Chapter 11
Iconicity and Diagrammatic Reasoning in Meaning-Making

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Abstract The focus of this chapter is twofold. The first is a semiotic description of the nature of diagrams. The second is a description of the type of reasoning that the transformation of diagrams facilitates in the construction of mathematical meanings. I am guided by the Peircean definition of diagrams as icons of possible relations and his conceptualization of diagrammatic reasoning. When a diagram is actively and intentionally observed, perceptually and intellectually, a manifold of structural relations among its parts emerges. Such relations among the parts of the diagram can potentially unveil the deep structural relations among the parts of the Object that the icon plays to represent. An Interpreter, who systematically observes and experiments with diagrams, mathematical or not, also generates evolving chains of interpretants by means of abductive, inductive and deductive thinking. Using Stjernfelt's model of diagrammatic reasoning, which is rooted in Peircean semiotics, I illustrate an emergent reasoning process to prove two geometric propositions that were posed by means of diagrams.

Keywords Triadic sign · Iconicity · Diagram · Diagrammatic reasoning Proving

11.1 Introduction

Borrowing from Kant and Peirce, I first present a theoretical rationale to justify that perceptual and logical judgments are not only essential for diagrammatic reasoning but that they also go hand in hand with the active and passive workings of the mind in the construction of objects of knowledge. I also justify both why mathematical diagrams have important iconic characteristics that facilitate the elicitation of inferential thinking and why they evolve in the mind of the Interpreter to acquire

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symbolic levels that are essential for abstract mathematical thinking. I then present a rationale to justify that diagrammatic reasoning is essentially an inferential process and that mathematical diagrams serve as epistemological tools in the learning-teaching of mathematics. I finally analyze the proofs of two propositions, posed in the form of geometrical diagrams, using Stjernfelt's (2007) model for diagrammatic reasoning. In the conclusion I highlight the importance of diagrams, diagrammatic reasoning and the interpretation process they entail, for the learning-teaching of mathematics.

11.2 The Passive-Active Actions of the Mind in the Construction of Objects of Knowledge

For Kant (1781/2007), the objects of knowledge have two independent sources of representations, namely, sensibility and intelligence. He calls the first source *phenomena* or 'things-as-they-appear' and the second *noumena* or 'things-as-they-are'. Therefore, an object of knowledge is both sensible and intellectual or rational. It is sensible insofar as it is the product of the laws of sensibility, and it is intellectual or rational insofar as it is the product of the laws of intelligence.

Kant also argues that the mind can be influenced by a *thing* or can create an *object of thought*. When the mind establishes a relation with a *thing*, insofar as it is affected by it, then the mind is *passive* with respect to that experienced thing. He calls this relation *sensible intuition.* In contrast, when the *object* depends upon the mind, then the mind is *active* creating that object. He calls this relation *intellectual intuition.*

On the one hand, *sensible intuition* is the receptivity of the person through which it is possible that his/her power of representation is affected in a certain manner by the presence of some object of experience. The *object of sensibility* is the phenomenon. When an *object* affects the senses directly, it produces a variety of sensible intuitions—a manifold of sensations and perceptions. This manifold carries with it two kinds of elements: (i) a subjective or material element (colors, taste, hardness, etc.), which has no cognitive value; and (ii) a formal or knowledge-giving element, which is the spatiotemporal organization and ordering of sensations that facilitates the formation of *perceptual judgments* (Wolff 1973).

On the other hand, *intellectual intuition* is the faculty of the person that enables the representation of things, which cannot act upon the senses by their own character. The *object of intelligence* is the intelligible which contains nothing except what must be known through intelligence—the noumenon. Kant contends that *intelligence* can have two uses: the real use and the logical use. The real use generates representations of objects or relations out of inner resources, thereby giving concepts to the mind. In contrast, the logical use orders and compares concepts, whatever their origin, in systems of species and genera according to the laws of logic. Both the real and logical uses of intelligence require the formation of *conceptual judgments* to complement and expand perceptual judgments (Wolff 1973).

What is a *judgment* for Kant? It is an act of the intellect in which two ideas, comprehended as different, are compared for the purpose of ascertaining their agreement or disagreement (Wolff 1973). Judgments are usually expressed in propositions composed of subject, copula or linking verb, and predicate.

Borrowing from Kant, Peirce argues that perceptual judgments on the particular and concrete contain general elements from which one can intuit general patterns, universal propositions, and principles. Perceptual judgments, he writes, are also related to the more deliberate and conscious processes of inferential reasoning, and this reasoning is continuous and carries with it the vital power of self-correction and refinement (Peirce 1992). That is, for Peirce, *all* knowledge is the product of the self-corrective activity of the mind. He also contends that there is nothing in the intellect that has not been first in the senses (CP 8.738) and that

> realities compel us to put some things into very close relation and others less so; but in the end, it is only the genius of the mind that takes up all those *hints of sense*, adds immensity to them, makes them *precise*, and shows them in an intelligible form of intuitions of space and time. (CP 1.383)

Both Kant and Peirce contend that *observation* has epistemological value and power because it genuinely depends on both sensible and intellectual intuitions. They argue that observation is tied to judgment, and that judgment is tied to intentionally planned reasoning. Peirce also contends that any inquiry activity, fully carried out by a person, is rooted in observation. For example, he writes, when different people observe a geometric diagram, they are able to *see* different relations, some *perceived* by the senses and some *inferred* with the aid of collateral knowledge. He also adds that collateral knowledge is a prerequisite in the apprehension and construction of new meanings (Peirce 1998).

Consequently, it can be said that in the *observation of geometric diagrams*, sensible and intellectual intuitions, collateral knowledge, perceptual and intellectual judgments, altogether, trigger abductive, inductive, and deductive inferences. Then it can also be said that the transformation of a diagram-token (an object of perception) into a diagram-symbol (an object of thought) is the product of the intertwined passive-active actions of the mind. Thus, a person's perceptual and intellectual judgments with and through diagrams are mediated by both sensible and intellectual intuitions.

11.3 Diagrams Initially Seen as Sign Vehicles of Iconic Nature

To say that 'diagrams are icons' is a very general and even strange statement when considered in colloquial speech. Nonetheless, it makes sense in the Peircean semiotics in which the notion of 'sign' is a manifold of elements and relations among them. Here I use only capital letters for the word SIGN to unambiguously signify not only its three constituent elements but also the three possible dyadic relations among them. The three constituent elements are: the *sign vehicle*,[1] the *interpretant*, and the *Object*. The three dyadic relations are: (1) between the sign vehicle and the Object it plays to represent; (2) between the sign vehicle and the interpretant it determines; and (3) between the interpretants generated and the dynamic object they progressively construct (i.e., an *object* which approximates the Object that the sign vehicle purports to represent and which is in a continuous state of refinement as new and more sophisticated interpretants are gradually generated). These three relations are not isolated from each other but interdependent on one another.

It is important to note that the *interpretant* is not the *Interpreter*. The interpretant is the effect of the sign vehicle on the mind of the Interpreter. The Interpreter, instead, is an agent who takes part in and presumably exerts control over the process of interpretation. Colapietro (1993) argues that the interpretant is not just any other result generated by a sign vehicle since this could also produce unrelated results. For example, he says, a fire indicating the presence of survivors of an airplane crash might also set a forest ablaze. The forest fire would be an incidental result but not an interpretant of the sign vehicle calling for help or indicating the whereabouts of the survivors.

Thus, the interpretant of a sign vehicle depends on what the Interpreter 'makes of it' and it is not just any co-emergent secondary result the sign vehicle might produce. In fact, the interpretant of a sign vehicle is another sign vehicle which is a transformation of the former; thus we could also denominate the interpretant as a sign-interpretant. This transformation enhances the initial sign vehicle and develops, in the mind of the Interpreter, a *dynamic object* which is more meaningful and more closely related to the Object that the initial sign vehicle plays to represent. Thus the interpretants are the product of acts of interpretation that progressively can move forward the conceptualization of the hidden Object that the initial sign vehicle stands to represent. As long as the Interpreter so desires, the interpretants become more sophisticated and abstract and the process of approximation of the *dynamic object* towards the Object continues through the process of interpretation which could be a never ending process of semiosis for the individual.

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[1]The word SIGN, in capital letters, is used here to refer to the Peircean notion of 'sign' defined as a system constituted by a set of three elements and the dyadic relations among the three elements. The Peircean triadic notion of 'sign' was and continues to be a historically new conceptualization of 'sign' for which he is famously known (see Vasco et al. 2009). In other words, we could symbolize his triadic notion of 'sign' as a system constituted by a set and the relations governing the elements of the set in the following way:

SIGN = ({sign vehicle, interpretant, Object}, Dyadic relations among the three elements of the set).

The *sign vehicle* is only one of the elements of the set that stands as a representation of another element in the set, namely the *Object*. Most of the time, Peirce used the word 'sign' for sign vehicle without advising the reader about the use that he meant; the meaning has to be decoded from the context in which the words were used. However, sometimes he clearly uses the words sign vehicle and representamen to refer to the representation of the Object.

For Peirce a sign vehicle is “anything which, being determined by an Object, determines an interpretation to determination, through it, by the same Object” (1906, p. 495). He also adds that “a sign [sign vehicle] is not a sign [sign vehicle] unless it translates into another sign [sign vehicle]” (CP 5.594) and that “a sign [sign vehicle] is anything which determines something else (its *interpretant*) to refer to an *object* to which itself refers in the same way, the interpretant becoming in turn a sign [sign vehicle], and so on ad infinitum” (CP 2.303, italics added). He goes even further to say that the *relation* between the sign vehicle and its Object could be of iconic, indexical, or symbolic nature.

When is the relation between a sign vehicle and its Object of iconic nature? The *icon* is a sign vehicle determined by its Object by partaking in certain characteristics of that Object. In other words, the *icon* is a sign vehicle that bears some sort of resemblance or similarity to its Object. Peirce subdivides the icons into three types: diagrams, images, and metaphors. The *diagram* is characterized by some kind of similarity with its Object in the sense that it displays somewhat the existing relations between the parts of the Object in a skeleton-like manner (Stjernfelt 2007). In contrast, the *image* represents the Object through simple qualities, and the *metaphor* represents the Object through a similarity found in something else.

When is the relation between a sign vehicle and its Object of indexical nature? The *index* is a sign vehicle determined by its Object by being in its individual existence and connected with it. The *index* has a cause-effect connection to its Object, and it directs the attention to that Object by blind compulsion that hinges on association by contiguity (CP 1.558, 1867). An example of an index is the connection between the letter ‘x’ and an unknown variable quantity.

When is the relation between a sign vehicle and its Object of symbolic nature? The *symbol* is a sign vehicle determined by its Object by more or less approximate certainty that it will be interpreted as denoting the Object as a consequence of a habit. The symbol hinges on intellectual operations, cultural conventions, and habit (CP 3.419).

Fisch (1986) argues that these three relations between the sign vehicle and its Object are not independent of each other and that they also evolve in the mind of the Interpreter. In fact, these relations constitute a nested triad in which the more complex sign vehicle involves specimens of the simpler ones. In other words, symbols typically involve indices which, in turn, involve icons. This also means that icons are incomplete indices which, in turn, are incomplete symbols. This relation between the sign vehicle and its mathematical Object also depends on what the Interpreter ‘makes of it’. For example, when a mathematician reads the institutionalized definition of limit, he can ‘see’ symbols hinting at relations, among others infinite embedded intervals of real numbers on the x- and y-axis. In contrast, students can only ‘see’ awkward mathematical marks (iconic sign vehicles with no clear meaning). Thus the teacher has to guide the evolution of the students’ interpretations of these ‘icons’ so that they can advance their understanding of them as symbolic sign vehicles that carry with them the rich meanings of the notion of limit.

From this categorization of sign vehicles into icons, indices, and symbols, we learn that different sorts of sign vehicles can represent, in different ways, the Object that reasoning is concerned with. Now, since reasoning has to make its conclusions manifest, to oneself and to others, it also has to be concerned with the *dynamic objects* of perceptual and rational insights, objects which are evolving in the mind of the Interpreter. Therefore, reasoning has to be concerned with the interpretation of sign vehicles and their transformation, through the interpretants, into mental sign vehicles.

Mathematical diagrams, as icons, implicitly represent the structural features of the mathematical Object through some kind of similarity. Thought-experimentation on the diagrams facilitates the perceptual and intellectual progress of such evolution in the mind of the Interpreter. The Interpreter may see a diagram merely as a diagram-token (a pure icon), or as a diagram-icon or schema (an icon with indexical traits), or as a diagram-symbol (an icon with iconic-indexical-symbolic traits). Then, what type of icon is a diagram for the Interpreter? It depends on what the Interpreter 'makes of it.' This means that it depends both on prior and collateral knowledge that the Interpreter has and is able to draw into the situation at a particular point in time, and on his/her own ways of *observing* and *interpreting* by means of perceptual and intellectual judgments. This is also to say that the Interpreter, in the process of observation and interpretation, simultaneously plays the perceptual elements in thought and the thought elements in perception in order to mediate them.

11.4 Mathematical Diagrams Elicit Deductive Reasoning

Peirce (1906) argues that symbols afford the means of thinking about thoughts in ways in which we could not otherwise think of them; they enable us to create abstractions, which are the genuine means of discoveries. Knowledge is habit and symbols rest exclusively on already well preformed habit; thus symbols do not furnish any self-observation and so they do not enable addition to our knowledge. On the other hand, indices provide only positive assurance of the reality and nearness of their Objects. This assurance does not give any insight into the nature of those Objects. In contrast to symbols and indices, icons partake in the more or less overt character of their Objects and therefore they do not stand unequivocally for this or that existing thing. As a consequence, the Object "may be a pure fiction as to its existence, … but there is one assurance that the icon does afford in the highest degree; namely, that which is displaced before the mind's gaze—the Form of the icon, which is also its Object—must be logically possible" (1906, p. 496).

This is to say that diagrams, being icons, implicitly present by analogy, perception or inference the structural characteristics of the Object that they play to represent. Therefore, the perceptual and intellectual observation of diagrams, on the part of the Interpreter, has the potential to bring to the fore possible logical relations that have the effect of unveiling the structural elements of the Object (the object-as-it-is). Mathematical diagrams such as geometric figures, mathematical formulas and equations, graphs, tables, maps, etc., are essentially icons that also carry with them potential indexical and symbolic features that can guide perceptual and intellectual intuitions. These features of diagrams in general, and of mathematical diagrams in particular, pertain to the *forms* of the relations that structure the parts of the Object. Peirce also argues that diagrams are necessary for deductive reasoning; nonetheless, this necessity only means that the conclusion follows from the premise(s):

> Deduction is that mode of reasoning which examines the state of things asserted in the premises, forms a diagram of that state of things, perceives in the parts of the diagram relations not explicitly mentioned in the premises, satisfies itself by mental experiments upon the diagram that these relations will always subsist, or at least would do so in a certain proportion of cases, and concludes the necessary, or probable truth. (CP 1.66)

Given that diagrams present only a skeleton representation of the relations among the constituent parts of their Objects, they trigger through observation and experimentation abductive, inductive and deductive inferential processes.

11.5 Diagrammatic Reasoning as an Inferential Process

Peirce argues that the structure of a diagram has structural similarities with the abstract and hidden structure of its Object. This similarity warrants that the purposeful observation, perceptual and intellectual, of the structural relations among the parts of the physical diagram (the phenomenon or the Object-as-it-is-perceived) will enable thought-experimentation to infer the structural relations among the parts of the Object (the noumenon or the Object-as-it-is) by means of inferential reasoning. Peirce calls this amalgamated thinking process *diagrammatic reasoning*:

> By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, *performs experiments* upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses it in a *general form*. (CP 2.96, italics added)

The aim of diagrammatic reasoning is the construction of a mathematical argument that warrants the abstract structure of the mathematical Object. This argument is constituted not only by the construction of isolated inferences but also by the logical and cohesive concatenation of them. Each inference is the result of evolving related interpretants to form a logical assertion. In contrast, the argument is the concatenation of logical inferences that lend themselves to form a coherent chain of mathematical inferences. In this chain, any inference is sustained by prior ones. The formation of the argument is, for Peirce, the formation of a logical rule that has coherence and completeness. When Peirce uses the terms logical or logic, he means it in the sense of 'logic' as the study of thought insofar as it is subject to self-control with the aim of developing good habits of reasoning.

The mathematical argument, once formed, has to be expressed in complete sentences or mathematical statements. Each sentence or statement (subject, copula or verb, and predicate) uses mathematical terms or notations in order to convey a unit of thought. This is to say that sentences or statements can encode logical inferences in linguistic and mathematical terms. For a collection of sentences or statements to constitute a written mathematical argument they have to be combined and concatenated in a logical and linear manner to unveil the holistic abstract structure of the mathematical Object.

The analysis of the parts of a diagram, the relations among those parts, and the synthesis expressed in the argument reflect not only the perceptual elements in thought and the thought elements in perception but also the audacity of the mind to bring into play relevant collateral knowledge to aid in the justification of certain assertions. From this nonlinear activity of the mind, the holistic unity of the argument emerges out of the formation of a diversity of perceptions and percepts, perceptual and logical judgments, and conceptions. The differentiation between perception, percepts, conceptions and concepts will be made in a later section. Altogether, they come to unveil, in no uncertain terms, the structure of the mathematical Object and the validity of the conclusion.

It comes then as no surprise that Peirce appropriates the triad *term/noun, proposition,* and *argument* as the triad that reveals the connection between the sign vehicle and the nature of the interpretants it produces in the mind of the Interpreter. Peirce expands the meaning of this triad and he considers it as the triad *seme/possibility/concept/term, pheme/actuality/proposition,* and *argument*. A seme/term is "anything which serves for any purpose as a substitute for an Object"; it is, after all, "the Immediate Object of all knowledge and all thought" (1906, pp. 506–507). The pheme/proposition is "intended to have some sort of compulsive effect on the Interpreter of it" (*ibid*.). The argument tends "to act upon the Interpreter through his own self-control, representing a process of change in thoughts or signs [sign vehicles], as if to induce this change in the Interpreter" (*ibid*.).

It will be worthwhile here to summarize that the SIGN, for Peirce, is a triadic relation between its constitutive elements, namely, the *sign vehicle*, its *Object*, and the *interpretant* it provokes in the mind of the Interpreter. He unfolds the relation between the sign vehicle and the Object in the triad (icon, index, symbol); the relation between the sign vehicle and the interpretant in the triad (seme/term, pheme/proposition, argument) assigning to the argument a high logical standing; and the relation between the sign vehicle and its own internal nature in the triad (qualisign, sinsign, legisign). A qualisign is a quality and it cannot act as a sign vehicle until it is embodied; however its embodiment has nothing to do with its character as a sign vehicle. A sinsign (meaning being only once) is an actual existing thing or event. A legisign is a law; it is not a single object but a general type. Each legisign signifies through an instance of its application and each instance is a replica or a sinsign (Peirce 1998).

Although the latter triad does not have an immediate impact on the main goal of this paper, the coordination of the above three triads are at the root of Peirce's tenfold classification of sign vehicles. They can support a better analysis of diagrammatic reasoning in the classroom and a better analysis of the epistemological process necessary for the learning-teaching of mathematics. In this chapter, I concentrate on the chains of interpretants that geometric diagrams prompt when they are intentionally observed and experimented with.

11.6 Mathematical Diagrams as Epistemological Tools

Stjernfelt, a semiotician who has dedicated several books and articles to the analysis of Peirce's diagrammatic reasoning, extensively argues about the benefits of his non-trivial definition of icon. He argues that Peirce's definition avoids the weakness of most definitions of iconicity by similarity because of its connection with observation and thought-experimentation to discover additional pieces of information about the Object that the icon stands to represent. Peirce argues that "a great distinguishing property of the icon is that by direct *observation* of it other properties concerning its Object can be discovered than those which suffice to determine its construction" (CP 2.279, quoted in Stjernfelt 2007, p. 90; italics added).

In other words, diagrams, as icons, afford the formation of perceptions and conceptions that the grammar and syntax of their construction permit. However, while physical diagrams remain in the field of the senses, new logical relations among their parts can possibly emerge by means of imagination, manipulation, observation, and thought-experimentation. After all, a diagram can be characterized in one's mind in a variety of ways, "as a token, as a general sign [sign vehicle], as definite form of relation, as a sign [sign vehicle] of an order in plurality, i.e., of an ordered plurality or multitude" (Robin 1967, p. 31).

Peirce argues that "both the iconic diagram and its Initial Symbolic Interpretant constitute what… Kant calls *schema*, which is, on one side, an *object* capable of being observed while, on the other side, is a *General*" (1976, p. 316, italics added). He also argues that more can be learned about the Object of the diagram by the contemplation of explicit and implicit relations hidden in the physical structure of the diagram. He also adds that "all necessary reasoning is diagrammatic" and that "the diagram is an icon of a set of rationally related objects, a schema which entrains its consequences" (Robin 1967, p. 31). Furthermore, it can be said that diagrams are *epistemological tools* for inferential thinking.

Peirce adopts and adapts Kant's cognitive notion of schema for the inner workings of the mind of the Interpreter that a geometric diagram can produce, and gives an operational definition for this cognitive activity. A geometric diagram

> is a construction formed according to a precept furnished by the hypotheses; being formed, the construction is submitted to the scrutiny of *observation*, and *new relations* are discovered among its parts, not stated in the precept by which it was formed, and are found, by

> a little *experimentation*, to be such that they will always be present in such a construction. (CP 3.560, italics added)

This operational definition of diagram and diagrammatic reasoning entails that once a geometric diagram is constructed, it can be observed, manipulated physically and mentally, and transformed through physical and intellectual experimentation. As a result, what follows is the formation of dynamic interpretants that sooner or later become logical interpretants. The latter will become logically harmonized to contribute to the formation of chains of inferences that, in time, come to be expressed, organized and synthesized in coherent and logically coordinated sentences or mathematical statements. This is to say that the formation of mathematical arguments, geometric or otherwise, is an evolving cognitive process that is by no means linear but that will be presented as linear in the written argument.

Euclidean geometry is a classic example of physical and intellectual manipulation, and of thought-experimentation that can be performed on geometric diagrams. "Euclid first announces, in general terms, the proposition he intends to prove, and then proceeds to draw a diagram, usually a figure, to exhibit the antecedent condition thereof" (Peirce 1976, p. 317). Peirce's assertion reminds us of Polya's heuristics for solving problems and the formation of the mental schema to support construction of solutions. This is to say, understanding the problem and constructing a figure or diagram, devising a plan by means of thought experimentation on the diagram and bringing into play appropriate collateral knowledge, carrying out the plan within the logic of mathematical systems, and retrospectively and prospectively reflecting on possibilities for generating new problems or generalizing the one at hand (Polya 1957).

Nowadays, given the dragging mode of dynamic geometry environments, the manipulation of geometric figures is expedited, and with it, the observation of intentional manipulation and planned experimentation. Thus, the observation of variant and invariant relations among the elements of a geometric figure facilitates the formation of conjectures as well as their validation. The role of diagrams in deductive reasoning is well argued by Peirce:

> All deductive reasoning, even simple syllogism, involves an element of *observation*; deduction consists in constructing an icon or diagram the relations of whose parts shall present a *complete analogy* with those of the parts of the *object of reasoning*; in experimenting upon this image in the imagination; and in observing the result so as to discover unnoticed and hidden relations among the parts. (CP 3.363, italics added)

Netz (2014), who has dedicated himself to studying the evolution of Greek mathematical thinking, describes the Greeks' history of reasoning with geometric diagrams as follows: (a) Greek mathematical diagrams shaped deduction in mathematics; (b) mathematical objects were determined through diagrams; (c) letters inserted in diagrams were indices, not symbols; (d) diagrams are the metonymy of the propositions; (e) the writing of a proof was preceded by an oral rehearsal. He then concludes that, in general, the emergence of mathematical thought requires an inter-subjectively given object.

All in all, in this and prior sections we have a semiotic and a historical assurance that mathematical diagrams are tools that mediate deductive thinking. Given that deductive thinking is a source of knowledge, we can conclude that mathematical diagrams serve as epistemological tools for the learning-teaching of mathematics.

11.7 Visualization as an Indispensable Element in Diagrammatic Reasoning

For centuries it has been recognized that sense perception is an essential element of cognition. Kant (1781/2007) asserted that perception without conception is simply blind and that conception without perception is merely empty. For him, the thought elements in perception and the perceptual elements in thought are complementary and both make human cognition a unitary process that leads the way from the elementary acquisition of sensory information to the most generic theoretical ideas (Wolff 1973).

Following Kant, Peirce differentiates between immediate perceptions and percepts. Kant argues that perceptions are the effect of the *empirical object* (thing-as-it-appears) upon bodily sense organs and that by subjective association the mind forms a manifold of perceptions that are coordinated by means of empirical judgments—this manifold of perceptions is the percept. Percepts need to be transcended by means of mental operations for a conception to be formed.

> The synthetic unity of a manifold, therefore, is the characteristic possessed by a collection of mental contents by virtue of their having been produced by the imagination in accordance with a single rule. The consciousness of that synthetic unity is the conception of the rule by which it has been produced. (Wolff 1973, p. 130)

In general, it can be said that percepts are concrete instantiations of the Object (phenomenon or the Object-as-it-appears) and whose gradual and continuous differentiation proceed toward conceptions. Peirce argues that this "continuity is a special kind of generality among the relation of all of a certain kind of parts of one whole and that this continuity implies a passage from one percept to a contiguous conception" (CP 7.535). In fact, this continuity is the essence of the process of unlimited semiosis which mediates the transformation of perceptions into percepts, of percepts into conceptions, and of conceptions into concepts or the conceptual Object (noumenon or the Object-as-it-is).

During the last decades, cognitive scientists, by means of experimentation, have reinforced the notion that to perceive the external environment "our brain uses multiple sources of sensory information derived from several different modalities, including vision, touch and audition. All these sources of information have to be efficiently merged to form a coherent and robust percept" (Ernst and Bülthoff 2004, p. 162).

In the classroom, dynamic geometric environments offer not only the visual modality but also the simultaneous touch modality during the dragging process, which is mediated by the mouse or by the touch-pad. The integration of these two sources of information not only reduces perceptual estimations produced by paper-pencil drawings of geometric figures but it also enhances the formation of more reliable perceptions. The transformation of these perceptions entails the formation of percepts and empirical and logical judgments to produce all sorts of interpretants that sustain the formation of conceptions in the process of reasoning through diagrams.

During the last decades, there has been a history of research in mathematics education focusing on visual perception and its close connection with conceptualization. Seminal theoretical works from cognitive scientists were and continue to be influential in mathematics education. Among some of these works could be mentioned, with apologies to all who are not mentioned here but who are nonetheless recognized in our minds, Arheim's book on *Visual thinking* (1969), Johnson-Laird's book on *Mental models* (1983), and Davis and Anderson's (1979) article on *Nonanalytic aspects of mathematics and their implication for research and education*. To place specific emphasis on visual perception, Arheim (1969) rephrased Kant and wrote, "vision without abstraction is blind and abstraction without vision is empty" (p. 188).

For a synthesis of research on visualization in the learning and teaching of mathematics, the reader is referred to Presmeg's (2006) handbook chapter. In what follows I present a bird's eye view of some of the seminal notions that emerged in the 1980s.

Bishop (1989) proposed two types of ability constructs: *integrating figural information* and *visual processing*. The first is described as the ability to relate to a particular situation presented in some form of visual representation. The second is described as the ability to translate abstract relations and non-figural information into visual terms. The notions of *concept definition*, *concept image*, and *image schemata* also contributed a good amount of research in mathematics education. *Concept definition* refers to the definition institutionalized by the mathematics community at large (Tall and Vinner 1981). *Concept image* refers to subjective construction of meaning that corresponds to the concept definition of institutionalized mathematics (Arcavi 1999; Tall and Vinner 1981). *Image schemata* refers to the individual's mental constructions connecting those related concept images subjectively constructed by the individual (Dörfler 1991). Both the concept image and the image schemata provide a non-verbal, non-propositional component of cognition that contrasts with the notational and/or verbal descriptions of the concept definition. These notions, directly or indirectly, somewhat address the abstract and analytic nature of inferential reasoning (abductive, inductive, and deductive).

Skemp's (1987) book, *The psychology of learning mathematics*, also has been very influential in mathematics education research. Skemp goes beyond the above mentioned notions when he links perception and symbolic systems (symbolic mathematical notations or what he calls surface structures) as mediators in the construction of concepts (or what he calls conceptual structures or deep structures). His diagram shows this connection (p. 177) (Fig. 11.1). Figure 11.1 is Skemp's diagram complemented with other definitions in his book.

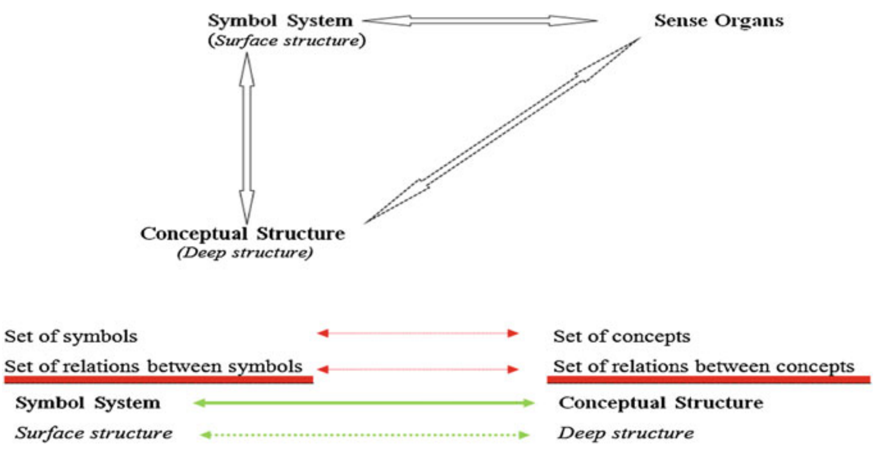


Fig. 11.1 Skemp's connection between perceptions and concepts

Skemp's diagram is a clear illustration of Goodman's (1978) argument in the introduction to his book, *Ways of worldmaking*. He writes,

> Kant exchanged the structure of the world for the structure of the mind; C.I. Lewis exchanged the structure of the mind for the structure of the concepts; and this book makes the argument for the exchange of the structure of the concepts for the structure of the several symbol systems of the sciences, philosophy, the arts, perception, and everyday discourse. (p. x)

In summary, all symbolic mathematical notations (surface structures) are essentially special kinds of diagrams; diagrams with iconic aspects and with the potential to unveil the indexical and more general symbolic aspects of the mathematical Objects that these diagrams stand to represent. The relations among the parts of the diagram (as-they-appear-to-the-senses) are physical representations of possible logical structural relations (as-they-appear-to-the-mind's eye) among the parts of the conceptual Object (the-Object-as-it-is). Reasoning through diagrams is essentially a process of unlimited semiosis, in which the interpretants mediated by diagrams are inferentially constructed and should also be logically connected to produce the conceptualization of the deep structure of the conceptual Object or an acceptable approximation of it. This semiosis is unlimited in the sense that even when an approximation to the conceptual object has been reached, a better and more sophisticated approximation could be produced in the future. For the notion of approximation we refer the reader to the article on inter-intra interpretation (Sáenz-Ludlow and Zellweger 2016).

11.8 Stjernfelt Model for Diagrammatic Reasoning

Stjernfelt (2007) captures, in Fig. 11.2, the essence of the process of diagrammatic reasoning, a process rooted in perceptual and mental activity to produce chains of inferences. This figure, which is itself a diagram, is a useful tool for thinking about the processes of proving and problem-solving. It synthesizes a manifold of relations that integrates the construction of the diagram, the observation of structural relations among its parts, and the perceptual manipulation and thought-experimentation to infer new possible relations conducive to the attainment of a logical conclusion or a solution.

He also describes this process in terms of the transformation of diagrams co-emerging with the formation of evolving interpretants. In this transformation, the implicit deep structural aspects of the Object (the Object-as-it-is) can be unveiled because of their analogy with the relations among the parts of the diagram. This is to say that, during the process of interpretation, the Interpreter mentally refurbishes the given or initially constructed diagram (transformand diagram) into more meaningful diagrams (transformate diagrams). In this process, a given or constructed diagram-symbol is interpreted by an Interpreter as a diagram-token and transformed into a diagram-icon or schema that is transformed again and again until the structural relations among the parts of the mathematical Object are unveiled by the Interpreter so as to see it as a diagram-symbol with the deep and hidden meaning that the initial diagram-symbol played to represent. Each time, new transformate diagrams reveal more and deeper structural relations among the parts of the mathematical Object that hinge on mental operations and inferential reasoning. It is in this sense that iconic sign vehicles grow into symbolic sign vehicles in the mind of the Interpreter. Although guided by the initial diagram-symbol, the Interpreter is giving freedom to think and to be creative within the context of a particular mathematical situation.

The sequence of interpretants that co-emerge with the transformations of the initial diagram is summarized by Stjernfelt (2007) using the letters ***a, b, c, d, e, f, g***. to describe the progressive steps in the development of deeper meanings constructed with and through new transformate diagrams.

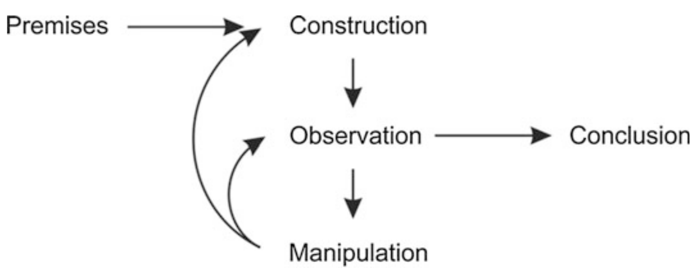


Fig. 11.2 Diagrammatic reasoning as a process (Stjernfelt 2007)

a. ***Symbol (1)***: Diagram-symbol [i.e., transformand diagram or mathematical symbol in the mind of the proposer of a problem or proposition].
b. ***Immediate Iconic Interpretant (b < a)***: Diagram-token [a rule-bound diagram]. An initial interpretation of the diagram-symbol ***a.***
c. ***Initial interpretant (b + c < a)***: The diagram-token is transformed into a diagram-icon [schema or skeleton relations among the parts of the diagram emerge in the mind of the Interpreter]. Initial transformate diagram.
d. ***Middle Interpretant ((b + c) + d < a)***: A diagram with three sources, **a**, **b**, and **c**. An emergent symbol-governed diagram equipped with possibilities of transformation [diagram-symbol, a more advanced transformate diagram with possibilities of further mental transformations].
e. ***Eventual, Rational Interpretant***: New emergent transformate diagram-symbol.
f. ***Symbol (2)***: Concluding transformate diagram-symbol or conclusion.
g. ***Post-Diagrammatical Interpretant (different from b)***: This interpretant is an interpretant of ***a*** as well; however, now the diagram-symbol produced is enriched by the total interpretant *of* ***Symbol (1).***

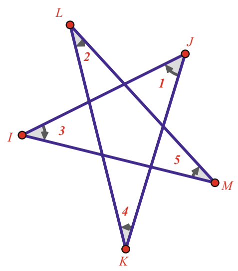
It is important to note that transformate diagrams are substantially embedded in the transformand diagram with all their unveiled significant features. That is, diagrammatic reasoning is the mental process of the Interpreter who intentionally endeavors both in the observation and in the manipulation of an initial diagram [transformand diagram/symbol (1)]. He first interprets this diagram as a diagram-token and then progressively enriches it and transforms it into diagram-icon and diagram-symbol (transformate diagrams). The final diagram-symbol [transformate diagram/symbol (2)] is the Interpreter's construction of the symbolic meaning of the initial diagram-symbol [transformand diagram/ symbol (1)]. This is to say that the Interpreter finally unveils, as best as he/she can, the structure of the *Object* that the transformand diagram [symbol (1)] stands for.

It can be said that diagrammatic reasoning is a process by which the Interpreter intentionally endeavors in a process of inter-intra interpretation (Sáenz-Ludlow and Zellweger 2016) to enhance both the observation and manipulation of an initially proposed diagram-symbol in the mind of the proposer but only perceived by the Interpreter as a diagram-token. The Interpreter then transformes it into a diagram-icon, which in turn is transformed into a diagram-symbol enriched with new inferred general relations that contribute to unveil the hidden structural relations of the mathematical Object implicit in the proposed diagram.

11.9 Examples of Diagrammatic Reasoning

11.9.1 The Five Point Star

The geometric diagram of a five-point star was taken from Nelsen's (1993) book *Proofs without words* (p. 14). This example was modified to ask for the formulation



m∠IJK	m∠KLM	m∠JIM	m∠LKJ	m∠LMI	m∠IJK + m∠KLM + m∠JIM + m∠LKJ + m∠LMI
44.75°	31.99°	32.95°	28.60°	41.71°	180.00°
44.75°	28.19°	38.30°	28.60°	40.16°	180.00°
39.43°	28.19°	39.63°	32.59°	40.16°	180.00°
39.07°	28.19°	46.16°	32.59°	33.98°	180.00°
46.38°	28.19°	41.34°	30.11°	33.98°	180.00°

Conjecture:
The sum of the five point-angles of any five-point star is 180°

Fig. 11.3 Transformand diagram and the conjecture

of a conjecture with the use of the Geometer's Sketch Pad (GSP). Below I first present the transformand diagram and the formulation of the conjecture. Then I present a sequence of interpretants and transformate diagrams which aided in the proof (Fig. 11.3).

Given the dragging and measuring modes of the GSP, a conjecture can be inductively constructed: the sum of the five point-angles of a five-point star is 180°. A first observation reveals a pentagon with five non-overlapping triangles (here called point-triangles) formed by the extension of each of its sides. A second observation reveals implicit overlapping triangles constituted by the pentagon and two point-triangles. A third observation indicates that there are five overlapping triangles, each sharing one vertex with the pentagon.

Now it is necessary to bring to the fore some sort of appropriate collateral knowledge to make some sense of the observation. For example, the measure of the interior and exterior angles of triangles, the measure of straight angles, the measure of exterior and interior angles of pentagons, and the measure of vertical angles. The ensuing mental operations are to inquire about the possible connections among the collateral knowledge, the point-triangles, and overlapping triangles to prove the conjecture. Figures 11.4, 11.5 and 11.6 present a sequence of transformate diagrams.

In what follows I describe the sequence of interpretants that enabled transformations of the given diagram-symbol which was initially interpreted as a diagram-token.

Immediate interpretant. Visual perception of the five-point star as constituted by a pentagon and five point-triangles. The Interpreter mentally creates a transformate diagram-token representing the initial relations among the parts of the given diagram (see Fig. 11.4).

Initial interpretant. The visualization of the five-point star as constituted by a pentagon with five point-triangles formed by the extension, in both directions, of each of its sides. The Interpreter transforms the prior diagram-token into a diagram-icon. The latter indicates new relations and new possibilities for the construction of regular and irregular five-point stars.

Middle interpretant. The visualization of five implicit overlapping triangles: Δ**W**JK, Δ**Z**MI, Δ**Q**KL, Δ**N**IJ, and Δ**P**LM, each of which shares a vertex (**W**, **Z**, **Q**, **N**, or **P**) with the pentagon and two corresponding vertices of the star. Thus each

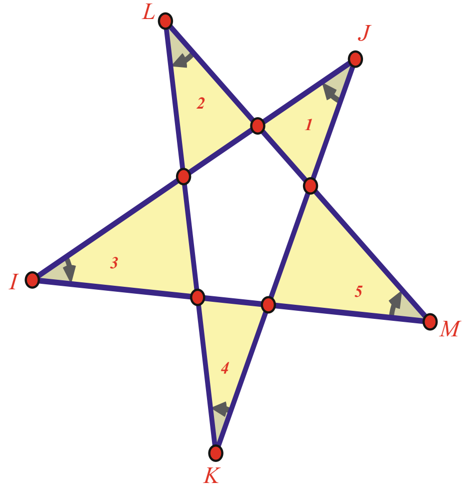


Fig. 11.4 Point triangles and pentagon

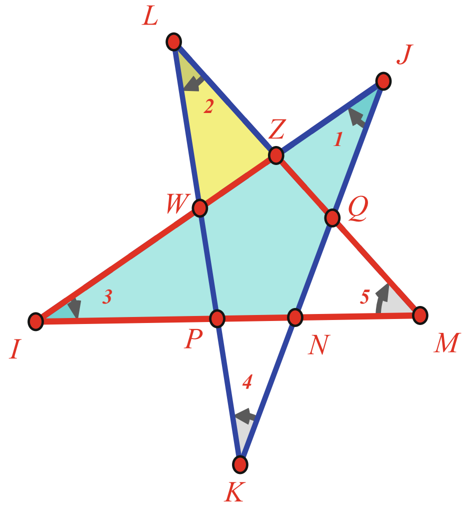


Fig. 11.5 Two of the five overlapping triangles: Δ**I**N**J** and Δ**I**Z**M**

triangle has already two point-angles of the star that when added give the measure of an external angle of that particular triangle. The Interpreter transforms the prior diagram-icon into a diagram-symbol. The latter enables the selection of appropriate collateral knowledge for the attainment of the proof of the conjecture (see Fig. 11.5).

Eventual rational interpretant. How are the angles of any point-triangle related to the angles of overlapping triangles that share a vertex with the pentagon? Focus on one point-triangle at a time, for example Δ**WZ**L. Which overlapping triangles will be related to this triangle? To make a decision, it would be useful to consider

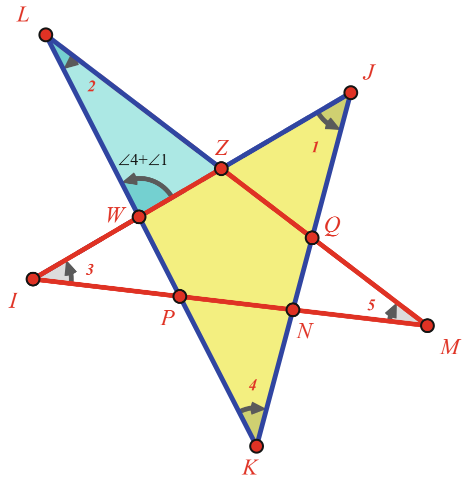


Fig. 11.6 Overlapping ΔJ**W**K and ΔI**Z**M correlated to point-triangle **WZ**L

the vertices **W** and **Z** of this triangle because they are also vertices of the pentagon. This leads one to consider the overlapping triangles ΔJ**W**K and ΔI**Z**M (see Fig. 11.6).

Consider triangles ΔJ**W**K and ΔI**Z**M

∠L**WZ** = ∠1 + ∠4 (external angle of ΔJ**W**K)

∠L**ZW** = ∠3 + ∠5 (external angle of ΔI**Z**M)

(∠1 + ∠4) + (∠3 + ∠5) + ∠2 = 180° (the sum of angles of Δ**WZ**L is 180°).

Since ∠4, ∠1, ∠3, ∠5, ∠2 are the point-angles of the five-point star, they add up to 180°.

Post-diagrammatical interpretant. The systematic dragging and observation of the five-point star guided the reasoning to formulate a conjecture about the sum of its point-angles and its proof. Several questions come to mind. Is this conjecture true for regular five-point stars? Can this conjecture be proved using the interior and exterior angles of the pentagon? Can this conjecture be proved using straight angles? Can the conjecture be generalized for n-point regular and irregular stars? The answers to these questions are in the positive. Unfortunately, space limitation does not allow for the presentation of the other proofs and the generalization but the reader is invited to try them. Nonetheless, the generalization arrived at is that the sum of the point-angles of any n-point star, regular or irregular, is (n − 4) 180°.

11.9.2 Trigonometric Functions of the Sum and Difference of Angles

The second example presents a geometric diagram proposed by Nelsen (2000, pp. 46–47) for proving the six trigonometric identities sin(α + β), cos(α + β), sin (α − β), cos(α − β), tan(α + β), and tan(α − β). The diagram proposed is an ingenious partition of a rectangle into four non-overlapping right-triangles. This partition is a creative abduction on the part of the proposer of the problem. Nonetheless, the inferential reasoning that follows from the observation of this transformand diagram is rooted in overcoded abductions as opposed to genuinely creative abductions (Sáenz-Ludlow 2016).

The analysis of the diagrammatic reasoning for each proof is presented visually by a sequence of transformate diagrams (Figs. 11.7 and 11.8) followed by a brief description of the co-emerging interpretants. The first four proofs are very similar in nature due to the strategic position of the length 1 for the hypotenuse of the most interior right-triangle.

Immediate interpretant. Visual perception and collateral knowledge aided the identification and justification of the positions of the angles α, β, α + β, α − β as well as the strategic position of β as one of the acute angles of the most interior right-triangle with hypotenuse of length 1.

Initial interpretant. In Fig. 11.7 the angle α with vertex C repeats with vertex B due to their perpendicular sides, and the angle (α + β) with vertex C repeats with vertex A due to their position as alternate interior angles between parallel sides of the rectangle and the transversal AC. In Fig. 11.8 the angle α with vertex C repeats with vertex B due to their position as alternate interior angles between parallel sides of the rectangle and transversal CB, and it also repeats with vertex A due to perpendicular corresponding sides. Angle (α − β) is directly given because α and β overlap and α > β. The right-triangle AFC in Figs. 11.8 and 11.9 facilitates the visual and determination of sine and cosine of (α + β) and (α − β) in terms of segment-lengths due to the length 1 of the hypotenuse AC of right-triangle AFC.

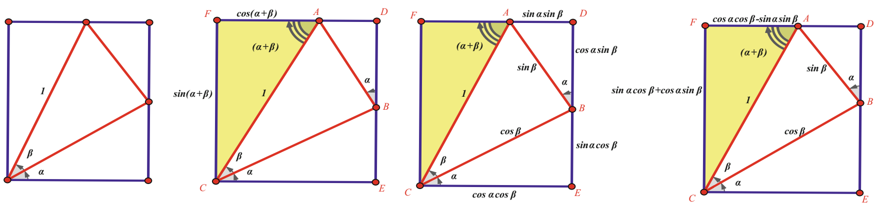


Fig. 11.7 Transformand and transformate diagrams for sin(α + β) and cos(α + β)

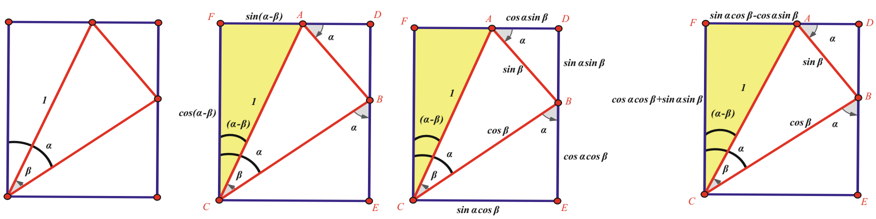


Fig. 11.8 Transformand and transformate diagrams for sin(α − β) and cos(α − β)

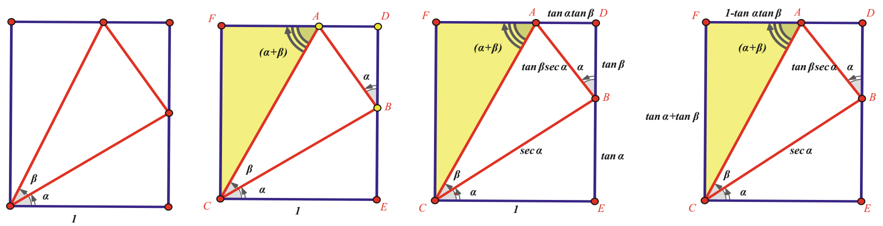


Fig. 11.9 Transformand and transformate diagrams for tan(α + β)

Middle interpretant. The right triangles ADB and CEB with hypotenuses sin β and cos β, respectively, enable the visual determination of the sine and cosine of their acute angles. The addition and subtraction of length-segments give the formulas for sin(α + β) and cos(α + β) (Fig. 11.7), and sin(α − β) and cos(α − β) (Fig. 11.8) from the respective transformate diagrams.

Eventual rational interpretant. By the definition of the sine and cosine functions of the angle α + β in the right-triangle AFC in Fig. 11.7, we have that sin(α + β) = sin α cos β + cos α sin β and that cos(α + β) = cos α cos β − sin α sin β.

Also by the definition of the sine and cosine functions of the angle α − β in the right-triangle AFC in Fig. 11.8, we have that sin(α − β) = sin α cos β − cos α sin β and cos(α − β) = cos α cos β + sin α sin β.

In the following two figures, Figs. 11.9 and 11.10, I present two sequences of diagrams to deduce the formulas for tan(α + β) and tan(α − β). The transformand diagrams (first diagram in each sequence) are the proposer's creative abductions. They are essentially the same transformand diagrams as in Figs. 11.7 and 11.8 but assigning the length 1 to the side CE in Fig. 11.9 or to side BE in Fig. 11.10. Clearly, the interpretants for these proofs incorporate the immediate and initial interpretants generated during the first four proofs shown above. Thus, this process is initiated with middle interpretants.

Middle interpretant. This is a realization that tan(α + β) and tan(α − β) can be determined from the ratios of the sides of the right-triangle FAC in Figs. 11.9 and 11.10, which have (α + β) and (α − β) as acute angles respectively. Thus, the task now is to use the trigonometric definitions of tangent and secant to determine the length of the sides of the other right-triangles ADB and CEB.

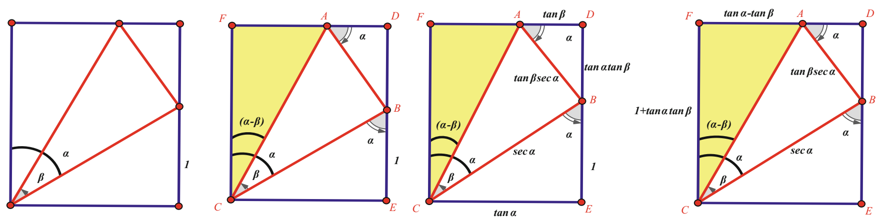


Fig. 11.10 Transformand and transformate diagrams for tan(α − β)

Rational interpretant. The lengths of the sides AD, DB, BE and CE of triangles ADB and CEB in the third transformate diagrams in Figs. 11.9 and 11.10 are the results of calculations with the definitions of the trigonometric functions. Once they are performed, the lengths of the sides FA and FC of right-triangle AFC are determined by simple addition or subtraction of segment-lengths.

Eventual rational interpretant. By the definition of the tangent function in the right-triangle AFC in Fig. 11.10, we have that $\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$. Also by the definition of the tangent functions in the right-triangle AFC in Fig. 11.10, we have that $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$.

Post-diagrammatical Interpretant. What is the significance of the position of the length 1 in each of these proofs? Suppose that in Fig. 11.9 we make EB = 1 or BD = 1 and in Fig. 11.10 we make AD = 1. Will the proof hold? The answer to this question is in the positive. The reader could make an effort to arrive at the same formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ in these cases. These changes are motivated by the systematic manipulation of the transformand diagram and corresponding calculations.

11.10 Conclusion

Mathematical diagrams, such as geometric figures, mathematical formulas, equations and graphs, serve as epistemological tools to mediate not only the formulation and validation of conjectures but also the conceptualization of well-established mathematical ideas. The above examples illustrate that a systematic observation of a transformand diagram assists not only in the visual perception of different relations among its parts but also in the inference of new relations among them. Some of these relations are perceived by the senses and still others are inferred with the aid of perceptual and logical judgments as well as collateral knowledge. Both explicit and implicit relations embedded in the diagram enable the construction of new transformate diagrams to make explicit new relations among their parts; relations that, by analogy, will unveil the structural relations of the mathematical Object that the initial diagram plays to stand for.

The evolving perceptual and mental transformation of diagrams co-emerges with the formation of new interpretants in the mind of the Interpreters and with their progressive inferential reasoning process to reach the desired goal. This means a progression in the conceptualization of the deep structure of the mathematical Object that diagrams purport to stand for. It goes without saying that, when working with diagrams, the interpretants generated by different Interpreters, although they should be somewhat similar in nature, may have different iconic, indexical, and symbolic features according to their personal knowledge and level of sophistication of their mathematical thinking. The wanted outcome is that these interpretants share a tendency to converge to the desired mathematical Object (Sáenz-Ludlow and Zellweger 2016). This clearly indicates that teachers need to develop two simultaneous and parallel types of awareness: (1) awareness of the teachers' own evolving interpretations of transformate diagram(s) and (2) awareness of the students' interpretations of those diagrams. This double awareness on the part of teachers will enable the guidance of students by using the students' current interpretations and understanding. It is in this sense that diagrams serve as tools for meaning-making—epistemological tools—for teachers and students alike, although such a meaning could be at different but compatible levels of understanding.

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