow diagrams for such situations are similar to those for constant change, as Fig. 4 demonstrates. For example if someone deposits $a(0) = 1000$ dollars into an account earning 3% interest per year, then

$$a(1) = a(0) + 0.03a(0), a(2) = a(1) + 0.03a(1), \ldots,$$

or more generally

$$a(n+1) = a(n) + 0.03a(n)$$

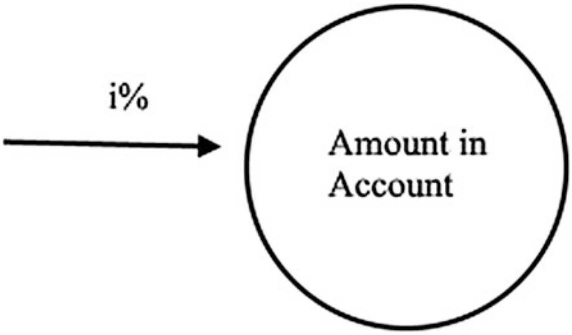

Fig. 4 Flow diagram for bank account earning *i*% interest per time period

Continued substitution leads to the closed form

$$a(n) = 1.03^n(1000)$$

illustrating that repeated multiplication is just exponentiation. Students should observe that the difference, $a(n+1) - a(n)$, is always the same proportion, 3 percent, of $a(n)$.

Simple situations can be given to students to study and explain, such as finding the number of layers when a sheet of paper is folded *n* times or the size of a slice of pie that is halved repeatedly. In the first case $i = 100\%$ while in the second case, the arrow in the flow diagram is going out, meaning you are subtracting 50%, leading to the equations $p_{n+1} = p_n - 0.5p_n = 0.5p_n$ and $p_n = (0.5)^n p_0$.

A context that often engages older students arises when they learn that about 13% of the caffeine in our body at the beginning of an hour is filtered out during the hour, leading to the recursive equation

$$c(n+1) = c(n) - 0.13c(n)$$

and the closed form

$$c(n) = (0.87)^n c(0)$$

where *n* represents the number of hours after the consumption of the caffeine. While the difference equation approach helps explain the process and is often easier to understand, the closed form expression can be used for fractions of an hour, so is more general.

Students' comprehension of exponential functions can be extended using the caffeine problem. For example, if students are asked to compute the amount of caffeine left 5 h after consumption, they will find that it is about half of what they started with. Thus, if we construct the function *c*(*j*) where *j* is the number of 5-h periods since initial consumption, then the corresponding equation would be $c(j) = (0.5)^j c(0)$. We note that 5 h is the approximate half-life of caffeine in the body. Students might be asked to compare the advantages and disadvantages of each formula. We would expect students to say that the second form is much easier to use for mental math; you can relatively easily estimate the amount of caffeine in your body at points of time throughout the day.

To extend this problem, students can be asked to find an estimate for the amount of caffeine in a beverage of choice. This is relatively easy to find online. Students can also be asked to estimate the amount of caffeine in a person's body at different times of the day, given that they drink several caffeinated beverages throughout the day. The idea is that at the point in time that a second beverage is consumed, you compute how much caffeine is left from the first beverage, add that to the second beverage and start the timer again. It is a good exercise for students to derive a piecewise defined function for this situation. For example, suppose 150 mg of caffeine is consumed at 9:00 AM and an additional 100 mg is consumed at 1:00 PM. Letting n be the number of hours after 9:00 AM, one possible function, assuming that the amount of caffeine in the body remaining from the previous day is negligible, is

$$f(n) = \begin{cases} (0.87)^n 150 & 0 \le n < 4 \\ (0.87)^{n-4} 186 & 4 \le n \end{cases}$$

Note that the exponent for the second condition must be $n - 4$ because n is the number of hours after 9:00 AM. An alternate form of this function is

$$f(n) = \begin{cases} (0.87)^n 150 & 0 \le n < 4 \\ (0.87)^n 150 + (0.87)^{n-4} 100 & 4 \le n \end{cases}$$

In this case, the second expression differentiates between the *old* caffeine and the *new* caffeine.

We note that in the case of proportional change, the difference equation can be written as $f(n+1) - f(n) = pf(n)$; that is, the change is proportional to the amount. The closed form is $f(n) = p^n f(0)$, so proportional change results in exponential growth or decay. Students should note that the slope of the secant line connecting $(n+1, f(n+1))$ and $(n, f(n))$ is $(p-1)f(n)$; that is, the slope is proportional to the value of the function. For the caffeine example, the slope of this secant line is therefore $-0.13f(n)$, meaning during that hour, 13% of the caffeine is eliminated. Students might be asked to graph the caffeine function for some initial amount of caffeine, and then be asked to draw secant lines at different points and compute the slopes of these lines. This will reinforce that the amount of caffeine being eliminated per hour is decreasing, which can be seen from the graph. They will see this again in a calculus course when they take the derivatives of exponential functions. This concept will also help them understand that the derivative is not just the slope of a tangent line, but is a rate of change of some quantity.

5 Constant and Proportional Change Together

In the examples previously discussed, we have used the recursive form to help students develop an understanding of the mathematics of change and apply it to the closed form. In each case it was relatively easy to develop a closed form model of the situation. More advanced students can be exposed to situations where the change is more complex, such as when change results from two or more aspects of the situation. In such cases, the recursive form is easier to develop. One such situation is a relatively simple extension of the caffeine model, in which there are repeated doses of whatever the chemical is. In the following situation, we assume we are studying an antihistamine, but numerous chemicals that we put into our bodies behave similarly (alcohol is not one of them).

Situation 3: Suppose a person takes a dose of 16 ml of antihistamine medicine at the beginning of a day. At the beginning of each of the following days, the person takes an additional dose of 16 ml. Each day, the kidneys eliminate 25% of the drug that was in the body at the beginning of the day. Let D_n represent the amount of drug in this person's body after n days, just after taking that day's dose. This means $D_0 = 16$ ml. Also, $D_1 = 28$ ml, the 12 ml from the previous day that were not eliminated plus that day's dose of 16 ml. Develop a recursive formula for D_{n+1} in terms of D_n and explain your formula. See if you can find a closed formula for D_n and explain that formula.

After constructing a flow diagram similar to Fig. 5, students can relatively easily develop the recursive formula

$$D_{n+1} = 0.75D_n + 16$$

which represents the 75% of the previous day's amount that was not removed, plus that day's dose of 16 ml. Using $D_0 = 16$, students can use a spreadsheet to see the buildup of the drug in the body over time, similar to Table 2. This leads to the discovery that the amount of drug in the body seems to level off at about 64 ml.

Fig. 5 Flow diagram describing change in drug in body

Table 2 Amount of antihistamine in body in ml at beginning of nth day

$n =$	0	1	2	…	20	21
$D_n =$	16	28	37	…	63.85	63.89

The closed formula is

$$D_n = -48(0.75)^n + 64.$$

A student with no background in discrete mathematical systems is unlikely to discover this formula without help and is perhaps even more unlikely to be able to explain why it is correct for all n. Students familiar with induction may be able to prove this formula works, but we believe that few will have insight into why the formula works or how to develop a formula for similar problems.

On further examination of the recursive form, students may be able to see why the closed form formula is reasonable for large n because the spreadsheet showed that D_n approaches 64 and they will be able to determine that $(0.75)^n$ approaches 0 as n gets large.

Students familiar with translations of functions should be encouraged to explore the difference equation $D_{n+1} = 0.75D_n + 16$ using a spreadsheet and a variety of initial values. If they observe that the graphs generated using this difference equation look like translations of an exponential function by 64, they may be able to come up with the translation

$$D_n = A_n + 64.$$

Notice that as D_n approaches 64, A_n approaches 0. Substitution into the difference equation and simplification gives the difference equation

$$A_{n+1} = 0.75A_n$$

which is already known to give rise to $A_n = c(0.75)^n$. Substitution back results in the closed formula $D_n = c(0.75)^n + 64$. Using the initial condition $D_0 = 16$ allows us to solve for c which results in the formula given above.

Most chemicals in the body have interactions, such as being absorbed into the liver or other organs, then being released from them, or being converted to a second chemical, which is then converted back to the first chemical, a process called *interconversion*. We can model one such interaction with a variation of our antihistamine model.

Situation 4: Suppose a person takes a dose of 16 ml of antihistamine medicine at the beginning of each day, which we assume is immediately absorbed into the plasma. Each day, the kidneys eliminate 25% of the drug from the plasma, 25% of the drug in the plasma is absorbed into the liver, and 50% of the drug in the liver gets released back into the plasma.

We note that this is similar to what happens to vitamin A in the body, except with different percentages; we are using percentages that work well in class. Let D_n represent the amount of drug in this person's plasma at the beginning of the nth day, just after taking that day's dose, and let L_n represent the amount of the drug in the liver at the beginning of the nth day. Thus, $D_0 = 16$ ml and $L_0 = 0$ ml. If students have not appreciated the use of flow diagrams before, they will for this problem. Figure 6 shows a flow diagram for this situation, which is easily constructed by going through the statement of the problem, one step at a time.

Using the flow diagram, we add quantities corresponding to arrows in and subtract quantities corresponding to arrows out to describe the change in the amounts of the drug in both the liver and plasma. Note that some arrows both go out of one circle and into the other circle. This gives the pair of equations

$$\begin{aligned} D_{n+1} - D_n &= 16 - 0.25D_n - 0.25D_n + 0.5L_n \\ L_{n+1} - L_n &= -0.5L_n + 0.25D_n \end{aligned}$$

These equations simplify to

$$\begin{aligned} D_{n+1} &= 0.5D_n + 0.5L_n + 16 \\ L_{n+1} &= 0.5L_n + 0.25D_n \end{aligned}$$

While we would not expect students to be able to come up with closed formulas for D_n and L_n, they can explore using a spreadsheet, as before. In this case, no matter the starting amounts, D_n will approach 64 ml and L_n will approach 32 ml. These values are called equilibrium values. If we designate the equilibrium amounts by D and L, they are the solutions to the pair of equations

$$\begin{aligned} D &= 0.5D + 0.5L + 16 \\ L &= 0.5L + 0.25D \end{aligned}$$

Thus, while students may not be able to find a formula, they can use their methods for solving systems of linear equations to solve for equilibrium values for the drug problem, which is the primary result of interest.

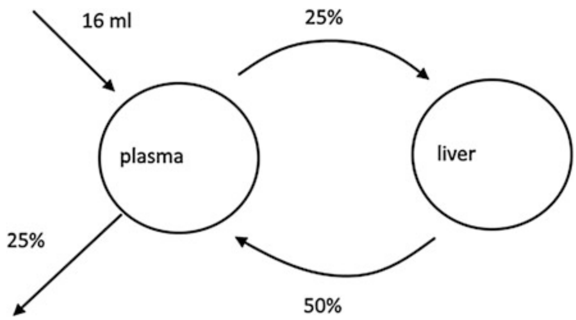

Fig. 6 Flow diagram of plasma/liver drug model

While the antihistamine model involves a more complicated closed formula, we note that there are some relatively simple difference equations, such as $s_{n+1} - s_n = a\, s_n^2$, for which a closed form solution does not even exist for most values of a, making the recursive approach necessary.

These important insights build a foundation for new mathematics. The recursive and closed form models combine to give students deeper mathematical insights and connections between different algebraic and geometric topics.

Having given a number of writing assignments asking students whether they prefer the recursive or closed formula, we believe that students generally prefer the recursive formula after they begin to understand what it says and how it is derived. Students tend to prefer closed formulas when they are easily obtainable, but develop the understanding that for more complex situations, recursive formulas may be more easily obtained and more easily used, provided a tool such as a spreadsheet is available.

The antihistamine and caffeine examples help us see the importance of providing examples using meaningful contexts in mathematics classrooms. The recursive formula is easily understood by students within the context given, and students gain appreciation that they can develop their own models.

If instead, students are simply given the closed formula for the drug problem, they could answer questions about the limit as n goes to infinity, or how many days it will take for the amount of drug to exceed 60 ml. This gives them practice in working with exponential functions, but the function itself is generally a mystery to them. They learn *about* mathematics, but they do not learn to *do* mathematics. Deep mathematical insight can result from development of the recursive formula and use of a spreadsheet.

6 Conclusion

At an early level, students learn to think recursively, such as when they learn that repeated addition is multiplication

$$a(n+1) = a(n) + b \text{ is equivalent to } a(n) = bn + c$$

and repeated multiplication is exponentiation

$$a(n+1) = b\,a(n) \text{ is equivalent to } a(n) = c(b)^n.$$

At this level, students are beginning to learn to use covariational reasoning, but this approach usually ends there. We should continue to support covariational reasoning among our students by building on recursive thinking in algebra, as recommended by Carlson (2002) and others. This will help students understand the implications of how quantities are changing in our closed formulas. This is a useful understanding in its own right, and a precursor to a deep understanding and appreciation of calculus, should the students study that at a later point in time.

Recursive thinking and problems in contexts of a recursive nature provide excellent opportunities for this at an elementary level. As an introduction to more advanced mathematics, the students' experience of writing recursive equations to model a situation themselves and exploration of the closed forms that represent the same situations, help to increase students' mathematical self-efficacy. Introducing recursive thinking early in students' educational careers also makes the introduction of the powerful tool of mathematical induction seem more natural and easier to comprehend, as suggested by Harel (2001).

In summary, we are suggesting that recursive reasoning should permeate the middle and secondary math curriculum, and not be taught all at once, or omitted. Models similar to the garden example can be introduced in middle school. Problems similar to the handshake problem and the caffeine problem could be introduced in late middle school and continued through early secondary school. The antihistamine problems could continue in later secondary school. We suggest that a little recursive thinking integrated over several years of mathematics learning may lead to a deeper and more robust mathematical understanding for our students. It will not take too much time when taught in this manner, and in fact may save time because of students' better understanding of topics in the current curriculum: Students will gain from this second, recursive, look at topics that many currently understand at only a superficial level. Used in an integrated fashion as we have described here, the study of recursion can enhance and deepen students' algebraic understanding.

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Part IV
Networks and Graphs

Food Webs, Competition Graphs, and a 60-Year Old Unsolved Problem

Margaret (Midge) Cozzens and Pratik Koirala

Abstract Food webs describe the flow of energy through an ecosystem. Understanding food webs can help to predict how important any given species is and how ecosystems change with the addition of a new species or the removal of an existing species. This paper indicates how teachers can challenge students to solve real world problems and, in the process, provides ways to model food webs with directed graphs and model competition by creating competition graphs. It has students consider a long standing conjecture that competition graphs derived from real food webs are interval graphs. The last section considers this sixty-year old unsolved problem and introduces the weighted model of a food web to better understand competition in an ecosystem.

Keywords Food webs · Competition graph · Predator-prey · Interval graph

1 Background

Food webs, through both direct and indirect interactions, describe the flow of energy through an ecosystem, moving from one organism to another. Understanding food webs can help to predict how important any given species is, and how ecosystems change with the addition of a new species or the removal of an existing species.

The study of food webs has occurred over the last sixty plus years, primarily undertaken by ecologists working in natural habitats. Mathematicians subsequently became interested in the graph- theoretical properties of food webs and their corresponding competition graphs. This paper provides a research problem that high school

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students can work to solve by creating (1) **discrete models** (graphs), (2) **creating hypotheses**, (3) **experimentation**, (4) **coming up with alternative hypotheses**, and (5) **repeating the process**. In this way, mathematics looks more like an experimental science, and discrete mathematical techniques allow this to happen.

The first section of this paper introduces graph models: food webs, together with associated directed graphs and parameters that play an important role in linking mathematics and ecology; competition graphs related to food webs; and interval competition graphs. The second section provides ways of generating reasonable hypotheses and experimenting to see if these hypotheses are true. The last section indicates how to generate an alternative hypothesis with an example.

Teacher and student notes and questions are indicated in boxes throughout the paper. The goals of this material for teachers and students are to:

- Recognize various relationships between organisms.
- Look for patterns across food webs.
- Use graphs to model complex trophic relationships.
- Make conjectures and hypotheses.
- Test conjectures and create alternative hypotheses.

2 Modeling with Directed Graphs and Graphs

2.1 Predator Prey Relationships Modeled with Directed Graphs

A food web is represented by a directed graph (digraph) $D = (V, A)$ with vertex set V and arc set A. Each vertex represents a species in the ecosystem and the arc (x, y) is directed from a prey species *x* to a predator *y* of that prey. Food webs by their very nature are directed graphs with no cycles, called **acyclic** (i.e. (a, b), and (b, c) and (c, a) is not allowed). No cannibalism implies that food webs contain no loops, no (a, a) arcs. A **basal species** is one that does not depend on any other organism in the ecosystem for food. That is, these are species usually located at the bottom of the food web.

> Have students draw a food web to model the following predator prey relationships: Sharks eat sea otters who eat sea urchins and large crabs. Sea urchins eat kelp and large crabs eat small fishes, who in turn eat kelp. Questions: What are the basal species? Is there a species that is not preyed upon, sometimes called a dominant species?

Figure 1 depicts a food web in which sharks eat sea otters, sea otters eat sea urchins and large crabs, large crabs eat small fishes, and sea urchins and small fishes

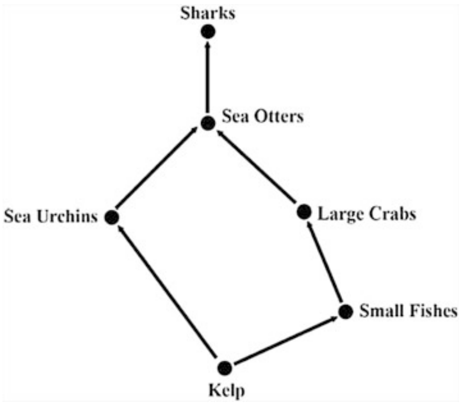

Fig. 1 Small food web

eat kelp. Equivalently, sea urchins and large crabs are eaten by sea otters (both are prey for sea otters) and sea otters are prey for sharks. Kelp is the only basal species, whereas sharks are at the top of this food web with no prey, no outgoing arcs.

In the early 1960s, when food webs were first used to model predator-prey relationships, arcs were directed from predator to prey. The current usage, which tracks the flow of energy from prey to predator, reverses this earlier convention. The interactions of species as they attempt to acquire food determine much of the structure of a community. Food webs represent these feeding relationships within a community. Basal species correspond to vertices with no incoming arcs: vertices with **indegree 0**. Species at the top of the food web correspond to vertices with **outdegree 0**. The digraph of a food web contains no directed cycles (since a species does not prey upon itself, either directly or indirectly). A larger food web is shown in Fig. 2.

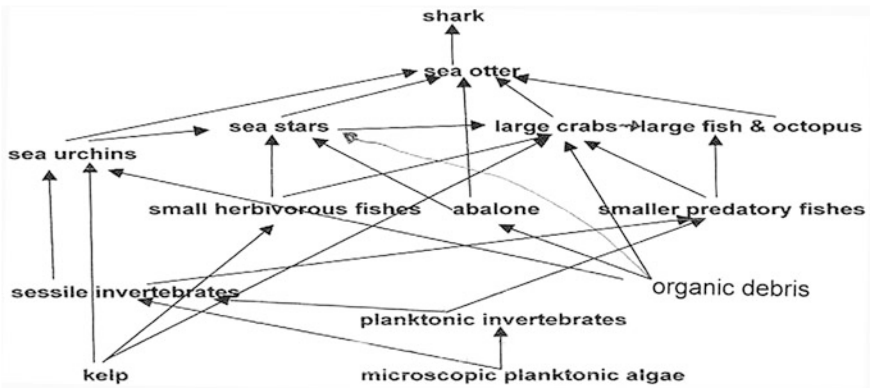

Fig. 2 Large shark food web

Question 1: *What are the basal species?* Kelp, organic debris, and microscopic planktonic algae, each of which has indegree 0.

Question 2: *What species are at the top of the food web?* Sharks are the only species at the top of this food web with no outgoing arcs, thus sharks have outdegree 0.

Question 3: *Find a path of length four in the food web: (i.e. a species is eaten by another, is eaten by another, and in turn by another, or a path with four arcs on it.)* One example is: kelp → sea urchins → sea stars → sea otter → shark.

Scientists use trophic levels in food webs to provide a way of organizing species in a community food web into feeding groups. They have used various measures to classify species in a food web into these various feeding groups, typically based on the positioning of species in the food web (the number of arcs between species). The food web shown in Fig. 1 is drawn indicating the relative positioning of the various species, whereas the food web drawn in Fig. 2 is not that precise.

In addition, not all ecological relationships have the same strength. Species may consume much more of one prey species than another. To model this, we can assign weights to the arcs of a food web to indicate food preferences, or the percentage of a species' diet that comes from a particular prey. This weighted model of food web will be described in detail in the next section. Also, for more on this topic see Cozzens (2011) and Cozzens et al. (2015).

2.2 Modeling with Competition Graphs and Interval Graphs

There has been considerable attention paid lately to creating graphical models for better understanding predator-prey relationships, especially to inform conservation policy makers. This section introduces several undirected graphs and parameters that are useful in understanding the competition of species.

Have students talk about how they could model competition given they had a particular food web—describe who competes with who for food. Try drawing a graph depicting the competition among species in Fig. 1.

Suppose a food web is represented by the directed acyclic graph D with n vertices and m arcs. The **competition graph** associated with D is an undirected graph G whose vertices are the species in D. There is an edge in G between species a and species b if and only if a and b have a common prey: i.e., there is some vertex x such that there exist arcs (x, a) and (x, b) in D.

Suppose we have a graph G, can we find a directed graph (food web) associated with G? For example, suppose we have the triangle in Fig. 3a.

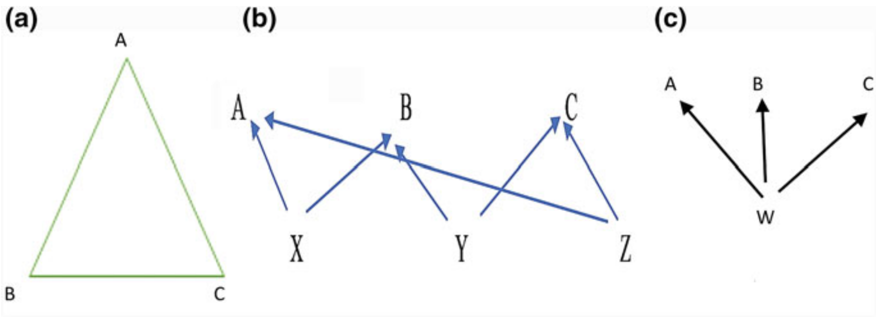

Fig. 3 **a** Triangle graph AB. **b** Option 1. **c** Option 2

> Question 4: Can you find a directed graph whose competition graph is the triangle ABC in Fig. 3a?

One option is that there is a species X such that both A and B eat X, and a species Y such that both B and C eat Y and a species Z such that both A and C eat Z represented in Fig. 3b. Option 2 is that A, B, and C eat species W, as indicated in Fig. 3c.

> Question 5: Are there other possibilities?

For a graph G, the **competition number** $k(G)$ of G is the fewest number of isolated vertices that need to be added to G so that G is the competition graph for some acyclic directed graph. In our example of a triangle, for option 1, we had to add 3 vertices X, Y, and Z to G to make it the competition graph of the food web, but in option 2 we only needed one, so $k(G) = 1$.

> Are you sure you need any at all?

Competition graphs are also known as niche overlap graphs and predator graphs, while common enemy graphs are also known as prey graphs. If D is a directed acyclic graph, then there must exist an isolated vertex in its corresponding competition graph G. One example would be a vertex having no incoming arcs in D. (Every directed acyclic graph contains at least one vertex of indegree 0.) In the case of food webs, these vertices represent the basal species. An interesting note is that

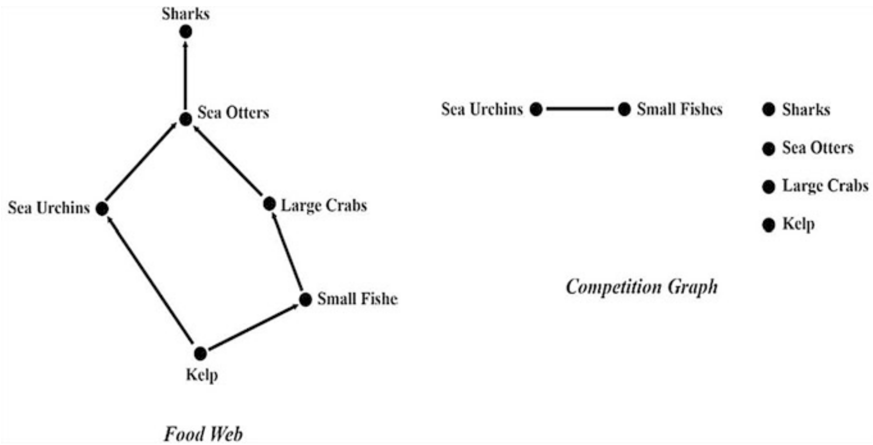

Fig. 4 A food web and its corresponding competition graph

any graph G can be the competition graph for some directed acyclic graph D by adding a sufficient number of isolated vertices to G.

> *Draw a competition graph for the food web of* Fig. 1. *Who competes with whom?* Sea urchins and small fishes both have kelp as a common prey, so (sea urchins, small fishes) is an edge of the competition graph. This simple competition graph, has one edge and four isolated vertices, as shown in Fig. 4.

So far, we have constructed models of food webs and competition using graphs and directed graphs. We will next describe a specific type of graph as an interval graph and we give some characterizations of interval graphs.

2.3 Models that Are Interval Graphs

A graph is an **interval graph** if we can find a set of intervals on the real line so that each vertex is assigned an interval and two vertices are joined by an edge if and only if their corresponding intervals overlap.

If G is the competition graph corresponding to a real community food web and G is an interval graph, then ecologists believe that the species in the food web have one-dimensional habitats or niches. That is, each species can be mapped to the real line with overlapping intervals if they have common prey, and this single dimension applies to each species in the web. This single dimension might be determined by temperature, moisture, pH, or a number of other factors. An example is shown in

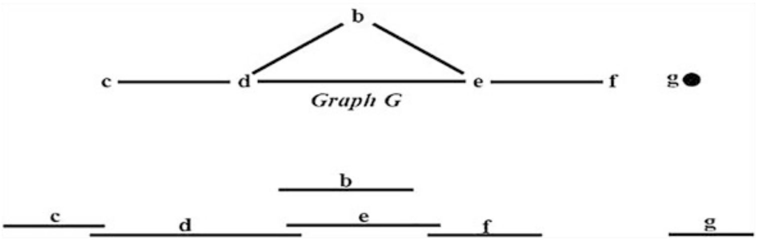

Fig. 5 A set of intervals, which correspond to graph G

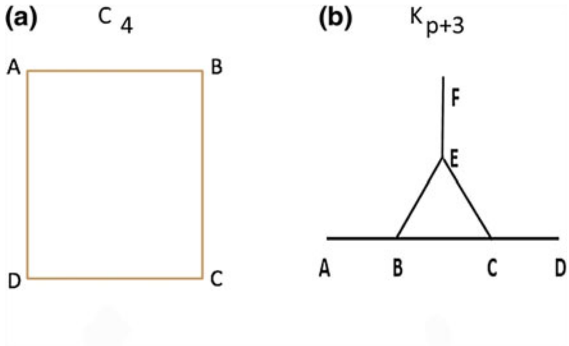

Fig. 6 Two forbidden subgraphs of interval graphs

Fig. 5 in which the intervals in Fig. 5 overlap if and only if they correspond to an edge in G. Therefore, G is an interval graph.

> Question 6: Give an example of an interval graph with 5 vertices and 6 edges.
>
> Question 7: Is the square, C_4 (left graph in Fig. 6) an interval graph?

2.4 *Characterizing Interval Graphs*

In the 1960s, Joel Cohen found that food webs arising from *single habitat ecosystems* (homogeneous ecosystems) generally have competition graphs that are interval graphs. In order to study this relationship between food webs and interval graphs arising from competition graphs, we need ways to easily tell if a graph is an interval graph. In the following, we describe several ways in which this has been done.

There are a number of characterizations of interval graphs, including a forbidden subgraph characterization: A cycle C_n is a graph with n number of edges connected in a closed chain. C_4 shown in Fig. 6a is a cycle with 4 edges. Formally, a cycle in a graph can be identified by finding a sequence of vertices v_1, v_2, … v_n, v_1 such that $\{v_i, v_{i+1}\}$ and $\{v_n, v_1\}$ are edges and $\{v_i, v_j\}$ are not edges for all $j \neq i + 1$. Graphs with no cycles of 4 or more vertices are called **chordal graphs** (Gilmore and Hoffman 1964; Lekkerkerker and Boland 1962).

An independent set of three vertices such that each pair is joined by a path that avoids the vertices connected to the third, such as vertices A, F and D in K_{3+p} in Fig. 6b, is called an **asteroidal triple**. It has been shown that a graph is an interval graph if and only if it does not contain a subgraph that is either cycle with 4 or more vertices or an asteroidal triple: the *forbidden* structures. Therefore, we can say that a graph is an interval graph if it is chordal and contains no asteroidal triple (Cozzens 2015).

A second characterization of interval graphs relates to maximal cliques. A clique of the graph G is a subgraph of G in which every pair of distinct vertices are connected by an edge in the subgraph. We call graphs complete graphs on n vertices if every pair of the n vertices in the graph is connected by an edge. For example, a triangle graph is a clique, as is a square with its diagonals. A **maximal clique** is a clique (all possible edges in the subgraph) that cannot be extended by including additional vertices. The graph K_{3+p} in Fig. 6b contains a maximal clique of size 3, the subgraph triangle BCE and three maximal cliques with two vertices each, edges AB, CD, and EF. So K_{3+p} has one maximal clique on three vertices and three maximal cliques on two vertices.

Maximal Clique Characterization of Interval Graphs: A graph is an interval graph if and only if one can order the maximal cliques C_1, C_2, …, C_k such that $C_i \cap C_{i+1} \neq \Phi$ (Fulkerson and Gross 1965).

Question 8: Identify the maximal cliques in the graph in Fig. 5. Do they satisfy the maximal clique characterization of interval graphs?

Question 9: Apply the Maximal Clique characterization to K_{3+p}.

If we try to order the maximal cliques, and start with AB, proceed to BCE then CD, there is no placement of EF. Similarly, if we order them as AB, BCE and EF then there is no place for CD. We have a further indication that K_{3+p}. is not an interval graph.

2.5 *Transitive Orientation*

An undirected graph G has a transitive orientation if its edges can be oriented in such a way that if edges (x, y) and (y, z) exist in the directed graph then there is an edge (x, z) in the directed graph. For example, Fig. 7 illustrates a transitive orientation of C_4.

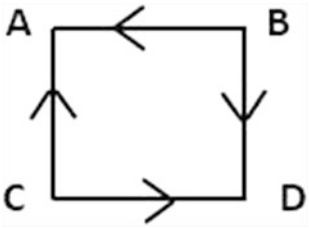

Fig. 7 Transitive orientation of the C_4 graph in Fig. 6

Question 10: Are there any other transitive orientations of C_4?

Question 11: Can you find a transitive orientation of K_{3+p}?

You should have discovered that K_{3+p} in Fig. 6 with an asteroidal triple doesn't have a transitive orientation. However, the other forbidden subgraph (C_4) in Fig. 6 does have a transitive orientation.

The complement of Graph G, call it G′ has the same vertex set as G and (a, b) is an edge in G′ if and only if it is not an edge in G. Analyzing 20 real food webs from an online food web database (globalwebdb.com), we see that the corresponding competition graphs and their complements were found to have transitive orientations.

An alternative characterization of interval graphs is that a graph G is an interval graph if it contains no cycles size 4 or greater **and** there is a transitive orientation of its complement G′. For example, K_{3+p} contains no cycle of size 4 or greater, but there is no transitive orientation of its complement.

2.6 *Consecutive Ones Property*

Recall that a clique is a complete subgraph and a maximal clique is clique where no more vertices can be included. A clique incidence matrix is a binary matrix where rows are maximal cliques and columns are vertices of a graph. Presence of a vertex in a clique is represented by 1 in the respective entry of the matrix, and 0 for absence. Consider the graph in Fig. 8 and its corresponding clique incidence matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that the columns of this incidence matrix consists of the vertices 1 through 12, listed in order. You can now read off the cliques from each row. For example,

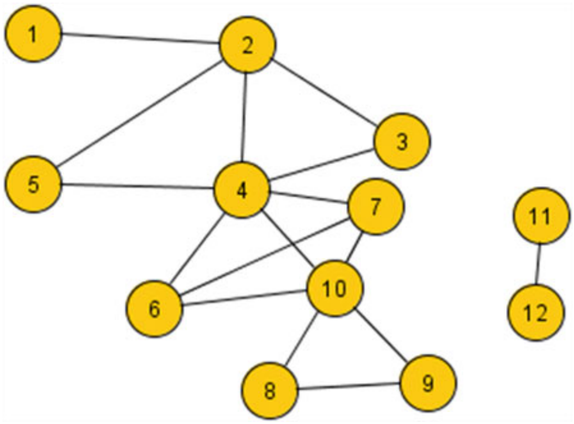

Fig. 8 Graph G with 6 cliques and 12 vertices

the second row consists of 1's in columns 2, 3, and 4, corresponding to the triangle formed by 2, 3, and 4.

A clique incidence matrix has a *consecutive ones* property if the rows of the matrix can be organized to get consecutive 1's in each column as indicated in matrix M. Note that matrix M is the clique incidence matrix of graph G depicted in Fig. 8.

Question 12: Construct the clique incidence matrix for G. Does it have the consecutive ones property?

A graph is an interval graph if its clique incidence matrix has the consecutive ones property, in that if its cliques can be represented overlapping along a real line, then its vertices can as well.

Question 13: Give the clique incidence matrix for K_{3+p}. Try to order the cliques so that the matrix has the consecutive ones property. Hint: You can't do it.

The competition graphs of the 20 food webs obtained from an online food web database (globalwebdb.com) showed the consecutive ones property, proving that they were in fact interval graphs. In a real food web, the competition graph has cliques with only a few vertices. This is the reason why the clique incidence matrix has a lot of 0 entries in the rows. This further explains the opportunistic nature of the species in a community.

Big Question: Are all competition graphs of real food webs interval graphs?

3 How Do We Think About (Open) Questions?

The remarkable empirical observation of Cohen that real-world competition graphs are usually interval graphs, has led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. It has also led to a great deal of research in ecology to determine just why this might be the case (Cohen 1978). Using randomly generated food webs (digraphs), Cohen et al. showed that the probability that a competition graph is an interval graph goes to 0 as the number of species increases. In other words, it should be highly unlikely that competition graphs corresponding to food webs are interval graphs (Cohen et al. 1979).

3.1 Generate a Hypothesis

For example: All competition graphs of real world food webs are interval graphs. What are ways to test this hypothesis?

One way is to try and characterize the acyclic directed graphs whose corresponding competition graphs are interval graphs? Once we have a characterization we can then see if it *fits* all food webs. *Is there such a characterization?* has been a fundamental open question in applied graph theory since the 1960s. Indeed, no one has found a forbidden list of directed graphs (finite or infinite) such that when these directed graphs are excluded, one automatically has a competition graph that is an interval graph. A second way is to look at the hypothesis from the ecology vantage point: what are the ecological characteristics of food webs that seem to lead to interval competition graphs? This latter question has puzzled ecologists and conservation biologists as well. As Cohen indicated, most (random) acyclic directed graphs do not have corresponding interval competition graphs. A third way is to find large classes of acyclic directed graphs that have corresponding interval competition graphs and hope that these classes cover all of the food webs. Indeed, researchers have tried to find classes of acyclic directed graphs with interval competition graphs, but these classes are very limited (Cho and Kim 2005). As it turns out understanding a little graph theory and the ecology of predator prey relationships is at the crux of solving both open problems. Both the relevant discrete mathematics and ecology are accessible to high school students.

Where do we start to try and find a characterization of acyclic directed graphs (food webs) whose corresponding competition graph is an interval graph? One way to start is to look at what kinds of things are forbidden for a graph to be an interval graphs, and back each type of graph up to a directed graph (a food web). For example, to get a forbidden C_4 in the competition graph corresponding to an acyclic diagraph (food web), you would need an acyclic subdigraph that looks like the one in Fig. 9. A and B have a common prey, B and C have a common prey, C and D have a common prey, and D and A have a common prey, but A and C have no

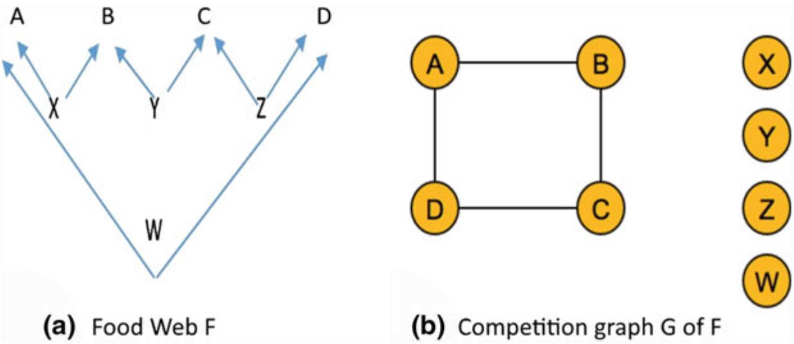

Fig. 9 A directed graph F (food web) with a competition graph G that contains a C_4

common prey and B and D have no common prey. The resulting digraph is acyclic. F is then a forbidden subdigraph for food webs when considering interval competition graphs.

3.2 Test the Hypothesis

Note that the above approach is not a test of the hypothesis, but an attempt at a characterization of food webs with interval competition graphs. We would then see if any food webs had D as a subgraph. That, however, would require knowing all food webs.

In addition to pursuing this graph theoretical approach let's look at a second approach, one from an ecological standpoint. Consider a predator's territory (home range) and potential results of competition for the same prey (Dobson 2009; May 2009). An interesting example to consider is the community that includes predators: cougar (mountain lion), grizzly bear (brown bear), and grey wolves. Each has a common prey of deer (white tail and mule deer). Therefore, the competition graph component containing these three species is a triangle and there is an isolated vertex deer as shown in Fig. 10.

A logical ecological question, would be within that territory are there other predators A, F, and D such that A competes with cougars and not grey wolves and not grizzly bears; F competes with cougars and grizzly bears and not grey wolves, and D competes with grizzly bears and wolves and not cougars? If this were the case then each of the vertices in the triangle would have a pendant and look like K_{3+p}, shown in Fig. 6, a forbidden subgraph for interval graphs. Using the example of the community which includes wolves, cougars, grizzly bears and coyotes, coyotes compete with mountain lions and grey wolves but not grizzly bears for elk

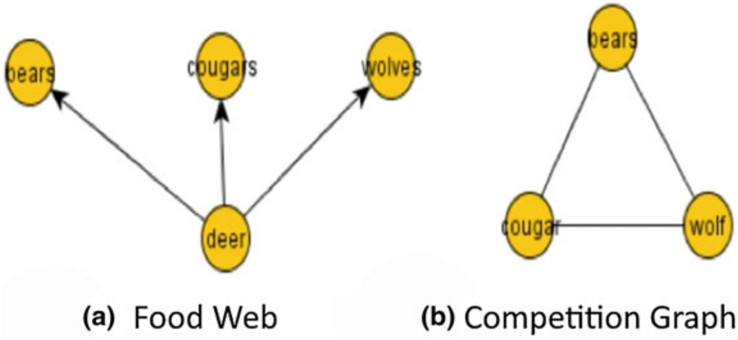

Fig. 10 An example of a part of a food web containing cougars and the corresponding competition graph

or anything else. Thus, there is an additional triangle of coyote, cougar, and grey wolf attached to the original triangle in the competition graph, a C_4 with diagonal.

Let's consider the clique characterization of interval graphs. Real food webs have very few maximal cliques in their corresponding competition graphs (Yodzis 1989). These constraints on the number of maximal competition cliques in a habitat or community in turn may account for the presence of interval competition graphs detected by Cohen and others. Perhaps these elements of internal structure represent "small, functionally coevolved guilds or component communities," which Colwell (1979) suggested.

Predators are first and foremost opportunistic. Large obligate carnivores will choose rabbits over deer if the rabbits are plentiful and other larger species like deer are not. Does this make a difference?

Possible suggestions for thinking about why competition graphs are generally interval graphs: (1) When there are four species, three of which compete for a specific prey, the fourth predator competes with at least two of them, not just one. (2) When species A competes with B and B competes with C then does A compete with C, a transitivity condition? It is hard to test as general a hypothesis as stated here, one that requires knowing all food webs. As we will see in the next section, it is more likely one finds a counterexample by accident.

4 Alternative Hypothesis—Use a Weighted Model Food Web

Why might we want to use a weighted model food web?

Like we mentioned earlier, not all ecological relationships have the same strength. By adding weights on the edges of the , the resulting competition graph may better model the competition in the community. A weight of an edge represents the percentage of predator's diet. So, we have an added complexity in the definition of a competition graph. Two species have an edge in the competition graph if they share a common prey and their edges from the common prey in the food web have weights greater than a threshold value. The threshold value, essentially determines if there is a competition between two species. The need to use a threshold value in order to determine if there is a competition between species comes from a simple example. If there are 10 species in an ecosystem and 5% of their diet contains dead organic matter then the species aren't really competing for the dead organic matters. So, putting edges between the species in their competition graph would be an inaccurate model of the competition in the ecosystem.

> Draw the competition graph for the Yellowstone wolf food web. Is it an interval graph, if not why not?

The competition graph for the food web in Fig. 11 is shown in Fig. 12. This graph is not an interval graph as the triangle coyote-bighorn sheep-elk has three pendant edges one to wolf from coyote, one from bighorn sheep to deer mouse, and one to beaver from elk, thus forming a K_{3+p}, a forbidden subgraph for interval graphs.

Alternative Hypothesis: *If predator A consumes only a small amount of a prey B then exclude the arc from B to A in the food web and construct the corresponding competition graph for the new food web.*

The wolf food web shown in Fig. 11 is an interesting food web of the predator prey relationships for wolves in Yellowstone National Park. As indicated in the last section, the competition graph of this food web has a forbidden subgraph K_{3+p} as shown in Fig. 12. The vertices joined by the bold edges and a dotted edge forms an asteroidal triple as shown in Fig. 6, which is a forbidden subgraph of interval graph.

Now we test our alternative hypothesis using a weighted model of the Yellowstone wolf food web and compute its competition graph. To do this, we approximate the percentage of the diet consumed of each species by each species. While it is impossible to know these percentages exactly, an ecologist working in Yellowstone gave the following estimates:

- Wolf eats 0.3 bison, 0.5 elk, 0.1 bighorn sheep and 0.1 beaver
- Bear eats 0.03 pronghorn, elk 0.05, deer mouse 0.02, pine 0.3 moths 0.2, berries 0.4
- Coyote eats pronghorn 0.3, elk 0.4, deer mouse 0.2, grasses 0.1
- Big horn sheep eats grasses 0.9 and willow 0.1
- Beaver eats pond lily.5 and pine 0.5

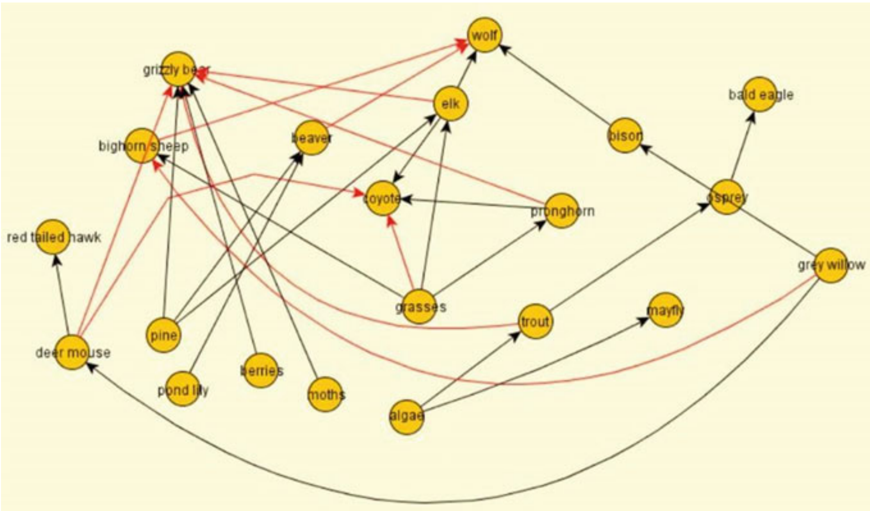

Fig. 11 The directed graph representing the wolf food web in the Yellowstone National Park with 21 species (it represents the predator prey relationships for wolves and their prey and competitors. It does not include species who do not interact with wolves or their competitors or prey)

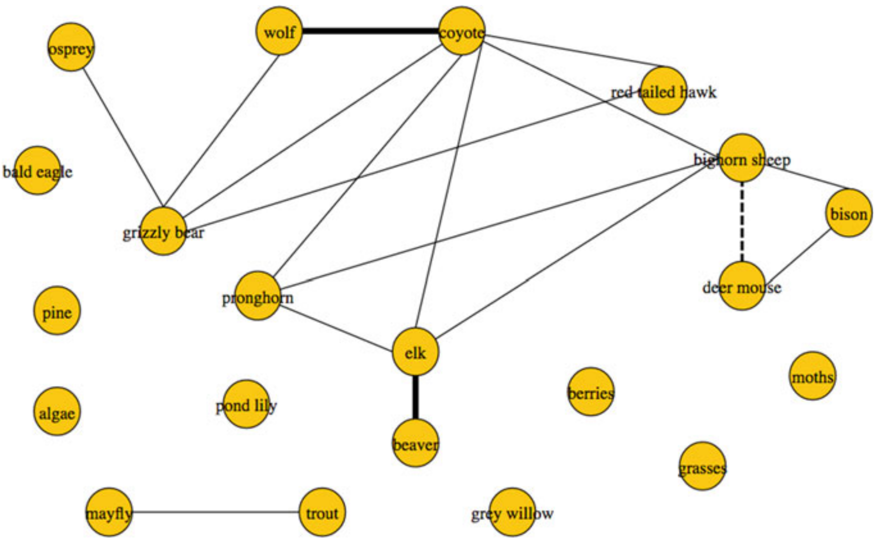

Fig. 12 This undirected graph represents the competition graph of the wolf food web in the Yellowstone National Park

- Elk eats grasses 0.4, pond lily 0.3, pine 0.3
- For example, 30% of the wolf diet is bison, 50% elk, 10% bighorn sheep, and 10% beaver

Label the edges of the food web in Fig. 11 with these weights. Now set a threshold of 10% and eliminate all arcs in the food web with weights less than or equal to 0.1. Redo the competition graph with these arcs removed.

Setting a threshold above 10% means the edge between big horn sheep and deer mouse in the competition graphs, the red edge in Fig. 12, is eliminated, as is the K_{3+p}. Now the competition graph of the weighted wolf food web in Yellowstone is an interval graph.

In summary, food webs and their corresponding competition graphs are easy to understand, and their characteristics evoke many questions that are also easily understood even if not easily answered. A remaining question raised by the Yellowstone National Park wolf food web is whether there is a way of setting threshold values that insures all resulting competition graphs will be interval graphs and can we prove that is the case?

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Graph Theory in Primary, Middle, and High School

Daniela Ferrarello and Maria Flavia Mammana

Abstract In this paper we present an experimental teaching activity conduced in some primary, middle and high schools in Sicily. The activity concerned several topics of graph theory. Here we highlight, in particular, the approach to teaching Eulerian graphs. The aim of the whole project was to present a fun, easy approach to mathematics in order to promote a good attitude towards mathematics in primary school children and to improve it in middle school kids and in high school young people. This goal is pursued also by showing some connections of mathematics with real life, making mathematics less abstract than the topics too often taught in school. Through this activity we also reach mathematical knowledge and practical abilities (related to graph theory), and above all mathematical competencies related to reasoning and mathematization, in particular by the use of graphs in mathematical models to solve problems. The teaching experiments were different, according to the different school level, but unified by the method, based on laboratorial activities, by presenting a problem to be solved together with classmates, by manipulating objects and guided by the teacher. These activities were realized by the use of artefacts: in the sense of Vygotskijan semiotic mediation, we used signs, symbols, maps, language and, in many cases, new technology's artefacts, to mediate mathematical concepts. Lessons involved also the body as a mean to learning, especially with children, according to embodied cognition theory.

Keywords Graph theory · Mathematical modelling · Eulerian trail
Eulerian cycle

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1 Introduction

The capability of using mathematical knowledge in solving real life problems is widely tested in various national and international tests (INVALSI, PISA, TIMSS) and, since last year, in the Italian final mathematics' exam, at the end of high school. But, according to the results obtained, students have difficulties in solving these kinds of problems.

Mathematics is more often seen as the *subject of numbers and rules* that is hard to understand and difficult to study: lots of students see it as something of no use and cannot see the connection with everyday life. The study of *real life problems* is becoming central in teaching at all levels:

> The National Council of Teachers of Mathematics (NCTM) is providing leadership in communicating to teachers, students, and parents what mathematical modeling looks like in K–12 levels. The 2015 Focus issue of NCTM's Mathematics Teaching in the Middle School was about mathematical modeling and the 2016 Annual Perspectives in Mathematics Education also focused on the topic (Levy 2015).

The idea is, given a real life problem, to translate it into a math problem, solve it with mathematical knowledge, and interpret the solution in terms of the given problem.

In this context graph theory is a good tool for modeling problems. Even if it is a quite new branch of mathematics (it was born at the end of 1700), over the years it has acquired a leading role for its use for applications in areas such as transportation, telecommunications, science experiments …

Graph theory presents, in mathematics education, several advantages: it permits students to *see applications of mathematics*, to start some argumentation and it boosts reasoning.

Graph theory is easy to understand, fun to use and intriguing to use in modeling real situations. It can be used to develop "a suitable vision of mathematics, not reduced to a set of rules to be memorized and applied, but recognized as a framework to address significant questions, to explore and perceive relationships and structures recurring in nature and in the creations of mankind" (MIUR 2012).

With this respect, in this paper we discuss an approach to graph theory that we carried out in primary, middle and high schools over the past 12 years. Countries other than Italy have a structured approach to Discrete mathematics (including graph theory, counting methods, recursion, iteration, induction, and algorithms), see DeBellis and Rosenstein (2004), Rosenstein (2014). The introduction of Discrete mathematics in school is, in fact, recommended by the National Council of Teachers of Mathematics (NCTM) in the USA since 1989, while in Italy it is not explicitly asked for in the curricula.

2 Königsberg Bridges Problem

In 1736 the Academy of Sciences in Petersburg published an article in which Leonard Euler solves the problem of the *bridges of Königsberg*: The city of Königsberg in Prussia (now Kaliningrad, Russia), crossed by the Pregel River, includes two large

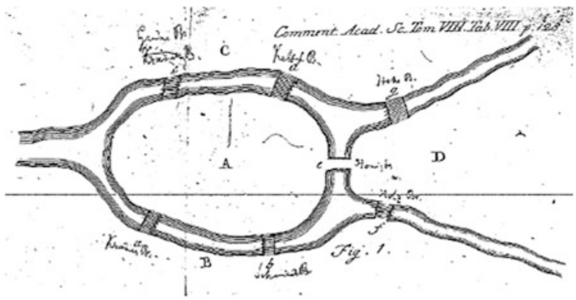

Fig. 1 Figure retrieved from Euler (1741)

islands. The islands and the mainland are connected to each other by seven bridges (Fig. 1). In 1700, the citizens of Königsberg used to have a nice walk around the city and wondered if they could find a walk that would cross each bridge once and only once, with the condition (optional) that they would be back to the same point they started.

Euler proposes to denote by A, B, C and D the four parts of the city, A and D are the islands and B and C the mainland (Fig. 1). He denotes a bridge between A and B by AB, one between B and D by BD: so, if a citizen starts from A, moves to B and then to D we can denote his/her walk by ABD. If the citizen then moves from D to C the whole walk would be ABDC. Then, the *word* ABDC states which walk the citizen did and how many bridges he crossed: three bridges in this case, one less than the number of letters, in general. And vice versa, if the citizen crosses n bridges then the number of letters of the word describing his/her walk is $n+1$. Then, the word solving the problem should have 8 letters (among A, B, C, D) corresponding to the 7 bridges. Solving the problem means to find the *correct* word.

Euler showed that this problem cannot be solved by essentially using the following reasoning. If the region A is connected to another area with only one bridge, then the correct word contains A only once, no matter if the walk starts in A or not. If the region A is connected with another area with three (five, seven…) bridges, then the correct word contains A twice (respectively three times, four times…), no matter if the walk starts in A or not. In general, if an area is connected to other areas with an odd number of bridges, say n, then the letter associated to that area appears exactly $(n+1)/2$ times. Then, in the specific case of Königsberg, the letter A must appear three times, the letter B must appear two times, so as letters C and D: so we should have a word with $3+2+2+2=9$ letters, other than the 8 letters that the correct word must have. The Königsberg citizen then cannot find the walk they were looking for. This problem was basically modelled with the aid of what has become known as a *graph* (Fig. 2).

A graph is a pair $G=(V,S)$, with V a finite set, whose elements are called vertices or nodes, and S is a set of pairs of elements of V, called edges with endpoints the vertices of the pair.

The Königsberg bridges graph (Fig. 2) is a graph with vertices A, B, C and D and edges represented with the drawn lines, that are $[A,B]_1$ $[A,B]_2$, $[A,C]_1$, $[A,C]_2$, $[A,D]$, $[B,D]$ and $[C,D]$. The problem of Königsberg's citizens can be seen as the problem of drawing the graph in Fig. 2 without lifting the pencil from the paper and without passing twice on the same line.

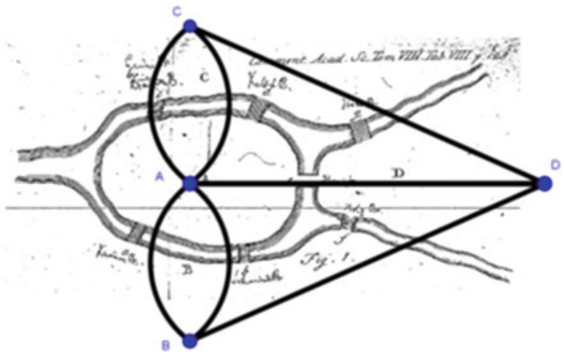

Fig. 2 Graph superimposed on map of Königsberg

3 Theoretical Framework

Several activities have been developed by the authors to introduce graph theory at all school levels.

The activities are realized in the perspective of horizontal teaching (Ferrarello et al. 2014) and with the use of technology. In horizontal teaching, the teacher enters the realm of the students' real life and offers activities fitting with the students' age, needs and the needs of the whole class. Technology helps to model situations by using "ad hoc" applications.

The activities on graphs we carried out are all based on a common theoretical framework: semiotic mediation in a Vygotskijan perspective supporting the activity of construction of mathematical concepts, through laboratorial activities, based in embodied cognition theory.

Mathematical content is mediated by the use of artefacts, in the Vygotskijan sense (Vygotskij 1981, p. 137): writing, speaking, using mathematical symbols, maps, diagrams are all involved in the mediation process that transforms situated signs into mathematical signs whenever a task is given. Our students (from children to adults) used written signs such as words and figures to represent the situation asked for by the task, to conjecture a possible solution, to test it, and then they used words to argue about possible solutions and to communicate it to the others, and finally transformed situated signs into mathematical meanings (Bartolini Bussi and Mariotti 2008).

Graphs perfectly fit in the *embodied mind* framework (Lakoff and Johnson 1999). Graphs can be used not only drawn on paper, but also to be manipulated as real objects: real strings in primary school, for instance, but also constructed by using special software; in such a way that nodes can be dragged and edges can be warped, as they were real. Body and mind, as a whole, participate in the construction of mathematical meaning, by using grounding metaphors (Lakoff and Nunez 2000), strings as edges, for instance.

Manipulation of objects is one the four components of a laboratory activity (Anichini et al. 2004; Reggiani 2008): (1) A problem to be solved; (2) Objects to be manipulated; (3) Interaction with people; (4) Role of the teacher.

1. A problem to be solved is a task that has to be not too hard (in the non-competence area of Vygotskij) nor too easy (in the competence area of Vygostskij), but accessible: a problem that can be solved in the interaction with others (peers or teacher), i.e. in the Zone of Proximal Development;
2. Objects, real or virtual, are to be manipulated;
3. Interaction with people refers to collaboration with mates to solve problems and mathematical discussion among teacher and students to strengthen concepts.
4. Role of the teacher is as a trainer, guiding and encouraging students to discover, to argue, to conjecture, to prove.

In fact,

> we can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together […] to the communication and sharing of knowledge in the classroom, either working in small groups in a collaborative and cooperative way, or by using the methodological instrument of the mathematic discussion, conveniently lead by the teacher (Anichini et al. 2004).

All the activities are oriented toward a good approach to mathematics supported by a positive interaction of affect and thinking in the learning process, because "affect influences thinking, just as thinking influences affect" (Brown 2012, p. 186).

The same topic (graphs) is treated, in our activities, in very different contexts by using different levels, as suggested by Van Hiele (1986), from visualization and description to rational and logical, in the view of vertical curriculum. In primary school, students receive tools to be able "to represent relationships and data and, in meaningful situations, use representations to obtain information, to express judgments and make decisions"—goals to be reached at the end of the fifth grade, in the "Relations, data and forecasts" part (MIUR 2012). In addition, significant time is given to playing, as a means for the "development of strategies suitable to different contexts." In middle school more attention towards formalization, generalization, argumentation is given, in order to "develop the ability to communicate and discuss, to properly argue." In high-school simple proofs and algorithms can be introduced (MIUR 2012). For high-school students it is important to understand the concept of a mathematical model. Generally, in Italian schools, it is proposed as a connection between mathematics and physics (for example the physical concept of velocity through the mathematical concept of derivative). It should also be used as an approach to reasoning with the help of graphical representation and referring to real contexts, as other mathematical models.

4 Topics

In this paper we present some activities we carried out in recent years that have been developed. The topic is the same for primary, middle and high schools, Eulerian graphs,[1] but the approach changes depending on level (Van Hiele 1986). We briefly present the contents here in an intuitive way. For a rigorous approach refer to West (2001), Wilson (1996).

Recall that a *graph* is a pair $G = (V, S)$, V being a finite set whose elements are called vertices or nodes, and S is a set of pairs of elements of V, called edges with endpoints being the vertices of the pair.

A graph is *connected* if from each vertex you can always reach any other vertex through adjacent edges. In this paper we always deal with connected graphs.

The *degree* of a vertex is the number of edges for which that vertex is an endpoint. For example, the graph in Fig. 2 is connected and the degree of C is 3, the degree of A is 5.

SemiEulerian graphs are those that *you can draw without lifting the pen from the paper and passing through each edge exactly once*. The taken walk is called an Eulerian trail. Eulerian graphs are SemiEulerian graphs such that there exists a closed Eulerian trail, i.e. such that the first vertex and the last vertex are the same. Such trail is called an Eulerian cycle.

The Königsberg Bridges Problem, mentioned at the beginning of the paper, consists in finding an Eulerian cycle/trail in the graph in Fig. 2. Euler found a necessary and sufficient condition for the existence of an Eulerian trail/cycle in a given graph, referring to *words* (Euler 1741). Referring to graphs, there is an Eulerian cycle in a connected graph if and only if each vertex has even degree; there is an Eulerian trail in a connected graph if and only if there are at most two vertices of odd degree.

Fleury's algorithm produces an Eulerian cycle (trail) in an Eulerian graph. The algorithm works as follows: if the graph is connected and with all vertices of even degree (at most two of odd degree), choose any vertex (a vertex of odd degree, if any) as starting vertex and select successively adjacent edges choosing a bridge only if there is no other choice, where a bridge is an edge which, if removed, produces a disconnected graph.

5 Eulerian Graphs in Primary, Middle and High School

Several activities have been carried out in several schools, but in different years.

In 2007, for the first time, we tested a graph theory activity in middle school, 8th grade, (Mammana and Milone 2009a, b) and the following year, a different one in 6th grade (Mammana and Milone 2010). In 2012 we brought graphs in primary

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[1]Other graph theory topics have been proposed to students but are not presented here.

schools, 3rd, 4th and 5th grade. Then, a proposal for high-school was written (Aleo et al. 2009) and tested in 9th and 10th grade (Ferrarello and Mammana 2017).

The way of conducting the activity was the same for all levels: laboratorial activities (Chiappini 2007). We also used some technology. Specifically, we used technological artefacts to support teaching and learning at every level, and we employed both old and new technology, namely paper, pencils, strings, but also software and online games. The new technologies used were:

- *yEd*, graph editor (https://www.yworks.com/products/yed);
- *Cabri* (http://www.cabri.com/);
- *Icosien* (http://www.freewebarcade.com/game/icosien/);
- *Planarity* (http://planarity.net/);
- *Fly tangle* (http://www.giochigratisonline.it/giochi-online/giochi-puzzle/FlyTangle3/).

yEd is a free software developed to draw and manipulate graphs. With yEd we can import images or our own data from existing spreadsheets, easily create diagrams via an intuitive user interface, automatically (or manually) arrange our diagrams elements, and export images of the created graphs (see Fig. 3). yEd has been used in primary school and in high school, while in middle school we used Cabri geometre. We used Cabri just to draw graphs, not as a dynamic geometry system. Any other Dynamic geometry software can be used. Of course, the first mission of a DGS is not drawing graphs, but back then (2007) yEd was not developed yet.

Icosien is an online game. It is not an educational game, but we used it with didactic purpose. In fact, the aim of the game is to wrap a string around some nails to create the given shape in each level, by constructing Semieulerian and Hamiltonian graphs (graphs in which a trail can be constructed that uses every vertex exactly once, but not necessarily every edge) (see Fig. 4).

Planarity and *Fly-tangle* are games in which the player has to arrange the vertices of a graph so that none of the lines intersect, except at the vertices.

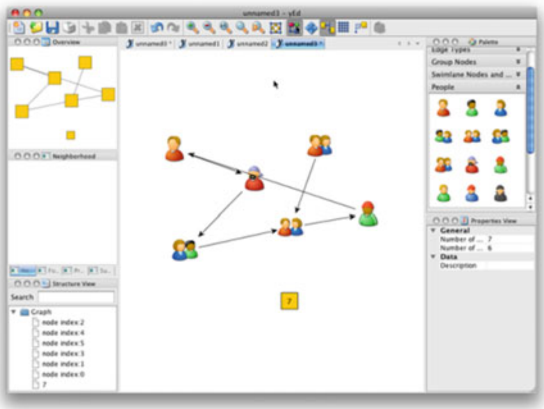


Fig. 3 Screen of a *yEd* developed graph

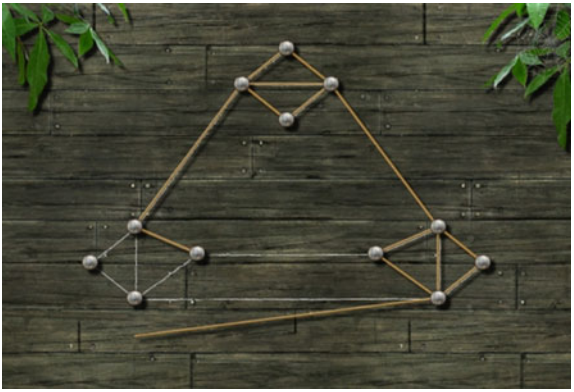


Fig. 4 Example of *Icosien* game

- Contents of the activities in primary school are: definition of graph, planar graph, vertex colouring, Eulerian graph, Hamiltonian graph (these activities lasted between 30 and 40 h).
- Contents of the activities in middle school are: definition of graph, graphs to model situations, Eulerian graphs (these activities lasted between 15 and 20 h).
- Contents of the activity in high school are: definition of graph, Eulerian graphs, Fleury algorithm, Spanning threes, Kruskal's algorithm, applications (these activities lasted about 20 h).

At all levels we proposed the Eulerian graph topic, that is the one we concentrate in this paper from this point forward.

In the activities we developed we had the following goals (in the following P stands for Primary, M for middle, H for High-school):

- recognize that a graph provides a possible modelling of a problem; (P, M, H)
- know how to go from a problem to its model as a graph; (P, M, H)
- recognize the essential elements of a graph; (P, M, H)
- identify similarities and differences between graphs; (P, M, H)
- recognize same graphs but with different representations; (P, M, H)
- know how to draw and represent a graph both with paper and pencil and by means of a suitable software; (P, M, H)
- formulate hypotheses on the characteristics of an Eulerian/SemiEulerian graph; (M, H)
- test formulated hypotheses on the characteristics of an Eulerian/SemiEulerian graph; (M, H)
- formulate conjectures, discussion supervised by teachers; (M, H)
- independently develop conjectures, argumentation; (H)
- compare hypotheses with classmates to achieve shared results; (P, M, H)

- use the condition of existence of a Semieulerian/Eulerian path for solving problems; (P, M, H)
- knowing how to apply an algorithm in order to find an Eulerian/Semieulerian path; (M, H)
- think about the fact that mathematical objects are "hidden" in various everyday situations. (P, M, H).

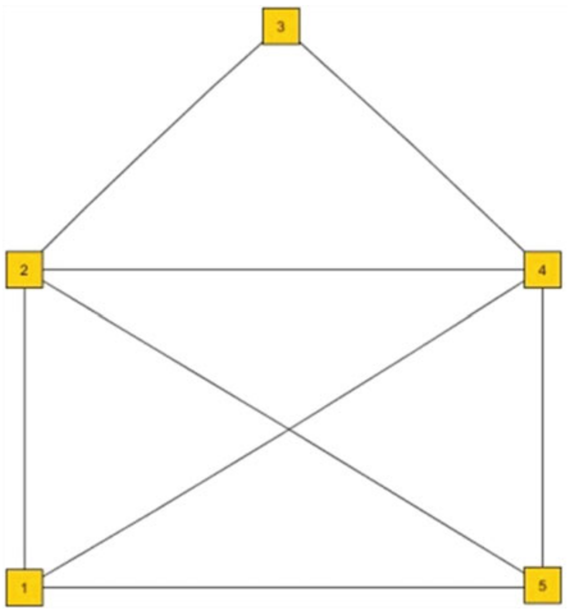
5.1 Eulerian Graphs in Primary School

Activities with 8 and 9 years old students were carried out (Ferrarello 2014, 2017). We aimed to make children enjoy the topic, rather than focus on mathematical competencies (that were achieved anyway). So the Eulerian activity was related to storytelling, playing with games, and a little of argumentation, as described below.

The first approach, using storytelling, was the problem of Königsberg seven bridges. Children tried to solve the problem by using a map of Königsberg, and then they were guided to construct the model of Fig. 2, by using *yEd*. We did not use standard definitions, but we called Eulerian and Semieulerian graphs *walkable* graphs, with the aim to recall in students' minds the activity to walk around the graph, by touching, just once, all the edges.

Together with the *impossible* graph of Königsberg, other *solvable* problems were given, as the classical *cabin* graph of Fig. 5.

Fig. 5 SemiEulerian cabin graph



In such a way pupils did not give up and carried on, trying and playing with solvable and non-solvable problems.

After nodes, edges and nodes degree were introduced, children were guided to notice similarities among walkable graphs: they discovered that such graphs had only nodes of even degree, we called them closed walkable, or just two nodes of odd degree, we called them open walkable. Students practiced with several graphs, with paper and pencil, by using the online game Icosien and with real string (see Fig. 6).

Activities with paper and pencil also included games of words and sentences, by using graphs whose nodes are letters and two letters are joined by an edge if you can use the two letters consecutively in a word. Semieulerian graphs hiding sentences were given: children found the right sentence by walking on each edge just once.

The use of the *Icosien game* was useful: it lets you know if you are wrong and pupils could try by themselves, even at home without the teacher. Pupils were happy to learn how to win the game, but they did not care at all about the motivation of why the strategy worked.

To argument about the reason behind the winning strategy we used the graph in Fig. 7, by orienting the edges in the direction of a *winning* path and by colouring every source (where the oriented edge starts) with a green chalk and every sink (when the oriented edges finish) with a red chalk. Then we focused on the first node of the winning path and reasoned on the number of going out and going in edges: two of them are used to pass through the path and one of them is used to go out at the beginning. Similarly, an even number of edges (two) are used in the last node to cross the trail and one is used to go in at the end of the trail. The other nodes are all used to cross the path, so they must have even degree. Similar considerations were given for Eulerian graphs.

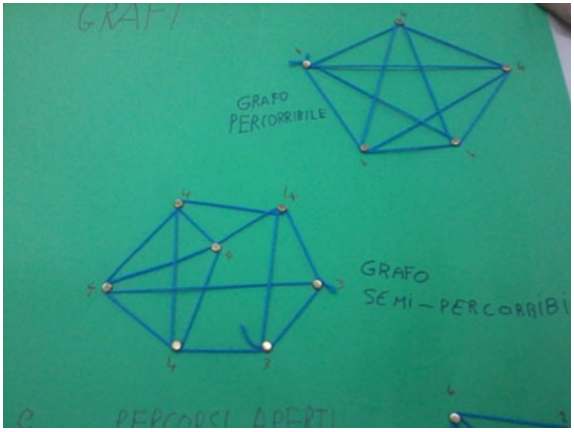


Fig. 6 Eulerian graphs made of string

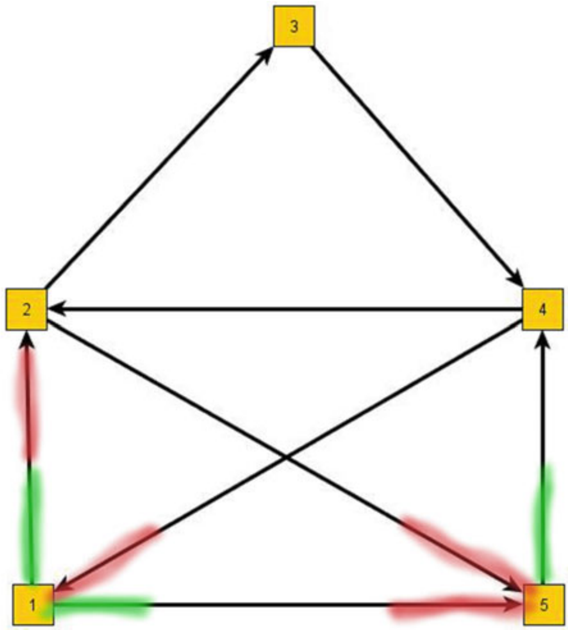


Fig. 7 Color-coding *in* and *out* parts of edges in a path

Table 1 Description of major components of Königsberg bridges activity

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Necessary and sufficient condition for the existence of Eulerian cycle/trail	Paper and pencil; Maps; Strings; Icosien	Collaboration among students; Mathematics discussion; Groups formed by students themselves: Each group had a "task": writers, drawers, thinkers	Prepare the activity; Coordinate the groups and the Math discussion

The whole phase of argumentation was led by the teacher, who stimulated pupils with questions, encouraging them to express their thoughts, making them think on their own actions and claims. And finally exulting for their good insights and reasoning.

In Table 1, the major components of the lab (described in Sect. 3), related to the activity, are described.

5.2 Eulerian Graphs in Middle School

Again, to arouse students' interest, they were immediately given the real problem situation of *The Königsberg bridges problem*, which they modelled using the graph that simplifies the analysis of the problem. They were then asked to solve the same problem on other graphs, obtained from the Königsberg graph by adding or deleting or moving an edge (Fig. 8). Through the identification of similarities and differences, students were lead to discover the conditions that must be satisfied in order to solve the problem, that is, so that there exists an Eulerian cycle.

This condition is then applied to polygons: precisely, students were asked to find polygons that are possible to draw, together with their diagonals, without lifting the pencil from the paper. At the end of this phase, the algorithm of Fleury is presented. For a better understanding of the algorithm and for catching students interest, the

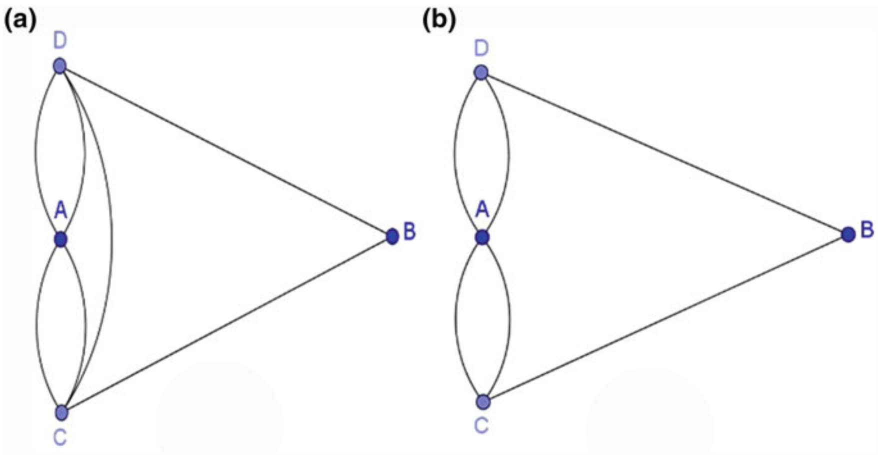
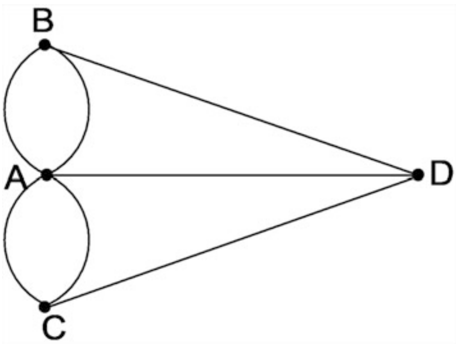


Fig. 8 Figure 2 with edges added and deleted

Fig. 9 Graph for problem 2 of worksheet 3



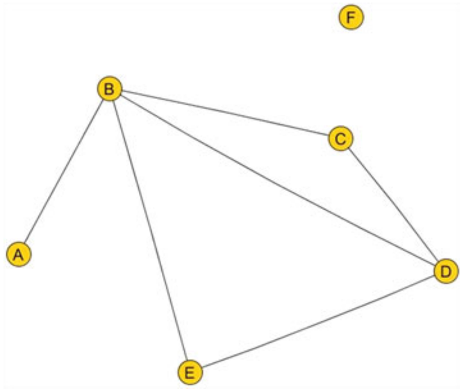


Fig. 10 Graph for high school activity

Table 2 Connections between various cities

	Florence	Pisa	Arezzo	Livorno	Lucca	Siena
Florence		YES		YES	YES	YES
Pisa	YES					YES
Arezzo				YES	YES	
Livorno	YES		YES			
Lucca	YES		YES			
Siena	YES	YES				

Cabri software was used—of course, Cabri was not used as a dynamic geometry software but for visualizing, step by step, the algorithm. They were also posed the problem of the existence of an Eulerian trail, and students were lead to discover the conditions that must be met for such a trail to exist. By using Cabri Geometre, students did find an Eulerian cycle/trail in several graphs. At the end of the whole activity students were asked to solve a problem (Table 2) that they had to model using a graph and to solve with graph theory tools.

Alice goes Florence, in Tuscany. She wants to visit some cities in Tuscany, Florence, Pisa, Arezzo, Livorno, Lucca, Siena, returning to Florence airport. Alice also wants to see as much scenery as possible along all the connections between the cities one and only one time because she does not have much time. Can you help Alice?

The table shows the existing connections between the various cities. (The empty boxes indicate that between the two cities there is not a direct link.)

Table 3 Worksheet 3 for Königsberg problem

WORKSHEET 3
(INDIVIDUAL WORK IN CLASS) Problem 3: It follows the graphs of Problem 1 (see Fig. 8a) and Problem 2 (see Fig. 9): For graph of Problem 1 you can find a solution and for graph of Problem 2 no. Why is that? What is the difference between the graphs? Try to give some answer: a) …………………………………………………………………………… …………………………………………………………… b) …………………………………………………………………………… ……………………………………………………………………………..

We have chosen to provide concepts through descriptions in natural language using gradually increasingly formalized language: in this way some linguistic difficulties could be avoided. For example, we did not define the *degree of a vertex*, but we talked of *number of edges for a vertex*: in this manner, although not correct from the graph language point of view, we used an expression that is nearer to the natural language of the pupils.

The whole activity has been carried out by means of worksheets, prepared by the teacher. The students worked on the worksheets by themselves, in class or at home, but there was always a class discussion to make sure that everybody got to the same point.

Here we report Worksheet 3 (in Worksheet 1, the Königsberg problem was introduced and students were asked if it was solvable or not, and in Worksheet 2 students were asked if, given the graph on the left in Fig. 8, it was possible to find a walk passing through all the edges exactly once, starting and finishing at the same vertex) (Table 3).

In Table 4, the major components of the lab related to this activity are described.

Table 4 Description of major components to middle school lab

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Finding the necessary and sufficient condition for the existence of Eulerian cycle/trail; Application to polygons; Application to "Alice's Tuscany tour"	Paper and pencil; Cabri geometre; Guided worksheets	Collaboration among students; Mathematics discussion; Groups works	Prepare the activity; Coordinate the groups and the Math discussion

5.3 Eulerian Graphs in High School

The activity was carried out in a high-school by following the path described in Aleo et al. (2009). The contents were the same as in middle school but the approach slightly different. Since the whole terminology is new for the students and to list all definitions can be quite boring, we decided to have the students discover them, with the aid of a guided activity (an example in Table 5). A final worksheet, called Trip logbook, collects all definitions as a reference for students during their *tour* around graphs. Afterwards, after modelling some problems in terms of graphs and after understanding that their solution consists in finding an Eulerian cycle, we present and use the Fleury algorithm for finding possible solutions.

At the end of the activity the following real life problem was given:

> Air Transportation is a complex activity: there are big money investments (aircrafts and infrastructure), highly qualified workers (pilots and staff), real time information (booking service for example). Expenses on air traffic are huge and we have to avoid waste. For example, an airplane on the ground does not result in any revenue for the company: therefore, no-fly time has to be reduced. To this end, some airline companies, once the routes are decided, study «circular trails» for each aircraft (a circular trail is a trail that covers all the routes only once).
>
> Eurofly, has to organize a circular route for an aircraft that has to cover the routs indicated in Table 6. If there is an empty cell then there is not a direct link between the cities.

Most of the students were able to model the given problem with the aid of a graph and understood that they were asked to find an Euler cycle of that graph. Students were very happy to have mathematics *touch* real life. One of them thought to use graph theory to model some problem of his father's company (a logistic transportation company).

Table 5 Guided activity for high school

Observe the following graph (see Fig. 10):		
	We have: Degree of A = 1 Degree of B = 4 Degree of C = 2 Degree of D = 3 Degree of E = 2 Degree of F = 0	
Definition: The Degree of a vertex is the number of …… whose the vertex is an endpoint.		

Table 6 Routes for Eurofly aircraft

	Rome	Paris	London	Athen	Milan	Madrid
Rome		YES		YES	YES	YES
Paris	YES					YES
London				YES	YES	
Athen	YES		YES			
Milan	YES		YES			
Madrid	YES	YES				

Table 7 Description of major components of high school activity

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Represent the network of an airline company with a graph; Finding necessary and sufficient condition for the existence of Eulerian cycle/trail; Application to Airline companies problem	Maps; yEd; Icosien; Worksheets	Collaboration among students; Mathematics discussion	Prepare the activity; Coordinate the groups and the Math discussion

In Table 7, the major components of the lab, related to the activity, are described.

6 Brief Conclusions

We strongly believe that graph theory deserves space in school teaching, because it permits an approach to modelling, argumentation, and connection with reality and, it may foster affection to mathematics from those students that have had a bad experience with it.

Students we worked with did benefit from this experience both for mathematical skills developed and a positive affective point of view. In particular:

Mathematical knowledge: Students, from primary to high school, were able to recognize the essential elements of a graph, to know how to draw, represent and manipulate a graph both with paper and pencil and by means of software, to identify similarities and differences between graphs and recognize the same graph drawn with different representations, to know the condition of existence of Semieulerian/ Eulerian paths in a graph.

Graphs were easily understood and manipulated as real and everyday-life objects (from cartoon princesses for pupils to social network friendships for teenagers), as suggested by horizontal teaching and embodied cognition theory.

Practical abilities: All the students used the condition of existence of a Semieulerian/Eulerian paths for solving problems, and most of the students in middle and high school knew how to apply an algorithm in order to find an Eulerian/Semieulerian path: Because when you discover properties, by touching them through laboratorial activities, instead of listening to the teacher, you get better and remember those properties.

Mathematization: Students, from primary to high school, achieved the ability to recognize in a graph a possible model of a problem, to know how to go from a problem to its model through a graph and to see mathematical objects hidden in various situations and everyday objects.

The mathematization process was driven by the use of speech, diagrams, gestures, … as situated signs later on transformed into mathematical signs.

Reasoning: Students of all grades formulated and tested hypotheses on Eulerian/SemiEulerian graphs and compared their own hypotheses with classmates to achieve shared results. In middle and high school, they formulated conjectures, with argumentation guided by teachers. In high school some students independently developed conjectures, through argumentation.

Affect: The most enthusiastic were primary school children who saw a beautiful and different mathematics, rich in games and cartoon characters. Middle school students organized an exhibition of the activity they did in class and invited parents and future students of the school to visit it.

Some of the high school students in the last experimentation came to the graphs meetings *forced* by a project carried out by their own school, but in the end they were happy to have participated and brought some modeling results home (one of them, for instance, thought to use graph theory for modelling problems of the transportation company of his father).

For all of them, fun and motivation was the first step towards learning. We aimed to promote a vision of mathematics not made of cold calculations, or of pseudo-problems to solve, which is often thought just to blindly apply those rules you read about some pages before.

The vision of mathematics we want to achieve is a mathematics included in everyday life, but hidden: to see it you should raise the veil that covers it.

Despite all of its usefulness and beauty, teachers are not prepared to teach it in class. Sometimes they do not even know the topic itself. Some teacher-training programs may include these topics in the future. More to come!

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Part V
Fair Decision-Making and Game Theory

Fairness

Sol Garfunkel

Abstract In this paper we discuss a number of fairness models related to fair division and bankruptcy problems, including several with a very long and colorful history. We emphasize the role of mathematical modeling in solving such fairness and equity problems. In addition to showing how accessible these discrete models can be, we attempt to show their intrinsic interest and the fact that they can and should be introduced in students' high school and even middle school mathematics curricula.

keywords Fairness models · Bankruptcy · Discrete models

1 Introduction

For the past 45 years I have worked to bring mathematical modeling and applications of mathematics into the mainstream mathematics curricula at all grade levels. This work has continuously (pun intended) bucked up against those who believe that analysis is mathematics and therefore courseware must be designed to prepare students for continuous mathematics. Hence the emphasis on algebraic manipulative skills. And even those who give a nod to modeling see it in terms of physics and engineering, reinforcing their belief in the calculus escalator. I have actually heard people say that 'if it doesn't involve a differential equation, it's not really modeling.'

We know better. Discrete models of very real and important phenomena abound. But due to the prejudice just described, these are generally discussed in separate discrete mathematics courses at the high school and undergraduate level often aimed at underachieving students. As with modeling, the mistake is made in somehow distinguishing them from 'mathematics'. Having a mathematics course called discrete math or mathematical modeling is like having an English language course called nouns or one called verbs. I realize that I come at this from a U.S.

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perspective, where we are still living with a layer-cake approach to secondary school mathematics, but I believe I am describing a worldwide phenomenon.

I believe that we need to make the case that discrete models of important and interesting real world situations can and should be introduced throughout a student's mathematics education—at ALL levels. Let me first discuss material at the secondary level, because it is where I have done a significant amount of work. As indicated, I believe in an integrated approach to teaching mathematics, not in making a big deal of sub disciplines, certainly not at the high school level. But while courses titled Mathematics 1, 2, or 3 make eminent sense, it also makes sense to have reasonable themes in order to organize the instruction. Themes such as Fairness and Equity, Risk, Growth and Change, Shape and Space, Conflict and Cooperation, Optimization, Information…are all possibilities—and many involve discrete models!

As an example let's consider some Fairness and Equity problems. Among the reasons that I find this theme compelling is that the problems are immediately recognizable to students, and much of the mathematics needed to get deeply into the analysis is essentially arithmetic and therefore universally accessible. And yet, the reasoning is mathematically sophisticated and the choice of models not at all obvious. Students can see how mathematicians really work—exploring, conjecturing, inventing representations. In addition there are many historical references, as these problems have been around for a long, long time.

2 Fair Division

Consider an inheritance problem where there is one asset, say for example a house, to divide between two heirs. Further imagine that both heirs want the asset and do not wish to have it sold and then to split the proceeds. We proceed by conducting an auction. In this simple case, the heirs write down what they think the property is worth to them. Let us imagine that the first heir thinks the house is worth $500,000 and the second thinks it's worth $400,000. In this case the house is awarded to the heir who values it most highly, heir 1. The question is how much money should heir 1 give to heir 2 to be 'fair'.

The logic is as follows. Heir 1 believes the house is worth $500,000, therefore he believes that his share is worth $250,000. Heir 2 believes the house is worth $400,000 so she believes her equity is $200,000. If we split the difference in half and Heir 1 gives Heir 2 $225,000 look at what happens—Heir 1 gets the house for $25,000 less than he thought it was worth and Heir 2 receives $25,000 more than she thought she deserved. A similar tactic can be used when there are more items to divide. This works with more 'players' as well.

Suppose for the moment that there is a house and four heirs—Bob and Carol and Ted and Alice. We assume that all four want the house (rather than selling it and dividing the money). We begin with an auction, i.e. each heir writes down what they would pay for the house. Let's say that the bids are as in Table 1.

Table 1 Four heirs' bids on a house

Bob	Carol	Ted	Alice
$120,000	$200,000	$140,000	$180,000

Because she is the highest bidder, Carol gets the house. The question is how much should she pay each of the other heirs. Carol values the house at $200,000, so she thinks her share should be worth $50,000. She now puts $150,000 in a kitty. Each of the other heirs withdraws what they think their share was worth from the kitty. So,

- Bob withdraws 120,000/4 or 30,000
- Ted withdraws 140,000/4 or 35,000
- Alice withdraws 180,000/4 or 45,000

That is a total of 110,000, leaving 40,000 in the kitty. Each heir (including Carol) gets their share or 10,000 each and thus Carol gets the house, Bob gets 40,000, Ted gets 45,000, and Alice gets 55,000. Note that each heir is $10,000 to the good.

2.1 The Adjusted Winner

In 1991 (I know, it's a long time ago, but distressingly current) Donald and Ivana Trump went through a celebrity divorce which received a great deal of media attention. However, despite the hype they decided to attempt an out of court property settlement. Of course, we have no way of really knowing what went on in their minds while going through the negotiations and can only give a rough estimate of the actual items involved. But we do know a fair amount about their assets.

To attempt a realistic illustration we take the following list of marital property: a 45-room mansion in Greenwich, Connecticut; the 118-room mansion in Palm Beach, Florida; an apartment in the Trump Plaza; a 50-room Trump Tower triplex; and just over a million dollars in cash and jewelry.

We begin the adjusted winner procedure by giving each party 100 points to distribute over the items in a way that reflects their relative worth to that party. Again we can't get into the minds of our divorcing couple, but given what we know, let's assume that Donald and Ivana used the point assignments in Table 2.

Table 2 Point assignments for marital assets by two parties

Marital asset	Donald	Ivana
Connecticut estate	10	38
Palm beach mansion	40	20
Trump plaza apartment	10	30
Trump tower triplex	38	10
Cash and jewelry	2	2

The adjusted winner procedure now allocates the property as follows:

1. Each party is initially given each asset for which he or she placed more points than the other party. Thus Donald receives the Palm Beach mansion (40 of his points) and the Trump Tower triplex (38 of his points), while Ivana initially receives the Connecticut estate (38 of her points) and the Trump Plaza apartment (30 of her points). Donald now has 78 points and Ivana only 68. Because she has fewer points Ivana gets the asset they both valued the same, i.e. the cash plus jewelry. She now has 70 points.
2. Now we have to transfer assets from Donald to Ivana until they have the same point total. The process is as follows.

Donalds's assets are arranged from left to right so that the fractions (Donalds's point value of the asset)/(Ivana's point value of the asset) increase or stay the same. For our example, the fractions for Donald's two assets are

- 40/20 (Palm Beach) and 38/10 (triplex)

We now transfer assets or fractions of assets until the points are equal. We begin with the Palm Beach mansion since he values it relatively less.

Note that if we were to transfer all of the Palm Beach mansion (worth 40 points to Donald and 20 points to Ivana) then Donald would have 78 − 40 or 38 points, while Ivana would get 70 + 20 or 90 points. So, we want to transfer only a fraction of the mansion to Ivana from Donald.

Let's call the fraction of the mansion that Donald keeps, x. Then the fraction that Ivana gets will be $1 - x$. Since Donald values the mansion at 40 points, his points from the mansion become $40x$. Ivana on the other hand values the mansion at 20 points, so her points from the mansion become $20(1 - x)$. Therefore, since Donald starts with 38 points (before the mansion) and Ivana with 70 points, we want

$$38 + 40x = 70 + 20\,(1 - x).$$

Solving for x, we get

$$\begin{aligned} 38 + 40x &= 70 + 20 - 20x \\ 38 + 40x &= 90 - 20x \\ 60x &= 52 \\ x &= 52/60 \end{aligned}$$

So, equality is reached when Donald retains 52/60 or about 87% ownership of the Palm Beach mansion and Ivana gets the remaining 13%. What is interesting here is that the actual settlement they reached is very close to that given by the adjusted winner procedure: Donald received the Trump Tower triplex, and Ivana received the CT estate, the Trump plaza apartment and the cash and jewelry. As to the split of the Palm Beach mansion, Ivana was awarded use of it for one month a year as a vacation home—pretty close to what we came up with!

Not only are variants of this procedure actually used in divorce courts, it is clear that examples of this type can be used to emphasize both arithmetic and elementary algebraic skills.

2.2 Claims

There is a class of inheritance problems first mentioned in the Talmud of the following kind. A father has two sons. At his passing the first son says that his father informed him that he would be left the entire estate. The second son claims that his father told him that the estate would be divided equally. How should we divide the estate?

In this simple example there are two apparent solutions. The first is proportionality, i.e. we divide the estate in the same proportion as the claims. In this case since one son claims all and the second claims half, the proportion of the claims is two to one and hence we give the first son two-thirds of the estate and the second one-third. This in fact was the form of solution used by the secular courts. However, the rabbis had a different interpretation. They argued that by claiming only half the estate, the second son gave up all claims on the second half that should go therefore to the first son. If the disputed half was then divided equally, the first son would be given three-quarters of the estate and the second son one-quarter. This is sometimes called the nested claims solution. Those of you who would expect more complication from a Talmudic discussion will be pleased to know that the problem as put forth there had ten sons—the first who claimed all, the next a half, the next a third, then a fourth and so on.

3 Bankruptcy[1]

This class of problems deals with the dissolution of a firm with a collection of assets that are insufficient to pay off the total claims against those assets. In the simplest form of a bankruptcy problem we have a collection of claimants: $C_1, C_2, C_3, \ldots, C_n$, with verified claims $c_1, c_2, c_3, \ldots, c_n$. The remaining assets E have also been verified and are to be distributed by a “wise person” or judge. (The letter E is used to suggest the word “estate.”) If someone leaves a will, it may turn out that the estate is not large enough to make the suggested dispersals. In this case, we have a “bankruptcy” problem where we treat the desired amounts to be dispersed as the claimant amounts and E is to be used to pay off these claims.

Assume that the amount to be distributed is strictly less than the amount which is being claimed. Thus, $c_1 + c_2 + c_3 + \cdots + c_n > E$. We would like to be able to

[1]This section is liberally adapted with permission from the AMS online feature column by Malkevitch 2015.

advise the judge about how to distribute the money using the best insights about fairness and equity principles. We are assuming that the claimants are isolated from each other and do not "bargain" or "negotiate" with each other regarding the amounts they might get from the judge. One can imagine that one could have a "game" where the claimants would get nothing if they could not agree how to split the estate, but if they could agree to share all of E in some way, they would be collectively allocated this amount. In this type of situation negotiations among the claimants would be required.

3.1 Total Equality

Let's begin with an example. Consider the case of two claimants.

Suppose E (remaining assets with which to settle claims) = $210
Claimant 1 has verified claims of $300
Claimant 2 has verified claims of $60.

Perhaps the first approach to solving equity problems is to treat individuals with *total equality*. It is this thought that governs the famous dictum with regard to voting: *one person, one vote*. If we do this in our example, we take the estate E, which amounts to $210 and divide by 2, giving $105. This amount would be given to each of the claimants. This may seem strange because we are not taking into account the size of the claims to get this value, only the number of claimants and the value of E being used. In particular, for the numbers here this means giving Claimant 2 more than he/she claimed!

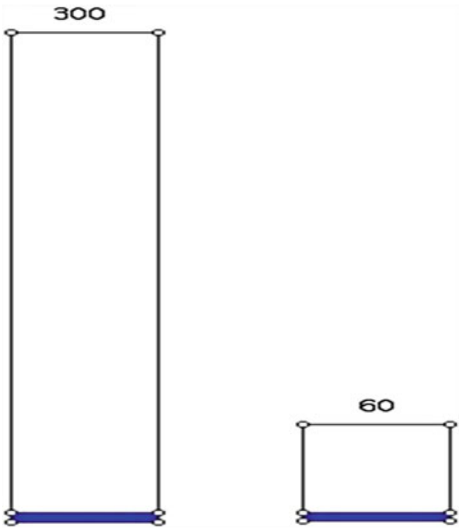
Should we decide that the above method is unreasonable, we might adopt the following second method: Equalize the claims of the claimants as much as possible but never give a claimant more than is requested. This notation of solving a bankruptcy has very old roots, having been in essence suggested by the great medieval philosopher Moses Maimonides (1135–1204) (often referred to only as Maimonides or as Rambam) (Fig. 1).

3.2 The Method of Maimonides

In modern mathematical terminology what we have here is a *constrained optimization* problem. Our desire is to make the amount given to each of the claimants as equal as possible but not to have any claimant receive more than his/her claim. This means in the mathematical formulation that certain inequalities would have to hold for a solution. Here is a geometrical way to solve this kind of problem easily

Fig. 1 Moses Maimonides

Fig. 2 Begin filling two bins to illustrate the method of Maimonides



without converting it to highly symbolic mathematical form. Imagine that the money in the estate to be distributed is a blue fluid. We begin to fill up two *bins* or *tanks* of size 300 and 60 (the claim sizes) with a small bit of fluid in each, keeping the amounts as equal as possible, as illustrated in Fig. 2. Remember that the size of the tanks corresponds to the size of the claims.

We keep filling the two containers equally until we fill up the smaller of the two bins, which amounts to completely filling this claim. The situation is now as shown in Fig. 3.

How much of the estate has been used up at this stage? The answer is 2(60) or \$120. This leaves \$210 − \$120 = \$90 to distribute and this all goes to Claimant 1

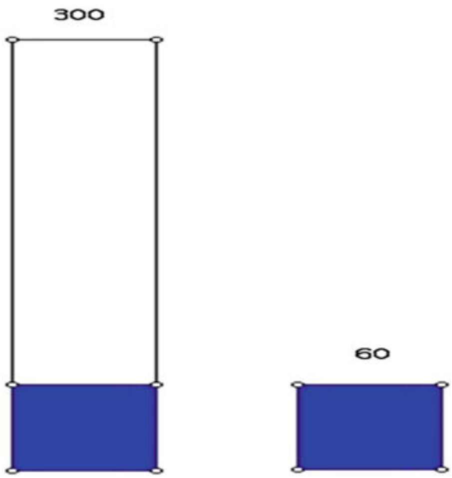


Fig. 3 Continue filling the bins equally until the smaller is filled

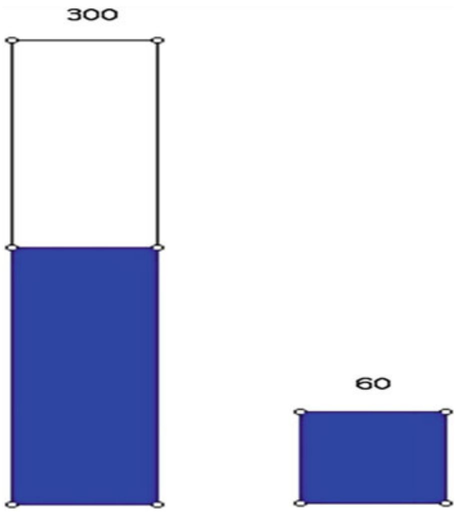


Fig. 4 Bins showing the Maimonides solution

since the complete claim of Claimant 2 has been met. Since \$60 + \$90 is \$150 the final settlement gives Claimant 1 \$150 and Claimant 2 \$60. Figure 4 shows this solution.

This geometric approach works very well for a large number of claimants. Typically, the claim of the smallest claimant can be fulfilled and then if there is more “estate fluid” to distribute this is done until the next smallest claimant’s claim is fulfilled. The process continues until all the estate “fluid” is gone.

3.3 Loss Methods

Are you happy with the Maimonides solution to the bankruptcy problem? Unlike the total equality solution, it takes into account the size of the claims. However, let us look, for example, at how much of what each claimant hoped for, failed to get recovered. For C_2 this amount is 0, while for C_1 this amount is \$150. This does not seem to spread the pain of loss very fairly. How much do the claimants collectively lose? Since E = \$210 and the claimants are claiming \$300 and \$60 respectively, the loss L = \$360 − \$ 210 = \$150. This notion suggests a new method. Why not spread the loss equally?

This would mean assigning a loss of \$75 to each claimant. For C_1 this amounts to giving him/her \$300 − \$75 or \$225, while for C_2 this amounts to giving him/her \$60 − \$75 = − \$15! Although \$225 + (−\$15) adds to \$210, the amount E the judge must distribute, something seems wrong here! The problem is that Claimant 2 is being asked to "subsidize" the settlement. The −\$15 that Claimant 2 coughs up is given to Claimant 1 along with all of the \$210 available to the judge. This total of \$225 makes it possible to cut Claimant 2's loss to \$75, which is equal to that of Claimant 1. However, many people will consider this unfair because the pain of Claimant 2 is made worse by having to subsidize the settlement.

Like the contrast between *total equality* and Maimonides, one can consider the analogue for loss using Maimonides. The idea is to equalize loss as much as possible without any claimant's loss becoming negative as a result.

To do this we must reduce the loss of the player with the largest claim to that of the person with the second largest claim, if this is possible. In this case, if we give C_1 \$210 this will bring his loss to only \$90; to reduce the loss further requires more money than is available in E. Thus, we accept the solution of c_1 = \$210 and c_2 = \$0.

Suppose we have three claimants with claims of \$100, \$80, and \$60, and there is an estate E of \$210. We can give \$20 to the first claimant reducing his/her *current loss* to \$80. Now we can give \$20 to each of Claimants 1 and 2 which reduces all the claimants to a current loss of \$60. At this point \$60 of the estate has been used. This leaves \$210 − \$60 = \$150. By giving each claimant \$50 of this we can equalize the losses. Thus Claimant 1 gets \$90, Claimant 2 gets \$70 and Claimant 3 gets \$50. These numbers add to \$210 as required and give each claimant a loss of \$10. Of course, in this problem one can also conceptualize as follows. Since the claims are \$240, and E = \$210, each claimant of the three will sustain a loss of \$10. This means that \$10 less than each claim is given to the claimant.

3.4 Proportional Methods

Another natural approach to settling a bankruptcy is to award the claimants an amount proportional to their claims. This would entail in our prime example giving

C_1 the amount (300/360)(210) = \$175 and C_2 the amount (60/360)(210) = \$35. This seems a very natural approach because it uses the size of the claims to decide how to divide what is given to each claimant.

We might also look at settling the bankruptcy by proportionality of loss. The loss in this example is \$150. Computing C_1's loss we get (5/6)(150) = \$125 and C_2's loss would be (1/6)(150) = \$25. Thus, we would have c_1 = \$300 − \$ 125 = \$175 and c_2 = \$60 − \$25 = \$35. This is the same solution as when we assign the gains proportionally. Is this an accident? No! We can use a bit of algebra to see that this result holds in general.

3.5 Contested Garment Rule

Finally, let us jump to a solution concept that goes back hundreds of years. This solution idea is discussed in a "document" known as the Babylonian Talmud, which initially consisted of oral materials handed down from one generation to another. (There is also a Jerusalem Talmud.). It was Barry O'Neil who called attention in modern times to the fact that the Babylonian Talmud treats various examples of bankruptcy problems. In modern accounts the technique described in the Talmud has come to be known as the *contested garment rule*, since the method was applied to a situation where two individuals claimed portions of a single garment.

How does the contested garment rule work? The basic idea is that depending on the amounts of the claims and size of E, sometimes one or both of the claimants can argue that some of the money "belongs" to that claimant. For example, in our situation Claimant 1 goes to the judge and says, since the only other claimant is only asking for \$60 and you have \$210 available, \$150 of your \$210 should be awarded to me. Suppose we refer to this \$150 as Claimant 1's *uncontested* claim against Claimant 2. Claimant 2 might try to argue in a similar vein, but in this case since Claimant 1 is claiming \$300, Claimant 2 has no non-zero uncontested claim against Claimant 1. Now there remains \$60 that both are claiming. Thus, the judge splits this amount equally between them. Hence, Claimant 1 gets \$150 + \$30 = \$180 and Claimant 2 gets \$0 + \$30 = \$30. On first hearing, this seems like a strange approach but it also has a certain appeal! (Suppose Claimant 1 and Claimant 2 claim respectively \$100 and \$80 and E = \$140. Claimant 1's uncontested claim against Claimant 2 is \$60, while Claimant 2's uncontested claim against Claimant 1 is \$40. Hence, the total of uncontested claims is \$60 + \$40 which means there is \$40 remaining that both claim. The judge splits this evenly. Claimant 1 gets \$60 + \$20 = \$80 and Claimant 2 gets \$40 + \$20 = \$60.) It may not even be obvious that the sum of the uncontested claims is always less than the amount E but it always will be.

There are a number of additional methods that can be described. It is rather remarkable that so seemingly simple and classical a problem can give rise to so much good mathematics—so accessible at early levels. And that is our main point. Students need to see that mathematics is useful in a variety of domains. They need to understand that it is not simply the application of the machinery of mathematics which accounts for its usefulness, but the power of mathematical reasoning. And as beautiful as continuous mathematics may be, it is only one piece of what we call mathematics and what we should teach as our subject. For a detailed survey of bankruptcy results see Thompson (2003).

4 Conclusion

I want to conclude with a quote from the recent *GAIMME Report, Guidelines for Assessment and Instruction in Mathematical Modeling* (Garfunkel and Montgomery 2016), published jointly by the Society for Industrial and Applied Mathematics (SIAM) and COMAP.

> The authors of this report firmly believe that mathematical modeling should be taught at every stage of a student's mathematical education. After all, why does society give us so much time to teach mathematics? In part, it is because mathematics is important for its own sake, but mostly because mathematics is important in dealing with the rest of the world. Certainly mathematics will help students as they move on through school and into the world of work. But it can and should help them in their daily lives and as informed citizens. It is crucial that students' experiences with mathematical modeling, as they progress through the grades, give them exposure to a wide variety of problems — how do we determine the average rainfall in a state? Where's the best place to locate a fire station? What is a fair voting system? How can I hang pictures along a staircase so they look straight? As we demonstrate in subsequent sections of this report, students can learn and appreciate the importance of modeling in their lives at all educational levels.

If mathematical modeling is a life skill to be taught and nurtured throughout a student's educational experience, then discrete mathematics and its applications must be at the heart of that experience.

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Mathematical Research in the Classroom via Combinatorial Games

Ximena Colipan

Abstract In this paper we present a summary of the results of our research concerning 2-player combinatorial games and its applications used to teach the know-hows of the mathematical activity via certain a-didactical research situations, called SiRCs, that transpose to the classroom the activity of an actual researcher in mathematics. We use a specific kind of combinatorial game called Nim-type games and here we only present in some detail a game called the chocolate game. Our main conclusion is that SiRCs based on Nim-type combinatorial games are effective tools to introduce a genuine (but not necessarily original) mathematical research activity to students from high school and above.

Keywords Combinatorial games · Nim-type games · Winning position
Losing position · Chocolate game

1 Introduction

This work is based on the Ph.D. thesis of the author (Colipan 2014) and is centered on combinatorial games and the role they may play in learning the fundamental *know-hows* of mathematical activity. We understand by know-how the knowledge, methods and techniques that are the base of all mathematical activity such as experimentation, particular case studies, building models, construction of proofs and definitions, etc.

In the interest of bringing problem solving abilities into the main focus of mathematics classes, several groups around the world have centered their research on *playful problems* as a device that may play an important role in learning and teaching mathematics. Several authors (Coppé and Houdement 2002; Godot 2005,

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2006; Giroud 2011; Gravier and Ouvrier-Buffet 2009) have shown that they can be used not only to introduce a new concept, but also to stimulate the learning of the know-hows related to the mathematical activity.

As a framework for our work we apply the model, *Situation de Recherche pour la Classe* (SiRC), created in the French research group math-à-modeler (Grenier and Payan 2002). This model has been shown to be effective in providing such playful problems from the field of discrete mathematics (Godot 2006; Giroud 2011). SiRCs are a-didactical situations transposing to the classroom the activity of an actual researcher in mathematics. Some of their characteristics are that they are directly related to a real research problem in mathematics, the initial problem can be easily accessed, elementary mathematical knowledge is enough to understand and attempt a solution to the problem, it is possible to apply several different solving strategies, a solution to a problem leads to a new problem, and the didactical variables of the problem are left open for the student.

In our research we have created several SiRCs based on 2-player combinatorial games (Colipan 2014, 2015; Colipan and Grenier 2015; Colipan 2016). In particular, we use Nim-type games, i.e., games played with several stacks of objects. In each turn a player must remove at least one object according to rules of the game. The game ends when all stacks are empty (Grundy 1939; Sprague 1935; Berlekamp et al. 2001). Our choice to use Nim-type games is justified since the competitive nature makes them attractive and playful, they require little material making them appropriate for the classroom, the rules are simple and few, and games are usually short (Delahaye 2009; Rougetet, this volume).

Our main conclusion is that such SiRCs based on Nim-type games indeed show promise to be effective tools to introduce an authentic mathematical research activity to students from high school and above. This research activity may or may not correspond to actual research problems as we will explain in the main body of the paper, but the goal is to put the students in the position of a researcher. We believe that mathematical research should be at the core of all mathematical training since it is by doing *what mathematicians do* that the students can genuinely learn mathematical thinking rather than by training procedures.

2 Nim-Type Combinatorial Games

As stated in the introduction, our aim is to provide playful problems that would develop problem solving skills by recreating with the students the activity of an actual researcher in mathematics. For such an aim, we have chosen Nim-type combinatorial games. In all its generality, a Nim-type game is a game where two players have a number of objects arranged in stacks in front of them. In each move, a player must take a certain number of objects from the stack according to a rule agreed upon before the game. The loser is the first player who is unable to play

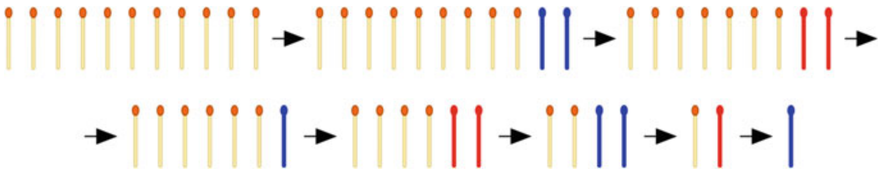
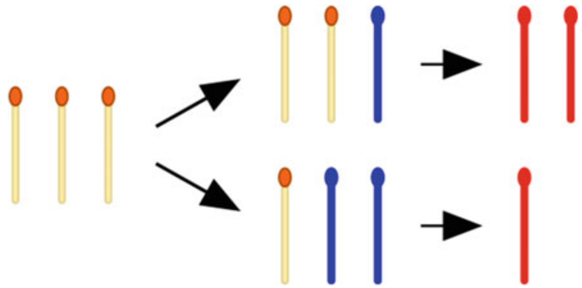


Fig. 1 Example of a game

Fig. 2 Winning strategy with three matches



since no more valid moves are available. The usual Nim game is the one where the rule is to take any positive number of objects from one and only one of the stacks (Grundy 1939; Sprague 1935; Berlekamp et al. 2001).

We give the following example of a Nim-type game. Two players have in front of them a stack with 11 matches. In each move, a player must take 1 or 2 matches from the stack. This game is called *la course à n* in (Brousseau 1998). In Fig. 1 we show an example where the two players are designated with blue for the first player and red for the second one.

In the interest of winning the game, a player will inevitably try to find a *winning strategy*, i.e., a recipe/method/algorithm that will allow the player to always make a move leading to victory regardless of what the opponent plays (Duchêne 2006). For example, in the above game, after playing for a while it is easy to realize that leaving three matches to the opponent will always allow us to win. See Fig. 2.

A position in the game such that a winning strategy exists is called a *winning position*. In the other case it is called a *losing position*. For instance, the position with 3 matches in the above game is a losing position as shown in Fig. 2. Some authors call such positions N and P, respectively. Here N stands for next and P stands for previous. The meaning for such notation is that for a winning position the next player to play has a winning strategy and for a losing position the previous player, i.e., the last player to have played, has a winning strategy. A more careful analysis of the above game shows that losing positions are exactly those where the number of matches is a multiple of 3. For any other position, the winning strategy is to remove the appropriate number of matches to leave a multiple of 3 matches to the opponent.

3 Research Situations for the Classroom (SiRC)

Research situations for the classroom are usually called SiRCs as an acronym for its French name *Situations de recherche pour la classe*. The main goal of SiRCs is, building on elementary mathematical knowledge, to bring the students, from elementary school to undergraduate, to a real mathematical practice giving them the opportunity to develop research in an autonomous way.

SiRCs are a-didactical situations in the sense of Brousseau (1998) that transpose to the classroom the activity of an actual researcher in mathematics. The SiRC model was described by Grenier and Payan (2002) as a research situation having the following properties.

> The situation is included in a research problem. Even if the problem has been already completely solved by professional mathematicians, it must be close to unsolved problems. The hypothesis is made that this proximity to unsolved questions, not only for the students but also for the teacher and the presenter of the situation is crucial for the way the students will face the situation.

The initial question is easy to access. For the question to be easily accessed by the students, the problem must lay outside over-formalized branches of mathematics. It must be the situation itself that brings the students into the mathematical aspects of the problem.

Initial solving strategies for the problem exist without the need of mathematical knowledge out of reach of the students. Such initial solving strategies are required not to bring a complete resolution of the problem, but some particular cases should be easy to handle with such strategies. The mathematical knowledge needed to approach the situation must be kept as elementary and reduced as much as possible, even if more developed techniques may be required to reach a full solution.

Several strategies to go forward into the research are available and multiple, possibly conflicting, developments are possible. This, from both the point of view of the activity (new constructions, proofs, computations) and the point of view of the mathematical notions required.

A solution to a particular case immediately brings out a new question. The problem may be extended without limit by the variables left open for the students. A counter-example does not finish the problem. Instead, it simply changes the question.

3.1 The Position of the Actors in a SiRC

The didactic contract in a SiRC is not usual since the actors (students and teacher) are in different positions than in the case of a usual didactical situation. The students are in the position of the researcher and they have as the main task to solve the problem and to produce results that are new to them. The teacher is in a double position of researcher and manager. He is in the position of a researcher since the students can ask him questions that he cannot answer easily. In this case he can join them to look for answers. He is in the position of manager since he has to control the activity of the students in regards to the learning objectives of the activities: the fundamental know-hows of mathematical activity.

In the remainder of this paper, we will describe a study of the environment, the position of the actors and the management of a SiRC in the case of the Nim-type game known as the chocolate game.

4 The Chocolate Game

The rules for the chocolate game are the following. The players are provided with a chocolate bar. At each turn, the players have to break the bar following a vertical or horizontal line and eat one of the pieces, but one of the chocolate squares is made out of soap. The winner is the player who avoids eating the soap square.

To be able to describe the positions of the game during our mathematical and didactic analysis below, every position will be denoted by a quadruple $P = (a, b, s, t)$ with a, b, s, t integers such that $s \leq a$ and $t \leq b$, corresponding to a chocolate bar of dimensions $a \times b$ where the soap square is located in the position (s, t) as shown in Fig. 3.

It is easy to show that the chocolate game is a geometrical incarnation of the usual Nim game with 4 stacks of sizes $a - s$, $s - 1$, $b - t$, and $t - 1$ (Colipan 2014). As an example, the successive plays in a match of the chocolate game starting with a bar of dimensions 4×3 with the soap square at $(s, t) = (2, 1)(s, t) = (2, 2)$ are shown in Fig. 4.

Light gray color denotes the part of the chocolate that has already been taken. This match consists in 5 plays since the first image shows the initial position.

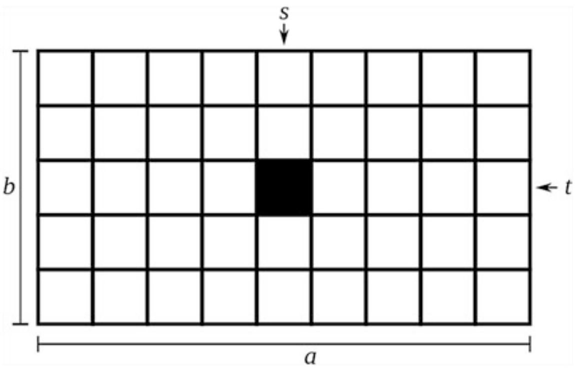


Fig. 3 Chocolate bar given by the position $P = (a, b, s, t)$

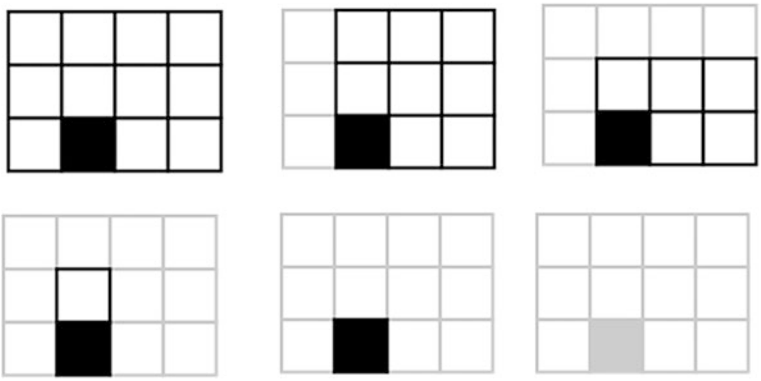


Fig. 4 Example of a match of the chocolate game

The first player is forced to take the soap square and therefore losses the match. With best play, the second player can be forced to take the soap square.

5 A Priori Analysis of the Chocolate Game

At first sight, the study of the SiRC, the chocolate game, can be developed in two stages: the search for easily treatable particular cases and the validation of the conjectures obtained by studying such particular cases.

On solving the game, we can observe that the size of the chocolate bar and the position of the soap square constitute *research variables* of the SiRC. Indeed, the problem is proposed in an open way and such values are left for the students to choose while progressing in their research. The direction that the research will follow will strongly depend on the choices the students make for these variables.

To make their first attempts to solve the problem (initial strategies), the students have several choices that will lead them to uncover different aspect of the game. One possible choice for the research variables is to fix (s, t), the position of the soap square, for instance, a usual choice is to fix the soap square in a corner of the chocolate bar. Another initial strategy is to fix (a, b), the size of the soap bar, for instance, work with a bar of size $a \times 1$. The mathematical description of the winning and losing positions of the chocolate game were described in the author's thesis (Colipan 2014). We will not repeat this analysis here.

5.1 Mathematical Notions in the Game

In the SiRC model, the mathematical notions that come into play are not considered learning goals, but each SiRC certainly brings into play mathematical notions that appear in the resolution process. In the chocolate game we can find the following notions that we classify by their nature.

Mathematical notions: these may appear depending on the strategy chosen to solve the problem. Among them, we have the development and application of basic notions of number theory concerning prime numbers, factor, multiples and Euclidean division; the use of equivalence classes in the integers modulo p, the application of symmetry, etc.
Notions specific to combinatorial games: notions described above such as winning strategy, winning position and losing position.
Fundamental know-hows of the mathematical activity: the chocolate game brings into play the element needed to formulate and validate results and hypothesis. We give a detailed account of such know-hows in the remainder of this section.

5.2 Formulation of Conjectures

To reach conjectures, the students are likely to make two different kinds of experimentations. First, they will randomly play the game in the phase we call random experimentation. During this phase, the students will just play on any bar chosen without meaning. The manager of the situation may later propose that the students, if they do not overcome this phase on their own, choose certain particular cases to bring them forward into their research.

In this experimentation the students will appropriate the game and they will begin to discover its first properties. For instance, they may discover that the losing positions are all symmetric in the cases where the chocolate bar is of dimension $a \times 1$ or whenever the chocolate bar is in a corner. Such a conjecture can actually be proven by a symmetry argument. Indeed, a symmetric position is a losing position since the opponent may imitate the moves of the first player to play until the position where only the soap square is left and so winning the match. Another particular case that can be analyzed with this kind of reasoning is the case of a bar of dimensions $a \times 2$ after the case of a bar of dimension $a \times 1$ is solved.

After this phase of random play, we make the hypothesis that the students will follow into a new phase that we call inductive experimentation. In this phase, the students will study particular cases specifically chosen to produce more elaborate conjectures about the winning strategy regarding more general cases.

The nature of the experimentation in this phase will strongly depend on the properties identified in the random phase. For instance, the students that have realized that symmetrical positions are losing in the special cases shown above may analyze larger bars to try to find more intricate symmetries.

5.3 Validation of Conjectures

We expect the students to produce three different kinds of arguments to validate the conjectures they may find during the experimentation phase.

Iterative and validative experimentations: a group of experiments is called repetitive when after issuing a conjecture the students play several matches in chocolate bars corresponding to their conjecture to test its correctness. As stated by Giroud (2011), it is not absurd to think that a result that repeats itself over and over is true. Nevertheless, this is not enough in mathematics.

Experimentation is called validative if it is aimed at finding a counter-example for a given conjecture. If such counter-example is not found, the conjecture is considered valid. Such an argument is not a valid proof. Nevertheless, this kind of reasoning can lead to a proof by exhaustion of cases in the case where we restrict the research variables so that there are only a finite number of cases. For instance, the match starting with the position $P = (6, 2, 3, 1)$ is interesting since it provides new insight on the game while still possible to be handled by exhaustion of cases.

Mathematical arguments: such arguments appear when attempting to research more general values for the research variables. A full set of valid mathematical arguments on different levels were presented in (Colipan 2014) in a language that may be out of reach of the students. Nevertheless, the contents of such arguments may be used in demonstrations in a less elaborated form.

Generic examples: A generic example for a SiRC is one that will reveal all the features of the game needed to come to a conjecture and proof for a winning strategy. Lacking the formalism to provide an actual proof, students may resort to such examples as a device that shows *what must be done* to prove their conjecture. In the particular case of a chocolate bar with the soap square in a corner, a generic example is any rectangle large enough that it will allow identifying that symmetric positions are the losing positions, for instance, the game with initial position $P = (6, 7, 1, 1)$. The same happens for the game with a bar of size $a \times 1$. A generic example that allows recognizing the symmetric nature of losing positions is $P = (12, 1, 6, 1)$. A complex enough example may allow the students to realize that they can work out particular examples using a backward case-by-case approach. Nevertheless, we believe that for the general case, no particular example is enough to convey all the information contained in the situation.

5.4 Didactic Variables of the Chocolate Game

There are two kinds of didactic variables in the SiRC, the chocolate game. First, we have the variables related to the physical support for the game. We can play the game with several supports such as the obvious pen and paper version but other supports are also possible and will influence the way the SiRC is perceived. The game can also be played with cardboard and scissors or even in an online support.

The main research variables for the SiRC are the values of (a, b) and (s, t). Starting from the values (a, b) and (s, t) we can extract all the necessary information to play and even to find a winning strategy for the game. Different values of $P = (a, b, s, t)$ lead to different conjectures inherent to the identification of winning and losing positions for each subproblem studied.

There is yet another research variable which is related to the number of chocolate squares that the players are allowed to take at every turn. Even if the statement of the SiRC we provide introduces no restriction on this number, a restriction would lead to a development of this SiRC into the grounds of subtraction games that have also been studied by the author in her thesis (Colipan 2014). Such problems are, in general, unsolved from the point of view of research in mathematics.

6 Description of Student Work

An experimentation of the chocolate game as a SiRC was carried out with first year students at the Universidad Católica del Maule whose studies are leading to becoming mathematics teacher at the high school level. Here page restrictions only allow us to give a brief account. Detailed analysis can be found in (Colipan 2015).

A total of fifty students participated in the experiment for a total time of four hours each divided into two sessions. The participants were organized in groups of three or four students. The material the students had at their disposal was controlled. This was done since previous studies by the author (Colipan and Grenier 2015) suggest that for such geometrical versions of Nim-type games the availability of squared paper distracts the students from the research nature of the problem and they restrict their analysis to cases bounded by the number of squares in their paper sheets. Each group had at their disposal the statement of the chocolate game as stated above but translated into Spanish, white paper and pens of different colors.

All the interactions in five of the groups were recorded via audio and video recorders. Moreover, to gather as much information as possible, four observers took selected notes on each of the groups developments. The experiment was managed by two researchers including the author of this paper. Both researchers were observing every group to check for understanding. They specifically did not provide answers based on their previous knowledge of the game, but could provide neutral answers to encourage the students in their research.

Fig. 5 Winning strategy for a chocolate bar of dimensions $a \times 1$

The devolution of the game was rapidly developed by the students and remained active through all two sessions. Once the game description was delivered to the students they all quickly started looking in a playful way for winning strategies for the game. This led them to perform autonomous research without any difficulty and, at times, without even realizing they were indeed doing mathematical research.

The notions of winning and losing positions were well assimilated by the students during their research and especially in the elaboration of conjectures. In particular, the students were able to capture the difference between a win due to random play and a win due to a winning strategy. They also grasped the idea that a winning strategy depends only on the current position of the game and not on the previous moves. Figure 5 shows the results by a group that managed to obtain a valid winning strategy for the case of a chocolate bar of dimensions $a \times 1$. On the left side drawing, they conclude that whoever leaves equal number of chocolate squares on each side of the soap square wins. On the left side drawing, they conclude that if there is equal number of chocolate squares on each side of the soap, then the first player to play loses.

All groups were able to formulate some conjectures regarding the situation, even if not all of them were correct. On the other hand, the search for winning strategies was not the same in all groups. Some groups remained in the local level for their conjectures by working on particular cases while other groups tried to generalize their observations by attempting to find winning strategies for some families of chocolate bars as shown in Fig. 5.

The elements of validations of conjectures were mainly examples and counterexamples. The only formal mathematical proofs that were observed during this experimentation were done by exhaustion of cases. In all other cases the validation was done in an informal way. For instance, in Fig. 6, the student gives an informal proof by symmetry of a winning strategy for the chocolate bars of dimensions $a \times a$ and $a \times 1$.

Shortcomings in oral and written expression by the students were a problem for them to be able to communicate their ideas. Most groups had serious problems in finding words that described their actions already at the level of oral communication. We believe this may happen due to lack of practice in group interaction in their mathematical training.

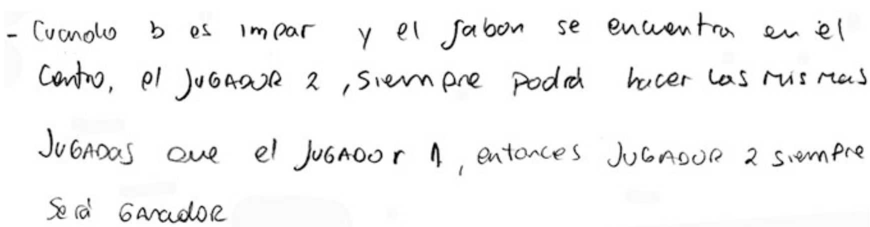

- Cuando b es impar y el Jabon se encuentra en el Centro, el JUGADOR 2, Siempre podrá hacer las mismas JUGADAS que el JUGADOR 1, entonces JUGADOR 2 siempre será GANADOR

Fig. 6 Informal proof of a winning strategy by symmetry

7 Conclusion

The SiRC called the chocolate game, is a playful situation that brings into play the fundamental know-hows of mathematical activity such as experimentation via trial and error (by random and validative experimentation), experimentation on particular cases to reach conjectures about more general cases (in the search for a winning strategy), ruling out conjectures by counter-examples or the validation of conjectures. This allows us to reach the conclusion that research situations based on Nim-type combinatorial games may induce a genuine mathematical activity that goes beyond the development and practice of techniques inherent to current mathematical instruction. Indeed, such SiRCs show themselves as a source for learning the fundamental know-how of all mathematical activity. Recall from the introduction that we understand by fundamental know-how of the mathematical activity the knowledge, methods and techniques that are the base of all mathematical activity such as experimentation, particular case studies, building models, construction of proofs and definitions, etc. This conclusion agrees with our previous study based on a different combinatorial game called the Euclid game (Colipan and Grenier 2015).

On the other hand, the perception that the manager has in regard to the situation is of fundamental importance. The manager must be convinced that in the context of a playful situation it is possible to create the environment for an a-didactical situation in mathematics. The manager must also understand that the main goal is to bring the students into the resolution process of a mathematical problem and that the research for solutions, and not the solution itself, is the interesting part in the process. This requires the acceptance of contributions by the students into the research process that comes only from intuitive or even personal grounds.

As a byproduct of our approach, it is possible that students improve their mathematical writing techniques while experimenting with a Nim-type SiRC. Indeed, it is only by having the need to communicate their findings to others that students can develop such skills. The SiRC model has embedded in its core the need to communicate results and Nim-type games can be particularly suited for this due to the little formal mathematical language needed to write statements.

We can say that such situations created around combinatorial games can induce in the students a significant learning of the fundamental know-hows of mathematical activity, as long as we accept that the objective in studying mathematics is to learn mathematical thinking and processes rather than to cover a fixed mathematical curriculum. For this learning to take place, the goal of such activities must be made clear to all involved: get the students to participate in an autonomous and effective way in a research situation without constraining them to learn any particular notion or procedure. We believe that mathematical teaching should have as main goal to develop critical thinking and this can better be achieved by bringing the student into a genuine mathematical activity rather than by learning a specific set of skills and notions.

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Machines Designed to Play Nim Games (1940–1970): A Possible (Re)Use in the Modern French Mathematics Curriculum?

Lisa Rougetet

Abstract The latest reform of the French high school education system leads to changes in the content of the curricula. In mathematics, a new theme entitled *algorithmic and programming* aims at initiating pupils (7th–9th grades) to "write, develop and run a simple program." To achieve this, the curriculum offers several class activities centered on "games in a maze, …, Nim game and Tic-Tac-Toe." As the mathematical solution of Nim relies on the binary system, easily characterized by bistable circuits, the first electromechanical Nim playing machines were built in the 1940s, followed later by smaller and purely mechanical machines. This article presents these inventions—which claimed pedagogical purposes—and considers their use in class as a recreational application to tackle the algorithmic and programming theme.

Keywords Algorithmic · Boolean algebra · Combinatorial games
Nim game · Nim-like games · Nimatron · Nimrod · Geniacs · Dr. Nim

1 Introduction: High School Curriculum Reform in France and Nim Games

Up to September 2016 in France, the high school mathematics curriculum was structured around four main themes: functions and data management, numbers and calculus, quantity and measurement, and geometry. Since the latest reform, a new theme has been added, entitled *algorithmic and programming*.[1] The resources

[1]The content of this new theme can be found on the Ministry of National Education website: http://cache.media.eduscol.education.fr/file/Algorithmique_et_programmation/67/9/RA16_C4_MATH_algorithmique_et_programmation_N.D_551679.pdf.

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provided by the government point out that “It [the teaching] makes possible to acquire methods that build the algorithmic thinking […]. Having an expert knowledge of computing languages is not a purpose, but using them is a way to acquire other processes of investigation, other methods to solve simulation or to model problems.” The expected skills at the end of the 9th grade are:

- breaking down a problem: analyzing a complex problem, and dividing it into sub problems;
- recognizing patterns: looking for invariant and repetition;
- generalization and abstraction;
- designing algorithms.

To do so, one of the proposed activities is to work on games: “to program playful applications (mazes, pong, battleship, nim, tic-tac-toe …)”. And one of the suggestions is that it can be done “unplugged”, i.e. without any computer devices.

This idea raises the question of using Nim-like games, and the machines that were built to play them, in mathematics teaching; how can they motivate students, which skills do they develop, could the machines be (re)-used in class to help pupils develop algorithmic thinking? As the reform is brand new, the considerations developed in this chapter remain theoretical, especially concerning the construction and the use of mechanical Nim playing machines.[2]

The first part of this chapter underlines the pedagogical aspects of Nim-like games and presents a short history of combinatorial game theory. We will see, as Ximena Colipan highlighted it in her Ph.D. thesis (2014) and in her chapter in this book, that combinatorial games may play a role in developing certain skills of mathematical practice and that they offer research activities for students from high school to university.[3] Knowing their history and the development of their mathematical theory is important to understand the epistemic issues that students face when they play the game and try to solve it.

The second part focuses on the first machines that were built to play Nim against a human being between the 1940s and the 1970s. They were designed not only to entertain, but also to explain concepts in mathematics, algorithmics, and computer science to a general public, when exhibited during fairs or science shows. During the 1950s, to reduce production costs, purely mechanical or slightly electrical machines were manufactured for personal use, with claimed pedagogical benefits to understand elementary level instructions in computing as well as the rules of the binary system and notions of Boolean algebra.

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[2]A preliminary version of the present chapter can be found in Radford et al. (2016).

[3]As far as I know, Colipan and the group she worked with during her Ph.D. are the only ones, in France, who experimented combinatorial games in class and published didactical results on it. The federative structure “Maths à Modeler” (whose aim is to propose workshops to the general public to discover fundamental computer sciences and mathematics) and “Plaisir Maths” (structure of mathematics popularization which gather animators, teachers and researchers to create and organize playful and didactical mathematical projects) also use combinatorial games in their actions of scientific dissemination.

I have not been able to find any documents or evidences that could show if these machines were actually used in class, or in other teaching contexts, to teach students some particular notions of discrete mathematics (this work is still in progress). Therefore, the discussion presented in this chapter about the educational potential of Nim game playing machines—both in their building and their practicing—to acquire algorithmic knowledge and develop skills in problem solving remains embryonic. As of this writing, I do not know of any classroom implementations that were led on this matter, but hope such experiments will be recorded soon.[4]

2 Nim-Like Games: History and Didactical Features

Nim-like games fall within the class of *combinatorial games*. In a combinatorial game, there are only two players, playing alternately. Usually, there are a finite number of positions and the information is complete—which means both players know what is going on at any moment of the game. There are no chance moves such as rolling dice or shuffling cards and in the *normal* play convention the player who finds himself unable to play loses.[5]

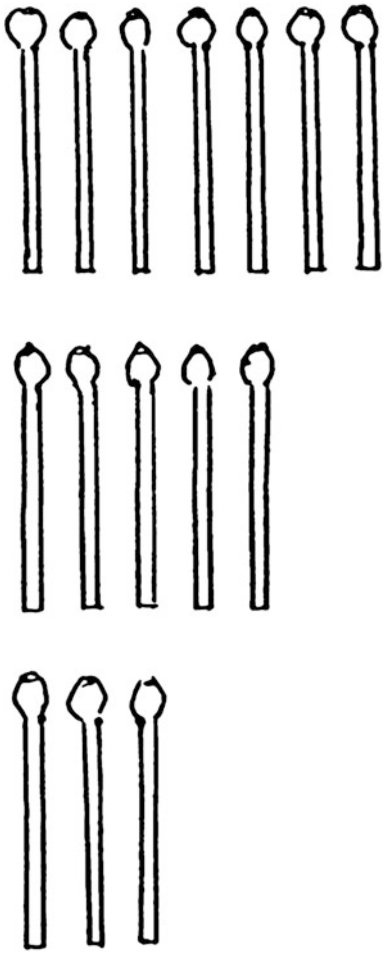
The Nim game—as it was introduced for the first time in 1901—is considered a take away game: usually three (or more) piles of counters are set on a table; each pile contains a different number of counters. Alternately, both players select one of the piles and remove as many counters as they want: one, two … or the whole pile. The first player who takes the last counter(s) wins the game. An example of a possible initial position is shown in Fig. 1. There are many different versions of Nim, depending on the initial number of piles and counters, and how they can be removed. Thus, any initial position poses a new problem, which needs to be analyzed using the results obtained for the previous configurations. This characteristic leads to developing pattern recognition skills (to find out how we attain a position already reached, which made us win), and to work on generalization (to find out if a general method can be developed to win the game in any position).

Being familiar with the history and the mathematical development of Nim-like games has a double interest: on one hand, it provides material to set up interdisciplinary activities (required in the curriculum) that combine history, mathematics, and computer science. On the other hand, playing Nim-like games in order to solve them (finding a winning strategy) places students in an epistemic approach through a singular experimental situation "that they can live and in which the knowledge will appear as the optimal solution to the problem" (Brousseau 1998, p. 49).

[4]In the same way that there are objects in physics and chemistry, there exist mathematical objects whose handling gives a meaning to theoretical mathematical concepts, and there exist studies on the construction and analysis of such mathematical objects, for instance Caroline Poisard's thesis on calculation instruments (Poisard 2005, p. 9).

[5]In the *misère* play convention the player who finds himself unable to play wins.

Fig. 1 Example of an initial position of the Nim game: three piles containing respectively 7, 5 and 3 matches



2.1 A Brief History of Combinatorial Game Theory

The starting point of the history of combinatorial game theory is commonly dated to 1901, when a Nim game was first mentioned under this name in an article published in the *Annals of Mathematics* by Charles Leonard Bouton (1869–1922), a mathematician from Harvard.[6] In this article, Bouton gives the complete mathematical solution to Nim (Bouton 1901), that is, a strategy to win every game. First, the number of counters in each pile must be written in binary. Then these binary numbers are placed in three horizontal lines so that the units are in the same vertical

[6]Actually, things are not so definite: we have found earlier analyses of combinatorial games in recreational mathematics books from the 16th century and thereafter. See, e.g. Rougetet (2014, 2016).

column. The sum of each column is calculated and if all of them are congruent to 0 mod 2, the position left on the table is called a *safe combination*. Such positions should be reached at each move in order to win the game, and as soon as we reach one, it is possible to obtain another one on our next move, but not for our opponent.

Indeed, safe combinations have the following properties: "I. If A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move. II. If A leaves a safe combination on the table and B diminishes one of the piles, A can always leave a safe combination" (Bouton 1901, p. 36). The general theory of Nim using the binary system may not often be discovered by pupils, but the aforesaid properties of safe combinations, defined in a recursive manner, are often understood rather well, even if not explicitly assimilated (Colipan 2014, p. 140 in the context of the geometrical Euclidean game).

Bouton's article is considered as a cornerstone of the development of combinatorial game theory, because, unlike recreational mathematics books of the 16th, 17th and 18th centuries, it gives a solution to any possible initial position, no matter the number of piles and objects in each pile. Admittedly, one can find the earliest trace of combinatorial game in Luca Pacioli's *De Viribus Quantitatis* (1508), a collection of arithmetical and geometrical problems, in which the following problem is proposed: "alternatively, two persons sum up numbers between 1 and 6, the first who reaches 30 wins. Does the first or the second player win?" We can recognize here a variant of "la course à 20"[7] that Brousseau extensively analyzed while presenting the frame of his theory of didactical situations (Brousseau 1998). But it should be noted that those recreational problems were solved only in a particular case; only one solution was given to a specific problem with particular numbers and there was no explanation using arbitrary numbers or variables. This remained true until the end of the 19th and the early 20th century, probably because of the relatively late development of algebra (Rougetet 2016).

After Bouton's article, combinatorial game theory was developed within the field of mathematics and became a beautiful abstraction with John Conway's surreal numbers in 1976. Conway's construction admirably generalizes Dedekind cuts and his theory distances itself from its original subject of study, namely games. Nowadays, combinatorial game theory is a branch of mathematics that connects mathematics (graph theory, set theory) and computer science (programming, artificial intelligence).

2.2 Didactical Aspects of Nim-Like Games

Nim-like games, and combinatorial games in general, provide a *winning strategy* for one of the two players. It means that, theoretically, it is possible to know the

[7]In "la course à 20" ("the race to 20") each player adds 1 or 2 to the previous result and the first who reaches 20 wins.

nature of any arbitrary position in the game (*winning position* or *losing position*), if we assume that both players play optimally.[8] In this chapter, a position is said to be a *winning position* if there exists a winning strategy for the player who is about to play. A position is a *losing position* if there exists a winning position for the previous player who just played.

2.2.1 Mathematical Practice, Institutional Knowledge, Non-institutional Knowledge

Different kinds of knowledge are associated with combinatorial games. Colipan (2014, p. 36) distinguishes non-institutional knowledge, which includes game-specific notions, such as winning and losing positions and winning strategy, and institutional knowledge, which consists of mathematical notions that can be found in the usual curriculum, for example properties of integers, induction, recursion, etc. The association of the latter type of knowledge indicates that the use of combinatorial games in class is appropriate for learning fundamental notions in mathematics through a playful situation.[9]

Playing and analyzing Nim-like games is also appropriate to develop the skills inherent in the “algorithmic and programming” theme of the new French curriculum reform. First, to understand how a game can be won in a general case, one needs to analyze simpler positions, i.e., positions that occur at the end of game.[10] This requires a player to break down the initial problem, which is quite complex, into simpler sub-problems and to try to generalize some configurations.[11] Then, even if games differ in their design from one to another, the ideas implemented to develop winning strategies are the same, centered on the properties of winning and losing positions. Thus, they stimulate pattern recognition (to identify a configuration seen before) and abstraction (to understand the underlying strategy regardless of the form of the game).

Beyond these notions connected to combinatorial games properties, Nim-like games could be good training to develop abstract thinking (to imagine what the opponent will play to better counter him). They could be used to introduce

[8]In practice, the analysis can be very complex, because of the high number of possible positions in most games.

[9]This approach is not new: the oldest analyses of games—which we would qualify nowadays as combinatorial—have been found in recreational mathematics books in the 16th century. Their main purpose was to “tickle curiosity” (Barbin 2007, p. 22), but also to use a playful dimension to acquire mathematical knowledge.

[10]For instance, when they play Nim game, pupils quickly start to analyze whether (1, 1), (1, 2) or (1, 1, 1) are winning or losing positions. These observations have been made on a group of 12 pupils (14–15 years old) in the context of a “mathematical summer camp” organized by Plaisir Maths in June 2016.

[11]E.g., once pupils have understood that (1, 1) and (2, 2) are losing positions, they can figure out that (n, n) is also a losing position, for any n. The same occurs with $(1, 1, n)$, which is a winning position for any n.

enumeration (given a position, what are the possible moves?) and to broach graph theory through game trees, which represent the possible connections between the positions in the game. But, as we said before, since the reform is brand new, the use of combinatorial games in class is still in its early stages.

Furthermore, combinatorial games enable the creation of situations of mathematical research through an experimental approach. For example, Nicolas Giroud (2011) describes an experimental approach focused on the implementation and analysis of three kinds of actions in problem solving: “presenting new problems, experimenting-observing-confirming, and trying to prove” (Giroud 2011, p. 7), all of which actions can arise when playing combinatorial games.

More generally, combinatorial games stimulate an interaction with an environment suitable for developing mathematical practices: discovery phases, conjectures, trial and error periods, reformulation, proof arguments, etc. For instance, it is interesting to submit a given position of a game to pupils and to ask them to determine its nature (winning or losing). They can put their ideas to the test and confirm or invalidate their hypothesis by playing directly.

Combinatorial games also enable pupils to become more autonomous, and improve their personal relationship with mathematics. In this kind of playful activity, pupils’ involvement is different, because it stimulates group work and communication, prevents from determent, and can motivate pupils with learning difficulties: wanting to find how to win is a strong pedagogical lever (Pelay 2011, p. 199). Thus, these games help address the affective domain of learning mathematics, as discussed more fully in Goldin’s chapter in this book.

2.2.2 Playfulness and Learning

Playing Nim-like games—first of all meaning having a playful activity—is directly connected with the mathematical issue of the situation: to find a winning strategy. Brousseau (1998) fully describes this aspect when he presents “la course à 20” (“the race to 20”) to explain his theory of didactical situations. This idea is in accordance with Nicolas Pelay’s main thesis: “it is possible to play and learn mathematics simultaneously and without inconsistency in an activity” (Pelay 2011, p. 53). The background of his work is slightly different from the regular school environment—scientific activities with teenagers in holiday camp—but his fundamental hypothesis “games are a decisive motivation of devolution in an a-didactical situation” (Pelay 2011, p. 52) seems completely compatible with playing combinatorial games in class to discover mathematical notions of game theory and other associated mathematics.

One factor that makes games enjoyable is the manipulation of objects: “Manipulation of objects belongs both to a playful environment and to a mathematical environment. The playful enjoyment is a lot into the manipulation of the game in itself (pieces, dice, counters, cards, etc.)” (Pelay 2011, p. 204). Thus, pupils may be more focused on the task and may less directly feel the pressure of an explicit mathematical approach that will enable them to find the winning strategy.

Moreover, using counters in Nim-like games can help to picture the problem better and enables a quick representation of a position at any moment of the activity (if the teacher is in a devolution or institutionalization phase).

Godot (2005) shows that concrete materials, even very simple ones such as those required for combinatorial games, stimulates manipulation, which is useful to investigate problems. Concrete material helps pupils to exhibit their solutions, their methods, to formulate their conjectures, without using any complex mathematical notions. Action is an essential part of mathematics learning for it gives meaning to the mathematical activity through its experimental dimension. While playing, pupils enter an activity of investigation and are involved in the game at the same time, in order to develop new strategies (Pelay 2011, p. 256).

Finally, combinatorial games present an important cultural dimension; they have endured over time, and the interest they arouse among mathematicians is a witness of their richness. The teacher willing to organize activities and experiments centered on combinatorial games in class should be informed by epistemic and historical knowledge (Durand-Guerrier 2007, p. 17) in order to understand the epistemic approach that students face when they play the game and try to solve it. Thus, a historical perspective is essential to maximize the mathematical and play potential of combinatorial games.

3 Machines Designed to Play Nim Games (1940–1970)

As outlined above, (Sect. 2.1) combinatorial game theory was developed within the mathematical field. Meanwhile, the Nim game was introduced to the general public through machines designed to play against human players. The initial ambition was to entertain, but also to explain to people how machines operated, and what mathematics was involved. We will see that the first machines were so big that they were exhibited during fairs or science shows. Then, personal machines, totally mechanical or partially electronic, were designed, especially to serve as learning tools to help understand some basic ideas in computer science (and also to reduce the production costs).

3.1 Machines Exhibited During Fairs and Science Shows

In the spring of 1940, an electromechanical Nim player machine weighing a ton, called *The Nimatron* (see Fig. 2), was exhibited at the Westinghouse Building of the New York World’s Fair and played more than 100,000 games (and won 90,000 of them). Two members of the Westinghouse Electric Company staff invented it during their lunch break. Condon, the signatory of the US Patent (Condon et al. 1940), underlined the entertaining purpose of the Nimatron, but also specified that it could illustrate “how a set of electrical relays can be made to make a decision in accordance with a fairly simple mathematical procedure.” (Condon 1942, p. 330).

The Nimatron was set to play Nim with 4 piles containing up to 7 counters. The human player began the game, and only 9 initial configurations were possible (due to spacelimitations), each of them being unsafe combination (using Bouton’s vocabulary), so that the human player had a chance to win. The lamps (which can be seen on Fig. 2) *a*1 to *a*7 (7 lamps for the 7 counters of the column *a*), *b*1 *to* b7, *c*1 to *c*7 and *d*1 to *d*7 were connected in circuits, which were controlled by relays A1 to A7, B1 to B7, C1 to C7 and D1 to D7 respectively, each of them controlled by a master relay A, B, C and D (one for each column). Other relays, AZ, BZ, CZ and DZ were activated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a zero power of 2; the relays AF, BF, CF and DF were activated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a first power of 2, and the relays AS, BS, CS and DS were activated when the number of energized lamps in the corresponding columns *a*, *b*, *c* and *d*, respectively, contained a second power of 2 (the maximal number of lamps being 7, 111 in binary). To play the game properly, the machine had to determine whether any power of 2 was contained in the number of energized lamps in an even or an odd number of columns. If the three powers of 2 appeared in an even number—which meant that the human player left a safe combination—the machine played randomly; otherwise it analyzed in which column a change should be made to obtain an even number of the three powers of 2.

In 1942, the Nimatron was exhibited for the last time at the convention of the Allied Social Sciences associations in New York City under the sponsorship of the American Statistical Association and the Institute of Mathematical Statistics. Then the machine was added to the scientific collections of the Buhl Planetarium in Pittsburgh (Condon 1942, pp. 330–331).

A few years later, Ferranti, the electrical engineering and defense electronics equipment firm, designed the first digital computer dedicated to play Nim, *The Nimrod*. It was exhibited at the Festival of Britain (Exhibition of Science) in May 1951 and afterwards at the Berlin Trade Fair (Industrial Show) in October of the same year. These exhibitions were a great success and many witnesses related that the most impressive thing about the Nimrod was not to play against the machine, but to look at all the flashing lights which were supposed to reflect its thinking activity. It had even been necessary to call out special police to control the crowds (Gardner 1959, p. 156). This particular display was built for the purpose of illustrating the algorithm and the programming principles involved.

The instructions followed by the Nimrod were written on the left side, as shown in Fig. 3. Moreover, a booklet was available for visitors,[12] for the price of one shilling and six pence, which contained a lot of information about automatic digital computers in general. The introduction states:

[12]The booklet, released in 1951, *The Ferranti Nimrod Digital Computer*, is available at the following website: http://goodeveca.net/nimrod/NIMROD_Guide.html.

Fig. 2 A young lady playing against Nimatron (Condon 1942, p. 330)

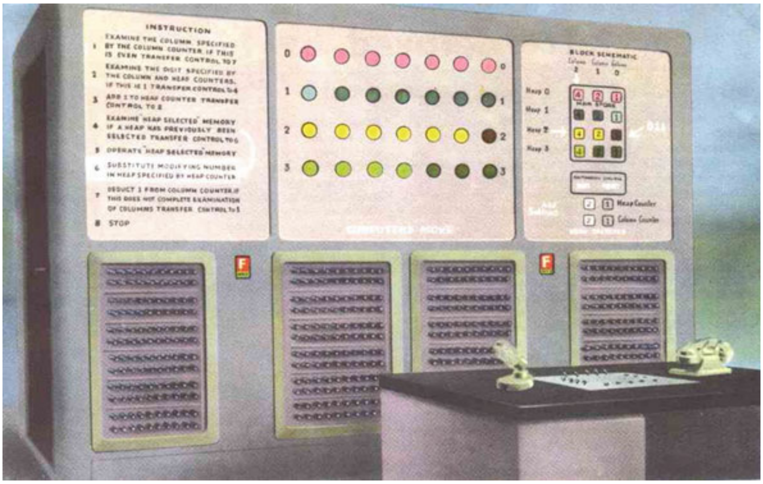


Fig. 3 Drawing of the Nimrod, with instructions on the left side. "The display in front of the machine is used to demonstrate the process involved when Nimrod carried out a move" (Nimrod 1951)

> the machine has been specially designed to demonstrate the principles of automatic digital computers […] the booklet has been prepared for those persons who desire to learn a little more about computers in general and of Nimrod in particular.

Explanations of main notions, such as *electronic brains*, *calculating machines*, *automatic computers*, *automatic sequence control* and *characteristics of automatic computers* (calculation, memory, and making decisions) are given in the first part of the booklet. The second part is devoted to Nimrod functioning ("some details of the machine", "the way in which Nimrod plays Nim"…). "Nimrod has been designed so that it can play the game of Nim with an opponent from the general public or it can be given a "split personality" so that for demonstration purposes it will play a game without an opponent" (Nimrod 1951).

For instance, the instructions run by the program while Nimrod plays Nim are the following:

1. Examine the column specified by the column counter. If this is even, transfer control to 7.
2. Examine the digit specified by the column and heap counters. If this is 1, transfer control to 4.
3. Add 1 to heap counter. Transfer control to 2.
4. Examine heap selected memory. If a heap has previously been selected, transfer control to 6.
5. Operate heap selected memory.
6. Substitute modifying number in heap specified by heap counter.
7. Deduct 1 from column counter. If this does not complete examination of columns transfer control to 1.
8. STOP.

This terminology is specific to algorithmics and programming: a set of instructions is provided, ordered in a logical way and described with conditional statements proper to Boolean data type. In a common language that everyone can understand, the procedure executed by the machine is thus explained (the earliest high-level programming languages with strong abstraction were written in the 1950s). The study of the program above with pupils in class could be used as an introduction to explain how to design an algorithm (as a systematic set of instructions), which is a required skill in the new French curriculum.

The Nimrod booklet also contains a glossary with definitions of the terms used. The authors highlight the quick evolution of automatic digital computers during the 1950s and want to clarify the new terminology that arises for describing the machines. Once again, there was a real will to use the simplest possible explanations in order to embrace the widest audience. We do not know for sure if the Nimatron or the Nimrod had a pedagogical impact in mathematics education, but obviously their exhibition attracted a lot of people. A few years later, smaller machines, cheaper to produce, appeared for pedagogical purposes, and were intended for a wide audience.[13] They are the subjects of the next section.

3.2 Machines Designed for Personal Use

Since 1945 there has been interest in helping people understand how automatic machines reason, calculate, and function. For example, consider Geniacs:

> And we know that equipment that you can take into your hands, play with, and do exciting things with, will often teach you more, and give you more fun besides, than any quantity of words and pictures. (Geniacs, 1955a, p. 2)

This promotes the educational toy *Geniac* designed and marketed by Edmund Berkeley and Oliver Garfield between 1955 and 1958.

Berkeley (1909–1988) was a mathematician, insurance actuary, inventor, publisher, and one of the founders of the Association for Computing Machinery (ACM) (Longo 2015). In 1949, he published a book titled *Giant Brains or Machines That Think*, which was the first explanation of computers intended for a general readership.[14] In the 1950s, Berkeley developed mail-order kits for small, personal computers such as *Simple Simon* and the *Brainiac*. At a time when computer development was on a scale barely affordable by universities or government agencies, Berkeley took a different approach and sold simple computer kits for middle income Americans. He believed that digital computers, using mechanized reasoning based on symbolic logic, could help people to make more rational decisions. These considerations show that the idea of handling objects to help the teaching of mathematical (here, logical) concepts is not new.

It has been suggested (Brougère 1995) that until the 20th century, games were not considered as direct educational tools. "Recreation is essential but game has no status beyond it" (Brougère 1995, p.135). This position has been evolving and now it is recognized that to construct a mathematical concept, a first phase of action is essential to build a mental representation (even if this handling phase alone cannot

[13]Other machines designed to play Nim were created between 1941 and 1958, but in a more mathematical sphere. In 1941, an assistant professor of mathematics at the University of California in Los Angeles (Gardner 1959, p. 156), Raymond Moos Redheffer, improved considerably the Nim-playing machine (Redheffer 1948). To our knowledge, Redheffer's machines were not exhibited to a broad public, consequently they were less known. In 1952, engineers from W.L. Corporation, Hubert Koppel, Eugene Grant and Howard Bailer, developed a lighter machine than Nimatron or Nimrod, as it weighed less than 25 kg and cost $2000 to build. We would like to note that the machines mentioned in this note had no clearly expressed pedagogical or educational aspirations and were probably not much widespread. Nevertheless, Pollack's DEBICON (1958) can be found on *Popular Electronics* magazine cover, which soon became the "World's Largest-Selling Electronics Magazine" (see Fig. 4, left).

[14]His journal *Computers and Automation* (1951–1973) was the first journal for computer professionals.

Fig. 4 *Popular Electronics* cover, January 1958 (left) and *Radio-Electronics* cover, October 1950 (right)

be enough to learn, and a mediation with the teacher, or someone who knows, is necessary). Construction kits such as *Simple Simon* and *Geniac* provide the necessary material to grasp mathematical and easy computer science notions and could be revisited nowadays in mathematic classes to illustrate the procedure of an algorithm. This idea will be further developed in the section devoted to Machines or Computer Type Devices for Educational Use Patented, with *Dr. Nim*.

When it was released in 1950, *Simple Simon* was the "World's Smallest Electric Brain" (see Fig. 4, right). It weighed 39 lb and showed how a machine could do long sequences of reasoning operations. "The machine itself has been demonstrated in more than eight cities of the United States"[15] (Geniacs 1955a, p. 2). "He will be useful in lecturing, educating, training and entertaining" (Berkeley and Jensen 1950, p. 29). By 1959, more than 350 sets of Simon plans had been sold, but it cost over $300 for materials alone, and Berkeley and Garfield admitted, "it is therefore too expensive for many situations in playing and teaching" (Geniacs 1955a, p. 2). That is the reason why they worked four years long to develop a really inexpensive electric brain: *Geniac*, a construction kit costing less than $20.

[15]Simple Simon was exhibited in New York, Seattle, Philadelphia, Boston, Washington, Detroit, Minneapolis, Pittsburgh, and other smaller cities. The fact that Berkeley could take Simon from place to place meant that students and other non-experts could have firsthand contact with automatic computing equipment "for real".

3.2.1 Geniacs

The name Geniac stood for “Genius almost-Automatic Computer” (Geniacs 1955a, p. 2). The construction kit consisted of 30 small electric brain machines—each one being a Geniac—which could be made with very simple electrical equipment. The guide supplied with the kit first gave a general description of the material and the way the different components worked.

One of the proposed problems was to design a Geniac that could play Nim in normal convention with four piles of matches, containing respectively 4, 3, 2 and 1 matches. The solution of the wiring is shown in Fig. 5.

Besides the construction of machines to play games such as Nim, Tic-Tac-Toe or to answer recreational mathematical riddles such as the Two Jealous Wives, the kit provided other Geniacs to illustrate more purely mathematical problems such as the adding machine, the multiplying and the dividing machines, or the machine for arithmetical carrying (Geniacs 1955a, p. 4). A 1958 advertisement explained all the interesting aspects of Geniac and highlighted its popularity, its pedagogical interest and its low price (see Fig. 6).

For only $19.95, Geniac offered a complete course in computer fundamentals used by thousands of colleges, schools and private individuals. It seems that in October 1958, more than 30,000 Geniacs kit were in use by satisfied customers. The advertisement clearly underlined the pedagogical purposes of Geniac for understanding notions of mathematics and computer engineering:

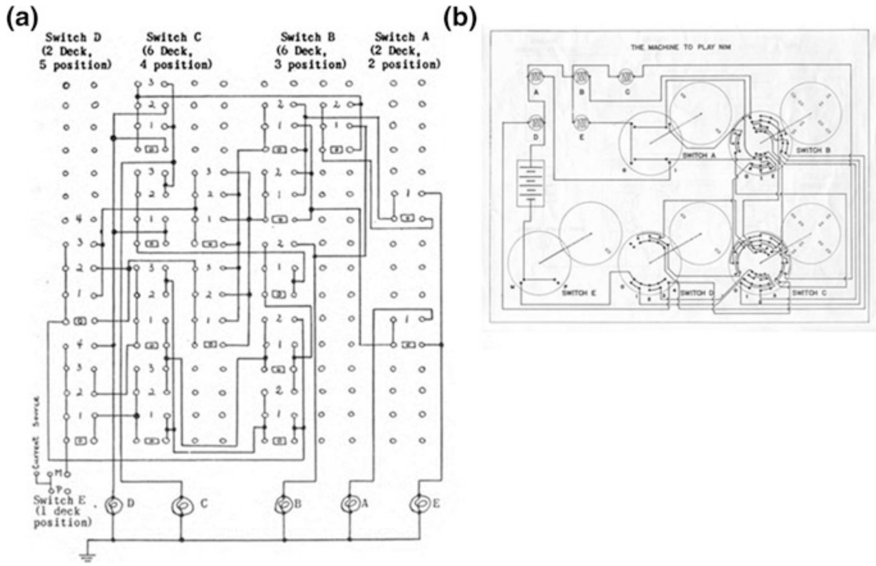


Fig. 5 **a** Geniacs (1955a, p. 36), **b** Geniacs (1955b p. 15)

Fig. 6 Advertisement for Geniac (Geniac 1958, p. 29)

> Geniac is a genuine electric brain machine, not a toy. The only logic and reasoning machine kit in the world that not only adds and subtracts but presents basic ideas of cybernetics, Boolean algebra, symbolic logic, automation, etc. So simple to construct that a twelve-year-old can construct what will fascinate a Ph.D. (Geniac 1958, p. 29)

Berkeley designed Geniac to be a tool for educators and it seemed it had some success in this area (Longo 2015). In 1958, the *Mathematical Gazette* published an article of a mathematics teacher, Martyn H. Cundy, who developed plans for a binary adding machine for classroom use. The machine could add two binary numbers of two digits or three digits (with or without carry). Cundy credited his work to Geniac that taught him the fundamentals for building his own machine and demonstrated that some knowledge of binary arithmetic should be part of the mathematical knowledge of the normal grammar-school pupil (Cundy 1958, p. 272). In 1956, in the magazine about education *Phi Delta Kappan*, Daniel

Davies[16] covered "breakthroughs" in educational administration, in which he detailed areas where new developments were having impacts on education. These included mathematics, for example game theory or binary numbers systems, and Davies claimed: "Boolean algebra is already at work in problem solving. One firm is advertising a kit for setting up an ingenious device known as Geniac which can quickly solve a wide range of problems involving multiple choices" (Davies 1956, p. 276). Moreover, as the 1958 advertisement stressed: "In addition to its value as a source of amusement and education the kit exhibits certain technological features that may have widespread implications in other areas" (Geniac 1958, p. 28).

The use of Berkeley small electrical brain machines in classrooms and how they impacted the teaching of mathematics is difficult to ascertain and this part of the work is still in progress. However, the popularity of Geniac during the 1950s is seen through many electronics magazines and science journals.[17]

3.2.2 Machines or Computer Type Devices for Educational Use Patented

We have already mentioned that the main problems of electromechanical machines such as *Nimrod* and *Nimatron* were, first of all, their size, and also their expensive production cost: "Standard electronic computers […] have been both bulky and expensive" (Du Bosque 1962). That is why during the 1960s, smaller electric or purely mechanical inventions, in the same vein as Geniac,[18] were patented for their "durability and reliability in use" (Weisbecker 1968). And they were explicitly designed to have educational value.

For example Joseph Weisbecker's invention, related to a unique mechanism in the nature of a computer for use as a toy, game, puzzle or educational device, was advertised "to illustrate computer operation and logical techniques […]" (Weisbecker 1968). Moreover, these inventions permitted "the achievement of elementary level instructions in computers" (Godfrey 1968). "[…] the invention is an educational device for indicating the best play to be made […]" (Du Bosque 1962). "It is a principal object of the present invention to provide improved educational game apparatus which permits the learning of strategy techniques, logic methods, and mathematical systems" (Morris 1971). Authors of such patents also justified the interest of their machines by filling the gap left "in the ability of the

[16]Daniel R. Davies was executive director between 1954 and 1959 of the UCEA (University Council for Educational Administration), an organization aimed to improve the professional preparation of educational administrators.

[17]For example, *Popular Electronics* or *Galaxy Science Fiction* magazines. A Ngram research in the English Google books corpus shows a net increase of the use of the term "Geniac" between 1955 and 1960: https://books.google.com/ngrams/graph?content=Geniac&year_start=1940&year_end=2000&corpus=15&smoothing=3&share=&direct_url=t1%3B%2CGeniac%3B%2Cc0.

[18]For example, during the early 1960s, Berkeley created other electric brain machines with Brainiacs and Tyniacs kits.

student to understand and comprehend what a computer is all about" since the venue of high-speed electronic digital computers (Godfrey 1968). They emphasized the importance of presenting an invention that would provide a game and a teaching aid "so as to attract persons of higher intellectual level, while maintaining relative simplicity for attention arresting use as a toy by relatively young children" (Weisbecker 1968).

Some of these inventions were marketed and sold as family parlor games. *Dr. Nim* is one of them: it was manufactured by E.S.R. Inc., a company specialized in education toys during the 1960s, and was equivalent to a single pile Nim game where 1, 2 or 3 counters could be removed at each move. Dr. Nim was played by one player—against the machine—and offered several starting positions: the game could be played in normal or in misère convention (the last player to move loses) and the initial number of marbles could vary between 9 and 20 (see Fig. 7). Here again the diversity of initial configurations encouraged the investigation of a strategy that could lead to a win for any game.

The Dr. Nim device included a number of flip-flops—bistable circuits (see Fig. 8)—being moved by marbles when they fell down, "so as to allow mathematical computations to be effected upon binary numbers to which the flip-flops are set" (Godfrey 1968).

> Inclusive among the concepts which may be explained and understood by this computer invention are the following: the binary number system; the simplicity for machine design using binary arithmetic; […] the rule for binary counting and addition; modular arithmetic; the use of two's complement arithmetic to achieve subtraction with only its add capability; […] binary multiplication. (Godfrey 1968)

Like the Nimrod, Dr. Nim provided a manual (Nim 1966) of fully detailed instructions (23 pages, A4 paper sized) with the rules of the game, its variations, how it was programmed, and also deeper considerations such as "can machines really think?" A few pages were devoted to the explanation of Boolean algebra in use behind flip-flops mechanisms. A capital letter was assigned to every flip-flop

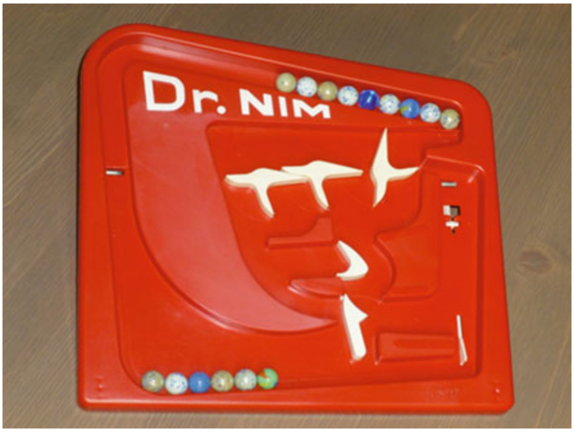


Fig. 7 Dr. Nim red plastic board with white flip flops, owned by the author

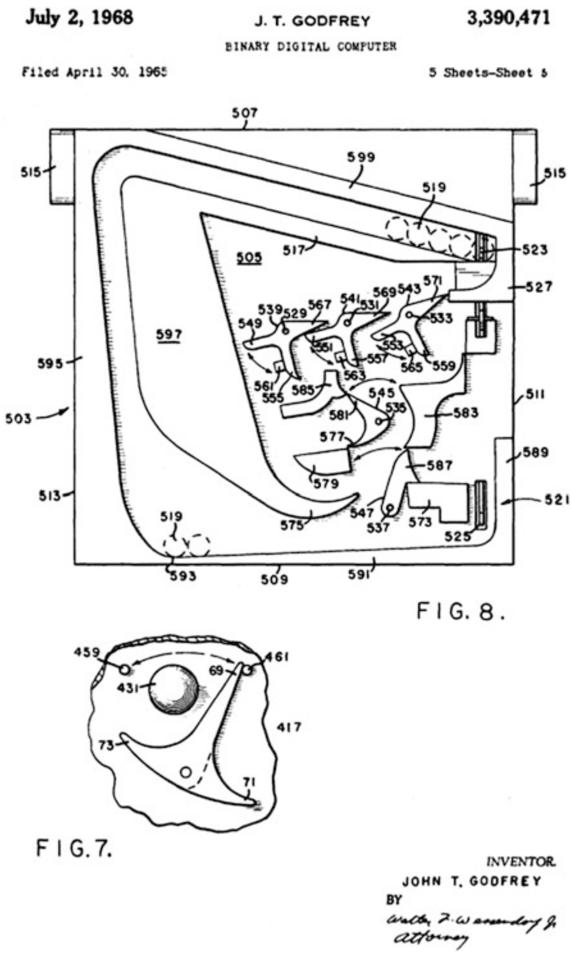


Fig. 8 Patent of a binary digital computer (Godfrey 1968)

(A, B, C, D and E) and a bar was put over the letter when the flip-flop was open, for instance Ā. Then, every number of marbles left in the top row of the machine was written in the form of equation; for example, when 13 or 9 or 5 or 1 marbles are left in the top row, the corresponding flip-flop configuration is Ā B C D E. Instructions given to the machine could be expressed with Boolean algebra operations *and* and *or*.

The program to play Dr. Nim against one person was also provided (see Fig. 9), such that one could understand how it operated only by counting the number of remaining marbles. This shows an example of how algorithmic and programming thinking could be approached in a classroom without necessarily using a computer.

At the beginning of the 1970s, other inventions were patented that "relate to educational game apparatus," for example a "computer-controlled apparatus for playing the game of NIM" (Morris 1971), but with the advent of electronic toys, the

Fig. 9 Program to play Dr. Nim against a person provided in the manual (Nim 1966)

PROGRAM TO PLAY DR. NIM AGAINST A PERSON

(Whoever takes last marble, loses.)

Instruction Number	Instruction
1	Let's start a new game with 15 marbles.
2	Write a 15 in Column "M" and a 2 in Column "R".
3	If you wish to take first turn, go to Instruction 6.
4	I will take 1 marble, write a "1" in Column N, subtract "1" from "M" and write back in Column "M". Also subtract "1" from "R" and write back in Column "R".
5	If "M" is now zero go to Instruction 15. Otherwise continue.
6	How many marbles do you wish to take? Write answer in Column "P".
7	Subtract "P" from "M" and write back in "M".
8	If "M" is now zero go to 16. Otherwise continue.
9	If "P" is less than or equal to R go to 11.
10	Add 4 to "R" and write back in "R".
11	Subtract "P" from "R" and write back in "R".
12	If "R" is not equal to zero go to 14. Otherwise continue.
13	Write a "4" in "R". Go to 4.
14	I will take "R" marbles. Write "R" in "N". Write "4" in "R". Subtract N from M. Go to 6.
15	You win! Go to 1.
16	You lose! Go to 1.

number of purely mechanical inventions declined. Moreover, the increase of personal computers favored the development of programs, and by the middle of the 1970s one could find game programming books.

In 1973 one of the first compilations of computer games in BASIC programming language was published: *101 BASIC Computer Games*, in which the Nim game was presented. The author, David Ahl, explained this interest in computer games by the expansion of minicomputers and timesharing networks that enlarged an emerging group of "computer hackers and of people who were furtively writing and playing game at lunchtime, before and after work on their employers' computer" (Ahl 1978, p. x). The PCC (People's Computer Company),[19] created in the early 1970s, was one of the first organizations to recognize and advocate playing as a legitimate way of learning. PCC recognized the potential of BASIC and helped install computers for children in libraries or schools to encourage a hands-on learning approach.

[19] In honor of Janis Joplin's rock group Big Brother and the Holding Company (Levy 1984, p. 136).

4 Conclusion

In the 1980s, the construction of mechanical Nim playing machines was still of interest, as "there are no commercial teaching materials that provide concrete modeling of Boolean algebra" (Cohen 1980) and such mechanical models could render this algebra intelligible. Nowadays, computers are part of our everyday life and most of us use them without necessarily knowing which operations and logical techniques underlie their functioning.

We believe that these Nim playing machines have great potential to improve skill in mathematics reasoning and problem solving. Furthermore, new theme "algorithmic and programming" in the reformed French curriculum recommends games to help pupils to break down a problem (such as when analyzing simpler positions), develop recursive thinking (analyzing the recursive properties of winning and losing positions), develop logical thinking (devising a winning strategy with conditional statements), and develop algorithmic thinking (creating an algorithm that solves the problem). The creation of machines such as Geniacs or a simpler Dr. Nim could help pupils to acquire such knowledge and skills.[20]

More generally, combinatorial games offer situations that enable pupils to take control of their own learning; they directly confront it by playing the game to confirm or invalidate their theory, and in this way build their knowledge. Games are also effective tools that can help students develop the skill of generalizing (an expected skill of the new curriculum) as they try to solve the games. In particular, by generalization we mean engaging in "at least one of the three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases" (Ellis 2011, p. 311).

We are confident that the strength of games for teaching, and the features they share with devolution (involvement, action, freedom, responsibility) make them essential tools for mathematics education.

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[20]Dr. Nim has been tested with 14–15 years old teenagers during the math summer camp organized by Plaisir Maths in June 2016, and we noticed that, after they studied and understood the strategy of the Nim game, they were curious to know what were the mechanisms of the Dr. Nim machine.

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Part VI
Logic and Proof

Mathematics and Logic: Their Relationship in the Teaching of Mathematics

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Abstract Since the time of the ancient Greeks, the concepts of mathematics have been closely associated with the notions of logical reasoning and proof. After analyzing various aspects of this interaction, the author identifies four features that are important for the education of prospective mathematics teachers: learning the structure of mathematical sentences, learning the concept of mathematical proof, learning methods of mathematical proof, and learning the structure of mathematical theories. Discrete mathematics offers an exceptional means through which students can learn these four features.

Keywords Logic of mathematics · Definitions and theorems · Mathematical proof Methods of proof · Mathematical theories · Education of prospective mathematics teachers

1 Introduction

Mathematical logic occupies a unique place among the disciplines studied by prospective mathematics teachers. Logic, as a science, emerged in Ancient Greece through the work most notably of Aristotle. Greek mathematicians such as Thales, Pythagoras, and Euclid, transforming mathematics from an empirical and descriptive science to one that is deductive, requires proof, and is based on the laws of logic.

In the nineteenth century, English mathematician George Bool first started to apply mathematical methods in logic, and mathematical logic became a branch of mathematics. In 1931 the Austrian mathematician K. Gödel achieved a result which became triumphal for mathematical logic. He proved, figuratively speaking, that the method of establishing the truth based on reasoning in accordance with the laws of

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logic is not omnipotent: there is truth that by this method cannot be proven. Mathematics by its own methods established limitations and boundaries on its applicability. This it is possible to do only by means of mathematical logic.

So, mathematics and logic have become inseparable. Therefore, because of the role that logic plays in mathematics as a science, logic should have a special role in the teaching of mathematics. In the preparation of future teachers of mathematics, particular attention should be paid to their training in logic. This idea was noted by the famous Dutch mathematician and pedagogue H. Freudenthal:

> It is an axiom of teachers' training that the teacher should know more than merely what he teaches. This "more" does not only aim at the subject matter. The teacher has to know the things he knows in a form different from that in which he is teaching them. He shall not only stand above the subject matter which he teaches but also above its logical form. To reach this goal he shall be able to fathom the logical depth of subject matter. Logic can help him to do so if it is more than indirect proof, conversion of theorems, equivalence, and so on. Rather than teaching logic, the mathematics teacher shall use logic and he shall make conscious to the learner that logic the learner is using.
>
> The teacher should be able to do more. He should also stand above the method he has chosen of presenting subject matter, and be able to make this method conscious to himself. In this task, too, logical analysis can be helpful. Not in the trivial sense that it is the logical structure that determines the method, but rather because by logical analysis one can discover the level of understanding and its logical relation." (Freudenthal 1977 vol. 2, p. 181)

Thus, the preparation of prospective teachers of mathematics in the field of logic should be professionally-pedagogically directed. To identify a creature of this kind, we first consider the question of how logic interacts with mathematics in learning mathematics and in learning mathematics, i.e. how is the didactic interaction of mathematics and logic.

1.1 The Didactic Interaction of Mathematics and Logic

Math and logic work closely together in the process of learning mathematics. The notion of mathematical rigor associated with logic is closely related to the level of development of logic. Analyzing the merits of this interaction, we identify four logical components inextricably linked with mathematics, and call them principles of logic in mathematical didactics.

These four components are: (1) the principle of learning the structure of mathematical sentences, (2) the principle of teaching the concept of proof of a mathematical theorem, (3) the principle of training methods for the proof of mathematical theorems, (4) the principle of learning the structure of mathematical theories. These are general provisions related to logic that are fundamental to methods of teaching mathematics. The fundamental nature of these principles in methods of teaching mathematics is that by failing to comply with them in the process of teaching mathematics, mathematics loses its main features as a science, i.e. qualities which distinguish it from the system of other sciences. In the end, the learner receives a distorted view about the general picture of mathematics and its particular parts.

2 The Principle of Learning the Structure of Mathematical Sentences—Definitions and Theorems

First of all, it is necessary to be able to see the logical structure of a mathematical sentence, both a definition and a theorem, and where and what logical connectives and quantifiers are involved in its formulation. In rewriting a mathematical sentence in the language of logical, mathematical logic is really helpful in the analysis of the structure of the mathematical sentence. The rewritten form of the sentences represents formulas (or correctly constructed expressions) of propositional algebra or predicate logic. Moreover, it is important to not only know how to convert these sentences (formulas), but also be able to write the negation of the mathematical sentences using the appropriate logical laws of propositional algebra and predicate logic.

Suppose we have a statement $A \rightarrow B$ (direct). As we know, the converse, $B \rightarrow A$, of a particular theorem, is not always a theorem, i.e. $A \rightarrow B$ can be true while $B \rightarrow A$ is false. Discrete topics offer easy examples of demonstrating this. For example, if n is divisible by 6, then n is divisible by 3 is true but the converse, if n is divisible by 3, then n is divisible by 6 is false.

In the following, we will discuss direct statements, converse statements, inverse statements, $\neg A \rightarrow \neg B$, and contrapositive statements, $\neg B \rightarrow \neg A$, which are the inverse of the converse.

2.1 Analysis of the Structure of Mathematical Propositions by Writing It in a Logical, Mathematical Language

Prospective mathematics teachers need to understand formal logical definitions such as those for the properties of binary relations. For example, they should know that the transitive property of a relation, written as ‘for every, x, y, and z, if $x = y$ and $y = z$ then $x = z$’ can be written as

$$(\forall x)(\forall y)(\forall z)[((x = y) \ \wedge \ (y = z)) \ \rightarrow \ (x = z)].$$

Consider a graph in the graph-theoretic sense of edges and vertices. Define the relation that $x = y$ if there exists a path from x to y. Then clearly this statement is true since if $x = y$ and $y = z$, then $x = z$, the path from x to z consisting of the path from x to y combined with the path from y to z.

Similarly, a linearly ordered set satisfies the condition

$$(\forall x)(\forall y)[(x \leq y) \ \vee \ (y \leq x)]$$

and that for a linearly ordered set to be discrete means that for each element x there is a next element, written logically as

$$(\forall x)(\exists y)[(x<y) \ \wedge \ (\forall z)((z\leq x) \ \vee \ (y\leq z))].$$

They should also be able to use formal logical notation to express fundamental geometric notions, such as the definition of when two straight lines are parallel or when they overlap or are skew. Similarly, they should know the definition of when two planes are parallel or when they overlap, and they should use logic to recognize that the relation of being parallel for straight lines or for planes or for direct line segments is an equivalence relation.

2.2 The Ability to Take Negation of Mathematical Sentences

When working with the definitions of mathematical notions, we often have to figure out whether a particular object satisfies this concept. For this we have to formulate the negation of the definition of the concept, and then bring this negation to such equivalent sentences in which the negation applies only to elementary propositions. Such a transition is carried out using equivalent transformations of definitions of mathematical notions. So an important skill in the process of equivalent transformations of the structure of mathematical propositions is the ability to take the negation of the logical form of propositions using the laws of propositional and predicate logic.

For instance, prospective mathematics teachers need to understand that each of the ordered sets $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ is linearly ordered. At the same time the set of all subsets of some set A, ordered with respect to inclusion of subsets, $\langle \boldsymbol{P}(A);\ \subseteq\rangle$ is not linearly ordered. In particular, it is easy to find two sets X and Y that satisfy negation of linear order $\neg(\forall X)(\forall Y)[(X\subseteq Y) \vee (Y\subseteq X)]$, i.e.

$$(\exists X)(\exists Y)[\neg(X \subseteq Y) \ \wedge \ \neg(Y \subseteq X)].$$

Also, they should be able to write the negation of being discrete,

$$\begin{aligned} &\neg(\forall x)(\exists y)[(x<y) \ \wedge \ (\forall z)((z\leq x) \ \vee \ (y\leq z))] \\ \cong \ &(\exists x)(\forall y)[(x<y) \ \rightarrow \ (\exists z)((x<z) \ \wedge \ (z<y))]. \end{aligned}$$

Students often confuse the negation of a property with an alternate property, for example distinguishing between functions that are not increasing versus those that are decreasing. For f to be non-increasing means that

$$\begin{aligned} &\neg(\forall x_1)(\forall x_2)[x_1<x_2 \ \rightarrow \ f(x_1)<f(x_2)] \\ \cong \ &(\exists x_1)(\exists x_2)[(x_1<x_2) \ \wedge \ f(x_1) \ \geq f(x_2)], \end{aligned}$$

whereas for f to be decreasing means that

$$(\forall x_1)(\forall x_2)[x_1 < x_2 \ \rightarrow \ f(x_1) > f(x_2)].$$

Students can work with rewriting sentences to see the difference in properties defined two different ways. For example, parallel lines can be defined in a broad sense as that they are either equal straight lines or lines that do not intersect, or in the narrow sense that the lines lie in the same plane and do not have common points. The relation of parallelism in the narrow sense is

- *anti-reflexive*: $(\forall a)(\neg(a \parallel a))$,
- *symmetric*: $(\forall a)(\forall b)[(a \parallel b) \ \rightarrow \ (b \parallel a)]$,
- *transitive*: $(\forall a)(\forall b)(\forall c)[((a \parallel b) \ \wedge \ (b \parallel c)) \ \rightarrow \ (a \parallel c)]$,

while the relation of parallelism in broad sense is *reflexive* $(\forall a)[(a \parallel a)$, *symmetric*, and *transitive*, i.e. it is an equivalence relation.

2.3 Equivalent Transforms of the Structure of Mathematical Statements—Definitions and Theorems

It is crucial here to learn how to determine which mathematical statements are equivalent to what, i.e. to learn how to transform the structure of mathematical statements to equivalent forms. Examples of such equivalents at the level of propositional logic are as follows:

$$\begin{aligned} A \rightarrow B \ &\cong \ \neg B \rightarrow A \ ; \\ A \rightarrow (B \wedge C) \ &\cong \ (A \rightarrow B) \wedge (A \rightarrow C) \ ; \\ (A \vee B) \rightarrow C \ &\cong \ (A \rightarrow C) \wedge (B \rightarrow C) \ ; \\ A \rightarrow (B \vee C) \ &\cong \ (A \wedge \neg B) \rightarrow C \ \cong \ (A \wedge \neg C) \rightarrow B. \end{aligned}$$

The discrete topic of number theory can frequently be used to help students understand the truth of seemingly abstract statements such at these, as the following example demonstrates.

An example of the statement $(A \wedge B) \rightarrow C \ \cong \ (A \wedge \neg C) \rightarrow \neg B$ is that, *If n is divisible by 3 and is divisible by 5, then n is divisible by 15,* is logically equivalent to: *If n is divisible by 3 and is not divisible by 15, then it is not divisible by 5.*

Geometry also offers a good means for making these statements clear, as the next example shows.

The sentence *If two lines lie in parallel planes, they are either parallel or skew lines* is logically equivalent to the sentence *If two lines lie in parallel planes and are not parallel, then they are skew lines*, that is, $A \to (B \vee C) \cong (A \wedge \neg B) \to C$.

The transformation ability of theorems is essential for teachers of mathematics. The transformation procedure helps one to consider the same one theorem per se from various logical points of view. Such consideration gives an opportunity, firstly, to replace the proof of the theorem in one formulation with its proof in the equal one, and then return to the original formulation by a purely logical method. Secondly, such consideration helps us involve mental mechanisms of the subconscious (intuition) to search for ways to prove the theorem. The more logic equivalences are acquired, the higher is the logical culture of the teacher.

Consider the following sign of parallelism of two planes: "*If every plane that crosses one of the two planes α and intersects the other, then planes α and β are parallel*".

Written at logical-mathematical language, this statement becomes:

$$(\forall\gamma)[(\gamma \times \alpha) \to (\gamma \times \beta)] \to (\alpha \parallel \beta). \quad (*)$$

The entry $\gamma \times \alpha$ means that planes γ and α intersect, i.e. γ and α are different and have at least one common point. Then the denial of intersecting two planes $(\gamma \times \alpha)$ means that the planes γ and α are either the same or have no common points, i.e. $\gamma||\alpha$. Converting the original theorem to its equivalent using the law of contraposition, $A \to B \cong \neg B \to \neg A$, gives:

$$\begin{aligned}(*) &\cong \neg(\alpha \parallel \beta) \to \neg(\forall\gamma)[(\gamma \times \alpha) \to (\gamma \times \beta)] \\ &\cong \neg(\alpha \parallel \beta) \to (\exists\gamma)[\neg((\gamma \times \alpha) \to (\gamma \times \beta)] \\ &\cong \neg(\alpha \parallel \beta) \to (\exists\gamma)[\neg(\neg\gamma \times \alpha) \vee (\gamma \times \beta)] \\ &\cong \neg(\alpha \parallel \beta) \to (\exists\gamma)[(\gamma \times \alpha) \wedge \neg(\gamma \times \beta))].\end{aligned}$$

So, instead of (*) we can prove the equivalent statement: *If the planes α and β are not parallel, then there exists a plane γ that intersects one of them and does not intersect the other*. It's enough to hold the plane γ through any point of the plane α parallel to the plane β, which is always possible if α and β intersect, which in our case takes place.

2.4 Analyses of Structure of Converse Sentences

The ability to equally transform the logical structure of mathematical sentences helps to better understanding the essence of relationships between a direct theorem and its converse sentences of various kinds.

So, for theorems that have a logical structure $(A_1 \wedge A_2) \to B$, we can specify two equivalent forms that represent conditional statements: $A_1 \to (A_2 \to B)$ and $A_2 \to (A_1 \to B)$. This can be clearly using graph theoretic examples, such as, the statement, if there exists a path from x to y and a path from y to z, then there exists a path from x to z, is logically equivalent to the statement, if there exists a path from x to y, then if there exists a path from y to z, then there exists a path from x to z.

Each of these forms has a converse statement:

$$B \to (A_1 \wedge A_2)\ ,\ \ (A_2 \to B) \to A_1\ ,\ \ (A_1 \to B) \to A_2.$$

It would be illuminating for students to write these converse statements using the previous graph theoretic example and then discuss their meaning.

In addition, we can consider the following converse forms of these theorems, taking the converse statements for $(A_2 \to B)$ and $(A_1 \to B)$:

$$A_1 \to (B \to A_2)\ \cong\ B \to (A_1 \to A_2)\ \cong\ (A_1 \wedge B) \to A_1,$$
$$A_2 \to (B \to A_1)\ \cong\ B \to (A_2 \to A_1)\ \cong\ (A_2 \wedge B) \to A_1.$$

It is easy to see that, among these five forms of converse statements, no two are equivalent. For each direct theorem of the form $(A_1 \wedge A_2) \to B$, not all of these forms will be true statements.

For example, consider the following theorem, having the structure $(A_1 \wedge A_2) \to B$: «*If the circles are equal* (A_1) *and belonging to them chords are equal too* (A_2), *then chords are equidistant from the centers of their circles* (B)». We formulate for it the converse all five of these forms.

1. $B \to (A_1 \wedge A_2)$: *If two chords of two circles equidistant from the respective centers of the circles, then these circles are equal and the chords are equal to each other.*
2. $(A_2 \to B) \to A_1$: *If equal chords of two circles implies these chords are equidistant from the centers of their circles, then these circles are equal.*
3. $(A_1 \to B) \to A_2$: *Being chords in equal circles, they are equidistant from the centers of these circles, these segments are equal.*
4. $A_1 \to (B \to A_2)$: *In equal circles, chords equidistant from a respective center equal.*
5. $A_2 \to (B \to A_1)$: *If the chords held in the two circles are equal, then the equidistance of these chords from the centers of the corresponding circles implies the equality of the circles.*

It is easy to show that the first inverse assertion is the only one that is true, and the rest are false.

3 The Principle of Teaching the Concept of Proof of the Mathematical Theorem

A mathematical theorem is a statement that can be proven true using what we call a proof. In turn, a proof is a kind of reasoning undertaken by one person and designed to convince other people of the truth of the assertion.

Mathematical logic allows us to characterize the logical form of arguments in order to determine whether our reasoning is correct. So, in the proof of a mathematical theorem, having the structure $A \to B$: *If A*, *then B*, we assume that there is a (true) condition of the theorem, A. Then we start to talk, bringing a sequence of claims $C_0, C_1, \ldots, C_m$. Each of these statements follows from statement, an axiom, or is derived from preceding statements of the sequence by the rule of inference **Modus Ponens (MP)**: *from statements P and* $P \to Q$ *it follows statement* Q, and the final statement C_m is the statement B.

Constructing such a chain, we prove that A deduces B, with the result that we conclude that the theorem $A \to B$: *If A, then B* is true. The rationale for this transition is a logical theorem of deduction. When proving the theorem, we need to strive to ensure that the chain of successive claims looms clearly in the mind of the learner.

Let's consider an example. A parallelogram is a quadrilateral whose opposite sides are parallel. We prove the theorem: *In a parallelogram the opposite sides are equal*. We give first the proof of this theorem, available in the usual geometry textbook:

> Let *ABCD* be a parallelogram, as seen in Fig. 1. Let the diagonals be *AC* and *BD* and let *O* be the common point of intersection. Then the triangles *AOB* and *COD* are equal because their angles at vertex *O* are equal, and $OA = OC$ and $OB = OD$ which is a property of the diagonals of a parallelogram (the point of intersection divides each diagonal in half). From the equality of triangles the sides: $AB = CD$ are equal. Similarly, from the equality of triangles *AOD* and *BOC* from which follows the equality of the other pair of opposite sides: $AD = BC$

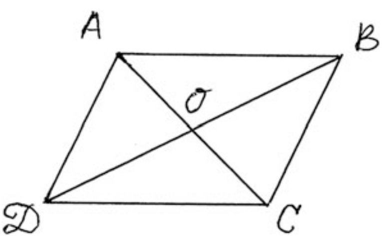

Fig. 1 Parallelogram *ABCD*

We present this proof as a sequence (chain) of claims that make it up:

(a) *ABCD*—parallelogram (the condition);
(b) *O*—the intersection point of its diagonals (to build);
(c) $\angle AOB = \angle COD$, $\angle AOB = \angle COD$ (as vertical angles, the vertical angles property);
(d) $OA = OC$ (property of diagonals of a parallelogram);
(e) $OB = OD$ (property of diagonals of a parallelogram;
(f) $\Delta AOB = \Delta COD$ (from the c, d, e by the basis of equality of triangles);
(g) $AB = CD$ (from the e by definition of congruent triangles).

Extend this sequence of statements to a sequence of statements that provide the evidence:

1. *ABCD*—parallelogram (hypothesis)
2. *O*—the intersection point of its diagonals (hypothesis)
3. $\angle AOB$ and $\angle COD$ are vertical angles (hypothesis);
4. If angles $\angle AOB$ and $\angle COD$ are vertical angles then $\angle AOB = \angle COD$ (previously proven theorem)
5. $\angle AOB = \angle COD$ (MP): (3), (4)
6. *ABCD* parallelogram and *O*, intersection point of diagonals $\wedge$-intr.: (1), (2)
7. If *ABCD*—parallelogram and *O*—the intersection point of its diagonals then $OA = OC$ (previously proven theorem)
8. $OA = OC$ (MP): (6), (7)
9. If *ABCD*—parallelogram and *O*—the intersection point of its diagonals then $OB = OD$ (previously proven theorem)
10. $OB = OD$ (MP): (6), (9)
11. $\angle AOB = \angle COD$ and $OA = OC$ and $OB = OD$; $\wedge$-intr.: (5), (8), (10)
12. If $\angle AOB = \angle COD$ and $OA = OC$ and $OB = OD$, then $\Delta AOB = \Delta COD$ (proven earlier, the sign of equality of triangles)
13. $\Delta AOB = \Delta COD$ (MP): (11), (12)
14. If $\Delta AOB = \Delta COD$, then $AB = CD$ (by definition of equal triangles);
15. $AB = CD$ (MP): (13), (14).

Similarly we can prove that $AD = BC$.

Statements (5), (8), (13) and (15) use a rule of inference, Modus Ponens (MP, or rule of removal of implication): $P, P \rightarrow Q \models Q$, and statements (6) and (11) use the rule of introduction of conjunction ($\wedge$-intr.): $P, Q \models P \wedge Q$, or $P, Q, R \models P \wedge Q \wedge R$.

So, from the hypothesis "*ABCD is a parallelogram*" we prove the statement "$AB = CD$". As a result, we conclude that we have proved the theorem: *If ABCD is a parallelogram, then AB = CD*. This conclusion is based on the logical rule of introduction of implication: *If* $\Gamma, P \models Q$, *then* $\Gamma \models P \rightarrow Q$. Here .$\Gamma$ is some set of known (previously proven) theorems, in our case, the theorems on the equality of vertical angles, about the point of intersection of the diagonals of a parallelogram, the equality of triangles; P: *ABCD is a parallelogram*; Q: $AB = CD$.

In connection with this example, let us note two circumstances. First, when building a chain-proof $C_0, C_1, \ldots, C_m$, we can use those two rules of logical inferences (MP) and $\wedge$-intr that we used in the previous example, but also a number of others. Here are some of them:

1. $P \wedge Q \models P$; $P \wedge Q \models Q$ (removal of conjunction)
2. $P \models P \vee Q$; $Q \models P \vee Q$ (introduction of disjunction)
3. $P \to Q,\ Q \to R \models P \to R$ (rule of the syllogism, or chain conclusion)
4. $P \to Q \models \neg Q \to \neg P$ (rule contraposition)
5. $P \to Q$, $\neg Q \models \neg P$ (rule Modus Tollens)
6. $(P \wedge Q) \to R \models (P \wedge \neg R) \to \neg Q$ (rule extended contraposition)
7. $P \to (Q \to R) \models (Q \to P) \to R$ (rule rearrangement of parcels)
8. $P \to R$, $Q \to R \models (P \vee Q) \to R$ (rule parsing cases)
9. $P_1 \to Q$, $P_2 \to Q$, $P_1 \vee P_2 \models Q$ (simple constructive dilemma)
10. $P_1 \to Q_1$, $P_2 \to Q_2$, $P_1 \vee P_2 \models Q_1 \vee Q_2$ (complex constructive dilemma)
11. $P \to Q_1$, $P \to Q_2$, $\neg Q_1 \vee \neg Q_2 \models \neg P$ (simple destructive dilemma)
12. $P_1 \to Q_1$, $P_2 \to Q_2$, $\neg Q_1 \vee \neg Q_2 \models \neg P_1 \vee \neg P_2$ (complex destructive dilemma)
13. $\neg\neg P \models P$ (strong deletion of negation)
14. P , $\neg P \models Q$ (weak deletion of negation).

Here is an example of inference performed by the rule of extended contraposition, where P represents "*The straight line l is perpendicular to two straight lines a and b lying in the plane* π"; Q represents "*Straight lines a and b are not parallel*" and R represents "*The straight line l is perpendicular to every straight c lying in the plane* π".

$(P \wedge Q) \to R$: *If a straight line l is perpendicular to two straight lines a and b lying in the plane* π*, and lines a and b are not parallel, then line l is perpendicular to any line c lying in the plane* π.

$(P \wedge Q) \to R (P \wedge \neg R) \wedge \neg Q$: *If a straight line l is perpendicular to two straight lines a and b lying in the plane* π*, and l is not perpendicular to some line c lying in this plane, then the straight lines a and b are parallel.*

The given inference rules are a mathematical formalization of those thought processes that occur in our brain when reasoning in the search for proofs of mathematical theorems.

Secondly, if we have already proven any theorems, they can of course be included in the proof $C_0, C_1, \ldots, C_m$, as in the example.

4 The Principle of Teaching the Ways and Methods of Proving Mathematical Theorems

4.1 Direct (Synthetic) and Backward (Analytical) Ways of Searching for Proofs of Mathematical Theorems

So, a proof of the theorem $A \to B$ is a chain of claims $C_0, C_1, \ldots, C_m$. How to find this chain? There are two methods of finding this chain. The first is synthetic or direct which is the construction of the chain in the forward direction, i.e. from A to B. The second is analytical or ascending analysis in which the construction of the chain is in the opposite direction, from B to A.

In Sect. 3, the proof of $A \to B$ applied the synthetic method. Let's see how we can reason when we build a proof of the same theorem $A \to B$ in the analytical method, bottom-up parsing.

As the first step in proving that in a parallelogram $ABCD$, the opposite sides are equal $AB = CD$, we need to represent the segments AB and CD as the corresponding sides of equal triangles. This can be done by noting the parallelogram diagonals AC and BD intersect at point O. This forms two triangles ΔAOB and ΔCOD.

The second step is to prove that $\Delta AOB = \Delta COD$. It is enough to prove that they satisfy the conditions of one of the theorems for the equality of triangles. This we can do by noting that (Step 3) $OB = OD$, by the property of the diagonals of a parallelogram, the point of intersection bisects the diagonal. Similarly (Step 4), $OA = OC$, by the same property. Finally (Step 5), $\angle AOB = \angle COD$ by the vertical angles property.

The analytical way allowed us to clearly find a way to prove the theorem. Now, to give (synthetic) proof of the theorem, we need to travel back: 5-4-3-2-1. □

From this example we can see that the analytical and synthetic ways are reversible. In the analytical method, we turn to the opposite claims in relation to the method that we operate in a synthetic proof. The analytical method is a way of searching for proofs; it allows the teacher and students to work together to find a way to prove the theorem, with the result that students become active participants in the learning process. The synthetic method is more prescriptive, algorithmic; analytical—more heuristic.

The logic of these methods of constructing proofs is the following. With synthetic, the way of finding a proof searches for steps in the forward direction, from the condition to the conclusion, and are carried out according to the rules of logical connections. These rules serve as prompts to the steps of the proof in the forward direction, which helps to move from top to bottom. The analytical way is finding evidence in the opposite direction, from the conclusion to the condition, *bottom-up*. The steps are in the opposite direction, from the conclusion to the condition, and are carried out by rules of introduction of logical connections. Using these rules, there is the need to find sufficient conditions for the execution of the assertion, and the challenge for the proof of the approval is reduced to the task of proving more simple assertions.

4.2 Additional Methods of Proving Mathematical Theorems

The most important method for proving mathematical theorems, dating back to Euclid, is the method of *contradiction*. It is based on the following equivalence of the propositional algebra (the law of contraposition): $X \to Y \cong \neg Y \to \neg X$.

It consists of the following. Let it be required to prove the assertion (theorem) "If X, then Y", which in symbolic form is represented as $X \to Y$, i.e., from the assumption X to draw the conclusion of Y. This is equivalent to assuming the negation of Y, $\neg Y$, is true and hence proving $\neg X$, i.e. we prove theorem $\neg Y \to \neg X$: "If $\neg Y$ then $\neg X$." Hence the conclusion that the theorem $X \to Y$ is true. This conclusion is based on the logical law of contraposition $\neg Y \to \neg X \cong X \to Y$ establishing equivalence of these allegations. Thus, the proof of the theorem $X \to Y$ actually is replaced by the proof of theorem $\neg Y \to \neg X$, opposite inverse (or inverse opposite) for this theorem.

The method of proof by contradiction can also be given such an interpretation. We assume true X and $\neg Y$, and deduce hence $\neg X$, i.e. prove the theorem $(X \wedge \neg Y) \to \neg X$. Hence we conclude the original theorem $X \to Y$. It is easy to see that the basis for this conclusion is easy to check the logical equivalent: $X \to Y \cong (X \wedge \neg Y) \to \neg X$.

Discrete math offers many easy to follow examples of the use of contradiction, one famous one being the following example.

A famous result from graph theory is that if every vertex in a connected graph has an even number of edges incident to it, then there exists a path that uses every edge exactly once, returning to the vertex at which it started. One proof of this result is to assume it is not true. We then assume we have a path that uses as many edges as possible, but does not use all of them. This path must return to the first vertex used in the path, since it is the only vertex left with an odd number of incident edges: For every other vertex, if there is an edge in, there must be an edge out because there are an even number of incident edges. So our largest path is a cycle. We now argue that there must exist a larger path. This is because since our largest cycle does not use all edges and the graph is connected, there must be an unused edge with one vertex on our cycle. Starting at this vertex and using this unused edge, we continue to use unused edges until we cannot go any further. We must have returned to the original vertex of this path, since it would be the only vertex with an odd number of unused edges. This forms a cycle which can now be inserted into the original cycle, forming a larger cycle, which contradicts that our original cycle was the largest.

Consider this final interpretation of method of proof by contradiction. To prove the assertion $X \rightarrow Y$, we again assume true X and $\neg Y$, but instead derive the assertions Z and $\neg Z$ (not related to the theorem $X \rightarrow Y$). Thus prove the theorem

$$(X \wedge \neg Y) \rightarrow (Z \wedge \neg Z).$$

Hence we conclude the original theorem $X \rightarrow Y$. Again it is easy to see that the basis for this conclusion corresponds to the logical equivalent:

$$X \rightarrow Y \cong (X \wedge \neg Y) \rightarrow (Z \wedge \neg Z).$$

We now consider the following method of proving theorems—*the method of reduction to absurdity* (Latin *reductio ad absurdum*). It has two modifications, which, as we shall see below, are substantially different in form and in substance. This is the method of bringing the opposing claim to absurdity and the method of bringing data claim to absurdity.

The method of *bringing opposing statement to absurdity* consists in the following. Let it be required to prove the statement X. Consider the opposite statement $\neg X$ and from it derive two contradictory statements (i.e. a statement and its negation) Y and $\neg Y : \neg X \rightarrow Y$ and $\neg X \rightarrow \neg Y$. From this it is concluded that the original assertion X is true. This proof-logic can be represented as a formula of the proposition algebra about which it is easy to prove that it is a tautology: $(\neg X \rightarrow \neg Y) \rightarrow ((\neg X \rightarrow Y) \rightarrow X)$.

The method of *bringing statement to absurdity* consists in the following. Suppose that we want to disprove the statement X, i.e. to prove a negative statement $\neg X$. In this case, two contradictory statements Y and $\neg Y$ are derived from X, the denial of $\neg X$, that is, we show the statement $X : X \rightarrow Y$ and $X \rightarrow \neg Y$. The conclusion that the statement $\neg X$ is true, i.e., statement X is refuted. This logic proof can be represented as a formula of the proposition algebra: $(X \rightarrow \neg Y) \rightarrow ((X \rightarrow Y) \rightarrow \neg X)$. It is easy to check that this formula is a tautology. We emphasize again that this method is applicable for the proof of negative statements.

In school geometry courses, it is shown that in any triangle: (1) *the square of the length of the side lying opposite an acute angle is less than the sum of the squares of the lengths of the other two sides*; (2) *the square of the length of the side opposite a right angle is equal to the sum of the squares of the lengths of the other two sides* (theorem of Pythagoras); (3) *the square of the length of the side lying opposite an obtuse angle is greater than the sum of the squares of the lengths of the other two sides of this triangle*. We introduce the following notation for statements:

A_1: «*In the triangle the angle* α *is acute*»;
A_2: «*In the triangle the angle* α *is right*»;
A_3: «*In the triangle the angle* α *is obtuse*»;
B_1: «$a^2 < b^2 + c^2$»;
B_2: «$a^2 = b^2 + c^2$»;
B_3: «$a^2 > b^2 + c^2$»;
where a, b, c are the lengths of the sides of a triangle, α is the angle opposite side

a. Then the three theorems can be written symbolically as: $A_1 \to B_1$, $A_2 \to B_2$, $A_3 \to B_3$. It is clear that of the three assumptions A_1, A_2, A_3, at least one true, that is, the angle α in the triangle must be either acute or right or obtuse. The consequences B_1, B_2, B_3 are pairwise mutually exclusive due to the properties of trichotomy for real numbers: for any two real numbers r and s, one, and only one of the following three relations is true: $r < s$, $r = s$, $r > s$.

If B_1 is true, then $\neg B_2$ and $\neg B_3$ so by contraposition $\neg A_2$ and $\neg A_3$. This means A_1 is true, leading to $B_1 \to A_1$, Similarly, we can show $B_2 \to A_2$ and $B_3 \to A_3$. In particular, theorem $B_2 \to A_2$, the converse of the Pythagorean theorem, reads as follows: If in a triangle the square of the length of one side is equal to the sum of the squares of the lengths of the other two sides, then the triangle is rectangular, and the right angle is the angle lying opposite the first side.

This is an example of *the principle of full disjunction*. Assume all the following:

$$(m > 2): \quad A_1 \to B_1 \ , \ A_2 \to B_2, \ldots, A_m \to B_m,$$

and from the assumptions $A_1, A_2, \ldots, A_m$, at least one is true. Also assume that the consequences $B_1, B_2, \ldots, B_m$ are pairwise mutually exclusive (i.e. no two different consequences not be true simultaneously. Then all inverse implications

$$B_1 \to A_1 \ , \ B_2 \to A_2, \ldots, B_m \to A_m$$

are true. This is essentially a logical theorem about mathematical theorems. Such theorems in logic are called metatheorems to distinguish them from the theorems of mathematical theories.

The essence of the principle of full disjunction is that it guarantees the truthfulness of converse statements to a special set of direct statements of one or another theory. In this case, it is not necessary to prove the converse statements of this theory if corresponding direct statements proven.

The principle of full disjunction has wide applications in mathematics. Classic is its application to the proof of the converse of the Pythagorean theorem, given above. Students often confuse the Pythagorean theorem and the converse, especially when it is necessary to use to prove other theorems or find the solutions to problems. This approach will make the students more closely relate to the merits of the concept of the converse theorem.

5 The Principle of Learning the Structure of Mathematical Theories

This refers to both an understanding of the axiomatic idea for the construction of a mathematical theory and teaching it. This includes the comprehension of the initial, undefined concepts of the theory, its axioms and theorems, including the meta-theory. In other words, the properties of this theory; consistency, completeness, categoricity, and independence of the axiom-system. The subject matter of *Axiomatic theory* as part of a course on mathematical logic must be continued in the pedagogy of all other mathematical courses at the university level.

In each of these courses the relevant axiomatic theories which are at the base of the relevant mathematical discipline must be consider in terms of mathematical logic. These mathematical principles will naturally extend into the foundation of the relevant school mathematics discipline. So, the axiomatic theory of numerical systems is the basis for the school course of algebra and elements of analysis, while the axiomatic construction of geometry based on the axiom systems of D. Hilbert and H. Weyl are the grounds of a school course of geometry.

Almost all mathematicians and methodologists agree that a purely axiomatic teaching of geometry in school is pointless and not necessary. "*Should axiomatics be taught in schools?*" asks Freudenthal (1977 p. 451), and says, "If it is taught in the form it has been in the majority of projects in the last few years, I say 'no'. Prefabricated axiomatics is no more a teaching matter in school instruction than is prefabricated mathematics in general." Arguing this thesis, he notes: "Euclidean geometry is acted out unaxiomatically by all reasonable people… As a full-fledged mathematician I am allowed to exercise geometry unaxiomatically because this is the indispensable preliminary stage of the axiomatic organization of the subject matter." (Freudenthal 1977 p. 449). "But what is judged to be essential in axiomatics by the adult mathematician, I mean *axiomatizing*, may be a teaching matter." (Freudenthal 1977 p. 451). Every adult mathematician knows that "axiomatic systems of Euclidean geometry are not created for exercises in Euclidean geometry but for metageometric exploration, for research into the foundations of geometry… If the axiomatic system of Euclidean geometry are considered, the business that matters is to reason *about* the axioms, to explore their mutual relations, their dependence and independence, their completeness." (Freudenthal 1977 p. 449).

Prospective mathematics teachers need to learn all these issues related to the axiomatic construction of mathematical theories. Propositional calculus can serve as an excellent model for studying all these methodical issues.

Consider the following formulas of proposition algebra (as we have said above that some of them model or simulate some methods of proving mathematical theorems):

(A3) $(\neg G \rightarrow \neg F) \rightarrow ((\neg G \rightarrow F) \rightarrow G)$ (the method of bringing inverse statement to absurdity);

(A3′) $(\neg G \rightarrow \neg F) \rightarrow (F \rightarrow G)$ (the method by contraposition);

(A3″) $(G \rightarrow \neg F) \rightarrow ((G \rightarrow F) \rightarrow \neg G)$ (the method of bringing data statement to absurdity).

Let us consider three formulas of proposition algebra further:

(A1) $F \rightarrow (G \rightarrow F)$,

(A2) $(F \rightarrow (G \rightarrow H)) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow H))$,

(A4) $\neg\neg F \rightarrow F$.

Formed from these formulas are four systems of axioms:

$$\sum_1 = \{(A1),\ (A2),\ (A3')\},$$
$$\sum_2 = \{(A1),\ (A2),\ (A3)\},$$
$$\sum_3 = \{(A1),\ (A2),\ (A3'')\},$$
$$\sum_4 = \{(A1),\ (A2),\ (A3''),\ (A4)\}.$$

On the basis of these systems of axioms will build a formal propositional calculus: $T_i = Th\left(\sum_i\right)$, $i = 1, 2, 3, 4$. Regarding them, we prove the following theorems (meta-theorems).

Theorem 1 $T_1 = T_2$.
Theorem 2 T_3 is included in T_2, but $T_3 \neq T_2$. (A formula belonging to T_2 but not by T_3, is the formula (A4).)
Theorem 3 $T_4 = T_2$.

These formal-logical results can be meaningfully interpreted in the following way:

- Method of proof by contradiction is equivalent to the method of bringing opposing statement to absurdity (Theorem 1)
- Method of proof by bringing this claim to absurdity is a weaker method of proof then by bringing opposite statement to absurdity (Theorem 2), and therefore the method of proof by contraposition;
- Method of proof by bringing data statement to absurdity is comparable in deductive power with the method of proof by bringing inverse statement to absurdity, as well as with the method of proof by contraposition, if there is added the opportunity to use in the process of proving a rule of removing a double negation (A4) (Theorem 3).

6 Educating Mathematics Teachers in Accordance with the Principles of Logic

The considered principles of logic indicate the main directions for implementation of logic into mathematics teaching. When these principles are absent in the learning of mathematics, mathematics loses its basic features as a science, i.e. those qualities which actually distinguish it from the system of other sciences. In the end, the learner receives a corrupted overall perception of both mathematics and its parts.

Therefore, in educating and developing would-be mathematics teachers, it is necessary to pay particular attention to their training in the field of logic. Taking into consideration the importance of logic in mathematics as a science and in teaching mathematics in general, the expertise in logic for would-be mathematics teachers is a most essential part of their training. This training must consist of two parts which are actual logical training per se and logical-didactic training.

Firstly, the foundation of training for would-be mathematics teachers is a course on mathematical logic which is aimed at teaching as their future main professional activity (Igoshin 2005, 2010, 2014, 2016, 2017). Here students acquire the knowledge and develop the skills in logic that will be in demand in their future teaching.

Secondly, the concepts, ideas and methods of mathematical logic from the fundamental course of mathematical logic are implemented in higher mathematical courses in the school curriculum such as geometry, algebra, number theory, mathematical analysis, number systems, discrete mathematics, theory of algorithms (Igoshin 2013, 2016), as well as into disciplines of psychological and pedagogical basis of teaching mathematics, methods of mathematics teaching, history and methodology of mathematics. In these courses students' attention is focused on those issues which are of fundamental logical value.

Thirdly, through higher mathematics courses, the ideas and methods of mathematical logic are supposed to naturally extend into the base of the relevant school mathematics discipline. The courses on teaching methods at universities aim to demonstrate how the knowledge of logic is used in teaching specific topics of school mathematics. Moreover, it is crucial to analyze in terms of logic the entire school course of mathematics both in general and some of its particular details (Igoshin 2012).

7 Concluding Remarks on the Teaching of Mathematics: Logic (Rigor) and Intuition (Visualization)

Despite its logic and rigor, mathematics could not develop without intuition and visualization. As we know, in mathematics, known facts are proven using logic, but these facts are discovered with the help of intuition and visualization. Logic is the tool of proof, intuition is the tool of invention. Logic and intuition are inseparable components of mathematical creativity. Consequently, in teaching mathematics, both of these components must be used. The importance of intuition in both science and education was emphasized by H. Poincare: "We need an ability that would allow us to see the target from a distance, and this ability is intuition. It is required by the researcher in selecting the way, it is no less necessary for the one who follows in his footsteps and wants to know why he chose it." (Poincaré 1983 p. 166).

In teaching mathematics, the issue of the relationship between logic and intuition is particularly vital when it is necessary to define the standard and criteria of assessment for teaching different parts of the course. One of the main tasks of any teacher is to achieve *conscious* acquisition of the presented material by the students.

Absolutely logically rigorous and perfect step-by-step proof of the theorem does not always lead to understanding of this proof per se. Poincare (1983 p. 311) describes this circumstance in this way:

> A mathematical proof is not just some conglomeration of syllogisms: *a syllogism is located in a known order*, and the order of the elements is much more important than the elements themselves. If I have a sense, so to say the intuition, of this order, so that I can see at a glance all the arguments as a whole, I don't have to fear that I will forget any one of the elements; each of them itself would take its assigned place without any effort of memory on my part.

The same idea is expressed by Hadamard (1970 p. 63): "…any mathematical argument, however complicated, must appear to me as a unique thing. I do not feel that I have understood it as long as I do not succeed in grasping it in one global idea…"

Thus, understanding the proof of the theorem is not reduced to the understanding and validation of each step of the formal proving, but is achieved by understanding the general idea that has led to this sequence of steps. To clarify this idea, it is impossible to do without non-strict, intuitive thoughts and images. Intuitive aspects of the proof of a particular theorem, and moreover, a mathematical theory, help students better understand their rigorous logic and are critical for teaching. "Of course, we will learn to prove, but also will learn to guess," urged Polya (1954 pp. 15–16).

Many discrete math topics offer a method for helping students develop their intuition as the discreteness allows them to visual and demonstrate the statements within numerous examples. This is in contrast to many analysis steps, such as proving limits, in which visualization is much more difficult and the prover must resort more directly to the logical statements without developing as clear an intuition.

However, here we need to keep in mind the following. It may seem paradoxical, but in mathematics, both in science and in school subjects, in fact the proofs used are far from strict logical canons, and represent arguments, designed to convince certain people that this or that statement is true. These circumstances impose a certain subjectivity on such understanding of proving: what convinces one may be unconvincing for another. In this sense, it may seem that in mathematics, the notion of proof is vague and uncertain, and has some in common with proofs in the humanities. This is not true. Since Ancient Greece, mathematics has been distinguished from all other sciences by the fact that it is a convincing science. Mathematicians at all times more or less have agreed on whether this statement proves or not. As the history of mathematics shows, this sort of rigorous standard changed alongside the development of mathematics, and what seemed strict, for example, in XVII–XVIII centuries, was criticized in the XIX century. Many of the arguments given by the mathematicians of the XIX century were considered to be totally unconvincing to mathematicians, and especially logicians, of the XX century.

Therefore, although in modern mathematics there is no strict definition of a rigorous proof of a mathematical theorem, however, each mathematician understands rigor intuitively. In every mathematical proof he/she feels the majestic shadow of the immutable logical laws and criteria elaborated by the logic throughout the twenty centuries. Although the proof in all its details may not be given, intuitively, he/she realizes that it can be converted into such, though this will need a great deal of time and effort. It is important to feel the possibility of such transformation in principle. This intuitive sense of rigor must be developed in a prospective scientist/mathematician and in a prospective teacher of mathematics. Each of them will use fundamental logic-training gained in adolescence as the basis for this education. It will serve as a source of intuition about logical rigor and will feed this intuitive sense.

This concept of logical and logical-based didactic training of would-be teachers of mathematics makes it a strategic part of the whole training of would-be teachers of mathematics. Namely, this approach to logic makes students feel as clearly as possible the pervasive influence of logic on mathematics. Due to this approach, logical knowledge will become the foundation of a scientifically oriented pedagogical outlook for the would-be teachers of mathematics. This approach to logical training, of course, requires a high level of teachers` professional qualification at teacher training departments, and their considerable effort and cooperation, but these efforts will result in significantly improving the quality of training.

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