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Norma Presmeg
Luis Radford
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Signs of Signification

Semiotics in Mathematics Education Research

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Norma Presmeg · Luis Radford
Wolff-Michael Roth · Gert Kadunz
Editors

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Editors
Norma Presmeg
Illinois State University
Maryville, TN
USA

Wolff-Michael Roth
Faculty of Education
University of Victoria
Victoria, BC
Canada

Luis Radford
Faculty of Education
Université Laurentienne
Sudbury, ON
Canada

Gert Kadunz
Fakultät für technische Wissenschaften, Institut für Mathematik
Alpen-Adria-Universität Klagenfurt
Klagenfurt
Austria

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Contents

Contributors

Yasmine Abtahi University of Ottawa, Ottawa, ON, Canada; L'Université du Québec à Montréal, Montreal, Canada

Mike Askew University of the Witwatersrand (WITS), Johannesburg, South Africa

Gert Kadunz Department of Mathematics, Alpen-Adria Universitaet Klagenfurt, Klagenfurt, Austria

Barbara M. Kinach Mary Lou Fulton Teachers College, Arizona State University, Mesa, AZ, USA

Christina M. Krause Faculty of Mathematics, University of Duisburg-Essen, Essen, Germany

Corin Mathews University of the Witwatersrand (WITS), Johannesburg, South Africa

Petra Menz Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

Isaias Miranda Instituto Politécnico Nacional, CICATA-Legaria, Mexico City, Mexico

Candia Morgan Department of Curriculum Pedagogy and Assessment, University College London Institute of Education, London, UK

Michael Otte University of Bielefeld, Melle, Germany

Norma Presmeg Mathematics Department, Illinois State University, Maryville, TN, USA

Uta Priss Ostfalia University, Wolfenbüttel, Germany

Naomi Prusak School of Education, The Hebrew University of Jerusalem, Jerusalem, Israel

Luis Radford École des sciences de l'éducation, Université Laurentienne, Sudbury, ON, Canada

Wolff-Michael Roth Applied Cognitive Science, University of Victoria, Victoria, BC, Canada

Jesus Victoria Flores Salazar Pontificia Universidad Catolica del Perú, Lima, Peru

Ulises Salinas-Hernández Cinvestav-IPN, Mexico City, Mexico

Alexander Salle Department of Mathematics/Computer Science, Institute of Mathematics, University of Osnabrück, Osnabrück, Germany

Anna Shvarts Lomonosov Moscow State University, Moscow, Russia

Nathalie Sinclair Faculty of Education, Simon Fraser University, Burnaby, BC, Canada

Debbie Stott South African Numeracy Chair Project, Education Department, Rhodes University, Grahamstown, South Africa

Osama Swidan Ben-Gurion University of the Negev, Beer-Sheva, Israel

Adalira Sáenz-Ludlow Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, USA

Hamsa Venkat University of the Witwatersrand (WITS), Johannesburg, South Africa

Chapter 1
Introduction

Norma Presmeg, Luis Radford, Wolff-Michael Roth and Gert Kadunz

Abstract This introductory chapter presents an overview of the role of semiotic issues in the teaching and learning of mathematics, as these issues are characterized and elaborated in the chapters of this monograph. Several threads are represented in the four sections of this book: the evolving sociocultural perspective is addressed in Sects. 1 and 4; Sect. 2 addresses linguistic and textual aspects of signification, and Sect. 3 represents Peircean perspectives that were recognized as important in our field more than two decades ago, which continue to have relevance.

Keywords Signs · Semiotics · Semiosis · Signification · Gesturing

1.1 Introduction

All the chapters in this monograph grew out of presentations by the authors in Topic Study Group 54 (TSG 54), *Semiotics in Mathematics Education,* of the Thirteenth International Congress on Mathematics Education (ICME-13), held in

N. Presmeg (✉)
Mathematics Department, Illinois State University,
5160, Candlewood Court, Maryville, TN 37804, USA
e-mail: npresmeg@msn.com

L. Radford
École des sciences de l'éducation, Université Laurentienne, Sudbury
ON P3E 2C6, Canada
e-mail: lradford@laurentian.ca

W.-M. Roth
Applied Cognitive Science, University of Victoria, MacLaurin Building A567,
Victoria, BC V8P 5C2, Canada
e-mail: mroth@uvic.ca

G. Kadunz
Department of Mathematics, Alpen-Adria Universitaet Klagenfurt,
Universitaetsstrasse 65, 9020 Klagenfurt, Austria
e-mail: gert.kadunz@aau.at

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Hamburg, Germany, 24–31 July 2016. The four regular sessions of TSG 54, and the associated Oral Communications in the Congress timetable, were well attended by scholars who had been working in the field of semiotics in mathematics education for decades, as well as interested newcomers to the field. This variety of experience in the topic is reflected in the chapters in this book, which is thus intended both as an introduction to the field, and a reasoned appraisal of research on semiotics in mathematics education, its significance, old and new theoretical developments, where it has been and where it might be going.

1.2 What Is Semiotics and Why Is It Significant for Mathematics Education?

Semiotics is related to *semantics*, according to one dictionary (Funk and Wagnall 2003), and may be defined as "The relation between signs or symbols and what they signify or denote." The same dictionary defines the verb *signify* as follows: "1. To make known by signs or words; express; communicate; announce; declare. 2. Hence, to betoken in any way; mean; import. 3. To amount to; mean." The adjective s*ignificant* has this definition: "1. Having or expressing a meaning; bearing or embodying a meaning. 2. Betokening or standing as a sign for something; having some covert meaning; significative. 3. Important, as pointing out something weighty; momentous."

These definitions point to a number of elements intrinsic to the nature of the activity with which semiotics is concerned, *semiosis*: it involves *signs*; these signs have *meanings*, which suggests that there are interpretations, and that consciousness is involved in some way—and also communication. Semiosis is "a term originally used by Charles S. Peirce to designate any sign action or sign process: in general, the activity of a sign" (Colapietro 1993, p. 178). A sign is "something that stands for something else" (p. 179); it is one segmentation of the material continuum in relation to another segmentation (Eco 1986). *Semiotics*, then, is "the study or doctrine of signs" (Colapietro 1993, p. 179). Sometimes designated "semeiotic" (e.g., by Peirce), semiotics is a general theory of signs or, as Eco (1988) suggests, a theory of how signs signify, that is, a theory of signification (see Presmeg et al. 2016, p. 1).

The significance of semiosis for mathematics education lies in the use of signs; this use is ubiquitous in every branch of mathematics. It could not be otherwise: the objects of mathematics are ideal, general in nature, and to represent them—to others and to oneself—and to work with them, it is necessary to employ sign vehicles,[1]

[1]A note on terminology: The term "sign vehicle" is used here to designate the signifier, when the object is the signified. Peirce sometimes used the word "sign" to designate his whole triad, object [signified]-representamen [signifier]-interpretant; but sometimes Peirce used the word "sign" in designating the representamen only. To avoid confusion, "sign vehicle" is used for the representamen/signifier.

which are not the mathematical objects themselves but stand for them in some way. An elementary example is a drawing of a triangle—which is always a particular case—but which may be used to stand for triangles in general (Radford 2006).

Semiotics has long been a topic of relevance in connection with language (e.g., de Saussure 1959; Vygotsky 1997). However, it is in the last few decades that its potential has been realized for mathematics education research. In the early 1990s, David Kirshner may be credited with the introduction of semiotics, in the form of Saussurean semiology, Peircean semiotics, and semiotic chaining, to many researchers in mathematics education in the USA. He organized the Annual Meeting of the North American chapter of the International Group for the Psychology of Mathematics Education in Baton Rouge, Louisiana, and Whitson (1994, 1997) delivered the keynote address, in which he also provided an entry to the semiotic activity of Walkerdine (1988). Semiotics gained the attention of many researchers interested in furthering the understanding of processes involved in the learning and teaching of mathematics (e.g., Anderson et al. 2003; Presmeg 1997, 1998, 2003, 2006a, 2006b; Radford 2013; Radford et al. 2008, 2011; Sáenz-Ludlow and Kadunz 2016; Sáenz-Ludlow and Presmeg 2006a). In the Vygotskian tradition, *semiotic mediation* was used as a powerful research lens (Mariotti and Bartolini Bussi 1998).

As characterized by Presmeg et al. (2016), the study of signs has a long and rich history. However, as a self-conscious and distinct branch of inquiry, semiotics is a contemporary field originally flowing from two independent research traditions: those of Peirce (1931–1958), the American philosopher who originated pragmatism, and de Saussure (1959), a Swiss linguist generally recognized as the founder of contemporary linguistics and the major inspiration for structuralism. In addition to these two research traditions, several others implicate semiotics either directly or implicitly: these include *semiotic mediation* (the "early" Vygotsky 1978), *social semiotics* (Halliday 1978), various theories of representation (Goldin and Janvier 1998; Vergnaud 1985; Font et al. 2013), relationships amongst sign systems (Duval 1995), and more recently, theories of embodiment that include gestures and the body as a mode of signification (Bautista and Roth 2012; de Freitas and Sinclair 2013; Radford 2009, 2014; Radford et al. in press; Roth 2010). Components of some of these theories are elaborated in this book.

As a text on the origin of (Euclidean) geometry suggests, the mathematical concepts are the result of the continuing refinement of physical objects that Greek craftsmen were able to produce (Husserl 1939).[2] For example, craftsmen were producing rolling things called in Greek *kylindros* (roller), which led to the mathematical notion of the cylinder, a limit object that does not bear any of the imperfections that a material object will have. Children's real problems are in

[2]Husserl distinguished two aspects of signs, namely *expression* and *indication* (Husserl 1970; Zagorianakos 2017). It is beyond the scope of this monograph to explore the implications of Husserl's phenomenological distinction here; however, both of these aspects of signs are highly relevant in the issues addressed in this book. Expression relates to intention and the grounding of ideation, whereas indication relates to communication and is the essence of semiotics.

moving from the material things they use in their mathematics classes to the mathematical things (Roth 2011). This principle of "seeing an A as a B" (Otte 2006; Wartofsky 1968) is by no means straightforward and directly affects the learning processes of mathematics at all levels (Presmeg 1992, 2006a; Radford 2002). Thus semiotics, in several traditional frameworks, has the potential to serve as a powerful theoretical lens in investigating diverse topics in mathematics education research.

1.3 Sociocultural Perspectives on Semiosis

Sociocultural perspectives on semiosis emphasize the social, cultural, and historical dimension of signs. In these perspectives signs are understood not as artifacts to which an individual resorts to represent or present knowledge, but as artifacts of communication and signification. Signs are not considered as mere expressions of the individual's thought; they appear rather as entities through which the individual orients her actions and reflections and shapes her experience of the world.

The origin of this non-representational view of signs goes back to the early Vygotsky, who considered the sign as a sort of psychological instrument deeply related to the way we conduct ourselves in society. The essence of sign use, Vygotsky argued, "consists in man's [sic] affecting behavior through signs" (1978, p. 54). Vygotsky was particularly interested in the role of language. In a notepad dated 1926 he defines language as follows:

> Language is not the relation between a sound and the denoted thing. It is the relation between the speaker and the listener, the relation between people directed toward an object, it is an interpsychic reaction that establishes the unity of two organisms in the same orientation toward an object. (Vygotsky in Zavershneva 2010, p. 25)

More than a representation device, language is a *nexus* between individuals.

At the end of his life, Vygotsky was moving away from the instrumentalist view of signs to a view where meaning and signification acquired a more prominent role and where consciousness was understood in semiotic terms. Vygotsky wrote: "Consciousness as a whole has a semantic structure" (1997, p. 137). By this, Vygotsky meant that consciousness is not something metaphysical but our actual link to the world. He continues: "*We judge consciousness by its semantic structure, for sense, the structure of consciousness, is the relation to the external world*" and concludes that "*Speech produces changes in consciousness.* Speech is a correlate of consciousness, not of thinking" (p. 137; emphasis in the original).

Vygotsky's non-representational view of signs leads to a conception of semiotics that opens an interesting path in which to investigate the problems underlying education in general and mathematics education in particular. Consciousness and thinking are not merely the production of the individual. Consciousness and thinking come into life against the backdrop of their sociocultural context. But this context is not a mere facilitator of consciousness and thinking. Consciousness and thinking do not merely *adapt* to the context, they are modified by it and, in a dialectic movement, they come to modify the context from which they emanate. In the dialect materialist framework in which Vygotsky sets the problem of consciousness and thinking, both the individual and culture are coterminous entities in perpetual flux, one continuously becoming the other and the other the one.

This is so because signs and semiotic systems more generally are bearers of a worldview that includes mathematical, scientific, aesthetic, legal, and ethic components through which individuals organize their world (Radford 2008). As a result, the apparently transparent and neutral manner in which students encounter mathematics and other disciplines in the school has an unavoidable ideological valence. For instance, the Cartesian graph, which is featured in several chapters in this monograph, conveys a conceptual view according to which things in the world can be related and referred to the same point (the Cartesian origin). It stresses a relational view of phenomena attended to in terms of variables and their relationship. In opposition to other kind of graphs, such as maps, what a Cartesian graph depicts is not the elements of the considered phenomena but specific mathematical relationships between them—their covariation. Behind a Cartesian graph lies thus a general view of the world, where things are thought of in certain culturally and historically constituted ways. Implicitly, they organize and orient the kind of experience that students and teachers make of the world, creating thereby sociocultural conditions for the emergence of specific forms of mathematical thinking and learning. The same can be said of other semiotic systems too (e.g., the alphanumeric symbolism of algebra). Through them, our view of the world becomes *naturalized*. The world appears in specific ways. This is what the ideological valence of signs means.

Of course, the ideological valence of signs expressed in the worldview that the signs unavoidably carry and the concepts to which they refer cannot be revealed to the students spontaneously, that is, in an immediated or unmediated manner. A student can spend hours looking at a Cartesian graph without necessarily understanding what this complex mathematical sign means and is meant for. A sign *as such* is no more than that: a sign. To signify, to reveal its conceptual power, a sign has to become part of an *activity*. It is not hence through signs as such that students make the experience of mathematics their own, and that they encounter the culturally and historically constituted forms of mathematical thinking in the school or the university. Mathematics can only be disclosed to the students through sign-based activity. It is through material and concrete sign-based activity that students learn mathematics and that teachers teach it.

One of the differences between sociocultural perspectives on semiotis resides in how they conceptualize the learning activity and the role they ascribe to signs. Some perspectives emphasize the *discursive* dimension of activity, while others emphasize its *intersubjective* and *ethical* relational dimension and the evolving object/motive of the activity. Some perspectives consider signs as *mediators* of activity (Bartolini Bussi and Mariotti 2008), others consider signs as *part* of activity and as part of the material texture of thinking (Radford 2016a, b).

1.4 Language and Text Orientations

The importance of language for the learning of mathematics is a widely studied subject (Hoffman 2005; Sáenz-Ludlow and Presmeg 2006b; Schreiber 2013). A search query on the subject "language" in the journal *Educational Studies in Mathematics* resulted in more than a thousand responses (June 2017). As an example, reference should be made here to two anthologies published in recent years or forthcoming (Moschkovich 2010; Barwell 2017) and to a recent review of language in mathematics education published during the past 10 years in the Proceedings of PME (Radford and Barwell 2016). A much smaller result followed the request for "semiotics and language," to which the contributions in the second section of this volume are devoted. What is the relationship between text and language, the written and the spoken (Radford 2002; Kadunz 2016)?

In the brevity of this introduction, let us concentrate on two of the most important authors. On the one hand, we focus here on Peirce (see also the next section of this introduction), whose semiotics can be seen as paradigmatic for the importance of the written when doing mathematics. On the other hand, in contrast let us consider the work of the philosopher Ludwig Wittgenstein, who, with his theory of language play, has made a significant contribution to the philosophy of language, as used also in didactics of mathematics (Vilela 2010; Knijnik 2012).

A semiotic view of the learning of mathematics—if one chooses a Peircean orientation—is mainly determined by interpreting the use of visible signs. What approaches and questions open up when a linguistic approach is added to this theory of signs? Which parallels can be found between the formulations of Peirce and Wittgenstein? A simultaneous use of both approaches for questions involving the didactics of mathematics took place a few years ago (Dörfler 2004, 2016).

Documented parallelism between the central concepts of Peirce and Wittgenstein can be found mainly outside mathematics didactics. In this respect, starting several years ago, Gorlée (1994, 2012) presented a series of publications dealing with Peirce's semiotics as a tool for the analysis of translation questions. In "Semiotics and the problem of translation" (1994), she dedicated a chapter to the relationship between notions of Peirce and Wittgenstein. What are the similarities between Peirce's semiotics and Wittgenstein's philosophy of language? Wittgenstein presented one of his most far-reaching tools, namely, the *language game*, in his philosophical investigations (1953–1968). In this formulation, among others, he included the following:

> Giving orders, and obeying them;
>
> Describing the appearance of an object, or giving its measurement;
>
> …
>
> Forming and testing a hypothesis;
>
> Presenting the results of an experiment in tables and diagrams;
>
> …

> Solving a problem in practical arithmetic;
>
> Translating from one language into another.
>
> … (Wittgenstein 1953–1968 Part 1 paragraph 23; Gorlée 1994, p. 97)

In particular, he counted mathematical activities among these games. On the one hand the participation in a language game is characterized by the obeying of rules. On the other hand, the language game is embedded in a form of life: "Though primarily language-based, language-games do not function in a social vacuum, but are inscribed in so-called 'forms of life'." A form of life is "… a pattern of meaningful behavior in so far as this is constituted by a group" (Finch 1977, p. 91; Gorlée 1994, p. 99).

Practicing a language game is practicing an activity within a form of life which, according to Wittgenstein, combines language and reality. If we follow Umberto Eco (1979), the concept of the form of life provides a first bridge between Wittgenstein and Peirce. "Eco identifies the cultural system as a whole with the dynamic process of semiosis, and therefore, cultural units with Peircean interpretants." (Gorlée 1994, p. 100). In Peirce's terminology, the meaning of a sign is another sign which leads to a never ending process of interpretation embedded in our sociocultural life, which can be seen as a certain kind of practice.

Another similarity between the theories of Peirce and Wittgenstein can be found when we look at Peirce's concept of "ground." For Peirce, ground seems to be a kind of context that determines how a character represents a designated object. As Gorlée portrays it, then, "this ground is an abstract but knowable idea serving as justification for the mode of being manifested by the sign" (1994, p. 101). This embedding in a context corresponds in some respects to Wittgenstein's concept of "inner motivation". This motivation is the "ground," in which a language game has to be played within the framework of the corresponding rules. Similar comparisons, which can be only hinted at here, concern the Peircean concepts of firstness, secondness and thirdness, contrasted with concepts from Wittgenstein`s language games (Gorlée 1994). What is common to these opposites is the fact that for Peirce and for Wittgenstein, the interpretation of signs (semiosis) as well as the activities within the context of a language game are more focused on the process than on the result. After this excursion into a certain philosophy of language let us return to pure Peircean semiotics, in the next section.

1.5 Peircean Semiotics, Including Semiotic Chaining and Representations

Throughout the 1990s and in the early 2000s, the issue of how representations of various kinds played a role in mathematics education was a significant focus for researchers. There were Working Groups on this topic in the meetings of the International Group for the Psychology of Mathematics Education (PME) and its North American affiliate (PME-NA), resulting in an edited volume of papers from these conferences (Hitt 2002). Several authors in the current monograph were represented in this volume (Otte, Presmeg, Radford, Sáenz-Ludlow). In this early work, researchers were groping for theoretical formulations that went beyond a dualistic view of mathematical representation and captured the complexity involved in learning mathematics using its signs. A Peircean semiotic perspective provided one such conceptual lens.

Although a representational perspective on semiotics has largely given way to evolving sociocultural views (see Sects. 1 and 4 of this monograph), Peircean semiotics still has a foundational role to play in semiotics defined as "the relation between signs and symbols and what they denote" (Funk and Wagnall 2003). Further, semiotic chaining based on Peirce's semiotics has historical significance in the field of mathematics education on account of its contribution to research in this field since the early 1990s (Whitson 1994, 1997; Presmeg 1997, 1998, 2003, 2006a, b; Sáenz-Ludlow and Presmeg 2006a), and it still continues to provide a viable research lens for the teaching and learning of mathematics (Sect. 3 of this monograph).

The essence of Peirce's semiotics is his use of triads (see Chap. 11 by Sáenz-Ludlow for a fuller treatment of this topic). "But it will be asked, why stop at three?" Peirce asked, and he replied that unlike a triad, which adds something to a pair, "four, five, and every higher number can be formed by mere compilations of threes" (Peirce 1992, p. 251). His triad of signs as composed of *object* (signified), *representamen* (signifier) and the essential component of *interpretant* made possible the chaining of signs, since each sign as a whole is subject to further representation and interpretation, in a never-ending process of potential signification. Presmeg (1998, 2002, 2006b) used the metaphor of Russian nested dolls to describe this process, which was useful in linking home cultural practices of students with the mathematics that they learned in school (see also Presmeg 2007). Sáenz-Ludlow and colleagues used elaborated versions of Peirce's triads and the chaining of signs in fine-grained analyses of the processes involved in teaching and learning geometry (Sáenz-Ludlow and Kadunz 2016, and see Chap. 11 of this monograph). Sign vehicles characterized by Peirce's triad of *icon*, *index*, or *symbol* were the basis for an analysis of connections among early processes in the teaching and learning of trigonometry at high school level, obviating the compartmentalization that is often a hindrance in such learning (Presmeg 2006a).

Semiotic resources including gesturing and tools; developments in theoretical frameworks that involve these aspects, are discussed in the next section.

1.6 Semiotic Resources Including Gestures and Tools

In most general terms, the sign has been defined as a relation between one portion of the material continuum, which serves as sign vehicle for a relationship with other portions of the continuum (Eco 1986). Thus, any material thing—scribbles with pens, characters printed by a machine, sounds coming from a mouth, or tools used for doing things—constitutes a segmentation of matter that may be part of a relation between things. Such relations among material things indeed reflect relations between persons; in turn, human relations are reflected in the relation between things (Marx and Engels 1978). The relation between material things comes to be attributed to one of these as a suprasensible (ideal) characteristic, namely, its value in economic (Marx and Engels 1962) or verbal exchange (Rossi-Landi 1983; Roth 2006); human relations, once represented in the individual, have become higher psychological functions and personality (Vygotsky 1989). Indeed, the human body, being material, may be part of a material configuration and thus serve as the sign vehicle for other things. It is therefore not without surprise to read that "as subjects, we are what the shape of the world produced by signs makes us become" (Eco 1986, p. 45).

Any human action may become significant and, thereby, become part of a signifying relation. Thus, for example, when asked to describe and explain an experiment they have done, physics students invited the teacher, "Look!," and then redid the experiment (Roth and Lawless 2002). When students in a mathematics class have their heads down, writing in a notebook, this bodily configuration and writing itself may be treated as the sign of their engagement with the task. Work-related body movements that are taken to stand for something are denoted by the term ergotic gestures [gestes ergotiques] (Cadoz 1994; Roth 2003). Interestingly, in speaking, more is happening than the production of sound words that are somehow referring to or invoking something else. The very act of speaking may be significant as an act generally or as a speech act specifically (Schütz 1932). "Did you say something?" is a query to find the significance in the former case, whereas "What did you say?" is a query to find out the significance in the latter case.

In individual development, there actually is a movement from ergotic gestures to symbolic gestures [gestes symboliques], which, in humans, may morph into or be replaced by productions such as sound-words or hand/arm movements that take their place (e.g., the stinky finger, a square formed by an appropriate configuration of thumbs and index fingers of two hands). This movement first was described in the case of children learning to gesture: an infant may be seen by the mother as reaching for an object; she takes the object and puts it into the hand of the infant; and finally, the infant moves hand and arm intentionally to point at objects (Vygotsky 1989). The same transformation also was reported among the bonobo chimpanzees, where part of the infant's movement involved in the mother's picking up the infant later, deployed in a frozen form, is treated by the mother as a sign that the infant wants to be picked up (Hutchins and Johnson 2009). In school science, the same trajectory has been described beginning with the initial ergotic gestures, which then turned into symbolic gestures using part of the equipment or substituted artifacts (tools), all of which eventually were replaced in verbal descriptions and diagrams (Roth and Lawless 2002). Materials, artifacts and tools have communicative and thus both social and psychological function. In the context of using graphs as part of lectures, hand/arm movements initially appeared to carve out the space, exploring possible placements of curves, before some of these movements actually produced the graph expressing a mathematical function (Roth 2012). That is, gestures may indeed pave the way for linguistic and conceptual development (Iverson and Goldin-Meadow 2005). It also has been shown that the relation between symbolic gestures and what they stand for may change in the course of time, thereby changing (developing) signifier–signified relations.

In the mathematics education literature, we can find the term "body language" (e.g. Evans et al. 2006). But this notion, though "common in everyday language, is not a useful concept here because body movements and positions are neither structured nor used like language" (Roth 2001, p. 368). Hand/arm movements may be located somewhere along a continuum ranging from idiosyncratic movements that accompany speech (gesticulation) to highly structured sign language, with language-like gestures, pantomime, and emblems lying between the two extremes. Sign language consists of hand/arm movements that indeed have a relatively fixed syntax and semantics (lexicon): it is language in a strong sense. Emblems take specific places in and in lieu of linguistic expressions. Gesticulations accompany speech but are not subject to syntax and semantics, so that the same movements may appear in many different contexts contributing in very different ways to communication. Finally there are body movements that are completely incidental: grooming gestures and body positions and configurations. None of these forms deserve to be classified as language in the linguistic sense.

It is useful to distinguish different functions of hand/arm and other body movements. Movements may have deictic (pointing) function, stand in an iconic relation with something else, or constitute a rhythmic feature denoted as beat gesture (McNeill 2005). Although a pointing gesture does not have mathematical content, it may nevertheless have an important function in making manifest a gestalt in the environment that is part of the sense-making process (e.g. Radford 2009). Iconic gestures may be sign vehicles for the relation with other concrete portions of the continuum, such as when a lecturer moves a hand in a straight line while talking about a linear function drawn on a chalkboard (Núñez 2009). Or they may stand for an idea, such as that of a mathematical limit (itself modeled on material limits), when the speaker holds one hand still while approaching it with the other (McNeill 1992). Although not immediately apparent, beat gestures, too, may have important functions in mathematical teaching|learning events, for example, supporting grouping and counting (Roth 2011).

In the past, mathematical knowing was considered in terms of mental constructions and conceptual frameworks. More recently, it was recognized that body movements generally and hand/arm-produced gesticulations more specifically manifest knowing. Some research is based on the conviction that there are two underlying cognitive systems, whereas others consider there to be one cognitive system that manifests itself in two different, sometimes contradictory ways (see the review by Roth 2001). Even more recently, embodiment and enactivist perspectives have attempted to emphasize the role of the body in human communication and knowing (e.g. Núñez 2009; Proulx 2013). Both approaches, however, have been subject to critique because of the underlying Cartesianism (Sheets-Johnstone 2009). This Cartesianism is overcome by a Marxian-Spinozist approach in which body and mind (thought) are two manifestations of the same underlying substance (Roth 2017). Consider the example of a circle. By inscribing a circle with a pen on a piece of paper, the body is in a state identical with the circle outside of the body (Spinoza 2002). Indeed, the body comes “into a state of real action in the form of a circle” (Il’enkov 1977, p. 69), and the associated awareness (consciousness) is the idea of and fully adequate to the circle. Even seeing a circle as such is fully adequate, because the eyes have to move along (but saccading to and away from) the circular line to produce the visual experience of a circle (Yarbus 1967). Drawing a circle and knowing a circle have become indistinguishable.

1.7 Conclusion

This introductory chapter gives the reader a preliminary overview of the diversity of theoretical formulations of semiotics as a field of scholarship, and of the power of semiotics as a research lens in investigating the complexities of learning and teaching mathematics. At present, semiotic theories have proved valuable mainly in fine-grained qualitative research studies involving students’ learning of mathematics, the relationships involved in activities towards this end, and the role of teachers and teaching in this regard, at various levels and in diverse social contexts. The chapters in this book exemplify the efficacy of semiotic theories as lenses in such research. However, as attested by Morgan’s chapter, there is also the potential for semiotics research in the wider fields of institutional contexts and policy research. Because of its relevance in the human endeavor of creating and learning mathematics, and all that entails, semiotics will continue to have significance in mathematics education and its research.

References

Anderson, M., Sáenz-Ludlow, A., Zellweger, S., & Cifarelli, V. (2003). *Educational perspectives on mathematics as semiosis: From thinking to interpreting to knowing*. Ottawa: Legas.

Bartolini Bussi, M., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York: Routledge, Taylor and Francis.

Barwell, R., et al. (Eds.). (2017). *Mathematics education and language diversity: The 21st ICMI study*. New York: Springer.

Bautista, A., & Roth, W.-M. (2012). Conceptualizing sound as a form of incarnate mathematical consciousness. *Educational Studies in Mathematics, 79*(1), 41–59.

Cadoz, C. (1994). *Les réalités virtuelles [Virtual realities]*. Paris, France: Flammarion.

Colapietro, V. M. (1993). *Glossary of semiotics*. New York, NY: Paragon House.

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics, 83,* 453–470.

de Saussure, F. (1959). *Course in general linguistics*. New York, NY: McGraw-Hill.

Dörfler, W. (2004). Diagrams as means and objects of mathematical reasoning. In *Developments in mathematics education in German-speaking countries. Selected papers from the annual conference on didactics of mathematics* 2001 (pp. 39–49). Hildesheim: Franzbecker.
Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt, & W. Dörfler (Eds.), *Theories in and of mathematics education. Theory strands in German speaking countries* (pp. 21–30). New York: Springer.
Duval, R. (1995). *Sémoisis et pensée humaine [Semiosis and human thinking]*. Bern: Lang.
Eco, U. (1979). *A theory of semiotics*. Bloomington: Indiana University Press.
Eco, U. (1986). *Semiotics and the philosophy of language*. Bloomington, IN: Indiana University Press.
Eco, U. (1988). *Le signe (The sign)*. Brussels: Éditions Labor.
Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics, 63,* 209–226.
Finch, H. L. (1977). *Wittgenstein—The later philosophy: An exposition of the "philosophical investigations"*. New York: Humanities Press.
Font, V., Godino, J., & Gallardo, J. (2013). The emergence of mathematical objects from mathematical practices. *Educational Studies in Mathematics, 82*(1), 97–124.
Funk & Wagnall (2003). *New international dictionary of the English language*. Chicago, IL: World Publishers Inc.
Goldin, G. A., & Janvier, C. (1998). Representations and the psychology of mathematics education. *The Journal of Mathematical Behavior*, *17*(1&2).
Gorlée, D. L. (1994). *Semiotics and the problem of translation*. Amsterdam, Atlanta: Rodopi.
Gorlée, D. L. (2012). *Wittgenstein in translation: Exploring semiotic signatures*. Berlin: Walter de Gruyter.
Halliday, M. (1978). *Language as social semiotic*. London: Arnold.
Hitt, F. (2002). *Representations and mathematics visualization*. Mexico City: Cinvestav IPN.
Hoffmann, M. H. G. (2005). *Erkenntnisentwicklung*. Frankfurt am Main: Vittorio Klostermann.
Husserl, E. (1939). Die Frage nach dem Ursprung der Geometrie als intentional-historisches Problem [The question about the origin of geometry as intentional-historical problem]. *Revue Internationale de Philosophie, 1,* 203–225.
Husserl, E. (1970). *Logical investigations*. (1901, D. Carr, Trans.) New York: Humanities Press.
Hutchins, E., & Johnson, C. M. (2009). Modeling the emergence of language as an embodied collective activity. *Topics in Cognitive Science, 1,* 523–546.
Il'enkov, E. (1977). *Dialectical logic: Essays in its history and theory*. Moscow: Progress.
Iverson, J. M., & Goldin-Meadow, S. (2005). Gesture paves the way for language development. *Psychological Science, 16,* 367–371.
Kadunz, G. (2016). Diagrams as means for learning. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics: How to describe the construction, visualisation, and communication of mathematics concepts*. Rotterdam: Sense Publishers.
Knijnik, G. (2012). Differentially positioned language games: Ethnomathematics from a philosophical perspective. *Educational Studies in Mathematics, 80*(1), 87–100.
Mariotti, M. A. & Bartolini Bussi, M. G. (1998). From drawing to construction: Teachers' mediation within the Cabri environment. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME International Conference* (Vol. 1, pp. 180–195).
Marx, K., & Engels, F. (1962). *Werke Band 23* [*Works vol. 23*]. Berlin: Dietz.
Marx, K., & Engels, F. (1978). *Werke Band 3* [*Works vol. 3*]. Berlin: Dietz.
McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.
McNeill, D. (2005). *Gesture and thought*. Chicago, IL: University of Chicago Press.
Moschkovich, J. N. (Ed.). (2010). *Language and mathematics education: Multiple perspectives and directions for research*. Charlotte, NC: Information Age Publishing.
Núñez, R. E. (2009). Gesture, inscriptions, and abstraction: The embodied nature of mathematics or why mathematics education shouldn't leave the math untouched. In W.-M. Roth (Ed.),

Mathematical representation at the interface of body and culture (pp. 309–328). Charlotte, NC: Information Age Publishing.
Otte, M. (2006). Mathematics epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics, 61,* 11–38.
Peirce, C. S. (1931–1958). *Collected papers* (CP, Vols. 1–8). Cambridge: Harvard University Press.
Peirce, C. S. (1992). In N. Houser & C. Kloesel (Eds.), *The essential Peirce* (Vol. 1). Bloomington, IN: Indiana University Press.
Presmeg, N. C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23,* 595–610.
Presmeg, N. C. (1997). A semiotic framework for linking cultural practice and classroom mathematics. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey (Eds.), *Proceedings of the 19th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 151–156). Columbus, Ohio.
Presmeg, N. C. (1998). Ethnomathematics in teacher education. *Journal of Mathematics Teacher Education, 1*(3), 317–339.
Presmeg, N. C. (2002). A triadic nested lens for viewing teachers' representations of semiotic chaining. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 263–276). Mexico City: Cinvestav IPN.
Presmeg, N. C. (2003). *Ancient areas: A retrospective analysis of early history of geometry in light of Peirce's "commens".* Paper presented in Discussion Group 7, Semiotic and socio-cultural evolution of mathematical concepts, 27th Annual Meeting of the International Group for the Psychology of Mathematics Education, Honolulu, Hawaii, July 13–18, 2003. Subsequently published in the *Journal of the Svensk Förening för MatematikDidaktisk Forskning* (*MaDiF*) (Vol. 8, pp. 24–34), December, 2003.
Presmeg, N. C. (2006a). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. In J. Novotná, H. Moraová, M. Krátká, N. Stehliková (Eds.), In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19–34). Prague: PME.
Presmeg, N. C. (2006b). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61,* 163–182.
Presmeg, N. C. (2007). The role of culture in teaching and learning mathematics. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 435–458). Charlotte, NC: Information Age Publishing.
Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in mathematics education.* ICME-13 topical survey. Hamburg: Springer Open.
Proulx, J. (2013). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics, 84,* 309–328.
Radford, L. (2002). The seen, the spoken, and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22,* 14–23.
Radford, L. (2006). How to look at the general through the particular: Berkeley and Kant on symbolizing mathematical generality. In S. Sbaragli (Ed.), *La matematica e la sua didattica* (pp. 245–248). Roma: Carocci Faber.
Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishers.
Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70,* 111–126.
Radford, L. (2013). On semiotics and education. *Éducation & Didactique, 7*(1), 185–204.
Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM—The International Journal on Mathematics Education*, *46*, 349–361.
Radford, L. (2016a). Mathematics education as a matter of labor. In M. A. Peters (Ed.), *Encyclopedia of educational philosophy and theory.* Section: Mathematics education

philosophy and theory. (P. Valero and G. Knijnik, Editors). Singapore: Springer. https://doi.org/10.1007/978-981-287-532-7_518-1.

Radford, L. (2016b). The theory of objectification and its place among sociocultural research in mathematics education. *International Journal for Research in Mathematics Education—RIPEM*, *6*(2), 187–206.

Radford, L., Arzarello, F., Edwards, L., & Sabena, C. (in press). The multimodal material mind: Embodiment in mathematics education. In J. Cai (Ed.), *First compendium for research in mathematics education.* Reston, VA: NCTM.

Radford, L., & Barwell, R. (2016). Language in mathematics education research. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education. The journey continues* (pp. 275–313). Rotterdam: Sense.

Radford, L., Schubring, G., & Seeger, F. (2008). *Semiotics in mathematics education: Epistemology, history, classroom, and culture*. Rotterdam: Sense Publishers.

Radford, L., Schubring, G., & Seeger, F. (2011). Signifying and meaning-making in mathematics thinking, teaching and learning: Semiotic perspectives. *Educational Studies in Mathematics, 77* (2–3), 149–397.

Rossi-Landi, F. (1983). *Language as work & trade: A semiotic homology for linguistics & economics*. South Hadley, MA: Bergin & Garvey.

Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research, 71,* 365–392.

Roth, W.-M. (2003). From epistemic (ergotic) actions to scientific discourse: Do gestures obtain a bridging function? *Pragmatics & Cognition, 11,* 139–168.

Roth, W.-M. (2006). A dialectical materialist reading of the sign. *Semiotica, 160,* 141–171.

Roth, W.-M. (2010). Incarnation: Radicalizing the embodiment of mathematics. *For the Learning of Mathematics, 30*(2), 8–17.

Roth, W.-M. (2011). *Geometry as objective science in elementary classrooms: Mathematics in the flesh*. New York: Routledge.

Roth, W.-M. (2012). Tracking the origins of signs in mathematical activity: A material phenomenological approach. In M. Bockarova, M. Danesi, & R. Núñez (Eds.), *Cognitive science and interdisciplinary approaches to mathematical cognition* (pp. 209–247). Munich: LINCOM EUROPA.

Roth, W.-M. (2017). *The mathematics of mathematics: Thinking with the late, Spinozist Vygotsky*. Rotterdam: Sense Publishers.

Roth, W.-M., & Lawless, D. (2002). Signs, deixis, and the emergence of scientific explanations. *Semiotica, 138,* 95–130.

Sáenz-Ludlow, A., & Kadunz, G. (2016). *Semiotics as a tool for learning mathematics: How to describe the construction, visualisation, and communication of mathematics concepts*. Rotterdam: Sense Publishers.

Sáenz-Ludlow, A., & Presmeg, N. (2006a). Semiotic perspectives in mathematics education. *Educational Studies in Mathematics. Special Issue*, *61*(1–2).

Sáenz-Ludlow, A., & Presmeg, N. (2006b). Guest editorial: Semiotic perspectives on learning mathematics and communicating mathematically. *Educational Studies in Mathematics, 61,* 1–10.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies in Mathematics, 82*(1), 51–73.

Schütz, A. (1932). *Der sinnhafte Aufbau der sozialen Welt [The phenomenology of the social world]*. Vienna: Julius Springer.

Sheets-Johnstone, M. (2009). *The corporeal turn: An interdisciplinary reader*. Exeter: Imprint Academic.

Spinoza, B. (2002). *Complete works*. Indianapolis: Hackett Publishing.

Vergnaud, G. (1985). Concepts et schèmes dans la théorie opératoire de la representation [Concepts and schemas in the operatory theory of representation]. *Psychologie Française, 30* (3–4), 245–252.

Vilela, D. S. (2010). Discussing a philosophical background for the ethnomathematics program. *Educational Studies in Mathematics, 75*(3), 345–358.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.

Vygotsky, L. (1997). *Collected works* (Vol. 3). New York: Plenum.

Walkerdine, V. (1988). *The mastery of reason: Cognitive developments and the production of rationality*. New York: Routledge.

Wartofsky, M. (1968). *Conceptual foundations of scientific thought*. New York: Macmillan.

Whitson, J. A. (1994). Elements of a semiotic framework for understanding situated and conceptual learning. In D. Kirshner (Ed.), *Proceedings of the 16th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 35–50). Baton Rouge, LA.

Whitson, J. A. (1997). Cognition as a semiosic process: From situated mediation to critical reflective transcendence. In D. Kirshner, J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 97–149). Mahwah, NJ: Lawrence Erlbaum Associates.

Wittgenstein, L. (1953–1968). *Philosophische Untersuchungen* [*Philosophical investigations*] (G. E. M. Anscombe, Trans.). Oxford: Basil Blackwell.

Yarbus, A. L. (1967). *Eye movement and vision*. New York: Plenum.

Zagorianakos, A. (2017). *Phenomenological analysis of the consequences of the overcoming of a cognitive dead-end of a prospective teacher of mathematics.* Paper submitted for the conference of the (Greek) Union of Researchers of Mathematics Education, December 3–5, 2017.

Zavershneva, E. (2010). The Vygotsky family archive (1912–1934). New findings. *Journal of Russian and East European Psychology, 48*(1), 14–33.

Part I
Sociocultural Perspectives on Semiosis

Introduction to Section 1: Luis Radford

This part deals with semiosis understood as the continuous production of signs and significations. The part comprises five chapters. There are two elements that tie them together. First, they share a general interest in exploring the social, cultural, and historical dimension of semiosis. Second, they resort to a Vygotskian concept of the sign.

It is worth recalling that, in the main Western semiotic traditions, signs have been understood as entities that *represent* things. Ferdinand de Saussure (1959), for instance, suggested that a sign is the union of a *signified* (i.e., the meaning or the concept) and the *signifier* (the sound-image).Vygotsky took a completely different route. Although Vygotsky's concept of signs evolved from a rather instrumental view—developed at the end of the 1920s where the sign was conceptualized as a psychological tool (Vygotsky and Luria 1994)—to a more social-relational one envisioned at the end of his life, a common point in Vygotsky's concept of signs was that signs are not entities to represent things or ideas or knowledge. In Vygotsky's view, the most streaking feature of signs is the orientating role that they play in the social life of the individuals, allowing them to organize and reorganize their interactions with other individuals and their deeds in the historical world. In a notebook containing notes written toward the end of his life, and referring in particular to language, Vygotsky's noted that

> Language is not the relation between a sound and the denoted thing. It is the relation between the speaker and the listener, the relation between people directed toward an object, it is an interpsychic reaction that establishes the unity of two organisms in the same orientation toward an object. (Vygotsky in Zavershneva 2010, p. 25)

Vygotsky's concept of sign leads to a complex and multilayered concept of semiosis. Revolving around the context of mathematics teaching and learning, each chapter explores different aspects of it.

Chapter 1, written by Luis Radford, explores a key question of subjectivity: how through cultural-historical activity people simultaneously are produced and co-produce themselves as individuals in the making, as unfinishable projects of life. The chapter starts with a metaphor where individuals are considered as dynamic agentic signs and goes on to explore the role of rules in a preschool setting around a mathematical game. Through an analysis of the children's game, where emotions and actions are fueled with tensions resulting from the cultural normative character of rules and the subjective enactment of them, the children come to position themselves and to be socially and historically positioned. The analysis suggests that responsibility and empathy (*pátheia*) appear as two essential intertwined threads of the fabric of subjectivity. While the former points to a culturally and historically evolved relation to the Other, the latter signals a truly human listening, based on a pre-conceptual, emotional understanding of the misery and agony of the other through which we recognize ourselves and the fragility of our human nature.

Chapter 2, written by Wolff-Michael Roth, deals with the later Vygotsky's conception of signs and delves into the relational nature it conveys. Like pointing gestures, the sign emerges as a relation where one person acts on another and only later as something to act upon oneself. Roth starts by drawing a parallel between commodity exchange and word use. He uses this parallel to illustrate, through an example with young children, the birth of a relation between a sign vehicle and its (ideational) content, while showing how this first relation yields place to a new relation. The parallel between commodity exchange and word use allows Roth to stress an often missed key point in semiotic analysis: that the relation between sign vehicle and its content or suprasensible character is not merely an idiosyncratic product but reflects a social relation: "a reflection of societal (i.e., universal) relations that exist as societal relation of material things (e.g., sign vehicles)." This conceptualization of signs is hence far away from the representational conceptualization. It is relational through and through. The chapter closes with an interesting suggestion, namely the possibility of abandoning the idea of sign mediation that Roth explores by resorting to a Spinozist-Marxian materialist approach.

Chapter 3 in this part, written by Ulises Salinas-Hernandez and Isaias Miranda, deals with the understanding of the Cartesian graph associated with the motion of a tennis ball in an inclined plane in a technological environment. The authors turn to the theory of objectification (Radford 2008) where the students' understanding is considered to be the result of the manner in which mathematical knowledge appears progressively in the classroom through the students' and teachers' sensuous, semiotic, and material activity. Through a fine-grained semiotic analysis, Salinas-Hernandez and Miranda show how the students' understanding of the graph undergoes a lengthy process of objectification that is correlated to various moments of classroom activity. These various moments of activity are not independent of each other. They can be conceptualized as the successive transformation of an initial moment. The initial moment is characterized by an *abstract* stance where the graph is imagined as a straight line. The initial moment of activity is transformed into another moment and so on until of more *concrete* moment of activity is reached: one where, through sensuous corporeal, symbolic, linguistic, and kinesthetic actions, the mathematical variables and the axes provided by the software become objects of attention and consciousness.

Chapter 4 in this part, written by Anna Shvarts, tackles the question of joint attention using a dual eye-tracking setting. Shvarts investigates the manner in which five pairs of a Grade 1 child and his/her parent inter-coordinate perception with other semiotic resources—gestures, oral language, and mathematical signs—in the processes of objectification involved in the understanding of Cartesian coordinates. In the beginning, the child and the parent see the graph but they do not attend to it in the same manner. Although the meaning of the Cartesian coordinates is *there*, as an "ideal form" in culture, the child does not perceive it yet. The disclosing of the meaning, its progressive grasping by the child, occurs in processes of objectification embedded in the participants' joint activity. In such a disclosing, the semiotic resources move from what Shvarts calls *pre-semiotic means of objectification* to proper semiotic means of objectification (Radford 2003). One of the major contributions of this chapter is to reveal the dynamics of such a movement—how, for example, progressively, through semiotic resources, the participants achieve joint attention, how they adjust their evolving interpretations, and how they assume the initiative in the task.

Chapter 5, written by Debbie Stott, revolves also around the question of joint attention. She resorts to the theoretical constructs of the *space of joint action* and *togethering* (Radford and Roth 2011). While the former refers to an "evolving, tuning, and reciprocating of the participants' perspectives, making thinking a collective phenomenon" (p. 232), the latter rests on an ethical commitment to produce joint activity out of which ideal cultural-historical mathematical forms are disclosed to the students' consciousness. Stott draws also on the concept of attention catching (Meira and Lerman 2009), the *moment* when attention is caught during a mathematical activity involving more than one participant. She presents two episodes from interviews in an after-school math club and investigates how joint action is (or is not) produced. In particular, she is interested in investigating the moments of *poēsis*—i.e., the creative moments in which an ideal mathematical form becomes an object of the students' attention and understanding (Radford 2015). Stott's analysis provides insights into the way attention catching and togethering occur and are sustained by the participants; her chapter also sheds light on the question of how spaces of joint action evolve.

References

Meira, L., & Lerman, S. (2009). Zones of proximal development as fields for communication and dialogue. In C. Lightfoot & M. C. D. P. Lyra (Eds.), *Challenges and strategies for studying human development in cultural contexts* (pp. 199–219). Rome: Firera Publishing.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning, 5*(1), 37–70.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishers.

Radford, L. (2015). Methodological aspects of the theory of objectification. *Revista Do Programa de Pós-Graduação em Educação Matemática da Universidade Federal de Mato Grosso do Sul (UFMS), 8*, 547–567.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*(2–3), 227–245.
Saussure, F. (1959). *Course in general linguistics.* New York: McGraw-Hill.
Vygotsky, L. S., & Luria, A. (1994). Tool and symbol in child development. In R. V. D. Veer & J. Valsiner (Eds.), *The Vygotsky reader* (pp. 99–174). Oxford: Blackwell Publishers.
Zavershneva, E. (2010). The Vygotsky family archive (1912–1934). New findings. *Journal of Russian and East European Psychology, 48*(1), 14–33.

Chapter 2
Semiosis and Subjectification: The Classroom Constitution of Mathematical Subjects

Luis Radford

Abstract In this chapter, I consider semiosis as the continuous production of signs and significations. However, I do not limit the scope of signs to marks or inscriptions. I consider individuals as signs too. Like signs, individuals come to occupy positions in the social world and behave in ways that are not at all different from signs in a text. A crucial difference between inscriptions and individuals, though, is that individuals are not merely signified through well-defined syntaxes as inscriptions and traditional signs are. The cultural syntaxes through which individuals come to be positioned in the social world are less visible: they are part of a dynamic cultural symbolic superstructure. Another crucial difference is that, unlike inscriptions and marks, individuals co-produce themselves—even if it is within the limits of the aforementioned symbolic superstructure. Individuals co-produce themselves in what in this chapter I term processes of subjectification. This chapter is an attempt to study the processes of subjectification in the mathematics classroom. To do so, I analyze a classroom episode with pre-school children.

Keywords Semiosis · Being and becoming · Subjectification · Subjectivity Ethics · Pre-school mathematics · Pre-school games

2.1 Introduction: Life as a Semiotic Zone

Traditionally speaking, signs have been considered as marks or inscriptions. The inscriptions on Mesopotamian clay tablets or footprints on the sand are examples of signs. In the first case, the signs are produced intentionally by an individual or group of individuals. In the second case, the production of signs may not be the result of an intentional act. Yet, by being noticed and accentuated someone may

L. Radford (✉)
École des sciences de l'éducation, Université Laurentienne,
Sudbury, ON P3E 2C6, Canada
e-mail: lradford@laurentian.ca

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interpret the footprint as the presence of human beings in the surroundings. In both cases, the signs signify something.

Now, we can also think of signs in a more dynamic way—as entities that unfold in time, as part of a semiotic process that generates and regenerates itself according to semiotic rules—e.g., implicit, explicit or partially explicit rules of syntax or rules of meaning production, such as politeness and social behavior more generally. The heroes or protagonists in a novel may be seen as signs in this sense. Heroes or protagonists evolve as they engage with others in various manners and activities. They evolve in accordance with semiotic rules that are historically and culturally situated—an idea that brought Bakhtin (1981) to talk about *literary genres*. Bakhtin considered literary genres as being framed by three driving vectors: ideology, differentiation, and polyglossia. I would like to go a step further and argue that life can also be seen as a semiotic zone: the confluence and interaction of various activities in which, through semiosis—i.e., through processes of signification—individuals come to agentically position themselves in differential, polyglossic, and ideological manners.

Polyglossia (the acknowledgment of a variety of ways of thinking/speaking) and differentiation ($A \neq B$ as well as $A \neq A$) refer to an always unique individual who, through her engagement in social activities, continuously positions herself through other individuals in the cultural-historical world as an unrepeatable entity always in flux—an entity in perpetual be-*coming*. Ideology refers to an already present system of ideas (scientific, ethical, aesthetic, legal, etc.) that subsumes the individuals and that transcends the individual *qua* individual.[1] Ideology ubiquitously operates in the agentic processes of differentiation and polyglossia so that, as different and unique as individuals are, their uniqueness is nonetheless shaped by the common cultural ground of ideology. Similarity and dissimilarity, likeness and unlikeness, difference and uniqueness are cultural concepts whose content does not come from the individual's own interior but from the situationally operating ideology, history, and culture.

This way of conceptualizing individuals is certainly at odds with the conception of the individual that we have inherited from the philosophy of the Enlightenment that has informed modern Western thought. We have become used to the idea that we are equipped with an interior from where our true Being emanates. It is in this interior that our deepest feelings and meanings are allegedly formed, so that what we need in order to grow as human beings is simply a stimulating environment. The transposition of this idea to children during the 19th and 20th centuries has allowed educators and psychologists to imagine and formulate education as a project that consists precisely in getting rid of the multitude of interferences that impede the natural growth and self-fulfillment of the student. The conceptualization of

[1]Hence, my use of ideology has nothing to do with something such as false consciousness or with claims of mistaken outlooks of reality. I use ideology as Bakhtin (1981) and Voloshinov (1973) used it: as a system of ideas that operates in a culture at a certain historical moment and that unavoidably embodies and refracts the contradictions of the various voices and theoretical and practical dominant and non-dominant attitudes of the individuals.

individuals that I am suggesting here, as reflective signs in an unfinished process of becoming, subsumed within a dynamic symbolic ideological superstructure, goes in another direction. It is a direction that does not posit the student as the origin of knowing and becoming, as an essentially already given and already made entity. It rather conceives of the student as a continuously moving sign in the making.

This chapter is organized into three parts. The first part is a continuation of this introduction. In this part I continue to problematize the question of the subject and to show how the manner in which we come to conceive of ourselves is always culturally and historically situated. In the second part I suggest a definition of Being and subjectivity. Subjectivity refers to the idea of the subject as a dynamic sign, an unachievable project of life. The way subjectivity moves and how it is subjectively and societally produced, are accounted for in terms of a conception of life as the confluence and interaction of activities embedded in an always changing semiotic zone full of contradictions and agonies, hope and laughter, whose syntactic and semantic (explicit and implicit) rules (more often implicit than explicit) I explore through the concept of *semiotic systems of cultural signification*. In the third part I discuss a classroom episode in a pre-school setting that intends to show a concrete example of the processes through which subjectivities are societally co- and self-produced.

2.2 The Question of the Subject

To a very large extent, mathematics education research has drawn—rather implicitly—on the concept of the subject that philosophers of the Enlightenment articulated in the 18th century. As I argue elsewhere (Radford 2012), the Enlightened philosophers sought to build, in the overcoming of fear, tradition, and feudal hierarchies, their idea of the new subject. They found in freedom the subject's most fundamental trait (Adorno 2006). The idea of subject that they envisioned is someone who is not there to follow what others say or do, but one who has to think and reason by him or herself. Kant illustrates this idea perhaps better than anyone else: the Kantian subject is a subject of reason, the crafter of its own destiny, the architect of its own projects of life, the origin and source of meaning and knowledge. From the Kantian perspective, to be a subject is to be free. And to be free is not to be subjected to anything other than one's reasons. The result is a self-sufficient, humanist, and substantialist idea of the subject. In this chapter I explore the question of the subject from a different viewpoint. I suggest that the subject is a cultural-historical entity in perpetual transformation—i.e., a subjectivity-in-the-making. My starting point is the rather banal—although rarely considered—fact that mathematics classrooms are not only producers of knowledge but of subjectivities as well. Drawing on the theory of objectification (Radford 2008a), I am particularly interested in investigating the processes of subjectification out of which subjectivities produce themselves and are at the same time produced by the activities in which they engage.

2.3 Being and Subjectivity

To move beyond the Enlightened substantialist concept of the subject, I suggest distinguishing between Being and subjectivity. The concept of Being that the theory of objectification brings forward highlights what we may call the *being's cultural nature*. What this means is that our idea of what an individual and his/her power of action and will (i.e., agency) are, are relative to their historical moment. If we were born in ancient Greece or another historical period, we would have conceived of ourselves in a very different manner from the way we do today. In the Athens of Plato, for example, in the midst of a society articulated around the distinction between free citizens and slaves, with a negative valence to manual work and a positive valence to intellectual work, our sense of individuality would have been embedded in a political-geographical criterion of inclusion/exclusion—Athenians versus foreigners—and defined in terms of the opposition between passion and temperance and the struggle for "self-mastery" (Taylor 1989). Very different is the contemporary concept of the individual, defined as a private owner (Radford 2014)—an individualist subject, drawn by possessive, consumerist and instant gratification drives who is continuously urged to express herself "creatively and authentically" (Illouz 1997, p. 35).

Taylor's and Illouz's remarks reveal not only the political-cultural axis that structures the subject, but also the fact that the sense of ourselves is ineluctably embedded in an *ethical* axis: how we conduct ourselves in the social world, how we show ourselves to others, and how we are expected to behave socially and to be recognized by others. These examples show that culture provides the "raw material" from which subjects draw the ideas of what they are (their meaning, their identity, their power of action, etc.). This "raw material" is part of what in the theory of objectification is termed *semiotic systems of cultural significations* (SSCS). SSCS are dynamic symbolic superstructures (Radford 2008a) that include cultural conceptions about the world and the individuals. They comprise (a) ideas about things in the world (e.g., the nature of mathematical objects and their way of existing), (b) ideas about truth (e.g., how truth is and can be established), and (c) ideas about the individuals. SSCS are full of tensions, as are the activities from which they emanate, and have a normative function (which may be explicit, or implicit, or partially implicit).

Clearly, the relationship between the cultural "raw material" included in the SSCS and the concrete individuals cannot be seen as a logical, or causal, or mechanical relationship. This is so because the account that I am sketching here adopts a Spinozist ontological nature about humans according to which humans are unavoidably and profoundly affected by their context (Spinoza 1989). But they are affected in a *reflexive* manner. What emerges from this affection bears the imprint of the culture, but it is always an entity in flux impossible to anticipate and predict. The relationship between the cultural "raw material" included in the SSCS and the concrete individuals is in fact *dialectical*. It is a dialectical relationship between what I mean by "Being" and "subjectivities."

The adjective "dialectical" does not refer here to a simple reciprocal influence between two given entities—in this case Being and subjectivity. Although, in usual parlance 'reciprocal influence' is a common meaning of dialectic and its adjective dialectical, I consider the relationship between Being and subjectivity along the lines of dialectical materialism. In the naïve use of dialectics, the entities in relationship appear as already formed. In Hegel's dialectic, which is at the basis of Marx's dialectical materialism, in contrast, the relation is one of transformation. The transformation of something dynamic, general, and virtual into something concrete that affirms its source and at the same time negates it. In his *Phenomenology of Spirit*, Hegel considers the example of a bud: "The bud disappears in the bursting-forth of the blossom, and one might say that the former is refuted by the latter" (Hegel 1977, p. 2). The bud is sublated in the blossom, and as such the blossom affirms the bud. But at the same time, the blossom (which is the transformation of the bud) refutes or negates the bud, as it does not coincide with it. It is something else, yet something that could only come to life through the movement and transformation of the bud. The key idea in the Hegelian conception of dialectic is *movement*. "It is of the highest importance," says Hegel, "to interpret the dialectical [moment] properly, and to [re]cognise it. It is in general the principle of all motion, of all life" (Hegel 1991, p. 128).

In the Hegelian train of thought, metaphorically speaking, it might be useful to consider Being as something like the bud—something out of which subjectivities emerge. Each subjectivity is different from other subjectivities (as blossoms are never identical to each other), yet all subjectivities are the transformation of something (i.e., Being) that is never fully given, but always in the process of change, something indefinite, potential.

Being, I suggest, is a general, cultural, dynamic (that is, always changing) non-metaphysical, ontological category. It is constituted of historically coded forms of conceptions about the individuals and the ways in which individuals are called to present themselves to the world and to interact with other individuals. More precisely, Being is constituted of cultural ways of living (i.e., *be-ing*) in the world: ways of conceiving of oneself and of being conceived, ways of positioning oneself and of being positioned, and forms of self and otherness (i.e., relationships with oneself and others). In this account, Being is *potentiality* (what Aristotle called δύναμις, *dunamis*); that is, something whose mode of existence is not *actual* but *potential*. What is *actual* is subjectivity.

Subjectivity is a process: the always ongoing instantiation or materialization of Being. This unachievable and always ongoing instantiation is a unique, *concrete* subject (a *subjectivity*), whose specificity results from the fact that it is a reflexive sentient entity always in a process of *be-com-ing*: an unfinished and unending project of life. Or to say it in terms of the ideas presented in the introduction, a subjectivity is a sign perpetually coming into life, a sign that agentically appears and co-produces in the social world through the *materialization* of the cultural-historical possibilities available to it (Radford 2008a). In this coproduction, the individual as subjectivity becomes aware and conscious of itself and from there can project and present herself to herself and others in new ways.

Subjectivities coproduce themselves not in contemplation but in the course of a process whose name is *human activity*. Following dialectical materialism, human activity is not a mere set of actions: activity is a *system* (Leont'ev 2009, p. 84) in constant development, incessantly affecting the individuals participating in the activity and at the same time affected by those individuals. We can ask: What is the nature of the human activity that, at school, produces teachers and students? What is its specificity? These questions orient the discussion of the classroom example that I discuss in the next section.

2.4 The Classroom Constitution of Mathematical Subjectivities

The example that I discuss comes from a pre-school classroom of combined pre-kindergarten and kindergarten children (4–6 years old). The content of my example is mathematics, which occupies an extremely important part of the Ontario pre-school curriculum. Naturally, the emphasized presence of mathematics at the pre-school level is coherent with what the children will find later on in primary and junior and senior high school: a curriculum formed around the axis of mathematics and language. Since the dawn of the 20th century, mathematics came to occupy a privileged position in the school curriculum of those countries that saw in industrialization the path towards modern society. Mathematics became the ally and support of the new capitalist forms of production. Many early 20th century pedagogues understood civilization as synonymous with industry (Radford 2004). To a large extent, the main problem of 20th century educational reform was the problem of massive schooling to train the young in the participation and development of a technological society. One century later, things have not changed much. Capitalism has not vanished. It has become trans-national, diversified, and globalized. It is hence not surprising that the preschoolers I see entering the school every morning start the day with activities around counting. If the school has to produce consumers and technologically oriented minds, counting has to be the starting point.

At first sight, counting may seem to be a *natural* activity: the same activity regardless of place and time. On a closer look, however, as anthropological and ethnomathematical research shows, not all cultures have counted in the same way and not all cultures have counted the same things (e.g., Lancy 1983). Counting can be better conceptualized as a culturally codified numerosity-oriented way of thinking and acting to make sense of the world (Radford 2008b). Classroom activity is an attempt to provoke the encounter of children with a specific culturally constituted form of thinking about numbers, figures, chance, information, etc. Rather than natural and conceptually neutral, the form of thinking favored in the Ontario and in other curricula around the world conveys a specific *worldview*. In the case of arithmetic and numerical literacy (or numeratie, as it is sometimes called), this worldview entails positions about what counts as counted, and operates within a particular *rationality* (e.g., how things *should* be counted) thereby creating a specific *regime of truth*. The worldview, its rationality, and concomitant regime of truth are all central elements of the semiotic systems of cultural significations that organize the school as a social institution and sanction the kind of knowledge to which students are exposed. To avoid believing that cultural mathematical forms of thinking are purely conceptual and to better grasp their economical-political substratum, it is worth recalling the political struggle between merchants and the feudal aristocracy in 13th century Florence about the legitimate way to count—with Arabian numerals or counters—and the prohibition issued by the Guild of the Money Changers against the use of Arabian numerals (Struik 1968). And all this would have very little bearing on our discussion were it not for the formidable fact that subjectivities are not merely produced as a by-product of learning: on the contrary, *what* is learned and *how* it is learned are the threads out of which subjectivities are made. From the theoretical perspective that I am sketching here, it would be a mistake to conceive of the child as already equipped with her proclivities, tastes, and personality—as idealist and rationalist pedagogies do. The child is an individual in an unending process of becoming dialectically entangled with *what* she learns and *how* she learns it.

To sustain the previous ideas, I discuss a classroom activity in the rest of the chapter. The activity is about an arithmetic game played between two children. The object of the activity, as discussed with the teacher and our research team, was to offer the children an occasion to become acquainted with cultural forms of counting and thinking about numbers as targeted by the curriculum. A plastic sheet contained two rows made up of 10 squares with space enough for the children to place a small plastic bear in each (see Fig. 2.1, pic 2 below). One child received 10 bears of one colour, and the other child received 10 bears of another colour. They received one dice. I focus here on the second part of the game. In the second part of the game, the children started with empty rows. The rules were as follows: taking turns, each child had to place on her/his row the number of bears that corresponded to the number shown by the dice after the child rolled the dice. The winner is the child who fills her/his row first. To fill the row, the child has to roll the dice and obtain the exact number of points on the dice as the number of spaces left on her/his row. To demonstrate the rules, the teacher played a game with a child in front of the class. Then, the class was divided into groups of two.

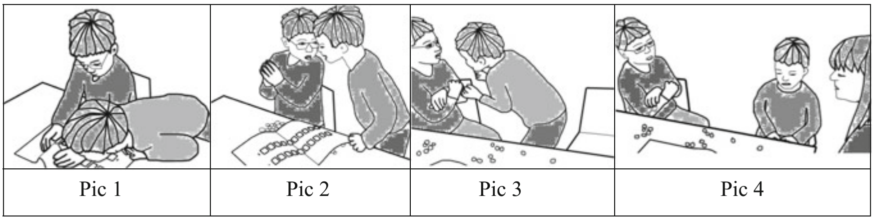

Fig. 2.1 Pic 1–3: Jack and Carl discuss the game. Pic 4: The teacher and the children

In terms of the mathematical notions involved, the students had to deal with: (a) producing a numerosity (the points shown by the dice); (b) counting the numerosity (quantity) either perceptually or with their fingers and/or words; (c) determining the number; (d) choosing a quantity of bears that corresponds to the number; (e) and placing the bears on the row and determining whether or not the game has been finished. In terms of the social dimension of the game, the game required the students to subject themselves to the rules of the game, to articulate their actions with those of the other child, and to pay attention to the various phases of the game. Here is an account of the game played between Carl and Jack.

Jack rolls the dice and gets 6. He says “six!” and places six bears on his row, one bear at a time while counting aloud “1, 2, 3, 4, 5, 6.” Carl closely follows Jack’s actions; he counts and says “six!” He waits for Jack to finish. Then he says “OK. My turn, my turn!” Jack responds “OK. I’ll just put this [the dice] there for a, for now” and places the dice close to Carl’s row. Carl takes the dice, rolls it, and says “Oh! 2!” He takes one bear at a time and places them on his row, while counting aloud “1, 2.” Jack follows Carl’s actions, counting closely. Jack seems to have forgotten how many bears are already on his row. He takes three bears from his row and counts those that remain, pointing to them successively, and says “1, 2, 3.” Then, he proceeds to put back those that he just removed. As he puts them back one after the other, he says “4, 5, 6.” The page moves a bit; the bears move from their square and now appear not properly placed. Several are on the same square. In the meantime, Carl moves the dice close to Jack’s row and says “Ok, it’s your go.”

Up to this point, the children follow the rules of the game. Following the rules, I want to suggest, is an important moment in the children’s process of subjectification; that is, in the process through which they co-produce themselves as subjects of mathematics and subjects of education, more generally. The children enact the game’s rules, which means that they have to subject themselves to the same *regime of truth*: they have to count following a *same* culturally and historically constituted way of counting that, despite the presence of the bears, the dice, etc., targets an abstract form of arithmetic thinking that will be required in the abstract commercial exchange network that the children will find in society. The children also enact a culturally and historically constituted form of living in the world that expresses itself in the forms of collaboration (e.g., turn taking) and social behavior.

However, by following the rules the children are not merely materializing a human form of living in the world or practicing a theoretical way of counting. They are doing much more than that. They are positioning themselves in a social world where their actions, regardless of their *difference* and their *polyglossic* nature, are recognized as *legitimate*. The apparently unimportant attention that Carl pays to Jack’s actions when Jack rolls the dice, obtains 6, picks up 6 bears and places them on his row serves indeed to legitimize Jack’s deeds. Legitimation is a joint endeavor, not a solitary one. It is a social concept. It requires cooperation. The children’s cooperation, as it will turn out later, is still very fragile.

To enter the social world, the children also have to *control themselves*. In our video, we see that they wait impatiently for the other player to finish placing his bears. As Vygotsky noted, “A very young child tends to gratify his desires immediately. Any delay in fulfilling them is hard for him and is acceptable only within certain narrow limits" (1967, p. 7). This is why, Vygotsky contends, it is a mistake to conceive of the child "as a theoretical being" (p. 7) moving emotionless from one cognitive stage to another.

Let me continue with my account of the game. Jack rolls the dice again. This time the upper face shows two points. Jack is not happy with the result, picks up the dice again, puts it in his hands, shakes his hands vigorously, and lets the dice fall. He utters "5!" Satisfied with the result, he starts adding bears while counting "1, 2, 3, 4." He runs out of bears. Carl has been looking at what Jack does, apparently without fully understanding Jack's actions. Carl does not seem perturbed by the fact that Jack has ignored the first result (the dice showing 2 points). At this moment a child from another group calls the teacher and Carl's attention moves to that group. In the meantime, Jack is busy reordering his bears on his row. Thirteen seconds later, Carl's attention comes back to Jack. Jack is still reordering his bears on his row. Carl stretches his arm and tries to get the dice, which is in front of Jack. Jack prevents Carl from taking the dice, and says "So, it's … wait! Ok, it's …." Carl does not pay attention to Jack and says "Ok my [turn], I …" Jack interrupts and says "No, wait! Wait! Wait!" After some physical struggle Carl succeeds in getting the dice. Jack continues "So, it's 1, 2, 3, 4, 5, 6" and keeps on placing and counting bears: "1, 2, 3, 4." Carl is not paying attention to what Jack does. Carl rolls the dice twice. Jack finishes counting and puts his arms in a victory position. He utters "I won! I won! I won! I won! I won! I won! Look!" Carl turns the dice in his hand, and when he finds the 6-point face, he stops and starts counting the points: "1, 2, 3, 4, 5, 6 … 6!" He tries to start putting six bears on his row. Jack puts his arms on the page covering all the bears to impede Carl from placing his bears. Jack says "I won! … Me, I won!" Carl moves his body towards the page and in a very frustrated tone says "Ughhhhhh!" (see Fig. 2.1, pic 1). Jack insists "Me, I won!" Carl replies "Me is getting mad at you!" Jack responds "Me, I won! Won!" Jack takes the dice and shakes it vigorously as if to start a new game (see Fig. 2.1, pic 2). Carl exclaims "No! JACK … Ughhhhhhh! No! This is enough!" He succeeds in getting the dice. "My was only when [I] have this" (he points to 6 on the dice) "So my turn." Jack answers: "No, you didn't get that! … You did like (he pretends to hold a dice in his hand and to move it around) flip, flip, flip and then you found 6! Um, Carl cheated, he does like flip, flip, flip, flip! … (pointing at Carl) Cheater! Cheater! Cheater! Cheater!" Carl reacts with his body. He comes very close to Jack as if he is going to hit him (see Fig. 2.1, pic 3).

The game turned very bad. At the beginning of this episode, Carl did not react to Jack's rolling the dice again after Jack got the discouraging 2 points. By rolling the dice twice, Jack transgresses the social dimension of the rule. To some extent, he is aware of it: when he picks up the dice the second time and shakes his hands vigorously, there is a sneaky smile on his face, which may mean something like: "You know, I know that I should not be doing this, but …" Maybe he interprets Carl's silence as a kind of complicity and continues playing seriously as if nothing had happened. Right after, Carl got distracted and his attention moved to another group. The result is a rupture in the children's collaboration that was present in the early part of this game. The collaboration includes a *coordination* of actions (e.g., taking turns) but also *paying attention* to what each player does. Part of collaboration is indeed to pay attention to others, even if it is not one's turn. To maintain his attention on the game is a tremendous task for Carl who is one year younger than Jack. In turn, although Jack's attention is on the dice and his bears, he does not realize that Carl is not paying attention. Jack is focused on his own actions. When Carl's attention comes back to the game, it is focused on taking his turn, regardless of the position of the game. The *regime of truth* that holds the children together in the first part of the game is no longer there. The social and theoretical common ground embodied in the rules of the game has disappeared. Without a common ground, the connection between the children is lost. Impulse and desire seem to drive the children's deeds. The other has become an impediment to one's own actions. Jack disqualifies Carl by treating him as a cheater. Carl, who exhibits a lesser mastery of the language than Jack, responds with unarticulated phrases and with frustrating emotions expressed verbally ("Ughhhhhhh!") and with threatening body language.

At this point, the teacher (T) comes to see Carl (C) and Jack (J):

T: (She positions herself close to C and talks to him in a calm tone.) Sit down.
J: (Furiously, points at C) Cheater!
C: Me no cheater. (Turns to the teacher.) He does not want to listen to me!
T: (Talks to C in a patient, supportive and comforting tone.) He doesn't listen to you? (See Fig. 2.1, Pic 4)
C: No!
T: What are you trying to tell him?
J: (Points at C) He, he cheats!
T: (Talks to J in the same calm tone she talked to C.). OK. Stop saying that.
J: He was doing like (Makes some gestures with his hands.) … and found 6.
T: (Talks to C in a calm tone.) What … what do you want to tell him?
C: Uh…
T: (Talks to both children.) Whose turn is it?
C: Me, me, me rolled like that but he didn't listen.
T: OK. Roll it [the dice] again. We'll restart.

At this point, the children started collaborating again. They started taking turns, paying attention to the other, showing solidarity, putting the bears on their row and counting aloud. The teacher remained with them for 12 s and left to see another group. The teacher succeeded in calming both children. Through her second and third utterance the teacher shows empathy; that is, as the Greek term *pátheia* intimates, the acknowledgment of the suffering of the other. Carl responds positively to the teacher's empathic attitude. The teacher also politely asks Jack to stop calling Carl a cheater. Not without effort, Jack acquiesces to the teacher's request, controls himself and calms down. The teacher is now in a position to restore the children's attention. The children can now move beyond accusatorial body, hand, and verbal actions and can focus on the game and its rules.

This episode shows the tensions that underpin the processes of subjectification out of which subjectivities are being produced. These tensions are not defects of a task design. Nor do they derive from a pedagogy that has gone wrong. They are part and parcel of the processes of subjectification and the disclosing of Being. Through these processes the children encounter forms of Being that have been culturally and historically constituted. These forms of Being are coded forms about the ways in which individuals are called to present themselves to the world and to interact with other individuals. These coded forms are what Hegel (2009) calls *generals* (as opposed to *singulars*). They are archetypes of living in the world and, as archetypes, they cannot be perceived or sensed directly. For instance, we cannot perceive or sense fairness as such—as we cannot perceive or sense directly, responsibility as such. To be sensed, to become objects of consciousness and reflection, to be idiosyncratically incorporated in the individuals' own repertoire, these coded forms of Being (e.g., the fair player and all the social, conceptual, cognitive, and ethical attributes that come with it, like, in our case, self-control, collaboration, solidarity, responsibility, sustained attention, theoretical perception, cultural-historical methods of arithmetic counting) need to *appear* in the concrete material world of action, sensation, language, feeling, and thought, where then they can be *cognized*. These coded forms of Being appear, in our case, in trying to follow the rules of a game. They appear first punctually, then in a more generalized way, as the children co-position themselves in the social world and become conscious of what entails behaving socially and intellectually in a given culture, and succeed progressively in incorporating these attributes in their own actions, always in unique and idiosyncratic ways and with unique and idiosyncratic results.

Talking about the development of self-control and voluntary direction of one's own actions, Vygotsky (1998) pointed out that they

> develop in the process of children's group games with rules. The child who learns to conform and coordinate his [sic] actions with the actions of others, who learns to modify direct impulse and to subordinate his [sic] activity to one rule or another of the game, does this initially as a member of a small group within the whole group of playing children. Subordination to the rule, modification of direct impulses, coordination of personal and group actions initially … is a form of behavior that appears among children and only later becomes an individual form of behavior of the child himself. (p. 169)

This transformation of self-control and interpersonal coordinated action, however, does not occur naturally. The classroom activity accomplishes that. It is the classroom activity that moves Being from its state of generality or potentiality to actuality through semiosis—i.e., semiotic collective processes of meaning-making. In this sense, the activity embeds the children and the teacher. But at the same time, the activity is produced by the deeds of the children and the teacher. This is why, methodologically speaking, we cannot attend to the participants without attending to the activity in which they are immersed, and reciprocally, we cannot attend to the activity without attending to the participants that produce the activity. This is the dialectic nature of activity and participants through which Being is disclosed. In its disclosing, the children and the teacher feel and socially experience anger, frustration, empathy, collaboration, responsibility, solidarity, etc.

These phylogenetically constituted attributes of Being become crucial elements of the always evolving Semiotic Systems of Cultural Significations that subsume the classroom, the school, and the educational system. These phylogenetically constituted attributes of Being become pointers of action and *be-com-ing*. Of course, such attributes of Being are not encountered, sensed, and experienced equally by the children. In the course of activity, they are understood (not necessarily at the conceptual level) in varied ways. They occur in an emerging interpersonal ethical attitude that, in previous work, we have termed *togethering*; that is to say, a relational attitude based on a not necessarily explicit "ethical commitment participants make to engage in and produce activity" (Radford and Roth 2011, p. 227). And as we have seen, the encounter with the aforementioned attributes of Being is deeply entangled with the manner in which the teacher interacts with the students. The teacher draws on developed features of Being that, in the course of classroom activity, come to interact with the children's emerging conceptual and emotional understanding of the situation. As Vygotsky (1989) once noticed, cultural forms of knowing and Being (voluntary attention, arithmetic thinking, forms of human collaboration and ethical dispositions) do not result from mere interaction. Contrary to other living species, humans are not pure biotypes. Instead, human cultural forms of knowing and Being result from the interaction between phylogeny (that is, the evolutionary development of a cultural group) and ontogeny (that is, the life-span development of individuals) in human activity (Moretti and Radford 2016).

2.5 Concluding Remarks

This chapter has been an attempt at investigating the production of subjectivities in the classroom. To do so, I resorted to semiotics, which is usually understood as a discipline dealing with the production of signs and their meanings. In the introduction I suggested considering signs in a more dynamic way and submitted that individuals can also be considered as signs. As a result, from this perspective, semiotics also deals with the manner in which individuals are signified and signify themselves. The key idea behind this theoretical stance is to conceive of the individuals as entities in the making who come to agentically position themselves—and are positioned by their actions and discourse—in differential, polyglossic, and ideological manners. While difference and polyglossia operate as centrifugal forces, ideology operates centripetally. I called these entities in the making *subjectivities*. They are co-produced in activity-bound processes of subjectification dialectically entangled with semiotic systems of cultural significations. These systems offer an ubiquitous symbolic superstructure to the individuals' actions and reflections. They include the ontological category of Being, which I formulated as *potential* ways of living in the world and encompassing a sense of ourselves—how we conduct ourselves in the social world, how we show ourselves to others, and how we are expected to behave socially and to be recognized by others. Since this sense of ourselves is ineluctably embedded in an ethical view, the production of subjectivities is always an ethical act.

These ideas were illustrated through a classroom activity game around counting. Although games and play in young children have been recently the object of research in mathematics education (see, e.g., Meaney et al. 2016; van Oers 2013), my interest focused on the manner in which a counting game opens up space for children to encounter historically and culturally constituted ways of knowing and Being—in this case, theoretical counting and living in the world, respectively. More specifically, the children were exposed to numerosity recognition (arithmetically interpreting the points on the dice), numbering (naming the numerosity), numerosity production (choosing a quantity of bears in accordance to the named numerosity), counting actions (placing the bears on the row), as well as sophisticated forms of cooperation, action coordination, but also empathy, respect, solidarity, and responsibility.

The classroom activity game allows us to see some of the dynamics and tensions that underpin classroom processes of subjectification. The game was based on a rule that, as all rules, is much more than conceptually following a sequence of instructions. The rule appears indeed as a social and conceptual structuring element through which the children become endowed with social *generality*: they recognize themselves and are socially recognized as players with the ensuing collective entailment of actions and expectations. In short, the children become endowed with a generic element that applies to them, as well as to others or to any player of the game for that matter. Rules in play (as in life), as well as other more implicit regulative mechanisms of behaviour, extricate the individual's experience from its pure subjective and private significance. They bring experience to a new realm. They position individuals into a cultural, historical, and social world. This is why it is a mistake to consider games from an educational viewpoint as mere promoters of conceptualizations. This is a reductive cognitive view of games.

In the game analysis we saw Carl, despite all his frustration, positively *responding* to the call of the teacher. To answer to the call of the other is part of *responsibility*. In his book *Éthique et infini*, Lévinas (1982) notes that responsibility is "the essential, primary and fundamental structure of subjectivity … It is in ethics, understood as responsibility, that the very node of the subjective is knotted" (p. 91). He goes on to say that "[r]esponsibility in fact is not a simple attribute of subjectivity, as if the latter already existed in itself, before the ethical relationship. Subjectivity is not for itself; it is, once again, initially for another" (pp. 92–93).

In short, in the example here discussed, the classroom activity opened up a space for the children in which existential and ethical areas of human life appeared. The teacher's intervention brought to the fore, in a decisive manner, forms of living in the world that restored the flow of the interaction of the children. Her intervention emphasized an all-encompassing ethical disposition to the other oriented towards a reciprocated listening that is much more than conceptual and that is also much more than a mere negotiation of social positions. It was oriented towards a truly human listening, based on a pre-conceptual, emotional understanding of the misery and agony of the other, an emotional understanding through which we recognize ourselves and the fragility of our human nature. These considerations may lead us to envision classroom activity in new non-individualist, communitarian based, aesthetic forms of human collaboration (Radford 2014) out of which we may be able to move towards what has been called "the liberation of both the senses and reason from their present servitude" (Marcuse 2007, p. 227).

References

Adorno, T. (2006). *History and freedom*. Cambridge: Polity Press.

Bakhtin, M. M. (1981). *The dialogical imagination*. Austin: University of Texas Press.

Hegel, G. W. F. (1977). *Hegel's phenomenology of spirit* (1st ed.). Oxford: Oxford University Press. (Original work published 1807)

Hegel, G. (1991). *The encyclopaedia logic.* (T. F. Geraets, W. A. Suchting, & H. S. Harris, Trans.). Indianapolis, IN: Hackett Publishing Company.

Hegel, G. (2009). *Hegel's logic.* (W. Wallace, Trans.). Pacifica, CA: MIA. (Original work published 1830).

Illouz, E. (1997). *Consuming the romantic utopia*. London: The University of California Press.

Lancy, D. F. (1983). *Cross-cultural studies in cognition and mathematics*. New York: Academic Press.

Leont'ev [or Leontyev], A. N. (2009). *Activity and consciousness.* Pacifica, CA: MIA.

Lévinas, E. (1982). *Éthique et infini [Ethic and infinity].* Paris: Fayard.

Marcuse, H. (2007). *Collected papers. Vol 4: Art and liberation.* Abingdon, Oxon: Routledge.

Meaney, T., Helenius, O., Johansson, M., Lange, T., & Werberg, A. (Eds.). (2016). *Mathematics education in the early years*. Cham, Switzerland: Springer.

Moretti, V., & Radford, L. (2016). Towards a culturally meaningful history of concepts and the organization of mathematics teaching activity. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME satellite meeting of the international study group on the relations between the history and pedagogy of mathematics* (pp. 503–512). Montpellier: IREM de Montpellier.

Radford, L. (2004). From truth to efficiency: Comments on some aspects of the development of mathematics education. *Canadian Journal of Science, Mathematics and Technology Education/Revue Canadienne de L'enseignement des Sciences, des Mathématiques et des Technologies, 4*(4), 551–556.

Radford, L. (2008a). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishers.

Radford, L. (2008b). Culture and cognition: Towards an anthropology of mathematical thinking. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 439–464). New York: Routledge, Taylor and Francis.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics, 80*(1), 101–118.

Radford, L. (2014). On teachers *and* students. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 1–20). Vancouver: PME.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*(2–3), 227–245.

Spinoza, B. (1989). *Ethics including the improvement of the understanding* (R. Elwes, Trans.). Buffalo: Prometheus. (Original work published 1667).

Struik, D. (1968). The prohibition of the use of arabic numerals in Florence. *Archives Internationales d'histoire des Sciences, 21*(84–85), 291–294.

Taylor, C. (1989). *Sources of the self*. Cambridge, Ma: Harvard University Press.

van Oers, B. (2013). Challenges in the innovation of mathematics education for young children. *Educational Studies in Mathematics, 84,* 267–272.

Voloshinov, V. N. (1973). *Marxism and the philosophy of language*. New York: Seminar Press.

Vygotsky, L. (1967). Play and its role in the mental development of the child. *Journal of Russian and East Psychology, 5*(3), 6–18.

Vygotsky, L. (1989). Concrete human psychology. *Journal of Russian and East European Psychology, 27*(2), 53–77.

Vygotsky, L. S. (1998). *Collected works* (Vol. 5). New York: Plenum Press.

Chapter 3
Birth of Signs: A (Spinozist-Marxian) Materialist Approach

Wolff-Michael Roth

Abstract The existence of "the sign" generally tends to be taken without questions. In this study I show that signs are born in and for activity, where they are part and constitutive of the relation with others. I use a brief case study as the empirical basis to suggest an approach grounded in cultural-historical and pragmatic approaches that abandon the notion of signs as mediators and instead focus on the developing communicative field that is common to the participants in relation. Abandoning the sign as mediator allows us to pursue studying the lines of becoming that were so dear to the late Vygotsky, who was in the process of replacing his earlier mediational approach to the sign by a Spinozist-Marxian approach.

Keywords Sign · Commodity · Sign vehicle · Materialism · Sensible Suprasensible

3.1 Introduction: A Cultural-Historical Materialist Take on the Sign

> Sign operations are the result of a complex process of development, in the full sense of the word. (Vygotsky 1994, p. 151)

Semiotics has gained considerable traction in the field of (mathematics) education. Educational semiotics can be defined as the amalgamation of learning theory and semiotics; it is "a form of inquiry into how humans shape raw sensory information into knowledge-based categories through *sign interpretation* and *sign creation*" (Danesi 2010, p. ix, emphasis added). Absent in this definition is the fact that in fluent practice, signs are not *interpreted* at all but are *read transparently* thereby

W.-M. Roth (✉)
University of Victoria, MacLaurin Building A567, Victoria, BC V8P 5C2, Canada
e-mail: mroth@uvic.ca

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giving access to phenomena directly (e.g. Roth 2003). There are thus three possible areas of investigation constituted by how people relate to the pertinent signs: creation, transparent reading, and interpretation. Much of mathematics education research tends to be concerned with the third. This chapter, on the other hand, takes the creation of signs as its topic, or, more correctly, it is concerned with the birth of signs in society-specific relations.

An important early articulation of the importance of the sign and its birth in social relations can be found in the work of the societal-historical psychologist Lev S. Vygotsky (e.g., 1997). Accordingly, the sign is first a means of and for acting on others before it is used for acting on the self; it first *is* a social relation before it exists for the person (Vygotskij 2005a). Vygotsky uses the birth of the intentional pointing gesture, an indexical sign, as an example for the birth of the sign generally. Thus, the process emerges from a child's failed grasping movement aimed at some object, which the mother treats as an instruction by handing the object to the child; and the movement has become a sign when the child actively uses the demonstrative gesture. Here, the hand and finger movement is bodily and real, therefore the *sensible* aspect of the sign is apparent. The *suprasensible* aspect, its ideality or "meaning" (the hand is pointing rather than moving arbitrarily) is relational. It arises from the fact that the movement (i.e., the sign) is a reflection of the mother-child relation. When it has become a sign, the hand movement has synecdochical function in that it stands for the relation with an Other as a whole. But prior to having become sign, there already is a hand movement that is part of a transactional event. Clearly, it is not yet a sign and thus does not *mediate* anything. The hand movement is a constitutive part of the relation; without the movement, the relation does not exist.

The theoretical development of the preceding point goes back to the cultural-historical analysis of the emergence of (exchange-) value as the suprasensible side of a sensible commodity (use-value) (Marx and Engels 1962). In the simplest version of an economic exchange, a commodity obtains its *value*, short for "exchange-value," in terms of the amount (quantity) of another commodity (e.g., 20 yards linen = 1 frock). In such an exchange, the commodity exists both for "buyer" and "seller," as use-value and exchange-value, respectively. Exactly the same can be shown to be the case in the exchange of words, verbal symbols, or signs (Roth 2006), or, for that matter, "any combination of such signs and the syntactical pattern of this combination" (Il'enkov 2012, p. 174). If the word does not exist for two, there is no exchange at all. The analysis can be taken further. The same producer may exchange the same amount of linen for other goods, so that its value comes to be expressed in terms of different quantities of different goods (i.e., 20 yards linen = 1 frock = 40 lbs. coffee etc.). All of these exchanges occur at small scales, *within* groups. This is equivalent to the case where students do mathematics in small groups. But what they come up with does not have to—and in many case does not—conform to standard mathematics. Something else has to happen, which the example of the commodity renders apparent. The general, universal value form emerges only in exchanges at the level of the society, when the various commodities are produced for the generalized satisfaction of human needs. In this case, the value of all commodities is expressed in the same good (e.g.

1 frock = 20 yards linen; 40 lbs. coffee = 20 yards linen, ¼ wheat = 20 yards linen etc.). All ideals (universals) of mathematics arise and exist in this latter form (Il'enkov 2012). Consciousness, thus, is not merely and accidentally social (as any relation between two people) but always and essentially societal, determined by "the societal being of man" (Marx and Engels 1978, p. vii). The suprasensible aspect of the material-sensible commodity, as for the material-sensible sign, simply is a manifestation of the societal [gesellschaftliche] relation constituted in and by the (commodity, sign) exchange. That is, the commodity (material sign) is a synecdoche of the exchange relation, its ideal form (i.e. the exchange-value) being a transposed and translated manifestation in consciousness of the material exchange relation.

From this ever-so-brief analysis, we can take three aspects of the birth of signs. First, there is a relation between two things, which, below, are identified to be the sensible sign vehicle and suprasensible sign content (mapping onto use-value and exchange-value, respectively). Second, there is a human relation in which the sign is exchanged and that is constituted (in part) by the exchange. Third, the peculiarity of the materialist approach in the Spinozist Marxian tradition lies in the fact that it takes the first relation also as a reflection of the second (i.e. the relation between things exchanged, objects or signs, reflects the relation between people). This cultural-historical materialist framework considerably expands currently available semiotic studies generally and semiotic studies in mathematics education particularly. Especially in mathematics education, children come to be confronted with phenomena and sign vehicles that they have not encountered before. Our field (mathematics education) therefore is an ideal test bed for investigating "*the natural history of the sign*" (Vygotsky 1994, p. 151), ranging from *birth* to *becoming* and to the *death* of the sign.

3.2 The Structuring Work on Which the Birth of Signs Is Founded

The preceding section exhibits sign use as involving a first relation between sign vehicle and sign content. Studies in the philosophy of language show that to interpret any linguistic or other "sign means to define the portion of continuum which serves as its vehicle in its relationship with the other portions of the continuum derived from its global segmentation by the content" (Eco 1986, p. 44). The "continuum," of course, is material. However, its portions and segmentations are not given; instead, they are the results of human labor (Jornet and Roth 2015). For example, in one think-aloud study, which asked practicing PhD biologists to interpret graphs from an introductory biology text, participants in many instances failed to give the generally accepted responses (Roth and Bowen 2003). While gazing at a figure plate—in which a birthrate ($b(n) \sim -n^2$) and death rate graph ($d(n) \sim n$) intersect at two points—to make a statement about what happened to the population n, one scientist said that she was looking at the slopes of the lines. A standard reading of the graph, however, requires looking at the values of the functions $b(n)$ and $d(n)$. On the same task, only 1 out of 17 biologists correctly replied to the question at which point the increase in the population size was largest—which is neither at the highest point of the birthrate graph nor at the point where the difference between birth and death rate is largest; instead, it is largest where the function $(b(n) - d(n)) \cdot n$ is maximized. Thus, most of the scientists' failures in interpreting the graph in the standard way can be traced to the non-standard structuring of the material display. Simultaneously, many participating scientists did not uncover the second part of the underlying sign relation, that is, they did not identify the content of the graph (i.e. what it was intended to be about).

This phenomenon can be understood if we think about the sign in the way Eco suggests: it consists of a material configuration that comes to stand out against everything else that configures the ground. There are two types of work required in the production and use (interpretation) of signs (Fig. 3.1) (Jornet and Roth 2015). First, each portion (signifier) is the result of a structuration of matter yielding form—e.g., something in the world needs to be perceived *as* some thing that has form (percept). Second, any sign is the result of the relation between two structured portions, each of which constitutes a signifier (segmentation of the continuum). Because these are separated by an ontological gap, relating percept and signifier or two signifiers requires gap-crossing work. In other words, there is a transactional relation between the two, each being integral to the definition of the other. Take Vygotsky's pointing example. To know that a hand movement constitutes an act of pointing requires the presence of an object pointed to. There may be two cases of trouble. If there is an intended pointing movement but no effect on the part of the intended recipient, we may hear, "Can you look over there!" In the reverse case, we may hear, "Are you trying to show me something?" or "What are you trying to show me?"

In the case of the scientists discussed above, their first type of problems arose when the form they perceived in the population graph plate was not the one they needed to perceive in the standard approach; and they were hampered in making the connections to the natural phenomena that the graphs are taken to stand for. Their work is to be theorized less as a process of sensory cognizing (Danesi 2010) and more in terms of a phenomenology of movement that does not initially need to or require units of knowledge, schema, and the likes (Vygotskij 2005a). In the

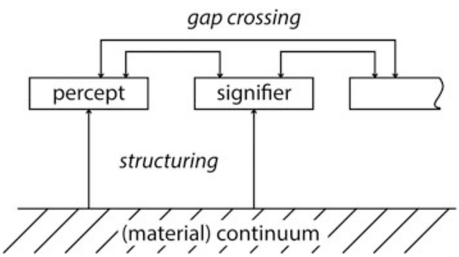

Fig. 3.1 Segmentations of materials, such as percept and signifier, have to be produced before gap-crossing work can link them up

following, I focus on how a new sign—a percept–signifier relation—is born in exploratory activity and then I provide a societal-historical account, which also leads us to abandon the idea of sign mediation. Vygotsky already moved in this direction, for during the last period of his work, he "came to understand that notions he had been using so far, such as 'sign mediation' or 'functional system,' could not fully explain the whole complexity of human consciousness" (Zavershneva 2014, p. 74).

3.3 The Genetic Origin of Signs

3.3.1 Case Study from an Elementary Mathematics Classroom

With the following lesson fragment from a second-grade mathematics classroom I exemplify and theorize the dawning of signs in human activity. The episode saliently exhibits (a) the birth of a first sign vehicle–sign content relation and, as if in slow motion, (b) the eventual death of the first relation and the birth of a second one. These events unfold as part of a second development: the emergence of a vehicle–content relation that has *common* currency within a group of students. Easily missed is the third aspect of signs: that the relation between vehicle and content also reflects the social relation. Only the second aspect tends to be studied in mathematics education research under such topic titles as "social construction" and "negotiation" of "meaning" and "socio-mathematical norms." The case study presented here, however, highlights all three aspects.

In the episode, the children have been asked to model a mystery object hidden in a shoebox (one portion of the continuum serving as the sign content) using plasticine (the second portion of the continuum serving as the sign vehicle). They are allowed only to reach into the shoebox through a hole—covered by a baffle so they cannot see the mystery object—and feel it. All three girls (Jane, Melissa, Sylvia), seen in that videotape in turn, repeatedly reach into the shoebox, apparently feeling out the hidden object, and then begin to shape the mass of plasticine that they each previously received for the purpose of the task. The task, therefore, is a completely material equivalent of situations where there is a (material) sign vehicle but the sign content is ideal (i.e. "meaning"), and, therefore, inherently inaccessible (which leads Lacan 1966, to a critique of purely ideal "meaning"). It is a situation where, "in order to become an object's (word's) sign, the stimulus [plasticine model] finds support in properties of the designated [mystery] object itself" (Vygotsky 1994, p. 151, content in square brackets added). The teacher eventually comes to the table where the children are working and groups together two of the plasticine shapes (made by Jane and Sylvia) while pointing out that these are similar; and they are marked to be different from the shape of the third model (Melissa's), which is kept apart. The girls are asked to settle on one shape (model) because there is only one mystery object and there cannot therefore be different shapes modeling it. There is a long debate because the models of different group members vary significantly (cube vs. slab).

Melissa first shapes her plasticine into a cubical form. She claims the mystery object to be a cube because it has the same three sides. While speaking, she uses her thumb and index finger configured into a caliper and holds it to three different orthogonally positioned edges of her plasticine shape (Fig. 3.2a). But her peers challenge this statement, therefore treating it as a claim. Over the course of the 16-minute episode, Melissa will have reached into the shoebox eight times, feeling (transitive) the object for a total of more than 3 min. As part of the debate, Sylvia repeatedly instructs Melissa in what the mystery object feels like (intransitive)—i.e. flat as if two palms rubbed together—and how to take the mystery object between index finger and thumb to feel (in/transitive) its thinness (Fig. 3.2b). But Melissa maintains that she feels three equal sides, which is well represented in the plasticine cube she has formed. At one point, Melissa has her right hand in the shoebox and the left hand on her model (as in Fig. 3.2c) and, based on this situation, continues to maintain that there is a cube inside the shoebox.

Jane then shows Melissa a test: one hand apparently feels the mystery object while the other simultaneously rotates and feels the plasticine model. Melissa eventually accepts the invitation to do likewise. The video shows her with the right hand in the shoebox while gazing intently at the left hand, which repeatedly and slowly feels (transitive) Jane's plasticine model and turns it over (Fig. 3.2c). After completing four feeling/turning cycles, Melissa looks up, puckers her lips, grabs her plasticine cube and begins to reshape it into a flat rectangular prism. What has been a very limited, local, and individual sign—the relation between the plasticine cube

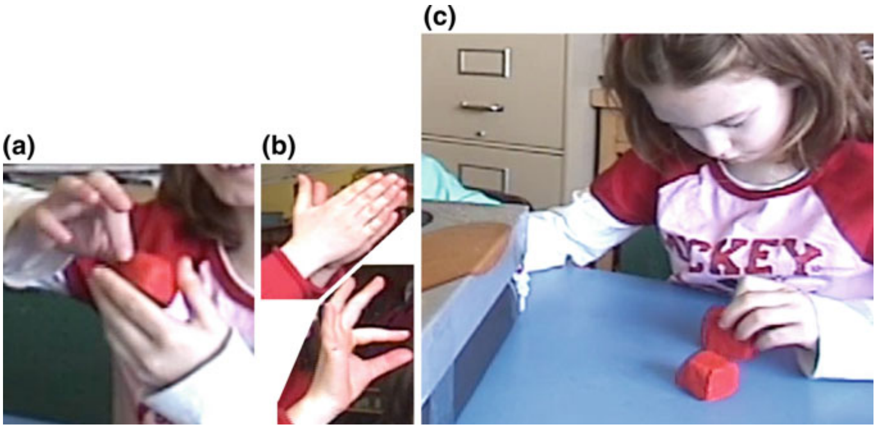

Fig. 3.2 Images from an event in which a sign is born. **a** Melissa shows why the hidden object is and feels like a cube. **b** Sylvia configures her hand/s to show why the object should be a flat slab. **c** By means of touch, Melissa directly compares the hidden object with a cube

and the mystery object—has died before it ever could become exchange currency in and constituent of the relations involving the three girls.

Here we observe how both sides of a sign relation come into being as a result of the structuring and restructuring work, and how they are related through the bodily work of a child. While the children are reaching into the box and feeling the object, what initially only is an obstacle opposing touch eventually comes to be felt (intransitive) as a particular form: from touching emerges a percept or worldly matter takes form (i.e., "percept"). Based on their respective percepts, each girl then shapes a piece of plasticine, Melissa, based on prior experiences, calling hers "a cube," whereas Jane and Sylvia do not have a name but are shaping the plasticine into a form that "feels" the same as the mystery object. That is, they use another part of the material continuum (plasticine), to which they give shape through molding so that it may serve as a sign vehicle for the mystery object (sign content). The relationship between the two is based on the similarity between the feel of the object (signified) and that of the plasticine serving as a signifier (Fig. 3.1). In this effort, the children are engaged in semiotic processes involving physical materials (mystery object, plasticine) that parallel other processes involving images.

3.3.2 The Societal Nature of the Phenomenon: A Historical Connection

The preceding episode should not be reduced to little girls' play. Instead, we observe here a societal phenomenon that evidences itself in many different guises. Sometimes the staff of the same social phenomenon—the birth between a natural phenomenon and a (mathematical) model—may be mature scientists. For example, in the 19th century, physicists were debating whether "light" (sign content) was a "wave" or a "particle" (sign vehicle) and, therefore, what the appropriate mathematical model was. Although the physicists agreed to be looking at the same phenomenon, light (corresponding to the mystery object), what they "understood" light to be, its (invisible) nature, was related to the model. They later realized that "light" *is neither* particle *nor* wave—how light manifests itself depends on the observation situation (operation). In the debate between two different symbol systems for describing quantum mechanical phenomena, there existed two models (sign vehicles), Schrödinger's wave mechanics and Heisenberg's matrix mechanics. These two, later were shown to be equivalent in the works of Dirac and von Neumann (like the teacher grouping of Jane's model and Sylvia's model). Similarly in the case of the three girls: It is immediately apparent that all three initial models would have been appropriate if the topic of investigation had been topology (they all have the same Euler number $n = 2$).

3.4 The Sign *in* Relation, the Sign *as* Relation

The case study exhibits the birth (and death) of sign vehicle–sign content relations. The description shows that even in the case of a material rather than ideal sign content, differences and disagreements may exist. This is consistent with observations in controlled studies showing that in teacher demonstrations, what students see—i.e. the sign content that results from their structuring work—may differ (Roth et al. 1997). This is significant in a discipline such as mathematics education, where students are shown demonstrations (e.g. proofs) or engage in hands-on investigations (e.g., Dienes blocks, Cuisenaire rods) and where they are to learn disciplinary practices and facts through sensual experiences. Such disciplinary practices are not simply "copied" by observing others. Instead, they always require students to act so that they may discover these practices in their own actions, much in the way that the child learning to point discovered this pointing in its own failed acts (Vygotskij 2005a). Considerable work is involved in relating the two portions of the continuum, some students establishing a relation whereas (many) others may fail.

The very notion of the sign implies that it is a possibility for two people in a verbal exchange; an object, a sign or word, cannot exist for one person only (Feuerbach 1846). This is immediately apparent when we consider the analogy with the commodity in an economic exchange relation: if there is no one commodity that changes hands, and therefore is common to two persons, there is no exchange. When a sign appears to exist for one person only—e.g. the proverbial knot in the handkerchief as reminder for something—then (a) it inherently could exist for another person and (b) it exists for that one person *as if* it were an Other within the self (Mikhailov 2001; Vygotskij 2005b). The sign thus emerges as a relation where one person acts on another, and only subsequently serves as a means to act upon oneself. Melissa's offer to take the cubical shape as the sign vehicle for the mystery object (sign content) is modeled on all other experiences with signs and social relations. That is, a sign exists only as a commodity in verbal exchanges between two or more people, where it has use- and exchange-value (expressed by another commodity, sign). As Marx showed, the relation between people in the exchange comes to be grafted onto the relation between things; and that relation then is reduced to constitute the exchange-value of one of these things. The suprasensible aspect of the commodity (sign)—i.e. the "pointing" function of a hand movement—is a reflection of an earlier, real relation between two people (mother and child). In other words, "the thing that is expressed in [the objective ideal form], 'represented' by it, is a definite societal [obščestvennoe] relationship between people which in their eyes assumes the fantastic form of a relationship between things" (Il'enkov 2012, p. 176). As a result, a sign, as any higher psychological function, initially did not simply exist *in* a relation with another person but more importantly *was* a social relation first; and, once it has been fused into one, the psychological can at any time be unfolded into the relation between two people (Vygotskij 2005a). Thus, all three girls articulate for others with their plasticine what they feel when touching the mystery object. But initially, the three models differ. No model therefore obtains the status of a commodity exchangeable for the mystery object.

In that episode, a crucial turning point in the events arises from the exchange with the teacher, who requests that the girls arrive at one and the same model (vehicle, signifier) because there is only one mystery object (content, signified). It is an explicit request to establish one rather than multiple sign relations and, therefore, to establish a situation where there is one model to be used by all in and as their communicative exchange. In this specific case, the nature of the tasks requires the establishment of an iconic relation, which means that the children have to come to agree on what can be felt in the box. But this is precisely where they initially disagree. Nobody can make Melissa change what she feels (intransitively); the results of her feeling (transitive) have to come to feel different. This is not easy, in the way it would not be easy to change into feeling happy over the comment of another when we initially felt hurt by it. In the episode, concordance of the (intransitive) feelings in the left and right hands is established through multiple tries of simultaneously feeling both parts of the continuum, one within the box, the other outside.

In this, the exchange with a teacher is crucial, as it is in many cases in mathematics education. This is so because when a teacher already expresses standard mathematical ideals in her behavior, these can be unfolded again into joint behavior (Vygotskij 2005a). There is a "renewed division into two of what had been fused into one" (p. 1023). The constraint requested by the teacher to have but one model constitutes a nudge towards the production of one model that is common to all, at least within the group, so that it can then be brought up for more general discussion at the whole-class level. In this, then, the possibility for a trajectory for local exchange-value of the model to a *universal* value is opened up. When everything is said and done—when the whole-class discussion begins, and especially after the girls have presented their work—we will have witnessed not just the birth of the relationship between the model (sign vehicle) and mystery object (sign content) and not just the birth of an agreement about a common model to be used in conversations about the mystery object. Instead, we observe how the social relations become absorbed into the material relation between things. The sign—the relation between vehicle and content—has "swallowed" one of its parents, the social relation, thereby mystifying its own nature. The sign, like any other material thing, "once it appears as commodity, it transforms itself into a sensible-suprasensible thing" (Marx and Engels 1962, p. 85). It "not only stands with its feet on the ground," that is, has material character, "but also stands on its head," that is, has ideal character (p. 85). In the Spinozist-Marxian analysis, the sign form no longer is mystical and mysterious. This is so because the sign form *appears to have* "meaning" as a "natural property" when in fact the suprasensible character of the sign merely is a reflection of societal (i.e. universal) relations that exist as societal relation of material things (e.g. sign vehicles).

The task of the three girls also presupposes the intelligibility and therefore societal nature of the condition: only one model (sign vehicle, signifier) can exist when there is only one object (sign content, signified). In the lesson fragment, the teacher's intervention directs the children towards establishing a single sign relation. The specific sign relations that are to emerge for school students—e.g., in mathematics lessons—already exist for practitioners in the field of study (e.g., mathematics). These practitioners, including mathematicians, are embedded in material cultures and act/reason in ways that are universally recognizably mathematical. Whereas the children might come up with all sorts of relations—for example, the three girls might have ended up retaining all three models or they might have retained the cube as their sign vehicle—the presence of the teacher affords a universally valid and used relation to re/emerge *as* and *in* the children's relation to each other. The task of the teacher is to assist students in gaining entry to the field of mathematics by creating conditions that allow for the rebirth of sign relations that already exist for others in the society. That is, her presence affords the re/birth of the universal in the labor of these children. In the episode, we do in fact see the foundations of a mathematical culture: all three children engage proof procedures (providing reasons for their models). We also observe that the children (Jane, Sylvia) themselves evolve an educational strategy for stimulating a process that allows a peer (Melissa) to perceive (feel) the mystery object differently and, therefore, to establish a new form of sign relation.

Some semiotics scholars focus on relationships not merely between materials but more importantly between some material and a non-material mental entity, a "concept," "meaning," or "construction." From the perspective of a *concrete human psychology*, this is not very useful—not in the least because humans engaged in communication always lack the second part of a sign relation when mathematical universals are involved (Roth 2016). A pragmatic take on the sign explicitly rejects the mediational function for accessing some "meaning" and consider only its use (Dewey and Bentley 1999; Wittgenstein 1997). The sign then is a manifestation of the very societal relation of which it is an integral part and that it contributes to constituting. The three girls are seen in the process of establishing that part of the sign relation that would stand for the (mystery) object of interest *in their future relations and communicative exchanges*. If the task had continued in the future, what would have emerged as the signifier (sign vehicle) could have been used to make present the mystery object even when it was no longer available. In use, the sign never is disconnected from its material and social basis, even and precisely when the sound-word is involved; and it is never disconnected from the soci(et)al relation that it has come to stand for. This is important for appropriately responding to the statement "cube," which might be raising a question concerning the nature of thing involved, a descriptive statement, or a request for some cubical object (depending on the intonation). That is, not some "meaning" of "cube" matters primarily but the function of its use. In the absence of accounting for the sound, verbal intercourse is incomprehensible; not accounting for the sensible material basis of the word is a fundamental shortcoming (Vygotskij 2005b).

The materialist approach to the sign offered here therefore allows us to overcome an often-observed dichotomy between body and mind. Signs are not merely "'representational glue' that interconnects [children's] body, their mind, and the world around them in a holistic fashion" (Danesi 2010, p. ix–x). Instead, they are concrete (always and necessarily partial) manifestations of a child-acting-with-others-in-environment unit that has practical, intellectual, and affective dimensions (Dewey 2008a; Vygotskij 2001). In use, the material sign vehicle is a means of affecting others first, and subsequently a means of affecting oneself as an Other. The effect on others initially is not intended. Instead, a sign (e.g. sound-word) emerges *in* and, more importantly, *as* a social relation when others treat it in a certain way (Vygotskij 2005a). Everything required in learning and development occurs out in the open and no recourse to inner (invisible) mental processes is necessary: everything that is taken to be internal in psychological functions is necessarily external. This is why we can research learning and development anthropologically: by investigating what appears in the public sphere. This does not mean that there is nothing happening in the mind; instead, it means that anything relevant is public and researchable as such. Similarly, the signifier (sign vehicle) to be used in the lesson fragment arises from the exchanges and the level of approval or disapproval different proposed sign relations receive (e.g., the girls' debate about the appropriate model). The ultimate sign relation therefore also reflects the relation between people and the function within and constitutive of this relation. Vygotsky's concrete human psychology takes all higher psychological functions to *be* social relations with others first before they become relations of the person with herself. Even writing personal notes was a relation with others first, now directed to and being for the person as "'the Other' within" (Mikhailov 2001). Viewed from the late Vygotskian Spinozist-Marxian perspective, the sound- (ink-) word has use- and exchange-value, which manifests itself in the differences of (mostly inaccessible) intent and observable effect on others.

3.5 On the Possibility of Abandoning the Idea of Sign Mediation

The soul knows no mediator. (Mikhailov 2006, p. 36)

Most scholars will easily accept—and hold up Vygotsky's work to support their position—that the sign functions as a *mediator* making possible the relation between two or more people or assisting a person to access memory (Middleton and Brown 2005). However, the late Vygotsky had abandoned this, his earlier approach and was beginning to work out a Spinozist-Marxian position according to which there is only *one* substance (Mikhailov 2006). Because there is only one substance, there cannot be mediators because everything already is part of everything else. This is a point that Vygotsky was taking up from the materialist, Spinozist philosopher L. Feuerbach (1846), who inspired Marx and his analysis provided above. Feuerbach states: "only that exists, which is for me and the other at the same time, wherein I and the other agree, what is not only mine—what is *general* [allge*mein* = mine of *all*]" (p. 308). As the case of the commodity in economic exchanges clearly exemplifies, it does not mediate the exchange relation but in fact constitutes the relation. There is no exchange relation if the commodity does not change hands, resting for an instant in the hands of seller and buyer.

The idea of the sign as taking a mediator position already has been critiqued within American pragmatist philosophy. From a pragmatist *trans*actional perspective, the often-invoked triangular relation between sign (i.e., representamen in Peirce's triad), referent, and interpretant is a false and misleading appropriation of Peirce's work on the sign (Dewey and Bentley 1999). Instead, signs testify to the mutual implication of one in the other in the course of historically situated practical, "socio-cultural" activity, where word and sign each have their histories (Dewey 2008b). The environment, including language, is a "medium or milieu, in the sense in which a *medium* is *inter*mediate in the execution or carrying *out* of human activities, as well as being the channel *through* which they move and the vehicle *by* which they go on" (p. 185). Instead of writing about medium, those pursuing the path taken by the late Vygotsky and like-minded Russian scholars write about a *communicative field*—variously referred to as *semantic field* [semantičeskoe pole] (El'koninova 2008; Mikhailov 2001), *sense-giving field* [smyslovoe pole] (El'konin 1994), or *imaging field* [pole izobraženija] (Bakhtin 1975)—that has changed because something in the visible environment writ large has been changed through accentuation. The signs (models) make visible, allow to be seen, in and as communication, characteristic features of the mystery object. This idea of language (sign) as the "*accented visible*, different from the visible … where something is made manifest and something, on the contrary, is covered over, as if withdrawn" (El'konin 1994, p. 23) is one of the central ideas in the notes of the late Vygotsky (Mikhailov 2001). The analogy between a communicative field and a force field in physics may help. Like a physicist's heavenly bodies, which produce the gravitational field within which they move and that acts upon them, humans and their world constitute and relate through a communicative field that is common to them. The force field is thus not external, between, and separating the entities.

As noted above in reference to Feuerbach, Marx, and Vygotsky, the thing, commodity, and word is impossible for one but is a reality for two persons. In the case of the word (language), this is immediately apparent from the following revised transcription, where what Sylvia (turn 2) and Jane (turn 3) say only makes sense when we also consider, as transcribed, what Melissa has said before (turn 1).

1	M:	((says)) feel it eh? I have felt it's a cu:be, HU:::::ge	
2	S:	((hears)) feel it eh? I have felt it's a cu:be, HU:::::ge	no it's not a cube
3	J:	((hears)) feel it eh? I have felt it's a cu:be, HU:::::ge	i didn't feel a cube

Only because what Melissa says is a reality also for Sylvia and Jane does the saying of the latter constitute a continuation of the talk. That "feel it eh? I have felt it's a cube, HU:::::ge" is the result of and constitutes their common communicative (semantic) field: it is not a mediator. The field is constituted by and constitutes the relation. Each turn pair, therefore, has both a sociological dimension cutting across speakers and a psychological dimension that exists in the unfolding of speech and hearing (Roth 2014). Of course, in Jane's response, we should also have included the import of the statement "no its not a cube" that had come from Sylvia's lips, and which, in Jane's statement "I didn't feel a cube" finds further (empirical) support.

It has been noted that the materialist analysis of the commodity has a direct equivalence in the materialist analysis of the sign (Roth 2006). Thus, the sensible sound-word (sign) has (suprasensible) exchange-value for the speaker and use-value for the recipient. These exist not inherently in the sign, that is, these are not properties that inhere in the sign. Instead, both are tied to events. It appears obvious that exchange-value is tied to exchange and does not exist otherwise; something that is not exchanged has no value given in terms of another thing. Furthermore, "use-value is actualized only in use or consumption" (Marx and Engels 1962, p. 50). We cannot consider "the sign" as a Kantian "thing in itself." In the transcription, the exchange event exists across two turns, two phrases or statements. The first phrase simultaneously has exchange-value (Melissa) and use-value (Sylvia, Jane); the two reply phrases (at the ends of turn 2 and turn 3) also have exchange value (Sylvia, Jane) and use-value (Melissa). Incidentally, the Russian term značenie, generally translated as "meaning," also translates "value" and "magnitude"; in the materialist approach of the Spinozist-Marxist tradition, it should be read and heard as "function and rôle" (Il'enkov 2012, p. 178).

The traditional view of the sign as standing between the subject and her world is a structural (logical) perspective that lists elements of an event and connects them. Unsurprisingly, when there are two people, the sound-words seem to be between and connecting them. From this perspective, therefore, the word (sign) plays the role of an intermediary—expressed in the well-known triangle of Vygotsky's earlier work that makes thematic that a direct connection between two people ($subject_1$ and $subject_2$) is impossible thus requiring something else connecting them. In the Spinozist-Marxian materialist approach, commodity and verbal exchange is viewed as an event that has a dynamic, developmental direction running transversally to the opposition of two self-contained individuals. Those interested in development and becoming focus on continued change. Thus, there are *lines of becoming*. Whereas the consideration of a word to be said to another focuses on the logical relations between speaker, listener, and word (language), lines of becoming focus our attention on (a) change occurring in participants and language while talk is occurring (Bakhtin 1975); (b) how thought, language, and situation are developing during speech activity (Vygotskij 2005b); and (c) how maker, materials, and artifacts are changing in the productive process (Ingold 2011). Thus, the three girls and the materials they use and work and their English language are changing while the lesson fragment is unfolding, whatever the individual intentions of the participants. This is so, for example, because in *actively* attending to what Melissa is in the process of saying (turn 1), Sylvia and Jane *are affected*, that is, they are changed. Moreover, because their response—the diastatic event encompassing actively receiving and replying—is already under way, so is their thinking. An event invisible to others and the self (Vygotskij 2005b), thinking unfolds and therefore changes the person. Not only the participants are changing, but also the material forms (here the plasticine). The children thus are changing physically (and presumedly physiologically) as well as psychologically and sociologically.

The radical difference of this approach is apparent from the analogy with a river (Deleuze and Guattari 1980; Ingold 2011). The transitive subject–object or subject–subject connection mediated by a sign is like the relation between two opposite points of a river linked (mediated) by the bridge (sign, tool) (Fig. 3.3a). However, the flow of the river is perpendicular; it is intransitive, continuing despite the theorists' inclinations to focus on the transitive relations between opposites. In being part of the field, all parts of activity are thought historically, as ever becoming and ever changing even after the mutually implicating encounter that leaves its traces in all of them (Fig. 3.3b). The Spinozist-Marxian approach is designed to understand phenomena culturally and historically. The present study makes thematic the entire life cycle of the sign, from its birth to its universal use.

3.6 Implications for Theorizing the Sign

The societal-historical approach to the sign that the late Vygotsky started has considerable implications for educational practice and research. In traditional cognitive scientific and constructivist theory, there are two problems with signs. The first concerns the relationship between thought and things in the world (how are thoughts related to material things?); the second pertains to the question of intersubjective agreement (e.g. "What is the 'meaning' of a sign vehicle?").

Taking up a late Vygotskian, Spinozist-Marxian agenda on the sign allows us to see both questions as artifacts of theory and method. Vygotsky may indeed be the most important forerunner of educational semiotics, for he recognized that signs are manifestations of societal relations, embedded in *dynamical* historical and cultural relations. In this approach, the suprasensible moment of the sign is not some purely ideal "meaning," "socially constructed" and internalized by learners. Instead, the suprasensible—značenie = value, function ("meaning")—moment is a reflection of the original, constitutive relation with another person. The higher, supersensible

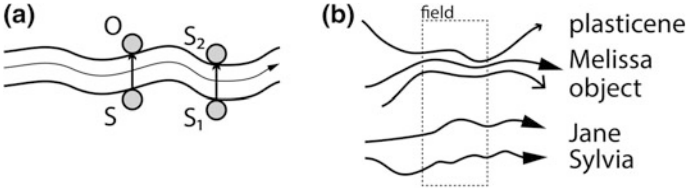

Fig. 3.3 **a** Traditional transitive subject–object and subject–subject relations require mediation (a bridge) to be linked. **b** Lines of becoming are intransitive, running transversally and constituting a field

function of the word was a social relation first and later appears as reflection of the relation in consciousness, where it is attributed to the natural property of it (i.e. "meaning" as natural property of the word).

In his late work, Vygotsky drew radical implications for his method. Accordingly, the sign in itself cannot be the unit of analysis; instead, the sign is only a manifestation of a larger *unit* that has to have all the characteristics of human society, societal relations, which constitute the very possibility of human sign forms. This unit, as is characteristic for educational semiotics, integrates humans and their environment and manifests itself in practical, intellectual, and affective ways. Because all that matters is the *use* of inherently material sign vehicles, humans become literate sign users by participating with others in societal activity where signs (language) also are part; one may say that persons become signs themselves (Fig. 3.3b). We no longer need to ponder the "meaning" of a sign—one of the mediating third parties that pragmatic philosophers (e.g. Dewey and Bentley 1999; Wittgenstein 1997) explicitly reject as useful to theorizing—because the sign always will be an integral part of some irreducible whole: a means of acting upon others or the self. As one Spinozist-Marxian philosopher suggests, "the ideal form of a thing is a form of societal-human life-activity, which exists not in that life-activity, but, namely, as a form of the external thing, which represents, reflects, another thing" (Il'enkov 2012, p. 184). In fact, ideality only exists in the continuous transformation of the two, "forms of activity and forms of things in their dialectically contradictory mutual transformations" (p. 192).

Educational practitioners following this approach might set up classroom contexts that afford the birth of (inherently shared) signs for communicative purposes. This is important because the birth and becoming of signs, as described and theorized here, is future oriented, irreducible to direct instruction or to the models of unidirectional transmission of factual knowledge. Students in mathematics, for example, may then be involved in producing and changing signs and sign relations while part of dialogical relations involving others (peers, teacher).

3.7 Conclusion

For the late Vygotsky, the Spinozist-Marxian, materialist logic implicit in Fig. 3.3b integrates the method of the sciences, nature, and philosophy. There is only one substance; and it manifests itself materially, biologically, and culturally. Materialism of the Spinozist-Marxian type is the only non-reductive human endeavor because it develops categories, the classical domain of philosophy, and their testing and further refinement in the same object of inquiry (Il'enkov 1960). Without doubt, Vygotsky, if he lived today, would have said that educational semiotics *is* nothing other than the ultimate science—and in that science there is no place for mediating thirds because the historical culture of mankind always and already is *immediately* given to all the participants in a communicative field (Il'enkov 2012). To grasp the idea of a field, an analogy with classical physics is offered above, where the relationship between two entities—two masses, two charges—is conceptualized not in terms of a third thing but in terms of a force field. Both entities constitute the field and, in turn, the field shapes their movements. The resulting field is common to the all entities in and constitutive of the field (Fig. 3.3b), but the effects on each are different. The relationships are dynamic—as exemplified in the relation of heavenly body pairs, such as sun–earth or earth–moon. We may usefully conceive of human relations in a similar way, shaping and being shaped by the exchange of signs, each of which is reflected in consciousness. As a result, "consciousness is reflected in the word like the sun is reflected in a droplet of water" and "the word … is related to consciousness like a living cell is related to an organism″ (Vygotskij 2005a, p. 1018).

References

Bakhtin, M. M. (1975). *Voprosy literatury i estetiki* [Problems in literature and esthetics]. Moscow: Xudoš. Lit.

Danesi, M. (2010). Foreword: Edusemiotics. In I. Semetsky (Ed.), *Semiotics education experience* (pp. vii–xi). Rotterdam: Sense Publishers.

Deleuze, G., & Guattari, F. (1980). *Mille plateaux: Capitalisme et schizophrénie* [A thousand plateaus: Capitalism and schizophrenia]. Paris: Les Éditions de Minuit.

Dewey, J. (2008a). *Later works vol. 10: Art as experience*. Carbondale: Southern Illinois University Press.

Dewey, J. (2008b). *The later works, 1925–1953 vol. 16, 1949–1952: Essays, typescripts, and Knowing and the known*. Carbondale, IL: Southern Illinois University.

Dewey, J., & Bentley, A. F. (1999). Knowing and the known. In R. Handy & E. E. Hardwood (Eds.), *Useful procedures of inquiry* (pp. 97–209). Great Barrington, MA: Behavioral Research Council.

Eco, U. (1986). *Semiotics and the philosophy of language*. Bloomington: Indiana University Press.

El'konin, B. D. (1994). *Vvedenie v psixologiju razvitija: B tradicii kul'turno-istoričeskoj teorii L. S. Vygotskogo* [Introduction to the psychology of development: In the tradition of the cultural-historical theory of L. S. Vygotsky]. Moscow: Trivola.

El'koninova, L. I. (2008). Rol' skazi v psixičeskom razvitii doškol'nikov [The role of the fairy tale in the psychological development of preschool children]. In *Psixologija igry i skazki* [Psychology of games and tales]. Moscow: ANO «Psixologičeskaja eletronnaja biblioteka».

Feuerbach, L. (1846). *Sämmtliche Werke: Zweiter Band* [Complete works vol. 2]. Leipzig: Otto Wigand.

Il'enkov, E. V. (1960). *Dialektika abstraktnogo i konkretnogo v «Kapitale» Marksa* [Dialectics of the abstract and concrete in Marx's "Kapital"]. Moscow: Akademii nauk CCCP.

Il'enkov, E. (2012). Dialectics of the ideal. *Historical Materialism, 20*(2), 149–193.

Ingold, T. (2011). *Being-alive: Essays on movement, knowledge and description*. London: Routledge.

Jornet, A., & Roth, W.-M. (2015). The joint work of connecting multiple (re)presentations in science classrooms. *Science Education, 99,* 378–403.

Lacan, J. (1966). *Écrits*. Paris: Éditions du Seuil.

Marx, K., & Engels, F. (1962). *Werke Band 23: Das Kapital* [Works vol. 23: Capital]. Berlin: Dietz.

Marx, K., & Engels, F. (1978). *Werke Band 3* [Works vol. 3]. Berlin: Dietz.

Middleton, D., & Brown, S. D. (2005). *The social psychology of experience: Studies in remembering and forgetting*. London: Sage.
Mikhailov, F. T. (2001). The "Other Within" for the psychologist. *Journal of Russian and East European Psychology, 39*(1), 6–31.
Mikhailov, F. T. (2006). Problems of the method of cultural-historical psychology. *Journal of Russian and East European Psychology, 44*(1), 21–54.
Roth, W.-M. (2003). *Towards an anthropology of graphing: Semiotic and activity-theoretic perspectives*. Dordrecht: Springer.
Roth, W.-M. (2006). A dialectical materialist reading of the sign. *Semiotica, 160,* 141–171.
Roth, W.-M. (2014). Science language *Wanted Alive*: Through the dialectical/dialogical lens of Vygotsky and the Bakhtin circle. *Journal of Research in Science Teaching, 51,* 1049–1083.
Roth, W.-M. (2016). *Concrete human psychology*. New York: Routledge.
Roth, W.-M., & Bowen, G. M. (2003). When are graphs ten thousand words worth? An expert/expert study. *Cognition and Instruction, 21,* 429–473.
Roth, W.-M., McRobbie, C., Lucas, K. B., & Boutonné, S. (1997). Why do students fail to learn from demonstrations? A social practice perspective on learning in physics. *Journal of Research in Science Teaching, 34,* 509–533.
Vygotskij, L. S. (2001). Lekcija četvertaja: Problema sredy pedologii [Lecture four: The problem of the environment in pedology]. *Lekcii po pedologii* (pp. 70–91). Izhevsk: Udmurdskij University.
Vygotskij, L. S. (2005a). Konkretnaja psixologija čeloveka [Concrete human psychology]. In *Psixologija razvitija čeloveka* (pp. 1020–1038). Moscow: Eksmo.
Vygotskij, L. S. (2005b). Myšlenie i reč' [Thinking and speech]. In *Psixologija razvitija čeloveka* (pp. 664–1018). Moscow: Eksmo.
Vygotsky, L. S. (1994). The problem of the environment. In R. van der Veer & J. Valsiner (Eds.), *The Vygotsky reader* (pp. 338–354). Oxford: Blackwell.
Vygotsky, L. S. (1997). *The collected works of L. S. Vygotsky, vol. 4: The history of the development of higher mental functions*. New York: Springer.
Wittgenstein, L. (1997). *Philosophische Untersuchungen/ Philosophical investigations*. Oxford: Blackwell.
Zavershneva, E. I. (2014). The problem of consciousness in Vygotsky's cultural-historical psychology. In A. Yasnitsky, R. Van der Veer, & M. Ferrari (Eds.), *The Cambridge handbook of cultural-historical psychology* (pp. 63–97). Cambridge: Cambridge University Press.

Chapter 4
Relating Computational Cartesian Graphs to a Real Motion: An Analysis of High School Students' Activity

Ulises Salinas-Hernández and Isaias Miranda

Abstract In this chapter we apply the recent findings of the Theory of Objectification (TO) to analyze grade 12 students' processes of interpretation of Cartesian graphs that are displayed by software—that manages real time video and reproduces it frame by frame in a computer—and are related to an experiment of a free falling tennis ball's motion across an inclined plane. We specifically are interested in analyzing how students' understanding of the mathematical relationships of the physical variables (space and time) precedes students' understanding of how real motion occurs. Our data analysis shows that students firstly associated the Cartesian graphs with the linear trajectory of the motion; secondly, with the researcher's intervention, students focused their attention on the software to deal with the shape of the Cartesian graph and the physical phenomena; thirdly, by simulating the experiment with artifacts and gestures, they established a functional relationship between the spatial (vertical and horizontal) and time variables. By doing so, they were able to describe the trajectory of the tennis ball in terms of various mathematical relations (vertical position and time, horizontal position and time, vertical position and horizontal position). We conclude that the opportunity the students have to manipulate (more or less on a whim) the software, is what affords students' processes of interpretation of a specific characteristic.

Keywords Theory of objectification · Activity · Computational cartesian graphs Real motion

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U. Salinas-Hernández (✉)
Cinvestav-IPN, Av. Instituto Politécnico Nacional 2508,
Col. San Pedro Zacatenco, 07360 Mexico City, Mexico
e-mail: asalinas@cinvestav.mx

I. Miranda
Instituto Politécnico Nacional, CICATA-Legaria, Av. Luis Enrique Erro S/N,
Unidad Profesional Adolfo López Mateos, Zacatenco, Gustavo A. Madero,
07738 Mexico City, Mexico
e-mail: imirandav@ipn.mx

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4.1 Introduction

The description of objects' motion may be one of the oldest intellectual activities in human history. In his *Physics*, Aristotle himself reports the methods used by naturalist philosophers who had preceded him in order to describe the nature of motion. These methods have changed over time: For example, while Aristotle's method consisted in describing in prose the space traveled by an object in a determined amount of time, the method commonly described in current science and mathematics textbooks consists in using both algebraic expressions—as functions of time—and space-time Cartesian graphs. However, what remains unchanged in the dialog—in the sense given by Bakhtin (1981) to this term—established by different scholars in different historical epochs, is the necessity of paying attention to the relationship between two variables (for instance, the relationship between the horizontal space traveled by an object during a certain amount of time).

With the introduction of technology into teaching/learning environments, efforts have been made to document how students make sense of motion (Burke 2010; Speiser et al. 2003). Some researchers have noted that students' success in understanding the nature of motion is influenced by the use of educational computer programs for plotting functions on position-time Cartesian graphs and of computer simulations for depicting data in these types of graph (Cicero and Spagnolo 2009; Miranda et al. 2007; Nemirovsky et al. 1998; Tao and Gunstone 1999; Thornton and Sokoloff 1990). Science and mathematics educators have mainly analyzed how this influence occurs by paying attention to the way (a) students progressively appropriate the technology so that they change their understanding of Cartesian graphs (Doerr and Zangor 2000; Goos et al. 2003), (b) technology takes part in students' making sense of mathematical symbols (Nemirovsky 1994; Urban-Woldron 2015), and (c) students deal with the discordance between what they predict and what computers display (Monaghan and Clement 1999, 2000). In these studies, a variety of theoretical approaches have implicitly or explicitly been used to analyze the role of technology in students' learning. For instance, while Monaghan and Clement (1999) used Clement's (1994) cognitive model to claim that computational images trigger students' mental simulations so that students are able to use them in non-technology problem solving situations, Doerr and Zangor (2000) and Goos et al. (2003) respectively used social (Lave and Wenger 1991) and sociocultural (Vygotsky 1978) theoretical approaches to go into details about the students' processes of appropriating technological devices in teaching and learning interactions.

Our study takes a somewhat different theoretical approach in order to contribute to the discussion pertaining to students' interpretation of Cartesian graphs that are displayed by graphing software. This approach is Radford's (2014a) Theory of Objectification (TO). Specifically, we use Radford's current considerations regarding the mediated role of activity—in the sense given by Leont'ev (1978) to this term—in students' mathematical knowledge. In this approach, students become aware of the meaning of the mathematical objects by means of their activity (in which the use of artifacts and signs are included). By applying this theoretical approach, our study may shed some light on clarifying how students' process of graph interpretation evolves. We believe it is possible to answer two questions previous researches have not tackled in regard to technology-enriched motion problems: In what way do students' understanding of the relationship between the variables involved in computational Cartesian graphs precede students' understanding of the characteristics of real motion? Given the fact that a motion of an object often occurs in a certain way and not in another (for instance, a ball released from a high position always falls), how do students modify—with the use of software—their interpretation of computational Cartesian graphs to comprehend real motion?

4.2 Theoretical Framework

The key premise of the TO lies in the following assumption:

> [T]hat the mathematics the students encounter in the school is a mathematics with a long history, shaped by individuals, cultures and institutions. Within this context, objectification refers to the social processes through which students become progressively conversant with cultural mathematical ideas and modes of thinking. (Radford 2009, p. 467)

Based on Hegel's (1977) dialectical approach and on Leont'ev's (1978) theory of activity, in the TO mathematics learning, as the previous quotation highlights, is an active and creative process that slowly takes place when students seek to understand the meaning of mathematical objects. In other words, the learning of mathematical objects is the process of making-sense of the cultural forms of thinking and understanding the world around us (Radford 2015a). The Cartesian graph, for instance, is included in that cultural form of understanding the motion of objects. As we highlighted in the introduction of this chapter, students do not learn the Aristotelian explanation about how objects move, but the current production of graphing motion included in the textbooks and shown in computational programs. Learning then takes place within a sociocultural context mediated by signs and artifacts, as Radford (2009) assures us based on his early findings.

However, in his current research, Radford (2014c) proposes that signs (verbal or gestural) and artifacts should be considered not as mediators but as a constitutive part of the activity—as this term is conceived in Leont'ev's (1978) theory. Resonating with the TO, students' interpretation of Cartesian graphs produced by graphing software is in line with what Roth and Lee (2003) also noticed regarding the interpretation of unfamiliar graphs made by scientists:

> [A]lthough [the Cartesian graph] is given materially in an instant, it is not apperceived in its detail in the same instantaneous way. The elements of the graph are not given to [students] once-and-for-all at the beginning, but emerge into their consciousness in the course of their interpretation. (p. 285)

Taking into account the TO, the previous quotation can be reformulated as follows: it is the activity that mediates the knowledge of the meaning of Cartesian graphs.

Since the concepts of knowledge and activity play an important role in our analysis, let us briefly describe how they relate to each other in the TO. Such description will show, at the same time, how this relationship serves as a category to analyze students' process of interpretation of Cartesian graphs.

According to the TO, knowledge is not an entity that can be directly observed or transmitted. Following Hegel's philosophy, in the TO knowledge is a process; it "has a *genesis* and a *movement*" (Radford 2009, p. 468); that is, it evolves. This means that knowledge cannot be constructed. Due to this *ever-changing* characteristic, knowledge does not exist in actuality; that is, it is not something that is given and therefore able to be observed. On the contrary, in accordance with Aristotle's (trans. 2004: 1019a 15) definition of potentiality, in the TO knowledge is something that does not occur yet, but is in the process of occurring. In other words, by not being able to be transmitted, knowledge is pure possibility; that is, "it can only become an object of thought and interpretation *through specific problem-posing and problem-solving activities*" (Radford 2015b, p. 136). In the research problem of this chapter, the knowledge of the characteristics of real motion becomes an object of thought through the way students interpret the Cartesian graphs produced by computational software.

Thus, according to the TO, it is in the specific *problem-solving activities* that knowledge is an actuality. Here is where the concept of activity plays an important role in the TO. Keeping the mediated characteristic of this concept, as Leont'ev (1978) himself proposed it, Radford's TO assumes that it is precisely individuals' activity that mediates knowledge and makes it to be concrete (Radford 2013). This assertion means that, since not all individuals' activities are equal to one another, different individuals made knowledge concrete in different ways. In saying "individuals' activity," we emphasize, as the TO does, that activity unfolds through all individuals that are presented with a specific context (e.g., classroom context) and through all material and non-material objects that these individuals use. The activity then mediates specific knowledge. In our study, students' activity may be observed when they try to explain a real motion with their interpretation of the computational Cartesian graphs. The explanation resulting from this interpretation would be specific knowledge of how to associate real motion with Cartesian graphs displayed by software. Other students, by means of their activity, would make knowledge concrete differently.

How students' knowledge of computational Cartesian graphs is related to the students' knowledge of the characteristics of real motion, and how students' activity in a graphing software enriched-environment is developed, are the subjects of the next sections of this chapter.

4.3 Method

This is a qualitative research study conducted in a high-school physics laboratory class. The teacher and 11 students (16–18 years old) of this class were videotaped during five work sessions of two hours each (once per week). The students were freely grouped in teams of 3–4 participants. We designed five tasks. Some tasks were solved in more than one session. Once the data collection was finished, we analyzed the questionnaires and videos of each session.

Previous to the data collection phase, we taught students how to use the AviMeca software, which had been installed on all computers in the laboratory. This software is able to manage real time video data and to reproduce it frame by frame. With this characteristic, students were able to put points on a moving object. It displays an *xy* Cartesian coordinate system that it is possible to move all over the computer screen. AviMeca also displays, in a three-column table, the values of time during the motion and values of position. Students then were able to copy this table into an Excel worksheet and were able to produce three different Cartesian graphs with three function tables: *x*-*y*, *x*-*t* and *y*-*t* With the four different orientations of the *x*-*y* Cartesian coordinate system (axes) (see Fig. 4.1, right), the AviMeca software allows students to obtain upwards concave and downwards concave curves for one motion.

For the purpose of this chapter, the data analysis comes from the answers to the first task by a team of 3 students (S1, S2 and S3). The students had to do the following steps to complete the task: (a) set up the inclined plane and place it at 1.10 *m* above ground level (see Fig. 4.2, left); (b) one teammate must drop a tennis ball from the highest point of the inclined plane; meanwhile, another teammate should video record the movement of the moving object; (c) with the aid of the AviMeca software, copy the three-column table into an Excel worksheet and plot

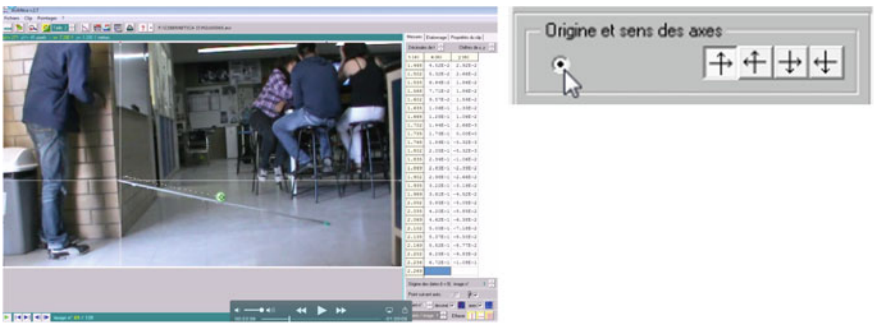


Fig. 4.1 Left. Screenshot of what the AviMeca software shows once it runs a video. At the same time, students can see one frame of the video and the three columns shown at the far right of the screenshot. Right. Screenshot of the Cartesian axes and their orientations as suggested by AviMeca. The arrows in the buttons indicate the positive way in which the software considers the values of the spatial positions

the vertical *y-t* function table to obtain a *y-t* Cartesian graph of the motion; (d) describe the motion of the tennis ball by using this Cartesian graph.

4.4 Data Analysis

In data analysis we fuse students' evolution of their interpretation of the Cartesian graphs displayed by the AviMeca software and students' activity that mediates their knowledge of the meaning of these graphs in the context of a real free fall motion. We divided students' activity into five excerpts. Every excerpt has the intention of showing a specific part of the process of the students' graph production and interpretation. Altogether, the five excerpts illustrate how students' activity was developed during their intention specifically to accomplish step (d).

The analysis starts right after the students completed steps (a) and (b) and before starting step (c).

Excerpt 1. Students' spontaneous prediction and their interpretation of linear trajectory as a Cartesian graph

This excerpt shows how, after having recorded the motion of the tennis ball and before using the software, S3 conjectures that the form of the graph must be a straight line (L1). However, when plotting the values in the *y-t* function tables, the software did not confirm this conjecture (L2).

L1 S3: [*Talking to the researcher while the other teammates are setting up the software*] The most general [*graph*] we can obtain is a straight line, right? […] Because it is an inclined plane (…) and we unite two points [*he refers to the points the software places when clicking on the tennis ball during the frame-by-frame reproduction of the video*] and having two points is a straight line to the nearest point. Well, that's my theory. [*Working with the software and looking at the screen*] I thought that it was a straight line [*referring to the graph obtained in* Fig. 4.2, left]

L2 S2: Me too!

L3 S1: Well, it is the straight line [*referring to the trajectory of the tennis ball during its free-falling motion*], but why does it come out like that? [*Points at the Cartesian graph obtained in* Fig. 4.2, left].

L4 S3: But there, you don't see the [*inaudible*] in the graph [*the student does not see the straight line in the graph*].

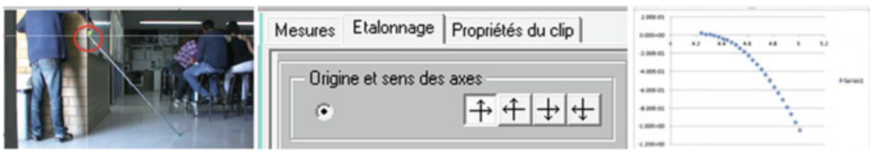


Fig. 4.2 Left. Screenshot of a student releasing the tennis ball. Center. Screenshot of the coordinate axes the students chose. Right. Screenshot of the Cartesian graph students obtained

S3 spontaneously predicted the shape of the Cartesian graph the software was about to depict (L1). Although this prediction was not part of what students were required to do, it was indeed part of the students' activity. By predicting the shape of the graph, they started the process of graph interpretation. But S3's prediction was not yet a description of the relationship between the two spatial (*x* and *y*) variables. The sentence "the most general [*thing*] we can obtain is a straight line" (L1) indicates that S3 was only connecting the points of each frame without making a relationship between the *x* and *y* variables. At this moment of the activity, the graph they expected (*predicting* graph) was in fact a copy of the phenomenon they observed and not the representation of the mathematical relationships between *y* and time. The knowledge about this mathematical relationship was still a mere possibility (Radford 2014b). It was through the comparison between their *predicting* graph and the computational Cartesian graph that this knowledge became an object of thought. In other words, the act of comparing between what they expected and what they really saw was the *problem-solving activity* the students created in order for the Cartesian graph to become an object of thought. We can notice that, although the students completed the step c—with the aid of the AviMeca software, copied the three-column table into an Excel worksheet and plotted the vertical *y-t* function table to obtain a *y-t* Cartesian graph of the motion—of the task, the relationship between the two variables was not yet an object of thought. The students needed more actions (talking to each other, interacting with the software, etc.) to become aware that the reason for the difference between the graphs lay in the coordinate axes they chose (see Fig. 4.2, center). Certainly, one of those actions could not be to plot every ordered pair of the *y-t* function table in the coordinate plane to picture the quadratic function shown in Fig. 4.2 (right). The software performed this action by means of its computational algorithms (and they were not shown to the students): there is specific knowledge that the software hides from the students. However, this *hidden knowledge* does not have to become an object of thought, for the goal of the activity was not to describe the computational algorithms that created the *y-t* Cartesian graph but to explain the tennis ball's motion by using this graph (step d). In fact, during the students' activity, the computational software performed some actions (for example, to determine the Cartesian graph according to the orientation of the coordinate axes) so that the students did not have to. The computational graph of the Fig. 4.2 (right) is then a concrete part of the knowledge of relating the vertical position of the tennis ball to the time taken during its fall; that is, it is only one of many possible Cartesian graphs that can be depicted by the software.

To sum up, this excerpt is characterized by the conflict between students' spontaneous interpretation, and the graph obtained by the software. What emerged, then, was the necessity to develop some actions to uncover the reasons of the disparities among the graphs.

During the rest of the session, in order to match the predicting graph to the computational graph, the team manipulated the Cartesian coordinate buttons of the software by placing the origin (point of intersection of the coordinate axes; see Fig. 4.2, center) in different locations on the screen but without changing the direction of the axes; in fact, the students used only the first button (from left to right) of those available (Fig. 4.2, center). Therefore, the graphs the team obtained still had the same concavity as the quadratic function. In order to obtain a different function, the students had to become aware of the characteristics of the axes of the software as well as of the place they would choose to put the origin of the axes.

Excerpt 2. Students' manipulation of the software and their interpretation of the *y-t* Cartesian graph

In the next session, after reviewing the graphs obtained by the team, the researcher (the first author of this paper) asked the students to find out whether it was possible to observe a concave upwards graph by choosing another axis. In his question, the researcher referred to these axes as the *reference systems*. Thus, the researcher not only added the concept of reference system to the students' language but also encouraged the students to focus their attention on a new way of understanding the graph as the relationship between variables. At the beginning, this relationship was vague, as seen in the following excerpt.

L5 S1: [*After manipulating the first two buttons (from left to right) of the* Fig. 4.1] It is just that we will never get an upward curve if we only change the reference system. It is illogical because our ball will always go down.

L6 Researcher: You say that the graph will never go up because the ball always goes down.

L7 S1: As long as we keep changing the reference system, the reference system will give us the position of our graph; but it [*the graph in* Fig. 4.2, right] will always be a curve.

L8 Researcher: Downward curve?

L9 S1: Yeah.

Although the researcher did not define the concept of reference system, the students knew that the location of the axes modified the vertex of the parabola shown in Fig. 4.2 (right) but not its concavity. In fact, the concavity downward of the graph is associated with the free falling motion of the ball (L5). Once again, at this moment of students' activity, the tennis ball's motion is associated with its trajectory and not with a relationship between its vertical position and time variables. Although the students have plotted the values of the *y-t* table function obtained by the software, they have not become aware that what they have really been doing is to modify the orientation of the axes. They have not yet objectified that one of these axes is the vertical position variable and the other the time variable.

After his intervention, the researcher decided to stop asking the team, for he saw S1 manipulating the software on his own. In order to unveil the meaning of the axes of the software in the context of this task, S1 did not chose the *y-t* function table as he did previously. He selected the *x-t* function table instead. Then, he chose the first button of the axes (from left to right) shown in Fig. 4.1 (right); that is, the same axes he chose at the beginning of the activity: up and right (positive) directions of the axes. By doing so, he observed the software displaying an upward concave parabola that contrasted with the downward concave parabola they saw in L1.

The relevance of the excerpt 2 in students' activity lies in that the researcher's intervention was an action that served to reorient the students' attention towards the mathematical variables and its connection with the phenomenon. This situation highlights the fact that, in addition to the artifacts and signs, all the individuals present in the activity contribute to unfolding it.

Knowing the rapidly graphing process of the software, S1 preferred manipulating the buttons of the axes rather than depicting a point-to-point graph using the values of the function tables. The action of manipulating the buttons mediated the *hidden* knowledge of the software in regard to what it actually does when plotting the values of function tables. The knowledge about the mathematical characteristics of the tennis ball's motion acquired cultural determinations due to the fact that S1 decided to operate the buttons of the axes. We can claim that students, along with S1, are in the middle of the process of becoming aware of the functional relationships of the variables of the Cartesian graph. This process kept on developing in the following excerpt.

Excerpt 3. Students' identification of the x, y variables and their interpretation of the meaning of the coordinate axes

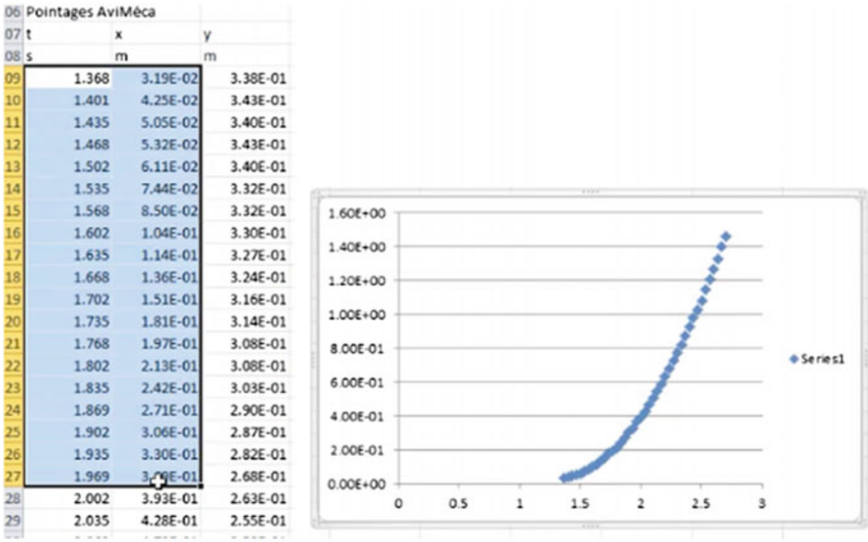
L10 S1: Look, if we plotted the time with respect to the variable *x*, we won't get the same as if we plotted the time with respect to variable *y*, because when we plot this [*referring to time*] with respect to *x*, it will be the curved line that is the trajectory. But, if it is this [*referring to time*] with variable *y* …

L11 S2: Let's see. Do it.

L12 S1: It will be the acceleration that is given by a curve. It's all calculation (…) Oh! It changes there [*referring to the concavity of the parabola; see* Fig. 4.3*, right*], but it is because we have to take the *x* variable.

In this excerpt, for the first time in the activity, the students incorporated the words *x* and *y* into the conversation and identified them as the variables of the coordinate axes of the software. Contrary to excerpts 1 and 2, in which the students struggled with the discordance between the trajectory of the motion of the tennis ball and the Cartesian graph of the software, in excerpt 3 the focus of the conversation was on the kind of variables that were represented in the axes. It is at this moment of the activity that the students started analyzing the Cartesian graph under the consideration that there should be a relationship between two variables and not under the sensorial perception of the real motion.

Unlike at the beginning of the team's activity, when the *y*-*t* Cartesian graph displayed in the computer screen turned out to be contrary to S3 and his teammates' expectations, in L10 S1 recognized that the *x*-*t* function table corresponded to the upward concave parabola shown by the software (see Fig. 4.3, right). The mathematical relationship between a spatial variable (in this case, horizontal position: *x*) and time has now become an object of students' thought. S1 is even able to synchronize the orientations of the axes with the concavity of the graph. In other words, S1 associated the vertical and horizontal axes with the time variable. More



06	Pointages AviMéca		
07	t	x	y
08	s	m	m
09	1.368	3.19E-02	3.38E-01
10	1.401	4.25E-02	3.43E-01
11	1.435	5.05E-02	3.40E-01
12	1.468	5.32E-02	3.43E-01
13	1.502	6.11E-02	3.40E-01
14	1.535	7.44E-02	3.32E-01
15	1.568	8.50E-02	3.32E-01
16	1.602	1.04E-01	3.30E-01
17	1.635	1.14E-01	3.27E-01
18	1.668	1.36E-01	3.24E-01
19	1.702	1.51E-01	3.16E-01
20	1.735	1.81E-01	3.14E-01
21	1.768	1.97E-01	3.08E-01
22	1.802	2.13E-01	3.08E-01
23	1.835	2.42E-01	3.03E-01
24	1.869	2.71E-01	2.90E-01
25	1.902	3.06E-01	2.87E-01
26	1.935	3.30E-01	2.82E-01
27	1.969	3.[illegible]E-01	2.68E-01
28	2.002	3.93E-01	2.63E-01
29	2.035	4.28E-01	2.55E-01

Fig. 4.3 Screenshots of the S1's selection of the *x-t* function table (left) and the graph obtained (right)

specifically, S1 was able to interpret the horizontal axis as the time and the vertical and horizontal axes as the positions of the ball. With these actions, the students have started developing the *hidden knowledge* of the software; they have started expressing the real motion of the tennis ball in terms of mathematical relationships, which is a cultural way of describing the phenomenon of motion.

It is worth noting that S2's expression: "Let's see. Do it" (L11), encouraged S1 to materialize what he (S1) had just said in L11. These two lines, L10 and L11, exemplified how knowledge, as possibility, is instantiated through activity. S1's supposition in L10 suggested how the knowledge of the variables involved in the parabola of Fig. 4.3 (left) would change if the *y* values were replaced by the *x* values. If S1 had not interacted with the software, this knowledge would have been a mere potentiality. Here is where S2's exhortation helps this knowledge to be concrete. Once again, by plotting the *x-t* function table, what S1 did was to show a particular instance of the knowledge of interpretation of Cartesian graphs, as accepted by professional scientists.

Summing up, the excerpt 3 shows the implications of the researcher's intervention made in the previous excerpt. Here, S1 took the initiative with the intention of obtaining an upward curve by choosing variables from the function tables. By doing so, he made the artifact relevant in the development of the activity.

Excerpt 4. Students' representation of the real motion and their mathematical interpretation of the Cartesian graph

After obtaining the upward concave graph, S1 conveyed his ideas regarding the meaning of the space-time relationship of the tennis ball's motion.

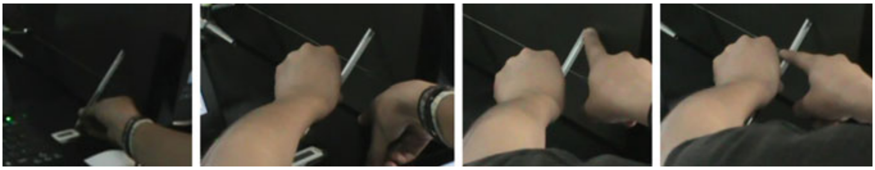


Fig. 4.4 From left to right, S1: uses a pen to represent the inclined plane (photo 1); points, with his right index finger, at the place where the origin of the reference system should be (photo 2); points at the place where the tennis ball started rolling (photo 3); moves his index finger along the pen, as representing the tennis ball's motion (photo 4)

L13 S1: I know why! Draw this: At the moment the ball moves down the inclined plane, it is like this [*represents the inclined plane using a pen; see* Fig. 4.4*, photo 1*]. Then, imagine that the reference system goes from this point [*pointed at the place where the origin of the reference system would be located; see* Fig. 4.4*, photo 2*]. This is *y* and this is *x*; they are the axes [*identifies the coordinate axes in his representation, by pointing at the place where the pen is lying*].

L14 S2: Yes.

L15 S1: And the ball is here. Then, in *t* equals zero, the *y* is at its maximum level [*points at the highest vertical position of the ball across the inclined plane, see* Fig. 4.4*, photo 3*], at its maximum point. And when time starts running, I mean, when we let the ball go, it will move [*represents the motion of the ball; see* Fig. 4.4*, photo 4*]. And at the moment it reaches, let's say, *t* equals ten [*points at the lowest vertical position of the ball across the inclined plane*] … *y* is going to be equals to zero. Why? Because it is on an inclined plane, I mean it was falling. But *x* was increasing [*points at where the x values increases*] because the trajectory of the ball was forward, inclined. Then, that's why the graph goes up or down. And if it's considered with respect to *y*, the graph will always go downwards, because the ball is decaying [moving downwards].

L16 S2: [*At the same time*] It is also considered with respect to *x*.

L17 S1: But, if it is considered with respect to *x*, what it's going to do is to increase exponentially.

The expression "I know why!" uttered by S1 reveals an understanding of what, up until this moment of the activity, had not been objectified: the functional relationship between variables. Although S1 added the term reference system to his explanation, he did not focus his attention on the meaning of this term; he rather paid attention to the mathematical representation of the Cartesian graph and its relation to the real motion of the tennis ball (L13). The meanings of both the reference system and the orientation of the axes, shown in the buttons of the software, acquired concrete determination through S1's gestures. The sentence "This is *y* and this is *x*" and the gestures that accompanied it, are altogether part of S1's actions through which he materialized the explanation of the motion from a mathematical point of view. Through gestures, S1 made visible not only the mathematical variables involved in the motion but also the relationship between them and the inclined plane (see Arzarello et al.'s 2009 study for more information about the role of gestures in students' processes of learning). More specifically, through his gestures, S1 reproduced the experiment of the inclined plane and added to his actions the vertical position vs. time and horizontal position vs. time relationships. First, the student used the pen to represent the inclined plane (Fig. 4.4, photo 1). By doing so, S1 transferred a digital image of the inclined plane into the real world. But he did not only constrain his actions in representing the inclined plane with a pen, he also started gesturing the mathematical relationship between spatial variables and time. In fact, S1 separated, through gestures and words, the horizontal motion of the ball from the vertical one, and related these two motions with time: On the one hand, he analyzed what would happen with two specific vertical positions (the *y* in S1's intervention) of the ball as time goes on (see the beginning of S1's intervention in L13 and the Fig. 4.4, photos 3 and 4); on the other hand, he analyzed what would happen with the horizontal position (the *x* in S1's intervention) as time goes on (see the end of S1's intervention in L13). It is worth mentioning that S1's precise words in expressing the mathematical relations involved in the tennis ball's motion, and S1's deictic gestures that further specify what he had said in his speech, indicate a proficiency in his narrative discourse (see more details of the role of language proficiency and gestures in the paper by So et al. 2013). It would not be accurate, however, to claim that S1's refined explanation in L13 was the final product of his internal cogitation. It was rather part of the activity that had been developing in the team. S1's actions obeyed his decisions to explain to his teammates what he had found when he clicked on the buttons of the axes. At the same time, while explaining, he forced himself to use the better words, gestures and even artifacts to make clear his point of view about the mathematical nature of the motion.

From a TO perspective, S1's intervention in L13 is an example of what it means to consider *the form in which knowledge is produced.* Theoretically speaking, S1's explanation refers to the way in which he uses words (such as inclined plane, position, time, and origin), gestures and artifacts (that is, S1's actions) that made the knowledge of describing a motion from a mathematical point of view concrete. Undoubtedly, the coordination of actions to do certain movements (gestures) had been developed by S1 during his life; however, it is only within the activity that this coordination of movements acquires a meaning. For instance, on the one hand, pointing at the origin of the reference system has a particular meaning: determining the place from which the vertical and horizontal positions of the tennis ball change as time passes. On the other hand, using the same finger, S1 pointed at another concept: the trajectory the object follows across the inclined plane. These gestures make sense within the specific context of the activity. To sum up, S1's gestures and words do not represent only an auxiliary collateral medium of his discourse, but also the form in which the knowledge of interpreting the real motion is materialized.

The relevance of excerpt 4 is that the activity has evolved with the concomitant students' interpretation of the phenomenon, which is now characterized in terms of the use of words related to mathematical objects (axes and reference system). This excerpt highlights that S1's awareness is perceived through the coordination of his gestures and language in order to organize his thoughts and to transmit them to his teammates.

With all the previous actions, the team was ready to take a step forward in its intention to relate the computational graphs to the real motion. In the next excerpt, S1 and his teammates went back to the first graph they saw (see Fig. 4.2, right) and were able to explain it in terms of the relational function of the two spatial variables: *x* and *y*).

Excerpt 5. Students' explanation of the meaning of the coordinate axes and their re-interpretation of their initial spontaneous prediction

L18 S1: Let's do this [*highlights the x* and *y columns shown in the software, see* Fig. 4.5, *left*] … only these two. If my mental ability is right… this is what has to come up [*At that moment, the graph of the* Fig. 4.5 *(right) comes up on the screen*] That is the straight line, that is our inclined plane. Then, if you take the time.

L19 S2: [*Interrupting*] It is just that here you get the inclined plane because you are not considering the time.

L20 S1: I only took the *x* variable and *y* variable.

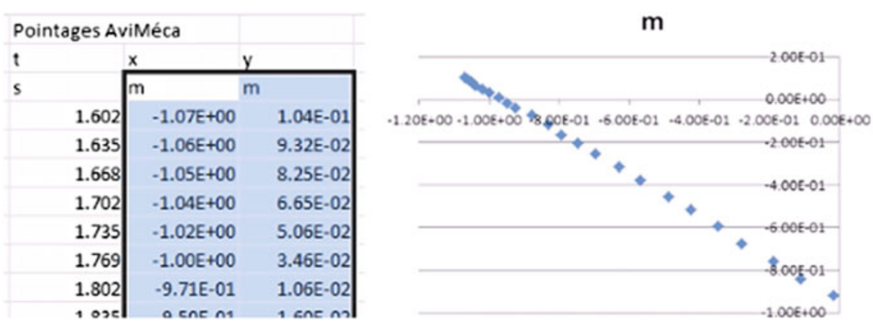
L21 S2: [*At the same time*] You are not considering time. It only gives you the coordinates you chose.

L22 S1: Yeah, it is only their behavior (…) this is our inclined plane [*refers to the graph obtained*]. But, if we consider it with respect to *y*, we will consider it as if the height is decreasing.

L23 S2: Yeah, that's right.

L24 S1: And if we consider it with respect to *x*, we are going to see the increase of the horizontal distance there was (…), but if we only have *x* and *y*, it is going to take our coordinate system and it is going to give us an inclined plane.

L25 S3: The straight line!



Pointages AviMéca		
t	x	y
s	m	m
1.602	-1.07E+00	1.04E-01
1.635	-1.06E+00	9.32E-02
1.668	-1.05E+00	8.25E-02
1.702	-1.04E+00	6.65E-02
1.735	-1.02E+00	5.06E-02
1.769	-1.00E+00	3.46E-02
1.802	-9.71E-01	1.06E-02

Fig. 4.5 Left: screenshot of the *x*-*y* function table chosen by S1. Right: the corresponding computational Cartesian graph of the *x*-*y* function table

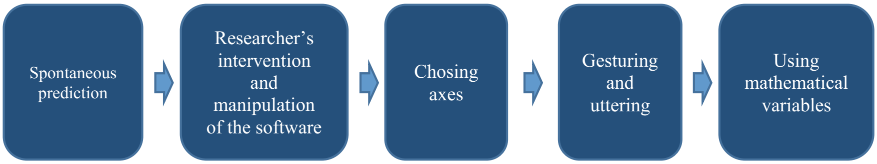


Fig. 4.6 Evolution of students' activity

Unlike the first excerpt, in which the team saw the software displaying a different graph (Fig. 4.2, right) from the one S3 had anticipated (L1), in this last excerpt of students' activity S1 recognized that that graph was due to the decision of selecting the table of *x* and *y* values. The phenomenon of the free falling motion was at this moment of the activity signified by a mathematical relationship between physical variables. S1 did not only trust in his "mental" (as he himself expressed it, L18) ability to agree with the straight line the computer displayed when he plotted the *x* and *y* values (L18), but also he was able to manipulate, more or less on a whim, the three variables involved in the phenomenon (vertical and horizontal positions and time). When S1 said: "I only took the variable *x*, and the variable *y*," he was conscious that the straight motion of the tennis ball could be mathematically explained with a straight line because it is able to choose a Cartesian graph that relates the *x* variable to the *y* variable. However, S1 was not the only student who was aware of the signification of the straight line of the Cartesian graph. S2 (L19 and L21) and S3 (L25) also objectified the mathematical relationship of the variables and were able to associate it with the real motion of the tennis ball. After all, S1's actions were part of a joint activity. The final conclusion made by S1, accepted by his teammates, condensed all the students' activity developed during the four previous excerpts.

In the attempt to summarize excerpt 5, we noticed that all the team members are aware that (a) his first prediction (made at the beginning of the activity) could be explained by using the mathematical variables of position (*x*, *y*), and (b) time variable is relevant in the shape of the graph.

To end this analysis, let us summarize the evolution of students' activity during all the five excerpts. In the next diagram (Fig. 4.6), each rectangle accentuates the main activity developed in each excerpt.

4.5 Discussion

In this chapter we analyzed the way grade 12 students became aware of the meaning of computational Cartesian graphs in relation to the motion of an object rolling across an inclined plane. This analysis was carried out with the formulations of the Theory of Objectification in regard to the role of activity in the mathematical learning process. Our analysis offers an alternative point of view to explain how the use of technology can be conceptualized as part of students' activity, in their intentions to achieve pedagogical goals. Data analysis suggests that the students' description of the free falling tennis ball's motion was preceded by students' understanding of the mathematical relation of the physical variables (space and time) involved in the Cartesian graphs the AviMeca software displayed. This understanding came about through processes involved in different actions. Firstly, the students expected that the computational Cartesian graph would look like the tennis ball's trajectory falling across an inclined plane (Excerpt 1). It was the fact that the *y-t* computational Cartesian graph shown in Fig. 4.1 (right) was not similar to students' expectations that triggered students' activity. Secondly, by presuming to justify that the downward concave graph (Excerpt 2) was due to the moving-down motion of the tennis ball, S1 decided to plot the *x-t* function table displayed by the software. These fortunate actions, along with the manipulation of the buttons of the axes of the software, made students think of the mathematical relation between the *x* and *y* (spatial) variables and time variables. Thirdly, when students returned to his first action (Excerpt 5)—when S1 predicted that the values of the *y-t* function table would correspond to a straight line (L1)—they had already understood the mathematical relation of the computational Cartesian graphs. Thus, at the end of Excerpt 5, the students realized that the linear trajectory of the tennis ball could be represented as a straight line only if the axes of the Cartesian coordinate system were spatial variables (positions *x* and *y*). The process of becoming aware we have described indicates that the way students understood the physical characteristics of the free falling motion of the tennis ball in the specific context in which a computational program is used, was certainly more fortuitous than planned. This does not mean that if S3 had not both spontaneously predicted the Cartesian graph at the beginning of team's activity and S1 deliberately chosen the *x-t* function table to obtained an *x-t* Cartesian graph, students would not have been able to explain the *x-y* Cartesian graph as a mathematical relationship between two spatial variables. What it means is that if S1 had solved the task in a different manner, the process of students' comprehension of the tennis ball's motion would have been different. It is precisely the opportunities the computational software offers of manipulating the buttons and the *x-y*, *x-t* and *y-t* function tables that led the students to a specific comprehension of the meaning of Cartesian graphs. The fact that the software did some calculations (e.g., plotting the function tables) caused students' activity to be reorganized in a different manner. It was enough for them to observe the graph displayed by the software without being concerned about the mathematical algorithms it uses. On the basis of our data analysis, we postulated that computational devices have some hidden knowledge that students do not necessarily have to become aware of; after all, students' final intervention in excerpt 5 was not about explaining those algorithms. With the use of the TO, students' final intervention can be considered as the final step in which their interpretation evolved. This is an evolution that found its roots in the way students' activity also evolved. The association between students' evolution of interpretation and students' evolution of activity we try to convey in this study means that excerpt 5 becomes fully intelligible if the previous excerpts are considered as the background in which students' activity had been developing.

We believe the TO offers a useful theoretical basis to explore students' learning of concepts that are in the border of the mathematics and physics fields. Such is the case for the concept of reference system. Although it was mentioned by the researcher in Excerpt 2 and implicitly used by the students during their activity, the reference system did not become an object of thought in all of the activity of the students: From the students' initial point of view, no matter what button is chosen, the *y*-*t* Cartesian graphs are always downward concave curves because the tennis ball will always move down the inclined plane. Since the concavity of quadratic functions is related to both the orientation of the coordinate axes and the point where these axes intersect (the origin), how students' activity would lead them to dissociate the mathematical representation from the trajectory of objects' motion, using technology to create teaching/learning environment, may need more attention for mathematics and science educators.

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References

Aristotle. (2004). *Metaphysics* (H. Lawson-Tancred, Trans.). New York: Penguin Group.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Bakthin, M. M. (1981). *The dialogic imagination* (C. Emerson & M. Holquirt, Trans). Texas, USA: University of Texas Press.

Burke, J. (2010). Fourth grade student conceptions of piecewise linear position functions. In P. Brosnan, D. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 227–233). Columbus, Ohio, USA.: The Ohio State University.

Cicero, M. L. L., & Spagnolo, F. (2009). *The use of motion sensor can lead the students to understanding the Cartesian graph*. CERME 6–WORKING GROUP 11, 2106.

Clement, J. (1994). Use of physical intuition and imagistic simulation in expert problema solving. In D. Tirosh (Ed.), *Implicit and explicit knowledge* (pp. 204–244). Hillsdale, NJ: Ablex.

Doerr, H. M., & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics, 41,* 143–163.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003). Perspectives on technology mediated learning in secondary school mathematics classrooms. *The Journal of Mathematical Behavior, 22*(1), 73–89.

Hegel, G. W. F. (1977). *Phenomenology of the spirit* (A. V. Miller, Trans.). Oxford, UK: Oxford University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, UK: Cambridge University Press.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality.* Englewood Cliffs, NJ: Prentice-Hall.

Miranda, I., Radford, L., & Guzmán, J. (2007). Interpretación de gráficas cartesianas sobre el movimiento desde el punto de vista de la teoría de la objetivación. *Educación Matemática, 19* (3), 5–30.

Monaghan, J. M., & Clement, J. (1999). Use of computer simulation to develop mental simulations for understanding relative motion concepts. *International Journal of Science Education, 21*(9), 921–944.

Monaghan, J. M. & Clement, J. (2000). Algorithms, visualization, and mental models: High school students' interactions with relative motion simulation. *Journal of Science Education and Technology, 9*(4), 311–325. ISSN: 1059-0145/00/1200-0311

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the velocity sign. *Journal of Mathematical Behavior, 13*, 389–422. https://doi.org/10.1016/0732-3123(94)90002-7

Nemirovsky, R., Tierney, C. & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16(2)*, 119–172. https://doi.org/10.1207/s1532690xci1602_1

Radford, L. (2009). "No! He starts walking backwards!": Interpreting motion graphs and the question of space, place and distance. *ZDM Mathematics Education, 41,* 467–480. https://doi.org/10.1007/s11858-009-0173-9.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education, 2*(1), 7–44.

Radford, L. (2014a). De la teoría de la objetivación. *Revista Latinoamericana de Etnomatemática, 7*(2), 132–150.

Radford, L. (2014b). On teachers and students: An ethical cultural-historical perspective. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Plenary paper) (Vol. 1, pp. 1–20). Vancouver (PME).

Radford, L. (2014c). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*(3), 405–422.

Radford, L. (2015a). Methodological aspects of the theory of objectification. *Perspectivas Da Educação Matemática*, *8*(18), 547–567.

Radford, L. (2015b). The epistemological foundations of the theory of objectification. *Isonomia*, 127–149.

Roth, W.-M., & Lee, Y. J. (2003). Interpreting unfamiliar graphs: A generative, activity theoretical model. *Educational Studies in Mathematics, 57,* 265–290.

So, W. C., Kita, S., & Goldin-Meadow, S. (2013). When do speakers use gestures to specify who does what to whom? The role of language proficiency and type of gestures in narratives. *Journal of Psycholinguistic Research, 42*(6), 581–594.

Speiser, B., Walter, C., & Maher, C. A. (2003). Representing motion: An experiment in learning. *Journal of Mathematical Behavior, 22,* 1–35.

Tao, P.-K., & Gunstone, R. (1999). The process of conceptual change in force and motion during computer-supported physics education. *Journal of Research in Science Teaching, 36*(7), 859–882.

Thornton, R. K., & Sokoloff, D. R. (1990). Learning motion concepts using real-time microcomputer-based laboratory tools. *American Journal of Physics, 58,* 858–867.

Urban-Woldron, H. (2015). Motion sensors in mathematics teaching: Learning tools for understanding general math concepts? *International Journal of Mathematical Education in Science and Technology, 46*(4), 584–598.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Chapter 5
Joint Attention in Resolving the Ambiguity of Different Presentations: A Dual Eye-Tracking Study of the Teaching-Learning Process

Anna Shvarts

Abstract This study explores the coordination of several semiotic presentations in the process of the objectification of Cartesian coordinates by first grade children. By means of novel dual eye-tracking technology, I investigated the micro-dynamics of a child's and her parent's attentions as they were involved in the teaching-learning process. The analysis of the synchronized data that were retrieved from two eye-trackers and from an external video camera showed ambiguity, not only of verbal terms and visual inscriptions, but also of pointing gestures. It brought out the joint attention moments as crucial for acquiring the culturally adequate meaning of the Cartesian plane. The active disclosure of different presentations' complementarity determined the moment when the child acquired some meaning, yet not always the cultural one. The joint attention (or the absence of it) allowed the adult to keep track of the understanding that was emerging in the child's mind, to compare it with her own perception of the cultural representation, and to rectify the child's perception.

Keywords Joint attention · Dual eye-tracking · Gestures · Multiple representations · Mathematics education

5.1 Introduction

One of the most prominent results of mathematics education research during the last couple of decades is the statement of the multi-representational nature of mathematical concepts and its significance for mathematics understanding (e.g., Duval 2006; Sierpinska 1994). Consequently, a concept in mathematics is not given by its definition or formal expression but it is determined by many different inscriptions.

A. Shvarts (✉)
Lomonosov Moscow State University, Mokhovaya st., 11–9,
Moscow 125009, Russia
e-mail: shvarts.anna@gmail.com

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The problem for the learning process is that the inscriptions are external to a learning person and their signification needs to be acquired in order to contribute to the mathematical meaning. Being obvious for algebraic equations, this statement was also shown to be true for visual materials: they are not self-explanatory and need to be introduced. Otherwise, for example, the canonical graph of a parabola might be formally associated with the $y = x^2$ equation, without any understanding that they do not correspond to each other (Aspinwall et al. 1996). Roth (2008) stresses joint orientation, namely an active attitude from both communicative partners, which is needed in order to make the meaningful features of visual representations salient.

In preparation for discussion of the methodology in the next section of this chapter, I introduce some material of this research and compare it with the results of our previous study (Krichevets et al. 2014). The figures represent the paths of eye-movements while participants were searching for a point with the given coordinates on the Cartesian plane. There are eye-movements of university students in Fig. 5.1 and we can see that they gaze along the axes, since it is exactly a movement along the axes that makes the diagram meaningful. In contrast, Fig. 5.2 represents eye-movements of a first-grade student who has never seen a Cartesian plane before and he is asked (in the same way as the university students in Fig. 5.1)

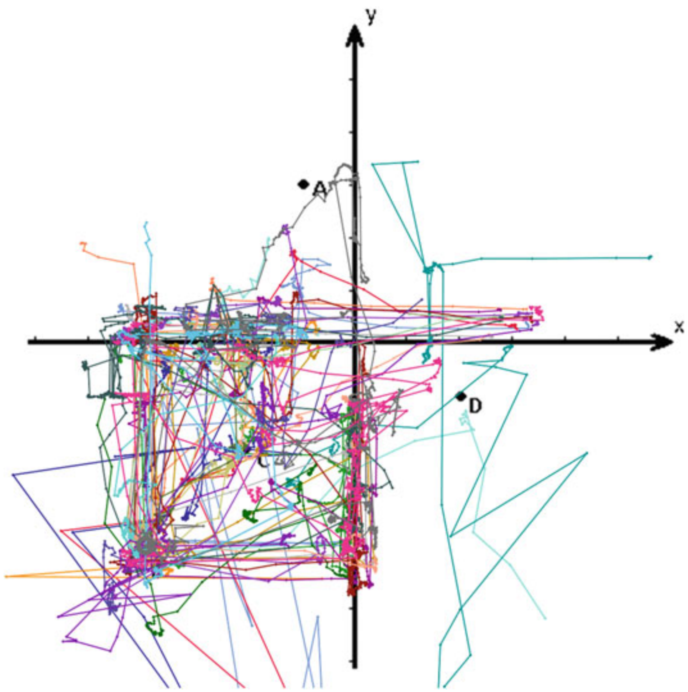

Fig. 5.1 Eye movements of the university students while they were searching for a point with (−4, −4) coordinates

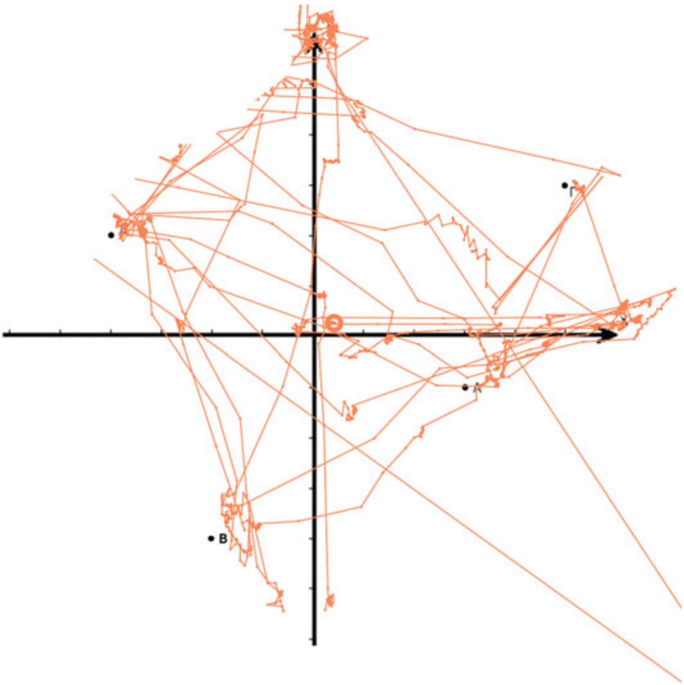

Fig. 5.2 Eye movements of a first grade student while he was searching for a point with (−2, −4) coordinates

to find a point with the given coordinates. His eye-movements do not follow the cultural mathematical sense of the diagram: he moves from one point to another, searching for an appropriate one between them. This simple example shows the ambiguity of the same inscription, which *looks* differently for the participants with different mathematical backgrounds. Both perceptions are appropriate and meaningful for the one who makes them: they suit the task to find a point, but somehow we, being the bearers of cultural educated perception, attribute the correct perception of the diagram to the paths of eye-movements in Fig. 5.1. The key question for educational science is the transformation of a student's perception from Fig. 5.2 to Fig. 5.1. How does this external inscription come to signify mathematical meaning for children during the learning process?

According to Vygotsky, the greatest distinctiveness of a child's development in comparison with any natural evolutional processes is the pre-existence of "ideal forms" or "final" forms (Vygotsky 1934/2001), which are special cultural ways of how to perceive reality and to perform actions that surround a child in her social environment. These "ideal forms" of knowledge or of 'how-to-do' abilities co-exist with "initial forms" of children's mental functions and interact with them. This pre-existence of ideal form works like a 'short-cut' in development: a child does not need to discover all cultural achievements on her own, but can acquire cultural forms (and theoretical perception in particular) at first in "a form of collective behavior of a child, forms of collaboration with others" (ibid, p. 90), and then these forms are transformed into individual functions. Although Vygotsky himself pays significant attention to the verbal components of internalization, embodied interaction and joint attention are logical continuations of his ideas concerning shared activity, as it is a necessary step in the internalization of 'ideal forms' (Krichevets 2014).

The introduction of a new mathematics concept cannot start from the signs (formulas, diagrams, etc.) themselves. Some culturally organized activity with the diagram is needed to determine the correct perception of diagrams. Following Marx, Radford (2010) refers to an eye as a theoretician: perception should function in a special enculturated way in order to perceive a diagram correctly. Radford supposes that this theoretical perception is formed during shared activity between the teacher and the students: following gestures, verbal explanations, prosody and other semiotic registers all together, the student is able to objectify the meaning of the diagrams and the other presentations and to establish the cultural way to perceive them. One of the ways to theorize this connection between representations is the "semiotic node", namely "pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification" (Radford et al. 2003, p. 56).

Radford (2008) points to the etymology of the phrase *to acquire,* which means *to seek,* as an important metaphor for the objectification of cultural meaning and the creative character of this process. I would add that this search for a new object as it emerges from different presentations is determined by a motivational aspect of the child's activity, since the object/motive may be defined as a need that has found the object which can fulfill this need (Leont'ev 1978; Roth and Radford 2011).

Further analysis confirms that the relation between different presentations is not set and these relations "involve *transactional coordination work*; this work produces the social relation between actors and, simultaneously, the sign relation between multiple forms of presentation" (Jornet and Roth 2015, p. 381). This transaction is a very first moment when visual or other materials become meaningful for a student by means of joint action, thus appearing to be *presentation* of mathematical meaning. Further, as the students discuss this first experience, they refer to some inscriptions as the *re-presentations* of the experience that they lived through. I point to the opposite process: at the moment of teaching intervention, the visual, verbal, gestural *re-presentations* (for an adult) become all together the *presentations* for a student. I claim that it is *joint attention* that plays a significant part in this transformation of perception processes and determines the interrelations of different *presentations* as meaning emerges for the student.

5.2 Joint Attention and Meaning of a Presentation

Joint attention is a specific situation when two people are focused on the same object and they are both aware of each other's focus of attention. Developmental psychology has distinguished joint attention between a child and an adult as an important mechanism that allows transfer of the cultural meaning during language acquisition (e.g., Tomasello and Farrar 1986). Unfortunately it has been rarely studied in educational research and I know only a couple of mentions of this phenomenon in mathematics education papers. Roth (2008) touches on joint attention as a part of joint orientation. Seeger (2008) emphasizes the role of shared intentionality and joint attention in acquiring meaning of the representations through social practices. At the same time, I suppose that joint attention is exactly a situation in which cultural perception is forming and mathematical objects are being shaped and then become referenced through a semiotic vehicle. The purpose of the inquiry reported in this chapter is to disclose the micro-mechanisms involved in how joint attention is forming, and how students are getting to perceive Cartesian coordinates in a cultural way when they are involved in shared actions of solving tasks.

From the point of view of Radford's objectification theory, I would say that a moment of joint attention is precisely the moment when an adult manages to guide the child's attention in a proper cultural way. But the situation is more complicated, since an adult does not just expose her "ideal form", her cultural way of perception (which is usually very quick and unarticulated (Krichevets et al. 2014)), but she unfolds a special cultural practice in front of a student in accordance with the student's needs. In research on language acquisition it is known that referencing a jointly attended object is effective in the case in which an adult follows the child's attention instead of guiding the child and forcing her to attend to another object (Tomasello and Farrar 1986). I suppose that this principle can be applied to mathematics education in such a way that a teacher might not only guide but also follow the student's attention and thus she might adjust her explanations—the unfolding cultural practice—in accordance with the child's focus of interest or/and the child's own strategies.

Developing the subtlety of joint attention further, I would like to stress the enactivist and phenomenological approaches in joint attention research. Joint attention is defined as a state in which two people attend to the same object and they are both aware of the fact that the other attends to the same object. However the role of awareness might be different. Gallagher (2011) brings an example of a football player who does not reflect on the others players' mental states and on the objects of their interest, but rather considers the football field together with its significance for his movements and sees intentions and dispositions of the other players directly from their gestures and movements without minding their minds. Hutto (2011) claims a possibility of joint attention without any "mind minding," which means to have a common object of focus without having any representation of another one's mind, but just getting involved in shared activity. It seems that a student discloses the mathematical objects in the same way: she does not try to guess what the teacher is attending to, but rather immediately takes into account the teacher's verbal and gestural expressions and constitutes meaning from them. Without mediation by any representation of another one's mind, this immediate reaction, which enactivism claims, turns analysis of shared activity towards the neurophenomenological notion of "extended body" (Froese and Fuchs 2012) that is expressed in the ability of humans to synchronize patterns of their behavior with others by the rhythms and dynamics of these patterns.

In some other cases a focus of attention might be noticed and taken into account in further shared actions. This pole of being aware of another person's focus versus merely being involved in common activity, I consider as helpful for understanding the structure of joint attention during the learning process, along with the opposition of following versus guiding the attention of a student.

This study is dedicated to the microanalysis of visual joint attention, as it can be accessed through dual eye-tracking technology, which allows the researcher to follow the eye-movements of two communicative partners while they are addressing the same diagram.

5.3 Eye-Tracking in Educational Research

The traditional way to study perception in a multi-representational situation is a detailed qualitative analysis of video records and audio waveforms of students' and teachers' behavior, which allow the researcher to retrieve synchronized data of gestures, verbal protocols and track intonations (e.g., Radford 2010; Roth 2008). My aim was to delve more deeply into individual processes of perception of each participant during the processes of teaching and learning as they develop through shared activity that unfolds by unification of a few representations. Thus I invoked eye-tracking technology. Eye-tracking technology allows the following of direction of an individual gaze; it is widely used in analysis of expert-novice differences of perception (for a review, see Gegenfurtner et al. 2011) and it becomes increasingly widespread in the investigations of learning processes (e.g., van Gog and Scheiter 2010). Some eye-tracking research studies in mathematics education are devoted to the investigation of one semiotic representation, such as diagrams (Epelboim and Suppes 2001), or texts (Peters 2010), or equations (Chesney et al. 2013). In other cases an analysis of eye-movements among formulas, texts and pictures was conducted (San Diego et al. 2006; Andrá 2009, 2015). Most of these papers include only analysis of eye-tracking data. An exception is the paper by San Diego et al. (2006), in which eye-movements were analyzed together with sketching and a verbal protocol of the student.

Concerning the transformation of the perceptive processes, it has been shown that perception of novices differs from experts' perception in many domains (Gegenfurtner et al. 2011), including mathematics (Andrá et al. 2015; Epelboim and Suppes 2001; Krichevets et al. 2014). The prominent result is experts' ability to focus on relevant areas and to ignore what is irrelevant. The educational question is how teachers may facilitate the transformation of students' perception to that of the experts. There is empirical evidence that looking at some particular areas correlates with more effective learning. Thus, with this result applied to the investigation of peers' communication, it has been shown that couples of students who have higher gaze similarity obtain higher learning outcomes (Belenky et al. 2014; Sharma et al. 2015). Analyzing eye-movements during learning by Massive Open Online Courses (MOOCs), Sharma and colleagues (Sharma et al. 2014, 2015) introduced a notion of "with-me-ness," which means an amount of time that a student attended such zones of the visual presentation that were associated with the words of the lecture at that moment. According to the data, the time spent in such areas correlates with the learning achievements. Remarkably, the correlations are stronger when the teacher starts from a blank board and writes during the lecture, as opposed to the teaching from a ready-made presentation. I suppose that writing during a lecture helps in the achieving of joint attention, since it guides the student's attention precisely according to the thought of the teacher.

In the meanwhile, a whole branch of educational research investigates a possibility of artificial guidance of a student's attention. It appears that direct guiding by visual cues does not necessarily lead to an improvement of the learning outcome. Although sometimes visual cues are effective (Rouinfar et al. 2014; Scheiter and Eitel 2010), in other cases the students did attend to the necessary areas, but there was no increase in the learning outcome (de Koning et al. 2010). The perceptual attendance does not mean transfer of the meaning. This phenomenon is also confirmed in experiments in which researchers purposed to hint a solution of the task by modeled eye-movements of experts: the participants would follow an expert way of perception and would see the solution. But it did not help with logical or geometrical material (van Gog et al. 2009; van Marlen et al. 2016). The positive result was achieved in a classification task from zoology that was accompanied by verbal explanations (Jarodzka et al. 2013).

Why was the guidance of attention by some formal system not effective, especially in mathematical tasks? Why does it lead to the visits of the essential areas without understanding of the meaning? In the case of locomotion perception (Jarodzka et al. 2013), there was no special meaning that needed to be uncovered by special cultural perceptive actions (like the meaning of the points' coordinates, for example). It might be that perceptual strategies of meaning elicitation are individual and consequently an attraction of the attention by universal cues does not fit everybody's understanding process. Empirical evidence of individuality was brought by an eye-tracking study of the very first embodied abstractions on the way from natural (uneducated) perception towards theoretical perception in research on proportional reasoning. The recurrent patterns of eye-movements that emerged during the experiment and helped to solve the task varied from subject to subject within the same task (Abrahamson et al. 2016). The challenge for a teacher is to direct a student towards mathematical meaning, but at the same time to leave space for the student's own strategies in perception.

5.4 Research Questions

The aim of this research is to analyze the micro-mechanisms of transformation of uneducated perception towards a mathematical one, as this transformation is based on the unification of a few representations in an ongoing teaching-learning process. I am focused on joint attention as a phenomenon that hypothetically allows transferring of cultural meaning from an adult to a child.

What are the micro-mechanisms that allow multiple presentations to coincide in one objectification? How does visual joint attention appear and what is the role of joint attention in acquiring mathematical meaning of different presentations, namely, gestures, verbal explanations, and visual inscriptions? Does a child perceive a visual diagram culturally following the adult's guidance? Does an adult take into account the child's focus of attention in her guidance? The main idea is to analyze focuses and dynamics of joint attention by means of dual eye-tracking.

5.5 Methodology and Method

5.5.1 Dual Eye-Tracking Technology

In this study I am interested in the exploration of teaching and learning processes in multi-representational situations, thus I combined eye-tracking data with synchronized verbal protocols, audio waveform and video materials. This technology allows the researcher to keep all of the advantages of traditional detailed qualitative analysis of video records and audio waveforms of students' and teachers' behavior (e.g., Radford 2010; Roth 2008), with additional access to the students' and the teachers' perceptual processes.

In dealing with teaching and learning as shared activity, I am particularly interested in the occurrence of joint attention. For this reason I used dual eye-tracking technology. This technology includes two eye-trackers and allows the detection of the moments when gazes of communicative partners are crossed on the same object. The simplest way to conduct dual eye-tracking is to record eye-movements of two people while each one is sitting in front of her own monitor and some synchronized pictures are presented (e.g., Belenky et al. 2014; Guo and Feng 2013; Sharma et al. 2013). The advantage of this method is precise information about the coordinates of the gaze on the screen, thus researchers can directly overlap two gaze paths and calculate gaze cross-recurrence (Richardson and Dale 2005) or gaze similarity (Sharma et al. 2015), which are similar measures that report the ratio of time when subjects were looking at the same area. By means of such data, the importance of gaze overlapping for productive communication between partners has been shown (Belenky et al. 2014; Sharma et al. 2015). But my aim is to investigate the communication, which is principally asymmetrical, when one of the communicative partners holds an "ideal form" (Vygotsky 1934/2001) of cultural perception, while the other needs to transform his or her ways of perception. We need to understand how a teacher manages, by an embodied dialogue including her gestures, intonations, questions, and verbal explanations, to transfer to the student a new mathematical way of perceiving. In the case of two separated monitors the communication is limited to the verbal channel: participants are able to speak and listen to each other but there is no actual shared space akin to that in a real teaching process where joint attention is used. Gestures are not recorded and are not analyzed in this version of dual eye-tracking.

Another approach involves two eye-tracking glasses, which allow freedom of movement in the shared space (e.g., Yu and Smith 2016). The problem emerges at the level of data analysis: each eye-movement track is presented within a video scene involving the particular participant. The most promising way to resolve this problem is to recognize areas of interest in each participant's video scene and to converge the gazes in these areas of interest (Pfeiffer and Renner 2014). Nevertheless, this process still does not allow the superimposing of one gaze path on the other.

In this research I introduced two remote eye-trackers, which were calibrated on the same monitor, so that two participants were sitting shoulder to shoulder while discussing the same diagram. They also were pointing at the diagram by means of a thin stick, which allowed natural gesticulation almost without harming the eye-tracking process. I collected not only eye-movement data, but also video and audio records of student-teacher interactions. During the analysis I synchronized all data sources into one video stream, which consisted of a screen record, eye-movements of a teacher and a student placed on the screen record, and an external video. There are two main advantages of this technical solution, in which two participants sit in front of the same monitor: (1) they have a common physical space and can communicate not only verbally but also by gestures; and (2) the common monitor allows the production of video of two gazes on the screen for further qualitative analysis. The second advantage is especially important when we are investigating an objectification of new mathematical objects in a dynamic scene and consequently cannot distinguish areas of interest before the analysis. The novelty concerns both the data analysis and the technical solution for eye-tracking equipment.

5.5.2 Participants

Five pairs of a parent and a first grade child took part in the research. Despite the unprofessional character of the parents' teaching we suppose that it is the best approach that could be achieved in an eye-tracking laboratory, since we are investigating one-to-one teaching, which is as unusual for regular teachers as for ordinary parents. We suppose that exploration of general mechanisms of shared activity between a parent and a child is helpful in understanding the teaching and learning process.

5.5.3 Material

Cartesian coordinates were chosen as appropriate material to study the development of theoretical perception of the first grades. Children were already freely counting within 10 but had never met the Cartesian plane. As the experiment showed, each of them was able to solve the tasks with the parent's help, and by themselves after the teaching. So this task could be considered as belonging to the proximal zone of their development (Vygotsky 2004).

5.5.4 Procedure

The procedure included three stages: a record of the adults' perception, a record of the teaching-learning process, a record of the children's perception. Each stage included 10 tasks on a number line from 0 to 20 (interfering tasks) and some tasks on the Cartesian plane (main tasks). Only performance of the main tasks will be analyzed in the current chapter. In the first and third (testing) stages the tasks on Cartesian coordinates were as follows. There were 4 points (A, B, C, D) presented on a diagram of the Cartesian plane, while the coordinates of one of the points were given in the upper left corner of the screen. The query was to choose the appropriate point on the plane. During the second (teaching) stage the set of tasks was slightly different. The first task was the same as one of the tasks in the testing stages and it was not allowed for the adult to intervene in the child's process of problem solving. During the next 13 tasks the adult was free to teach the child. The first 9 tasks had only one point on the Cartesian plane and the child had to figure out its coordinates: in the first 4 tasks the point belonged to the first quadrant, while the next 5 tasks had points in other quadrants. The final 4 tasks had the same structure as the tasks in the testing stages with 4 points in each task.

5.5.5 Apparatus

For data collection I used an SMI RED eye tracker with sampling frequency of 120 Hz, with participants seated at approximately 40–50 cm distance from the monitor. The data analysis was conducted by Begaze 3.5. The SMI Observational Package was used for synchronized external video record and analysis. In the teaching stage an additional EyeTribe eye tracker with a sampling frequency of 30 Hz was involved in the dual eye-tracking procedure. Our own software was elaborated to produce video with paths of two eye-movements from two eye-trackers, which were accompanied by audio waveform.

Thus, my data included four semiotic registers that were involved in teaching-learning processes and presented in synchronized data of video records and both participants' eye-movements:

(1) a visual inscription of Cartesian coordinates;
(2) verbal communication between the adults and the children;
(3) pointing gestures;
(4) the verbal task that was given in the upper left corner.

5.6 Data Analysis

The design of the research is a mixture of a laboratory experiment and a teaching experiment, since I capture the natural communication in experimental teaching settings within the zone of proximal development. Mainly following the methodology of a Vygotskian semiotic approach (Radford and Sabena 2015), I focus on semiotic nodes–the segment of the teacher-student shared activity as it involves a system of different semiotic means. I unpack the dynamics of their appearance, in order to show how different semiotic components of activity influence the meanings generated by one another as they emerge in the objectification process. Additionally, in congruence with my research questions, I analyze eye-movements of a teacher and a child. Taking into account the possibility of peripheral vision and being aware of the limits of such operationalization, I consider eye-movements as operationalization of attentional focus, and overlapping eye-paths as evidence of visual joint attention. In my qualitative analysis it is never only overlapping gazes that indicate a joint attention moment, but congruence of different data sources: a moment was considered as joint attention when eye-movement data, video and audio records of gestures and conversations evidenced that participants are really engaged with the same region of the visual inscription. While the strict definition of joint attention requires both participants to be aware of the other's focus (Tomasello and Farrar 1986), I follow enactivist and phenomenological approaches to joint attention, in which participants may be engaged with one object without representing another person's mind, although they may be taking it into account as witnessed by their behavior (Gallagher 2011; Hutto 2011). The possibility of theoretical interpretation of joint attention from the point of view of my research questions, namely the issue of intersubjective sense making through multimodal shared activity, makes joint attention moments the *units* of the analysis (Vygotsky 2004), in accordance with general Vygotskian methodology where method and theory are joined in revealing a phenomenon as it emerges in a special situation of provoking development (Radford and Sabena 2015).

In the following, I describe a few short episodes from one of the child-adult pairs. This pair was selected as one with the most complicated communication, and with a serious number of misunderstandings. The choice is deliberate, since I suppose that the unprofessional character of the teacher and the communicative difficulties bring forward, and let us analyze, processes which are normally very quick and unarticulated. In each episode I focus on the following aspects, and then integrate them all together into one system in discussion that summarizes the issues.

(1) Types of semiotic inscriptions (namely a gesture, a verbal term or a visual inscription);

(2) An initiative of the child and the adult in the shared activity: who is guiding and who is following;
(3) Joint attention appearance: how it emerges and how it is related to the child's objectification;
(4) A structure of joint attention: are the participants aware of the focus of the other?
(5) An objectification process under the lens of the culture: does the shared activity lead to the transfer of cultural meaning or not?

5.7 Findings and Discussion

5.7.1 Episode 1. An Ambiguity of Visual and Gestural Presentations. Part 1

This first episode corresponds to the very beginning of the teaching; it is the first task, in a series after a few tasks on the number line, in which a child (Kosta) needed to find the number that corresponds to a point. From the start the adult passes the initiative to the child, letting him guess what these two lines and a dot might mean.

Adult: What are the coordinates of the point A? What do you think?
Child: Mm-mm, (Fig. 5.3[1]).

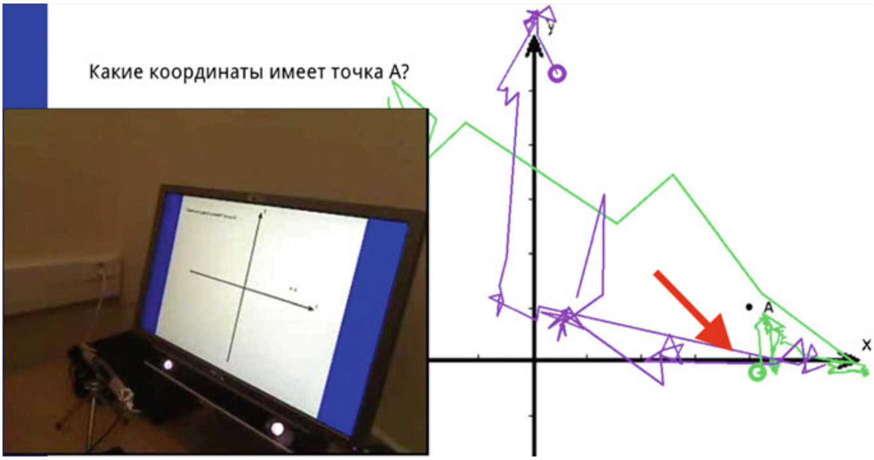

Fig. 5.3 The child determines the second coordinate by counting from the point A projection to the end of the X-axis's visible part

[1]These and following figures represent the paths of the eye-movements on the Cartesian coordinates (only the fragments are shown). The red arrows point at the positions of the gazes at these very moments. The green line represents the eye-movements of the child; the violet line represents the eye-movements of the adult. The pictures on the left show where the adult was pointing (the end of the pointing stick is marked by a red circle).

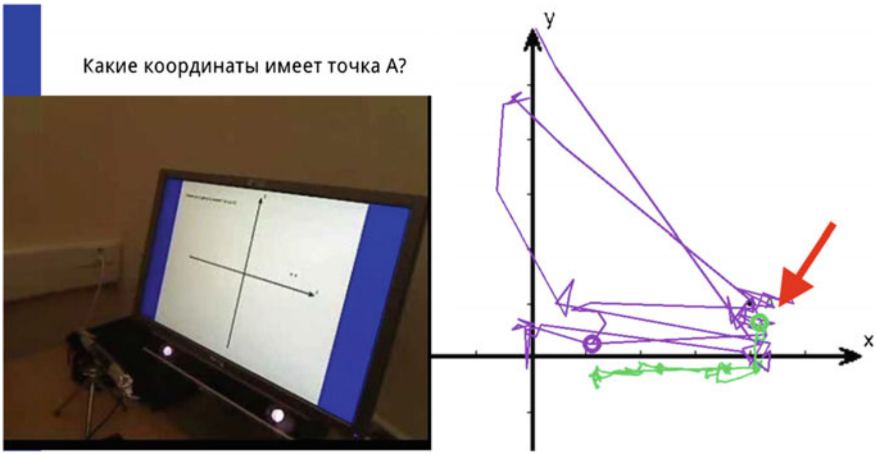

Fig. 5.4 The child finds the x-coordinate of the point A

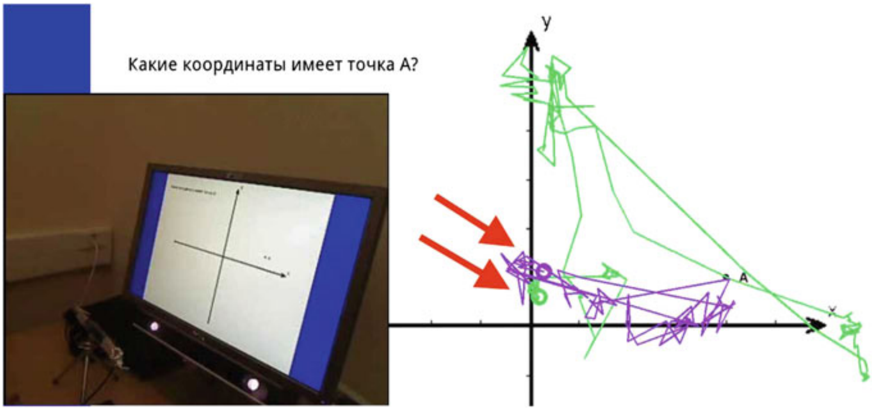

Fig. 5.5 The child doubts if he was supposed to use another strategy for the y-coordinate determination

Child: I think (Fig. 5.4) four… and two! (Fig. 5.5).

Adult: Why [is it] two? Do you count like this? Yes? Four? (*She counts as she makes bow-shaped movements, starting from the point* (*1*, *0*) *on the X-axis and following the dashes up to* (*4*, *0*), *and the child starts counting with her,* Fig. 5.6). One, two, …

Child: Aga [Yes].

Adult: three, four. And up.

Child: And up, (*immediately*) one. (*The adult moves the pointer to zero point and then she moves the pointer up, using an inaccurate* (*too long*) *gesture. The*

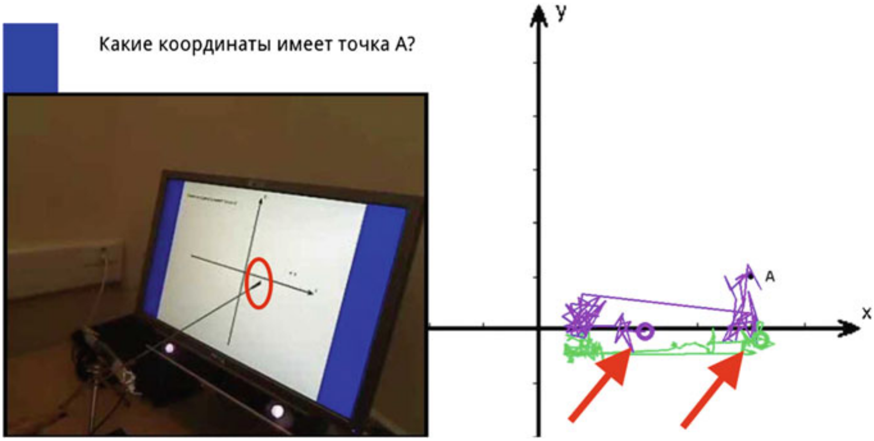

Fig. 5.6 The child and the adult are counting together along the X-axis

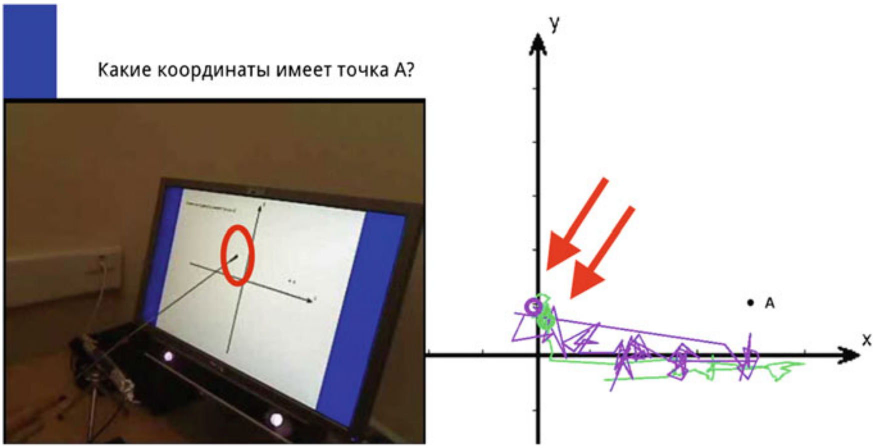

Fig. 5.7 The child and the adult are focused at the projection of the point A to Y-axis

child makes the same movement by his gaze, but he stops precisely at the point (0, 1), Fig. 5.7).

Adult 1: [Therefore] Four, one.

At first, the answer of the child seems to be mistaken to his mother since she expects the cultural answer (4, 1). Eye-movements of the child reveal that there was much reason under the wrong answer (Fig. 5.3). He tried to use another strategy than the cultural one: to count from the projection of the point to the end of the axis's visible part (to the arrow). At the same time he is obviously aware that there might be another strategy since he looks at the y-coordinate of the target point immediately after the answer (Fig. 5.5). (It needs to be noticed that Fig. 5.5 does not represent joint attention; both gazes appeared to be at the same place independently: the mother was waiting for a correct answer, while the child came there after his inadequate answer ‘two’.) So his answer is somehow a way to question: how should I do it, this or that way?

The adult takes initiative as soon as the wrong answer is given; she guides the child through the correct actions and shows how to count from the zero point. The ambiguity of the visual inscription, which creates a problem for the child, is unnoticeable for an adult who does not question the child’s answer and his focus of attention but just exposes the cultural strategy. The cultural strategy is presented by a gesture that is too long, and the word ‘up’, which could lead to misunderstanding, but being interconnected with a dashed structure of Y-axis and corresponding to the child’s expectations, it is understood correctly. While Jornet and Roth (2015) describe structuring and relating of presentations as articulated work during the learning, here we can see the same processes in about 300 ms of interaction. Moreover these activities cannot be separated and they influence each other: the gesture was structured in accordance with the dashes on the axis. The adult managed to guide the child’s attention and to form the joint attention, but we need to take into account, that this guidance was kindly understood in accordance with the idea that the child already had. The first task was successfully passed, but can we be sure that the child figured out the cultural way of perceiving the Cartesian coordinates? Let me continue with further material.

5.7.2 *Episode 1. An Ambiguity of Visual and Gestural Presentations. Part 2*

The target point of the next task is (2, 3).

Adult: What are the coordinates of the point A? Towards right. Count towards right (Fig. 5.8).

Child: Three.

Adult: (*pointing with the stick at the X-axis*) Two! Why [three]? One, two! (Fig. 5.9)

Child: Yes… But… One, two! But… (*he breifly looks back to the Y-axis from the projection of the point to the arrow*).

Adult: Then up!

Child: Two… and… a-a-a…

Adult: (*motion of the stick from zero point along the y-axis* (Fig. 5.10)) And? Child and adult (synchronously): Three… (Fig. 5.10).

Adult: So it is two and three.

From the very beginning the child has an idea that he needs to count from the point’s projection to the arrow, moreover, he chooses the y-axis despite the call to

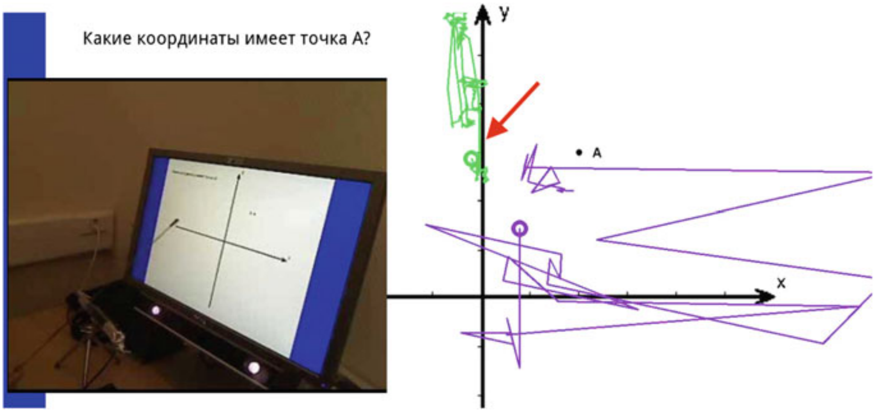

Fig. 5.8 The child determines the y-coordinate by counting from the point A projection to the end of the Y-axis's visible part

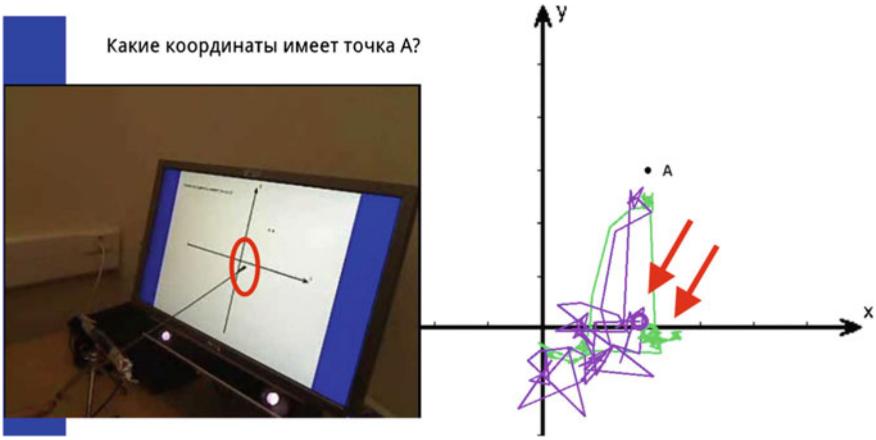

Fig. 5.9 The adult reminds that one needs to count along the X-axis for the first coordinate

"count towards right" that reveals the inefficiency of verbal guidance (Fig. 5.8). Thus in this case each participant has initiative and the child does not follow the directions of the adult while the adult does not question or pay attention to the source of the child's answers. Then the adult reinforces her guidance by gesture and the child follows her directions, thereby achieving joint attention (Fig. 5.9), and he comes to the idea that the x-coordinate needs to be counted first. Does it mean that a strategy of counting from the zero point is established? Nothing like this! The adult performs the next gesture–the long arc along the y-axis and the child indeed counts in a vertical direction, but he sticks to his initial misunderstanding of counting to the arrow and obviously interprets the gesture as a general direction to count along the

Fig. 5.10 The adult points along the Y-axis and the child counts from the point A projection to the arrow

y-axis, instead of counting from the zero point to the point's projection (Fig. 5.10). In comparison with the previous episode, we can see that the same structure of presentations (the visual inscription and approximate gesture) again leads to some interpretation; however, this time presentations are structured and related in a wrong (from cultural point of view) way, although again in accordance with the expectation of the child. This time joint attention was not achieved but, unfortunately, the answer was correct! (There were three intervals from both directions.) This coincidence of the adult's and the child's answers creates a mirage of joint attention, in which participants suppose that they look at the same object while looking at different areas, and, consequently, they have a mirage of understanding. In the next task the child again makes an attempt to count from the point's projection to the arrow, but, being interrupted by the adult, agrees to follow the cultural way.

Episode 1 shows the subtlety of guidance, since both the diagram and the gestures appear to be ambiguous and become meaningful only through congruence to each other. They form a semiotic node as they are involved in the shared activity of meaning disclosure or objectification (Radford et al. 2003; Radford and Sabena, 2015). It is important to notice that presentations acquire meaning only after understanding has happened and, while the diagram of the Cartesian coordinates is a traditional cultural semiotic means, the gestures unfold only in particular communication. The child's interpretation is the only act that makes the gestures meaningful in some intersubjective sense; it transforms them from *pre-semiotic* to *semiotic means* of objectification. The presence of these semiotic and pre-semiotic means in shared activity does not guarantee that any node and interpretation will appear. It is the need of the child (Leont'ev 1978; Roth and Radford 2011) to correlate his strategy with the cultural one, which orients him towards a search for interpretation of the adult's too long gestures in their connections with the diagram. Each time the interpretation corresponds to some anticipation of meaning on the part of the child, but while in part 1 of Episode 1 there was the culturally correct interpretation and the successful connection, in part 2 we can claim that there was just an occasional interpretation and only a mirage of understanding, despite the feeling of congruency by both the adult and the child. In the next episode I show a more developed and unfolded search for congruency, which finishes with a coincidence of different presentations that does not correspond to the cultural meaning.

5.7.3 Episode 2. Misunderstanding as an Active Coordination of Different Presentations

The next episode shows a deliberate search for connection between different presentations and a crucial role of joint attention in acquiring the cultural meaning.

The task had a target point (−4, 3) and it followed the task of Episode 1.

Adult: Let me remind you, there are minuses here. So it will be minus one, two, three… (Fig. 5.11) Minus.

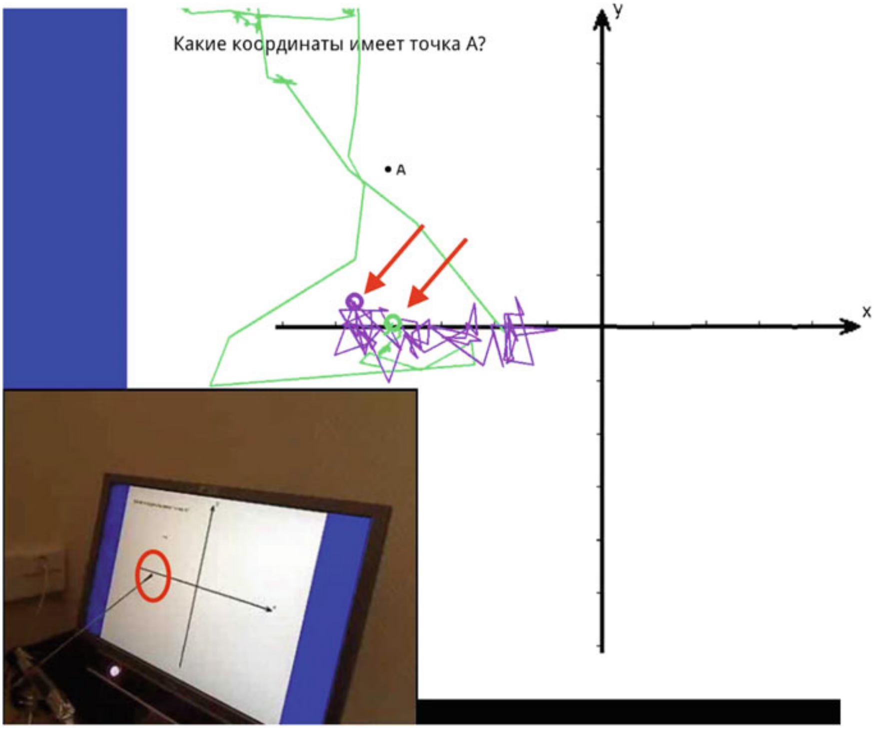

Fig. 5.11 The child follows the adult's gestures and counts along the X-axis. The result of counting contradicts the adult's verbal guidance

Child: Minus… (*He looks at the target point* (−4, 3), *then looks back to the X-axis, down to the Y-axis, which gave a negative sign in the previous task*).
Adult: It will be also by horizontal [axis] but with the sign ‘minus’. Minus… [how much?]
Child: Minus…
Adult: What is the number by horizontal [axis]?
Child: By horizontal… (*He looks again at the negative part of Y-axis and then somewhere else* (Fig. 5.12)).
Adult: The same way, you need to count. Where is the point? (*She moves the stick towards the point and back to the X-axis but the movement itself is directed along the vertical Y-axis* (Fig. 5.13). *It appears to be a decisive moment for the child: he determinedly counts along the Y-axis and finds the answer that he was searching for* (Fig. 5.14)).
Child: By horizontal we have three.
Adult: So… (*the adult misses the wrong answer*) and by vertical [axis]?
Child: Four. (*the child and the adult are focused on the different axes* (Fig. 5.15)).

Then the adult understands that there was a misunderstanding and explains everything from the beginning, this time she is guiding the child’s attention by gestures.

The explanations in this episode begin from the mother’s initiative and she guides the child’s attention on the visual inscription by both the gestures and verbal counting. The x-coordinate of the target point is four, but the adult counts up to

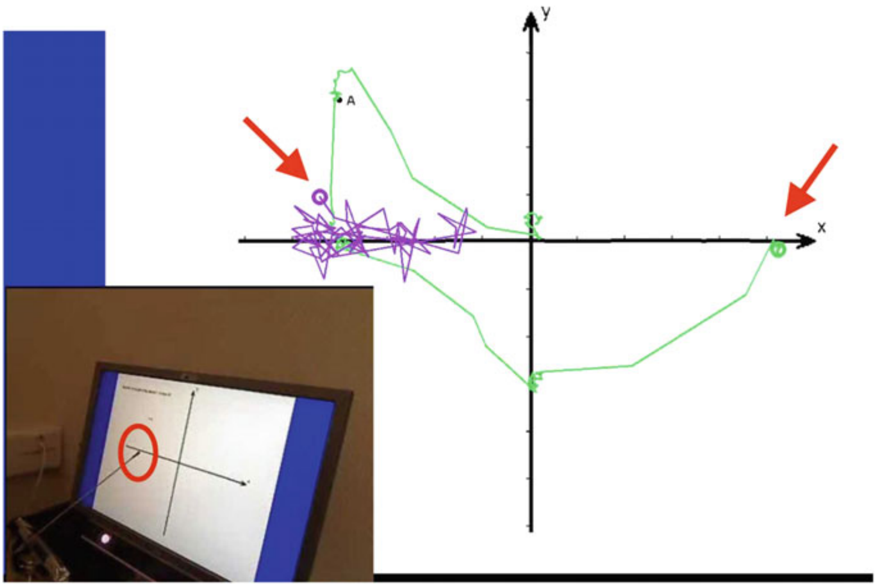

Fig. 5.12 The child searches for x-coordinate that would let him overcome the contradiction

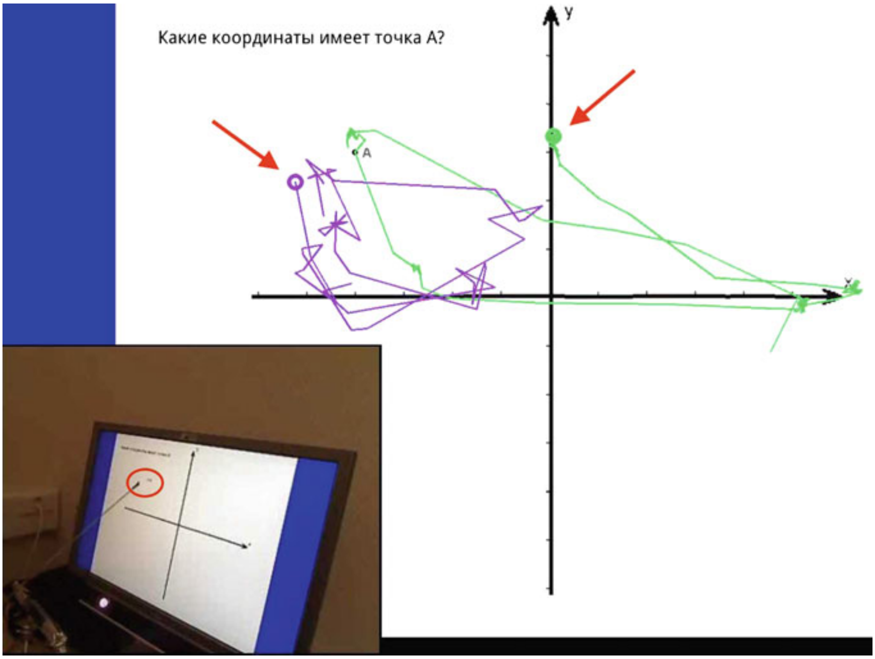

Fig. 5.13 The child misinterprets the adult's gesture

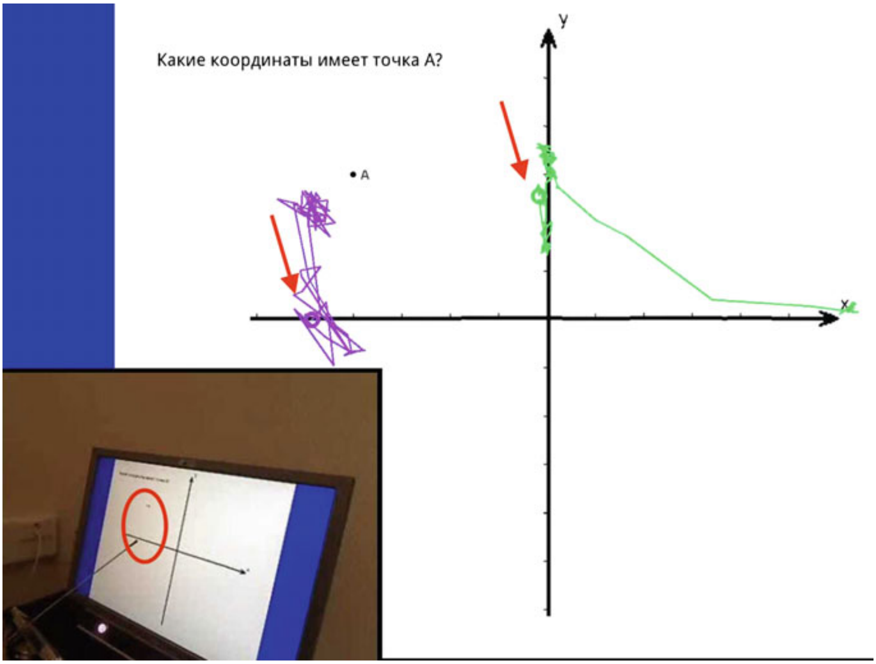

Fig. 5.14 The child counts along the wrong axis but finds an answer that is coherent with the previous verbal guidance

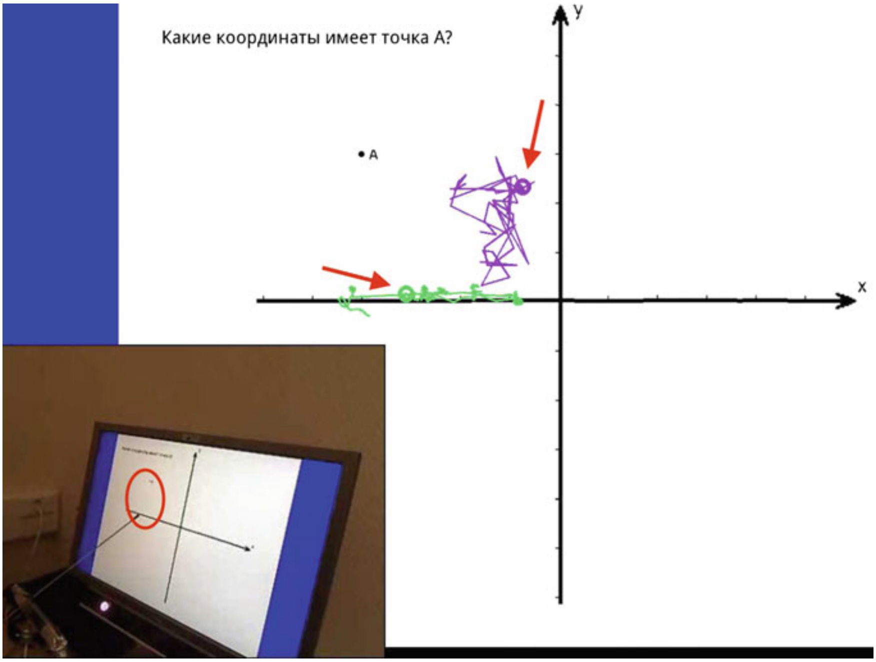

Fig. 5.15 The absence of joint attention helps the adult to distinguish misunderstanding between her and the child

three and leaves off counting; by an intonation she forces the child to continue on his own, thus she passes the initiative to the child. Unfortunately the child does not catch her intonation and the verbal counting sounds counter to the visual inscription and the pointing gesture, since the pointer is at number 4 (Fig. 5.11). The child starts to explore the whole scene trying to find a coinciding answer (Figs. 5.12 and 5.13). And suddenly a gesture of the adult helps him to resolve his confusion. The adult tries to show that he needs to count to the projection of the point on the X-axis and she makes the vertical move from the point's projection to the target point and then back. But Kosta interprets her gesture in a different way, which corresponds to the previous verbal directions ("One, two, three…"): verbal "three" corresponds to three intervals along the Y-axis. He starts to count along the Y-axis where the coordinate is exactly 3 (Fig. 5.14), and this interpretation leads him to balance all the inscriptions despite the fact that he comes to the wrong answer from the cultural point of view. At this moment the adult just does not notice the misunderstanding—she was giving such a full direction that it was hard to expect the wrong answer, and, despite the absence of joint attention, they move further in coordination. She directs him this time only verbally ("…by vertical [axis]?") and the child easily misinterprets this not very familiar term in accordance with his previous activity. Only the mismatch in verbal articulation of the second coordinate reveals for the adult the misunderstanding, which is corrected by the gestures, and the cultural answer is transferred to the child through the joint attention moment.

Thus, a misunderstanding is still a coordination of verbal, gestural and visual presentations for the child. Joint attention helps the adult to track whether the coordination of presentations by the child is adequate: supposing that they look at the same place (Fig. 5.15), she finds out that his answer does not correspond to what she is looking at, and, consequently, there is a misunderstanding.

The moments of joint attention in these two Episodes 1 and 2 are forced by the adult's explanations, which redirect the child from his own strategy, while the child kindly agrees to follow these explanations. In each case, the joint attention led to the culturally correct answer, although in some cases (Episode 1 part 2; Episode 2) there were misunderstandings, which were accompanied by the absence of joint attention when the coincidence of different presentations corresponded to the child's anticipations instead of the cultural perception of the adult. This tension between guided attention and anticipation of the answer shows the difficulty that is met in attempts to construct artificial guidance (e.g., de Koning et al. 2010; van Marlen et al. 2016), especially when we take into account that mathematical visual inscriptions always need some special work for their important features to become salient (Roth 2008).

5.7.4 Episode 3. Ambiguity of Verbal Expressions and a Following Strategy in Joint Attention

The next episode corresponds to the 5th task where the target point (2, −5) has a negative coordinate for the first time. At the beginning the child starts to read text of the task (which repeats from task to task without any change) and the mother stops him.

Adult: [Count] By horizontal. It is the same [task as before]. By horizontal!

Child: One, two, (Fig. 5.16, *The adult is waiting at the zero coordinate*), three… (Fig. 5.17, *the adult already starts an eye-movement*).

Adult: Two!

Child: Four, five (Fig. 5.18, *the adult has started the gesture*).

Adult: Kosta (*pseudonym*), here we have the point (Fig. 5.19, *directs the pointing stick at the target point*). Chiiik (Fig. 5.20).

This case is symptomatic of how verbal guidance can be inadequate and how helpful gestures are. The role of joint attention as a way to track misunderstanding by the child can be seen distinctly.

In this episode the adult tries to transfer the ruling agency to the child since she has shown the cultural strategy a few times in the previous tasks. She tries to do it by limitation of her guidance to verbal instructions instead of gestures, so she asks him to count "by horizontal". The child starts counting "One, two, three…". The correct answer is 2 (the x-coordinate goes first) and the adult recognizes that the

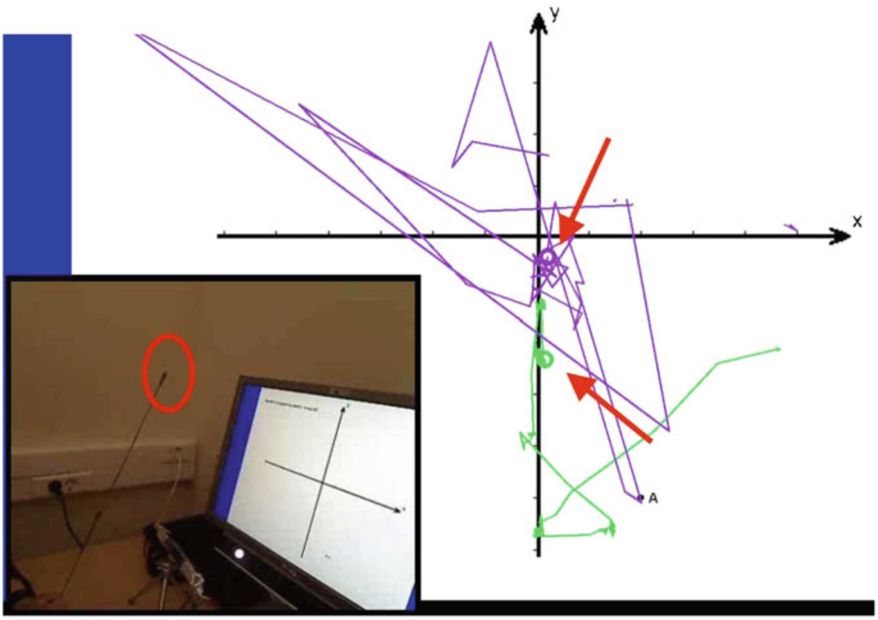

Fig. 5.16 The child starts counting along the Y-axis instead of the X-axis

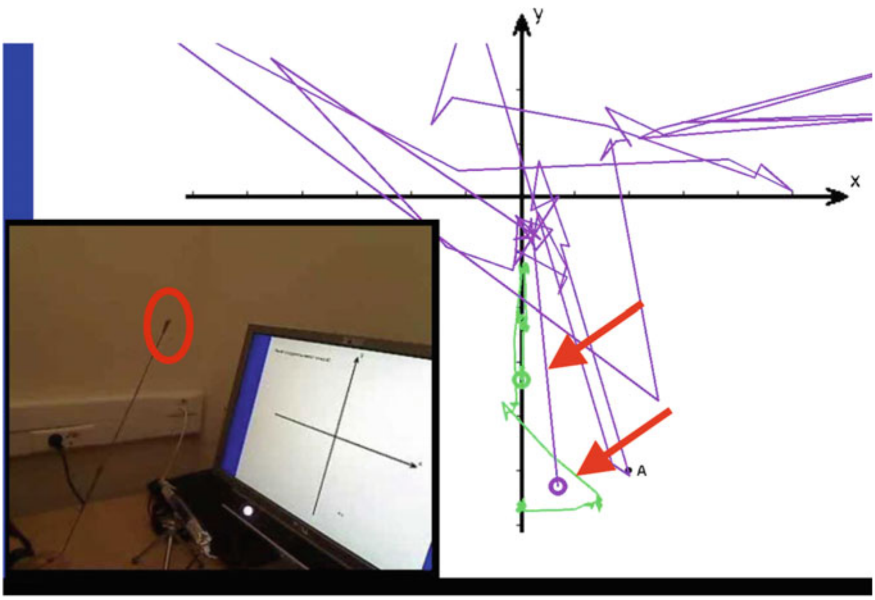

Fig. 5.17 The adult notices that the child mixed the axes at the moment when the child does not stop counting at 'two'

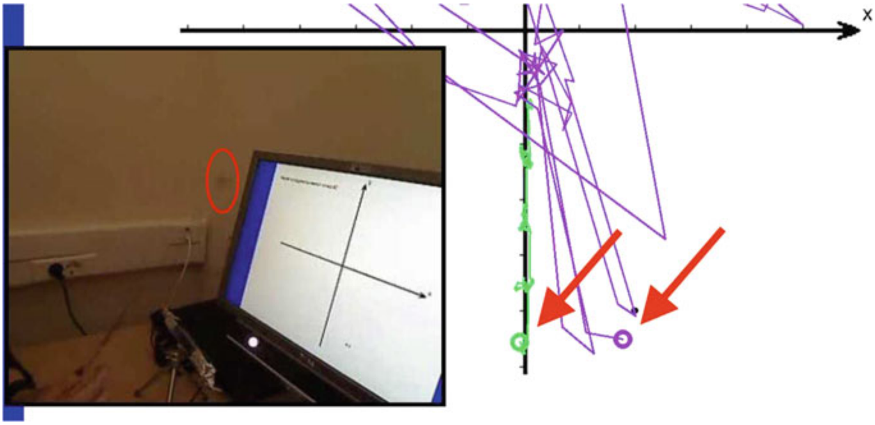

Fig. 5.18 The adult starts a gesture to correct the child while he is still counting

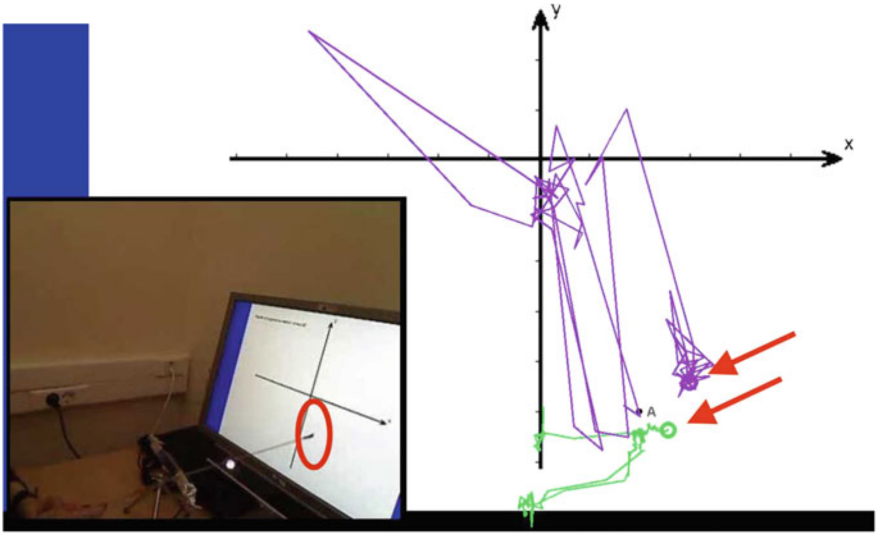

Fig. 5.19 The adult catches the child's attention and establishes joint attention

child is mistaken already between the words "two" and "three". She immediately looks along the Y-axis checking that she understood the child's mistake in a correct way. Then she establishes joint attention by pointing to the A and catching the attention of the child (Fig. 5.19) and then by a gesture shows him that he needs to count along the X-axis (Fig. 5.20). This time the meaning of her vertical gesture is different from the previous one: she does not point at the dashes on the axis, but

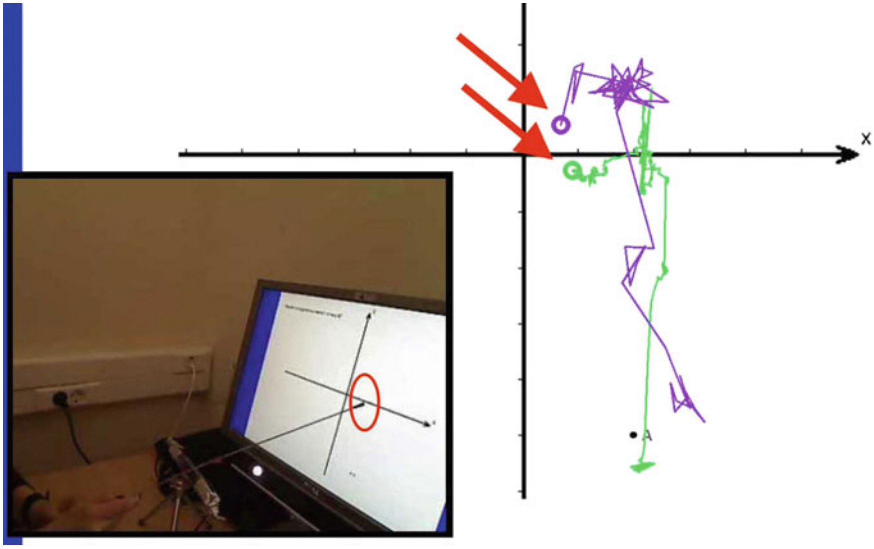

Fig. 5.20 The adult guides the child's attention from the point A to its projection on the X-axis

lines the projection from the target point to the axis. Now the vertical movement serves the need to count along the horizontal axis and we could expect a difficulty in interpretation of it. However, this is a rare case in this teaching session when the adult follows the attention of the child and adjusts her teaching according to it. By her brief gazes at the point and then to its projection and back to the point (Fig. 5.19) she finds a way of explaining that is best for the current focus of the child's attention, and she develops the communication taking the child's strategy and focus into account. Remarkably, as was predicted from the research on infants (Tomasello and Farrar 1986), this tactic appears to be quite successful despite the fact that the gesture has a different meaning in drawing projection. The child does not meet any problem in the interpretation of this gesture as it has developed from his focus and he immediately follows the adult, thus letting the joint attention and understanding emerge.

The time that was needed for an adult to distinguished the vertical counting of the child instead of the horizontal (about 250 mc), and then to elaborate the optimal strategy of explanation–before the child would finish counting–reveals the strongest embodied coordination between communicative partners, like the one that football players live through (Gallagher 2011), or it exemplifies the extended body (Froese and Fuchs 2012), as the adult adapted her intervention to the child's activity in the best possible way.

5.8 Concluding Remarks

As we have seen, different presentations supplement each other and clarify a way to structure themselves (Jornet and Roth 2015). For example the movement of gestures or lines of the Cartesian plane were structured by the child in accordance with each other. Thus, there was no meaning, no semiotic means for the child before a few presentations would merge in one node in order to be objectified as a meaningful piece of shared activity: the gesture was too long to be tied to a dash on the diagram; the diagram did not tell from where the counting should have been started. This node of *pre-semiotic* means does not emerge by chance for a child, but appears as an answer to the child's anticipation: the child counts along the Y-axis in Episode 1 part 1, Figs. 5.5 and 5.7, or counts towards the arrow of the axis in Episode 1 part 2, Figs. 5.8 and 5.9. In other words a new way of counting is an object/motive that fulfills a need for a new way of perceiving or for a solution of the task (Leont'ev 1978). Moreover, there is a strong intention to find this coordination between different presentations: in the case in which the presentations did not coincide, the child started an active search for a way to perceive them congruently (Episode 2, Fig. 5.12). Thus, objectification happens as unification of a few presentations, independently of the interrelation of emerging understanding to the cultural meaning. And then it is joint attention that helps to guide the new objectification into a cultural stream. In Episode 1, part 1 (Fig. 5.7) the joint attention appeared and cultural perception was acquired immediately. In Episode 1, part 2 (Fig. 5.10) on the contrary, there was no joint attention and no cultural perception, despite the illusion of understanding. In Episodes 2 (Fig. 5.15) and 3 (Fig. 5.17) it was the absence of joint attention that allowed the misunderstanding to be disclosed and it was joint attention that finally led to the establishment of cultural perception of Cartesian coordinates.

How does joint attention form? There are two participants who are actively engaged with an object in a joint attention situation, but their paths towards joint attention might be different. In Episode 1, part 1, and in Episode 2, the child was extensively guided towards joint attention and cultural perception. Then he gave the culturally correct answer. However, we have seen that verbal guidance [Episode 1, part 2 (Fig. 5.8) and Episode 3 (Fig. 5.16)] did not lead to the appropriate effect and needed to be supplemented by gesture. Moreover, even gestures and verbal guidance together could not guarantee that joint attention would appear, as it did not happen in Episode 1 part 2 and during Episode 2. Thus an adult may guide a child, but the child might anticipate something different, connect presentations into a different node than the adult would expect from her side of activity, miss the joint attention and acquire a different meaning. This phenomenon sheds light on the difficulties in artificial attention guidance (e.g., de Koning et al. 2010; van Marlen et al. 2016).

Another way for joint attention to emerge is when an adult follows a child instead of guiding him (see Episode 3). The adult appeared to be so sensitive to the child's answer that she could distinguish a mistake almost immediately and quickly find a way to approach this mistake and to form joint attention with the child. She followed the child's strategy and continued it; in this case there was no need to redirect the student's attention but rather to adjust it. In the episode in question, this strategy easily led to joint attention and understanding, which corresponds with the findings of Tomasello and Farrar (1986) that a following strategy is better than a redirecting strategy in the acquisition of words. This successful interaction was achieved so quickly and easily and the intervention of the adult was so clearly spatially incorporated into the way the child started to perceive the diagram, that I would claim it was immediate coordination of two participants in embodied dialogue through their extended bodies (Froese and Fuchs 2012).

Concluding, the system of representations does not exist independently of the learner and the teacher: any presentation (including a pointing gesture) is highly ambiguous and understanding appears as an active coordination of a few *presentations*. Nevertheless the coincidence might lead to misinterpretation, which is still based on the integration of the *presentations* by a child and his or her anticipation of the answer. The absence of joint attention, being noticed in time, is a way to confront the child's misunderstanding and the adult's anticipation of the correct answer, in order to establish joint attention and resolve this ambiguity of meanings. In the teaching situation, joint attention might be achieved by an adult's extensive guidance or by following a child's attention and her perceptual strategy. In the second case, the guidance might be embedded in the child's strategy; it seems this way might be preferable for successful communication and, possibly, for the learning outcome, although it needs further investigation.

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References

Abrahamson, D., Shayan, S., Bakker, A., & Van der Schaaf, M. F. (2016). Exposing Piaget's scheme: Empirical evidence for the ontogenesis of coordination in learning a mathematical concept. In C.-K. Looi, J. L. Polman, U. Cress, & P. Reimann (Eds.), *Transforming learning, empowering learners. Proceedings of the International Conference of the Learning Sciences* (*ICLS 2016*) (pp. 466–473). Singapore: International Society of the Learning Sciences.

Andrá, C., Arzarello, F., Ferrara, F., Holmqvist, K., Lindstroem, P., Robutti, O., & Sabena, C. (2009). How students read mathematical representations: An eye tracking study. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49–56). Thessaloniki, Greece: PME.

Andrá, C., Lindström, P., Arzarello, F., Holmqvist, K., Robutti, O., & Sabena, C. (2015). Reading mathematics representations: An eye-tracking study. *International Journal of Science and Mathematics Education, 13*(2), 237–259. http://doi.org/10.1007/s10763-013-9484-y.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1996). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*(3), 301–317. http://doi.org/10.1023/A:1002976729261.

Belenky, D., Ringenberg, M., Olsen, J., Aleven, V., & Rummel, N. (2014). Using dual eye-tracking to evaluate students' collaboration with an intelligent tutoring system for elementary-level fractions. In *CogSci 2014* (pp. 176–181). Quebec City, Canada.

Chesney, D. L., McNeil, N. M., Brockmole, J. R., & Kelley, K. (2013). An eye for relations: Eye-tracking indicates long-term negative effects of operational thinking on understanding of math equivalence. *Memory & Cognition, 41*(7), 1079–1095. http://doi.org/10.3758/s13421-013-0315-8.

de Koning, B. B., Tabbers, H. K., Rikers, R. M. J. P., & Paas, F. (2010). Attention guidance in learning from a complex animation: Seeing is understanding? *Learning and Instruction, 20*(2), 111–122. http://doi.org/10.1016/j.learninstruc.2009.02.010.

Duval, R. (2006, February 28). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*. Springer. http://doi.org/10.1007/s10649-006-0400-z.

Epelboim, J., & Suppes, P. (2001). A model of eye movements and visual working memory during problem solving in geometry. *Vision Research, 41*(12), 1561–1574. http://doi.org/10.1016/S0042-6989(00)00256-X.

Froese, T., & Fuchs, T. (2012). *The extended body: A case study in the neurophenomenology of social interaction*. *Phenomenology and the cognitive sciences* (Vol. 11). http://doi.org/10.1007/s11097-012-9254-2.

Gallagher, S. (2011). Interactive coordination in joint attention. In A. Seemann (Ed.), *Joint attention: New developments in psychology, philosophy of mind, and social neuroscience* (pp. 293–305). Boston, United States: MIT Press.

Gegenfurtner, A., Lehtinen, E., & Säljö, R. (2011). Expertise differences in the comprehension of visualizations: A meta-analysis of eye-tracking research in professional domains. *Educational Psychology Review, 23*(4), 523–552. http://doi.org/10.1007/s10648-011-9174-7.

Guo, J., & Feng, G. (2013). How eye gaze feedback changes parent-child joint attention in shared storybook reading? In *Eye gaze in intelligent user interfaces* (pp. 9–21). London: Springer. http://doi.org/10.1007/978-1-4471-4784-8_2.

Hutto, D. D. (2011). Elementary mind minding, enactivist-style. *Joint attention: New developments in psychology, philosophy of mind, and social neuroscience* (pp. 307–341). United States: MIT Press.

Jarodzka, H., Van Gog, T., Dorr, M., Scheiter, K., & Gerjets, P. (2013). Learning to see: Guiding students' attention via a model's eye movements fosters learning. *Learning and Instruction, 25,* 62–70. http://doi.org/10.1016/j.learninstruc.2012.11.004.

Jornet, A., & Roth, W.-M. (2015). The joint work of connecting multiple (re)presentations in science classrooms. *Science Education, 99*(2), 378–403. http://doi.org/10.1002/sce.21150.

Krichevets, A. N. (2014). Vygotsky and intersubjectivity. *Psychology in Russia: State of the Art*, *7* (3), 13–23. http://doi.org/10.11621/pir.2014.0302.

Krichevets, A., Shvarts, A., & Chumachenko, D. (2014). Perceptual action of novices and experts in operating visual representations of a mathematical concept. Psychology. Journal of Higher School of. *Economics, 11*(3), 55–78.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs, NJ: Prentice-Hall.

Peters, M. (2010). Parsing mathematical constructs: Results from a preliminary eye tracking study. In *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 30, pp. 47–52).

Pfeiffer, T., & Renner, P. (2014). EyeSee3D: A low-cost approach for analyzing mobile 3D eye tracking data using computer vision and augmented reality technology. In *Proceedings of the Symposium on Eye Tracking Research and Applications—ETRA'14* (pp. 195–202). New York, NY: ACM Press. http://doi.org/10.1145/2578153.2578183.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education* (pp. 215–234). Rotterdam: Sense.

Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics*, *30*(2), 2–7. Retrieved from http://www.jstor.org/stable/20749442.

Radford, L., & Sabena, C. (2015). The question of method in a Vygotskian semiotic approach. In *Approaches to qualitative research in mathematics education* (pp. 157–182). Utrecht: Springer. http://doi.org/10.1007/978-94-017-9181-6_7.

Radford, L., Demers, S., Guzmán, J., & Cerulli, M. (2003). Calculators, graphs, gestures, and the production meaning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *27 Conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 55–62). Honolulu, United States.

Richardson, D. C., & Dale, R. (2005). Looking to understand: the coupling between speakers' and listeners' eye movements and its relationship to discourse comprehension. *Cognitive Science, 29*(6), 1045–1060. http://doi.org/10.1207/s15516709cog0000_29.

Roth, W.-M. (2008). The dawning of signs in graph interpretation. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education* (pp. 83–102). Rotterdam: Sense.

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam: Sense.

Rouinfar, A., Agra, E., Larson, A. M., Rebello, N. S., & Loschky, L. C. (2014). Linking attentional processes and conceptual problem solving: Visual cues facilitate the automaticity of extracting relevant information from diagrams. *Frontiers in Psychology, 5,* 1094. http://doi.org/10.3389/fpsyg.2014.01094.

San Diego, J. P., Aczel, J., Hodgson, B., & Scanlon, E. (2006). "There's more than meets the eye": Analysing verbal protocols, gazes and sketches on external mathematical representations. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 17–24). Prague: PME.

Scheiter, K., & Eitel, A. (2010). The effects of signals on learning from text and diagrams: How looking at diagrams earlier and more frequently improves understanding (pp. 264–270). Berlin, Heidelberg: Springer. http://doi.org/10.1007/978-3-642-14600-8_26.

Seeger, F. (2008). Intentionality and sign. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 1–18). Rotterdam: Sense.

Sharma, K., Caballero, D., Verma, H., Jermann, P., & Dillenbourg, P. (2015). Looking AT versus looking THROUGH: A dual eye-tracking study in MOOC context. In *Computer Supported Collaborative Learning*. Retrieved from https://www.isls.org/cscl2015/papers/MC-0250-FullPaper-Sharma.pdf.

Sharma, K., Jermann, P., & Dillenbourg, P. (2014). "With-me-ness": A gaze-measure for students' attention in MOOCs. In *International conference of the learning sciences*. Retrieved from https://pdfs.semanticscholar.org/bf6d/bd6e7b1ccf2ea60cdf04eb437043154e81b2.pdf.

Sharma, K., Jermann, P., Nüssli, M.-A., & Dillenbourg, P. (2013). Understanding collaborative program comprehension: Interlacing gaze and dialogues. In *Computer Supported Collaborative Learning* (*CSCL 2013*). Madison, Wisconsin, USA.

Sierpinska, A. (1994). *Understanding in mathematics*. New York: Falmer.

Tomasello, M., & Farrar, M. J. (1986). Joint attention and early language. *Child Development, 57* (6), 1454. https://doi.org/10.2307/1130423.

van Gog, T., Jarodzka, H., Scheiter, K., Gerjets, P., & Paas, F. (2009). Attention guidance during example study via the model's eye movements. *Computers in Human Behavior, 25*(3), 785–791. https://doi.org/10.1016/j.chb.2009.02.007.

van Gog, T., & Scheiter, K. (2010). Eye tracking as a tool to study and enhance multimedia learning. *Learning and Instruction, 20*(2), 95–99. https://doi.org/10.1016/j.learninstruc.2009.02.009.

van Marlen, T., van Wermeskerken, M., Jarodzka, H., & van Gog, T. (2016). Showing a model's eye movements in examples does not improve learning of problem-solving tasks. *Computers in Human Behavior, 65,* 448–459. https://doi.org/10.1016/j.chb.2016.08.041.

Vygotsky, L. S. (2004). *The essential Vygotsky*. New York, NY: Kluwer Academic, Plenum Publishers.

Vygotsky, L. S. (2001). *Lektsii po pedologii [Lectures on pedology]*. Izhevsk: Izdatel'kii dom Udmurtskii universitet. (Original work published 1934).

Yu, C., & Smith, L. B. (2016). The social origins of sustained attention in one-year-old human infants. *Current Biology: CB, 26*(9), 1235–1240. https://doi.org/10.1016/j.cub.2016.03.026.

Chapter 6
Attention Catching: Connection the Space of Joint Action and Togethering

Debbie Stott

Abstract In the context of after school mathematics clubs in South Africa, in this chapter I seek to gain a firmer and deeper understanding of the theoretical concepts of 'space of joint action' and 'togethering'. I do this by connecting the notions of Meira and Lerman's attention catching and Radford's moments of poēsis, which are used as a combined lens to analyse data from two task-based interviews with 9 to 10-year old club learners. Using examples and non-examples, I analysed sustained sequences of attention catching as observed when participants paid attention to each other in a mathematical manner to enhance their understanding or sense making. I argue that the way in which participants take advantage of these sequences has a bearing on the way in which the space of joint action evolves and how togethering unfolds. The findings from this chapter contribute to calls for understanding the special work of mathematics teaching and may be pertinent in both classroom and out-of-school time contexts in South Africa and beyond.

Keywords Attention catching · Space of joint action · Togethering
Special work of mathematical teaching · Moments of poēsis

6.1 Introduction and Context

In her plenary presentation at the 13th International Congress on Mathematical Education, Ball (2016) asked, "What is the special work of mathematics teaching?". She argued that there is "something to the mathematically interactive, discursive, and performative work of mathematics teaching that is important to understand" (p. 24). She concluded by urging the mathematics education community to look at *how* mathematical listening, speaking, interacting, acting, fluency, and doing are part of the work of teaching, and to hear, see, and read students, in "real time."

D. Stott (✉)
South African Numeracy Chair Project, Room 34, Education Department,
Rhodes University, PO Box 94, Grahamstown 6140, South Africa
e-mail: d.stott@ru.ac.za

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If we are to understand how to hear, see and read students in real time, as Ball indicates, there is a need to examine every aspect of the mathematical activity from interaction, discussion and resources. One way to do this is to examine mathematical activity using the idea of semiotic resources that are made up of speech, hesitations, intonations, gestures, postures, touching and eye contact and so on. Globally, much work has recently been done on expanding the emerging field of semiotics in mathematics education (see Presmeg et al. 2016 for a useful summary).

In the South African context, multimodal approaches to research and pedagogy are increasing as an academic area of study. Archer and Newfield (2014) point out that with the diversity of language, culture, and ethnicity that characterises South African educational contexts, a multimodal approach potentially provides a beneficial alternative to "monolingual and logocentric approaches to meaning-making" (p. 3). Stein and Newfield (2006) argue that in the classroom, a multimodal pedagogy can inform a social justice and equity agenda and has the potential to make classrooms "more democratic, inclusive spaces in which marginalised students' histories, identities, cultures, languages and discourses can be made visible" (p. 11). I note a lack of research in the South African context with regard to multimodality pedagogies in mathematics education, particularly in the field of primary mathematics. This chapter sits in this broader landscape.

Since 2012 I have facilitated many after school mathematics clubs (as examples of out-of-school time programmes) with children aged between 9 and 12 years old. This chapter emerges from my longitudinal doctoral study within these clubs, which I both facilitated and researched. Two such clubs formed the empirical field for this chapter and were run within the context of the South African Numeracy Chair (SANC) Project at Rhodes University. This project focuses on a dialectical relationship between research and development in the field of numeracy education in South Africa. A key strand of the project's development work is these after school mathematics clubs. Graven (2011), at the start of her work as the Chair, argued that after school mathematics clubs hold the potential to address some of the challenges young numeracy learners face. The ongoing project clubs are conceptualised as communities where sense making, active mathematical engagement and participation, and mathematical confidence building are foregrounded. Activities in the clubs are designed for both the recovery and extension of learners (Stott and Graven 2013).

With this South African context in mind, this chapter is a response to Ball's call, specifically regarding hearing, seeing and reading students in real time, and how this may inform facilitation within such clubs and broader out-of-school time programmes. To analyse how this occurs during two paired task-based interviews with 9 and 10 year olds in after school clubs in South Africa, this chapter draws on the socio-cultural constructs of the *space of joint action*, *togethering, attention catching* and *moments of poēsis*. Focusing on a single activity from the task-based interviews, this chapter endeavours to connect the notions of attention catching and moments of poēsis as a way to make marginalised students' histories, identities, cultures, languages and discourses visible and to illuminate and gain a deeper understanding of how the space of joint action evolves and what it means for facilitators with regard to togethering in the context of these clubs and other out-of-school (OST) spaces.

More precisely, I investigate (1) if the space of joint action evolves by bringing forth something (sensible mathematics) to the realm of attention (moments of poēsis) and (2) how the participants take advantage of the moments when attention is caught, and how they subsequently respond, in order to sustain the mathematical engagement, understanding or sense making. I also consider the usefulness of space of joint action and togethering constructs as narrative tools in the presentation of data.

6.2 Space of Joint Action and Togethering

Drawing on Leont'ev's and Vygotsky's work in connection with knowledge objectification and consciousness, Radford and Roth (2011) present two concepts called the *space of joint action* and *togethering*. I use these concepts as narrative tools in the presentation of data for this chapter.

Radford and Roth define the *space of joint action* as a *single* teaching and learning space where interaction between teachers and students is based on an "evolving, tuning, and reciprocating of the participants' perspectives, making thinking a collective phenomenon" (p. 232). Through a "complex sensuous evolving coordination and tuning of speech, gestures, gaze, and actions, interaction in a space of joint action" (p. 235) can unfold. In and through joint action and sense-making, "passages" (Radford 2015, p. 560) will occur in the teacher-student interactions where students become progressively aware of the cultural-historical or "ideal forms" of mathematical knowledge. Joint activity is thus an opportunity for producing and ascertaining that the "ideal form" is reflected similarly in the consciousness of all participants. However, Radford and Roth (2011) caution that the evolution of a space of joint action does not guarantee a successful outcome in terms of the participants becoming aware of the ideal form. This is where the construct of *togethering* comes into play.

Drawing on the Russian term *obuchenie*, taken as meaning both teaching and learning, Radford and Roth point to the importance of the teacher in joint activity using the concept of *togethering*. Not to be taken as simply getting together to do something, the notion of togethering has two key aspects. The first is an ethical commitment on behalf of the participants to engage in and produce activity. Secondly, the purpose of that activity is for the teacher to bring the "ideal forms" of mathematical knowledge into activity by working together with the students, because the teacher cannot "inject such a structure into the student's consciousness" (p. 240), nor can she set knowledge in motion by herself. The purpose of the ideal object is seen differently by the teacher and the students and may not be clear to the students at the beginning. The result is that for students, joint action may bring forth the "ideal form", whilst joint action for the teacher involves constantly repositioning the "ideal form" so that it becomes an object of consciousness for the students.

Theoretically, these two constructs make sense. However, in several articles, Radford (2014, 2015) has stressed that the real focus needs to be on understanding *how* students become progressively aware of the culturally and historically constituted forms of thinking. In looking at how, this process becomes complex. As Roth and Radford (2011) point out in quoting Livingstone: “none of the participants in mathematical activity can see any hidden contents of the minds of others. What they act upon and react to is what the respective other makes available to them” (p. 25). So, how does one look for what is made available by the participants from an analytical perspective? How is the complex coordination of multimodal forms analytically observed in video data? How are they made observable? This being my concern, I turned to a construct I worked with in my doctoral study, Meira and Lerman’s (2009) notion of *attention catching*.

6.3 Catching Attention and Poēsis

Meira and Lerman (2009) proposed that the emergence of a Zone of Proximal Development (ZPD) is dependent on whether the participants in a learning activity catch the attention of the other(s). They argued that when someone or something captures a student’s attention their subsequent utterances and actions could be modified, creating a scenario where a ZPD might emerge. In my doctoral work, I developed this idea to some extent to help make the emergence of a ZPD observable. Although Meira and Lerman worked with this notion in connection with the ZPD, in this chapter my aim is to connect it to the concepts of the space of joint action and togethering.

Immediately one is struck by how this attention catching may differ from everyday attention catching. The wind blows, the phone rings, something moves–it catches our attention, but it does not mean that we are making meaning in mathematics. Similarly, in educational contexts, events such as a child shouting or crying, clicking his or her fingers, or dropping a book or a pencil, may not be related to mathematical sense making. The nature of attention catching proposed by Meira and Lerman is very specific. It focuses on the *moment* when attention is caught during a mathematical activity involving more than one participant. If attention is caught in the way they suggest, the expectation is that something will be said or done next that continually works towards mathematical communication or ‘catching each other’s thoughts’. Meira and Lerman indicate that attention catching directly affects what is subsequently said or done. More specifically, attention is caught when a participant explains his or her reasoning or talks about mathematical ideas. In a private conversation with Lerman,[1] he spoke further about how, once such attention is caught, it may or may not be sustained. The idea of sustained

[1]Steve Lerman was an advisor for my doctoral studies (2012–2014).

attention catching is an important one for this chapter, as it is a focus point for the analysis process discussed later.

Attention catching can be initiated by any participant using any semiotic resource; a child may catch the attention of an adult by asking a question, or an adult might say something to a child to catch his or her attention, to orient the child "towards what she wants the child to learn" (Lerman 2001, p. 8). Thus, in Lerman's view attention catching gives clues about the nature of communication between participants in a learning activity and allows us as researchers to make sense of this communication.

In using the notion of attention catching, I questioned how the participants in a learning activity take *advantage* of the moments when attention is caught and how they respond as a result. This brought me to Radford's (2015) "*moments of poēsis*" construct which provides some clarity as to how this may happen. Radford states that the process of learning (objectification) entails "moments of poēsis", which bring "forth something to the realm of attention and understanding" (p. 551). Poēsis, Radford (2010) claims, "is a creative moment of disclosure—the event of thing in consciousness" (p. 3). Poetic moments result from the complex connections between semiotic means of objectification that accompany and orient the students' "perceptual, aural, linguistic and imaginative activity" (p. 6). At the time (2010), he referred to these poetic moments in connection with Vygotsky's (1978) ZPD. However, his more recent work frames these moments in the theory of objectification within spaces of joint action and togethering (Radford 2015).

Poēsis, which originates from the Ancient Greek *poíēsis*, meaning production or composition, can be thought of as the process of making, producing or creating. Radford's poēsis highlights the complex notion of creativity, one that has its own domain in mathematics education research. Creativity for me is the generation of new and original ideas, but Radford mentions creativity as an 'event of the thing in consciousness'. Leikin and Pitta-Pantazi (2013) point to a distinction between relative and absolute personal creativity in school students, which is useful here. Relative creativity refers to a student's own educational history and comparison with other students and is the ability to "produce mathematical ideas/solutions in a new situation (to a new mathematical problem that was not learned previously) or to produce original solutions to previously learned problems" (p. 161). Absolute creativity, by contrast, is often evaluated by the professional community that regards it as a meaningful creation from an historical perspective. Thus, I infer that when Radford refers to moments of poēsis, he is perhaps considering a *personal* creative process, where students, in the moment, create mathematically meaningful new ideas, methods, interpretations, and so on for themselves.

The similarities between the notions of attention catching and moments of poēsis are noticeable, particularly regarding bringing something to the realm of attention and understanding. Both notions are focused on learning activity in mathematics classrooms; both talk of attention and understanding and both were originally conceived in connection with the Vygotskian ZPD. However, the definition of attention catching provided above mentions actions but does not specifically state perceptual, aural, linguistic and imaginative activity, whereas poēsis does. Perhaps, more interestingly, poēsis encompasses the idea of personal creativity, where something is brought into consciousness. Based on these similarities, I hypothesise that it is possible to link these two notions to create an analytical lens that will enable examination of *how* students become progressively aware of the culturally and historically constituted forms of thinking in the space of joint action.

In this chapter, I use "sensuous multimodal forms" (Radford and Roth 2011, p. 231) specifically to pay attention to and understand the interactions between participants in the space of joint action, when mathematical ideas and reasoning are communicated to each other. In doing so, I aim to clarify and connect the notions of attention catching and poēsis and hope to unpack how togethering takes place in the context of the after school mathematics clubs. I approach this by identifying places in the video data excerpts where sensuous multimodal forms such as hesitations, intonations, gestures, postures, touching and eye contact connect to *sustained* instances where the utterances or actions of one participant affected the other, which perhaps result in a modification of what was said or done as suggested by Meira and Lerman (2009). I elaborate on this analytical process and on the idea of sustained attention catching in the following section.

6.4 Methodological and Analytical Approach

The data presented in this chapter were drawn from two video-recorded task-based interviews with club learners (aged 9–10 years) who attended South African state schools and who spoke English as a second or third language. As both club facilitator and researcher, I conducted the two interviews reported in this chapter, each of which involved two pairs of female learners. Each interview was about an hour long and consisted of a range of activities. In the remainder of this chapter, I use the term *facilitator* to refer to myself. The unit of analysis is sustained sequences/moments of attention catching/poēsis as observed through the sensuous actions of the participants in the space of joint action during a learning activity.

The puzzle in Fig. 6.1 below was one of three activities used in the task-based interviews. Learners working in pairs, were asked to find the values that related to each shape to ensure that the row and column numbers would stand true. The 'ideal form' of this activity is to reason deductively and think algebraically. Ideally, by working through the puzzle, the club learners would begin to understand the need to work from the general to the specific, narrowing the range of options until they arrive at a conclusion that will ensure that the numbers they assign to the shapes will work for all the instances in the puzzle.

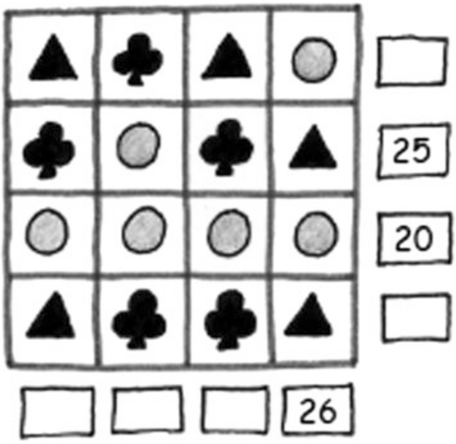


Fig. 6.1 Task-based interview activity

6.4.1 Connecting Attention Catching and "Moments of Poēsis" in the Data

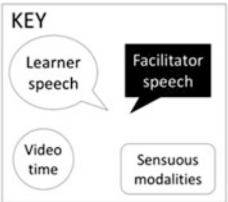
As noted, my focus was how to connect and operationalise attention catching and moments of poēsis for the purposes of analysing how the space of joint action evolves and what togethering might look like. Using an interpretive analysis approach, the transcript was initially populated with *verbal utterances* and then coded for the range of *multimodal forms* noted earlier, which connect to the perceptual, aural, linguistic and imaginative activity in moments of poēsis.

The final analytical step involved studying the video for attention catching and "moments of poēsis". In doing this I searched for places where there were *sustained* sequences of interaction. When participants engage with each other in a meaningful way, i.e., pay attention to each other in a mathematical manner to enhance their understanding/sense making or to progressively bring an idea or concept forth into consciousness, several sequential attention catching moments may occur. This can be considered as a sustained sequence of attention catching moments. The video data revealed examples when learners did not pay attention to each other, except for superficial glancing, where collaboration did not occur and where sharing of mathematical ideas did not happen. These, I argue would not be considered as sustained attention catching moments.

Once examples and non-examples of sustained attention catching were located, I analysed the sensuous multimodal forms to observe what, if anything, was coming into the realm of attention and more specifically for creative moments, where the learners created meaningful new ideas, methods, interpretations, and so on for themselves. Additionally, if utterances or other multimodal actions of one participant affected the other(s), perhaps resulting in a modification of what was said or done as suggested by Meira and Lerman (2009), this was shown as an attention catching moment or moment of poēsis. In this way, I brought together the notions of attention catching and moments of poēsis.

6.5 Data and Analysis

A ‘comic strip’, frame-by-frame representational format (derived from Plowman and Stephen 2008) is used below to present vignettes from the two interviews. A sequence of images is accompanied by the time it occurs in the video, learner and facilitator speech and sensuous actions (gestures, eye contact, movement, body posture, writing, and so on). The key below indicates the symbols used in the format. In the descriptions of sensuous modalities, “DAS” refers to myself as facilitator.



The first interview presents examples of sustained attention catching and personal creative moments, where it gradually comes to one learner’s attention (she becomes aware) that she can solve the puzzle using mathematics that she already knows. The second interview presents what I consider to be a counter-example. That is, there is no sustained attention catching and there is an absence of personal creative moments.

6.5.1 Interview 1: Examples of Sustained Attention Catching Moments

As we enter the story, Anathi and Thembela have already spent about 8 minutes working out the values for the circle (5) and the triangle (8). I have already explained that all the clubs would be the same value. Both girls were looking at the second row which has a circle (5), a triangle (8), two clubs (value unknown) and totals 25. They know that the circle and triangle have a total value of 13 and need to find the value of the two clubs to get to 25. Anathi breathed in and said to me: “Ooooh. Eish, hey, hey. You see what I did?” My attention was caught immediately.

She explained what she did (45:01), tapped the chart, counted on her fingers and finished off with an excited “Aaah!”. I was further attracted by her excitement and this determined what I said next. I asked her a question (45:14), introducing the

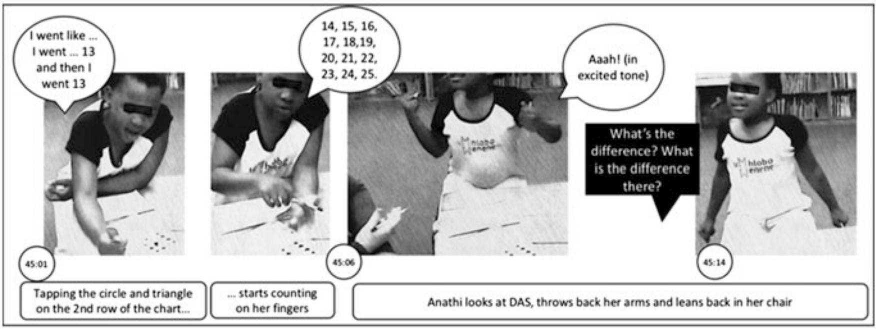


Fig. 6.2 Interview 1 (part one)

mathematical term of "difference". She reacted to this by looking at me, throwing back her arms and leaning back in her chair (Fig. 6.2).

Anathi did not verbally respond to my question (Fig. 6.3). Instead, she looked at the chart and covered her mouth with her hand (45:16). I asked the difference question again. She sat forward in her chair and wiped her face with her hands. I encouraged her to count again (45:21).

Her posture, hand gestures and gaze suggest some discomfort on her part, perhaps she did not know what I meant by 'difference'. By asking her to count again I encouraged her to continue trying. This is an example of the two of us working together in a space of joint action. I understood the term 'difference' but perhaps she did not. We would require togethering on my part and further joint activity to reconcile those differences. However, I claim that something had been brought into her realm of attention; attention has been caught through sustained instances of interaction.

The joint activity had become a space where Anathi and I exposed our ways of thinking through our speech and other multimodal forms. What is subsequently said in this space is adjusted so that each participant can understand the other, as is illustrated below. In other words, what is said and done is modified by *attention catching*.

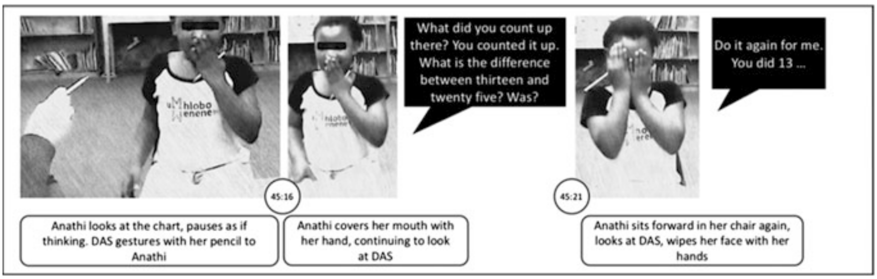


Fig. 6.3 Interview 1 (part two)

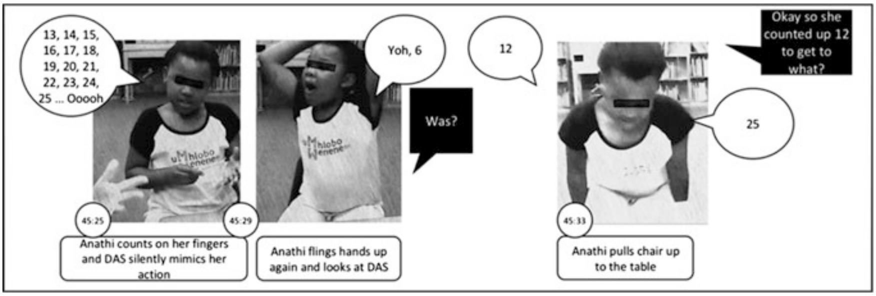


Fig. 6.4 Interview 1 (part three)

The sustained interaction continued here (Fig. 6.4) when I silently mimicked her actions as she re-counted “13, 14 … 25”. Then she said “Ooooh” (45:25), flung up her hands, looked at me and said “Yoh, 6” (45:29). I was not sure what she meant by this so I clarified with “was? She said “12” as she pulled her chair up to the table. Again, it was unclear what she meant by ‘12’, so I rephrased: “Okay so she counted up 12 to get to what?” (45:33). Her response was “25”.

Continued interaction signifies that Anathi and I were working (in a space of joint action) to understand each other. When I said something, it affected what Anathi and Thembela did next and ensured that mathematical communication was maintained. Similarly, Anathi’s sensuous actions modified what I said or did next.

Anathi started to write on her paper (Fig. 6.5), but then paused and looked directly at me (45:38). I asked “Okay, now so what do you think the club is?” She immediately answered “6”. Thembela, who had up to this point, been leaning back in her chair and apparently following our interaction, leant into the desk, which signified for me, her re-engagement and wrote “club = 6” on her paper. Both girls said “the club is 6”. I asked “why” (45:40). Anathi danced in her chair, clapped her hands and tapped the chart with confidence and then said “Because half of the twelve is 6. Oooh”.

Anathi made an expansive hand gesture (Fig. 6.6) as she gave her explanation about halving (45:45). I pointed to the club on the chart and in a rising intonation,

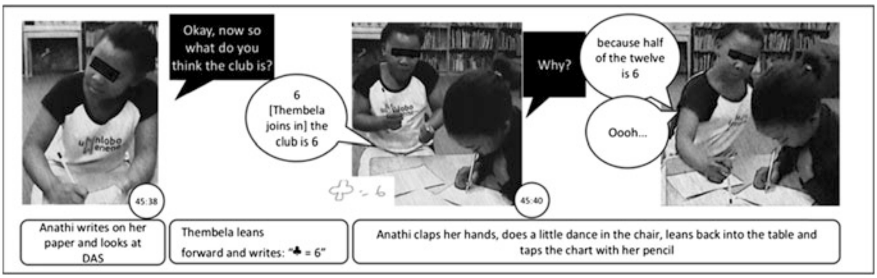


Fig. 6.5 Interview 1 (part four)

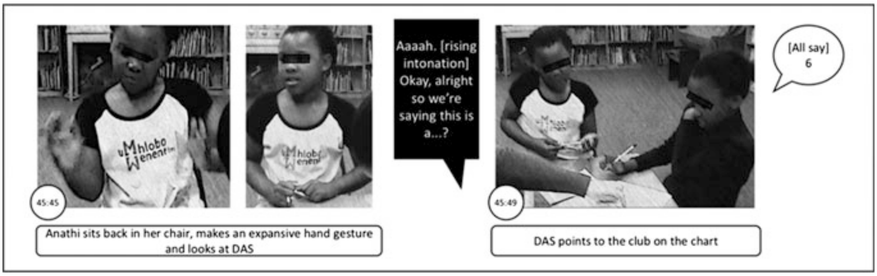


Fig. 6.6 Interview 1 (part five)

asked the girls to confirm the value of the club (45:49). They both said “6”. In the last part of the interview (not shown in the sequence above) I asked the girls to confirm if this value of six would work in another place on the puzzle and together they worked on completing this last piece of the puzzle.

6.5.1.1 Interview 1: Discussion

The combination of the different kinds of sensuous actions brought into being by Anathi acted as attention catching moments in various ways. Firstly, *my* attention was caught by these actions, which resulted in my responding in a mediational manner by asking questions and mimicking actions. Secondly, they portrayed Anathi’s emotions as shown in her sense of excitement as she worked through her thinking—it is revealed not only by what she says but in her whole being. Her postural actions (leaning back in the chair, flinging arms in the air), gestures and making eye contact with me, caught my attention, and togethering continued to take place between the two of us even though verbal utterances were not always well-defined.

I argue that through this complex sequence of sensuous actions between myself and Anathi something is progressively brought to Anathi’s realm of attention and understanding. Our actions and utterances were modified as a result of those moments. I submit that this excerpt exemplifies an example of where a space of joint action evolved, where we worked together so that Anathi became progressively aware of the mathematics in this activity. She may not have realised the ideal form, but she went some way to drawing on her own mathematical knowledge to make sense of the puzzle.

Furthermore, I argue, that in the final moments (see Figs. 6.5 and 6.6), Anathi realised that the 12 needed to be divided into two equal parts to compensate for the two clubs in the second row, i.e., she had to halve the 12. Her entire being was invested in this realisation of a mathematical structure (half of 12) and finally making sense of the value for the club, as evidenced by her dance and other sensuous actions. Anathi’s enactment of the concept of halving, using a hand gesture, suggests that the concept may be actualised for her through this multimodal activity. As Radford (2014) argues, the “sensual and the conceptual become entangled” (p. 354) in that moment.

This was a *moment of poēsis* for Anathi—it was her creative moment of discovery. She was drawing on knowledge that she already had and she was possibly making sense of it in this context. She had created something meaningful for herself. Although she would not be able to verbalise it as such, in this moment, she was perhaps just beginning to understand what it means to use existing mathematical knowledge to reason deductively, in a culturally and historically constituted way.

This episode is significant as it illustrates how through the sustained nature of the attention catching moments, Anathi and I took advantage of these moments and worked together to bring forward our mathematical interpretations and for a moment of poēsis to take place for Anathi. As the interview illustrates, I realised that she had arrived at an answer that would work but I invited her to clarify and think further about her contributions as a way of opening up possibilities for her to develop a more structured idea of the deductive reasoning required here. I suggest that this can be thought of as togethering.

Other examples of *togethering* occur in each part of this interview. Recall that one aspect of togethering is the constant repositioning of the “ideal form” by the teacher so that it becomes an object of consciousness for the learners. I am not just interested in the girls solving this single instance of the puzzle, but also in their ways of thinking generally about puzzles like this. Thus, rather than simply accepting the girls’ correct answers and moving on, I attempt to ensure that repositioning happens, using different approaches. For example, I ask Anathi about the ‘difference’ between the two numbers as a way of introducing a mathematical idea, I ask her to redo her counting again in order to help her reach an understanding of what she is trying to do, I ask both of them why the club is 6, and I finally get the girls to ensure that the value of 6 for the club will work in the rest of the puzzle.

6.5.2 Interview 2: Example of Absence of Sustained Attention Catching Moments and Poēsis

The extracts that follow are from a different task-based interview in another club in which myself, Akhona and Kuhle are engaged with the same activity. I include it to show a contrasting set of interactions between myself and the two girls. The lack of sustained attention catching moments and in-the-moment responses to those moments shows that when togethering is absent, no meaningful space of joint action evolves. It also exemplifies an absence of moments of poēsis.

After approximately one minute of starting the puzzle, the girls find the value of 5 for the circle. They are subsequently unsure about how to go about solving the next part of the puzzle. Kuhle says, “we don’t know these numbers” pointing to the other shapes on the chart. They make little progress in solving the puzzle for the next three and a half minutes. My role during this time was to guide the girls to find a way forward without explicitly telling them what to do. Here I have just directed them towards the total of 26 at the bottom of the fourth column and prompted the girls to think what the triangles might be if the 2 circles are 5 each. The extract below begins about 4½ minutes into the task.

Both girls began by pointing to the 26 at the bottom of the fourth column (Fig. 6.7), perhaps as a way of focusing their attention. Akhona immediately raised her arms and started to count silently on her fingers. Kuhle saw what she was doing and copied her. Kuhle voiced her counting from 10 to 20 (4:31).

Kuhle looked at the ceiling, then at me and said "13" (4:45) in a soft voice (Fig. 6.8). The soft voice may hint at an uncertainty of her answer. I responded with "Hmmm? 13?" which caused both girls to look at me (4:48). I pointed to the two triangles in column four (4:53). Akhona's gaze followed my gesture but Kuhle continued to look at me with her pen on her chin.

I asked Kuhle to clarify what the triangles were equal to (4:55). She said "13" again (Fig. 6.9). I wrote down her contribution. I gestured to Akhona, who frowned at me (5:00). I encouraged her to continue working.

So far, these three parts of this second interview uncover only one moment where attention is caught (4:48). I asked Kuhle a direct question and she responded. The question did result however, in a slight modification of what Akhona did. She stopped her counting, frowned at me and then continued to work on her own.

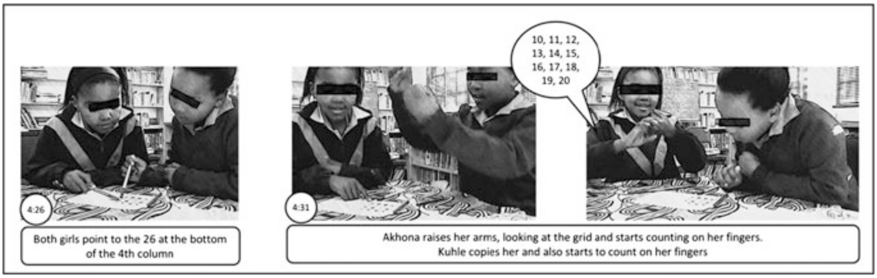


Fig. 6.7 Interview 2 (part one)

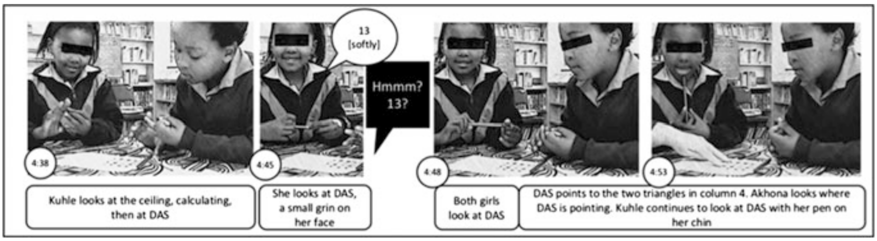


Fig. 6.8 Interview 2 (part two)

Fig. 6.9 Interview 2 (part three)

I suggest that it is likely that Akhona's attention was not sufficiently caught by anything that Kuhle or I did or said. After this brief attention catching moment, Akhona worked on her own to find the value for the triangle and after another 2½ minutes, offered the value of 8. We pick up the story at 7 min and 25 s.

I began by indicating Kuhle's previous answer (7:25) and say to her "Look here. You put 13. How did you get 13?" (Fig. 6.10). Kuhle leant into the desk (7: 29) and looked closely at her previous answer. She said "I halved the 26" (7:31) whilst gesturing and looking over at Akhona's workings.

I asked her if that would still be her answer now that she had seen the way Akhona had worked it out (Fig. 6.11). My intention was to reconcile the two different contributions for the value of the club by trying to reposition the activity for Kuhle's sense making. Kuhle responded with a soft "no" (7:42) whilst shaking her head and continuing to look at her previous answer. Now that it was clear to me what she had done, I asked again if she would change her previous answer. She responded with a soft "yes" but did not offer anything further (8:00).

From this point, Kuhle made no further progress on the puzzle. Akhona finished solving it on her own. Kuhle was silent, looked around the room and played with her face and mouth. She occasionally glanced at Akhona to see what she was doing but did not engage with her in any way. In this interview, we see much evidence of this. Kuhle drew on eye contact (gaze) as well as facial expressions and body posture as sensuous actions. The nature of these seem to signify a disengagement from the activity.

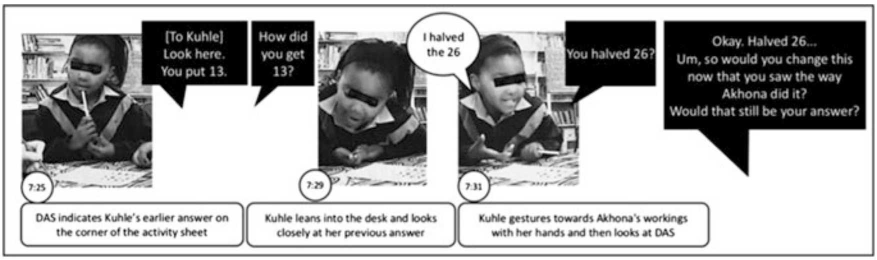


Fig. 6.10 Interview 2 (part four)

Fig. 6.11 Interview 2 (part five)

6.5.2.1 Interview 2: Discussion

The interview reveals evidence of many sensuous actions. However, the absence of *passages* of sustained attention catching suggests that Kuhle was not sufficiently engaged through this activity. Kuhle spoke very softly, responded to my questions with one word answers and without evidence of re-thinking her answer. Granted, my questions were not particularly well articulated. Nothing that I said or did modified what she said or did subsequently, evident in the absence of sustained attention catching moments. I did not provide further mediation that could have helped push the mathematical communication forward. Thus, I suggest that attention was not caught and it seems that nothing came into our realm of attention. Neither I nor Kuhle took advantage of the moments when attention was caught to further our engagement or mathematical understanding. I submit that I was unsuccessful in my attempt to reposition the activity for Kuhle and thus also unsuccessful in demonstrating togethering. Furthermore, there is little evidence that I encouraged the girls to work together or to assist in sense making. It is evident that a space of joint action did not evolve.

Additionally, there does not seem to be any connectedness in the actions, in other words we did not work together and we did not 'catch each other's ideas' and build on them. Interaction unfolded in a stilted way, there was not a sense of "togetherness" (Radford and Roth 2011). I argue that this is because the sensuous actions do not relate to each other through a sustained attention catching sequence. Each participant acted on her own, without collaboration. We were simply "doing things" (Radford 2016, p. 445) and each of us was doing something different. As noted, when attention is caught in a meaningful or noteworthy manner, modification of what is said or done is evident in the subsequent utterances and actions of the participants. We cannot see this occurring here in a continued manner.

Radford and Roth (2011) talk of the interaction in the space of joint action as one that evolves and where tuning of perspectives can occur. They argue that the individuals who participate in activity are different—both cognitively and emotionally. As a result, "ideal forms" are *refracted* differently in the individual consciousness of the teacher and the students. This is of consequence for the interaction in this interview. I have suggested that no space of joint action evolved here as meaningful attention catching did not take place; both learners and I would need to work together to ensure that the girls become progressively aware of the ideal form of this activity. This did not happen as there was no tuning of perspectives. It follows then, per Radford and Roth's argument, that the ideal form of this activity (reasoning deductively) may not be reflected (or refracted) similarly in the consciousness of myself, Kuhle or Akhona. I submit that the thinking is not collective and that by the end of the activity, each participant still had a different idea of the purpose of the activity: Kuhle's idea may have been to find the *right* answer as quickly as possible, Akhona's might have been an emerging understanding of how puzzles like this work through trial and error whilst mine was related to the emerging deductive reasoning that a puzzle such as this offers.

I therefore maintain that there were no moments of poēsis for the girls; there is no evidence of creative moments, where they created meaningful new ideas, methods or interpretations, during the interview.

6.6 Discussion

In pulling together and comparing the data from the two interviews, I have shown different ways in which attention catching and moments of poēsis come together in a space of joint action for the learners to progressively become aware of or make sense of the mathematics at hand. A space of joint action does not arise between the participants when there is no noteworthy sequence of attention catching moments. In the latter case, there is some doubt as to whether the encounter with the mathematics makes sense to the learners or if realization of the ideal form is possible. This is borne out by Shvarts (this volume) who argues that joint attention moments are necessary for learners to acquire the cultural meanings and forms of mathematical concepts.

Radford and Roth (2011) indicate that togethering will unfold in "unforeseeable ways" (p. 242) and that togethering can only happen through understanding the learners and their needs. Their point is noteworthy and seems to be borne out by the data presented here. Both interviews point to togethering unfolding in unforeseeable ways. In the first interview, in my role as facilitator, I seem to be more in tune with the learners and their needs. I also seem to be more confident in the way that I reposition the ideal object of the activity, evident in the way that I frame and ask questions and offer mediating actions in response to attention catching moments. In the second interview, my questions are often clumsy and I do not push the communication forward. I infer from these examples that my difficulty is in anticipating in advance how myself or the girls would interact and respond in the moment. I cannot pre-plan *how* I will reposition the ideal form of the activity but I do know the ideal object of the activity.

What is noticeable, especially in interview one, is that when my attention is caught by something one of the girls says or does, I offer mediation in some way to sustain the mathematical engagement, to encourage sense making or to further mathematical understanding. The interviews reveal that learners seem to respond in different ways to this offered mediation, in other words they take advantage of it differently. They respond *directly* in a meaningful or noteworthy way by trying to explain themselves mathematically or by carrying out a mathematical action. Alternatively, learners respond *indirectly*, evidenced by listening, making and maintaining eye contact with another, by nodding and so on. However, these actions are not active and do not contribute to moving the communication forward. Finally, learners do not respond at all, thus preventing sequences of sustained interaction and thus attention catching and moments of poēsis to arise, and this affects how the space of joint action evolves and how realization of the ideal form takes place.

Consequently, I offer that a key aspect of the notion of *togethering* includes a *mindfulness* on the part of the facilitator of the direct contributions that the learners make through all their sensuous actions. Togethering could thus include offering flexible, timely, in-the-moment responses in both speech and other sensuous forms to those contributions. Further, being mindful of the way in which learners respond to the mediation offered brings a consideration of the learners, their needs and current understandings. These give clues about how to constantly reposition the activity so that a progressive awareness or grasping of the ideal form is more likely to take place. Again, Shvarts (this volume) alludes to this idea that a teacher may follow a student's attention and subsequently adjust her explanations accordingly to guide the student towards the culturally accepted practise or ideal form. This idea of mindfulness and offering in-the-moment responses is not new in mathematics education (see Alibali et al. 2013; Watson 2007 for example). However, I argue that it is important to note it is as part of the notion of *togethering*, as it provides a tangible way for educators to think about togethering in their practice.

6.7 Synthesis and Concluding Remarks

In concluding, I return to my aims for this chapter and look at each one in turn. My broader aim was to consider if the theoretical concepts of space of joint action and togethering could be used as narrative tools for presenting data from a different context in after school mathematics clubs. I found the constructs to be useful as a narrative tool for presenting analysis as they provided the theoretical focus and descriptive language to describe how interaction unfolded and how mathematics progressively came to the attention of the various participants in the task-based interviews presented here.

I exemplified how I connected the notions of attention catching and moments of poēsis through analysis to illuminate and gain a deeper understanding of how the space of joint action evolves. I provided examples and non-examples from the data on how the participants take advantage of the moments when attention is caught and how they subsequently respond to sustain the mathematical engagement, understanding or sense making. In using the combined lenses of attention catching/

moments of poēsis, I have expanded on the notion of attention catching proposed by Meira and Lerman (2009).

I started this chapter with a request from Deborah Ball to understand the special work of mathematics teaching particularly regarding hearing, seeing and reading students in real time. The insights I have shown here concerning togethering may contribute to the body of work on the special work of mathematics teaching both in classroom and OST time contexts. Additionally, by using these for this analysis, I profess to having a more concrete understanding of the ways in which a space of joint action evolves and what togethering looks like from a facilitator's point of view. This is important for moving forward in my work with learners in the clubs who are most often part of the marginalised group noted in the introduction, as I will be able to apply these findings in that space. Furthermore, I will be able to communicate these ideas through my work with both in-service educators and OST facilitators.

In closing, I would like to use this deepened understanding to elaborate on a 'symphony' metaphor used by Radford[2] (2016) which describes the joint endeavour carried out by teachers and students to produce "sensible" mathematics. As I understand it, in this symphony, the conductor (teacher) and musicians (students) are immersed in the process of making music (coming to know mathematics) based on the musical work (culturally encoded forms of thinking and doing mathematics). Neither conductor nor musicians act individually but rather as a seamless whole to make the music. This is where I extend the metaphor to include the space of joint action, togethering, attention catching and moments of poēsis. The musical work in the form of the symphony is the ideal form of the mathematics. The role of the conductor is to lead the orchestra, to keep a collective focus and to interpret the music. Similarly, the role of the teacher in the mathematical activity described in this chapter is to work with the students to keep a focus on and to bring forth the ideal form of mathematics through togethering. The primary role of the musicians in the orchestra is to play the music, to engage in the act of making music and to follow the lead of the conductor. The role of the students is to be guided by the teacher in coming to know mathematics. The combined work of the conductor and musicians can be thought of as joint labour in the space of joint action.

How do the conductor and musicians connect and engage? Through their sensuous actions and by catching each other's attention in moments of poēsis. I suggest that this link between the sensuous actions and attention catching sequences is essential for progress in the mathematical activity.

[2]Personal conversation, ICME 13, July 2016.

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References

Alibali, M. W., Nathan, M. J., Church, R. B., Wolfgram, M. S., Kim, S., & Knuth, E. J. (2013). Teachers' gestures and speech in mathematics lessons: Forging common ground by resolving trouble spots. *ZDM—International Journal on Mathematics Education*, *45*(3), 425–440.

Archer, A., & Newfield, D. (2014). *Multimodal approaches to research and pedagogy: Recognition, resources and access*. New York: Routledge.

Ball, D. L. (2016). Uncovering the special mathematical work of teaching. In *13th International Congress on Mathematical Education (ICME)*. Hamburg, Germany: Plenary presentation. Retrieved from https://deborahloewenbergball.com/presentations-intro/.

Graven, M. (2011). Creating new mathematical stories: Exploring opportunities within maths clubs. In H. Venkat & A. A. Essien (Eds.), *Proceedings of 17th National Congress of the Association for Mathematical Education of South Africa (AMESA)* (pp. 161–170). Johannesburg: University of the Witwatersrand.

Leikin, R., & Pitta-Pantazi, D. (2013). Creativity and mathematics education: The state of the art. *ZDM—International Journal on Mathematics Education, 45*(2), 159–166.

Lerman, S. (2001). A cultural/discursive psychology for mathematics teaching and learning. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education: An international perspective* (pp. 3–18). New York: Routledge.

Meira, L., & Lerman, S. (2009). Zones of proximal development as fields for communication and dialogue. In C. Lightfoot & M. C. D. P. Lyra (Eds.), *Challenges and strategies for studying human development in cultural contexts* (pp. 199–219). Rome: Firera Publishing.

Plowman, L., & Stephen, C. (2008). The big picture? Video and the representation of interaction. *British Educational Research Journal, 34*(4), 541–565.

Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in mathematics education*. (G. Kaiser, Ed.). Hamburg: Springer Open.

Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics, 30*(2), 2–7.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM—International Journal on Mathematics Education, 46*(3), 349–361.

Radford, L. (2015). Methodological aspects of the theory of objectification. *Revista Do Programa De Pós-Graduação Em Educação Matemática Da Universidade Federal De Mato Grosso Do Sul (UFMS), 8,* 547–567.

Radford, L. (2016). Mathematics and mathematics classroom activity through the lens of a metaphor. In M. Iori (Ed.), *La Matematica e la sua Didattica/Mathematics and Mathematics Education. On the occassion of the 70 years of Bruno D'Amore* (pp. 439–446). Bologna: Pitagora Editrice.

Radford, L., & Roth, W. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*(2–3), 227–245.

Roth, W., & Radford, L. (2011). Toward a science of the subject. In A. Saenz-Ludlow & L. Radford (Eds.), *A cultural-historical perspective on mathematics teaching and learning* (Vol. 2, pp. 1–27). Rotterdam: Sense Publishers.

Stein, P., & Newfield, D. (2006). Multiliteracies and multimodality in English in education in Africa: Mapping the terrain. *English Studies in Africa, 49*(1), 1–21.

Stott, D., & Graven, M. (2013). The dialectical relationship between theory and practice in the design of an after-school mathematics club. *Pythagoras, 34*(1), 29–38.

Shvarts, A. (this volume). Joint attention in resolving the ambiguity of different presentations: A dual eye-tracking study of the teaching-learning process. In N. Presmeg, L. Radford, W.-M. Roth, & G. Kadunz (Eds.), *Signs of signification: Semiotics in mathematics education research*. Dordrecht: Springer.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds., A. R. Luria, M. Lopez-Morillas, M. Cole, & J. V Wertsch, Trans.). Cambridge, MA: Harvard University Press.

Watson, A. (2007). The nature of participation afforded by tasks, questions and prompts in mathematics classrooms. *Research in Mathematics Education, 9*(1), 111–126.

Part II
Language and Text Orientations

Introduction: Gert Kadunz

For some twenty years (see Otte, this section), the theory of signs has been known and used as a valuable tool for planning lessons and for describing and interpreting students' activities as well. Among all the research questions arising in this area, the focus of all of the four chapters in this part is on investigating the relationship between language and mathematics learning. Each of these chapters presents its own view on this relationship. However, they all use semiotics as a tool for investigating mathematics learning from a wide perspective. In this respect, the texts of Priss, Kadunz and Otte can also be read as papers following ontological interests, while Morgan's deliberation also concentrates on the sociopolitical impact of language on education. Hence semiotics appears as a versatile tool, as a kind of Swiss knife.

In his considerations, Gert Kadunz discusses the question of the ontological status of mathematical objects. For this, he investigates the different uses of language within mathematics at school and mathematics at the university. The gap between these uses is rather wide as Kadunz explains, using the example of the "limit of a sequence" concept. Hence he asks whether the way in which this concept is applied using lenses of school mathematics could be wrong in the sense of university mathematics? What can we say about the relationships between the languages in these two realms? Is mathematics at university a more correct version of a kind of independent mathematics? One way to answer such a question is to discuss the problem of translation between languages. However, answers to this issue can be found among others within the oeuvres of Walter Benjamin and Martin Heidegger. Both philosophers claim that translation cannot be seen as the transmission of a meaning located behind the text we wish to translate. Kadunz argues that the relationship between mathematics in school and at university could be seen similarly, and therefore the assumption of an independently existing mathematics is neither useful nor necessary. In this respect, the author is in agreement with Ludwig Wittgenstein and his anti-Platonistic view of mathematics. Kadunz closes his deliberation with a bridge to Peirce and his semiotics by following Peirce's idea to identify “meaning as a translation of one sign into another sign system” (Jakobson 1985, p. 251, cf. Otte 2011, p. 314). This translation, which can be seen as a transformation of visible signs, too, leads to the conclusion that mathematics in school and mathematics at university are not two sides of the same coin. On the contrary, these two parts of mathematics can be seen—metaphorically speaking—as models for each other.

Candia Morgan’s report provides a semiotic view on learning that does not concentrate on analysing single students’ or teachers’ activities but also offers views on mathematical learning within a wider context. Her approach is based on the social semiotics of Halliday (1978) and his suggestion that all modes of communication—language too—are functional and not representational. This functional approach includes the following:

- the field of discourse—the event being spoken about;
- the tenor—who the interlocutors are and what are their relationships to each other and to the event;
- and the mode of discourse—the role of the text itself within the event (Morgan this section).

Morgan presents two examples which demonstrate the strength of Halliday’s social semiotics. First she concentrates on the learning of geometry. Through analysing the structure of a given text concerning a geometrical problem, Morgan is able to describe how this text functions in three different aspects: ideationally, interpersonally and textually.

Her second example uses social semiotics as a tool to investigate patterns within texts in a context larger than a classroom. With this example, Halliday’s semiotics appears to be transformed into a kind of political semiotics. In her analysis, Morgan concentrates on texts in the “official fields of examinations and of school monitoring” (Morgan this section) to facilitate the role of such kinds of texts in the reader’s development of a critical awareness of how mathematics is discussed in a wider context.

Michael Otte’s paper appears like a flight over relevant statements of philosophers, starting in the seventeenth century with Descartes and ending in the twentieth century with Quine, Schlick and Chomsky. The aim of this journey lies in the author’s wish to present a brief outline of a complementary principle between meaning and reference. What are the reasons why semiotics arouses only a little interest among mathematics educators and mathematician? We can read the answer between the author’s lines. It is the misunderstanding of this “complementarity”. To name only a few of Otte’s examples, there is complementarity between epistemologically oriented semiotics (Peirce) and linguistically grounded semiotics (de Saussure). Further examples are the complementarity of function and metaphor or of mathematics as a language and as a system of visible signs. Semiotics in the sense of Peirce appears to be a valuable tool presenting a new view on the learning of mathematics, since this semiotics is a methodology suitable for investigating aspects of Otte’s idea of complementarity.

In her deliberations, Uta Priss concentrates on the use of Peirce and some relevant parts of his semiotics (see also Ludlow, section 3, this volume) and investigates the influence of the use of everyday language on the construction and interpretation of mathematical concepts. Together with Peirce's semiotics, Priss uses "Formal Concept Analysis" (Ganter and Wille 1999) to back her claim that the associative character of everyday language could hinder students in building robust mathematical concepts. However, the notation of mathematical concepts is not always complete and unambiguous by itself. Hence, the use of everyday language to interpret such notations could cause further confusion. The searching for the meaning of a mathematical concept, as we like to do it with concepts of everyday language, should be avoided. "One should not ask what a definition means, but instead what it says" (Priss, this section). In this respect, we should not try to look behind mathematical concepts but just ask how to use them. Here we find an anti-Platonistic view on mathematics, which can be found in the other papers of this section, too.

References

Ganter B., & Wille R. (1999). *Formal concept analysis: Mathematical foundations*. Berlin-Heidelberg-New York: Springer.

Halliday M. A. K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. London: Edward Arnold.

Otte, M. (2011). Evolution, learning, and semiotics from a Peircean point of view. *Educational Studies in Mathematics, 77*, 313–329.

Chapter 7
A Matter of Translation

Gert Kadunz

Abstract This chapter concentrates on the often formulated question about the ontological status of the concepts of mathematics. As an alternative to philosophical approaches this paper uses arguments from linguistics and semiotics. Through comparing mathematics in school and mathematics at the university we are confronting the question of how to translate between the two of them. The result of such deliberations is that at least the assumption of a Platonistic view on mathematical concepts is not fruitful.

Keywords Language · Translation · Ontology · Semiotics

7.1 Introduction

The question of the nature of 'mathematical objects' is discussed in the context of both the philosophy of mathematics (e.g., Shapiro 2000) and within mathematics education. Several of these texts use semiotic approaches and look for answers by combining semiotics with sociocultural theories (Radford 2013). Other authors (e.g. Duval 2006) attempt to describe the meaning and significance of mathematical objects with recourse to theories from structuralism. My paper takes a different approach. It focuses on the relationship of mathematical languages, namely, the languages of school mathematics and of university mathematics. "There is thus a good correspondence between the types of language employed in school mathematics and research mathematics …" (Ernest 2008, p. 44). This correspondence can be described by different means. Ernest refers amongst others to proposals of Halliday (1985) and Rotman (2000). Another way of investigating this relationship asks whether one of these languages is the 'correct' mathematical language and how a translation between these languages can be done successfully. The theories

G. Kadunz (✉)
Department of Mathematics, Alpen-Adria Universitaet Klagenfurt,
Universitaetsstrasse 65, 9020 Klagenfurt, Austria
e-mail: gert.kadunz@aau.at

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on the issue of translation I use in this text concentrate on Walter Benjamin and Ludwig Wittgenstein. Both authors suggest that a successful translation does not need the assumption of an objectively existing mathematical instance.

However, as mathematics is not just a spoken language, it depends heavily on the written forms. Numerous papers have investigated the use and the importance of the visible for the learning student and the mathematician as well (e.g., Presmeg 2006). Peirce's theory of signs offers a successful approach on how to investigate the importance of the visible, written part of mathematics. Using this semiotic theory the closing section presents a pragmatic answer to the question of the 'nature' of mathematical concepts.

The relation of mathematics at school and mathematics at the university—within the meaning of a research-based mathematics—has been discussed for more than 100 years, in contrast to the relation between mathematics education[1] and mathematics. In the case of mathematics education and mathematics, quite often methodological, psychological, or pedagogical differences are considered (Presmeg 2014) while in the case of mathematics in school and mathematics at university, questions relate to the formal correctness of concepts and propositions. In this regard, Biermann and Jahnke (2014) report Felix Klein's view of mathematics in the Prussian secondary schools (Gymnasien) of the 19th century. Klein (2007) postulated a "double discontinuity", and wrote as follows:

> It is remarkable that this modern development has passed over the schools without having, for the most part, the slightest effect on the instruction, an evil to which I often alluded. The teacher manages to get along still with the cumbersome algebraic analysis, in spite of its difficulties and imperfections, and avoids the smooth infinitesimal calculus …The reason for this probably lies in the fact that mathematical instruction in the schools and the onward March of investigation lost all touch with each other after the beginning of the nineteenth century … I called attention in the preface to this discontinuity, which was of long standing, and which resisted every reform of the school tradition… In a word, Euler remained the standard schools. (Klein, p. 155; Biermann and Jahnke 2014, p. 5)

The mentioned "double discontinuity" thus describes both the distance between inquiring mathematics and school mathematics and the continuation of an—in the view of Klein—outdated access to topics of calculus.

[1]The relationship of mathematics and mathematics education was not always stress-free and is still the subject of a series of lectures and symposia. Just to mention two examples, several texts in "Mathematics and mathematics education: Searching for common ground" (Fried and Dreyfus 2014) and papers in "Transformation—A fundamental idea of mathematics education" (Rezat et al. 2014) concentrate on this relationship. Both collections, interestingly, emerged as commemorative books for Theodore Eisenberg and Rudolf Sträßer respectively, two well-known however already retired researchers in mathematics education.

7.2 The Question

The above cited references claim a rather wide difference between mathematics as a science and mathematics in school. Perhaps minor differences can be expected if we concentrate on those differences that prevail between the content of mathematics that is taught in school and that which is presented to students of mathematics at the university during the first semesters. *Grosso modo*, the topics of school mathematics can be found within the table of contents of university textbooks of mathematics. However, has the word *mathematics* in both cases the same meaning? Or is the use of the term "same mathematics" misleading in the first place? A possible difference between mathematics in school and university mathematics—apart from the variety and width of the content—can become particularly clear in those chapters that deal with the infinite, in particular in chapters concentrating on calculus.

Let us take a look at the limit of a (real) number sequence as an essential term of calculus in school. The presentation of the concept of limit within textbooks used in schools or books on mathematics education (e.g., Dankwerts and Vogel 2006) concentrates on activity-oriented contexts and differs from the formal definition. The understandable reason for this kind of presentation lies in the interests of the authors to make the limit term easily constructible for learners. This action-oriented approach is often accompanied by certain verbalizations. Let us take a look at those verbalizations (Table 7.1).

Often such verbalizations are complemented by visualizations. A look into schoolbooks[2] shows that a special task-based teaching, which can also be seen in the above verbalizations, appears to be inevitable to introduce the learner to a successful use of the limit concept. However, Freudenthal (1983) emphasized that for teaching there is the need to demonstrate the discontinuity between formal mathematical explanation and illustrative interpretation. Such discontinuities were already known to mathematicians and philosophers in ancient Greece. Let us remember Zeno of Elea (480–430 B.C.) a student of Parmenides. Zeno tried to demonstrate that a premature use of the word infinity could lead to a number of logical inconsistencies. We all know his arguments, which should show that such inconsistencies lead to the logical conclusion that there is no movement (Pichot 1995, p. 441).

We emphasize that compared to the learning of the limit in school, mathematics at university concentrates on a formal logic definition and use of this concept. Referring to the Archimedean axiom, the limit of a series is presented using the "epsilon-language" and is practiced by formal examples (e.g., Fritzsche 2005; or Spivak 2006). As a rule these textbooks renounce the use of verbalizations mentioned above, in order to avoid possible misinterpretations (in the sense of the mathematics at university level). In particular, neither the image of moving

[2]The question of the equality of 0.9999…. And 1 is a problem often discussed in school. This example allows one to address a number of properties of the limit of a sequence.

Table 7.1 Some verbalizations (Dankwerts and Vogel 2006, p. 26)

"More or less well done … To find good verbalizations is an art if we simultaneously have to observe the formal definition of the limit and the requirements of teaching. For the fact that the sequence 1/n converges to 0, many ways of speaking are in use. Here are some examples. "1/n comes as close as possible to 0 if n increases." "1/n tends to 0 as n approaches infinity." "1/n is coming closer to 0 as n increases." "1/n is coming closer to 0 without ever reaching it." The wordings (1) and (2) are suitable because they reflect the content of the formal limit definition properly. Variant (3) is problematic because it does not contain the central aspect of the definition of the limit: If we follow variant (3) then 1/n is also coming closer to (−1). In formulation (4)—applied to any convergent sequence—the distortion of the concept of the limit of a sequence is clearly visible: Even constant sequences are convergent."

elements of a sequence nor the talk of elements coming nearer and nearer to the limit of sequence are applied.

So we have at least two different uses of the word limit. In school mathematics, it appears descriptive and action-oriented while at the university this word is used in a strictly formal sense. Is one of them the correct one and the other in a formal sense wrong? Which meanings can we ascribe to these two different ways of speaking?

As a second example, I refer to the different uses of the word *vector*. The question of what a vector is can be upset after the first classes of a course on Linear Algebra. However, the concise answer that a vector is an element in a vector space is not suitable for use in schools, where a more geometric view of vectors dominates. Teachers also like to introduce vectors by discussing particular applications, which allow the representation of contexts from economics and physics. In doing so, the precise definition of a concept is softened up and the structural mathematics is removed. Consequently, central concepts of linear algebra such as linear independency, dimension, subspace or matrix, to name just a few elementary ones, are motivated and defined in a geometric or application-based way. Hence, we ask, is linear algebra in school flawed, since it suffers from a lack of generality?

7.3 A Linguistic View

This question of the ways of speaking—formal logic or action-oriented—offers a possibility of providing information about the 'proper' use. For this purpose it is useful to turn our attention to the translation between these modes of speech and ask when a translation between languages generally, and particularly between the mentioned modes of speech in mathematics, is a successful translation.

A first hint to answer this question can be found in the considerations of Walter Benjamin[3] (Benjamin 1977, p. 140) concerning language and language comprehension. Let us follow Sybille Krämer and her interpretation of Benjamin (Krämer 2008, p. 41). Krämer reports that Benjamin concentrates in his remarks on the question of *how* something is expressed in language. However, we do not want to respond to Benjamin's sometimes dark and hard to understand descriptions of language, but focus on the message "*in* a language" as this is relevant to the following considerations. In Benjamin's eyes it is fundamental to know "that something is communicated *in* the language and not through language" (Krämer 2008, p. 44, my italics).

Benjamin complained that in the conventional view of language, a word communicates something that can be found outside the language. In this case language would be a means between something and the speaker. It is precisely this view of language that Benjamin wanted to avoid. He holds the view, to emphasize it, that something is communicated *in* the language, which means for him as equivalent to, that something appears or shows itself.

How can we use this view of language to answer our question of translation between mathematics in school and mathematics at the university? In a conventional approach, we could see this as a task to transfer a text from one language into another language. The purpose of this transfer would be to obtain the content of the text. The art of the translator appears in the neutralization of the differences between the languages involved. If this neutralization would succeed, then this would mean that we could say the same thing in different ways (Krämer 2008, p. 264). For Benjamin this is precisely the hallmark of a poor translator. The good translator does not look for the equality of the meaning located behind the languages but conversely he ensures deviations in their scope. That is the way differences between the languages show themselves. "If we translate in this sense then the diversity and the incommensurability of languages is not covered up but becomes clearer" (Krämer 2008, p. 265).

A similar picture concerning the question of how translation is possible can be found in Martin Heidegger's work. Let us use Krämer's comments on Heidegger again. For him translating is an event, which is determined by the distance between languages. A text from a different language is determined by an irreversible foreignness where the translator, in doing his or her work, has to dare to leap to the other riverbank. In his deliberations about the word translation ("Übersetzung" in German), Heidegger concentrates on the "Über" (across or over) aspect, meaning thereby not a transfer of sense which is independent of its linguistic formulation. "In translating the words of Heraclitus our need is great. Here trans*lation* is seen as a *trans*lation to the opposite bank, which is little known and far away on the other side of a broad, fast moving river." (Heidegger 1979, p. 45; cf., Krämer 2008, p. 176).

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[3]Hannah Arendt edited in 1968 in "Illuminations" a number of Benjamin's papers on art, literature and translation.

We have already observed a similar viewpoint on the relationship between languages in Walter Benjamin's work. His approach to our question on the relationship between school mathematics and mathematics at the university can be fruitfully applied. For Benjamin, Heidegger's leap becomes a transposition, a transformation of one language into another language. For Benjamin, register is the defining feature of "translatability", in which commonalities and differences are noted. In transforming a text, he requires concentration on the individual words and not on grammar. At the same time it is precisely this focus that determines the productivity of a good translation through the discussion of a word's meaning.

Benjamin distinguishes between "what we mean," which we can also call the word's meaning, and the "way in which it is meant." With regard to "what we mean"—in Benjamin's opinion—"bread" and "pain" are the same; whereas with regard to "the way in which it is meant" the two are different and mean very different things, embedded in history, culture and everyday use in the English or French language areas (Krämer 2008, p. 186).

Therefore, we must ask ourselves if the productivity referred to here, which feeds off the knowledge of numerous differences in the usage of words, is also a productivity that reveals itself in learning? At the very least, this is true for the learning of concepts which are transferred from the use case of a teacher into the use case of the learner. And can we not see the relationship between school mathematics and mathematics at the university in a similar light as a transformation of uses, which do not need any of the senses underlying the respective mathematics?

What does this view of language and of the task of a translator mean for our question of the possibility of translation between mathematics in school and at the university? When we first assume that a translation can succeed because both school mathematics and the university mathematics would correspond to a common content independent of them, then we would assume that there exists a mathematics independent of the particular representation. Then we would have a reference for the correctness of the respective mathematics. But that would mean that the mathematics taught in school is tainted by errors (see the references above on the limit of a sequence).

On the other hand if we renounce the assumption of the existence of a universal mathematics independent of the used signs, social practices and tools, and if we follow Benjamin's assertion about the use of language, then the relation of school mathematics and university mathematics manifests itself in a different light. In this case we do not have to transfer either the content postulated behind the words of a text or the mathematics existing behind the visible and audible signs. On the contrary, we could say that school mathematics and university mathematics each form something independent, embedded in their own particular social, historical and formal contexts. A successful translation in the sense of Benjamin would be a transfer in compliance of this context without assuming the existence of an independent mathematics. Such a transformation uses as a tool, among others, the means of school mathematics or those of the mathematics at university. In terms of model theory a structure A is interpreted by a structure B (Hodges 1997).

7.4 A Second Opinion

Abandoning the assumption of an independently existing mathematics is by no means justified only by the desire for translatability (for an anthropologically based view see Radford 2008). Let us supplement Benjamin's proposal, which opened this model-theoretic perspective, with a second theoretical approach. This approach also dispenses with the assumption of mathematics existing independently of human activity and supports a constructive synopsis of school and university mathematics. It was Ludwig Wittgenstein who took this view of mathematics in his numerous comments. Let us share the opinion of Wittgenstein using firstly the text "Mathematics without metaphysics" (Mühlhölzer 1999). In Wittgenstein's view mathematics is a form of life, which is grounded in a pragmatic but not ontological-objective sense. A question presented by Wittgenstein in this context concentrates on the issue of following rules when doing mathematics. How can one decide with a rule whether the application of a rule is a correct or incorrect application of the rule (Mühlhölzer 1999, p. 4)? Wittgenstein replies that a rule cannot do this on its own, but it is an intersubjectively controlled and accepted applying of this rule that decides on the validity of its use.

Closely related to the question of following rules is Wittgenstein's rejection of truth as an ontological orientation of mathematical activity. On this issue, Ramharter and Weiberg (2006) submitted a detailed text. They report that Wittgenstein explained in his book, "Remarks on the foundations of mathematics (Bemerkungen zu den Grundlagen der Mathematik, BGM)" (Wittgenstein 1988), alternatives to the claim that truth within mathematics can be ensured ontologically. As a first alternative, truth could be replaced by utility. Wittgenstein (1991) holds "that one cannot say that a series of natural numbers—as well as our language—is true, but they are useful and, above all, they will be used. (BGM I, §4/S. 37ff.)" (Ramharter and Weiberg 2006, p. 62). Since it makes no sense to look for the ultimate justifications of language games in general, he concluded that this search within mathematical language games is futile too.

We note from the indication on "*the use*" that Wittgenstein has a different proposal in mind. For him, it is the question itself that he considers to be misleading, as in his view this question cannot be answered meaningfully. So Wittgenstein renounces the claim of a truth existing behind mathematics by renouncing this question. As alternatives, Wittgenstein offers studies of functionalities of mathematical language games. "It is our mistake to look for an explanation where we should look at the facts as primary phenomena. That means that we should say this language game is played." (PU, §654; Ramharter and Weiberg 2006, p. 62).

The idea of the existence of mathematics as an independent entity could also have originated from a certain experience of mathematics. Wittgenstein examined in this context the "inexorability of mathematics." We experience this feeling clearly, for example, when we try to design or comprehend a mathematical proof. Is this relentlessness not an indication of an independent mathematics? As an example Wittgenstein presents a comparison of usual and unusual measuring instruments:

> What would happen then if we reason differently– …? How would we come into conflict with the truth, if our measuring rods were made from very soft rubber instead of wood and steel?
>
> – Now, we would not get to know the right measure of the table.
> – You mean, we would not receive, or at least not reliably, the measuring value, we receive with our hard measuring rods.
> – This person would be wrong who would have measured the table with the stretchy graduation and stated, that he would measure 1.80 m following our usual kind of measuring. But if he says that the table measures 1.80 m according to his method, so that's right …
> – However, that is not a correct measurement.
> – It is similar to our measuring and can meet practical purposes under certain circumstances.
>
> (BGM 1, §5/S. 38; Ramharter and Weiberg 2006, p. 62)

Hence Wittgenstein emphasizes that it is not the instrument itself that relentlessly forces us to a certain conclusion, but that we as users opt for a particular purpose. What does this mean for logical reasoning? We could imagine another kind of reasoning and also put it into practice, but then we would not use it in mathematics. Wittgenstein claimed that we would not even name it reasoning. In this sense, the supposed inexorability that would let us assume an independent mathematics, is nothing else than the acceptance of a particular practice. This *practice* is called mathematical or logical reasoning and there is neither an obligation nor an absolute necessity that could force us to follow a rule of inference. On the contrary the acceptance of this kind of reasoning is at the same time the acceptance of participation in mathematical activity. This means that we decided voluntarily to participate in the mathematical practice and simultaneously we can renounce the adoption of an independently existing mathematics.

Up to here the deliberations have followed concepts and ideas found within linguistics or the philosophy of linguistics. However, as already mentioned above, mathematics is not just a language, as mathematical activities are heavily determined by the use of visible signs. This assertion is confirmed by numerous papers from quite different research areas. To name just two, remember for instance Schmandt-Besserat (1997). In "How Writing Came About" she reports about the origin of writing in the time of ancient cultures of the Middle East. She claims that the origin of writing was not to reproduce the spoken, but the need for certainty of information based on numbers. Likewise, the famous mathematician Polya (2014) developed numerous suggestions on how to find a solution to mathematical problems, many of which concentrate on the visual.

For some twenty years now a new approach has been used, alongside psychological and pedagogical theories, to interpret the activities of students learning mathematics. This is semiotics, with its different forms, which are leading research in mathematics education. Among these different forms of semiotics, that of

Charles S. Peirce has been shown to be a successful tool for investigating the visual part of mathematics (Hoffmann et al. 2005; Sáenz-Ludlow and Kadunz 2015). Hence let us concentrate now on semiotics to find a possible answer to the main question of this paper.

7.5 A Semiotic View

As the final point of the deliberations let us consider briefly the question of the existence of mathematical objects from a semiotic point of view. It was Jerome Bruner who in 1969 formulated the question in this context: "What do we say to the young child, asking if concepts like force or pressure really exist?" (Bruner 1969, p. 54; cf., Otte 2011, p. 314). One way to respond to Bruner is to focus on the issue of the construction of meaning of signs in mathematics. Among other approaches, the already mentioned semiotics of Peirce provides an option. In the words of Jakobson (1985), "One of the most felicitous, brilliant ideas which general linguistics and semiotics gained from the American thinker is his definition of meaning as translation of one sign into another system of signs (CP 4.127)" (Jakobson 1985, p. 251, cf. Otte 2011, p. 314).

Let us note this observation and turn the issue of translation between languages into an issue of transformation between sign systems. Must we assume the existence of independent mathematics to speak about this transformation? Let us accompany Seeger (2011) in his semiotic explanations about the construction of meaning, where he focuses on the sign oriented discourse between students.[4] In accordance with Benjamin, Seeger (2011) claims, "On the other hand, the use of discourse is the necessary condition for understanding and manufacturing meaning —that we do not understand through discourse what is in the real world, but that we only understand in discourse" (p. 217).

In mathematics, this discourse is primarily determined by activities with the use of visible signs. The participants in this discourse experience the 'mathematical object' as if it were a kind of trademark of the signs they use successfully—use in the sense of Wittgenstein. The ontological positions of Peirce and Wittgenstein are rather different, however, but a connection can be made as the pragmatic maxim can be seen as a game with signs. This connection is based on action oriented learning of mathematics. What does this mean? Let us consider Vaihinger (2009/1911) and his philosophy of "as if" (Seeger 2011) in two statements,

– the educational slogan of "performance before competence … expresses very well that one has to pretend to know what the new is and how to do it, before the new is at one's disposal;

[4]Significant deliberations on this issue can also be found in the chapter by Sáenz-Ludlow and Zellweger (2015).

– in abductive reasoning … and problem solving, the discovery of the new is often attained if we pretend to know what the next step is. (Seeger 2011, p. 218)

Is it not a triumph, in accordance with the motto "performance before competence", if the construction of the meaning of mathematical terms succeeds without the assumption of the existence of independent mathematics? On the one hand, it is true that Peirce asserts that the proper use of a sign is the unreachable limit of a process, yet on the other hand "the relation to practice is providing an interpretation that allows pretending the final sum total has been reached" (Seeger 2011, p. 219). In this respect, the assumption of the existence of a mathematical object outside the world of man seems obsolete.

In conclusion let us refer to Rotman (2000) and his semiotic view of mathematics. In his text *Mathematics as sign*, Rotman concentrates on mathematics as a science which is essentially determined by the written (Dörfler 2016). To some extent the question of translation between languages would for him also be a question of translating only between the written texts.

> Far from being the written traces of a language that merely describes prior mental construction appearing as presemiotic events accessible only to private introspection, signifiers mark signs that are interpreted in terms of imagined actions which themselves have no being independent of their invocation in the presence of these very signifiers. In this dialectic relation between scribbling and thinking, whereby each creates what is necessary for the other to come into being, persuasion– …–has to be located. (Rotman 2000, p. 29)

This focus on the visible is what brings the science to life when it is used by a person doing mathematics in his or her social context. Such a context could be teaching in school but equally activity as a researcher at the university. "Numbers, then, appear as soon as there is a subject who counts" (Rotman 2000, p. 38). It is the person doing it who constructs mathematics according to the given needs and with reference to the prerequisites. It is not necessary to presume an independently existing mathematics.

7.6 Summary

Our starting point was the question of the relationship between school mathematics and university mathematics and their conjunction with the means of a translation. What conditions must be met in order that key terms can be translated? The question of translation led us to Walter Benjamin and his proposal, that a successful translator should not seek a meaning behind the language but that all the meaning is expressed *within* or *inside* the language. For our purposes, this means that translating does not need the assumption of an unalterable absolute and objective mathematics as the main criterion. The view of mathematics should always be a view that takes into account the context of mathematics as a human endeavor. This renunciation, as a consequence of Benjamin's proposal, is substantially supported by Ludwig Wittgenstein and his reflections on the foundations of mathematics.

Abandoning an objective standard—also in the semiotics view—allows us to talk successfully, for example, about school mathematics in the words of mathematics taught at university. With Gravemeijer et al. (2000), we could metaphorically say that the one mathematics is a model for the other mathematics.[5]

References

Arendt, H. (Ed.). (1968). *Walter Benjamin: Illuminations*. New York: Schocken Books.

Benjamin, W. (1977). Über Sprache überhaupt und über die Sprache der Menschen. In R. Tiedemann & H. Schweppenhäuser (Eds.), *Walter Benjamin, Gesammelte Schriften* (Vol. II.1, pp. 140–157). Frankfurt am Main: Suhrkamp.

Biermann, H. R., & Jahnke, H. N. (2014). How eighteenth-century mathematics was transformed into nineteenth-century school curricula. In S. Rezat, M. Hattermann, & A. Peter-Koop (Eds.), *Transformation–A fundamental idea of mathematics education* (pp. 5–27). New York: Springer.

Bruner, J. (1969). *The process of education*. Cambridge: Harvard University Press.

Danckwerts, R., & Vogel, D. (2006). *Analysis verständlich unterrichten*. München: Spektrum Akademischer Verlag.

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In W. Dörfler, A. Bikner-Ahsbahs, A. Vohns, & R. Bruder (Eds.), *Theories in and of mathematics education* (pp. 21–31). Berlin, New York: Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Ernest, P. (2008). Towards a semiotics mathematical text (Part 3). *For the Learning of Mathematics, 28*(3), 42–49.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, Boston, Lancaster: D. Reidel.

Fried, M. N., & Dreyfus, T. (Eds.). (2014). *Mathematics and mathematics education: Searching for common ground*. Dordrecht: Springer.

Fritzsche, K. (2005). *Grundkurs Analysis 1*. München: Elsevier.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel, & K. McLain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225–273). Mahwah, NJ: Lawrence Erlbaum Associates.

Halliday, M. A. K. (1985). *An introduction to functional grammar*. London: Edward Arnold.

Heidegger, M. (1979). Heraklit: 1. Die Anfänge des abendländischen Denkens. 2. Logik. Heraklits Lehre vom Logos. In M. Heidegger, *Gesamtausgabe* (Vol. 55). Frankfurt am Main: Klostermann.

Hodges, W. (1997). *A shorter model theory*. Cambridge: Cambridge University Press.

Hoffmann, M. H. G., Lenhard, J., & Seeger, F. (Eds.). (2005). *Activity and sign*. New York: Springer.

Jakobson, R. (1985). *Selected writings* (Vol. VII). Berlin: Mouton.

Krämer, S. (2008). *Medium, Bote*. Suhrkamp: Übertragung. Kleine Metaphysik der Medialität. Frankfurt am Main.

Klein, F. (2007). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (Reprint ed.). New York: Dover.

[5]However, there is an asymmetry from the technical view in favor of university mathematics here.

Mühlhölzer, F. (1999). Mathematik ohne Metaphysik. In J. Nida-Rümelin (Ed.), *Rationalität, Realismus, Revision* (pp. 416–423). Berlin: Walter de Gruyter.

Otte, M. (2011). Evolution, learning, and semiotics from a Peircean point of view. *Educational Studies of Mathematics, 77,* 313–329.

Pichot, A. (1995). *Die Geburt der Wissenschaft*. Frankfurt, New York: Campus.

Polya, G. (2014). *How to solve it: A new aspect of mathematical method*. Princeton: Princeton University Press.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Rotterdam: Sense Publishers.

Presmeg, N. (2014). Mathematics at the center of distinct fields: A response to Michael and Ted. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground* (pp. 45–53). Dordrecht: Springer.

Radford, L. (2008). Culture and cognition: Towards an anthropology of mathematical thinking. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed.). New York: Routledge, Taylor and Francis.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education, 2*(1), 7–44.

Ramharter, E., & Weiberg, A. (2006). *Die Härte des logischen Muß. Wittgensteins Bemerkungen über die Grundlagen der Mathematik*. Berlin: Parerga.

Rotman, B. (2000). *Mathematics as sign: Writing, imagining, counting*. Stanford: Stanford University Press.

Rezat, S., Hattermann, M., & Peter-Koop, A. (Eds.). (2014). *Transformation–A fundamental idea of mathematics education*. New York: Springer.

Sáenz-Ludlow, A., & Kadunz, G. (Eds.). (2015). *Semiotics as a tool for learning mathematics—How to describe the construction, visualisation, and communication of mathematical concepts*. Rotterdam, Boston, Taipei: Sense Publishers.

Sáenz-Ludlow, A., & Zellweger, S. (2015). Classroom mathematical activity. When it is seen as an inter-intra double semiotic process of interpretation: A Peircean perspective. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics.* Rotterdam: Sense Publishers.

Schmandt-Besserat, D. (1997). *How writing came about*. Austin: University of Texas Press.

Seeger, F. (2011). On meaning making in mathematics education: Social, emotional, semiotic. *Educational Studies of Mathematics, 77,* 207–226.

Shapiro, S. (2000). *Thinking about mathematics: The philosophy of mathematics*. Oxford, New York: Oxford University Press.

Spivak, M. (2006). *Calculus* (3rd ed.). Cambridge: Cambridge University Press.

Vaihinger, H. (2009). *The philosophy of "As If"*. Eastford: Martino Fine Books. Original work published 1911.

Wittgenstein, L. (1988). *Philosophische Untersuchungen (PU)* (Vol. 1). Frankfurt am Main: Suhrkamp.

Wittgenstein, L. (1991). *Bemerkungen über die Grundlagen der Mathematik (BGM)* (Vol. 6). Frankfurt am Main: Suhrkamp.

Chapter 8
Using Social Semiotics to Explore Institutional Assumptions About Mathematics Students and Teachers

Candia Morgan

Abstract In this chapter I present some theoretical principles and analytical tools of social semiotics, discussing how they may contribute to mathematics education research. Two examples are offered. The first shows how the way an examination question is written construes the nature of mathematical activity; I consider how this may function for different students. The second example uses thematic analysis to look at a corpus of official documents, asking what activities are made available in these texts for teachers and students.

Keywords Social semiotics · Thematic analysis · Systemic functional linguistics Official discourse · School mathematics practices

8.1 Introduction

Mathematics education as a field has a wide range of concerns, but most semiotic research has tended to focus on analyzing students' mathematical understanding and its development, including analysis of student-teacher-tool interactions. Radford's cultural historical semiotic approach situates sign use and knowledge production within a cultural context, but research from this perspective also focuses primarily on teaching and learning processes and interactions, on the nature of mathematical knowledge and on the development of mathematical thinking (e.g., Radford 2003). Less attention has been paid to how a semiotic perspective can provide insights into the wider context of mathematics education and how this may affect the experiences of student and teachers.

The teaching and learning interactions of students and teachers take place within schools and colleges that are themselves situated within local and national education systems, national and international policies and discourses about mathematics

C. Morgan (✉)
Department of Curriculum Pedagogy and Assessment, University College London Institute of Education, 20 Bedford Way, London WC1H 0AL, UK
e-mail: candia.morgan@ucl.ac.uk

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and about teaching, learning, curriculum and assessment. The field of mathematics education research needs to address all of these sites, seeking to understand the concerns and practices of all those involved in their various ways with mathematics education and the relationships between them—while recognizing that it is likely to be necessary to focus on only a small part of the complex at any one time (Morgan 2014; Valero 2010). In this chapter I hope to contribute to this project of researching the wider complex of social practices of mathematics education by discussing a semiotic perspective that offers some theoretical and analytical tools, and by offering examples of analysis that can provide insight into how teachers and students of mathematics and the ways they experience mathematics teaching and learning, may be shaped by the institutional context in which they are situated.

The approach that I am adopting here is founded in social semiotics (Halliday 1978; Hodge and Kress 1988; Morgan 2006). Fundamental to this theoretical perspective is the principle that language and other modes of communication (including visual images, gestures, specialized forms of mathematical notation, etc.) are functional, not representational. Any socially coherent unit of communication (a 'text' that plays a role within a given social practice), whether a linguistic utterance, a picture, a mime, a research paper, is not assumed to represent any objective feature of an independently existing 'real world' but functions to realize, that is, to 'make real' the following: the *field* of discourse—the event being spoken about; the *tenor*—who are the interlocutors and what are their relationships to each other and to the event; and the *mode* of discourse—the role of the text itself within the event. The way we experience the world is thus construed through the use we make of language and other means of communication (Halliday and Mathiessen 1999). This use of communication is shaped (though not determined) by the immediate context in which it takes place (the context of situation) and by the wider social practices, cultural norms and expectations with which the participants are familiar (the context of culture). Semiotic analysis that seeks to understand how meanings are formed thus needs to take these contexts into account.

While social semiotic theory and analytic approaches have been extended to encompass non-verbal semiotic systems (Jewitt et al. 2016; Kress and Van Leeuwen 2006), including specialized mathematical systems such as geometric diagrams (Alshwaikh 2011), algebraic notation and Cartesian graphs (O'Halloran 2005), verbal language still comprises the most sophisticated and differentiated semiotic resources and it is here that I focus my attention in this chapter. Within social semiotics, systemic functional linguistics (SFL) has developed descriptors that enable analysis to distinguish how specific lexicogrammatical characteristics of texts perform ideational, interpersonal and textual metafunctions, contributing to realizing the field, tenor and mode of discourse respectively. In this paper, I will focus primarily on the ideational metafunction, the way that language realizes the field of discourse. In the context of mathematics education research, analysis of how communication functions ideationally can address general questions such as the following: What is mathematics (in the context of the communication that is the current object of study)? Who or what does mathematics? What do teachers do? What do students do? (See Morgan 2006). In this chapter, the first example I present addresses the first two questions, interrogating the nature of mathematics and mathematical activity that may be construed by students faced with two slightly different texts. The second example focuses on the last two questions, considering how a set of texts current in school mathematics contexts in England creates possible positions, actions and relationships for teachers and students.

8.2 Example 1: Analyzing the Mathematics in School Mathematics

In my first example, my focus is on how students may experience and engage in mathematics itself, illustrating the use of SFL tools to analyze how some small extracts of school mathematics text function to realize mathematical activity and considering possible implications for students. Halliday identifies the transitivity system as a significant contributor to the ideational metafunction, enabling participants in communication to construe what is happening and who and what is involved. In traditional grammar, a verb is termed transitive or intransitive, depending on whether it can take a grammatical object. Halliday's transitivity system extends this notion to encompass all types of relationships among processes (often expressed as verbs), participants (e.g. subjects and objects of processes) and circumstances (e.g. the time, manner or place in which a process occurs). SFL describes in general terms the implications of particular lexicogrammatical choices within the transitivity system. Thus, taking a mathematical example, we can say that statements (1) and (2) below construe different versions of the nature and origin of a geometric figure:

$$\text{P lies on the line AB and CP is perpendicular to AB.} \tag{1}$$

$$\text{A perpendicular dropped from C meets AB at P.} \tag{2}$$

Statement (1) draws our attention to the properties of the points and lines and relations between them, using the atemporal present tense and an attributive relational process (*is perpendicular*), whereas in statement (2) the past participle *dropped* suggests an action taking place in time, although agency in this action is obscured (who or what dropped the perpendicular is unknown), and the figure may be construed as having been constructed rather than simply existing (see Morgan 2016, for a fuller discussion).

Analysis is not, however, a simple matter of reading off 'the meaning' or the specific way the text functions from the lexicogrammar. The descriptions above are not enough to allow us to draw conclusions about the significance of such differences for participants in particular practices. For this purpose, it is necessary to take account of the immediate context of situation within which the texts occur as well as the broader cultural context, including the social structures, assumptions, and values of the broader communities implicated in the practice. In this case, statement (1) formed part of a question set in a high-stakes examination taken by students in England at the end of compulsory schooling. Statement (2) is a transformed version of statement (1), incorporating grammatical forms used in a similar examination question set in another year.[1] Because of their origin in externally set, high-stakes examinations, these texts may be seen as part of an official mathematical discourse, likely to influence the forms of mathematics experienced in classrooms. The original context of situation of these texts and others like them thus included students' experience of school mathematics up to the point of examination, the physical and social environment of the examination room, the linguistic and mathematical resources brought by students to the examination, etc. The context of culture would include the following: the roles that such examinations and qualifications play in individual lives and in the broader society; public and academic views of the nature of mathematics; and, more generally, the values, norms and ways of communicating of the students' communities and of the dominant groups in national and international society.

These contexts are not homogeneous, nor will all students experience them, and hence relate to the texts, in similar ways. Given the examination setting, an important contextual consideration is the extent to which students are likely to be able to recognize and draw upon their previous mathematical experience in order to interpret such statements and the examination tasks of which they form part. The official discourse of education and examination in England and elsewhere also places value on equality of opportunity—a value that forms part of the context of culture. It is thus relevant to consider how these statements may function for different groups. For example, statement (1) has a simpler grammatical structure than statement (2) (consisting of two simple clauses joined by *and* rather than involving a complex nominal group with a dependent clause *dropped from C*); it may thus offer fewer linguistic obstacles for students with lower levels of literacy in English. On the other hand, statement (1) is also consistent with the tendency of academic mathematical discourse to be impersonal and to deal with the properties of mathematical objects and the relationships between them rather than on actions, construing "a world made out of things, rather than the world of happening—events with things taking part in them—that we were accustomed to" (Halliday 1993, p. 82). It may thus be less familiar than statement (2) for students whose everyday ways of communicating focus primarily on people and actions (see, for example,

[1]The analysis reported in Example 1 was undertaken as part of the project *The Evolution of the Discourse of School Mathematics*, funded by the UK Economic and Social Research Council (grant number ES/1007911/1). This project sought to describe changes over time in the discourse of school mathematics, seen through the lens of high-stakes examination.

Lunney-Borden 2011 on the ways of speaking and knowing of indigenous people in Canada[2]), and hence harder to engage with in the ways expected by the examiners.

Social semiotics considers language to be a semiotic system encompassing a ‘meaning potential’, that is, the set of ways of experiencing the world that may be construed through linguistic means. The use of SFL tools not only describes the lexicogrammatical structure of a given text but also, through analyzing this structure, enables us to describe how the text functions ideationally, interpersonally and textually to offer particular forms of experience. In this case, I have identified differences in the potential of the two short statements to allow readers to construe a geometric figure either as a set of objects and relationships between them or as the result of a process of construction. For those of us who have successfully learnt to participate in specialized mathematical discourse (including teachers, textbook authors, examiners, test designers), the significance of this distinction may not be immediately apparent, since, in coming to grips with the object-process duality of many mathematical constructs, we often fail to recognize it or to appreciate how novices may experience it in a less unified way. For less experienced students, however, the differences may have a more significant effect on how they relate to the mathematical situation and respond to the examination task.

In presenting this small example, I do not seek to make claims about the nature of school mathematics in general, or even within the English examination system. Such claims would require analysis of much more substantial samples of text. Nor do I seek to claim that student readers will necessarily respond to the texts in the ways suggested—this is also an issue that requires investigation. Rather, the analysis of the two texts and discussion of how some groups of students may relate to them has potential to inform the choices made by the authors of examinations, textbooks and other school mathematics texts.

8.3 Example 2: Official “Good Practice in Mathematics Teaching”

In my second example I move away from the immediate site of interaction between student and mathematical text to consider the domain of government and its interaction with schools and teachers. I illustrate one way in which a social semiotic orientation can illuminate patterns in larger texts and sets of texts. The example I discuss is an analysis of a set of five short documents (3–4 pages each) downloaded from the website of the UK government agency, the Office for Standards in Education (Ofsted). The page from which they were downloaded was entitled

[2]Thanks to Beth Herbel-Eisenmann (personal communication) for focusing my attention on linguistic variation as an aspect of context.

"Ofsted examples of good practice in mathematics teaching"[3] and each document presents a case study, describing the practice of a school or college.

Why are these documents of interest to me as a researcher in mathematics education? To answer this question, I have to delineate some context, which includes some key characteristics of the national education system in England and, in particular, the role that Ofsted plays in regulating schools. Every school and further education college in receipt of state funding is subject to regular inspection by Ofsted. The inspection encompasses consideration of national test and examination results as well as scrutiny of various aspects of management and observation of teaching and of student behavior. This results in allocation of grades for a number of aspects, ranging from 1 (outstanding) to 4 (inadequate). A grade of 4 in one or more aspect can result in 'special measures' and re-inspection. Failure to improve adequately can result in the removal of senior managers and even closure of the school. Inspection thus has very high stakes for school management. Ofsted also publishes periodic reports on various aspects of education, providing national overviews. These include descriptions of what the agency identifies as 'good practice', such as the documents considered here.

As with any high-stakes assessment regime, Ofsted has a strong influence on practice within many schools, not only dominating management priorities but also affecting the day-to-day practice of teaching, as the work of teachers and subject departments is subject to monitoring and regulation by school managers, using criteria shaped by their perception of what will be valued by Ofsted inspectors. This institutional environment constitutes part of the context within which the documents discussed here are located, affecting the ways that potential readers may read them.

My particular concern is with how these texts function ideationally, that is: What is the world of "good practice in mathematics teaching" that Ofsted provides for the reader to construe? As in the previous example, I attend to the transitivity system. In this case, the particular focus of my analysis is on how teachers and students feature in this world: the processes in which they are agents or which they are subjected to, their attributes and the ways they are classified. In this chapter, given the limited space available, I present only an analysis of teacher and student agency, addressing

[3]This webpage can be accessed at https://www.gov.uk/government/collections/ofsted-examples-of-good-practice-in-mathematics-teaching. The documents analyzed here were last accessed and downloaded on 14 December 2015 but the webpage is regularly updated to provide a recent set of examples. An attempt to access the documents on 8 March 2016 revealed that two of those included in the current data set had been removed. Attempts to access the two missing documents revealed messages to say they had recently been withdrawn. In one case (School L) the withdrawal was justified because it was over 3 years old. In the other case (School A) it was more strongly stated that: "This good practice example no longer reflects current government policy". Accessing again on 10 February 2017 found only one of the original case studies (School C) together with two new cases focusing on examples of "family learning", organised by adult education services, rather than on school practice.

the question of what teachers and students do within Ofsted's world of "good practice".

The analysis starts at the level of the clause, identifying in each case who or what are the actors in what kind of process. This process was supported by use of NVivo, coding each clause according to agency, identifying whether agency was ascribed to teachers, to pupils, to the school management or to an inanimate or abstract construct, or whether agency was obscured, for example by use of the passive voice. Codes were also allocated to clauses attributing qualities to teachers or to pupils but analysis of these attributes is beyond the scope of this chapter. All clauses in which agency was ascribed to teachers or to pupils were extracted in order to investigate the types of processes and circumstances involved.

In the previous example, analysis of isolated statements was used to highlight how specific linguistic choices have potential to make differences in the ways that the world of geometry may be construed. Here, however, my interest is in developing an understanding of the world of "good practice" that may be construed by readers of these official texts. In this case, it is necessary to look at patterns that recur within each text and across the full set of texts. For this, I have adopted a version of Lemke's (1983) thematic analysis, simplified to consider only the transitivity system of actors, processes and circumstances. This form of analysis identifies common semantic structures through the cohesive devices present in the text, in order to identify semantic themes that are established by recurrence within single texts or throughout the whole corpus. A common structure may be detected not only in the direct repetition of specific actor/process relations but also in lexical covariation, such as the presence of synonyms, and in grammatical transformations of similar relationships. In what follows I denote such recurring themes by the use of capitals, e.g. PUPILS NEED SUPPORT. (Note that these texts tend to use *pupil* rather than *student*. This choice of word suggests that one should construe young people in accord with their institutional relationships rather than as people who study.) The identification of such a theme summarizes an Actor-Process-Goal structure that recurs in a variety of forms. In the set of texts considered here, for

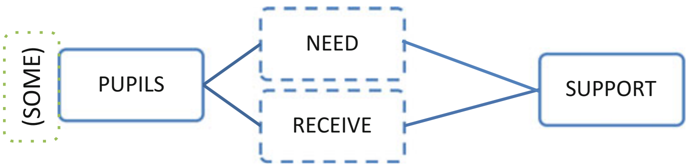

Fig. 8.1 Summary of recurrent themes relating "pupils" and "support"

example, the semantically related constructions PUPILS NEED SUPPORT and PUPILS RECEIVE SUPPORT are found in three of the five texts, both in this form (Actor-Process-Goal) and in other forms, including:

"Pupils receive the support they need"	The nominal group *the support they need* transforms the process *need* so that it qualifies the object *support*. This statement thus incorporates both PUPILS NEED SUPPORT and PUPILS RECEIVE SUPPORT
"Less able pupils benefit from expert support for their individual needs"	The process *benefit from* is considered here to be semantically related to *receive*, with the additional attribution of positive value The nominal group *support for their individual needs* is a grammatical transformation, objectifying the basic relationship PUPILS NEED SUPPORT. In this statement, the qualifiers *less able* and *individual* introduce an additional semantic component, constructing a classification of different types of pupil
"Pupils receive appropriate support"	The qualifier *appropriate* is considered to be semantically related to what pupils *need*

These recurrent semantic themes, summarized at the level of the set of texts, are shown in Fig. 8.1. The themes may be read directly from the figure:

(SOME) PUPILS NEED SUPPORT
and
(SOME) PUPILS RECEIVE SUPPORT,

noting that the qualification (SOME) does not occur in every instantiation of these themes.

In the next sections, I present the outcomes of the thematic analysis in order to describe those parts of the ideational structure of the world of "good practice in mathematics teaching" in which teachers and pupils are actors.

8.3.1 What Do Teachers Do?

The statements with teachers as agents were scrutinized in order to characterize the nature of teachers' role within Ofsted's construal of "good practice". In order to develop a thematic analysis of the nature of teacher activity presented in the texts, these statements were further categorized as being about teaching (43 statements), about assessment (20 statements) or involving an activity that was not directly related to student-teacher interactions (17 statements), including mainly collaborating with colleagues and developing their practice by receiving feedback or training. In this chapter, I present the analysis of teaching activity. Figure 8.2 shows the three main types of teaching activity, present in all five case studies.

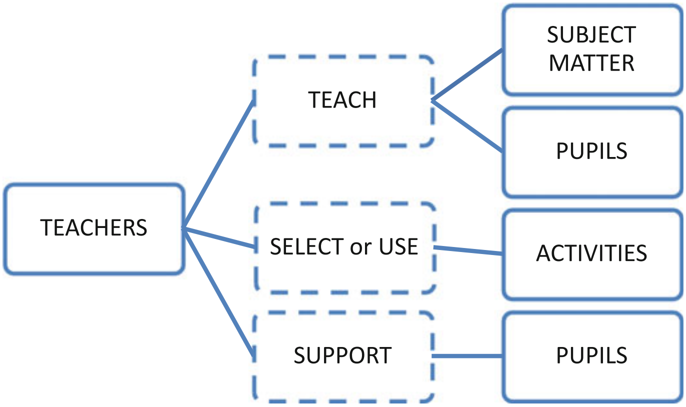

Fig. 8.2 Thematic analysis of teaching activity—dominant discourse across all schools

As might be anticipated given that PUPILS NEED and RECEIVE SUPPORT (as identified above), we find a reciprocal thematic structure TEACHERS SUPPORT PUPILS. Apart from supporting, TEACHERS TEACH both the subject matter and the pupils, though interestingly they do not appear to teach mathematics TO pupils. The teachers' role is perhaps made clearer by the additional thematic structure TEACHERS SELECT/ USE ACTIVITIES. Having chosen appropriate activities for their pupils, further actions taken by teachers in the classroom are absent or invisible.

Just two of the school case studies provide more detailed insight into specific aspects of teaching and how they are valued (Fig. 8.3). In the case of School A, positive value is ascribed to creativity, as an attribute of both teachers and teaching, while teachers are also construed as acting directly upon learning, for example, "making learning outstanding".

This valuing of creativity is apparent in the School A case study not only in the way that teacher activity is directly presented in the recurrent semantic theme TEACHERS USE (CREATIVE/EXCITING) STRATEGIES, but also in other ways, in particular by contrasting current practice in the school with "traditional lessons" and "textbook lessons", and by labeling the lack of prescription in the school's scheme of work as "surprisingly informal". It is worth noting that a more

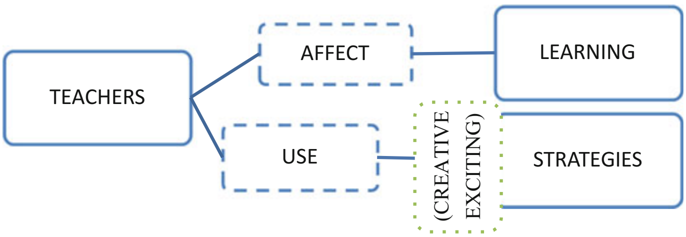

Fig. 8.3 Additional thematic structure of teaching activity in School A

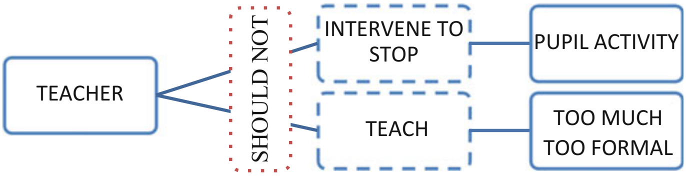

Fig. 8.4 Additional thematic structure of teaching activity at School B

recent attempt to access this slightly anomalous case study found that it had been removed from the Ofsted website with a statement that "This good practice example no longer reflects current government policy."

The second case study that provides more details about teaching processes is School B (Fig. 8.4). In this case, the kind of teaching that is valued is presented only by contrast with processes presented as negative or inappropriate, for example:

> All too often teachers risk intervening and doing the investigation for the children!

> We would never stop a child who has an efficient method, but often we find that if a child has been taught a formal written method at a very early age, it may be at the expense of their grasp of mental methods.

Interestingly, these negatively valued teaching processes are not contrasted with positively valued actions of teaching but with processes that are identified (and positively valued) as forms of assessment: rather than intervening while children investigate, "[t]eachers are encouraged on these occasions to listen and record."

8.3.2 *What Do Pupils Do?*

The processes in which pupils are presented as agents are rather more diverse. To some extent, this diversity is due to the way the texts ascribe attributes to pupils, distinguishing them into different groups, which are then ascribed different forms of

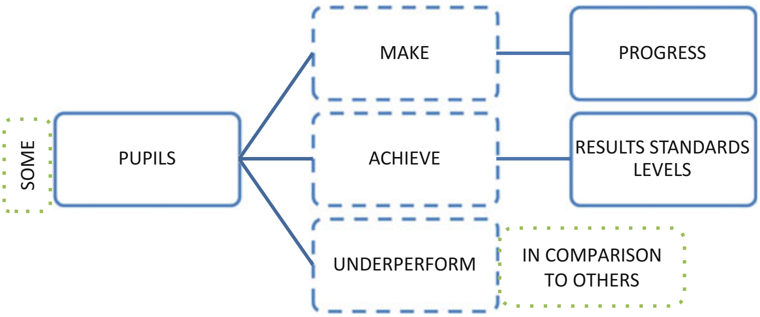

Fig. 8.5 Thematic structure of pupil achievement—dominant discourse across schools

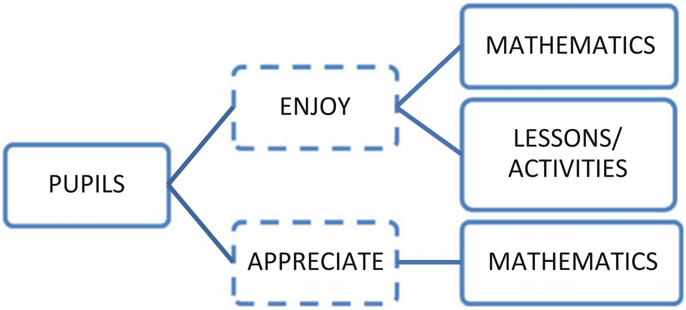

Fig. 8.6 Thematic structure of pupil affective response—dominant discourse across schools

agency. In particular, all the case study texts distinguish some pupils as being more or less "able" or as achieving more or less than their peers. A semantic grouping of statements involving pupils as actors resulted in four main categories, related to Support, Achievement, Affective response and Classroom activity. The results of thematic analysis regarding the category of Support were presented above (Fig. 8.1). The other three categories are presented below.

8.3.2.1 Achievement

Two ways of conceptualizing pupil achievement are thematized across the set of texts. On the one hand there is a dynamic process: (SOME) PUPILS MAKE PROGRESS. From this perspective, pupil achievement is recognized as taking place over time and measured relative to each individual's starting point. On the other hand, the theme (SOME) PUPILS ACHIEVE RESULTS/STANDARDS/LEVELS measures pupil achievement at a particular point in time against a fixed standard. The third theme identified in Fig. 8.5 also compares pupil achievement to a norm but in this case identifies failure rather than achievement: (SOME) PUPILS UNDERPERFORM.

Many of the occurrences of the themes grouped under this heading are qualified by an indication that different groups of pupils achieve in different ways or to different extents. This is consistent with the persistent dominance in the English education system of assumptions, institutional structures and practices that separate children according to 'ability' (Morgan 2017).

8.3.2.2 Affective Response

All the case studies include statements claiming that pupils have positive affect towards mathematics or towards their mathematics lessons (Fig. 8.6). In most cases, this is simply stated as liking or enjoying mathematics, PUPILS ENJOY MATHEMATICS. In other cases, this enjoyment is attributed to lessons or, in a few cases, specific forms of lesson activity, PUPILS ENJOY LESSONS/ACTIVITIES, e.g.:

> Students enjoy a range of sorting and matching activities, often in pairs or groups that promote discussion and help to develop their understanding.

A claim linking pupils' beliefs about mathematics to positive affect is made in two of the cases, PUPILS APPRECIATE MATHEMATICS. In particular, in School N, a post-compulsory college catering for young people aged 16-19, this theme is especially strong, focusing repeatedly on the 'relevance' of mathematics, rather than on pure enjoyment, for example:

> They also appreciate the way that numeracy development is made relevant to their main area of study and to real life.

8.3.2.3 Classroom Activity

As found in the analysis of teacher activity, little detail is generally included about the ways that students act in classrooms (Fig. 8.7). Two main types of theme are

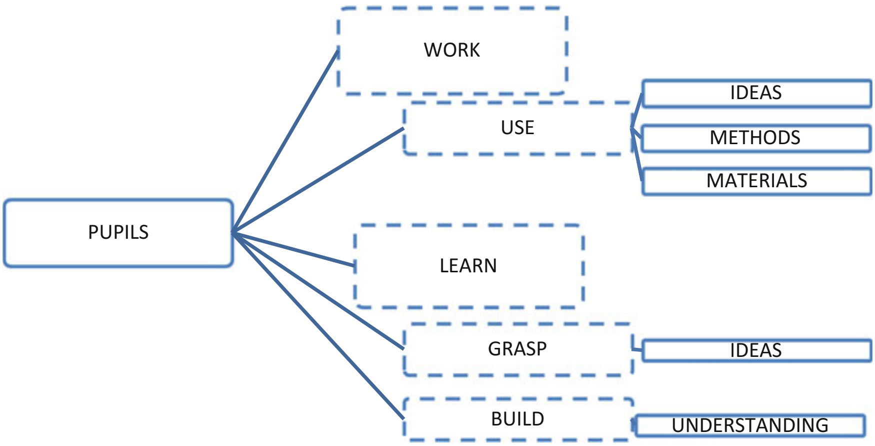

Fig. 8.7 Thematic structure of pupil classroom activity—dominant discourse across schools

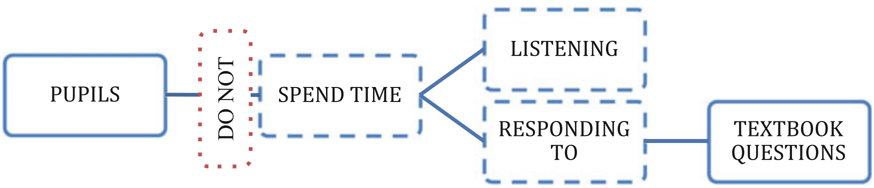

Fig. 8.8 Additional thematic structure of pupil activity in School A

identified in relation to classroom activity, each of which appears both in a generic form and in elaborated forms. On the one hand, PUPILS WORK. Where the nature of this "work" is elaborated, it involves resources that PUPILS USE … MATERIALS or previously learnt IDEAS or METHODS. The second type of pupil classroom activity is learning: PUPILS LEARN. Interestingly, the nature of learning activity is elaborated by two, slightly different, physical metaphors: PUPILS GRASP IDEAS and PUPILS BUILD UNDERSTANDING.

Just two of the case studies offer more detailed exemplification of classroom activity: in School A an extended description is given of a simulation activity involving pupils in buying, selling and negotiating discounts, while the School B case study includes reference to one pupil "working on an exercise practicing the meaning of 'is equal to'". While these two examples contribute to how readers may construe pupils as actors in classrooms, they are very limited and specific, making it unlikely that a reader could form a general impression of what pupils do in classrooms in either of these schools. The only more general descriptions of what pupils do are presented in the negative in the School A case study (Fig. 8.8). As noted in the analysis of teacher activity discussed above, for School A, current pupil activity is valued by contrasting it to a supposedly 'traditional' practice in which pupils are passive recipients LISTENING or RESPONDING.

Other references to more specific forms of classroom activity are objectified, using nouns or nominal groups such as *investigation*, *calculation*, *problem solving*, *discussion*, *practical activity*, *structured play*, *sorting and matching activities*. This objectification obscures the roles that students and teachers may play, providing no indication of where agency might lie in relation to these types of activity.

8.3.3 Reflecting on the Thematic Analysis of Official Discourse on Teacher and Pupil Activity

The dominant discourse, present in all five texts, does not problematize or elaborate the process of teaching. The few examples of classroom activities do not provide generalizable principles for devising or evaluating further activities. As the thematic structure offers little description of what is involved in teaching a topic or teaching students, the texts present a world in which everyone is assumed to know and agree upon what constitutes 'teaching'. Given that these texts are labeled as case studies of good practice, serving as models for development of practice in other schools, it appears that, in general, teaching is not to be seen as a site for development. This conclusion, drawn from texts published officially in 2015, is in stark contrast to the extensive guidance about "good teaching" in official documents published prior to the election of a new Conservative/Liberal coalition government in 2010 and since removed from official websites (see Morgan 2009 for an analysis of official discourse from the pre-2010 era). Rather than the development of teaching, the case studies place emphasis on ‘support’ for pupils. The broader analysis of actors and processes also identified ‘monitoring’ as a highly valued activity (22 occurrences across the five texts) to be undertaken by teachers and by school management. It appears that the policy reflected in this set of texts (accessed in December 2015) focuses on achievement (and the need to support some pupils to reach required levels of achievement) and progress (monitored to ensure that it is adequate) rather than on the processes of teaching that might enable these.

8.4 Implications of a Social Semiotic Approach to Research

The two examples presented in this chapter have illustrated the application of a social semiotic approach at two different grain sizes. The first example took single statements as its unit of analysis, while, in the second example, the unit of analysis was a corpus of five documents, comprising a total of 18 pages. In both cases, the analysis addressed how the texts function ideationally: how they realize the world of geometric objects, or of teachers, pupils and their activities. The starting point in each case was identification of actors and processes, though these were then examined using tools appropriate to the scope of the unit of analysis. To compare the two single statements in example 1, grammatical choices at the level of the clause were identified, revealing potential differences in the way readers might construe geometric objects and activity. In contrast, the thematic analysis used in example 2 worked at the level of the whole set of documents, identifying persistent patterns of actor-process relationships in order to identify how the activity of mathematics teachers and their pupils is construed in the discourse of a government agency. A clause-level analysis similar to that used in example 1 could also be conducted across a more substantial text or set of texts in order to characterize the quantitative distribution of particular grammatical choices.

As discussed earlier, taking a social semiotic perspective to analyze communication entails more than simply applying a set of linguistic tools or techniques. In order to consider how the communication may shape the ways participants experience the world, it is also necessary to take account of the context of situation in which the texts are produced and used and the broader context of culture shared by the participants in the communication. Of course, not all participants draw on identical experiences, cultural contexts or discursive resources; alternative interpretations and ways of construing the world are possible. In presenting the examples in this chapter, I have attempted to give enough of my understanding of these contexts (as an ‘insider’ in the education system in England) to provide backing for the interpretations I have made. Responses from colleagues with whom I have shared these examples provide support for my contention that the interpretations presented here are convincing to others with knowledge and experience of the English education system.

The examples presented in this chapter illustrate some of the power of a social semiotic orientation to provide insight into those aspects of mathematics education beyond the immediate context of teaching and learning processes and interactions. Understanding mathematics education as a complex of social practices situated within social structures draws attention to a wider range of relevant practices. It also enables analysis of the meanings that may be construed through texts produced within a given practice to be informed by knowledge of the contexts within which they are likely to be read. Focusing on texts produced in the official fields of examinations and of school monitoring and accountability has enabled identification of hegemonic discourses and hence of strong messages about teachers, pupils, mathematics, teaching and learning that permeate schools in England. Knowledge gained by this analysis enables critique and provides support for resistance to these messages. While the analysis reveals how the 'ideal reader' addressed by the text may construe their experience of mathematics education, the ways we experience the world are not determined by the hegemonic discourse. Our critical awareness of alternative ways of speaking about mathematics, teaching and learning makes it possible to challenge the institutional assumptions embodied in such texts and to change these assumptions by imagining and speaking of other ways of thinking and being. An important method of critique is to point not only to what *is* but also to how it might *be different*. Here SFL has much to offer as it is predicated on the notion of language as a *system*, conceptualizing specific instances of texts as formed by making (not necessarily deliberate or conscious) choices between paradigmatic and syntagmatic alternatives in the lexicogrammar, and on the theoretical principle that these choices affect how texts *function* in realizing the field, tenor and mode of discourse. This empowers us to consider, for example, what difference it might make to students if geometry were presented as actions and their outcomes instead of as a set of objects and relationships. It enables us to ask what difference it could make to teachers and students if the messages of the official discourse directed the attention of school management to, say, the quality of mathematics teaching rather than (or as well as) the quantity of student achievement.

References

Alshwaikh, J. (2011). *Geometrical diagrams as representation and communication: A functional analytic framework.* (unpublished PhD thesis), Institute of Education, University of London, London.

Halliday, M. A. K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. London: Edward Arnold.

Halliday, M. A. K. (1993). Some grammatical problems in scientific English. In M. A. K. Halliday & J. R. Martin (Eds.), *Writing science: Literacy and discursive power* (pp. 69–85). London: Falmer.

Halliday, M. A. K., & Mathiessen, C. M. I. M. (1999). *Construing experience through meaning: A language-based approach to cognition*. London: Continuum.

Hodge, R., & Kress, G. (1988). *Social semiotics*. Cambridge: Polity Press.

Jewitt, C., Bezemer, J., & O'Halloran, K. (2016). *Introducing multimodality*. London: Rutledge.

Kress, G., & Van Leeuwen, T. (2006). *Reading images: The grammar of visual design* (2nd ed.). London: Routledge.

Lemke, J. (1983). Thematic analysis: Systems, structures and strategies. *Recherches Semiotiques/ Semiotic Inquiry, 3,* 159–187.

Lunney-Borden, L. (2011). The 'verbification' of mathematics: Using the grammatical structures of Mi'kmaq to support student learning. *For the Learning of Mathematics, 31*(3), 8–13.

Morgan, C. (2006). What does social semiotics have to offer mathematics education research? *Educational Studies in Mathematics, 61*(1/2), 219–245. https://doi.org/10.1007/s10649-006-5477-x.

Morgan, C. (2009). Questioning the mathematics curriculum: A discursive approach. In L. Black, H. Mendick, & Y. Solomon (Eds.), *Mathematical relationships in education: Identities and participation* (pp. 97–106). London: Routledge.

Morgan, C. (2014). Understanding practices in mathematics education: Structure and text. *Educational Studies in Mathematics, 87*(2), 129–143. https://doi.org/10.1007/s10649-013-9482-6.

Morgan, C. (2016). Studying the role of human agency in school mathematics. *Research in Mathematics Education, 18*(2), 120–141. https://doi.org/10.1080/14794802.2016.1176595.

Morgan, C. (2017). From policy to practice: Discourses of mastery and 'ability' in England. In A. Chronaki (Ed.), *Mathematics education and life in times of crisis: Proceedings of the 9th international conference of Mathematics Education and Society* (pp. 717–727). Volos, Greece: University of Thessaly Press.

O'Halloran, K. L. (2005). *Mathematical discourse: Language, symbolism and visual images*. London: Continuum.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70.

Valero, P. (2010). Mathematics education as a network of social practices. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. LIV-LXXX). Lyon: Institut National de Rechérche Pédagogique.

Chapter 9
Semiotics, Epistemology, and Mathematics

Michael Otte

Abstract The relationship among language, thought, and knowledge has been perceived in different ways throughout the history of Western culture. In general, since the advent of the modern age, mathematical ideas have been considered as universal, objective and accurate, being detached from the contingencies of language and the flaws of communication. However, since the nineteenth century, with the failure of attempts to establish definitive foundations of mathematics, the advent of the expansion of education and the consequent view of knowledge as a social institution, new theoretical perspectives emerged, attributing importance to the relations and similarities between mathematics and language.

Keywords Semiosis · Peirce · Mathematics · Mathematics education
Complementarity

9.1 Introduction

The main impact of the Scientific Revolution of the 17th century came from a change in the habits of thought and, in particular, from a campaign for individual certainty. It was the central problem of Descartes and the general purpose of his *Discourse on Method*. "The one activity in the world, which really does concern Descartes, is thought and the pursuit of truth. Had he composed the Lord's Prayer, it would no doubt have contained the invocation 'and lead us not into error'" (Gellner 1992, p. 7).

Until the Renaissance, all knowledge seemed the result of an interpretation of the *Great Book of the Universe*. Then at a certain period in history it happened that words and things parted ways and direct interpretation of our sense impressions seemed to become utterly unreliable and the quest for individual certainty became a fundamental motif of the modern age. It provided impetus for the emergence of

M. Otte (✉)
University of Bielefeld, Neuenkirchenerstr. 6, 49324 Melle, Germany
e-mail: michaelontra@aol.com

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modern science and was very influential in shaping the debates about the role mathematics could or should play in science. Interpretation of Nature became replaced by experimentation and this opened the road towards the application of mathematical methods in science.

It is a merit of Foucault (1973) to have brought these facts to our attention. At the beginning of the 17th century,

> writing has ceased to be the prose of the world, resemblances and signs have dissolved their former alliance, similitudes have become deceptive. … Thought ceases to move in the element of resemblance. Similitude is no longer the form of knowledge, but rather the occasion of error. … The age of resemblance is drawing to a close. … And just as interpretation in the sixteenth century … was essentially a knowledge based upon similitude, so the ordering of things by means of signs constitutes all empirical forms of knowledge as knowledge based upon identity and difference. (pp. 47–51 and pp. 56–57)

For Aristotle, and up to the period we call the Baroque, things had essences; now words have meanings. Essence is the attribute or set of attributes that make up an entity, something Leibniz had called the "complete concept of a substance". And his *Principle of the Identity of Indiscernibles* establishes a one-one relation between concepts (ideas) and objects (substances). Leibniz's principle expresses the view that the universe is a system of signs, rather than a set of atoms. In this sense one might say that the Leibnizian project is situated "at the very heart of Classical thought" (Foucault 1973, p. 57). Leibniz does not admit atoms, indiscernible objects. Atoms are like the points of a line or the elements of a set—indistinguishable in themselves. In contrast a system of signs is established in terms of differences. In his fifth letter to Clarke, Leibniz (1956) stated:

> I said that in sensible things, two that are indiscernible from each other can never be found; for instance, two leaves in a garden or two drops of water, perfectly alike are not to be found. I believe that these general observations in things sensible, hold also in proportion in things insensible. (p. 61)

"Every thought is a sign. This is the doctrine of Leibniz, Berkeley and the thinkers of the years around 1700", wrote Peirce (CP 5.470). The insight that man is essentially a "symbolic being" slowly and gradually replaced the traditional Aristotelian characterization of man as a "rational animal".

And the break that Foucault describes has finally since the 19th century led to a *principle of complementarity* of sense and reference. The appearance of this principle is due to the fact that science and mathematics became social institutions and practices whose epistemic subject and theoretical object became social entities. Sense and reference of symbolic representations are distinguished by their complementary roles in the development of knowledge. We use our language terms attributively as well as referentially, because an entity is not just the sum of its attributes and we do in fact encounter things sometimes directly without knowing much of how to describe them.

At the turn to the 20th century Charles Peirce and Ferdinand de Saussure came independently forward with the idea of a general theory of signs. Peirce suggested to call the new discipline *semiotics* and Saussure named it s*emiology*. Both aimed at a general and unified theory of signs that could serve as a method and foundation for a wide variety of other scientific disciplines.

9.2 What is a Semiotic Framework?

What do people mean when talking about a semiotic framework? A semiotic framework is essentially a methodological device. Semiotics is a methodology, not a philosophical doctrine. However, questions of methodology gain continuously in weight and consume growing parts in all discussions about research and science, not least because of the omnipresence of scientific technologies in modern society.

Because all thinking occurs in terms of signs, to interpret something just means to represent it. The essence of something is nothing but the essence of a representation of that thing. We can ask neither for the ultimate referent, nor for the definite meaning of a sign. And therefore the semiosis stretches out in both directions, towards the object—there is no definite referent—as well as towards the interpretant—there is no final and definite interpretation either and the interpretant is just a translation or development of the original sign. Peirce accordingly distinguishes between two objects of the sign, "the *immediate object*, or the object as the sign represents it, from the *dynamical object* or really efficient but not immediately present object" (CP 8.343). And we should also distinguish between the immediate and the dynamical interpretant.

It is commonly assumed that signs are instruments invented and used for certain purposes. But, Peirce introduced the term "semiosis" to challenge such a perspective. In other words, there must be some general notion that establishes the semiotic process, because the sign stands for its object not in all respects, but in reference "to a sort of *Idea*, the ground of the representamen" (Peirce, CP 2.228). And the notion of a general involves the imagination of a continuum of possible individuals (Peirce, CP 5.102). There are all sorts of humans, good ones and bad ones. In mathematics and the natural sciences these continua are called "objectual variables", that is, incompletely specified objects (Quine 1974, p. 98).

The fundamental triad in Peirce's semiotics is thus "object—sign—interpretant" (Peirce CP 8.361). This triad indicates an inherently dynamic sign process that is not controlled by an external human agent according to his wishes. For example, the process of communication is not constituted by the encounter of independent actors, who decide to tell each other whatever comes to their minds. Rather communication is a social system, a system that does not directly interact with the person's minds or consciousness and that is not arbitrarily constructed by the participants. The human agents are subsystems, or rather they have to constitute themselves as such subsystems of the larger social system of communication. Peirce uses the phrase "man is a sign" to describe this situation (cf. Radford's chapter, this volume). He wrote as follows:

> Man makes the word. … But since man can think only by means of words or other external symbols, these might turn round and say: 'You mean nothing which we have not taught you, and then only so far as you address some word as the interpretant of your thought'. … The word or sign which man uses is the man himself. For as the fact that every thought is a sign, taken in conjunction that life is a train of thought, proves that man is a sign. (CP 5.313–314)

Semiotics is not reductionist, it flourishes on differences. The method of semiotics is characterized by a kind of structuralist or systems approach whose essential requirement consists in conceiving of reality in terms of the complementarity of structure and function. The fallacy of set-theoretical foundationalism in the sense of Cantor, as endorsed by the New Math consists in the belief that a thing is the sum of its elements. However, the parts of a system are systems again, and they become subsystems by the different functions they fulfill within the system. Scientists tell us that even the nervous cells within a single organ, such as the eye, are specialized: some cells are sensitive to color, others only to straight direction in the vertical sense, others in the horizontal, etc.

A system is not just the sum of its parts. It is always more than that, because its differentiation into parts exists only as a function of the whole. A swan has eyes to see and a long neck to reach for food into water and has wings to fly, etc. And when we try drawing a swan, in the end we must not just draw feathers, legs, wings, and a head, but must draw the ***Swan***!

The system as a whole can constitute itself as a particular system only in difference and interaction with the environment and by the specific functions it assumes within a larger system. We seem therefore to run into paradox when trying to describe the notion of the systems approach:

> Any given system can be adequately described provided it is regarded as an element of a larger system. The problem of presenting a given system as an element of a larger system can only be solved if this system is described as a system. (Blauberg et al. 1977, p. 270)

The systems approach requires a kind of evolutionary realism in the sense of Peirce and the paradoxes of the systems approach have to be interpreted as expressing the contradictoriness of a process evolving in time.

Concentrating on the semiotic nature of thought one might try to understand the acquisition of knowledge as a socially as well as objectively constrained process. To this end one should be able to fuse together the two poles of the classical semiotic heritage, the epistemologically focused tradition that emphasizes the importance of the indicative sign and the linguistically grounded tradition that studies the conventional symbol. The semiotics of Charles Peirce, on the one hand, and the linguistic work of Ferdinand de Saussure, on the other hand, represent these two different traditions, having developed their mature accomplishments during the same time period around the turn of the 20th century.

Peirce did emphasize mathematics as a semiotic activity—rather than as a mere language. Essential to this activity are *icons* and *indices,* so called "non-symbolic thought-signs" (Peirce CP 6.338). Peirce distinguishes them by saying that icons are "pictures or diagrams or other images such as have to be used to explain the meanings of words," whereas indices have the role of designating objects. These non-symbolic signs are particularly important with respect to the growth of knowledge. If all thinking would be conceptual, it would be a mere recognition of the implicitly already known, in the manner of Plato and all knowledge would become analytical.

No new knowledge is possible without icons and indices. The famous unknown "x" of symbolic algebra permits one to make the yet unknown an object of mathematical activity elaborating its relationships with the known numbers by a series of diagrams. In mathematical diagrams, be they algebraic or geometrical, iconicity prevails. Icons are the only type of sign possibly bringing about new insights. "A great distinguishing property of the icon is that by direct observation of it other truths … can be discerned than those which suffice to determine its construction" (Peirce CP 2.279). And Peirce saw, "as no one before him had, that indication (pointing, ostension, deixis) is a mode of signification as indispensable as it is irreducible" (Seboek 1994, p. 31). Peirce wrote:

> one might think, that there would be no use for indices in pure mathematics, dealing, as it does, with ideal creations, … But the imaginary constructions of the mathematician, and even dreams, are so far approximate to reality as to have a certain degree of fixity, in consequence of which they can be recognized and identified as individuals. (Peirce CP 2.305)

The indices occurring in pure mathematics refer to entities or objects that belong to a model, rather than to "the real world", that is, they indicate objects in constructed semantic universes. Think of Descartes' analogy between arithmetic and geometry and the introduction of coordinates, which provide a mechanism for introducing the calculational methods of arithmetic and algebra into geometry. One might, however, as well interpret this Cartesian analogy as a kind of geometrization of algebra. It allowed mathematicians, for example, to relate an algebraic formula to a geometric curve, which was of fundamental importance to the mathematization of physics. "When Descartes discovered analytical geometry… it inaugurated an uninterrupted series of reciprocal assimilations of branches of mathematics till then heterogeneous" (Beth and Piaget 1966, pp. 228–229).

Finally one might even see Peirce's statement that "all mathematical reasoning is diagrammatic" and "all necessary reasoning is mathematical reasoning, no matter how simple it may be" (Peirce CP 5.148) as a semiotic version of that analogy between geometry and algebra. Though Saussure admitted that the sign may have a referent, he took that to lie beyond the linguist's purview and in his system there exist therefore no indexical signs.

Now, Russell (1956) comments on the development of the evolution of the relations between logic, language and mathematics as follows:

> Mathematics and logic, historically speaking, have been entirely distinct studies. Mathematics has been connected with science, logic with Greek: But both have developed in modern times: logic has become more mathematical and, mathematics has become more logical. (p. 194)

However, the difference between mathematics and language, as Russell sees it, re-appeared during the 19th century as a difference between two conceptions of logic (Heijenoort 1967) and of mathematics itself. The advent of pure mathematics since the beginning of the 19th century was characterized, for example, by the opposition between two different forms or paradigms: axiomatization versus arithmetization. Semiotically speaking, the controversy was about what is to be defined in mathematics: concepts or their extensions. Consider for example the heated controversy about the foundations of arithmetic of natural numbers between Dedekind and Peano, on the one side, who wanted to base the concept of number axiomatically and by ordinal numbers, and Frege, or Russell, on the other side, who complained that axioms would not define numbers in terms of set-theory, that is, as objects, rather than concepts.

9.3 The Logic of Linguistic Communication

The logic of human communication and self-understanding is largely intensional. What matters primarily is meaning or meaningfulness, rather than objective reference. Perhaps your mother's birthday coincides with that of Hitler or any other horrible person. But, when reporting about the birthday celebrations, you surely would not like to have your story becoming rephrased, by saying, "we were all happy on Hitler's birthday", although such a reformulation would be extensionally equivalent. Or if a housewife comes back home from shopping and says to her husband "the shopkeeper told me that you both have birthday together", then that's probably literally untrue, although the shopkeeper and the husband might have been born on the very same day.

The greater part of people's conversation is taken up with matters of social import and common language is heavily oriented towards human cohesion and the management of social contacts. "We are all social beings and our world is cocooned in the interests and minutiae of every day social life" (Dunbar 1996, p. 4). In fact, it is believed today that the development of language owes much to sociocultural needs and the needs of cooperation. Besides, linguistic description is better suited for describing familiar situations or objects which people oversee and take in intuitively and at a glance.

There is a difference here because the growth of mathematical and scientific knowledge lacks the quasi-automatic character evident in the learning of our mother tongue. Already Galileo had pointed out the differences and inherent problems. In Galileo's "Assayer" (Il Saggiatore) of 1623, the difference is stated by comparing God's Word in the Bible, which is adapted to the frame and imagination of the people, on the one hand, and the Great Book of Nature, on the other hand, which presents the realities of Nature objectively as they are and without regard to human interpreters and their desires or preconceptions. Galileo made the point quite clear against Sarsi:

> I seem to detect in Sarsi a firm belief that, in philosophizing, it is necessary to depend on the opinions of some famous author …. Perhaps he thinks that philosophy is a book of fiction written by some man, like the *Iliad*, or *Orlando Furioso*—books in which the least important thing is whether what is written there is true. Mr. Sarsi, this is not how the matter stands. Philosophy is written in this vast book, which continuously lies upon before our eyes (I mean the universe). But it cannot be understood unless you have first learned to understand the language and recognize the characters in which it is written. It is written in the language of mathematics, and the characters are triangles, circles, and other geometrical figures. (Wikisource)

Language is not only a means to distinguish men from beasts, but it serves also to differentiate between cultural sectors and different purposes. Without the mathematical symbolism there is no science in the modern understanding. One might be justified to claim that the "new" algebra of the 16th/17th centuries has been brought about to a large degree by the opportunities offered by the printing press (Eisenstein 1979). Oral language is an analog system; written language speaks to the eye. It allows a meta-perspective and is thus capable of stimulating reflection.

The creation of a formal mathematical language was of decisive significance, not only for the growth of mathematics itself, but also for the constitution of modern science and technology. The concept of a mathematical function, on which the notion of natural law is based, "applied to physical phenomena, appeared for the first time in the literature of mankind in a prescription for gunners in 1546" (Zilsel 2003, p. 110), eighteen years before the birth of Galileo and exactly half a century before the birth of Descartes.

The prevalence of sense or meaning over reference and truth is often addressed under the label "functional semantics" What a pity that Galileo did not know this term! Everybody from booksellers and newspaper agents to real estate firms says nowadays: "Our goal: Making our customers, clients and their families happy, and *content is our top priority*". "Content" means presenting a picture of life and the world as it is designed in order to define the image of a company. The product is placed in a context with which a maximum large target group can identify. It has nothing to do with content really, it is just functional language. The more content, the less meaning!

An important starting point for functional semantics is the recognition that meaning making occurs in specific contexts and that language use is functional within those contexts (cf. Morgan's chapter in this volume). Individuals try by what they say to achieve effects in their social world. But, the most important prerequisite for learning and knowing is the possibility of simultaneously experiencing a body of knowledge, as well as its development or application. Strictly speaking, this possibility is provided by social cooperation only. But, signs and texts serve as substitutes for direct cooperation. They represent crystallized cooperation. A written text may serve even as a means of cooperation between my yesterday ego and myself to-day, by showing me the object of my own writings and thereby helping to correct the one-sided functional view. One might once more get an idea how important the printing press has been to the history of mathematics and of knowledge in general.

If one believes that communication is constituted by the encounter of independent actors who decide to utter their wishes or commands or whatever, rather than conceiving of communication as a social system, one might also come to believe that signs are essentially determined by the human subject, being just functions of their wishes and desires. However, we cannot 'not communicate' and we have no absolute control over what we do communicate in fact. From the perspective of the individual's interaction with the world, signs are undoubtedly essentially determined by their objects, like the smoke that indicates the fire, or the footprints on the beach that informed Robinson Crusoe of the arrival of another human being on his lonely island. And even if someone deliberately sets up a fire in order to give a signal to a partner, he must act according to the laws of nature.

Rhetoric had always conceived of meanings as functional. The sophists in Plato's *Athens,* being masters of rhetoric, boasted themselves of their capacity to promote any proposition alternatively as either being true or false. "Man is the measure of all things", said Protagoras, the most prominent of the sophists. Which man, one might ask? Now, consider this: Tversky and Kahneman posed the following problem to a sample of doctors:

> A new strain of flu is expected to kill 600 people. Two programs to combat the disease have been proposed. If program A is adopted 200 people will be saved If program B is adopted, there is one-third probability that 600 people will be saved and a two-third probability that no people will be saved. Which of the two programs would you favor? (quoted by Pinker 2007, p. 243)

Most of the doctors picked program *A,* motivated by pragmatic logic and professional attitude. But, suppose an analogous case in which you and various members of your own family were involved in a similarly precarious situation. And it would fall on your responsibility to choose a way out. We believe that in such a situation, few people would opt for strategy *A*. Every person wants to be part of some greater Being, some universal idea or belief system, some *Weltanschauung*. We want to know what it means to be human and we want to know what kind of obligations result from belonging to the same family, nation or from being human. The meaning of individual existence is invariably connected to the meaning of life and history. It follows that one does not require or expect the same kind of behavior from a scientist, a doctor a politician or a family member! As Michael Ignatieff once wrote, "It is difference which seems to rule our duties, not identity. … The lives of a father, a daughter, a son are precious to us; the lives of strangers count for little" (Ignatieff 1984, p. 29).

Peirce's so-called *Pragmatic Maxim* reproduces the inherent dilemma in epistemological terms. The original 1878 statement of the *Maxim* runs as follows: "Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object" (Peirce CP 5.18). Peirce commented on this about 25 years later, in 1902, in a contribution to Baldwin's "Dictionary of Philosophy and Psychology". The Pragmatic Maxim, there he says,

> might easily be misapplied, … The doctrine appears to assume that the end of man is action – If it be admitted, on the contrary, that action wants an end, and that end must be something of a general description, then the spirit of the maxim itself, which is that we must look to the upshot of our concepts in order rightly to apprehend them, would direct us towards something different from practical facts, namely, to general ideas, as the true interpreters of our thought. (CP 5.3)

9.4 Metaphor Revisited

One important consequence of extreme emphasis on specific applications and pragmatic functionality is the loss of metaphor. What Peirce describes in his maxim could semiotically be reproduced as the complementarity of function and metaphor. Metaphor requires a meta-perspective and it serves to generalize. Metaphors are arguments or even theories in a nutshell, because they connect different conceptual fields. However, the functional and the metaphorical are not completely independent from each other, at least not in our thinking.

Metaphor is the result of the creative capacity to see something as something else, the complex numbers as elements of vector-space, for example! Or, vice versa, to introduce algebraic structures into the realm of geometry. Rhetorically, metaphors serve to make one see something in a certain light or from a certain perspective, that is, to see A as B or even A as A: $A = B$ or $A = A$!

Because every two entities bear some likeness to each other it seems essential to consider how to choose the *terminus comparandi*. Before the modern revolution, described by Michel Foucault, sensible features of the plants were memorized, from which one sought to guess their medical effects. Paracelsus (1493–1541), a famous physician of his time, guessed from the form of a plant its possible healing effects. The foxglove (*Digitalis purpurea*), for example, was used to treat heart complaints, due to the fact that there is a kind of colored heart in its flowers. The walnut was used, because of its appearance, to treat diseases of the brain, etc. etc. But with the Scientific Revolution these views changed. Remember Galileo's famous dictum that mathematics is the language of science and that the Great Book of nature is written in a mathematical language.

When I was a mathematics student at the University of Munich, many years ago, the professor once drew an equilateral triangle on the blackboard and two algebraic formulae: $x^2 + y^2 + z^2$ and $xy + z$ and asked where the similarities are. Both formulae appear to be disconnected from the geometrical figure of the triangle, but in fact the first formula entails great similarity to the equilateral triangle: Both have the same type of symmetry, both instantiating a full permutation group.

"Colorless green ideas sleep furiously", is a phrase composed by the great linguist Noam Chomsky in 1957 as an example of a sentence whose grammar is correct, but whose meaning seems nonsensical (Chomsky 1957, p. 15). Chomsky's sentence may, however, when interpreted metaphorically, appear as a completely acceptable phrase within a piece of poetry. And it was accepted as such a piece in a poetry competition at Stanford in 1985.

The metaphorical is not in the things represented, but is in the representation, or ultimately, in the complementarity of sense and reference. Metaphors are intensional. When the artist presents Napoleon as Roman emperor the viewer must

> perceive the metaphor as an answer to the question why that man has been put by the artist in those clothes–a different question entirely from that which asks why Napoleon is dressed that way, the answer to which might not be metaphorical at all. … The locus of the metaphorical expression is in the representation–in Napoleon–as Roman-emperor–rather than in the reality represented, namely Napoleon wearing those clothes. (Danto 1981, p. 171)

As soon as one has realized the possibility of splitting up content into semantics and syntax one can match them up in more than one way. This is done in science as frequently as in poetry or art.

Bernhard Riemann (1826–1866), one of the greatest mathematicians of the 19th century, once wrote a small paper on the "Mechanism of the Ear" (Mechanik des Ohres), in response to a publication of Hermann Helmholtz. Riemann (1892) wrote as follows:

> The physiology of a sense organ requires … two particular foundations; one psychophysical, that is, the empirical verification of the achievements of the organ, and one anatomical, that is, the investigation of its structure. … Accordingly, there are two possible ways of gaining knowledge about the organ's functions. Either we can proceed from its construction, or we can begin with what the organ accomplishes and then try and explain these accomplishments. … Opting for the second possibility one is confronted with three tasks: 1. The search for an appropriate hypothesis to explain the achievements. 2. To find out about the sufficiency of the hypothesis for an explanation. 3. The comparison with experience. … One has to quasi rediscover the instrument and must consider its achievements as a goal and its creation as a means to this end. However, these achievements and therefore this goal are given by experience and not subject to speculation. (Riemann 1892, pp. 338–339)

Riemann does not rule out the structural approach. He wants to emphasize that structure and function are complementary to each other and this complementarity becomes effective from an evolutionary perspective. And it leads us to guess the general laws of nature, which in turn help to construct machines. For example, the metaphorical equation *Heat* = *Motion* is the basis of quite a number of technical constructions from the locomotive to the ultrasonic cleaner.

In summary we conclude that the *complementarity* or dynamical interaction between sense and meaning, or meaning and reference represents the most fruitful approach to all questions in the field!

9.5 Why Did Semiotics Arouse no Greater Interest Among the Communities of Mathematics and Mathematics Education for so Long?

Mathematics education or didactics is an interdisciplinary enterprise by necessity. This fact has been widely recognized since the years of the first International Congresses on Mathematical Education in the 1960s and 70s. But mathematics education has not been able so far to develop an integrated methodology. Semiotics might offer opportunities to do better in the future. Deely (1982) wrote as follows:

> Whereas the rise of modern science brought about the conditions requiring a new kind of specialization that gradually has led to an atomization of research and fragmentation of intellectual community, recognized by all as counterproductive, *semiotic* can establish new conditions of a common framework and cross-disciplinary channels of communication that will restore to the humanities the interdisciplinary possibilities that have withered so alarmingly when scientific specialization knew no check and alternative. (preface)

And Charles W. Morris, another father figure of modern semiotics wrote, “I share the view that insight into the nature of signs provides us with an instrument which improves our understanding of, and effective participation in, the whole of our contemporary intellectual, cultural, personal, and social problems” (Morris 1938, Introduction).

The reasons for the lack of interest in semiotics among mathematicians and mathematics educators, I think, are various and even somewhat contradictory. Mathematicians are individualists and Platonists. Peirce wrote as follows:

> All great mathematicians whom I have happened to know very well were Platonists… I believe that the great majority of them would regard the formation of such conceptions such as that of imaginary quantity and that of Riemann surfaces as mathematical achievements, and that, considering those hypotheses not as mere instruments for investigating real quantities, but in themselves. They would rank them as having much higher value than anything in the Arabian nights, for example. Yet why should they do so, if those hypotheses are pure fictions? There is certainly something to which modern mathematical conceptions strive to conform, be it no more than an artistic ideal. (ESP2, pp. 51–52)

Any serious research effort needs some objective resistance, some object about which one wants to find out new truths. Mathematics strives, like every science for truth. However, were theories identical with their languages there would be no false theories and no truth. Now mathematicians claim to obtain absolute knowledge and such a thing seems achievable only with respect to ideal objects (Platonic ideas). In consequence, Platonism, as well as intuitionism, negate the significance of symbols and representations. Boutroux (1920), the nephew of Henri Poincare, has expressed this thought as follows:

> The mathematical fact is independent of the logical or algebraic garb under which we seek to represent it: in fact, the idea we have of it is richer and fuller than all the definitions we can give, than all the forms or combinations of signs or propositions by which it is possible for us to express it. (p. 203)

One might add, however, that the number of different ways of representing that, which seems to be the same mathematical object, also increased beyond imagination. Pure mathematics is the child of an explosive growth of mathematical activity that started around 1800 and that, may be summarily characterized by observing that for the first time a great number of connections between apparently very different concepts and theories were discovered.

And it seems reasonable to guess that the ideal of a purely conceptual and analytical mathematics came from experiencing the advantage offered by a change of representational modalities, while following the very same ideas.

Mathematics educators, on the contrary, are socially more adjusted and are constructivists, prone to tinkering. They consider psychology the basic science of their trade and believe that the arbitrariness of signs makes people the source of symbols. They want to be masters of the words like Humpty-Dumpty in the story *Through the Looking-Glass* of Alice's travels in Wonderland. Humpty discusses semantics with Alice!

> "I don't know what you mean by 'glory'," Alice said. Humpty-Dumpty smiled contemptuously. "Of course you don't—till I tell you." … "When *I* use a word," Humpty-Dumpty said, in a rather scornful tone, "it means just what I choose it to mean—neither more nor less." (Carroll 1988, p. 196)

This is indeed the attitude of mathematics teachers and mathematicians. "Stick to the definitions and do not let your thoughts run around freely", the professor says in the classroom. And Leibniz exclaimed: "*Calculemus!*" Leibniz was an optimist and he considered calculation to be the key to settling all human conflicts or disagreements. However, as Lewis Carroll demonstrated in another little piece of dialog, Leibniz's dream could come true only if humans were programmable robots.

> Achilles had overtaken the Tortoise, and had seated himself comfortably on its back. Well, now, let's take a little bit of the argument in that First Proposition of Euclid, said Achilles: '(*A*) Things that are equal to the same are equal to each other'.
>
> (*B*) The two sides of this Triangle are things that are equal to the same.
>
> (*Z*) The two sides of this Triangle are equal to each other.
>
> Readers of Euclid will grant, I suppose, said Achilles, that ... anyone who accepts *A* and *B* as true, *must* accept *Z* as true? … Tortoise: And might there not *also* be some reader who would say 'I accept *A* and *B* as true, but I *don't* accept the Hypothetical?' Achilles: Certainly there might. … Well, now, said the Tortoise, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept *Z* as true. … 'I'm to force you to accept *Z*, am I?' Achilles said musingly. (Carroll 1988, pp. 1104–1105)

Achilles has, in fact, no means to make the Tortoise accept this mode of inference, and in particular to accept Z. The only thing he can do is to write down precisely this claim as a further request: "(*C*) If *A* and *B* are true, *Z* must be true."

Logicians have commented on Carroll's little piece by saying that one's assumptions must be explicitly augmented by the exact mechanisms by which one is to deduce consequences from those assumptions. This seems trivial! Remember: *Calculemus*! But, what Achilles learns, to his lasting regret, is that formal logic can never force us to do or to accept anything. Francis Bacon (1561–1626), a protagonist of the philosophical revolution that Foucault describes—being a little less optimistic than Humpty Dumpty and Leibniz–nevertheless, recommended mathematical interpretation as a means to unravel the meanings of words beyond doubt:

> Let us consider the false appearances that are imposed upon us by words. … Although we think we govern our words, … yet certain it is that words … do shoot back upon the understanding of the wisest, and mightily entangle and pervert the judgment. So as it is almost necessary in all controversies and disputations to imitate the wisdom of the mathematicians, in setting down in the very beginning the definitions of our words and terms, that others may know how we accept and understand them. (Bacon 1893, p. 70)

Bacon obviously believed that the mathematical language is capable of providing precise definitions, though not everybody might find them to his taste. But, in 1710, George Berkeley had argued convincingly that nobody could give a definition of the idea of a *general triangle*. It seems that not even geometry could define its objects, although they lie open to everybody's eyes. What does lie before our eyes, in fact, are continua, that is, incompletely specified universals. And we pick up a prototype from such a continuum, according to our present purposes. A general, like the "famous" general triangle, or the apple in a proposition like "an apple is a fruit", is a free or objectual variable that can be further specified as need might be (Quine 1974, p. 98).

The most fundamental aspect of Berkeley's alternative suggestion is to understand that symbolic representations enable us to make one idea go proxy for many others by treating it as a representative of a kind (Berkeley 1975, pp. 71–72). We need not have a definition of the general idea of a triangle, because any proof is based on some very specific premises, rather than on the intuition of a general idea, although such an intuition might be at work in stimulating further investigation. A general triangle is "general" only with respect to a certain purpose and within a certain context. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle, on which the proof is to be based, can be assumed to be *equilateral*, without loss of generality, because the theorem in case is a theorem of affine geometry and any triangle is equivalent to an equilateral triangle under affine transformations. Equivalence and universality are to be considered from the point of view of a theory.

And if we concentrate on the mathematical activity of constructing a proof we are led to conceiving of mathematical concepts in completely functional or instrumental terms. Moritz Schlick, founder of the *Vienna Circle* logical empiricists–following Berkeley's indications—has characterized the situation as a concentration on the conceptual function in the context of "intellectual labor." "There are no such things as general images. … It was Berkeley who first enunciated this proposition with full clarity" (Schlick 1985, p. 18).

A mathematical concept is to be defined in operative terms, as Schlick (1985) continues, having Hilbert's axiomatization of geometry in mind and observing that

> we employ none of its properties save the property that certain judgements hold with respect to it. … It follows that for a rigorous science, which engages in series of inferences, a concept is indeed nothing more than that concerning which certain judgements can be expressed. Consequently, this is also how the concept is to be defined. (p. 33)

Mathematics seems to become intensional and determined by syntax, a system of formal deduction and hypothetic-deductive reasoning. What the axioms describe are concepts or classes of objects, rather than particular objects themselves. Peano's axioms do not answer the question "What are numbers, what is the number 1 or 2? Numbers could be anything, even games (Conway-Numbers, Hackenbusch-Games, Chessboard-Computer, geometrical Vectors, etc.). And this is what Frege or Russell complained about. We want our numbers not merely to verify mathematical formulae, like 7 + 5 = 12, they say, "but to apply in the right way to common objects" (Russell 1956, p. 9). This "right way" requires a model-theoretical investigation, however, an investigation that has to find out about the equality of two entities of a model, two Hackenbusch-Games, for example. The basis of mathematical reference is the verification of A = B. This might not always be a completely simple and obvious matter. It certainly was not a straightforward and simple idea to think of calculating with geometrical segments (Descartes) or vectors and functions of vectors (Grassman) or matrices (Heisenberg).

Mathematics distinguishes itself from logic exactly by the fact that it has objects, but these objects belong to model worlds, or universes of discourse, which can be changed if necessity arises. These universes of discourse often are mathematical themselves. Think of the case of the vector space representation of the imaginary numbers! They might, however, also arise from pragmatic contexts of human activity.

And it might have taken time until such hypostatic abstractions as the concept of number, of energy or of economic value appeared. "It must have required many ages to discover," wrote Bertrand Russell, "that a brace of pheasants and a couple of days were both instances of the number two." And it is even more interesting to notice that freeing the number concept of definite reference came along with the advent of symbolic algebra. Putting *A* and, say, *Two* on the same conceptual level requires to see *Two* as a concept that no longer intends a definite number of specific things. *Two* means "in Vieta ... the general *concept* of twoness in general. … It no longer means or intends a determinate number of things, but the general number-character of this one number" (Klein 1985, p. 25).

9.6 Mathematics and Semiotics

The logicist conception of mathematics as endorsed by analytical philosophy conceives of mathematics as a universal analytical language. Mathematics "is only a manner of talking about objects, rather than about the objects themselves." It is part of the logic and the logic is created only by the language, "namely by the fact that the symbolism we use allows to say the very same in different ways", wrote the Vienna Circle Mathematician Hans Hahn (1988, p. 57).

The view of mathematics as language has especially been prominent in educational contexts. As symbolization seems a completely conventional matter, as Humpty-Dumpty has told us, there are no questions about mathematical objects. And if some student would pose such questions, it would be difficult to answer in an acceptable way. So, mathematics education might satisfy itself and attempt to teach students how to use a coherent system of mathematical reasoning or calculation. "Mathematics is a language since it provides both a conveyance for and a substantiation of our thoughts. It is that aspect of mathematics that explains the key role it plays in modern science" (Effros 1998, p. 132). As a consequence, mathematical fluency becomes considered most important. And further: "The pedagogical principles underlying mathematics instruction," wrote Effros, "are quite similar to those used in language instruction" (p. 135).

But neither mathematics nor logic are mere languages. Logic, wrote Peirce, is just

> another name for semiotic, the quasi-necessary doctrine of signs. By describing the doctrine as quasi-necessary, or formal, I mean that we observe the characters of such signs as we know, and from such an observation, … we are led to statements … as to what *must be* the characters of all signs used … by an intelligence capable of learning by experience. (CP 2.227)

And on some different occasion, but about nearly the same period in his life Peirce wrote,

> The work of the poet or novelist is not so utterly different from that of the scientific man. The artist introduces a fiction. … The geometer draws a diagram, which if not exactly a fiction, is at least a creation, and by means of observation of that diagram he is able to synthesize and show relations between elements which before seemed to have no necessary connection. (CP 1.383)

Consider geometry. Someone might wonder, for example: Is it possible to turn around at an arbitrary angle? Is it possible to move at all? Yes of course is it possible, will be answered! But, Zeno had denied the possibility of motion or change. And Lewis Carroll had shown that this disbelief cannot be defeated by linguistic or logical means. One cannot force anyone to see what he does not see, either, but mathematics at least shows how things are.

Mathematics shows, for example, that the axioms for the Euclidean plane can be formulated wholly in terms of the group of rigid motions. This is done by framing some simple concepts—the notion of an *involution*, for example—and then proving theorems, such as "every rigid motion of the plane is a composite of involutions." It sounds complicated and it would certainly leave us at a loss when attacked "freehandedly" and without semiotic support. The totality of all possible movements in the plane seems to be a very complicated matter after all. But, after we have framed the appropriate notions, things seem to clear up.

If we remind ourselves that an involution is either a reflection in some line or a half-turn about some point, we might even think of the possibility of operating with transformations rather than with points or point-sets, which has the advantage that groups are structured entities while point-sets are not. One might even try and transform geometrical proofs about points and lines into a system of formulas in terms of involutions (Reidemeister 1957, p. 100). Or, take the example of the imaginary numbers, which is perhaps more familiar.

The notion of complex number is an abstraction from possibility inasmuch as it is possible to extend the basic arithmetical operations to such entities. However, as long as the imaginary unit had gained admission to arithmetic as merely a calculatory symbol, it produced the some confusion and uneasiness. Only after Gauss had given a relational interpretation in the frame of the model of the so-called Gaussian number-plane (as a two-dimensional vector space), it became a legitimate mathematical object.

The mathematician constructs and manipulates or modifies a diagrammatic representation of the premises in order to find out that foreign idea—to use Peirce's expression—which must be added to the set of explicit premises already available. Peirce called such kind of reasoning theorematic reasoning. Theorematic reasoning implies generalization, that is, the introduction of new conceptions or ideal objects, which are a result of a process called "hypostatic abstraction" (with respect to the fundamentally important notion of hypostatic abstraction see also CP 4.235, 5.447).

9.7 Conclusion

Our considerations have essentially been stimulated by the following questions:

1. What do people mean when talking about a semiotic framework? And why should mathematicians and mathematics educators be more interested in the semiotic framework of Charles S. Peirce (1839–1914), rather than Ferdinand de Saussure (1857–1913)?
2. Why, did semiotics arouse no greater interest among the communities of mathematics and mathematics education for so long?

The conclusions that emerged during the discussion of these questions were already mentioned more or less explicitly in the text. Mathematics, unlike logic, possesses its own objects. But these objects are elements of a social-communicative and objective practice and not ideas in a Platonic sky.

Methodologically, a principle of complementarity of meaning and reference seems to be essential (Otte 2003). This principle can be expressed in many ways, depending on context, as the complementarity of

Intension and Extension,
Syntax and Semantics,
Tool and Object,
Function and Structure, and
Theory and Model or intended Application.

References

Bacon, F. (1893). *The advancement of learning*. London: Cassell & Co.

Berkeley, G. (1975). *Philosophical works*. London: Dent & Sons.

Beth, E., & Piaget, J. (1966). *Mathematical epistemology and psychology*. Dordrecht: Reidel.

Blauberg, I., Sadovsky, V., & Judin, J. (1977). *Systems theory*. Moscow: Progress Publishers.

Boutroux, P. (1920). *L'Idéal Scientifique Des Mathématiciens*. Paris: Alcan.

Carroll, L. (1988). *The complete works of Lewis Carroll*. London: Penguin Books.

Chomsky, N. (1957). *Syntactic structures*. The Hague/Paris: Mouton.

Danto, A. (1981). *The transfiguration of the commonplace*. USA: Harvard University Press.

Deely, J. (1982). *Introducing semiotic*. Bloomington, IN: Indiana University Press.

Dunbar, R. (1996). *Grooming, gossip and the evolution of language*. London: Faber and Faber.

Effros, E. (1998). Mathematics as language. In H. G. Dales & G. Oliveri (Eds.), *Truth in mathematics* (pp. 131–146). Oxford, UK: Oxford University Press.

Eisenstein, E. (1979). *The printing press as an agent of change* (Vol. 2). Cambridge, UK: Cambridge University Press.

Foucault, M. (1973). *The order of things*. New York: Vintage Books.

Galileo, G. (1623). *The assayer*. https://it.wikisource.org/w/index.php?title=Il_Saggiatore/6&oldid=818600.

Gellner, E. (1992). *Reason and culture*. Oxford: Blackwell.

Hahn, H. (1988). *Empirismus, Logik, Mathematik*. Frankfurt a.m.: Suhrkamp.

Heijenoort, J. V. (1967). Logic as calculus and logic as language. *Synthese, 17*, 324–330.

Ignatieff, M. (1984). *The needs of strangers*. London: Penguin Books.

Klein, J. (1985). *Lectures and essays*. Annapolis, Maryland: St. John's College Press.

Leibniz, G. W. (1956). *The Leibniz-Clarke correspondence.* In H. G. Alexander (Ed.). Manchester: Manchester University Press.

Morris, C. W. (1938). *Foundations of the theory of signs*. Chicago, IL: University of Chicago Press.

Otte, M. (2003). Complementarity, sets and numbers. *Educational Studies in Mathematics, 53,* 203–228.

Peirce, C. S. *CP: Collected papers of Charles Sanders Peirce* (Vol. I–VI). In C. Hartshorne & P. Weiß (Eds.). Cambridge, Mass: Harvard UP (1931–1935); (Vol. VII–VIII). In A. W. Burks (Ed.). Cambridge, Mass: Harvard UP (1958) (quoted by no. of volume and paragraph); ESP: *The essential Peirce* (Vol. 2). Bloomington, IN: Indiana University Press (1998).

Pinker, S. (2007). *The stuff of thought*. New York: Viking Press.

Quine, W. V. O. (1974). *The roots of reference.* LaSalle, IL: Open Court.

Reidemeister, K. (1957). *Raum und Zahl*. Heidelberg: Springer.

Riemann, B. (1892). *Gesammelte Mathematische Werke* (2nd ed.). In H. Weber & R. Dedekind (Eds.). Leipzig: Teubner.

Russell, B. (1956). *Introduction to mathematical philosophy*. London: Routledge.

Schlick, M. (1985). *General theory of knowledge*. LaSalle, IL: Open Court.

Seboek, T. H. (1994). *Signs*. Toronto: University of Toronto Press.

Zilsel, E. (2003). *The social origins of modern science*. Dordrecht: Kluwer.

Chapter 10
A Semiotic-Conceptual Analysis of Conceptual Development in Learning Mathematics

Uta Priss

Abstract This chapter investigates examples of mathematical concepts that students find difficult to learn from a semiotic-conceptual perspective. Semiotic and linguistic features that contribute to the perceived difficulty of mathematical language are identified. This chapter suggests that natural language and mathematical language differ with respect to their semiotic-conceptual structure: while natural language is interpreted using associative concepts, the meanings of mathematical concepts are formal and must not be interpreted in an associative manner.

Keywords Semiotic-conceptual analysis · Formal concepts · Associative concepts
Mathematics education

10.1 Introduction

This chapter proposes the use of semiotic-conceptual analysis (SCA) as a means of investigating why students appear to have difficulties with learning new mathematical concepts. The investigation is aimed at the first two steps in Middendorf and Pace's (2004) model for "decoding the disciplines" which are as follows: identifying particular difficulties that students encounter and comparing how experts approach these differently from students. The other steps of the model, in particular the design and evaluation of teaching materials, are not discussed in this chapter. This chapter suggests that mathematics may be difficult for students because it uses a language that differs structurally from natural language and requires a different form of conceptualization.

The investigation was carried out while teaching a class on discrete structures (DS class) to first year computer science students. Using a flipped classroom teaching method, the students were reading textbook sections and working on exercises before each class session. The students submitted answers to exercises

U. Priss (✉)
Ostfalia University, Wolfenbüttel, Germany
URL: http://www.upriss.org.uk/

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and also a short summary, reflection and possible questions about the weekly textbook pages via a Learning Management System (LMS), which enabled the lecturer to structure the materials discussed in the class sessions as needed. A large amount of class time was dedicated to group exercises and class discussions interspersed with short presentations. During the semester, a record was kept of any concepts from the LMS materials, observations during the class sessions and the final examination that seemed particularly difficult for the students. This chapter presents an attempt at structuring and analyzing observations from these data using SCA as a qualitative research method.

There appears to be an agreement among mathematics educators that mathematics at schools is different from mathematics at universities (e.g., Kadunz 2017). The didactic literature for school mathematics tends to focus on the children's first encounter with core mathematical concepts. At universities students are expected to use mathematics in a scientific manner that incorporates proofs. Thus students must transition from a pre-expert mode of using mathematics to an expert mode of proving mathematics. Moore (1994) suggests a model for analyzing this transition from school mathematics to using formal proofs at universities. He sees concept use as a third aspect of mathematical understanding in addition to Vinner and Dreyfus's (1989) model of concept definition and concept image. According to his analysis a correct use and understanding of mathematical definitions is crucial for mathematical proofs. Concept images without an understanding of definitions are insufficient for comprehending and writing proofs. Some proofs can even be written without concept images by simply manipulating formal notations. Edwards and Ward (2004) also analyze the role of mathematical definitions and observe that one of the problems that students are having is that they are relying on concept images instead of concept definitions when these two are in conflict. Definitions and proofs are a form of mathematical discourse carried out in a formal language. In my observation in the DS class, the students were having frequent problems with the understanding and use of mathematical language and formal notation in definitions and proofs. This raises the questions of why this mathematical language is special compared to other languages and why it is difficult to learn.

SCA was employed as a means for analyzing mathematical language. SCA provides a mathematical definition of signs and their relationships and a means for analyzing data that are collected in the context of teaching (Priss 2016). The focus of SCA is on detecting, visualizing and exploring the implicit structures within a corpus of recorded signs (such as video-taped utterances or student submitted exercises or examinations). It is then up to a researcher to draw conclusions from the data. For example a researcher could be constructing an SCA representation and then point to an element in this model that in his or her opinion corresponds to a concept image of a teacher and to another element corresponding to a concept image of a student. By formally representing the results of an analysis using SCA for modeling, this analysis becomes transparent and can serve as a basis for discussion with other researchers. Priss (2016) presents some examples of SCA representations of data obtained from the DS class. This chapter does not contain the mathematical details of such an analysis but instead focuses on semiotic and linguistic structures that were observed in the collected data.

The next section explains why a semiotic analysis is considered to be suitable in this context and explains some of the core SCA notions. Section 10.3 defines the conceptual models that support SCA. Sections 10.4 and 10.5 provide the main analysis of the examples from the DS class: Sect. 10.4 focuses on the linguistic and basic semiotic structures and Sect. 10.5 discusses the special nature of concepts underlying mathematical language.

10.2 Semiotic Analysis

The term "semiotics" has been used in a variety of meanings by different researchers. In this chapter, semiotics is used in the sense of SCA, which is inspired by Peirce. One of Peirce's definitions reads as follows: "A representamen is a subject of a triadic relation to a second, called its object, for a third, called its interpretant, this triadic relation being such that the representamen determines its interpretant to stand in the same triadic relation to the same object for some interpretant" (Peirce 1903, CP 1.541). It is not the aim of this paper to provide a philosophical discussion of this complex definition. SCA only adapts from Peirce the idea that signs are triadic and that some sign components determine others. The following description of SCA is adapted from Priss (2016). In SCA the three sign components are called "representamen", "denotation" and "interpretation". In SCA "representamen" refers to the sign vehicle or physical representation of a sign. Because signs are represented via their representamens the notions of "sign" and "representamen" may sometimes seem difficult to separate. In this chapter, "sign" always refers to the triple and "representamen" to the sign vehicle. It is not clear in Peirce's definition what precisely is meant by the word "relation". In SCA the word "relation" is used in its mathematical sense, which means that a triple is an instance or element of a relation but not a relation itself. In SCA a sign is a triple that is an element of a relation called "semiotic relation". A semiotic relation is therefore a set of signs.

The notion of "object" is not employed in SCA because it has too many different connotations in different contexts. SCA uses "denotation" in order to refer to the meaning of a sign without determining what kind of meaning it is. Peirce's definition appears to indicate that an interpretant directly corresponds to a single sign. In SCA an interpretation is not restricted to one sign and can be a component of several different signs. An interpretation is a partial function that maps representamens into denotations. For example, one interpretation could present the view of a student, another one the view of a teacher. Priss (2015) speculates that Peirce's "interpretant" corresponds to a pair consisting of representamen and interpretation in SCA. Several attempts have been made (for example by Marty 1992) to mathematize Peirce's semiotics. SCA only attempts to mathematize the core definition of "sign" but not any other aspects of Peirce's work.

Semiotics and at least modern linguistics appear to be quite similar and share many goals, theories and methods. Thus it should be clarified why SCA is called "semiotic" instead of "linguistic". Both disciplines investigate sign systems, structures, meanings and uses. The focus of linguistics appears to be on natural language, which is in its spoken or written form a linear structure because syntax and grammar are interpreted as linear structures or as syntax trees, which can be derived from linear structures. A mathematization of signs as suggested in SCA, on the contrary, is concerned with sign systems that need not be linear, such as graphical representations, diagrams and gestures. Although other attempts have been made to generalize linguistics to non-linear representations, for example by defining a grammar for graphs (Rozenberg 1997), such approaches do not seem to be as widely accepted. Thus by emphasizing semiotics over linguistics, SCA is not restricted to linear models or tree models and can be applied to a wider range of representations than just natural language.

Mathematical models tend to focus on structural relationships and provide abstract descriptions that ignore many concrete details. In mathematics only the explicitly defined relationships are relevant but not any other implicit associations. SCA provides a mathematical definition of "sign" in addition to a sense of how this should be used in semiotic applications. The definition of "sign" with respect to applications in SCA can be formulated as: "a sign is a unit of communication corresponding to a triple (i, r, d) consisting of an interpretation i, a representamen r and a denotation d with the condition that i and r together uniquely determine d." The following mathematical definition of signs in SCA is more abstract: "For a set R, a set D and a set I of partial functions from R to D a *semiotic relation* S is a relation $S \subseteq I \times R \times D$. A relation instance $(i, r, d) \in S$ with $i(r) = d$ is called a *sign*." The sets R, D and I are the (finite) sets of representamens, denotations and interpretations, respectively. The mathematical definition contains no further constraints for R and D other than being sets and lacks any reference to units of communication whereas the application-oriented definition is less precise with respect to which sets are involved. Thus, one has to distinguish the mathematical view of SCA where the notion of "sign" can be applied to any kind of data provided as sets and an application-oriented view according to which the data should have some additional features as described below.

From an application-oriented view, representamens are sign vehicles, which can be simple (such as a word or a graph) or compound (such as a sentence or paragraph of text). Denotations are any kind of description of meaning. Interpretations represent different viewpoints (for example a viewpoint of a student or teacher), contexts or states (in applications of SCA to programming languages). Whether some item is a representamen, denotation or interpretation is decided by a researcher who uses SCA in an application. Thus some item can be a representamen for someone in some context and also a denotation for someone else in some other context. The dynamics of the mathematical and the application-oriented view can be different. In mathematics the sets are defined first. In an application it is possible that signs are first identified as units of communication and it is then decided which aspects of these units are considered representamens, denotations and interpretations. Mathematically, a denotation is an element which is a component of a sign. In applications, whether something is a denotation can depend on other intrinsic features. Having these two views is not unique to SCA but affects all applied mathematics. The two views complement each other and are not contradictory as long as mathematical reasoning is clearly distinguished from application-oriented reasoning and not inappropriately mixed. This distinction between two views is similar and related to the two different types of conceptualizations discussed in Sect. 10.5. From a mathematical view, the SCA triple of representamen, denotation and interpretation can be used to model concept definition, concept image and concept use. From an application-oriented view both triples have slightly different meanings. Concept images refer to what people are thinking when they are conducting or learning mathematics whereas in SCA cognitive aspects need not be directly modeled. The next section discusses conceptual models that serve as a basis for SCA.

10.3 Conceptual Analysis

Similarly to semiotics, the term "concept" has also been defined in a variety of ways by different researchers. Formal Concept Analysis (FCA) provides a mathematical definition of concepts (Ganter and Wille 1999). From an application-oriented view, concepts are core units for the modeling of thought and linguistic processing. For example, the meanings of words (or signs in general) can be considered to be concepts. In FCA, a formal concept is a pair of two sets that uniquely determine each other (cf. Ganter and Wille 1999, for the exact definition). The two sets are called extension and intension. From an application-oriented view, an extension is a set of items that are described by the concepts and an intension is a set of features, attributes or classes that characterize the items. For example, a concept "bird" has examples or prototypes of birds in its extension and attributes of birds, such as "having wings", in its intension. According to Priss (2002) concepts can be associative or formal. A formal concept has necessary and sufficient attributes in its intension that determine precisely the items in its extension and vice versa as defined by FCA. Examples are concepts that are defined scientifically, such as "mammal" or "prime number". Associative concepts are grounded in the cultural knowledge and experiences of the person who is using the concepts. Their extensions and intensions are fuzzy and sometimes not very clear at all. For example, it is difficult to precisely distinguish between a seat, a chair, a bench and a stool or to determine what is or is not a pet. Natural language concepts tend to be associative. Even concepts that have a formal scientific definition tend to be used associatively in natural language as evidenced by the debate about whether a tomato is a fruit or a vegetable.

Figure 10.1 shows an example of a subconcept–superconcept reltionship. According to FCA, formal concepts can be structured into a precise hierarchy because a concept A is a subconcept of a concept B if the extension of A is a subset

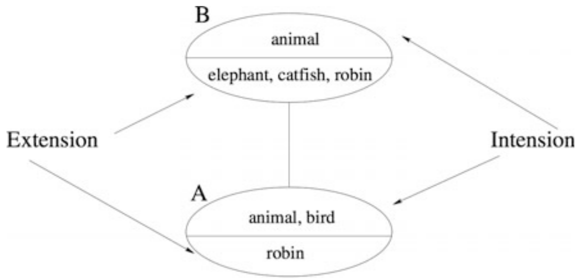

Fig. 10.1 A subconcept–superconcept relationship between two formal concepts

of B's extension and the intension of B is a subset of A's intension. In Fig. 10.1 this is the case because {robin} is a subset of {elephant, catfish, robin} and {animal} is a subset of {animal, bird}. This relationship corresponds to a mathematical Galois connection and results in mathematical lattices. The two concepts in Fig. 10.1 form a two-element lattice. Associative concepts are usually better modeled by networks of associative relations (similarity, part-whole, cause-effect, attribution and so on). Because FCA is a mathematical theory, it is a modeling tool that can be applied to any data of any binary relation. For example, a conceptual hierarchy can be developed from words and their parts of speech, students and their examination results, student comments and the time when the comments were made and so on. Thus as shown in the next example, all parts of a sign, the representamens, denotations and interpretations can also be modeled as formal concepts.

Figure 10.2 shows a conceptual model for parsing and understanding natural language words using SCA. Comprehending natural language words usually involves at least two steps that are each modeled as a sign. As a first step a word has to be recognized within a sentence, processed according to its part of speech and

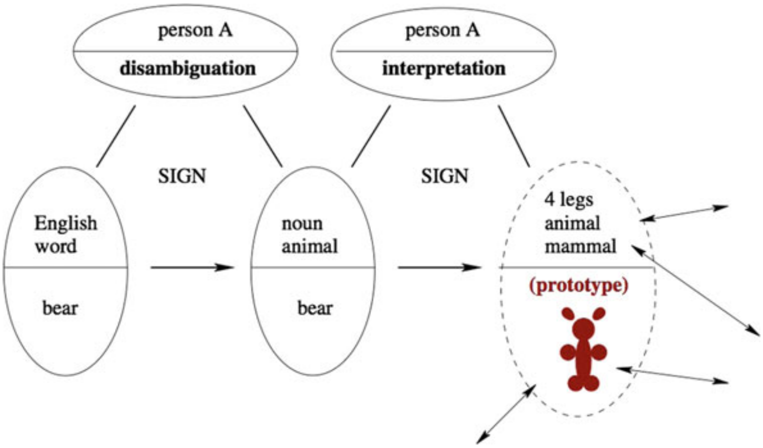

Fig. 10.2 Comprehending natural language words

potentially be disambiguated. For example, the word "bear" has two distinct meanings. It could either be a verb or a noun referring to an animal. A second step is required to assign meaning to a word. The meaning of a word is usually an associative concept consisting of some intensional features (such as a bear being a mammal and having legs) and some extensional prototypical or real examples. In Fig. 10.2 the first step is called "disambiguation" and the second one "interpretation" according to an application-oriented view. From a mathematical view they are both interpretations in SCA. The middle concept (bear/noun, animal) is a denotation of the first sign and a representamen of the second sign.

If the word "bear" is used by a biologist, then the last concept on the right is a formal concept. If the word is used in natural language then the last concept is most likely an associative concept (indicated by the dashed line in Fig. 10.2). Its extension tends to be a fuzzy prototype corresponding to all kinds of bear-like creatures possibly including teddy bears. Its intension contains prototypical features that apply to most items in the extension but not necessarily all. The arrows pointing away from this associative concept are meant to indicate that this concept is embedded in a network of other concepts and experiences. All the associated knowledge contributes to arguments and reasoning involving this concept. The meaning of this concept is not formally defined but is established in its context of use. Most likely two different speakers will have slightly different associative concepts for the same word. It should be emphasized that the model in Fig. 10.2 is meant formally and not cognitively. From a cognitive view the steps are not consecutive but instead knowledge about associative concepts influences how words are disambiguated.

10.4 Reading and Writing Mathematical Language

The use of SCA supports analyses of sign components, which are discussed in this section, and of conceptualizations of sign components, which are discussed in the next section. Because interpretations are a component of signs, and interpretations can include context, a representamen used with the same denotation but in different contexts might correspond to two different signs (depending on how the interpretations are defined in an application). Thus, equality of signs may not be of interest in applications. Furthermore, denotations are frequently only similar to each other but not equal (such as the denotations of "car" and "vehicle"). Representamens as physical manifestations are also frequently only equivalent to each other but not equal. For example, two handwritten instances of the same word are never totally equal. In mathematics, equality is usually evaluated with respect to denotations while ignoring representamens. For example, "x = 5" is an expression of three signs with representamens "x", "=" and "5" where different representamens ("x" and "5") have an equal denotation (value 5). Therefore, even though "x" and "5" are equal as denotations, they are neither equal nor equivalent as representamens. This results in a whole range of possible relationships between signs some of which are mathematically defined in SCA. For example, two signs are strong synonyms if their denotations are equal, synonyms if their denotations are similar and polysemous if they are synonyms with equivalent representamens. In this case, similarity is defined using tolerance relations, that is relations that are reflexive and symmetric. More details about this can be found in the paper by Priss (2016). Stating that "signs are synonyms if their denotations are similar" might not seem different from a usual linguistic definition, but in the case of SCA each of the core words used in this definition ("sign", "denotation" and "similar") has a precise mathematical meaning. Mathematical language can now be investigated using the precision provided by SCA.

It should be mentioned that the use of mathematical language has already been investigated by many other researchers. Schleppegrell (2007) provides an overview of the linguistic challenges of mathematics and its impact on mathematics education. According to Schleppegrell, Halliday coined the notion of a "mathematical register", which encompasses meanings and expressions used in mathematical language. Students need to learn to understand and use this register with its multiple semiotic features: symbolic notation, written and oral language, visualizations, specialized vocabulary and grammar, and implicit logical relationships. Some differences between mathematical and natural language are not structural but related to language use and style. For example, a statement such as "a set *S* of real numbers is called bounded from above if it contains an upper bound" is typical and acceptable as mathematical language. But in natural language repeated use of the same word ("bound") would be considered poor style. Furthermore non-mathematicians might consider the definition to be circular whereas mathematicians consider it non-circular because it defines a property based on the existence of a feature both of which happen to have a similar name. These are linguistic patterns that students need to learn or unlearn when they are studying mathematics. SCA is more focused on structural aspects than on usage, vocabulary, grammar, style and so on. The remainder of this section investigates four structural aspects (incompleteness, polysemy, synonymy and iconicity) of mathematical language.

The first example of a structural aspect is incompleteness, which refers to denotational content and structures that are insufficiently represented by representamens. One reason for incompleteness is that the foundations of mathematics tend to be too difficult to be presented to students in school or in the first years of university. Sets, logic and numbers are often introduced without providing details of the underlying theories and axioms. An example of incompleteness can be observed for the equals sign. Prediger (2010) describes an overview of the different uses of the equals sign. In my opinion they are partly due to incompleteness because of missing quantifiers. The difference between "a + b = b + a" as a general equivalence and "$a^2 + b^2 = c^2$" as a contextually used attribute of triangles would be obvious if the equations were presented with quantifiers stating explicitly for which variables they are valid. An example of incompleteness that I encountered in the DS class referred to the difference between logical Iff (if and only if) and logical equivalence. Iff can be represented by a simple truth table: p Iff q is true if either both are true or both are false. Logical equivalence, however, is a semantic concept.

Two statements are logically equivalent if they have the same truth value in every model. Because a discussion of model theoretic semantics is too difficult for students who have just started to learn logical symbols, the textbook that was used in the DS class vaguely defined logical equivalence as “something that is only written when it is always true”. This incompleteness gave rise to confusion that was difficult to resolve. For example, students asked what the truth table for logical equivalence looks like. It is unsatisfactory to answer that this is something they will learn later, in particular because logical equivalence is essential for mathematical proofs.

The second example is polysemy. The use of mathematical symbols often has historical reasons resulting in different notations for the same content (synonymy) and the same symbols having distinct meanings in different contexts (polysemy). As mentioned above, SCA defines polysemy as equivalent representamens with similar denotations. Therefore, two signs with the same representamen but slightly different denotations are polysemous. If the denotations are very different (not similar) then they would be homographs. An example of polysemy is the equals sign in the two statements $4 + 1 = 5$ and $2 + 2 + 1 = 4 + 1 = 5$. In the first statement, the equals sign can be interpreted as a binary operation resulting in true or false because $4 + 1 = 5$ or $= (4 + 1, 5)$ is true. If the same interpretation was applied to the second statement, it would result in “true = 5”, which is nonsensical. In this case the equals sign cannot be read as a binary operation unless it is assumed to be a case of ellipsis ($2 + 2 + 1 = 4 + 1$ and $4 + 1 = 5$). Otherwise it can only be read as the basis of an equivalence relation. Other symbols (<, >, <=>, =>, and so on) also have these two kinds of polysemous uses. Using the decoding model mentioned in the introduction, it is important for lecturers to decode such hidden structures in their discipline because while students might struggle with the ambiguity presented by polysemous signs, lecturers might not even be aware that there is an ambiguity at all.

Another example of polysemy that I encountered in the DS class was the modulo operation. The statement “12 mod 5” can be interpreted as an operation (resulting in 2) or as resulting in an equivalence class where 2, 7, 12 and so on are all equivalent. In the textbook this interpretation was notated using parenthesis (such as “7 = 12 (mod 5)”). One problem with this notation is that it is not symmetric. Furthermore the textbook also used the notation “(8 + 12) mod 9”. The students found this confusing. I suspect that the confusion was exacerbated by the fact that the students did not yet have a solid understanding of equivalence relations and therefore preferred the interpretation of modulo as an operation. In addition they appeared to rely more on the notation compared to the lecturer who initially overlooked the textbook use of parenthesis and interpreted modulo as indicating an equivalence relation independently of whether the parenthesis were provided or missing.

The third example is strong synonymy, which characterizes signs with equal denotations. In natural language, weak synonymy tends to be more common than strong synonymy because words tend to have at least slight differences in meanings or connotations, such as in the example of “car” and “vehicle”. In mathematics, however, strong synonymy is very common because there are usually many

Fig. 10.3 An example of a definition

2 1 5 6 4 3

The inverse image of $M \subseteq I$ is $\{x \in D \mid f(x) \in M\}$

different equal or logically equivalent means for representing any mathematical denotation. One of the in-class exercises that I used frequently in the DS class was to ask the students to translate between more verbose mathematical descriptions and more concise descriptions using formulas. These correspond to different synonymous signs with distinct syntactic rules for their representamens that the students need to learn. For example, Fig. 10.3 shows a concise definition of an inverse image of a set with respect to a function. Synonymous representations of that statement are as follows:

1. A more verbose definition: the inverse image of a subset S of the codomain is the set of all elements of the domain that map to the members of S.
2. A literal reading: the inverse image of a subset M of I is a set of elements x of D with f of x element of M.
3. A high-level interpretation: although an element of the domain that is mapped onto an element of the codomain need not be unique, if one considers the set of all elements of the domain that map to members of a set of the codomain, then this set is uniquely defined and called the inverse image. Thus if one generalizes the function from a relation between elements to a function between sets, then the function can be inverted.

In the DS class students appeared to find it equally difficult to produce a correct verbose description as to write a formula. In verbose descriptions students often used inappropriate vocabulary if they had not yet realized that mathematical terms can only be replaced with strong synonyms (such as "function" instead of "mapping") but not with weak synonyms (such as "function" instead of "relation"). Students might produce a literal reading, but that does not show that they understood anything other than how to pronounce the formal notation. High-level interpretations are to be expected only later in the semester and indicate a deeper level of understanding.

The fourth example is iconicity. SCA defines an icon as a sign where the representamen and denotation are similar to each other according to some defined similarity relation. For example, the representamen "$3 < 4 < 5$" presents a linear order in an iconic manner because the denotational content of "being larger" is similar to the representamen structure of "being further to the right". Difficulties with statements such as the one in Fig. 10.3 can also be caused by a lack of iconicity. The sequence in which the statement has to be read is not from left to right. As indicated by the numbers it has to be read by first selecting a codomain, then a subset of that codomain, then an element of that set, then considering the inverse function (or relation) and all the elements that are mapped onto the selected element and ensuring that these elements are in the domain. The last step is to assign all of this to the notion "inverse image" that is defined. Figure 10.4 shows a model for the statement in Fig. 10.3, which is just another representation that is weakly synonymous and more iconic but potentially less precise. Figure 10.4 shows that although it is not possible to construct an inverse image for individual elements if a function is not a bijection it is possible to do so for sets via the arrows in the figure. In order to understand the definition of inverse images one has to either know the model in Fig. 10.4 or know the sequence in which Fig. 10.3 is read. The lack of iconicity between the traditional reading sequence from left to right and the actually required sequence could be an added difficulty for Fig. 10.3. Natural language provides some means for representing sequences other than recounting events in the order in which they happen by using conjunctions, pronouns, sub-clauses and so on. Modern literature and poetry are often considered more difficult to read because they tend to diverge from the normal sequence. I would argue that therefore the statement in Fig. 10.3 is at least as difficult to read as modern literature and poetry.

It should be cautioned that iconicity does not mean that a representamen is intuitive. In the DS class I noticed that students were using incorrect assumptions about graphical representations that they were not familiar with. For example I observed that the students initially tended to interpret graphical representations of ordered sets (such as Hasse diagrams) as if they were vectors in a coordinate system. For example, one student tried to determine with a ruler whether two nodes were at the same level. When the student was told that levels did not matter, she did seem to change her interpretation of the representation. Thus iconicity can only function if it is preserved by interpretations and not misinterpreted.

In summary, mathematical notation is not always precise, complete and unambiguous. Incompleteness, polysemy and synonymy are potential sources of confusion for students who are learning new mathematical terminology. A first requirement for understanding a mathematical representamen is to complete it in the case of incompleteness and to disambiguate it in the case of polysemy. Being able to produce different synonymous representamens and, in particular, high-level representamens is an indication of understanding a concept. Using the structures provided by SCA as a guideline, lecturers can analyze mathematical signs with respect to decoding hidden difficulties, for example by looking systematically for cases of incompleteness, polysemy and so on. While iconicity has been investigated by other researchers (Radford 2008) and some of my examples are similar to ones from the literature (e.g., Moore 1994), it would be of interest to conduct a more complete investigation of cases of polysemy, synonymy, iconicity and so on in a larger mathematical context. In that case SCA could serve as a guideline for detecting examples instead of being used for analyzing examples as I have done in this chapter.

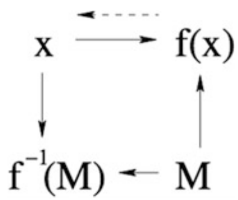

Fig. 10.4 A model for Fig. 10.3

10.5 The Denotation of Mathematical Concepts

A conjecture of this chapter is that in addition to mathematical language being incomplete, polysemous, full of synonyms and sometimes lacking iconicity, there is yet another more significant reason for why mathematics can be difficult to learn. The conjecture is that there is a semiotic-conceptual difference between natural and mathematical language. As explained above, Fig. 10.2 shows how natural language words are disambiguated and interpreted. But according to SCA as shown in Fig. 10.5, the second sign is missing in the case of mathematical language. A formal concept of the equivalence class of 12 modulo 5 has a precisely defined meaning that does not require further interpretation beyond what is defined. The denotations of mathematical representamens are formal concepts and not associative. They are not related to items in the external world and associations with external knowledge must not be used for mathematical reasoning. As soon as a mathematical concept has been defined everything that is known about the concept can be proven to be true. Anything that cannot be proven—or cannot yet be proven—is not considered to be part of the knowledge about the concept and is not admissible to be used in proofs. A mathematical concept has as its intension exactly those features that can be proven to be true and in its extension exactly those instances that are proven to have all the intensional features. All relationships with other concepts can be described using equivalence relations or implications. What mathematicians think about the concept, what associative concepts they might still have is irrelevant for the nature of the concept itself. The selection of topics to be studied in mathematics is socio-culturally influenced, but once a concept has been defined, its equivalences and implications are deterministic and not open to further interpretations.

Other sciences also sometimes employ mathematical language and formal concepts. Abstract concepts that are not formally defined, however, are different from formal concepts. A concept such as ‘mathematics’ that describes the discipline

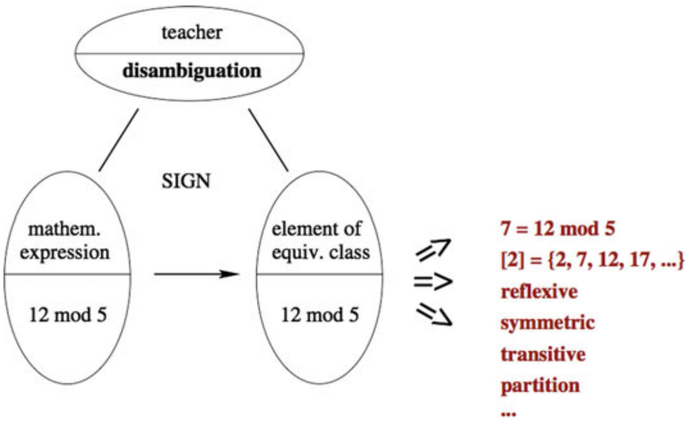

Fig. 10.5 Understanding mathematical concepts

of mathematics is associative and requires a second interpretation because it does not have a precise definition. Different people have slightly different interpretations of what it means. It is not possible to discuss the concept 'mathematics' without considering the context in which the word is used.

This conjecture about the difference between mathematical and non-mathematical language is related to how variables are used differently in mathematics and in programming languages (Priss 2015). Variables in programming languages are triadic signs because they have a name (representamen) and a value (denotation), which depends on the state of a running program (where the state corresponds to an interpretation). Mathematical variables, however, are usually just placeholders whose value does not change depending on a state. They are not fully triadic and only need to be disambiguated but not further interpreted. The conjecture states that the denotations of mathematical representamens are formal concepts that are fully specified via their definitions.

While the conjecture is supported by SCA, it is difficult to prove it in general. There is some evidence provided by the fMRI studies of Amalric and Dehaene (2016) that show that mathematicians use their brains differently depending on whether they are evaluating mathematical and non-mathematical statements. This is consistent with the conjecture that mathematical thinking involves a different form of conceptualization. The conjecture is also consistent with the analysis of Moore (1994) and Edwards and Ward (2004) about the importance of definitions and proofs for mathematics as mentioned in the introduction. SCA provides more depth to this analysis by highlighting the exact structural difference between natural and mathematical language as illustrated by Figs. 10.2 and 10.5. Last but not least, the conjecture serves as a suitable model for explaining some of the observations from the DS class as described in the remainder of this section.

In my experience students try to associate a concrete example with a concept as soon as possible and are then confused when they learn that the example is not identical to the concept. For example, after students learned in the DS class about the example of a Boolean algebra consisting of True, False and logical operators, they were very resistant to the idea that power sets are also Boolean algebras. It seemed that if they had to give up on the idea that Boolean algebras contain the values True and False, then they were left with a vacuum and a feeling that they no longer had an understanding at all. In the case of graph theory, students believed that graphical representation is an essential feature of graphs. If asked what a graph is they stated that it is something that is graphically represented because that seemed to be their core associative concept of graphs when in fact it is neither a necessary nor a sufficient condition. When asked to define a graph, an expert would cite or rephrase the definition that he or she has learned and would be very careful with statements that are not closely related to that definition. Students on the contrary appeared to base their arguments more on their associations and less on the provided definitions. As another example, students had much more difficulty with the concept of a ring than with groups or fields because groups are fairly easy to visualize and the examples of the fields of rational or real numbers are well-known from school. The examples of rings are not as easy to visualize. Furthermore students did not see a purpose in defining a structure (such as a ring) that has slightly different features than another structure (such as a field) because they did not see the role of definitions. In fact one student wrote "I don't understand why… needs to be defined. I understand how and for what it is used, but not why it needs to be defined with fixed rules."

Students also had problems with concepts where all or most of the instances in the extension appeared trivial. For example, showing that equality is an equivalence relation was difficult because the proof is almost trivial. Students did not clearly distinguish between the notions of equality and equivalence and thus did not know what it was that needed to be proven. In the case of group theory, understanding the importance of the group closure axiom appeared to be more difficult than the other axioms because all of the examples that the students were initially shown were closed. In these cases, formal reasoning based on definitions is straightforward, but only if one does not think about examples and focuses on applying rules.

If the conjecture is correct then the most significant source of errors for students who are learning mathematics is that they think they should interpret mathematical definitions in the same way as natural language words and believe that understanding means to generate an associative concept in their minds. I am proposing that there is a clear transition between non-expert and expert use of mathematics that occurs at the point when a student realizes that mathematical concepts are formal–although this does not mean that students consciously reflect on this. I observed during the DS class examples of students lacking a feeling for the nature and the limits of mathematical knowledge. They were puzzled when they heard that for some properties it is not known whether they are true or that no solutions exist for some problems. It appears that students think that if one has understood a mathematical concept then one should know everything about the concept and one should have an intuition for what is true and what is not. This view is in accordance with associative concepts and in contrast to formal concepts. Understanding a formal concept means being able to apply and use it correctly and being able to produce correct synonymous representamens for it. As mentioned before, SCA is not a cognitive theory. Most likely, expert mathematicians have associative concepts in their minds when they are thinking mathematically, but they know about the limitations of associations and use exclusively formal concepts in their definitions and proofs.

As mentioned above, mathematical variables are different from programming language variables. In the DS class I noticed that students focused too much on the names of variables and literals instead of their positions. For example, in one exercise students had to determine why a given structure did not fulfill an axiom. The challenge in this exercise was to discover that certain names were used differently than usual (such as 1 representing False) and had their positions switched. Students found this exercise difficult because they appeared to think that 1 has to mean True because of some attributes they associate with 1 and True. Presumably students interpret these representamens to stand for fixed associative concepts, possibly independently of the context in which they are used. Similarly, some students appeared to think that ½ always means 0.5 even in modular arithmetics groups. Variables and literals are also mathematical representamens, but they are even more shallow because in many cases they are not introduced via formal definitions. Instead they are just placeholders, such as x representing certain elements from a set. Students often associate meaning with the particular name that is used as a representamen even though all of the names can be changed without any impact on the mathematical content. For example, students tend to think that if elements x and y are selected from a set, then x and y must be different because they are named differently.

In my opinion, fMRI studies similar to the one conducted by Amalric and Dehaene (2016) might be a suitable means for supporting or rejecting the conjecture that was put forward in this section. If there is a transition from a pre-expert mode to an expert mode of mathematical thinking that involves some radically different manner of thinking and happens at some time while students are at university then this should be visible in brain scans. It would, however, still not provide insights into the exact types of conceptualizations that are employed. Further qualitative and quantitative studies could be designed to collect more data and to focus on a more specific research question about conceptualizations. Formally, the structures provided by SCA support the conjecture.

10.6 Conclusion

In summary, this chapter investigates mathematical language with semiotic means and with reference to conceptual structures. Mathematical language can be incomplete, polysemous, full of synonyms and lack iconicity. In this chapter, I conjecture that while the interpretation of natural language representamens results in associative concepts that link to other associative concepts and experiences, it is a source of error if students interpret mathematical language in that manner. In mathematics, definitions and proofs are crucial. One should not ask what a definition means, but instead what it says. This is because a mathematical definition specifies a formal concept exactly and without requiring further associations. The knowledge about a mathematical concept is precisely what can be proven to be equivalent to or implied by the concept. Understanding a definition involves translating its content into one or several synonymous representamens. In particular, graphical representamens are often useful. The translation must neither add nor delete features that cannot be proven. First year university students tend to think that understanding means forming associations with concrete and real world examples. They are not aware of the special role that definitions and proofs play in university mathematics.

I have not yet attempted to explore specific teaching methods that address these challenges. As mentioned in the introduction, designing and evaluating specific teaching methods for materials that the students find difficult are further steps in the decoding the disciplines model. Possible teaching methods could involve having students translate between different synonymous representamens, such as between formulas, graphical representamens and verbose descriptions. At least in the DS class, this appeared to be a suitable exercise for practicing the use of mathematical language and for detecting concepts that the students had not yet understood. It seems crucial to provide students with a sufficient variety of abstract examples so that they do not associate concrete and incorrect examples. It might also be helpful to discuss the special nature of mathematical knowledge with the students so that they can improve their metacognitive skills with respect to mathematical learning.

References

Amalric, M., & Dehaene, S. (2016). Origins of the brain networks for advanced mathematics in expert mathematicians. *Proceedings of the National Academy of Sciences, 113*(18), 4909–4917.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly, 111*(5), 411–424.

Ganter, B., & Wille, R. (1999). *Formal concept analysis: Mathematical foundations*. Berlin, Heidelberg, New York: Springer.

Kadunz, G. (2017). *A matter of translation* (This volume).

Marty, R. (1992). Foliated semantic networks: Concepts, facts, qualities. *Computers & Mathematics with Applications, 23*(6), 679–696.

Middendorf, J., & Pace, D. (2004). Decoding the disciplines: A model for helping students learn disciplinary ways of thinking. *New Directions for Teaching and Learning, 2004*(98), 1–12.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249–266.

Peirce, C. S. (1903). *CSP's Lowell lectures of 1903*. 2nd Part of 3rd Draught of Lecture III, MS [R] 465.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13,* 73–93.

Priss, U. (2002). Associative and formal concepts. In U. Priss, D. Corbett, & G. Angelova (Eds.), *Conceptual structures: Integration and interfaces. Proceedings of the 10th International Conference on Conceptual Structures*, Springer, Berlin, LNAI 2393, pp. 354–368.

Priss, U. (2015). An introduction to semiotic-conceptual analysis with formal concept analysis. In S. B. Yahia & J. Konecny (Eds.), *Proceedings of the Twelfth International Conference on Concept Lattices and Their Applications*, Clermont-Ferrand, pp. 135–146.

Priss, U. (2016). A semiotic-conceptual analysis of conceptual learning. In O. Haemmerle, G. Stapleton, & C. Faron-Zucker (Eds.), *Graph-based representation and reasoning. Proceedings of ICCS'16*, Springer, Berlin, LNCS 9717, pp. 122–136.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM—The International Journal on Mathematics Education, 40*(1), 83–96.

Rozenberg, G. (Ed.). (1997). *Handbook of graph grammars and computing by graph transformation* (Vol. 1). Singapore: World Scientific.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly, 23*(2), 139–159.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356–366.

Part III
Peircean Semiotics, Including Semiotic Chaining and Representations

Introduction: Norma Presmeg

All four chapters in this part address Peircean semiotics, semiotic chaining of various types, or representations, each in their individual ways. However, the first two chapters are more theoretical in nature, using empirical work as exemplars to illustrate theory, whereas the last two have a more empirical purpose, and semiotic theories serve as lenses in the accomplishment of the research goals.

In Chap. 11, Sáenz-Ludlow provides a masterful summary of some of the key elements in Peircean semiotics, particularly as they pertain to diagrammatic reasoning and iconicity. Following Peirce, diagrams are viewed as *icons of possible relations* and diagrammatic reasoning as an inferential process that encompasses both the nature of diagrams and their transformations. This reasoning is of various types; it entails chains of interpretants and may be abductive, inductive, or deductive in nature. Sáenz-Ludlow describes Stjernfelt's (2007) model of *diagrammatology*, thus linking nicely with Kinach's Chap. 13 in this part, in which diagrammatic reasoning is used to conceptualize a new kind of semiotic chaining.

Sáenz-Ludlow invokes Kant's two sources of representations, namely sensibility (phenomena—things as they appear) and intelligence (noumena—things as they are), in order to provide a foundation for Peirce's *perceptual judgments*: "all knowledge is the product of the self-corrective activity of the mind" (Sáenz-Ludlow). Observation depends on sensible and intellectual intuition, but for *new* meanings, collateral knowledge is required.

Sáenz-Ludlow's chapter is an excellent introduction to the theoretical elements used and developed in the other three chapters in this part. In Chap. 12, Salazar bases her theoretical analysis principally on the works of Duval and his Theory of Registers of Semiotic Representations. Salazar brings the gift of literature from Spanish, Portuguese, and French sources to English-speaking readers, whose familiarity with the theories of Duval might otherwise have been confined to his plenary presentation in the 1999 annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education and the publication that followed (Duval 2002). The three dynamic cognitive activities in every semiotic representation according to Duval, namely formation, treatment, and conversion, are linked in Salazar's account with elements in one of Peirce's triads, the classifications of signs according to their internal structure, namely as qualisign, sinsign, and legisign, respectively. This triad constitutes three elements of the ten relations that the sign vehicle can take, including the following:

- with its object—icon, index, or symbol;
- with its interpretant—seme, pheme, or argument; and
- in its internal structure—qualisign, sinsign, or legisign. (See Chap. 11)

There is thus good theoretical consonance between Chaps. 11 and 12. However, as Salazar points out, Duval did not develop his theory for Dynamic Representation Environments (DREs), and this area is the special contribution of Salazar in Chap. 12, in which examples from these environments are used to constitute a figural register.

There is also consonance between Chap. 11 and Kinach's framework in Chap. 13, in which she presents a new type of semiotic chaining based on Peirce's principles of diagrammatic reasoning and, in particular, the following:

Principle 1: copulating several propositions into one compound proposition;

Principle 2: omitting something from a proposition without introducing error;

Principle 3: inserting something into a proposition without introducing error. (Peirce 1998, p. 213)

However, Kinach's chapter has the pragmatic goal of enhancing prospective teachers' experiences with visual tasks in mathematics, and semiotic chaining is used as a tool in this regard. Thus, *mathematics teacher preparation* is the focus of the chapter, and theory is presented in a form suitable for the intended audience of preservice teachers. Also for this purpose, three of Krutetskii's (1976) series of tasks are mined as a source of tasks. However, this chapter entails careful analysis of only one elementary area concept set in an interactive computer environment, a digital game in five levels.

As in Chap. 13, the goal of Mathews, Venkat, and Askew in Chap. 14 is a pragmatic one. In view of the complexity involved in the teaching of division in elementary school, their goal is to elaborate dynamic pathways observed in the teaching of this topic. Four *signification pathways* were identified, based on the analysis of data collected for the first author's doctoral research. Influenced by theoretical models in the literature (Walkerdine 1988; Presmeg 1998, 2006; Ernest 2006), various sources addressing difficulties in the teaching and learning of division concepts, and the one–many relationship between division operations and situations (Askew 2015), the authors use *enacted semiotic systems related to division* to identify four signification pathways:

- coherent;
- coherent with limitations;
- coherent with instances of ambiguity; and
- mathematically incoherent.

The authors conclude that "Without careful selection of examples, actions, gestures, and other signs, teachers may produce multiple associations that may lead to the endorsement of unintended ways of making sense of division." Their analysis contributes to dissemination of knowledge that may help teachers and prospective teachers to construct more coherent signification pathways.

References

Askew, M. (2015). *Transforming primary mathematics: Understanding classroom tasks, tools and talk.* London: Routledge.

Duval, R. (2002). Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 311–335). Mexico: Cinvestav-IPN.

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics, 61*(1–2), 67–101.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children.* Chicago, IL: University of Chicago Press.

Peirce, C. S. (1998). *The essential Peirce* (Vol. 2, edited by the Peirce Edition Project). Bloomington, IN: Indiana University Press.

Presmeg, N. (1998). Ethnomathematics in teacher education. *Journal of Mathematics Teacher Education, 1*(3), 317-339.

Presmeg, N. (2006). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61*(1–2), 163–182.

Stjernfelt, F. (2007). *Diagrammatology: An investigation on the borderlines of phenomenology, ontology, and semiotics.* Dordrecht: Springer.

Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.

Chapter 11
Iconicity and Diagrammatic Reasoning in Meaning-Making

Adalira Sáenz-Ludlow

Abstract The focus of this chapter is twofold. The first is a semiotic description of the nature of diagrams. The second is a description of the type of reasoning that the transformation of diagrams facilitates in the construction of mathematical meanings. I am guided by the Peircean definition of diagrams as icons of possible relations and his conceptualization of diagrammatic reasoning. When a diagram is actively and intentionally observed, perceptually and intellectually, a manifold of structural relations among its parts emerges. Such relations among the parts of the diagram can potentially unveil the deep structural relations among the parts of the Object that the icon plays to represent. An Interpreter, who systematically observes and experiments with diagrams, mathematical or not, also generates evolving chains of interpretants by means of abductive, inductive and deductive thinking. Using Stjernfelt's model of diagrammatic reasoning, which is rooted in Peircean semiotics, I illustrate an emergent reasoning process to prove two geometric propositions that were posed by means of diagrams.

Keywords Triadic sign · Iconicity · Diagram · Diagrammatic reasoning
Proving

11.1 Introduction

Borrowing from Kant and Peirce, I first present a theoretical rationale to justify that perceptual and logical judgments are not only essential for diagrammatic reasoning but that they also go hand in hand with the active and passive workings of the mind in the construction of objects of knowledge. I also justify both why mathematical diagrams have important iconic characteristics that facilitate the elicitation of inferential thinking and why they evolve in the mind of the Interpreter to acquire

A. Sáenz-Ludlow (✉)
Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, USA
e-mail: sae@uncc.edu

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symbolic levels that are essential for abstract mathematical thinking. I then present a rationale to justify that diagrammatic reasoning is essentially an inferential process and that mathematical diagrams serve as epistemological tools in the learning-teaching of mathematics. I finally analyze the proofs of two propositions, posed in the form of geometrical diagrams, using Stjernfelt's (2007) model for diagrammatic reasoning. In the conclusion I highlight the importance of diagrams, diagrammatic reasoning and the interpretation process they entail, for the learning-teaching of mathematics.

11.2 The Passive-Active Actions of the Mind in the Construction of Objects of Knowledge

For Kant (1781/2007), the objects of knowledge have two independent sources of representations, namely, sensibility and intelligence. He calls the first source *phenomena* or 'things-as-they-appear' and the second *noumena* or 'things-as-they-are'. Therefore, an object of knowledge is both sensible and intellectual or rational. It is sensible insofar as it is the product of the laws of sensibility, and it is intellectual or rational insofar as it is the product of the laws of intelligence.

Kant also argues that the mind can be influenced by a *thing* or can create an *object of thought*. When the mind establishes a relation with a *thing*, insofar as it is affected by it, then the mind is *passive* with respect to that experienced thing. He calls this relation *sensible intuition.* In contrast, when the *object* depends upon the mind, then the mind is *active* creating that object. He calls this relation *intellectual intuition.*

On the one hand, *sensible intuition* is the receptivity of the person through which it is possible that his/her power of representation is affected in a certain manner by the presence of some object of experience. The *object of sensibility* is the phenomenon. When an *object* affects the senses directly, it produces a variety of sensible intuitions—a manifold of sensations and perceptions. This manifold carries with it two kinds of elements: (i) a subjective or material element (colors, taste, hardness, etc.), which has no cognitive value; and (ii) a formal or knowledge-giving element, which is the spatiotemporal organization and ordering of sensations that facilitates the formation of *perceptual judgments* (Wolff 1973).

On the other hand, *intellectual intuition* is the faculty of the person that enables the representation of things, which cannot act upon the senses by their own character. The *object of intelligence* is the intelligible which contains nothing except what must be known through intelligence—the noumenon. Kant contends that *intelligence* can have two uses: the real use and the logical use. The real use generates representations of objects or relations out of inner resources, thereby giving concepts to the mind. In contrast, the logical use orders and compares concepts, whatever their origin, in systems of species and genera according to the laws of logic. Both the real and logical uses of intelligence require the formation of *conceptual judgments* to complement and expand perceptual judgments (Wolff 1973).

What is a *judgment* for Kant? It is an act of the intellect in which two ideas, comprehended as different, are compared for the purpose of ascertaining their agreement or disagreement (Wolff 1973). Judgments are usually expressed in propositions composed of subject, copula or linking verb, and predicate.

Borrowing from Kant, Peirce argues that perceptual judgments on the particular and concrete contain general elements from which one can intuit general patterns, universal propositions, and principles. Perceptual judgments, he writes, are also related to the more deliberate and conscious processes of inferential reasoning, and this reasoning is continuous and carries with it the vital power of self-correction and refinement (Peirce 1992). That is, for Peirce, *all* knowledge is the product of the self-corrective activity of the mind. He also contends that there is nothing in the intellect that has not been first in the senses (CP 8.738) and that

> realities compel us to put some things into very close relation and others less so; but in the end, it is only the genius of the mind that takes up all those *hints of sense*, adds immensity to them, makes them *precise*, and shows them in an intelligible form of intuitions of space and time. (CP 1.383)

Both Kant and Peirce contend that *observation* has epistemological value and power because it genuinely depends on both sensible and intellectual intuitions. They argue that observation is tied to judgment, and that judgment is tied to intentionally planned reasoning. Peirce also contends that any inquiry activity, fully carried out by a person, is rooted in observation. For example, he writes, when different people observe a geometric diagram, they are able to *see* different relations, some *perceived* by the senses and some *inferred* with the aid of collateral knowledge. He also adds that collateral knowledge is a prerequisite in the apprehension and construction of new meanings (Peirce 1998).

Consequently, it can be said that in the *observation of geometric diagrams*, sensible and intellectual intuitions, collateral knowledge, perceptual and intellectual judgments, altogether, trigger abductive, inductive, and deductive inferences. Then it can also be said that the transformation of a diagram-token (an object of perception) into a diagram-symbol (an object of thought) is the product of the intertwined passive-active actions of the mind. Thus, a person's perceptual and intellectual judgments with and through diagrams are mediated by both sensible and intellectual intuitions.

11.3 Diagrams Initially Seen as Sign Vehicles of Iconic Nature

To say that 'diagrams are icons' is a very general and even strange statement when considered in colloquial speech. Nonetheless, it makes sense in the Peircean semiotics in which the notion of 'sign' is a manifold of elements and relations among them. Here I use only capital letters for the word SIGN to unambiguously signify not only its three constituent elements but also the three possible dyadic relations among them. The three constituent elements are: the *sign vehicle*,[1] the *interpretant*, and the *Object*. The three dyadic relations are: (1) between the sign vehicle and the Object it plays to represent; (2) between the sign vehicle and the interpretant it determines; and (3) between the interpretants generated and the dynamic object they progressively construct (i.e., an *object* which approximates the Object that the sign vehicle purports to represent and which is in a continuous state of refinement as new and more sophisticated interpretants are gradually generated). These three relations are not isolated from each other but interdependent on one another.

It is important to note that the *interpretant* is not the *Interpreter*. The interpretant is the effect of the sign vehicle on the mind of the Interpreter. The Interpreter, instead, is an agent who takes part in and presumably exerts control over the process of interpretation. Colapietro (1993) argues that the interpretant is not just any other result generated by a sign vehicle since this could also produce unrelated results. For example, he says, a fire indicating the presence of survivors of an airplane crash might also set a forest ablaze. The forest fire would be an incidental result but not an interpretant of the sign vehicle calling for help or indicating the whereabouts of the survivors.

Thus, the interpretant of a sign vehicle depends on what the Interpreter 'makes of it' and it is not just any co-emergent secondary result the sign vehicle might produce. In fact, the interpretant of a sign vehicle is another sign vehicle which is a transformation of the former; thus we could also denominate the interpretant as a sign-interpretant. This transformation enhances the initial sign vehicle and develops, in the mind of the Interpreter, a *dynamic object* which is more meaningful and more closely related to the Object that the initial sign vehicle plays to represent. Thus the interpretants are the product of acts of interpretation that progressively can move forward the conceptualization of the hidden Object that the initial sign vehicle stands to represent. As long as the Interpreter so desires, the interpretants become more sophisticated and abstract and the process of approximation of the *dynamic object* towards the Object continues through the process of interpretation which could be a never ending process of semiosis for the individual.

[1]The word SIGN, in capital letters, is used here to refer to the Peircean notion of 'sign' defined as a system constituted by a set of three elements and the dyadic relations among the three elements. The Peircean triadic notion of 'sign' was and continues to be a historically new conceptualization of 'sign' for which he is famously known (see Vasco et al. 2009). In other words, we could symbolize his triadic notion of 'sign' as a system constituted by a set and the relations governing the elements of the set in the following way:

SIGN = ({sign vehicle, interpretant, Object}, Dyadic relations among the three elements of the set).

The *sign vehicle* is only one of the elements of the set that stands as a representation of another element in the set, namely the *Object*. Most of the time, Peirce used the word 'sign' for sign vehicle without advising the reader about the use that he meant; the meaning has to be decoded from the context in which the words were used. However, sometimes he clearly uses the words sign vehicle and representamen to refer to the representation of the Object.

For Peirce a sign vehicle is “anything which, being determined by an Object, determines an interpretation to determination, through it, by the same Object” (1906, p. 495). He also adds that “a sign [sign vehicle] is not a sign [sign vehicle] unless it translates into another sign [sign vehicle]” (CP 5.594) and that “a sign [sign vehicle] is anything which determines something else (its *interpretant*) to refer to an *object* to which itself refers in the same way, the interpretant becoming in turn a sign [sign vehicle], and so on ad infinitum” (CP 2.303, italics added). He goes even further to say that the *relation* between the sign vehicle and its Object could be of iconic, indexical, or symbolic nature.

When is the relation between a sign vehicle and its Object of iconic nature? The *icon* is a sign vehicle determined by its Object by partaking in certain characteristics of that Object. In other words, the *icon* is a sign vehicle that bears some sort of resemblance or similarity to its Object. Peirce subdivides the icons into three types: diagrams, images, and metaphors. The *diagram* is characterized by some kind of similarity with its Object in the sense that it displays somewhat the existing relations between the parts of the Object in a skeleton-like manner (Stjernfelt 2007). In contrast, the *image* represents the Object through simple qualities, and the *metaphor* represents the Object through a similarity found in something else.

When is the relation between a sign vehicle and its Object of indexical nature? The *index* is a sign vehicle determined by its Object by being in its individual existence and connected with it. The *index* has a cause-effect connection to its Object, and it directs the attention to that Object by blind compulsion that hinges on association by contiguity (CP 1.558, 1867). An example of an index is the connection between the letter ‘x’ and an unknown variable quantity.

When is the relation between a sign vehicle and its Object of symbolic nature? The *symbol* is a sign vehicle determined by its Object by more or less approximate certainty that it will be interpreted as denoting the Object as a consequence of a habit. The symbol hinges on intellectual operations, cultural conventions, and habit (CP 3.419).

Fisch (1986) argues that these three relations between the sign vehicle and its Object are not independent of each other and that they also evolve in the mind of the Interpreter. In fact, these relations constitute a nested triad in which the more complex sign vehicle involves specimens of the simpler ones. In other words, symbols typically involve indices which, in turn, involve icons. This also means that icons are incomplete indices which, in turn, are incomplete symbols. This relation between the sign vehicle and its mathematical Object also depends on what the Interpreter ‘makes of it’. For example, when a mathematician reads the institutionalized definition of limit, he can ‘see’ symbols hinting at relations, among others infinite embedded intervals of real numbers on the x- and y-axis. In contrast, students can only ‘see’ awkward mathematical marks (iconic sign vehicles with no clear meaning). Thus the teacher has to guide the evolution of the students’ interpretations of these ‘icons’ so that they can advance their understanding of them as symbolic sign vehicles that carry with them the rich meanings of the notion of limit.

From this categorization of sign vehicles into icons, indices, and symbols, we learn that different sorts of sign vehicles can represent, in different ways, the Object that reasoning is concerned with. Now, since reasoning has to make its conclusions manifest, to oneself and to others, it also has to be concerned with the *dynamic objects* of perceptual and rational insights, objects which are evolving in the mind of the Interpreter. Therefore, reasoning has to be concerned with the interpretation of sign vehicles and their transformation, through the interpretants, into mental sign vehicles.

Mathematical diagrams, as icons, implicitly represent the structural features of the mathematical Object through some kind of similarity. Thought-experimentation on the diagrams facilitates the perceptual and intellectual progress of such evolution in the mind of the Interpreter. The Interpreter may see a diagram merely as a diagram-token (a pure icon), or as a diagram-icon or schema (an icon with indexical traits), or as a diagram-symbol (an icon with iconic-indexical-symbolic traits). Then, what type of icon is a diagram for the Interpreter? It depends on what the Interpreter 'makes of it.' This means that it depends both on prior and collateral knowledge that the Interpreter has and is able to draw into the situation at a particular point in time, and on his/her own ways of *observing* and *interpreting* by means of perceptual and intellectual judgments. This is also to say that the Interpreter, in the process of observation and interpretation, simultaneously plays the perceptual elements in thought and the thought elements in perception in order to mediate them.

11.4 Mathematical Diagrams Elicit Deductive Reasoning

Peirce (1906) argues that symbols afford the means of thinking about thoughts in ways in which we could not otherwise think of them; they enable us to create abstractions, which are the genuine means of discoveries. Knowledge is habit and symbols rest exclusively on already well preformed habit; thus symbols do not furnish any self-observation and so they do not enable addition to our knowledge. On the other hand, indices provide only positive assurance of the reality and nearness of their Objects. This assurance does not give any insight into the nature of those Objects. In contrast to symbols and indices, icons partake in the more or less overt character of their Objects and therefore they do not stand unequivocally for this or that existing thing. As a consequence, the Object "may be a pure fiction as to its existence, … but there is one assurance that the icon does afford in the highest degree; namely, that which is displaced before the mind's gaze—the Form of the icon, which is also its Object—must be logically possible" (1906, p. 496).

This is to say that diagrams, being icons, implicitly present by analogy, perception or inference the structural characteristics of the Object that they play to represent. Therefore, the perceptual and intellectual observation of diagrams, on the part of the Interpreter, has the potential to bring to the fore possible logical relations that have the effect of unveiling the structural elements of the Object (the object-as-it-is). Mathematical diagrams such as geometric figures, mathematical formulas and equations, graphs, tables, maps, etc., are essentially icons that also carry with them potential indexical and symbolic features that can guide perceptual and intellectual intuitions. These features of diagrams in general, and of mathematical diagrams in particular, pertain to the *forms* of the relations that structure the parts of the Object. Peirce also argues that diagrams are necessary for deductive reasoning; nonetheless, this necessity only means that the conclusion follows from the premise(s):

> Deduction is that mode of reasoning which examines the state of things asserted in the premises, forms a diagram of that state of things, perceives in the parts of the diagram relations not explicitly mentioned in the premises, satisfies itself by mental experiments upon the diagram that these relations will always subsist, or at least would do so in a certain proportion of cases, and concludes the necessary, or probable truth. (CP 1.66)

Given that diagrams present only a skeleton representation of the relations among the constituent parts of their Objects, they trigger through observation and experimentation abductive, inductive and deductive inferential processes.

11.5 Diagrammatic Reasoning as an Inferential Process

Peirce argues that the structure of a diagram has structural similarities with the abstract and hidden structure of its Object. This similarity warrants that the purposeful observation, perceptual and intellectual, of the structural relations among the parts of the physical diagram (the phenomenon or the Object-as-it-is-perceived) will enable thought-experimentation to infer the structural relations among the parts of the Object (the noumenon or the Object-as-it-is) by means of inferential reasoning. Peirce calls this amalgamated thinking process *diagrammatic reasoning*:

> By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, *performs experiments* upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses it in a *general form*. (CP 2.96, italics added)

The aim of diagrammatic reasoning is the construction of a mathematical argument that warrants the abstract structure of the mathematical Object. This argument is constituted not only by the construction of isolated inferences but also by the logical and cohesive concatenation of them. Each inference is the result of evolving related interpretants to form a logical assertion. In contrast, the argument is the concatenation of logical inferences that lend themselves to form a coherent chain of mathematical inferences. In this chain, any inference is sustained by prior ones. The formation of the argument is, for Peirce, the formation of a logical rule that has coherence and completeness. When Peirce uses the terms logical or logic, he means it in the sense of 'logic' as the study of thought insofar as it is subject to self-control with the aim of developing good habits of reasoning.

The mathematical argument, once formed, has to be expressed in complete sentences or mathematical statements. Each sentence or statement (subject, copula or verb, and predicate) uses mathematical terms or notations in order to convey a unit of thought. This is to say that sentences or statements can encode logical inferences in linguistic and mathematical terms. For a collection of sentences or statements to constitute a written mathematical argument they have to be combined and concatenated in a logical and linear manner to unveil the holistic abstract structure of the mathematical Object.

The analysis of the parts of a diagram, the relations among those parts, and the synthesis expressed in the argument reflect not only the perceptual elements in thought and the thought elements in perception but also the audacity of the mind to bring into play relevant collateral knowledge to aid in the justification of certain assertions. From this nonlinear activity of the mind, the holistic unity of the argument emerges out of the formation of a diversity of perceptions and percepts, perceptual and logical judgments, and conceptions. The differentiation between perception, percepts, conceptions and concepts will be made in a later section. Altogether, they come to unveil, in no uncertain terms, the structure of the mathematical Object and the validity of the conclusion.

It comes then as no surprise that Peirce appropriates the triad *term/noun, proposition,* and *argument* as the triad that reveals the connection between the sign vehicle and the nature of the interpretants it produces in the mind of the Interpreter. Peirce expands the meaning of this triad and he considers it as the triad *seme/possibility/concept/term, pheme/actuality/proposition,* and *argument*. A seme/term is "anything which serves for any purpose as a substitute for an Object"; it is, after all, "the Immediate Object of all knowledge and all thought" (1906, pp. 506–507). The pheme/proposition is "intended to have some sort of compulsive effect on the Interpreter of it" (*ibid*.). The argument tends "to act upon the Interpreter through his own self-control, representing a process of change in thoughts or signs [sign vehicles], as if to induce this change in the Interpreter" (*ibid*.).

It will be worthwhile here to summarize that the SIGN, for Peirce, is a triadic relation between its constitutive elements, namely, the *sign vehicle*, its *Object*, and the *interpretant* it provokes in the mind of the Interpreter. He unfolds the relation between the sign vehicle and the Object in the triad (icon, index, symbol); the relation between the sign vehicle and the interpretant in the triad (seme/term, pheme/proposition, argument) assigning to the argument a high logical standing; and the relation between the sign vehicle and its own internal nature in the triad (qualisign, sinsign, legisign). A qualisign is a quality and it cannot act as a sign vehicle until it is embodied; however its embodiment has nothing to do with its character as a sign vehicle. A sinsign (meaning being only once) is an actual existing thing or event. A legisign is a law; it is not a single object but a general type. Each legisign signifies through an instance of its application and each instance is a replica or a sinsign (Peirce 1998).

Although the latter triad does not have an immediate impact on the main goal of this paper, the coordination of the above three triads are at the root of Peirce's tenfold classification of sign vehicles. They can support a better analysis of diagrammatic reasoning in the classroom and a better analysis of the epistemological process necessary for the learning-teaching of mathematics. In this chapter, I concentrate on the chains of interpretants that geometric diagrams prompt when they are intentionally observed and experimented with.

11.6 Mathematical Diagrams as Epistemological Tools

Stjernfelt, a semiotician who has dedicated several books and articles to the analysis of Peirce's diagrammatic reasoning, extensively argues about the benefits of his non-trivial definition of icon. He argues that Peirce's definition avoids the weakness of most definitions of iconicity by similarity because of its connection with observation and thought-experimentation to discover additional pieces of information about the Object that the icon stands to represent. Peirce argues that "a great distinguishing property of the icon is that by direct *observation* of it other properties concerning its Object can be discovered than those which suffice to determine its construction" (CP 2.279, quoted in Stjernfelt 2007, p. 90; italics added).

In other words, diagrams, as icons, afford the formation of perceptions and conceptions that the grammar and syntax of their construction permit. However, while physical diagrams remain in the field of the senses, new logical relations among their parts can possibly emerge by means of imagination, manipulation, observation, and thought-experimentation. After all, a diagram can be characterized in one's mind in a variety of ways, "as a token, as a general sign [sign vehicle], as definite form of relation, as a sign [sign vehicle] of an order in plurality, i.e., of an ordered plurality or multitude" (Robin 1967, p. 31).

Peirce argues that "both the iconic diagram and its Initial Symbolic Interpretant constitute what… Kant calls *schema*, which is, on one side, an *object* capable of being observed while, on the other side, is a *General*" (1976, p. 316, italics added). He also argues that more can be learned about the Object of the diagram by the contemplation of explicit and implicit relations hidden in the physical structure of the diagram. He also adds that "all necessary reasoning is diagrammatic" and that "the diagram is an icon of a set of rationally related objects, a schema which entrains its consequences" (Robin 1967, p. 31). Furthermore, it can be said that diagrams are *epistemological tools* for inferential thinking.

Peirce adopts and adapts Kant's cognitive notion of schema for the inner workings of the mind of the Interpreter that a geometric diagram can produce, and gives an operational definition for this cognitive activity. A geometric diagram

> is a construction formed according to a precept furnished by the hypotheses; being formed, the construction is submitted to the scrutiny of *observation*, and *new relations* are discovered among its parts, not stated in the precept by which it was formed, and are found, by

> a little *experimentation*, to be such that they will always be present in such a construction. (CP 3.560, italics added)

This operational definition of diagram and diagrammatic reasoning entails that once a geometric diagram is constructed, it can be observed, manipulated physically and mentally, and transformed through physical and intellectual experimentation. As a result, what follows is the formation of dynamic interpretants that sooner or later become logical interpretants. The latter will become logically harmonized to contribute to the formation of chains of inferences that, in time, come to be expressed, organized and synthesized in coherent and logically coordinated sentences or mathematical statements. This is to say that the formation of mathematical arguments, geometric or otherwise, is an evolving cognitive process that is by no means linear but that will be presented as linear in the written argument.

Euclidean geometry is a classic example of physical and intellectual manipulation, and of thought-experimentation that can be performed on geometric diagrams. "Euclid first announces, in general terms, the proposition he intends to prove, and then proceeds to draw a diagram, usually a figure, to exhibit the antecedent condition thereof" (Peirce 1976, p. 317). Peirce's assertion reminds us of Polya's heuristics for solving problems and the formation of the mental schema to support construction of solutions. This is to say, understanding the problem and constructing a figure or diagram, devising a plan by means of thought experimentation on the diagram and bringing into play appropriate collateral knowledge, carrying out the plan within the logic of mathematical systems, and retrospectively and prospectively reflecting on possibilities for generating new problems or generalizing the one at hand (Polya 1957).

Nowadays, given the dragging mode of dynamic geometry environments, the manipulation of geometric figures is expedited, and with it, the observation of intentional manipulation and planned experimentation. Thus, the observation of variant and invariant relations among the elements of a geometric figure facilitates the formation of conjectures as well as their validation. The role of diagrams in deductive reasoning is well argued by Peirce:

> All deductive reasoning, even simple syllogism, involves an element of *observation*; deduction consists in constructing an icon or diagram the relations of whose parts shall present a *complete analogy* with those of the parts of the *object of reasoning*; in experimenting upon this image in the imagination; and in observing the result so as to discover unnoticed and hidden relations among the parts. (CP 3.363, italics added)

Netz (2014), who has dedicated himself to studying the evolution of Greek mathematical thinking, describes the Greeks' history of reasoning with geometric diagrams as follows: (a) Greek mathematical diagrams shaped deduction in mathematics; (b) mathematical objects were determined through diagrams; (c) letters inserted in diagrams were indices, not symbols; (d) diagrams are the metonymy of the propositions; (e) the writing of a proof was preceded by an oral rehearsal. He then concludes that, in general, the emergence of mathematical thought requires an inter-subjectively given object.

All in all, in this and prior sections we have a semiotic and a historical assurance that mathematical diagrams are tools that mediate deductive thinking. Given that deductive thinking is a source of knowledge, we can conclude that mathematical diagrams serve as epistemological tools for the learning-teaching of mathematics.

11.7 Visualization as an Indispensable Element in Diagrammatic Reasoning

For centuries it has been recognized that sense perception is an essential element of cognition. Kant (1781/2007) asserted that perception without conception is simply blind and that conception without perception is merely empty. For him, the thought elements in perception and the perceptual elements in thought are complementary and both make human cognition a unitary process that leads the way from the elementary acquisition of sensory information to the most generic theoretical ideas (Wolff 1973).

Following Kant, Peirce differentiates between immediate perceptions and percepts. Kant argues that perceptions are the effect of the *empirical object* (thing-as-it-appears) upon bodily sense organs and that by subjective association the mind forms a manifold of perceptions that are coordinated by means of empirical judgments—this manifold of perceptions is the percept. Percepts need to be transcended by means of mental operations for a conception to be formed.

> The synthetic unity of a manifold, therefore, is the characteristic possessed by a collection of mental contents by virtue of their having been produced by the imagination in accordance with a single rule. The consciousness of that synthetic unity is the conception of the rule by which it has been produced. (Wolff 1973, p. 130)

In general, it can be said that percepts are concrete instantiations of the Object (phenomenon or the Object-as-it-appears) and whose gradual and continuous differentiation proceed toward conceptions. Peirce argues that this "continuity is a special kind of generality among the relation of all of a certain kind of parts of one whole and that this continuity implies a passage from one percept to a contiguous conception" (CP 7.535). In fact, this continuity is the essence of the process of unlimited semiosis which mediates the transformation of perceptions into percepts, of percepts into conceptions, and of conceptions into concepts or the conceptual Object (noumenon or the Object-as-it-is).

During the last decades, cognitive scientists, by means of experimentation, have reinforced the notion that to perceive the external environment "our brain uses multiple sources of sensory information derived from several different modalities, including vision, touch and audition. All these sources of information have to be efficiently merged to form a coherent and robust percept" (Ernst and Bülthoff 2004, p. 162).

In the classroom, dynamic geometric environments offer not only the visual modality but also the simultaneous touch modality during the dragging process, which is mediated by the mouse or by the touch-pad. The integration of these two sources of information not only reduces perceptual estimations produced by paper-pencil drawings of geometric figures but it also enhances the formation of more reliable perceptions. The transformation of these perceptions entails the formation of percepts and empirical and logical judgments to produce all sorts of interpretants that sustain the formation of conceptions in the process of reasoning through diagrams.

During the last decades, there has been a history of research in mathematics education focusing on visual perception and its close connection with conceptualization. Seminal theoretical works from cognitive scientists were and continue to be influential in mathematics education. Among some of these works could be mentioned, with apologies to all who are not mentioned here but who are nonetheless recognized in our minds, Arheim's book on *Visual thinking* (1969), Johnson-Laird's book on *Mental models* (1983), and Davis and Anderson's (1979) article on *Nonanalytic aspects of mathematics and their implication for research and education*. To place specific emphasis on visual perception, Arheim (1969) rephrased Kant and wrote, "vision without abstraction is blind and abstraction without vision is empty" (p. 188).

For a synthesis of research on visualization in the learning and teaching of mathematics, the reader is referred to Presmeg's (2006) handbook chapter. In what follows I present a bird's eye view of some of the seminal notions that emerged in the 1980s.

Bishop (1989) proposed two types of ability constructs: *integrating figural information* and *visual processing*. The first is described as the ability to relate to a particular situation presented in some form of visual representation. The second is described as the ability to translate abstract relations and non-figural information into visual terms. The notions of *concept definition*, *concept image*, and *image schemata* also contributed a good amount of research in mathematics education. *Concept definition* refers to the definition institutionalized by the mathematics community at large (Tall and Vinner 1981). *Concept image* refers to subjective construction of meaning that corresponds to the concept definition of institutionalized mathematics (Arcavi 1999; Tall and Vinner 1981). *Image schemata* refers to the individual's mental constructions connecting those related concept images subjectively constructed by the individual (Dörfler 1991). Both the concept image and the image schemata provide a non-verbal, non-propositional component of cognition that contrasts with the notational and/or verbal descriptions of the concept definition. These notions, directly or indirectly, somewhat address the abstract and analytic nature of inferential reasoning (abductive, inductive, and deductive).

Skemp's (1987) book, *The psychology of learning mathematics*, also has been very influential in mathematics education research. Skemp goes beyond the above mentioned notions when he links perception and symbolic systems (symbolic mathematical notations or what he calls surface structures) as mediators in the construction of concepts (or what he calls conceptual structures or deep structures). His diagram shows this connection (p. 177) (Fig. 11.1). Figure 11.1 is Skemp's diagram complemented with other definitions in his book.

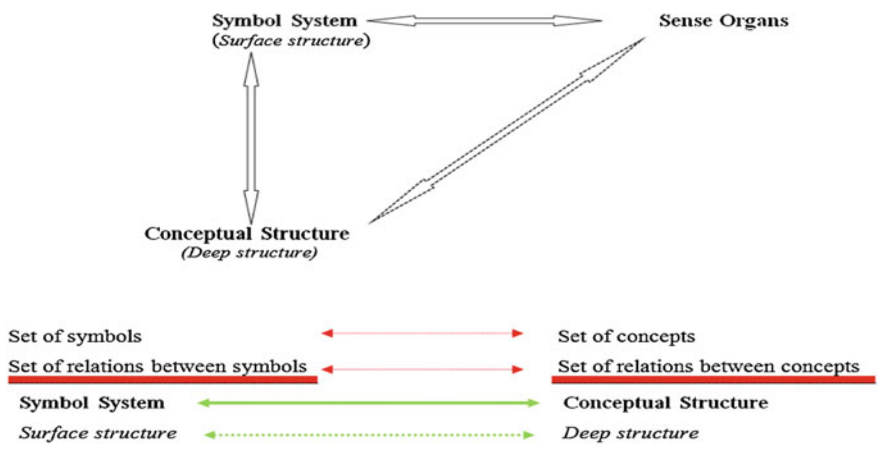


Fig. 11.1 Skemp's connection between perceptions and concepts

Skemp's diagram is a clear illustration of Goodman's (1978) argument in the introduction to his book, *Ways of worldmaking*. He writes,

> Kant exchanged the structure of the world for the structure of the mind; C.I. Lewis exchanged the structure of the mind for the structure of the concepts; and this book makes the argument for the exchange of the structure of the concepts for the structure of the several symbol systems of the sciences, philosophy, the arts, perception, and everyday discourse. (p. x)

In summary, all symbolic mathematical notations (surface structures) are essentially special kinds of diagrams; diagrams with iconic aspects and with the potential to unveil the indexical and more general symbolic aspects of the mathematical Objects that these diagrams stand to represent. The relations among the parts of the diagram (as-they-appear-to-the-senses) are physical representations of possible logical structural relations (as-they-appear-to-the-mind's eye) among the parts of the conceptual Object (the-Object-as-it-is). Reasoning through diagrams is essentially a process of unlimited semiosis, in which the interpretants mediated by diagrams are inferentially constructed and should also be logically connected to produce the conceptualization of the deep structure of the conceptual Object or an acceptable approximation of it. This semiosis is unlimited in the sense that even when an approximation to the conceptual object has been reached, a better and more sophisticated approximation could be produced in the future. For the notion of approximation we refer the reader to the article on inter-intra interpretation (Sáenz-Ludlow and Zellweger 2016).

11.8 Stjernfelt Model for Diagrammatic Reasoning

Stjernfelt (2007) captures, in Fig. 11.2, the essence of the process of diagrammatic reasoning, a process rooted in perceptual and mental activity to produce chains of inferences. This figure, which is itself a diagram, is a useful tool for thinking about the processes of proving and problem-solving. It synthesizes a manifold of relations that integrates the construction of the diagram, the observation of structural relations among its parts, and the perceptual manipulation and thought-experimentation to infer new possible relations conducive to the attainment of a logical conclusion or a solution.

He also describes this process in terms of the transformation of diagrams co-emerging with the formation of evolving interpretants. In this transformation, the implicit deep structural aspects of the Object (the Object-as-it-is) can be unveiled because of their analogy with the relations among the parts of the diagram. This is to say that, during the process of interpretation, the Interpreter mentally refurbishes the given or initially constructed diagram (transformand diagram) into more meaningful diagrams (transformate diagrams). In this process, a given or constructed diagram-symbol is interpreted by an Interpreter as a diagram-token and transformed into a diagram-icon or schema that is transformed again and again until the structural relations among the parts of the mathematical Object are unveiled by the Interpreter so as to see it as a diagram-symbol with the deep and hidden meaning that the initial diagram-symbol played to represent. Each time, new transformate diagrams reveal more and deeper structural relations among the parts of the mathematical Object that hinge on mental operations and inferential reasoning. It is in this sense that iconic sign vehicles grow into symbolic sign vehicles in the mind of the Interpreter. Although guided by the initial diagram-symbol, the Interpreter is giving freedom to think and to be creative within the context of a particular mathematical situation.

The sequence of interpretants that co-emerge with the transformations of the initial diagram is summarized by Stjernfelt (2007) using the letters ***a, b, c, d, e, f, g***. to describe the progressive steps in the development of deeper meanings constructed with and through new transformate diagrams.

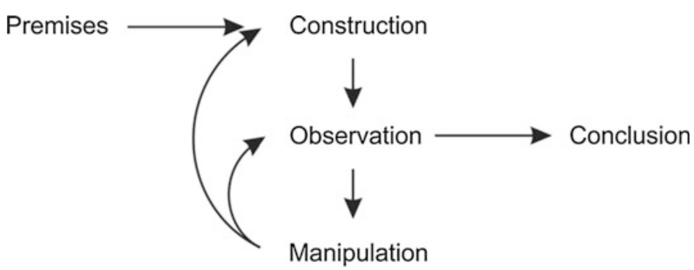


Fig. 11.2 Diagrammatic reasoning as a process (Stjernfelt 2007)

a. ***Symbol (1)***: Diagram-symbol [i.e., transformand diagram or mathematical symbol in the mind of the proposer of a problem or proposition].
b. ***Immediate Iconic Interpretant (b < a)***: Diagram-token [a rule-bound diagram]. An initial interpretation of the diagram-symbol ***a.***
c. ***Initial interpretant (b + c < a)***: The diagram-token is transformed into a diagram-icon [schema or skeleton relations among the parts of the diagram emerge in the mind of the Interpreter]. Initial transformate diagram.
d. ***Middle Interpretant ((b + c) + d < a)***: A diagram with three sources, **a**, **b**, and **c**. An emergent symbol-governed diagram equipped with possibilities of transformation [diagram-symbol, a more advanced transformate diagram with possibilities of further mental transformations].
e. ***Eventual, Rational Interpretant***: New emergent transformate diagram-symbol.
f. ***Symbol (2)***: Concluding transformate diagram-symbol or conclusion.
g. ***Post-Diagrammatical Interpretant (different from b)***: This interpretant is an interpretant of ***a*** as well; however, now the diagram-symbol produced is enriched by the total interpretant *of* ***Symbol (1).***

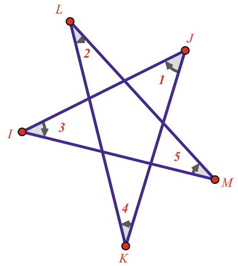
It is important to note that transformate diagrams are substantially embedded in the transformand diagram with all their unveiled significant features. That is, diagrammatic reasoning is the mental process of the Interpreter who intentionally endeavors both in the observation and in the manipulation of an initial diagram [transformand diagram/symbol (1)]. He first interprets this diagram as a diagram-token and then progressively enriches it and transforms it into diagram-icon and diagram-symbol (transformate diagrams). The final diagram-symbol [transformate diagram/symbol (2)] is the Interpreter's construction of the symbolic meaning of the initial diagram-symbol [transformand diagram/symbol (1)]. This is to say that the Interpreter finally unveils, as best as he/she can, the structure of the *Object* that the transformand diagram [symbol (1)] stands for.

It can be said that diagrammatic reasoning is a process by which the Interpreter intentionally endeavors in a process of inter-intra interpretation (Sáenz-Ludlow and Zellweger 2016) to enhance both the observation and manipulation of an initially proposed diagram-symbol in the mind of the proposer but only perceived by the Interpreter as a diagram-token. The Interpreter then transformes it into a diagram-icon, which in turn is transformed into a diagram-symbol enriched with new inferred general relations that contribute to unveil the hidden structural relations of the mathematical Object implicit in the proposed diagram.

11.9 Examples of Diagrammatic Reasoning

11.9.1 The Five Point Star

The geometric diagram of a five-point star was taken from Nelsen's (1993) book *Proofs without words* (p. 14). This example was modified to ask for the formulation



m∠IJK	m∠KLM	m∠JIM	m∠LKJ	m∠LMI	m∠IJK + m∠KLM + m∠JIM + m∠LKJ + m∠LMI
44.75°	31.99°	32.95°	28.60°	41.71°	180.00°
44.75°	28.19°	38.30°	28.60°	40.16°	180.00°
39.43°	28.19°	39.63°	32.59°	40.16°	180.00°
39.07°	28.19°	46.16°	32.59°	33.98°	180.00°
46.38°	28.19°	41.34°	30.11°	33.98°	180.00°

Conjecture:
The sum of the five point-angles of any five-point star is 180°

Fig. 11.3 Transformand diagram and the conjecture

of a conjecture with the use of the Geometer's Sketch Pad (GSP). Below I first present the transformand diagram and the formulation of the conjecture. Then I present a sequence of interpretants and transformate diagrams which aided in the proof (Fig. 11.3).

Given the dragging and measuring modes of the GSP, a conjecture can be inductively constructed: the sum of the five point-angles of a five-point star is 180°. A first observation reveals a pentagon with five non-overlapping triangles (here called point-triangles) formed by the extension of each of its sides. A second observation reveals implicit overlapping triangles constituted by the pentagon and two point-triangles. A third observation indicates that there are five overlapping triangles, each sharing one vertex with the pentagon.

Now it is necessary to bring to the fore some sort of appropriate collateral knowledge to make some sense of the observation. For example, the measure of the interior and exterior angles of triangles, the measure of straight angles, the measure of exterior and interior angles of pentagons, and the measure of vertical angles. The ensuing mental operations are to inquire about the possible connections among the collateral knowledge, the point-triangles, and overlapping triangles to prove the conjecture. Figures 11.4, 11.5 and 11.6 present a sequence of transformate diagrams.

In what follows I describe the sequence of interpretants that enabled transformations of the given diagram-symbol which was initially interpreted as a diagram-token.

Immediate interpretant. Visual perception of the five-point star as constituted by a pentagon and five point-triangles. The Interpreter mentally creates a transformate diagram-token representing the initial relations among the parts of the given diagram (see Fig. 11.4).

Initial interpretant. The visualization of the five-point star as constituted by a pentagon with five point-triangles formed by the extension, in both directions, of each of its sides. The Interpreter transforms the prior diagram-token into a diagram-icon. The latter indicates new relations and new possibilities for the construction of regular and irregular five-point stars.

Middle interpretant. The visualization of five implicit overlapping triangles: Δ**W**JK, Δ**Z**MI, Δ**Q**KL, Δ**N**IJ, and Δ**P**LM, each of which shares a vertex (**W**, **Z**, **Q**, **N**, or **P**) with the pentagon and two corresponding vertices of the star. Thus each

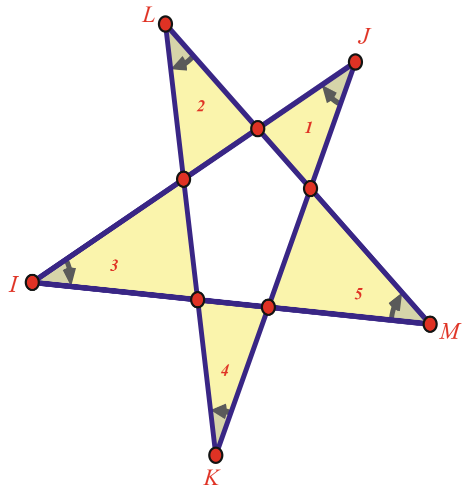


Fig. 11.4 Point triangles and pentagon

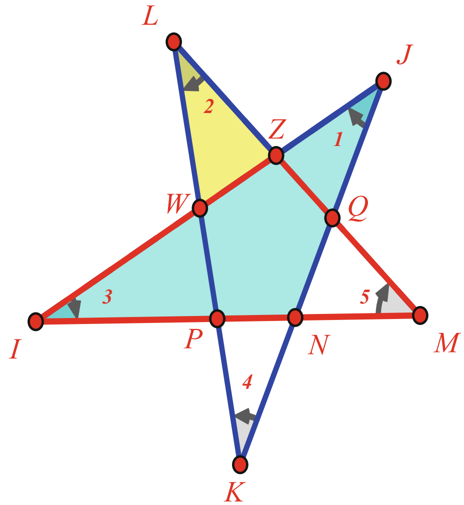


Fig. 11.5 Two of the five overlapping triangles: ΔI**N**J and ΔI**Z**M

triangle has already two point-angles of the star that when added give the measure of an external angle of that particular triangle. The Interpreter transforms the prior diagram-icon into a diagram-symbol. The latter enables the selection of appropriate collateral knowledge for the attainment of the proof of the conjecture (see Fig. 11.5).

Eventual rational interpretant. How are the angles of any point-triangle related to the angles of overlapping triangles that share a vertex with the pentagon? Focus on one point-triangle at a time, for example Δ**WZ**L. Which overlapping triangles will be related to this triangle? To make a decision, it would be useful to consider

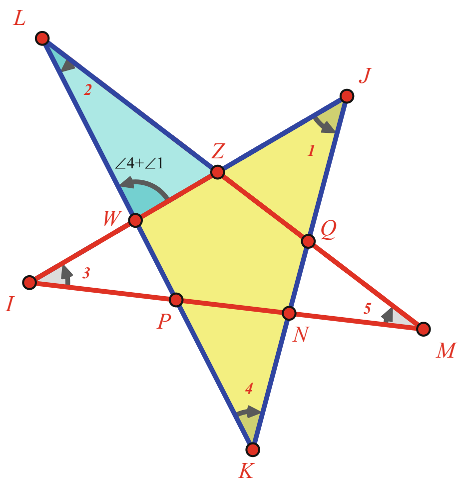


Fig. 11.6 Overlapping ΔJ**W**K and ΔI**Z**M correlated to point-triangle **WZ**L

the vertices **W** and **Z** of this triangle because they are also vertices of the pentagon. This leads one to consider the overlapping triangles ΔJ**W**K and ΔI**Z**M (see Fig. 11.6).

Consider triangles ΔJ**W**K and ΔI**Z**M

∠L**WZ** = ∠1 + ∠4 (external angle of ΔJ**W**K)

∠L**Z**W = ∠3 + ∠5 (external angle of ΔI**Z**M)

(∠1 + ∠4) + (∠3 + ∠5) + ∠2 = 180° (the sum of angles of **ΔWZ**L is 180°).

Since ∠4, ∠1, ∠3, ∠5, ∠2 are the point-angles of the five-point star, they add up to 180°.

Post-diagrammatical interpretant. The systematic dragging and observation of the five-point star guided the reasoning to formulate a conjecture about the sum of its point-angles and its proof. Several questions come to mind. Is this conjecture true for regular five-point stars? Can this conjecture be proved using the interior and exterior angles of the pentagon? Can this conjecture be proved using straight angles? Can the conjecture be generalized for n-point regular and irregular stars? The answers to these questions are in the positive. Unfortunately, space limitation does not allow for the presentation of the other proofs and the generalization but the reader is invited to try them. Nonetheless, the generalization arrived at is that the sum of the point-angles of any n-point star, regular or irregular, is (n − 4) 180°.

11.9.2 Trigonometric Functions of the Sum and Difference of Angles

The second example presents a geometric diagram proposed by Nelsen (2000, pp. 46–47) for proving the six trigonometric identities sin(α + β), cos(α + β), sin (α − β), cos(α − β), tan(α + β), and tan(α − β). The diagram proposed is an ingenious partition of a rectangle into four non-overlapping right-triangles. This partition is a creative abduction on the part of the proposer of the problem. Nonetheless, the inferential reasoning that follows from the observation of this transformand diagram is rooted in overcoded abductions as opposed to genuinely creative abductions (Sáenz-Ludlow 2016).

The analysis of the diagrammatic reasoning for each proof is presented visually by a sequence of transformate diagrams (Figs. 11.7 and 11.8) followed by a brief description of the co-emerging interpretants. The first four proofs are very similar in nature due to the strategic position of the length 1 for the hypotenuse of the most interior right-triangle.

Immediate interpretant. Visual perception and collateral knowledge aided the identification and justification of the positions of the angles α, β, α + β, α − β as well as the strategic position of β as one of the acute angles of the most interior right-triangle with hypotenuse of length 1.

Initial interpretant. In Fig. 11.7 the angle α with vertex C repeats with vertex B due to their perpendicular sides, and the angle (α + β) with vertex C repeats with vertex A due to their position as alternate interior angles between parallel sides of the rectangle and the transversal AC. In Fig. 11.8 the angle α with vertex C repeats with vertex B due to their position as alternate interior angles between parallel sides of the rectangle and transversal CB, and it also repeats with vertex A due to perpendicular corresponding sides. Angle (α − β) is directly given because α and β overlap and α > β. The right-triangle AFC in Figs. 11.8 and 11.9 facilitates the visual and determination of sine and cosine of (α + β) and (α − β) in terms of segment-lengths due to the length 1 of the hypotenuse AC of right-triangle AFC.

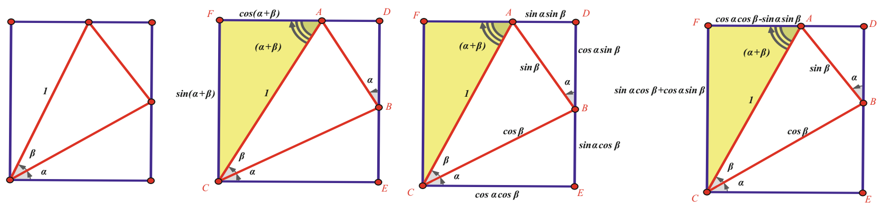


Fig. 11.7 Transformand and transformate diagrams for sin(α + β) and cos(α + β)

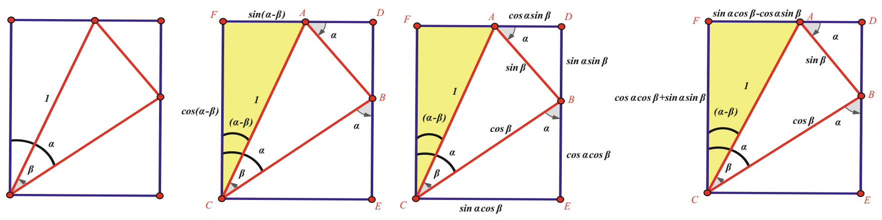


Fig. 11.8 Transformand and transformate diagrams for sin(α − β) and cos(α − β)

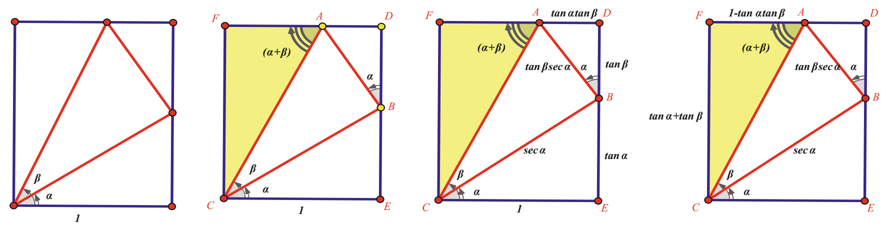


Fig. 11.9 Transformand and transformate diagrams for tan(α + β)

<u>*Middle interpretant*</u>. The right triangles ADB and CEB with hypotenuses sin β and cos β, respectively, enable the visual determination of the sine and cosine of their acute angles. The addition and subtraction of length-segments give the formulas for sin(α + β) and cos(α + β) (Fig. 11.7), and sin(α − β) and cos(α − β) (Fig. 11.8) from the respective transformate diagrams.

<u>*Eventual rational interpretant*</u>. By the definition of the sine and cosine functions of the angle α + β in the right-triangle AFC in Fig. 11.7, we have that sin(α + β) = sin α cos β + cos α sin β and that cos(α + β) = cos α cos β − sin α sin β.

Also by the definition of the sine and cosine functions of the angle α − β in the right-triangle AFC in Fig. 11.8, we have that sin(α − β) = sin α cos β − cos α sin β and cos(α − β) = cos α cos β + sin α sin β.

In the following two figures, Figs. 11.9 and 11.10, I present two sequences of diagrams to deduce the formulas for tan(α + β) and tan(α − β). The transformand diagrams (first diagram in each sequence) are the proposer's creative abductions. They are essentially the same transformand diagrams as in Figs. 11.7 and 11.8 but assigning the length 1 to the side CE in Fig. 11.9 or to side BE in Fig. 11.10. Clearly, the interpretants for these proofs incorporate the immediate and initial interpretants generated during the first four proofs shown above. Thus, this process is initiated with middle interpretants.

<u>*Middle interpretant*</u>. This is a realization that tan(α + β) and tan(α − β) can be determined from the ratios of the sides of the right-triangle FAC in Figs. 11.9 and 11.10, which have (α + β) and (α − β) as acute angles respectively. Thus, the task now is to use the trigonometric definitions of tangent and secant to determine the length of the sides of the other right-triangles ADB and CEB.

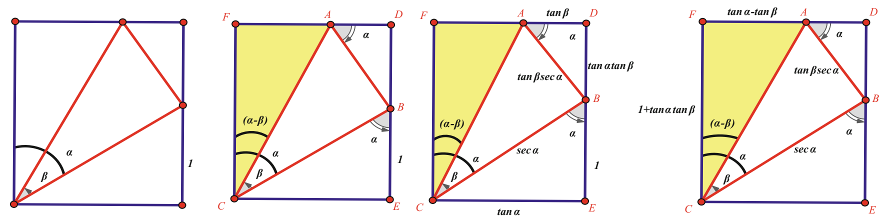


Fig. 11.10 Transformand and transformate diagrams for tan(α − β)

Rational interpretant. The lengths of the sides AD, DB, BE and CE of triangles ADB and CEB in the third transformate diagrams in Figs. 11.9 and 11.10 are the results of calculations with the definitions of the trigonometric functions. Once they are performed, the lengths of the sides FA and FC of right-triangle AFC are determined by simple addition or subtraction of segment-lengths.

Eventual rational interpretant. By the definition of the tangent function in the right-triangle AFC in Fig. 11.10, we have that $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$. Also by the definition of the tangent functions in the right-triangle AFC in Fig. 11.10, we have that $\tan(\alpha - \beta) = (\tan \alpha - \tan \beta)/(1 + \tan \alpha \tan \beta)$.

Post-diagrammatical Interpretant. What is the significance of the position of the length 1 in each of these proofs? Suppose that in Fig. 11.9 we make EB = 1 or BD = 1 and in Fig. 11.10 we make AD = 1. Will the proof hold? The answer to this question is in the positive. The reader could make an effort to arrive at the same formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ in these cases. These changes are motivated by the systematic manipulation of the transformand diagram and corresponding calculations.

11.10 Conclusion

Mathematical diagrams, such as geometric figures, mathematical formulas, equations and graphs, serve as epistemological tools to mediate not only the formulation and validation of conjectures but also the conceptualization of well-established mathematical ideas. The above examples illustrate that a systematic observation of a transformand diagram assists not only in the visual perception of different relations among its parts but also in the inference of new relations among them. Some of these relations are perceived by the senses and still others are inferred with the aid of perceptual and logical judgments as well as collateral knowledge. Both explicit and implicit relations embedded in the diagram enable the construction of new transformate diagrams to make explicit new relations among their parts; relations that, by analogy, will unveil the structural relations of the mathematical Object that the initial diagram plays to stand for.

The evolving perceptual and mental transformation of diagrams co-emerges with the formation of new interpretants in the mind of the Interpreters and with their progressive inferential reasoning process to reach the desired goal. This means a progression in the conceptualization of the deep structure of the mathematical Object that diagrams purport to stand for. It goes without saying that, when working with diagrams, the interpretants generated by different Interpreters, although they should be somewhat similar in nature, may have different iconic, indexical, and symbolic features according to their personal knowledge and level of sophistication of their mathematical thinking. The wanted outcome is that these interpretants share a tendency to converge to the desired mathematical Object (Sáenz-Ludlow and Zellweger 2016). This clearly indicates that teachers need to develop two simultaneous and parallel types of awareness: (1) awareness of the teachers' own evolving interpretations of transformate diagram(s) and (2) awareness of the students' interpretations of those diagrams. This double awareness on the part of teachers will enable the guidance of students by using the students' current interpretations and understanding. It is in this sense that diagrams serve as tools for meaning-making—epistemological tools—for teachers and students alike, although such a meaning could be at different but compatible levels of understanding.

References

Arcavi, A. (1999). The role of visual representations in the learning of mathematics. In *Proceedings of the 21st annual conference for the psychology of mathematics education* (Vol. 1, pp. 55–80).

Arheim, R. (1969). *Visual thinking*. Los Angeles, CA: University of California Press.

Bishop, A. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7–16.

Colapietro, V. M. (1993). *Glossary of semiotics*. New York, NY: Paragon House.

Davis, P., & Anderson, J. (1979). Nonanalytic aspects of mathematics and their implication for research and education. *Society for Industrial and Applied Mathematics Review, 21*(1), 112–126.

Dörfler, W. (1991). Meaning, image schemata and protocols. In *Proceedings of the 15th annual conference for the psychology of mathematics education* (Vol. 1, pp. 17–32).

Ernst, M. O., & Bülthoff, H. H. (2004). Merging the senses into a robust percept. *TRENDS in Cognitive Science, 8*(4), 162–169.

Fisch, M. H. (1986). *Peirce, semeiotic, and pragmatism*. Bloomington, IN: Indiana University Press.

Goodman, N. (1978). *Ways of worldmaking*. Indianapolis, IN: Hackett Publishing Company.

Johnson-Laird, P. N. (1983). *Mental models*. Cambridge, MASS: Harvard University Press.

Kant, I. (1781/2007) *Critique of pure reason* (M. Müller, Trans.). New York, NY: Penguin Books.

Nelsen, R. B. (1993). *Proofs without words* (Vol. I). Washington. D.C.: The Mathematical Association of America.

Nelsen, R. B. (2000). *Proofs without words* (Vol. II). Washington. D.C.: The Mathematical Association of America.

Netz, R. (2014). Greek mathematical diagrams: Their use and their meanings. *For the Learning of Mathematics, 18*(3), 33–39.

Peirce, C. S. (1906). Prolegomena to an apology for pragmaticism (PAP). *The Monist, 16*(4), 492–546.

Peirce, C. S. (1931–1966). In *Collected papers of Charles Sanders Peirce (CP)*. Edited by C. Hartshorne, P. Weiss, & A. W. Burks. Cambridge, MA: Harvard University Press.

Peirce, C. S. (1976). In C. Eisele (Ed.), *The new elements of mathematics* (NEM) (Vol. IV). The Hague: Mouton Publishers.

Peirce, C. S. (1992). In N. Houser & C. Kloesel (Eds.), *The essential Peirce (EP)* (Vol. I). Bloomington, IN: Indiana University Press.

Peirce, C. S. (1998). In The Peirce Edition Project (Ed.), *The essential Peirce (EP)* (Vol. II). Bloomington, IN: Indiana University Press.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Rotterdam, The Netherlands: Sense Publishers.

Polya, G. (1945/1957). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Robin, R. S. (1967). *Annotated catalogue of the papers of C.S. Peirce*. Worcester, MA: The University of Massachusetts Press.

Sáenz-Ludlow, A. (2016). Abduction in proving. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for the learning of mathematics* (pp. 155–179). Rotterdam: Sense Publishers.

Sáenz-Ludlow, A., & Zellweger, S. (2016). Classroom mathematical activity when it is seen as an inter-intra double semiotic process of interpretation. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for the learning of mathematics* (pp. 43–66). Rotterdam: Sense Publishers.

Skemp, R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Stjernfelt, F. (2007). *Diagrammatology: An investigation on the borderlines of phenomenology, ontology, and semiotics*. Dordrecht, The Netherlands: Springer.

Tall, D., & Vinner, (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151–169.

Vasco, C. E., Zellweger, S., & Sáenz-Ludlow, A. (2009). García de la Madrid: Ideas and signs in the Iberian Gray Zone (1650–1850) that follows the Black Hole (1350–1650). In J. Deely & L. G. Sbrocchi (Eds.), S*emiotics 2008: Proceedings of the thirty-third annual meeting of the Semiotics Society of Americ* (pp. 93–111).

Wolf, R. P. (1973). *Kant's theory of mental activity*. Gloucester, MA: Peter Smith.

Chapter 12
Semiotic Representations: A Study of Dynamic Figural Register

Jesus Victoria Flores Salazar

Abstract The aim of this chapter is to constitute the dynamic figural register. In order to do this, and based on the theory of Registers of Semiotic Representation and on the pertinence of the use of a Dynamic Representation Environment (DRE) for geometry teaching and learning, I revisit the three cognitive activities essential to every semiotic representation in the sense of Duval, which I call dynamic formation, dynamic treatment, and dynamic conversion, because it may be noticed that, when a person interacts with DRE, the three cognitive activities possess different features than when using pencil and paper.

Keywords Register of semiotic representation · Dynamic figure
Dynamic representation software

12.1 Introduction

This research is based on an issue that remained open in the investigation done by Salazar (2009), in which the need to configure the dynamic figural register is explicitly stated in the context of solving tasks involving geometry content in which one interacts with dynamic representation environments (DREs).

To pick up this issue, I present research related to semiotics, specifically the semiotics of Peirce, with Duval's theory of Registers of Semiotic Representation. I concentrate on geometry teaching, due to the importance of the use of figures, and on research in which there is evidence of the pertinence of DREs in geometry teaching and learning.

The research by Dionizio and Bandt (2012) aims at identifying the journey taken by semiotics in order to position the theory of Registers of Semiotic Representation. They present the relation between Duval's theory and aspects of Peirce's semiotics,

J. V. F. Salazar (✉)
Pontificia Universidad Catolica del Perú, Av. Universitaria 1801,
San Miguel, Lima 32, Peru
e-mail: jvflores@pucp.pe

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and they point out that, for Duval, Peirce's model classifying all types of representations is essential in terms of a triadic process of interpretation that leads to distinguishing several hierarchical levels of signs.

In that sense, the research by Silva (2013) is important because it also relates the theory of Registers of Semiotic Representation to Peirce's semiotics, and it shows the link between their concepts; that is to say, based on the hierarchical model of signs, she shows that the three cognitive operations essential to configure every register of semiotic representation—formation, treatment and conversion of an identifiable representation—are related to three out of the ten classifications of sign according to Peirce: *qualisign*, a quality that is a sign; *sinsign*, a real thing or event that is a sign; *and legisign*, a law that is a sign. Regarding mathematics teaching and learning, she identifies a comparative semiotic model, in which *qualisign* relates to the recognition of the characteristics of the given mathematical object in the formation of a representation, *sinsign* to the treatments in the representations of the mathematical object, and *legisign* to the changes made in the representations of the mathematical object through deductive reasoning.

The use of figures is fundamental in geometry teaching because it allows us to have access to the represented mathematical objects, infer properties, and solve problems. Further, as Duval (1995) states, the cognitive activity required in geometry demands more than other areas of mathematics considering that operations on figures, and their corresponding discourses, have to be simultaneous.

For these reasons, I looked for research such as that by Pavanello (2004), Marmolejo and Vega (2005), and Fortuny et al. (2010), which points out the importance of the use of figures in geometry, considering that they favor the development of the ability to abstract and generalize, because, from a level in which the represented geometric figures are recognized (by perception) despite perceiving them all as indivisible, a level is reached in which the properties of these figures are distinguished so as to establish relations between them with regard to their properties.

Also, on the one hand, researchers specify that the importance of geometric figures lies in the fact that they form an essential intuitive support for the development of geometric problems, because they allow students to see much more than what the formulations say; they allow the formulation of propositions and the heuristic exploration of complex situations. Therefore, it is necessary to differentiate the ways a figure may be seen, and to select those ways that are pertinent and powerful to solve the posed geometric problem.

On the other hand, the research by Miskulin et al. (2003) shows, from a semiotic perspective, the main characteristics and didactic-pedagogical potentialities of different technological environments used in mathematics teaching in a public university. They analyzed these environments through the syntheses done by students who used them in their classes, and they identified those potentialities. Another research study I consider important is the one by Mendes (2006), which presents a semiotic analysis of the game application called *Simcity*. The way of playing is subject to the interaction and/or interpretation each player attributes to the game in a given instant, for which he or she uses different representations. It is worth mentioning that, in both studies, it was perceived that the process of *semiosis* occurred at every instant during the interactions with the software.

Regarding DREs, the work of Veloso (2000) and Gravina (2001, 2008) points out that the DRE interactive interface allows the exploration and experimentation of the represented objects, and the immediate feedback helps students question the results of their actions/operations and verify the validity of their conjectures. In the same line of thought, Laborde (2001), Olivero and Robutti (2001), Restrepo (2008) and Grinkraut (2009) confirm that DREs possess a *visual* language that stimulates communication, and their *interactivity* stimulates students to be interested in the mathematical content that can be worked on with these environments.

Additionally, Artigue (2002) and Laborde (2005) maintain that, when DREs are used strategically, they can be really useful for students to justify the results of a given task, or they can use them as a tool to validate constructions, elaborate conjectures and inferences related to the properties of the represented mathematical objects. Further, in the construction of mathematical knowledge with the use of technology—with dynamic representation software in our case—DREs can facilitate the visualization of properties of different represented mathematical objects that could not be discovered with other resources.

Regarding the dragging function that facilitates the heuristic exploration of the figures because it allows the simulation of movement that facilitates the change of position, shape, and measure, Olivero (2003) states that it possesses three essential functions:

> *Feedback*: Since the student is in control of the constructions he or she performs, dragging offers a powerful means of feedback that helps the development of strategies to solve a problem situation […]
>
> *Mediator between figure and design*: It is given that it allows the invalidation of mistaken constructions […]
>
> *Analysis or search mode*: Dragging allows the student to examine his or her construction and search its invariable properties by exploring the figure. (pp. 56–59)

Likewise, researchers such as Salazar et al. (2012), da Silva and Salazar (2012), García-Cuéllar (2014), Salazar (2015), Gómez (2015) and Peñaloza (2016) show that the use of DREs in the processes of geometry teaching favors the interpretation of the properties that characterize represented geometric figures. This focus on the interpretation of properties happens because, in the sense of Duval, DREs accelerate their treatments and articulate their apprehensions. In addition, they invigorate conversions among different registers of semiotic representation, which confirms the need to characterize the dynamic figural register.

As a consequence, given the importance of the geometric figures in geometry teaching and the current benefit provided by the interaction with DREs, it is considered necessary to research how the figural register is configured when interacting with these environments.

Based on the aforementioned issues, in this chapter I aim to answer the following question: *How is the dynamic figural register constituted when interacting with DREs?* Therefore, I first present aspects from the theory of Registers of Semiotic Representation.

12.2 Registers of Semiotic Representation

Duval (1995) explains that semiotic complexity is behind the difficulties in mathematics learning and that the analysis of mathematical productions demands tools for semiotic analysis adapted to cognitive processes mobilized in every mathematical activity.

Regarding the mathematical activity, this is done necessarily in a context of representation, since there is no other way to have access to the object but by its representation. Semiotic representations are generally considered as a means to externalize mental representations for communication purposes, but they are also essential for the cognitive activity, since they play a fundamental role in the development of mental representations, the execution of different cognitive functions, and the production of knowledge.

In addition, all learning is related to the *semiosis* and *noesis* processes, where *semiosis* is the apprehension or production of a semiotic representation and *noesis* is the conceptual apprehension of an object; thus, noesis is inseparable from semiosis. According to Duval (1993), “they are productions constituted by the use of signs belonging to a system of representation, which have their own difficulties in meaning and functioning” (p. 39). Regarding mathematics, semiotic representations are the ones that allow conceptualization, reasoning and problem solving.

Duval explains that natural language, algebraic, graphic and figural representations are external, conscious semiotic representations; that is, they are representations produced by students, which can be executed when doing operations in a given semiotic system. Duval defines these systems as *registers of semiotic representation* since they allow the three cognitive activities that are essential in every register: formation, treatment and conversion.

The formation of a representation is constituted by a group of noticeable marks that can be identified as the representation of something in a given system, such as the design of a geometric figure, the elaborations of a diagram, etc. that have to respect certain rules, such as the rules of construction for figures, etc., which play two roles: on the one hand, these rules ensure the conditions of identification and recognition of the representation and, on the other hand, they ensure the possibility of using it for the treatments.

Regarding the treatment, it is “to transform representations according to the only rules typical of the system so as to obtain other representations that can constitute knowledge earning in comparison with the initial representations” (Duval 2004, p. 30). The nature and number of these rules may vary from one system to another; for example, reconfiguration is a typical treatment of geometric figures, which, depending on the problem, requires a specific number of configurations. The conversion of a representation is the external transformation of a representation into another representation in another semiotic system. Treatments and conversions are independent sources of incomprehension or mental block in students. However, Duval states that conversion is the most decisive transformation for comprehension in mathematics.

Regarding mathematics, and geometry especially, I know it demands a cognitive activity more complex than other domains of mathematics because it is necessary to construct, reason and see figures at the same time, since treatments in the figural register are simultaneous as there must be coordination between figural treatments and those of theoretical discourse in natural language. In that sense, figures can be seen and analyzed based on perception, which allows the distinguishing of their shape and the recognition of their geometric properties, not only of visually distinguished shapes, but also based on reproducing or constructing them with tools.

In that sense, Duval (2005) explains that, to be able to do a cognitive analysis of a figure, figural units must be considered. These figural units are the most basic shapes into which every figure can be analyzed. For example, they can be 3D (polyhedra), 2D (polygons), 1D (lines or curves) and 0D (points).

Regarding the construction or reproduction of a figure, Duval states that manipulative material (pieces of a puzzle, Tangram folding paper, etc.) can be used, allowing material objects to be manipulated, such as a ruler (calibrated or not), a compass, a mold, etc., since they allow the construction operation, and software, as shown in Fig. 12.1.

In addition, heuristically using figures in geometry to solve problems, explain properties, elaborate conjectures, etc., depends on two operations, namely, the division of the figure into figural units of the same dimension that rely on perception (operation of reconfiguration), and the dimensional deconstruction of figures. The significance of the latter resides in the fact that the mathematical way of seeing figures consists in decomposing them as *nD/2D* shapes, that is into figural units of inferior dimension in comparison with the given figure. For example, the figure of a cube (3D) is decomposed into a configuration of squares (2D figural units) and the squares are decomposed into lines or line segments (1D figural units). The lines or segments can be decomposed into points (0D figural units); in relation to the points, they are visible when they appear as the intersection of 1D figural units (secant lines or lines that form a corner: vertices, angles, etc.).

In relation to the figural register, Duval (2011) indicates that, from a cognitive point of view, in mathematics education it should be assumed that geometric figures form a register of semiotic representations. Likewise, based on the way one 'sees' a figure, Duval explains that it is possible to analyze it cognitively according to its shape, to the geometric knowledge it activates, and to the tools used for its construction. There are three aspects he considers to be important in analyzing a figure:

- Shape of the figure: based on the perception of the figure, whose contours are closed and where the figure can be seen as a group by juxtaposition, in which

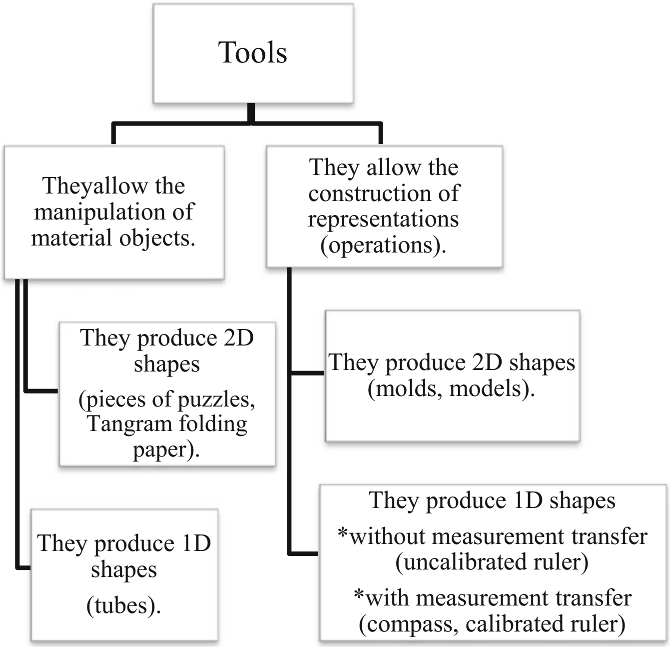

Fig. 12.1 Classification of tools to construct figures. *Source* Adapted from Duval (2005, p. 14)

there are as many shapes as closed contours; or by superposition, in which there are less shapes than closed contours.

- Knowledge of the geometric properties of the figure, which must be activated for their analysis and whose figural units must be considered, or visual variables which are related to the number of dimensions: 0D (a point), 1D (a line), 2D (a surface), and qualitative variations of shape, contour, orientation, etc.
- Tools used to represent the figure or to construct it, which may include the following: manipulative material (pieces of puzzles, Tangram folding paper, etc.) that allow the manipulation of objects, ruler (calibrated or not), compass, mold, etc. that allow construction operations, as seen in Fig. 12.1.

Regarding the classification of tools to construct geometric figures, I observe that Duval does not explore the use of DREs. In that sense, DREs must be considered because, as we know, they are useful tools for geometry teaching and learning since they allow the representation of geometric objects and they have been rekindling interest in studying geometry in the last few decades. Another reason is because, unlike tools, DREs allow the carrying out of operations of figure construction, whether by juxtaposition or superposition, which can only be done separately with the aforementioned tools.

Also, DREs facilitate seeing the figure geometrically as an assembly of 0D, 1D or 2D figural units; in other words, an analysis of the figure's dimensional deconstruction can be done. In that sense, Mithalal (2011) explains:

> Dimensional deconstruction consists in seeing the representation of an object as a set of figural units linked by geometric properties. This last point is fundamental because it designates the role of those properties in this process: it is not enough to isolate figural units; it is also necessary to organize them geometrically. (p. 115)

Therefore, dimensional deconstruction allows the studying of the transformation of a figure into a different one of the same dimension, and it allows the identification of its figural units. Then, it is conjectured that, when DREs are used, the dimensional deconstruction is accelerated because the student may quickly identify the figural units of which the figure is composed when using the tools in the software as well as the dragging function. Likewise, changing the position of the observer also allows him to identify the figural units of a given figure.

For example, the task I present, adapted from the research by Gómez (2015), consists in relating the measurement of the area of the FGHE quadrilateral to the measurement of the area of the new $FGG'F'$ figure.

To solve it, auxiliary lines are constructed (to deconstruct the figure); that is, figural units of a smaller dimension are used, namely 1D (lines, circumference networks) and 0D (points). In addition, the geometric properties used include the dragging function of DREs, so it is possible to analyze the transformation of the ABCD quadrilateral (see the left side of Fig. 12.2) into a different figure of the same dimension (see the right side of Fig. 12.2). Also, when doing these operations, it is validated that, since $F'F = 2EF$ and $F'G' = FG$ and since $F'F \parallel G'G$ and $F'G \parallel EH \parallel FG$, the new figure formed is a parallelogram $FGG'F'$ of the 2:1 ratio, by relating the two measurements of the area.

When figures are constructed in DREs, it is likely that they are cognitively different than when it is done in a non-dynamic environment, because treatments are accelerated and the coordination between the figure treatments and the discourse is different, as the discourse is in the figure.

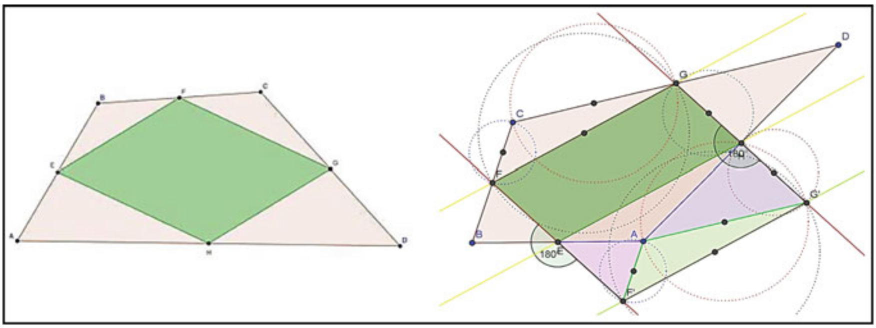

Fig. 12.2 Dimensional deconstruction

For these reasons, I point out that these environments make it viable to elaborate conjectures about the properties of the mathematical objects represented in them, whether by juxtaposition or superposition.

12.3 Dynamic Figural Register

Since the aim of the research is to identify how the three cognitive activities, namely, formation, treatment and conversion, occur when working with geometry content and interacting with DREs, each one of them is presented next with examples, taking the work of Gómez and Salazar (2016) as reference.

12.3.1 Dynamic Formation

According to Duval (1995), the formation of a semiotic representation consists in representing a given object in a certain semiotic system; it is essential for this formation to respect the rules of conformity of the semiotic system used. It is worth mentioning that these rules of conformity define a system of semiotic representation and the types of constitutive units of all possible representations. In addition, they allow the recognition of the representations as representations in a given register. Regarding the formation in the figural register, it occurs by identifying the figural units and their combination based on the geometric properties of the figure.

In that sense, Duval (2011) explains that there are several ways to recognize the shapes or figural units of a figure, though recognizing a shape excludes the possibility of recognizing others.

That is why, in DREs, forming a semiotic representation, which I call dynamic formation, at least three different ways to recognize a figure are identified. In the first one, to identify a geometric object in DREs, a given tool is activated, allowing the creation of the desired figure. In this case, the student recognizes the characteristics of the object he or she wishes to represent through its figural units.

For example, given the task to create any ABC triangle, it is necessary to recognize the characteristics of a triangle in order to perform the task. In this section, the student must activate geometry knowledge. Thus, in Fig. 12.3, I observe that the polygon tool makes it possible to create the requested figure since it shows the representation of the object associated with a "key" that allows the student to directly recognize in the tool the figure requested, which in this case is a triangle.

In the second way of dynamic formation (see Fig. 12.4), the rules of formation of semiotic representation also depend on the choice of elemental figural units and their combination. This sequence occurs because of the conditions of GeoGebra DRE (advantages and/or limitation of the same environment). In the case of the previous task, at the perceptive level, variations related to the number of

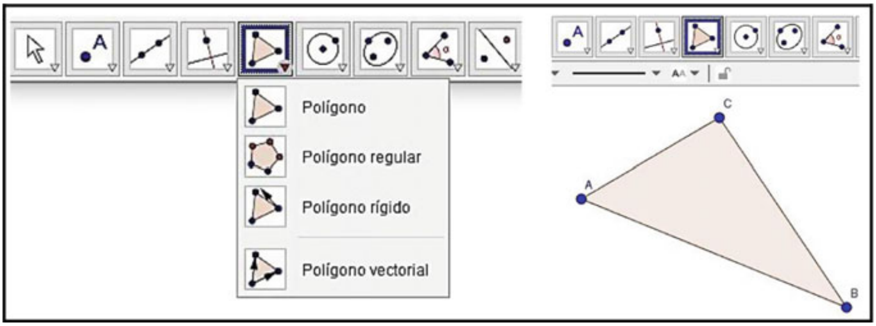

Fig. 12.3 Dynamic formation–shape 1

dimensions stand out, such as 0 (point), 1 (line/segment) or 2 (surface), and qualitative variations in shape, orientation, color, etc.

The third means of dynamic formation is the one that refers to a geometric construction in which properties stay invariant; that is, the variation of the parameters in Geogebra allows the discovery of invariants and the generalization of particular cases. For example, in the task to construct an ABCD square, the formation of the square is presented on the left of the DRE and, on the right, the protocol of construction that allows the showing of 0D and 1D (points and points of intersection, lines, segments) figural units and 2D units (circumference and polygon) in the construction, as shown in Fig. 12.5.

By using the dragging function, the constructed figure keeps its properties invariant. In addition, this type of formation allows the observation of the dimensional deconstruction of the figure since lines, segments and points are identified. Picking up aspects from Peirce's semiotics and from the theory of Registers of Semiotic Representation, I observe that the formation of a dynamic semiotic representation is related to the *qualisign*, which is the first classification of signs and which, in mathematics teaching, is related to the recognition of the characteristics of the mathematical object.

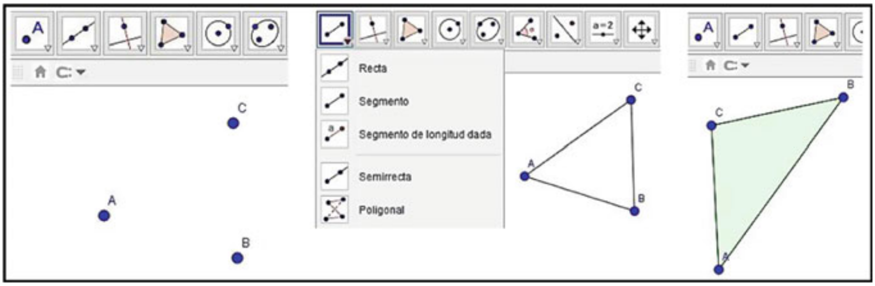

Fig. 12.4 Dynamic formation–shape 2

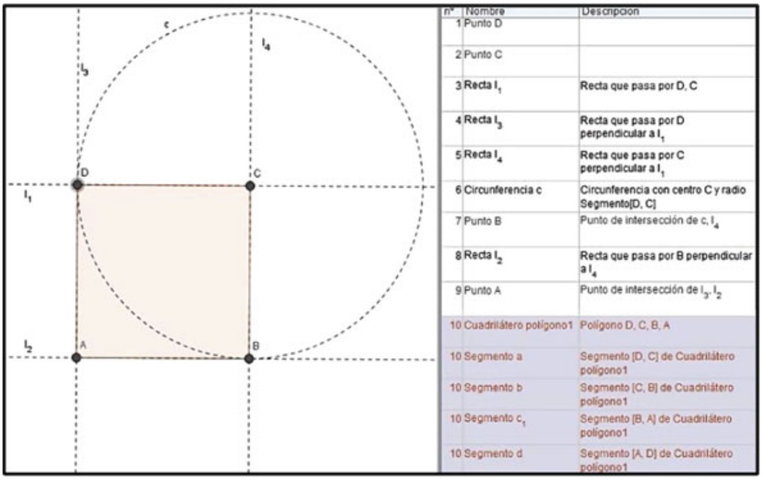

Fig. 12.5 Formation of a square

12.3.2 Dynamic Treatments

According to Duval (2005), treatments are executed in the figure when modifications are done, for instance: changing its position while preserving the same configuration (change of orientation, translation, figure rotation, length of the sides, etc.), and decomposing the figure into its figural units, combining these to form another figure or dividing it into other subfigures that might or might not be regrouped to form other figures.

I highlight that the treatments are accelerated when two fundamental functions of DREs are used, which are as follows: direct manipulation, which allows the changing of position of the figure or the position of the work area; and dragging, which allows the rapid carrying out of operations related to figure reconfiguration. This way, the types of figure treatment that I identified in this study are changing the position of the figure without modifying it by using the function of direct manipulation; changing the length of the sides of the figure by using the dragging function as well as the homothecy tool, which multiples every distance by the same factor from a fixed point; and reconfiguring the figure, in which the dragging function is used as well as other specific tools that depend on the constructed figure.

But I must also point out that these treatments are not necessarily done separately or in sequence because they can be done simultaneously in DREs. These treatments possess specific features; in this part, I focus on figure reconfiguration.

Reconfiguring a figure: to do this type of treatment (dividing a figure into other subfigures), the tools needed are used from DREs, depending on the figure and the problem to be solved in order to separate, regroup or form another figure from an initial configuration. In addition, with the dragging function, the heuristic function of the figure is improved since configurations can be done rapidly and dynamically.

Open the Activity file. Move the S point in such a way that the intersection of ABCD and EFSH squares forms the configuration of a square, a triangle and any quadrilateral, and answer: what is the relation between the surface formed by the intersection of the squares and the surface of the ABCD square? Justify your answer in GeoGebra.

Fig. 12.6 Task 1

To explain this type of treatment when using DREs, a task is presented (see Fig. 12.6), which was adapted from the research by Gómez (2015), Salazar and Almouloud (2015), Gómez and Salazar (2016), where GeoGebra DRE was used.

Figure 12.7 shows the file that has to be opened to do the requested activity.

I observe that, when using DREs, the discourse about the activity is different from that when the same activity is presented without the use of these environments. In addition, a file is presented in the activity, which is ready for the student to move the S point and change the position of the EFSH square, through direct manipulation and dragging, to configure the requested figures.

The following are three possible configurations that can occur when solving the task with dynamic treatments:

The intersection of the figures represents a square: to get this configuration and be able to establish the configuration and the requested relation, the student could use the tools and draw perpendicular lines L_1 and L_2, which intersect in the E

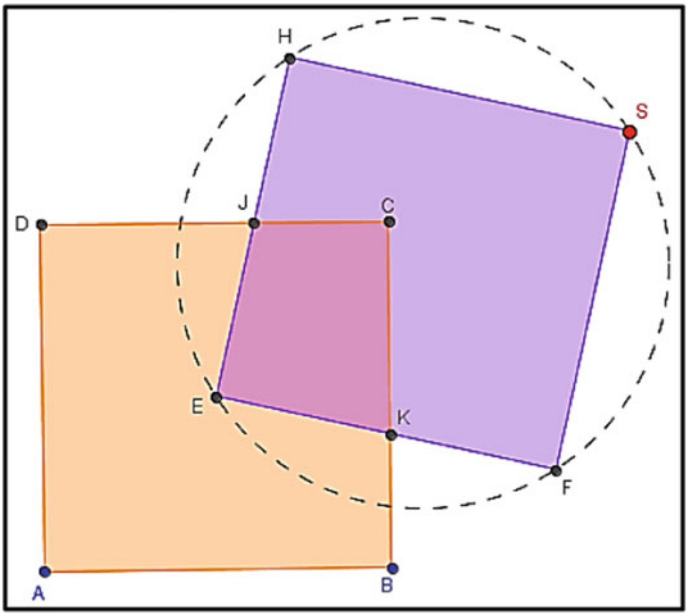

Fig. 12.7 Figure of the task. *Source* Salazar and Almouloud (2015, p. 933)

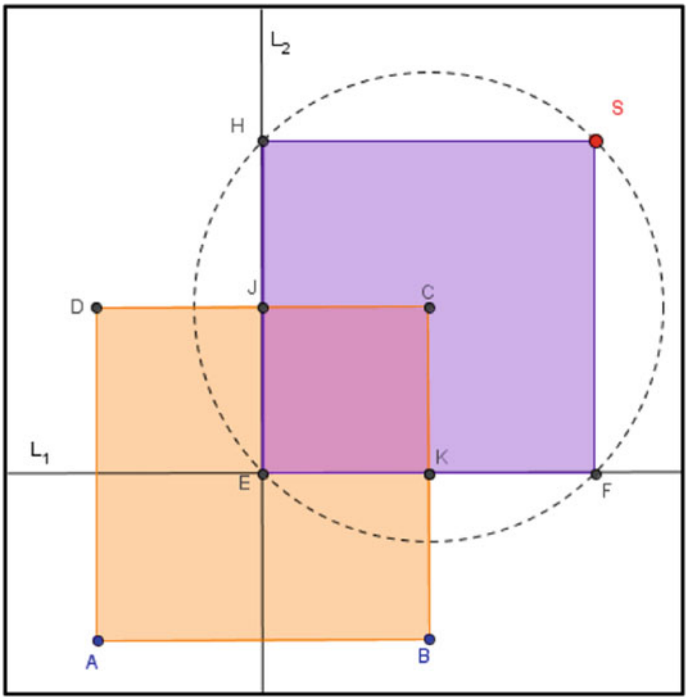

Fig. 12.8 EKCJ square. *Source* Salazar and Almouloud (2015, p. 934)

point (see Fig. 12.8), and then move (drag) the S vertex until getting the requested configuration.

In the previous figure, the $\overline{EH}$ *and* $\overline{EF}$ sides of the *EFSH* square overlap in L_1 and L_2, and the square figure is created by overlapping the two 2D figures (two squares), which creates another figure of equal dimension (EKCJ square). The student can easily perceive and state that the intersection of the figures represents a square that coincides with the fourth part of the ABCD square.

On the other hand, in Fig. 12.9, another solution to the configuration of the square is shown. In this case, the student drew perpendicular lines that cut in the E center, then he projected the lines on the sides of the ABCD square, and marked the M and N points of the $\overline{AB}$ *and* $\overline{BC}$ sides respectively (middle points of these sides). Afterwards, he dragged the S point to the position and verified with the *middle point* tool that the intersection of the $\overline{EF}$ side with the $\overline{AB}$ side is the M point, and the intersection of the $\overline{EH}$ side with the $\overline{BC}$ side is the N point.

I consider that the student did the dimensional deconstructions of the figure because he explored the relations between the 0D units (points, points of intersection), 1D (lines) and 2D (squares), the group being by superposition.

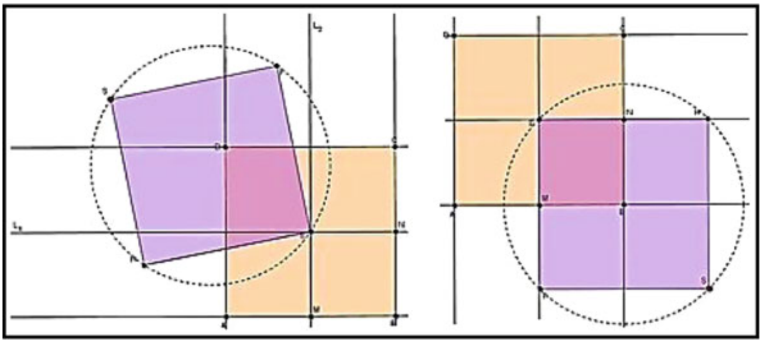

Fig. 12.9 The square–other solutions. *Source* Salazar and Almouloud (2015, p. 935)

The intersection of the figures represents a triangle: in this case, the student may draw the $\overline{AC}$ *and* $\overline{BD}$ diagonals of the ABCD square and, using direct manipulation, drag the S point until the requested figure is configured.

The position can change then until forming a *BEC* triangle over the ABCD square, for example (see Fig. 12.10). In this case, also by grouping and superposition, it is perceived that the intersection of the figures coincides with the fourth part of the ABCD square.

The intersection of the figures represents any quadrilateral: when dividing the ABCD quadrilateral into four congruent regions by the L_1 and L_2 perpendicular lines (Fig. 12.11) of the ABCD quadrilateral, with the use of the tools in the software it is possible to establish a relation between the subfigures that form due to

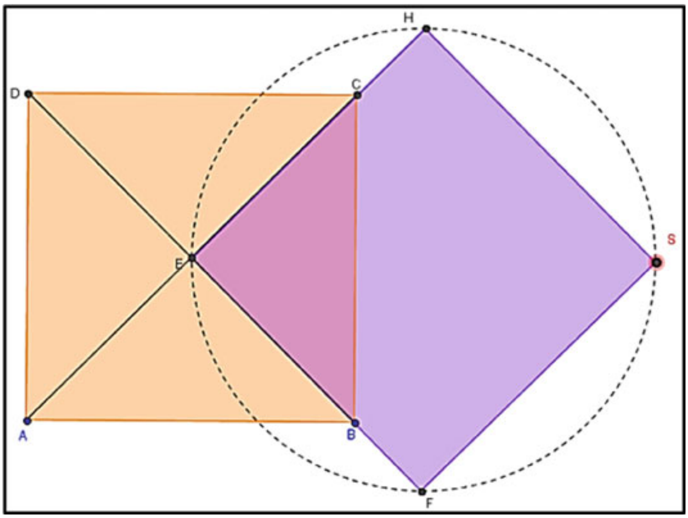

Fig. 12.10 Another configuration. *Source* Salazar and Almouloud (2015, p. 936)

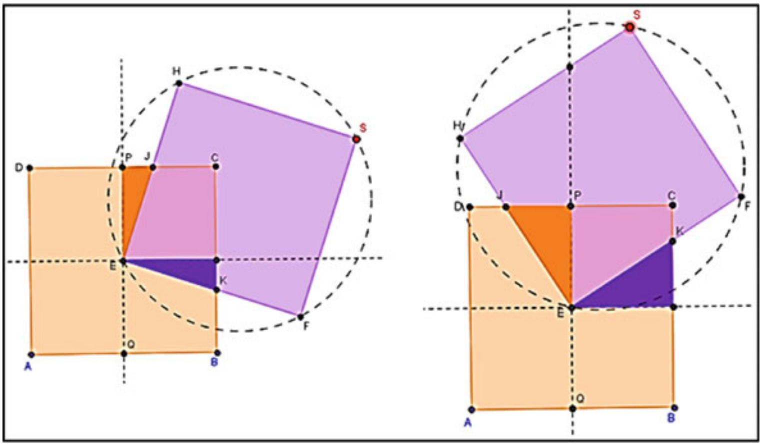

Fig. 12.11 Another configuration of any quadrilateral. *Source* Salazar and Almouloud (2015, p. 937)

the sequence of subfigures (triangles and squares); that is, the figure is deconstructed dimensionally since the following examples are identified: 1D figural units (points of intersection between the L_1 and L_2 perpendicular lines and the sides of the ABCD square), and 2D figural units such as triangles in the EKCJ and EKCP quadrilaterals.

In Fig. 12.11, it can be observed that, when the S vertex is dragged, the regions formed are easily distinguished, and the relation is identified between the surface of the ABCD square and the quadrilateral formed with the intersection of the EFSH square (activation of geometry knowledge, such as the congruence of triangles). In addition, with the dragging function, the heuristic function of the geometric figure is improved when using DRE since the student can do reconfigurations dynamically. Regarding the treatment of a dynamic representation based on the semiotic model for mathematics teaching and learning, this coincides with *sinsign*, which is the third type of Peirce's signs.

12.3.3 Dynamic Conversion

According to Duval (2011), a conversion constitutes an external transformation of a representation into another, preserving all of or part of the content of the initial representation. It demands that the student coordinate representations. In that sense, conversion requires us to perceive the difference between the content of the representation (noesis) and the signifier that represents it (semiosis), and it explains that if the connection between noesis and semiosis is not established, the conversion becomes incomprehensible or impossible to do.

After the treatments in the DRE are done, the following question is answered: *What is the relation between the surface formed by the intersection of the squares and the surface of the ABCD square?* In natural language, the surface formed by the intersection of the squares is the fourth part of the surface of the ABCD square.

The register of the figures is finally configured in DREs, which I call: ***dynamic figural register*** since the three cognitive activities that allow the configuration of a register of semiotic representation are developed.

I highlight that the DRE functions of direct manipulation and dragging facilitate the development of these cognitive activities. Moreover, they facilitate the processes of visualization, in the sense of Duval, of mathematical objects because they allow the student to treat the figure differently than in other non-dynamic environments. Thus they are powerful tools that must be used well in mathematics classes and when teaching geometry notions in particular.

12.4 Conclusions

The operations that are possible to do in the figure, such as modifying the position and reconfiguring when using DREs, are countless since the functions of direct manipulation and dragging allow the acceleration of figural treatments. This acceleration prompts the student to use geometric properties to construct a given figure, besides invalidating mistaken constructions and favoring the assimilation of the invariants related to the represented geometric objects.

The three fundamental cognitive activities required to construct the dynamic figural register in DREs are done differently than when working in non-dynamic environments because the functions of direct manipulation and dragging allow the building of relations between the figural and discursive treatments. In addition, treatments are accelerated in the dynamic figural register; and figure variations, such as changing the position of the length of the side of a figure and reconfiguring it, can be done economically.

Regarding the way to see a figure, in the sense of Duval, DREs allow the perception of the figure by juxtaposition or superposition, which allow figural operations (reconfiguration and dimensional deconstruction) to be done by students differently than when working with other tools.

Moreover, the dragging function, which allows the carrying out of treatments when using DREs, also serves as a feedback function because, since the student is 'in control' of the constructions he does, he can carry out strategies to solve a problem. It is important to highlight that, when using DREs, the dragging function allows the position and length of the sides of a figure to be changed simultaneously. This feature of these environments permits the student to explore the figure interactively and dynamically.

Regarding geometry teaching, I agree with Duval when he states that it is necessary to pose tasks that allow figure variation, other figures that help 'see' the solution and those that, on the contrary, make it difficult or impossible to see it.

Afterwards, organize tasks in which dimensional deconstruction is unpredictable, since this is contrary to perception and immediate recognition of the figural units.

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References

Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A:1022103903080.

da Silva, M. J. F., & Salazar, J. V. F. (2012). Cabri 3D na sala de aula. In *VI Congreso Iberoamericano de IBEROCABRI* (pp. 101–107). Lima, Peru: Ozlo.

Dionizio, F. A., & Bandt, C. F. (2012). O caminho percorrido pela semiótica e a importância dos Registros de Representação Semiótica para a aprendizagem da matemática. *Proceedings of the IX Seminário de pesquisa em Educação Matemática da região sul* (pp. 2–15). Caxias do Sul, Brazil.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives Université Louis Pasteur* (pp. 37–65). Strasbourg: IREM.

Duval, R. (1995). *Semiósis et pensée humaine. Registres sémiotiques et apprentissages intellectuelles*. Paris, France: Peter Lang.

Duval, R. (2004). *Los problemas fundamentales en el aprendizaje de las matemáticas y las formas superiores en el desarrollo cognitivo*. Cali, Colombia: Merlín.

Duval, R. (2005). *Les changements de regard nécessaires sur les figures*. IREM, Grenoble, *76*, 7–27. Available in: http://www-irem.ujf-grenoble.fr/revues/revue_n/fic/76/76n2.pdf.

Duval, R. (2011). *Ver e Ensinar a Matemática de outra forma: entrar no modo de pensar, os registros de representação semiótica*. São Paulo, Brazil: PROEM.

Fortuny, J. M., Iranzo, N., & Morera, L. (2010). Geometría y tecnología. In M. M. Moreno, A. Estrada, J. Carrillo, & T. A. Sierra (Eds.), *Investigación en Educación Matemática XIV* (pp. 69–85). Lérida, Spain: SEIEM.

García-Cuéllar, D. J. (2014). *Simetría axial mediada por el GeoGebra: un estudio con estudiantes de primer grado de educación secundaria*. Thesis (Mathematics Education Master). Pontificia Universidad Católica del Perú, Lima, Perú.

Gómez, C. (2015). *Proceso de visualización de cuadriláteros: un estudio con profesores de nivel secundario*. Thesis (Mathematics Education Master). Pontificia Universidad Católica del Perú, Lima, Perú.

Gómez, C., & Salazar, J. V. F. (2016). Visualización de cuadriláteros en el registro figural dinámico. In J. V. F. Salazar & F. Ugarte (Eds.), *Investigaciones en Educación Matemática* (pp. 189–204). Lima, Peru: PUCP.

Gravina, M. (2001). *Os ambientes de geometria dinâmica e o pensamento hipotético-dedutivo*. Thesis (Mathematics Education Doctorate). Pontificia Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil.

Gravina, M. A. (2008). Drawing in movement and insights for the proof process. *International Journal of Continuing Engineering Education and Life-Long Learning, 18*(5/6), 564–574.

Grinkraut, M. L. (2009). *Formação de professores envolvendo a prova matemática: um olhar sobre o desenvolvimento profissional*. Thesis (Mathematics Education Doctorate) Pontificia Universidad Católica de São Paulo, São Paulo, Brasil.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.

Laborde, C. (2005). Robust and soft constructions: Two sides of the use of dynamic geometry environments. In *Proceedings of the 10th Asian Technology Conference in Mathematics* (pp. 22–36). Seoul, South Korea.

Marmolejo, G., & Vega, M. (2005). Geometría desde una perspectiva semiótica: visualización, figuras y áreas. In *Proceedings of the XV Encuentro de Geometría y sus Aplicaciones y III Encuentro de Aritmética* (pp. 661–693). Bogotá, Colombia: Universidad Pedagógica Nacional.

Mendes, R. M. (2006). As potencialidades pedagógicas do jogo computacional. *Simcity 4*. Thesis (Education Master). Universidade de São Francisco, Itaba, Brazil.

Miskulin, R. G. S., Moura, A. R. L., & Silva, M. R. C. (2003). Um estudo sobre a dimensão semiótica da tecnologia na educação e na educação matemática. In *Proceedings of the II Seminario Internacional de pesquisa em Educação Matemática* (pp. 57–89). Santos, Brazil.

Mithalal, J. (2011). Vers la mobilisation d'une géométrie axiomatique et de la déconstruction dimensionnelle: intérêt de la géométrie dynamique tridimensionnelle. *Proceedings of the séminaire national de didactique des mathématiques* (pp. 113–126). Paris, France: IREM.

Olivero, F. (2003). *The proving process within a dynamic geometry environment*. Thesis (Education Doctorate) University of Bristol, Graduate School of Education, England.

Olivero, F., & Robutti, O. (2001). Measures in Cabri as a bridge between perception and theory. In *Proceedings of the 25th Conference of the International Group for Psychology of Mathematics Education* (pp. 9–16). Utrecht, Netherlands: Freudenthal Institute/Utrecht University.

Pavanello, R. M. (2004). Por que Ensinar/aprender Geometria?. In *VII Encontro Paulista de Educação Matemática.* Available in: http://miltonborba.org/CD/Interdisciplinaridade/Anais_VII_EPEM/mesas_redondas/.

Peñaloza, T. (2016). *Proceso de Visualización del Paraboloide en Estudiantes de Arquitectura mediado por el GeoGebra*. Thesis (Mathematics Education Master). Pontificia Universidad Católica del Perú, Lima, Perú.

Restrepo, M. A. (2008). *Genèse Instrumentale du déplacement en Géométrie Dynamique chez des élèves de 6^e*. Thesis (Mathematics and Computer Science Doctorate). Université Joseph Fourier, École Doctorale des Mathématiques, Sciences et Technologies de L'information, Informatique, France.

Salazar, J. V. F. (2009). *Génese instrumental na interação com Cabri 3D: um estudo de Transformações geométricas no espaço*. Thesis (Mathematics Education Doctorate). Pontificia Universidad Católica de São Paulo, São Paulo, Brazil.

Salazar, J. V. F. (2015). Génesis Instrumental: el caso de la función cuadrática. *Revista Iberoamericana de Educación Matemática UNION, 1*(41), 57–67.

Salazar, J. V. F., & Almouloud, S. A. (2015). Registro figural no ambiente de geometria dinâmica. *Educação Matemática e Pesquisa, 17*(5), 927–932.

Salazar, J. V. F., Gaita, C., Malaspina, U. J., & Ugarte, F. (2012). The use of technology and teacher training: An alternative for the teaching of spatial geometry. In *12th International congress on mathematical education* (pp. 3774–3781). Seoul, South Korea: ICME-12.

Silva, C. R. (2013). *Signos peircianos e registros de representação semiótica: qual semiótica para a matemática e seu ensino?*. Thesis (Mathematics Education Doctorate). Pontificia Universidad Católica de São Paulo, São Paulo, Brazil.

Veloso, E. (2000). *Geometria temas atuais materiais para professores (Desenvolvimento curricular no ensino secundário)*. Lisbon, Portugal: Instituto de Inovação Educacional.

Chapter 13
Progressive Visualization Tasks and Semiotic Chaining for Mathematics Teacher Preparation: Towards a Conceptual Framework

Barbara M. Kinach

Abstract Visualization plays an important role in mathematics learning, but in the United States where many prospective teachers (PTs) have few if any experiences learning mathematics through visualization, mathematics teacher educators are challenged to design tasks that generate within PTs' thinking an appreciation for the role visualization plays in mathematics learning. This chapter examines the affordances of progressive visualization tasks and semiotic chaining for use in mathematics teacher preparation. To the literature on dyadic and nested forms of semiotic chaining, data analysis in this chapter contributes a new type of semiotic chaining based on Peirce's three principles of diagrammatic reasoning.

Keywords Teacher education-preservice · Learning trajectories (Progressions) Technology

13.1 Why Visualization? … as a Semiotic System? … for Preservice Mathematics Teachers?

While recognition of visualization and its importance for learning is beginning to capture the attention of mathematics teachers in the United States, in part due to the conceptual demands of the national mathematics standards initiative (National Governors Association 2010), researchers in mathematics education have long recognized the role visualization plays in mathematics learning. Presmeg's chapter on visualization in the *Psychology of Mathematics Education Handbook* (2006a) traces the evolution of visualization research beginning with its roots in spatial and mathematical abilities research (Bishop 1980; Krutetskii 1976) during the period from 1976 to 2006. These studies have generated broad consensus on a definition of

B. M. Kinach (✉)
Mary Lou Fulton Teachers College, Arizona State University,
7271 E. Sonoran Arroyo Mall, Mesa, AZ 85212, USA
e-mail: barbara.kinach@asu.edu

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visualization. Since 2003 Arcavi's conceptualization of visualization as "both the product and the process of creation, interpretation, and reflection upon pictures and images" (p. 215) has been the prevailing definition in the field. More recently, the ZDM journal further advanced the field of visualization research by publishing a special issue on "visualization as an epistemological learning tool" (Rivera et al. 2013; Presmeg 2013). Examining the role of visual tools in mathematics learning, the ZDM papers range from studies of young students using the number line as a visual tool to older students comparing graphic representations of a function and its derivative. Despite researchers' attention to the role of visualization within school mathematics, relatively little work has been done on visualization in mathematics teacher preparation. With one exception, none of the visualization papers discussed above focus on teacher education.

This chapter aims to close this gap in the literature by taking up the question of visualization within mathematics teacher preparation. Specifically, the chapter examines the problem of preparing pre-service teachers (PTs) to teach mathematical concepts, processes, and relationships through progressive visualization with visual tools. To do this, for reasons that are described below, the chapter first defines *visualization* as a semiotic system. The chapter then employs Peirce's theory of signs and Krutetskii's system of problem types for developing mathematical reasoning to examine the chains of significance in a progressive visualization task from an innovative digital K–8 mathematics curriculum designed to teach mathematics concepts visually without words. To the literature on dyadic and nested forms of semiotic chaining (Presmeg 2006b), this analysis contributes a new type of semiotic chaining based on Peirce's principles of diagrammatic reasoning. Semiotic chaining of diagrams is proposed as a visualization-pedagogy for preparing PTs to foster deep understanding of mathematical notions through design of progressive visualization tasks that (1) make connections between visual and symbolic representations of the same mathematical notions and (2) promote mathematical generalization and abstraction. Semiotic perspectives on mathematics learning as a sign-interpreting game and on mathematics teaching as the art of chaining representations to foster generalization of mathematical concepts recommend the study of signs and their interpretation as a central component of mathematics teacher preparation (cf., Sáenz-Ludlow, this volume).

13.2 The Problem of Preparing Preservice Mathematics Teachers to Teach Through Visualization

From my practice as a mathematics teacher educator (MTE) in the mathematics methods course, I know that many PTs come to their study of mathematics learning and teaching with few if any experiences learning mathematics through visualization and visual tools (Kinach 2002). It is important, therefore, for MTEs to determine what types of tasks generate within PTs' thinking an appreciation for the role visualization plays in mathematics learning. A deep understanding of how students learn mathematics can generate such appreciation. For this, representational theories of mathematics learning are useful. In the methods course, PTs learn from these theories that the human mind learns abstract mathematical ideas not by reflection on mathematical symbols but by generalizing the abstract idea from multiple concrete representations of it (Dienes 1973). PTs also learn that it is advisable to sequence representations from concrete to pictorial to abstract, or as Bruner (1961) proposed—from enactive to iconic to symbolic—to facilitate students' ability to infer an abstract mathematical idea from a series of representations. Representational theories perturb conceptions of mathematics learning as memorizing rules and direct-instruction conceptions of mathematics teaching, by featuring the interplay between visual representations and inferential reasoning in the formation of mathematical notions.

Appreciation for the role of visualization in mathematics learning further develops as PTs internalize the necessity of implementing the mathematics standards in their future teaching. The Representation Standard from the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000) recommends that students use and connect representations to communicate abstract mathematical ideas and translate across representations to solve problems. The Lesh translation model (Cramer 2003) is a cognitive tool for guiding the design of learning sequences that build representational fluency across five modes of representation: manipulatives, pictures, written numerical and algebraic symbols, verbal symbols, real world contexts (Fig. 13.1). Sequencing representations logically and visually to facilitate generalization of mathematical ideas is an art that is challenging for PTs to learn and MTEs to teach.

As researchers increasingly suggest (Presmeg 2006b), thinking of mathematics semiotically as the study of signs is likely to generate new insight into how students make sense of mathematics and how teachers ought to sequence the signs of mathematics for optimal and meaningful learning. It is for this reason that in this chapter I choose to investigate visualization as a semiotic system by focusing specifically on semiotic chaining and its affordances for developing an appreciation for the role of visualization in mathematics learning and the ability to design progressive visualization tasks that foster mathematical abstraction and generalization.

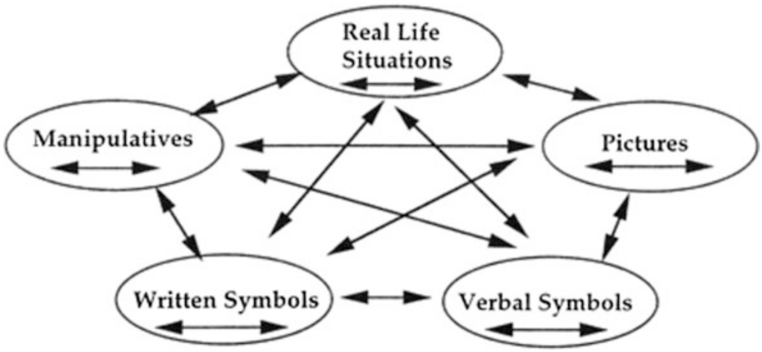


Fig. 13.1 Lesh translation model

13.3 Theoretical Framework

13.3.1 Peircean Theory of Signs

Semiotics is the study of signs and how they signify meaning. A semiotic system, according to Peirce's triadic theory of signs, consists of three parts: object, sign, and interpretant. The *object* is the thing represented; as conceptualized for this chapter the object is the abstract mathematical idea intended for the student to learn. PTs learning to use visualization and semiotic principles to foster mathematics learning might think of the object as the learning goal—the abstract mathematical notion to be signified through any of the five modes of representation in the Lesh translation model previously described. The *sign* in Peirce's triadic model is the representation of the object; the sign is the signifier, the vehicle through which the learner comes to comprehend the object. The sign is also known as the *representamen* in Peirce's theory. The *interpretant* is the idea about the object that the sign triggers in the mind of the interpreter (who in this chapter is the pre-service mathematics teacher or K–12 learner). The interpretant itself can be considered a sign for it represents the learner's understanding of the sign/object relation at a particular moment in the learning process. Once formed, the initial interpretant shapes a learner's understanding of the object providing a cognitive foundation for subsequent re-interpretation of the sign and formation of an evolved interpretant. This elaboration of Peirce's theory of signs is based on his definition of *sign* as "a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent.* This thing is called the object of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called an *interpretant* of the sign" (Peirce 1998, p. 13).

The above description of Peirce's theory of signs is, I believe, an accessible rendering for pre-service teachers. Mathematics teacher educators can employ it, along with his definition of sign, in the methods course to introduce pre-service teachers to (1) the role of visualization and representations in mathematics learning and (2) the practice of scaffolding visual representations to develop students' understanding of a mathematical concept, procedure, or relation. Peirce's definition of sign aligns with the previously discussed representational theories of mathematics learning that PTs study in the methods course. Moreover, his concept of interpretant is useful for directing teacher candidates' attention to the sense that students make of the representations, problems, and other curriculum materials teachers use to develop understanding of mathematical notions. Students' interpretations of learning materials are not always those intended by the teacher (Sáenz-Ludlow, this volume), implying a need for scaffolding learning experiences to allow for re-interpretation until the desired learning objective approximates the intended learning goal. This idea of learning as an interpreting game requiring informative feedback and learner reinterpreting toward the intended learning goal (Sáenz-Ludlow) is, in my experience as a MTE, an effective counter to beliefs about mastery learning gleaned from direct-instruction teaching practices.

Peirce's theory of signs is, in fact, much more complex than the elaboration above suggested for PTs. Indeed it is necessary for the semiotic analysis of the progressive visualization task examined in this chapter to delve more deeply into Peirce's triadic theory and to define how I will use his terms. One complexity is his use of the term sign. In the above definition, he conceptualizes sign as a signifier, specifically the thing signifying the object. In other instances (e.g., Peirce 1992), he uses the term sign to refer to the entire triad (object, sign, interpretant) in which case sign is being used in what may be thought of as both a macro-sense (SIGN) and micro-sense (sign). To avoid confusion, Peirce scholars tend to substitute the term sign vehicle for sign when discussing sign in the sense of signifier or representamen of the object. In this chapter I use the lower-case spelling of the term sign to refer to the sign in its micro sense as signifier of the object and use the upper-case spelling of the term SIGN to indicate the entire triad of object-sign(representamen)-interpretant. I employ the terms sign, representamen, sign vehicle, and representation interchangeably in this chapter for pedagogical reasons, in order to investigate whether my semiotic analysis of the visualization task is compatible with the Lesh translation model used in the methods course to motivate teaching mathematics through representations.

13.3.1.1 Sign as Icon, Index, or Symbol

Analysis of visualization tasks in this chapter also requires note of Peirce's three-part categorization of signs as icon, index, or symbol. Icons refer to their objects by similarity in features, or as Peirce described, icons are "*likenesses* … which serve to convey ideas of the things they represent simply by imitating them" (Peirce 1998, p. 5). Indices, which Peirce also called *indications* (p. 5), show something about the object or thing they represent in the sense that, for example, smoke indicates fire or a highway milepost indicates miles traveled from the origin. Symbols, having no likeness or natural connection to the object they represent, refer to the thing by virtue of consensus and use within a community. Words, algebraic characters, and mathematical symbols such as + or = are examples of symbols. The meanings of symbols exist by virtue of their common usage within a community of discourse.

13.3.1.2 Diagrams and Diagrammatic Reasoning

Icons are further categorized by Peirce into images, diagrams, and metaphors. A diagram as a category of icon is especially relevant for the analysis of visualization tasks in this chapter as these tasks consist almost entirely of iconic diagrams. According to Peirce, the diagram is a particularly useful kind of icon because it "suppresses a quantity of details, and so allows the mind more easily to think of the important features." For Peirce, this was the "great distinguishing property of this [type of] icon … that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction" (Peirce as quoted in Stjernfelt 2007, p. 358).

For Peirce all mathematical reasoning is diagrammatic. His conceptualization of diagrammatic reasoning will be instructive for the task analysis conducted later in the chapter:

> That is, we construct an icon of our hypothetical state of things and proceed to observe it. This observation leads us to suspect that something is true, which we may or may not be able to formulate with precision, and we proceed to inquire whether it is true or not. For this purpose it is necessary to form a plan of investigation and this is the most difficult part of the whole operation. We not only have to select the features of the diagram which it will be pertinent to pay attention to, but it is also of great importance to return again and again to certain features. Otherwise, although our conclusions may be correct they will not be the particular conclusions at which we are aiming. But the greatest point of art consists in the introduction of suitable *abstractions*. By this I mean such a transformation of our diagrams that characters of one diagram may appear in another as things. A familiar example is where in analysis we treat operations as themselves the subject of operations. (Peirce 1998, pp. 212–213)

Given this conceptualization of diagrammatic reasoning, Peirce (pp. 212–213) identifies three steps for designing such reasoning. I will argue that these steps are the design principles underlying the visualization task analyzed in this chapter:

1. "copulating separate propositions into one compound proposition",
2. "omitting something from a proposition without possibility of introducing error",
3. "inserting something into a proposition without introducing error".

13.3.1.3 The Triadic Nature of Interpretant and Object

Just as Peirce categorizes the icon into three types, so he conceives both the interpretant and object as triadic notions. The triadic nature of each component within Peirce's theory of signs is grounded in his conceptualization of the nature of reality: firstness, secondness, and thirdness. Firstness refers to the initial unreflected state of things. It is, for example, the first impression learners have when reading a mathematics problem—the immediate interpretant. Secondness refers to reflections on the object subsequent to the initial interpretant/impression. Secondness is an intermediate state between firstness and thirdness. During the problem solving process, secondness is the dynamic state of a learner's understanding as it continuously refines as learners delve deeper into the relationships within a problem setting. Secondness gives rise to what Peirce called the dynamic interpretant. Thirdness is the final interpretant, the notion that the sender intended to communicate. In learning situations, thirdness is the desired learning goal; it is the taken-as-shared understanding of the object. The above gives rise to three states of being and experience (ontology and phenomenology) of the interpretant and the object: immediate, dynamic, and real. Figure 13.2 (Sáenz-Ludlow and Kadunz 2016) summarizes the relations among object, sign, and interpretant emphasizing how the sign mediates between the object and the interpretant.

13.3.2 Semiotic Chaining

Semiotics is the study of signs and how they signify meaning. Semiotic chaining, as used in this chapter, is a process of sign-making involving the artful sequencing of signs to evolve understanding of an intended learning goal by the designer of the semiotic chain. Understanding of the intended object develops continually through a range of experiments on linked representations that manifest the intended qualities of the object. In mathematics lessons, it is recommended by Bruner and other representational learning theorists that the designer of the semiotic chain sequence representations from enactive to iconic to symbolic. Chaining therefore is based on the assumption that meaning and understanding develop over time, and are not limited to any one point in time.

Presmeg (2006b) cites two examples of the use of semiotic chains in mathematics teacher education research. Both are rooted in real world contexts. Adeyemi's (2004) use of chains with pre-service elementary teachers centers on linking K–12 students' activities with the mathematics curriculum while Hall's (2000) use of chains with in-service teachers involves use of an everyday activity to

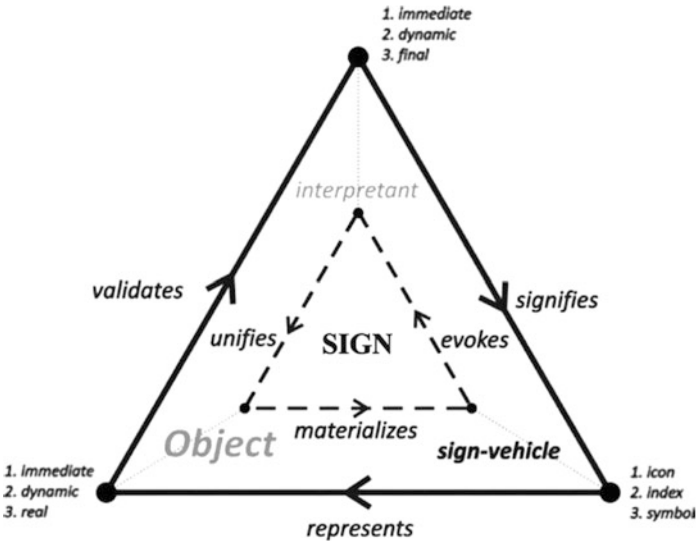


Fig. 13.2 Saenz-Ludlow and Kadunz (2016): Peircean triadic sign model

motivate a mathematics concept. Specifically, Hall's teachers evolved the concept of a batting average through the following semiotic chain:

> Baseball game → hits versus at bats → success fraction → batting average (Presmeg 2006b, p. 168). Both semiotic chains are characterized as Peircean nested semiotic chains, in contrast to Saussurean dyadic chains. The sequence of signifiers (representamen) is such that each sign (representamen) 'slides under' the subsequent signifier, in the sense that each triad is contained in the next, after the manner of Russian nested dolls (Presmeg 2006b, p. 165). In this formulation, the new signifier in the chain stands for everything that precedes it in the chain. The previous signifier, as well as everything that it represents, is now the new signified. (p. 169)

In contrast, the semiotic chains examined in this chapter employ interactive diagrams and Peirce's three principles of diagrammatic reasoning to foster generalization of a mathematics concept.

13.3.3 Krutetskii's Problem Types and Mathematical Abilities

Krutetskii's categorization of problem types and mathematical abilities also informs my analysis of visualization tasks for this chapter. Of the 26 problem series Krutetskii created to investigate, assess, and ultimately develop students' mathematical abilities, three are relevant for my purposes in this chapter: Series V, Series VII, and Series XVII (Table 13.2). Series V—*systems of problems of a single type* (p. 100)—was intended to develop a type of generalization that Krutetskii called "subsumption under a concept." This series introduces a new concept and provides students the opportunity to generalize the new concept from particular instances of it. The extent of students' ability to generalize is judged by three abilities: (1) how well a student can "see a general type in different problems," (2) how the student passes from "solving simpler problems to solving more complex ones of the same type," and (3) how the student can "differentiate problems of one type from externally similar problems of another type" (Krutetskii 1976, p. 115). Within Series V, it is possible to abstract, or generalize, specific features despite external differences among the problems because certain "associations" (to use Krutetskii's term) or commonalities exist across the system of problems. For Krutetskii, these associations are of two types. First, there are characteristics of the given geometric figure and its elements; and second, there is the recognition that solving the problem requires an *operation* or action (Krutetskii 1976, p. 52).

Series VII—*systems of problems with graded transformations from concrete to abstract*—was intended to develop perception, specifically the forming of a generalization. As described by Krutetskii,

> this series of problems shows how easily and quickly a pupil can translate the solution of a problem into a general scheme, how capable he is of transferring from operations with concrete quantities to operations with conventional symbols that enable him to express relations between quantities in a general, abstract form. (p. 125)

The structure and rationale for this problem series is instructive for analyzing how the visualization task analyzed in this chapter transitions from concrete to abstract variants through intermediate forms. Specifically, Krutetskii describes this series to be

> A system of ten problems, each of which is gradually transformed from a concrete to an abstract, general scheme. Each problem has from three to five variants. The first problem (*a*) is a problem on a completely clear concrete plan; the last (*e*) is the same problem translated onto an abstract, general level. Variants *b*, *c*, and *d* are intermediate representing gradual translations from *a* to *e*, with successive generalizations of increasingly greater number of elements of the problem. (p. 123)

Series XVII—direct and reverse problems—was intended to investigate the ability to "restructure the *direction* of a mental process, to change from a direct to a reverse train of thought" (Krutetskii 1976, p. 143). Acknowledging that not all relations are symmetric, Krutetskii separated the question into psychological and mathematical components. Primary was the question of reversibility. Was the pupil able, for example, to reverse the logical direction of a problem that asks for the area measurement of a rectangle given its dimensions to one that asks for the possible dimensions of a rectangle given its area measurement. Even with non-symmetric relations such as *cats are a subset of mammals*, the intention was to test ability to reverse the train of thought to mammals are a subset of cats. Mathematical accuracy, for Series XVII reversibility problems, was a secondary concern.

Table 13.1 summarizes the three Krutetskii problem series pertinent to the analysis of progressive visualization tasks in this chapter. Each series was designed to investigate certain mathematical abilities, which for the featured series are: (1) isolate form from content, (2) abstract from concrete (and I would add *digital*) spatial forms, (3) reverse a mental process, and (4) generalize mathematical material to see what is common in what is externally different (Krutetskii 1976). All but reversibility of thought (#3) relate to the ability to generalize mathematical objects, relations, and operations.

Table 13.1 Krutetskii's problem types and mathematical abilities operating within area of rectangle visualization task

Krutetskii problem type series		Krutetskii mathematical ability	Visualization task relevance
V	Systems of problems of a single type	Generalization (subsumption under a concept)	Applies within each game level 1 through 5
VII	Systems of problems of with graded transformations from concrete to abstract	Perception (forming generalizations)	Applies across levels when levels are considered a system of problems
XVII	Direct and reverse problems	Reversibility of mental process	Game levels 4–5 reverse levels 1–2–3

13.3.4 Visualization Task Context: The Mathematics Methods Course

In this section, I consider a digital visualization task that I have used with pre-service elementary teachers in the mathematics methods course. The task is based on the 5th grade *Area of Rectangle* game in the Spatial Temporal Mathematics (ST Math) curriculum. ST Math is a neuroscience-based supplementary mathematics curriculum for grades K through 8 that is available to schools through the MIND Research Institute (MRI 2014), a former research affiliate of the University of California at Irvine. This supplementary mathematics program, which develops mathematics concepts visually without words through interactive, animated digital game-like mathematics puzzles, mobilizes visual examples, animations, and informative feedback to facilitate evolution of students' understanding of a mathematics concept.

I think of these games as *progressive concept visualizations*. Since 2013, through a partnership with MRI, I have been conducting design research in the methods courses that I teach for elementary PTs to determine a role for this innovative technology in the preparation of mathematics teachers. Among other things, I have employed select games to assess PTs' mathematics content knowledge, develop their ability to write learning objectives, and model Bruner's representational theory of mathematics learning and instructional sequencing (concrete-pictorial-abstract).

Viewed from a semiotic perspective, the games are concerned with the construction of concept(s) through signs (Kadunz 2016). The signs employed within the games are collections of Peirce icons, symbols, and indices. Each game includes interactive iconic diagrams. Experimentation with the diagrams produces informative visual feedback providing further opportunity to reason with the diagram—to notice patterns, form inductive hypotheses, and conduct deductive experiments on the icon diagram—for the purpose of discovering the game's hidden mathematical learning goal (object). Reasoning elicited by the visual puzzles spans the reasoning types Peirce examines, from deduction to induction, hypothesis, and abduction.

Structurally each game design consists of multiple levels. Each level deploys a Series V Krutetskii *system of problems of a similar type* whose function it is to foster generalization of a concept, or what I identify as the Intended Interpretant for that level. Collectively the SIGNS across game levels form a Series VII Krutetskii *system of problems with graded transformations from concrete to abstract*, whose purpose is to facilitate generalization of the intended object (abstract mathematical learning goal) from the SIGN sequence. Levels 4 and 5 exemplify Krutetskii Series XVII *direct and reverse problems* for assessing ability to change the direction of a mental process. While it is impossible to depict the interactivity of these concept visualization games in print format, Fig. 13.3 provides a static visual overview of one game and its five levels.

Fig. 13.3 A bird's eye view of the visual flow of the 5-level ST Math *Area Game*

13.3.5 Analysis of Visualization Task

Figure 13.3 depicts the 5-level Area of Rectangle game, a progressive visualization task I use in the methods course to illustrate learning and teaching mathematics through visualization and representations. Below I examine the game within and across levels for chains of significance in search of design principles for a conceptual framework to guide prospective teachers' creation of progressive visualization tasks (semiotic chains) for the mathematics classroom.

The semiotic object of this ST Math game is *area.* Area is essentially a geometric concept; it is the two-dimensional (flat) space within a bounded region. A related concept is *area measurement,* or the number of square units covering two-dimensional space. The game relates the spatial concept area to its measurement; this relationship, which is illustrated particularly for a rectangle, applies as well to other geometric figures that may be regular or irregular.

Level 1. In level 1 of the game, the semiotic object is *area of a rectangle.* The sign (representamen), which signifies both the spatial quality of a rectangle's area and its measurement, consists of three parts: (a) rectangle icon, subdivided into unit squares, which initially appears static but breaks apart during the game to confirm the game player's area-measurement hypotheses, (b) two linked symbols (the word area followed by the mathematical equal sign), and (c) 10-by-2 highlightable interactive rectangular icon with numerical symbol label (index) that automatically pops up to indicate the number of highlighted squares (Fig. 13.4). To solve the interactive visual puzzle, which is without verbal directions, learners must click on various parts of the screen to see what happens. For each click the game generates visual feedback. Through trial and error guided by this visual feedback, students infer the level 1 puzzle's underlying mathematical concept, the *intended*

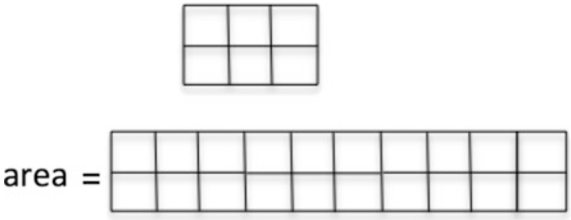


Fig. 13.4 Level 1 problem type

interpretant (II_1), namely that the interior space of the rectangle is called area and that its size can be determined by counting the unit squares in the interior of the rectangle.

Students infer the II_1 through noticing what happens to the highlighted unit squares of the dynamic 10-by-2 rectangle. Once highlighted, a numeral automatically pops up to indicate the quantity of highlighted squares after which the highlighted squares break up. The game then provides visual feedback to show whether or not the highlighted squares can be arranged into a shape that is congruent to the interior of the given rectangle. If this occurs, JiJi the penguin avatar walks across the screen to indicate mathematical success. If not, JiJi's path will be blocked and an opportunity to retry the visual puzzle provided.

Another important semiotic feature of level 1, and all levels of the Area of Rectangle game, is the Krutetskii Series V system of similar problems that comprise the level. Multiple examples and opportunities for experimentation on the representamen (diagrams) are required to generalize the intended game-level concept (Krutetskii 1976) or interpretant for, as Peirce maintained, with any communication, inferring the intended interpretant from one example is unlikely as there is always a certain range of interpretation (Peirce as quoted by Joswick 1996, p. 100). To arrive at the intended consensus interpretation, the interpreter (or in this case the learner) requires multiple signs exemplifying the concept (object) with opportunity to experiment on them in order to *experience* the concept, reason with it (Joswick 1996, pp. 100–101), and ultimately infer the qualities of the intended object from the sign (representamen). For example, a common initial interpretation of the sign in level 1 is that the aim of the visual puzzle is to highlight a shape congruent to the given rectangle. Through experimentation learners notice however that the highlightable squares activate linearly in order from 1 to 10 making it impossible to highlight a rectangle comprised of three squares in row 1 and three squares directly above in row 2. Thus, the learners' initial conjecture must be re-thought and a new hypothesis formed and tested. Further, as Peirce explains, even if a particular puzzle were to be solved correctly on the first attempt, it is unlikely that the desired level of generality for the intended interpretant will be noticed on the first inference (Peirce 1998, p. 212). The series of similar problems at game level 1 (as well as for the similar problem series at each of the other game levels) therefore provides, according to Peirce, opportunity for thought to develop about the intended object through experimentation on the level 1 sign. This experimentation produces a sequence of inductions and hypotheses that ultimately yield dynamic interpretants that tend toward the intended interpretant, that is, the desired ultimate consensual understanding of the initial intended object.

In sum, the intended interpretant (II_1) for game level 1 is the spatial concept of area of a rectangle, its measurement, and the following inductive argument: If highlighted squares can be arranged into a shape that is congruent to the interior space of the rectangle, then the rectangle interior is called the area of the rectangle and the rectangle's area measurement can be found by counting the unit squares of the given subdivided rectangle. This interpretant is a generalization derived from the series of eight similarly structured problems for which Fig. 13.4 serves as exemplar. The dimensions of other rectangles in the problem series vary from 3-by-2 to 4-by-3 to 3-by-4 to 5-by-2 to 2-by-4 to 1-by-6 to 2-by-3 to 2-by-5.

Level 2. The semiotic object for game level 2 is the same as for level 1, *area of a rectangle*. The sign (representamen) for level 2 is identical to that for level 1 with the exception that the numerical dimensions of the given rectangular icon are labeled (Fig. 13.5). Diagrammatically, therefore, the sign for level 1 is a subset of the sign for level 2. Generalizing from the Krutetskii Series V system of eight similar problems in level 2, and benefiting from the interpretive lens that the Intended Interpretant from level 1 (II_1) provides, students infer the level 2 puzzle's underlying mathematical concept, the *intended interpretant (II_2): the area, or interior space of this rectangle, can be measured in one of two ways, either by counting unit squares of the subdivided rectangle or by multiplying the numerical linear dimensions of the rectangle*. In fact, it is possible that level 2 game-interpreters may solve the visual puzzle but not notice that the product of the rectangle's dimensions equals the area measurement of the given rectangle. This inference, however, will be required for game level 3. Notice here how each level of the game changes the previous sign to open up a new perspective on the object (Kadunz 2016, p. 35).

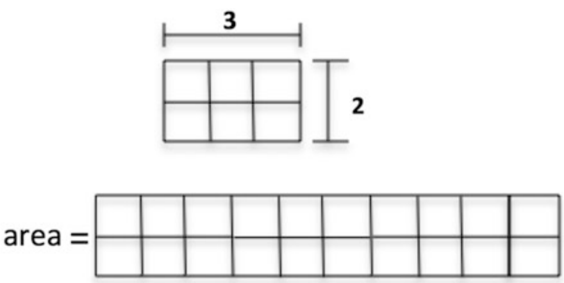


Fig. 13.5 Level 2 problem type

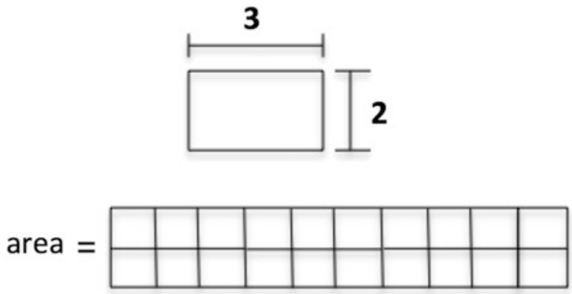


Fig. 13.6 Level 3 problem type

Level 3. The semiotic object for game level 3 is, again, *area of a rectangle.* As for prior game levels, the sign (representamen) for level 3 represents two qualities of the intended object—its spatial character and its numerical measurement. The sign for level 3 is nearly identical to that for level 2 with the exception that the given rectangle is not subdivided into unit squares (Fig. 13.6). The interpreter must either hypothesize the number of unit squares to highlight (and have this confirmed or rejected by the interactive puzzle) or recall the relation discovered in level 2 (i.e., area measurement of a rectangle can be determined by either counting the unit squares within the rectangle interior or multiplying the rectangle dimensions). With the benefit of the intended interpretant from level 2 (II_2), a reasonable conjecture for the area problem in Fig. 13.6 is to highlight 6 unit squares. The game will provide visual feedback identical to that previously described: highlighted squares will break apart and either cover (or be shown to not cover) the interior space (area) of the rectangle. From this experimentation on the level 3 sign/representamen, students infer the level 3 puzzle's underlying mathematical concept, the *intended interpretant (II_3): the area, or interior space of this rectangle, can be measured by multiplying the numerical linear dimensions of the rectangle.*

Level 4. The semiotic object for game level 4 is, as for previous levels, *area of a rectangle.* While the component parts of the sign (representamen) for level 4 are identical to that for levels 1 and 2, their order and relation are reversed (Fig. 13.7). Whereas levels 1, 2, and 3 posit a rectangle of given magnitude and require inference of the rectangle's area measurement, level 4 posits the rectangle area measurement pictorially as a collection of unit squares and requires creation of a rectangle whose magnitude equals the given area measurement. Through experimentation on the representamen students infer the level 4 puzzle's underlying mathematical concept, the *intended interpretant (II_4): area of any rectangle equals the product of its dimensions*. For the given area of 6 square units in Fig. 13.7 a variety of rectangles (1-by-6, 2-by-3, 3-by-2, or 6-by-1) could be drawn on the interactive digital grid to solve the visual puzzle.

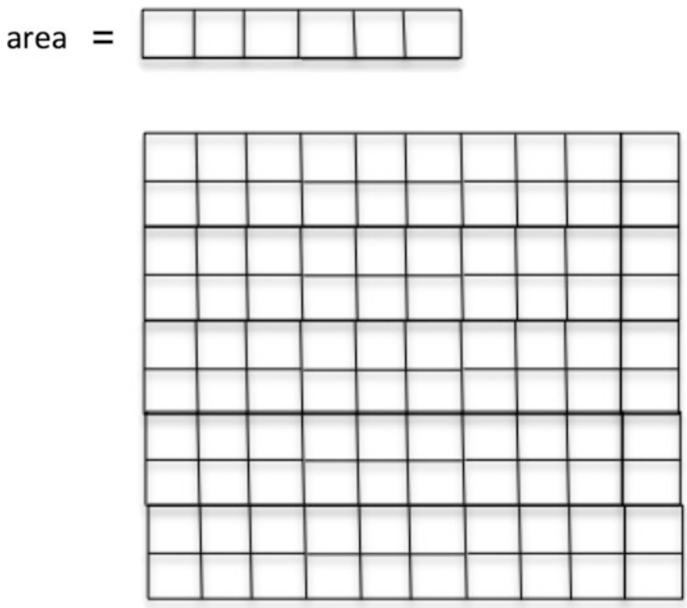


Fig. 13.7 Level 4 problem type

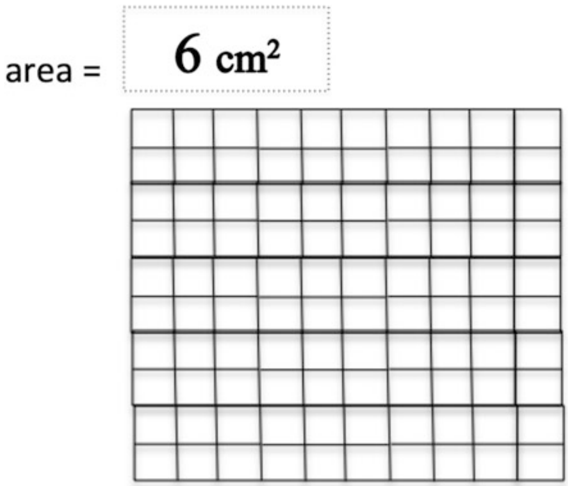


Fig. 13.8 Level 5 problem type

Level 5. Level 5 maintains the structure and reversed logic of level 4 but presents the area measurement of the rectangle numerically (e.g., 6 cm^2) instead of as a picture of unit squares (compare Figs. 13.7 and 13.8). Given the area measurement of a rectangle numerically, players must manipulate the representamen diagram to create a partitioned rectangle whose area measurement equals the given measure. Essentially, therefore, the *intended interpretant of level 5 (II_5) is the symbolic formula for area of a rectangle, Area = length times width, or* $A = L \times W$.

13.3.6 Semiotic Chaining Analysis

To determine whether the *area of rectangle* game is a semiotic chain, I analyzed the game from a variety of perspectives. First, I examined the representamens across game levels for a Peircean nested semiotic chain in the sense reported by Presmeg (2006b) for Adeyemi and Hall. Analysis of the game's representamen failed to produce such a chain. Second, I examined the argumentation that produced the Intended Interpretants for each game level. Finally, I analyzed the game according to Peirce's design principles for diagrammatic reasoning sequences.

13.3.6.1 Chaining by Argumentation Through Deductive Inference

Given that the SIGN at each level of the Area of Rectangle game includes representamens comprised of iconic diagrams that for Peirce invited deductive reasoning, it is a logical move to explore deductive inference as the source of a syllogism that might yield a semiotic chain within this game. Recalling Peirce's claims that all mathematical reasoning is diagrammatic and that reasoning with diagrams has the power to reveal things hidden from view about the logical structure of the diagram, about the relationship of its parts to one another, and about the analogical relation of these parts to the intended object, I examined the argumentation within the game. Table 13.2 illustrates the results of my analysis also elaborated below.

In the following argument, I define A, B, and C to hold the following meanings:

A = area = rectangle interior space

B = measure(A) = count A's unit squares

C = measure(A) = multiply dimensions of A

Level 1 diagram implies:

Through diagrammatic reasoning on the level 1 diagram, students discover:

measure(A) = B

Level 2 diagram implies:

Through diagrammatic reasoning, students discover:

measure(A) = B,

measure(A) = C.

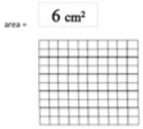
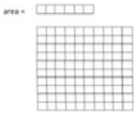
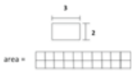
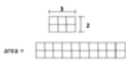
Through deductive reasoning, students discover that two things equal to the same thing are equal to each other, or B = C.

Level 3 diagram implies:

Through diagrammatic reasoning OR through application of relationships learned at level 2, students discover:

measure(A) = C.

Table 13.2 Application of Peirce three-step plan for design of diagrammatic chains of significant

Representamen	Argumentation	Intended interpretant	Level
 R5	m(A) = C, then create A	Given the area of any rectangle interior as a numerical quantity of unit squares, a rectangle of appropriate size can be created on a dynamic grid either by recalling from levels 3 or 4 that the product of the rectangle dimensions equals the area measurement or by applying the area of rectangle formula, Area = length × width	5
 R4	m(A) = B, then create A	Given the area of any rectangle interior as a picture of unit squares, a rectangle of appropriate size can be created on a dynamic grid either by trial and error or by recalling from level 3 that the product of the rectangle dimensions equals the area measurement	4
 R3	m(A) = C From level 2, we know m (A) = B and m(A) = C Since both members of the conjunction (or copulation per Peirce, 1998, p. 213) are true, one of the members of the conjunction can be dropped without changing the truth value of the conjunction from True to False	Interior rectangle space is called area; its size is measured by multiplying dimensions of rectangle	3
 R2	m(A) = B and m(A) = C	Interior rectangle space is called area; its size is measured by counting unit squares in the interior of the rectangle; its size is also measured by multiplying the dimensions of the rectangle	2
 R1	m(A) = B	Interior rectangle space is called area; its size is measured by counting rectangle unit squares	1
Object: area of rectangle	A = area = rectangle interior space B = measure(A) = count A's unit squares C = measure(A) = multiply dimensions of A		

Through inductive reasoning conducted on the level 3 system of similar problems, students discover the formula:

Area measurement of rectangle = length x width.

Level 4 diagram implies:

Through diagrammatic reasoning OR through application of relationships learned at levels 1 and/or 2, students discover:

if B = measure(A), then a rectangle interior (A) of appropriate size can be created to equal B square units.

Level 5 diagram implies:

Through diagrammatic reasoning OR through application of relationships learned at levels 2 and/or 3, students discover:

if C = measure(A), then a rectangle interior (A) of appropriate size can be created to equal C square units and the dimensions of this rectangle will be factors of C.

Argued more succinctly, it follows that if the area measurement of the rectangle subdivided into unit squares equals the number of unit squares covering the rectangle interior (II_1), and the number of unit squares covering the rectangle interior equals the product of the rectangle dimensions (II_2), then by deductive argument, the product of a rectangle's dimensions equals the area measurement of a rectangle subdivided or not (II_3). From an inferential reasoning point of view, the concepts and relations inferred across levels 1, 2, and 3 form a semiotic chain based on deductive inference in the sense that the argumentation for level 1 implies the argumentation for level 2 which in turn implies the argumentation for level 3 because it was already established in level 2. The model of chained inferential reasoning described above does not extend to levels 4 and 5. I argue, however, that a semiotic chain based on deductive inference exists across game levels 1–3.

13.3.6.2 Chaining by Design: Peirce's Three-Step Plan for Chaining Diagrams

At this point, it would be instructive for the reader to return to Peirce's description of diagrammatic reasoning quoted earlier in the chapter. In this passage, he alludes to the design of chains of signification with diagrams (Peirce 1998):

> We not only have to select the features of the diagram which it will be pertinent to pay attention to, but it is also of great importance to return again and again to certain features. Otherwise, although our conclusions may be correct they will not be the particular conclusions at which we are aiming. But the greatest point of art consists in the introduction of suitable *abstractions*. By this I mean such a transformation of our diagrams that characters of one diagram may appear in another as things. (p. 213)

To create such a sequence of diagrams, Peirce proposes a plan based on three basic principles of inference: principle 1, "copulating several propositions into one compound proposition;" principle 2, "omitting something from a proposition without introducing error;" and principle 3, "inserting something into a proposition without introducing error" (Peirce 1998, p. 213). I will argue that these principles undergird the Area of Rectangle game.

Table 13.2 illustrates application of Peirce's three principles to the design of semiotic chains based on diagrammatic reasoning in the representamen and argumentation columns. Beginning with the level 1 diagram (representamen), apply Peirce's third principle to yield the diagram (representamen) for level 2. This results in the addition of numerical dimensions to the level 1 diagram without introducing error. At level 2, experimentation with the diagram produces a diagrammatically reasoned and previously confirmed (true) proposition m(A) = B along with a newly inferred true proposition m(A) = C. Following Peirce's first principle, form these two propositions into a (true) compound proposition, m(A) = B and m(A) = C, using the laws of logic for truth tables. Next apply Peirce's second principle: drop one of the true propositions (m(A) = B) from the compound statement to infer the true proposition at level 3, namely m(A) = C: since both members of the conjunction ("copulation" in Peirce's terminology) are true, one of the members of the conjunction can be dropped without changing the truth value of the conjunction from True to False.

13.3.7 Towards a Conceptual Framework for Semiotic Chaining

Peirce's three design principles constitute the basis for a conceptual framework of semiotic chaining grounded in diagrammatic reasoning. The semiotic chain revealed by the above analysis is an application of Peirce's design principles for diagrammatic reasoning. I suggest that the innovative progressive concept visualization task analyzed above exemplifies a new type of semiotic chain that I characterize as a Peircean diagrammatic semiotic chain. As illustrated by the progressive concept visualization task analyzed in this chapter, the links within diagrammatic semiotic chains are Peircean iconic diagrams sequenced with the planned inferential logic of the Peircean design principles for true statements: A, A and B, B. Also noteworthy in the above analysis is the role that translation of representations may play in the design of diagrammatic semiotic chains. In this instance, the insertion without adding error of a numerical representation into a pictorial one makes application of Peirce's third principle possible. I suggest the three Peircean principles of diagrammatic reasoning provide mathematics teacher preparation with an accessible, conceptually clear visualization-pedagogy for use with pre-service teachers to guide their planning of progressive visualization tasks that (1) link visual and symbolic representations of the same mathematical notion and (2) foster mathematical generalization and abstraction. The diagrammatic reasoning structure proposed by Peirce, together with the principle of translating representations, provide the basis for a specialized conceptual framework of semiotic chaining grounded in Peircean iconic diagrams. Future reflection and comparative research on the affordances of the three types of semiotic chains—

dyadic, nested, diagrammatic—will help to close the gap in our understanding of visualization practices for mathematics teacher preparation.

References

Adeyemi, C. M. (2004). *Semiotic chaining: Preservice teacher belief and instructional practices* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses. (Accession Order No. AAT 3172873).

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241.

Bishop, A. (1980). Spatial abilities and mathematics education—A review. *Educational Studies in Mathematics, 11*, 257–269.

Bruner, J. S. (1961). *The process of education*. Cambridge, MA: Harvard University Press.

Cramer, K. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 35–58). Mahwah, NJ: Lawrence Erlbaum.

Dienes, Z. P. (1973). A theory of mathematics learning. In F. J. Crosswhite, J. L. Highins, A. R. Osborne, & R. J. Shunway (Eds.), *Teaching mathematics: Psychological foundations* (pp. 137–148). Ohio, OH: Charles A. Jones Publishing.

Hall, M. A. (2000). *Bridging the gap between everyday and classroom mathematics: An investigation of two teachers' intentional use of semiotic chains*. Retreived from ProQuest Dissertations and Theses. (Accession Order No. AAT 9971753).

Joswick, H. (1996). The object of semeiotic. In V. M. Colapietro & T. M. Olshewsky (Eds.), *Peirce's doctrine of signs: Theory, applications, and connections* (pp. 93–102). Berlin, Germany: Walter de Gruyter & Co.

Kadunz, G. (2016). Geometry, a means of argumentation. In A. Saenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics: How to describe the construction, visualization, and communication of mathematical concepts*. Rotterdam, The Netherlands: Sense Publishers.

Kinach, B. M. (2002). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education, 18,* 51–71.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago, IL: University of Chicago.

MIND Research Institute. (2014). *Spatial-temporal (ST) math*. Irvine, CA: The MIND Research Institute.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards (Mathematics)*. Washington, DC: Author.

Peirce, C. S. (1992). *The essential Peirce* (Vol. 1, 1867–1893). In N. Houser & C. Kloesel (Eds.). Bloomington, IN: Indiana University Press.

Peirce, C. S. (1998). *The essential Peirce* (Vol. 2, 1893–1903). In Peirce Edition Project (Eds.). Bloomington, IN: Indiana University Press.

Presmeg, N. (2006a). Research on visualization in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205–235). Rotterdam, The Netherlands: Sense Publishers.

Presmeg, N. (2006b). Semiotics and the “connections” standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61*, 163–182.

Presmeg, N. (2013). Contemplating visualization as an epistemological learning tool in mathematics. *ZDM—The International Journal on Mathematics Education*, *46*, 151–157.

Rivera, F. D., Steinbring, H., & Arcavi, A. (2013). Visualization as an epistemological learning tool. *ZDM—The International Journal on Mathematics Education*, *46*, 1–2.

Sáenz-Ludlow, A., & Kadunz, G. (2016). Constructing knowledge seen as a semiotic activity. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics: How to describe the construction, visualization, and communication of mathematical concepts* (pp. 1–21). Rotterdam, The Netherlands: Sense Publishers.

Stjernfelt, F. (2007). *Diagrammatology: An investigation on the borderlines of phenomenology, ontology, and semiotics*. Dordrecht, The Netherlands: Springer.

Chapter 14
Primary Teachers' Semiotic Praxis: Windows into the Handling of Division Tasks

Corin Mathews, Hamsa Venkat and Mike Askew

Abstract The teaching of division is a complex task: division is difficult to teach in a connected and coherent way, given the diversity of models of division, and differences in their associated actions and utterances. This chapter focuses on the signs produced by teachers in their attempts to explain division. The signs produced follow from unifications between signifiers and signifieds located within what we are calling signification pathways. Our focus is on signification pathways that are co-produced by teachers and learners, and endorsed by the teacher. The data presented in this chapter exemplify categories within an emerging analytical framework of signification pathways when teaching division, which vary in the coherence of the signs that are involved. In this chapter we consider the possibilities and constraints of these signification pathways in relation to the semiotic system related to division from a mathematical interpretant perspective. The analysis makes visible limitations, ambiguity and incoherence across the signification pathways. The conclusion examines how the production of signs and the connection of signs in the signification pathway may lead to certain meanings about division that the teacher endorses as valuable in the teaching of division.

Keywords Semiotics · Signification pathways · Division · Coherence
Foundation phase

C. Mathews (✉) · H. Venkat · M. Askew
University of the Witwatersrand (WITS), Private Bag 3, 2050 Johannesburg
South Africa
e-mail: corin.mathews@wits.ac.za

H. Venkat
e-mail: hamsa.venkatakrishnan@wits.ac.za

M. Askew
e-mail: michael.askew@wits.ac.za

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14.1 Introduction

Hoffman (cited in Radford et al. 2008) notes the usefulness of semiotic theory for focusing on "semiotics praxis", that is how actors assemble and sequence signs in teaching and learning. The focus of this chapter is to theorize South African teachers' modes of assembling and sequencing signifiers related to division in mathematics. Division, as international literature suggests, is replete with mathematical and pedagogical complexities. In mathematical terms, at first glance, the division relation is based on a simple triad of quantities: the dividend, the divisor and the quotient. But beneath this seeming simplicity lies complexity. The literature on early number observes that there are two basic phenomenological situations that lead to the need for division: partitioning, or sharing, situations and quotitioning, or measuring, situations. In pedagogical terms, complexity relates to a number of features. Among these are similarities in the phonology of words that may relate to different dividing actions and diagrammatic forms, and the "one-many" nature of the relationship between division as a mathematical operation and the situation types that this operation can index. This already complex picture is further compounded in primary schools in the South African mathematics teaching landscape, research into which raises concerns about coherence and connections in the presentation of mathematical ideas (Venkat and Naidoo 2012; Askew et al. 2012). This complexity suggests that a focus on teachers' semiotics praxis can provide valuable insights into teachers' methods of seeing and working with division.

The use of semiotics in this chapter rests on Peircian theory, contextualized in the work of Ernest (2006) that focuses on "semiotic systems". It is also based on Presmeg's (1998) approach involving the need to move beyond dyadic attention to signifier-signified chains and webs to take into account the inclusion of attention to interpretants. The chapter attends to the meanings of interpretants in two ways: firstly, by focusing on the logical implications of the semiotic systems—what we have called "*signification pathways*" co-produced by teachers and students, and endorsed by teachers in learning about division. Secondly, the chapter considers the possibilities and constraints of these signification pathways in relation to the semiotic system that concerns division from a mathematical interpretant perspective. In this respect, our interest goes beyond the idiosyncratic sign use of individuals; rather, we seek to understand patterns and rules of sign use to broaden teachers' semiotics practice in developmental ways.

This monograph as a whole is focused on semiotic theories, so first we explain our use of semiotic approaches. Second, we give a literature overview on division from a mathematical and a pedagogic perspective. We re-interpret some of this literature specifically to emphasize the implications for thinking in terms of semiotic systems. A short summary of the ways in which problems with coherence have been presented and analysed in the South African context follows. This is also re-interpreted in relation to signification pathways and semiotic systems. This scene setting provides an overview of an emergent framework of signification pathways that has been developed in the course of the first author's doctoral study (Mathews

2016). This framework presents a typology of signification pathways, ranging from coherence at the upper extreme and errors at the lower extreme, with two interim pathways of working with signification, which appeared to either limit, or disrupt, coherence in different ways. We present and discuss this typology with illuminating examples drawn from the dataset of the first author's doctoral study.

14.2 Semiotic Systems and Signification Pathways

What it means to learn mathematics is construed in a range of ways in the mathematics education literature (Presmeg et al. 2016): as internalization of a cultural-historical activity (Vygotsky 1979); as appropriation of discursive repertoires (Sfard 2008); or as expanding competence and confidence across example spaces (Watson and Mason 2005). From a semiotic viewpoint, a fundamental idea involving the learning of mathematics is the appropriation of the sign systems and meanings associated with mathematics as a discipline. Several writers emphasize that the mathematical use of symbolism makes the teaching and learning of mathematics particularly suitable for semiotic analyses (Presmeg et al. 2016). This literature, and other relatively recent overviews of the history and development of studies based on semiotic theory in mathematics education, note the disparities in the approaches to the key sources of semiotic notions—the writings of Saussure, Peirce, and Vygotsky. This chapter does not re-tread this ground but instead details the specific antecedents for the approach taken in our study, and explains the reasons for our choices.

Our reading of semiotic approaches starts from the work of Walkerdine (1988), which provided the theoretical basis for the data analysis in the first author's doctoral study, with data gathered from a series of Grade 3 and 4 lessons based on division in South African schools. Analysis of this dataset revealed frequent incidences of "slippage" or "ambiguity" in teachers' handling of key division ideas. While much writing in the field of semiotics has attended to the ways in which mathematical ideas are constructed through the coherent assembling and connecting of signifiers (for instance, Radford 2010), our analysis indicated the need to understand gaps in the concatenation of signifiers in classroom teaching. Here, Walkerdine's excerpts of interactions were particularly illuminating in showing that interpretations of the associations of signifiers to signifieds varied frequently between participants in a discourse. While at one level this is simply a re-assertion of the arbitrary nature of signifiers and associations, for us, Walkerdine's handling of empirical data showing that signification chains involved different interpretations, was useful in our research. Walkerdine (1988) mentions that

> semiotics has stressed the productive power of signification, that is, it is the very social production of relations of signification, which provides the possibility of subjectivity. […] If material phenomena are only encountered within their insertions into, and signified within a practice, this articulation is not fixed and immutable; it is slippery and mobile. It

seems then that signifiers do not cover fixed ‘meanings’ any more than objects have only one set of physical properties or function. (p. 30)

We initially adopted a similar approach; we tried to trace chains of signification within transcripts (that included all teacher-learner talk, gestures, artefact use and inscriptions) created from video-taped lesson observations. We found, however, that classroom activity and talk were rarely linear in terms of how they proceeded, that is, it was not the case that a single signified was linearly and tidily elaborated in classroom discourse to some end-point, prior to attention shifting to a new signified. Instead, what was common is the phenomenon described by Presmeg (2006) in which the teacher sometimes stepped “aside” to deal with an element nested within a task, before returning to deal with the broader signifier. Sometimes the teacher did not return to the original broader task at all. Presmeg (2006) described the “dyadic model of chaining” as involving a chaining of signifiers in which the “new signifier in the chain stands for everything that precedes it in the chain. The previous signifier, as well as everything that it represents, is now the new signified” (p. 169).

The language of a chain of signification draws one’s attention to a serial formation of signs within a practice. Within the process of forming chains of signification, initial signs are used in the production of new signs. For Kirshner and Whitson (1997), the autonomous nature of the signifiers allows for the formation of a chaining process. This chaining process is seen when the signifying term (signifier 1) in a preceding sign combination comes to serve also as signified term (signified 2) in a succeeding sign combination. In such a “chaining of signifiers” the preceding signifieds and sign combination are sometimes described as “sliding under” the succeeding. A chain of signification is dependent on the formation and connection of preceding signs so that signs are interpreted, and therefore provide meaning. Cobb et al. (1997) present an example of a coherent chain of signification involving a sequence of signifiers that denote packs of ten candies, beginning with bars of ten unifix cubes standing in for the outcome of packing ten candies into rolls of ten candies each. In examining the chain of signification that ensues, Cobb et al. conclude that “the bars of 10 unifix cubes came to signify at least a numerical composition of 10 for most of the children. The use of pictured rolls as signifying numerical composites, indicating the results of the packing activity were now taken as a given” (p. 183).

In contrast, our early data analysis indicated frequent ‘side-steps’ to new signified entities (often sub-elements of the initial signified entity) in the South African context. This led to our use of the notion of “signification pathways” to encompass episodes in which the teacher would begin an episode with a particular signifier and then shift focus to new signifiers, sometimes returning to the initial signifier, and sometimes continuing along the new signifier path.

While we have talked about signifiers as associated with signifieds in the process of elaborations of their meanings (rather than talking about signifieds that are elaborated via signifiers), our point of agreement with Presmeg relates to the need for a nested model containing multiple pathways of signifiers. We also concur with her approach of attending to meanings within these pathways. Our elaboration is to focus on meanings at two levels rather than one. Firstly, we explore the meanings that can be interpreted as endorsed by the teacher within the signification pathways associated with particular tasks and across sequences of tasks and lessons. Secondly, we consider these meanings in relation to the meanings of division that are advocated in the discourse of the mathematical community.

A further notion that was useful in the operationalization of this approach was the assertion that signs and sign use can only ever be understood as part of a larger *semiotic system* (Ernest 2006). Semiotic systems are described as comprising the following three components:

(a) There is a set of signs, each of which might possibly be uttered, spoken, written, drawn, or encoded electronically.
(b) There is a set of rules of sign production, for producing or uttering both atomic (single) and molecular (compound) signs. These rules concern much more than the definition and determinants of a well-formed, i.e., grammatically correct, sign. They also concern the sequencing of signs in conversation, i.e., what sign utterances may legitimately follow from prior signs in given social contexts. In mathematics this includes rules that legitimate certain text transformations, e.g., "cancelling", the common divisor of numerator and denominator, in fractions.
(c) There is a set of relationships between the signs and their meanings embodied in an underlying meaning structure.
(Adapted from Ernest 2006, p. 70)

Our interest was in exploring patterned sign use within lessons to infer the nature of enacted semiotic systems related to division, and to comment on this enacted semiotic system in relation to the semiotic systems that are advocated as mathematically and pedagogically useful in the mathematics education literature on division. Ernest's notion of semiotic systems takes into account our need to include atomic and molecular signification chains, as well as a range of signifiers that include, for us, the different forms stated above, and extend to include gestures and actions related to division, whose use we observed during lessons, and which have significance in Arzarello's (2006) descriptions of *semiotic bundles*.

14.3 Semiotic Systems Related to Teaching and Learning Division

The literature on early mathematical learning calls for formal mathematical learning built on children's everyday activities and actions involving quantities. Anghileri (1995b), writing about division, observed that everyday sharing activities provide a useful basis upon which one key interpretation of division can be constructed. Division interpreted in this "partitive" way involves one-by-one distributing gestures. Carpenter et al. (1999) note that with extended experiences over time, this one-by-one distribution comes to be truncated into grouped, that is, non-unit, distributing actions. Division though, is not reduceable to sharing situations. In situations involving division by fractions, such as: '*How many children can share 6 oranges if each child has half an orange*?'—division is amenable to being worked with as a quotitive interpretation, in which "6 oranges" are measured out using a "half-orange". Such situations involve measuring, rather than distributing gestures (Tirosh and Graeber 1991). This measuring can also be linked to a repeated subtracting of the measuring unit from the dividend. Whilst partitioning can also be viewed in terms of repeated subtraction of rounds of distribution, language selections have to be carefully differentiated if sense-making linked to the situation is to be kept in focus. In quotitioning situations, the subtraction quantity is the size of each group, or the measuring unit, whereas in partitioning, or sharing, situations the size of each of group becomes visible only after the distribution of items. At the heart of this difficulty is what Askew (2015) points to as the one-many nature of the relationship between symbolic mathematical operations and situations/actions. A consequence of this one-many structure is that while story situations need signification pathways that cohere with the situation, or narratives that acknowledge shifts, in the division model symbolic calculations can be solved using partitive or quotitive strategies with efficient strategic calculation involving judgements as to whether it makes more sense to interpret tasks partitively or quotitively, e.g., Anghileri's (1995a, p. 86) tasks: $6000 \div 6$ and $6000 \div 1000$. Furthermore, Anghileri (1995a, p. 86) points to the linguistic variety in division situations, with overlapping words pointing to different actions; for example, "shared into 3, shared into 3s, shared by 3, shared with 3" simply generates this language conundrum.

Other important aspects of the semiotic system of division are diagrammatic representations. Carpenter et al. (1999) present examples of the construction of "direct models" of partitive and quotitive situations, moving into combinations of oral skip counting in multiples, coupled with partial diagrams in which aspects of the situation (iconic images of the number in each group or the number of groups) and aspects of the count are jotted down. An important and more structured representation of division is the array format, consisting of a rectangular arrangement in which the transformable relationship between partitioning and quotitioning can start to be explained in ways that are more difficult to achieve in less structured iconic or indexical grouping diagrams.

The complexities noted above have led to a wide range of suggestions about how to begin the early teaching of division, and how to support progression and flexibility. Literature in the Realistic Mathematics Education (Gravemeijer 1997) and Cognitively Guided Instruction (Carpenter et al. 1999) paradigms point to the usefulness of beginning instruction in the context of real or realistic situations, with problems that can be modelled with division-related signifiers in the form of gestures, images and symbols. These studies also point to the ways in which progression occurs through the truncation of signifiers, and moves from concrete actions on objects to signifiers of these actions. Unit counting strategies give way to grouped counting strategies, which in turn give way to working with combinations of known and derived facts, alongside working with properties such as the commutativity and distributivity of multiplication, and the inverse relations of division and multiplication.

Of specific importance to us in the light of the above discussion is the need for careful signifier choices, assemblies and sequences that support students in making sense of situations in ways that allow access to increasingly formal, flexible and efficient ways of working. In relation to Mathews' (2014) analysis indicating offers of signifiers in instruction that assumed awareness of unknown quantities, we also looked, from a mathematical perspective, at coherent signifier productions that worked from givens to unknowns.

Pedagogic advice in mathematics education is strongly oriented to the need to ground assemblies of semiotic forms in meaning-making. In the context of two basic situations—partition and quotation—with distinct associated actions that give rise to division, this advice can raise tensions that have to be resolved in classroom working. We use a scenario to illustrate some of these tensions:

> A child responds thus to a task asking: "45 carrots are shared among 9 rabbits. How many carrots would each rabbit get?"
>
> Learner: "I can count in 9s for this. Nine, eighteen, twenty-seven, thirty-six, forty-five"–opening a finger as each word is said. "It's 5. The answer is 5 carrots each."
>
> Teachers: "Yes, that's correct. Well done".
>
> Some other children are busy drawing diagrams of carrots and rabbits and distributing them as this answer is given and accepted.

In this scenario, a sharing situation is presented, and a solution involving measuring 45 in 9 s is offered and accepted. As the mathematical discussion above indicates, the solution process is mathematically appropriate, due to the one-many relationship between division operations and situations, but the teacher does not draw learners' attention to an action that is normally associated within a sharing situation, requiring a little elaboration in the semiotic bundle to explicate it.

The literature points to the importance of incorporating sharing and measuring tasks in instruction, with signifiers building into both efficiency and properties related to multiplicative structure over time. As noted, limitations in both tasks and semiotic bundles can restrict the possibilities for well-grounded learning about division. Evidence of these limitations shows a range of localized "rules" that emerge in the pedagogic semiotic system. Tirosh and Graeber (1991) note the common expression that "division makes smaller", emanating from restrictions in tasks in the early grades to positive integer divisors. Allied with this restriction, Fischbein et al. (1985) point to other rules, such as: the divisor is always less than the dividend, and the divisor is always a whole number.

Several features emanating from this body of work were useful to our development of an emergent analytical framework in the broader study:

- The care needed to develop signification pathways that contain well-connected sign sequences with the potential to support sense-making;

- The need to attend to shifts in the interpretation of division in the mid-stream of working, and whether these shifts are made explicit in aspects of the semiotic bundle being worked with;
- Rules presented in the course of signification that, while appropriate in the localized task setting, are stated as though they had a more universal realm of application;
- The need to examine the range of division tasks presented across lesson sequences.

Given that our attention in this chapter is on exemplifying key signification pathways that emerged as a useful typology for considering division teaching, we focus here on the first three of the above features, with less attention to the last feature.

14.4 Methodology and Data Sources

The data drawn on for this chapter derive from the doctoral study of the first author, which explores these issues in the particular context of Grades 3 and 4 division teaching in three Johannesburg schools, involving six teachers. Given our aim in this chapter to exemplify and analyse the four signification pathways within the framework, we draw on data from one Grade 3 teacher in this dataset, whose episodes (based on data collected in term 2 of 2011) provided 'telling cases' of all the pathways. This teacher, like the others, had over fifteen years' teaching experience. She taught in a school located within a previously advantaged community which continued to attract relatively privileged cohorts of learners. In the broader study, all the division tasks introduced and explained by each teacher, and the learners' responses, were video-taped and transcribed, and included details of all teacher-learner talk, gestures, artifacts used, and inscriptions.

Data analysis was grounded in parsing lessons into episodes, with an *episode* marked as a period in which the teacher did whole class teaching or mat work (direct teaching of a few learners at a time on a mat) while using the chalkboard, overhead-projector, flashcards or charts to explain a task or calculation leading to division. A change in division task also marked a new episode. In some instances, the new division task would have multiple tasks related to it; episodes also included instances of the use of multiplication to verify the division number sentence answer. In other cases, references to prior division tasks or answers were made. Episodes involving skip counting, whilst not directly division tasks, were included in the listing of episodes given the usefulness of this fluency for division problem-solving. Group work (facilitated activities by the teacher) episodes were also left in the listing for completeness, but not analysed given our focus on whole-class instruction. The analysis of each episode focused on the signs produced, and the meanings that could be associated with these signs. The distinguishing and collating of episodes led to the identification of four signification pathways: mathematically coherent, coherent with limitations, coherence with instances of ambiguity, and mathematically incoherent. These are elaborated in Table 14.1.

14.5 Illustrating and Discussing the Signification Pathways

The four categories, rather than standing in progression, provide a way of looking at complexity related to the production of signs. The first signification pathway emphasizes coherence throughout the episode. In such episodes, teachers produced signifiers (Sf) in the form of utterances, symbolic number inscriptions, images or

Table 14.1 Four types of signification pathways

Signification pathways	
Category	Description
Category one Mathematically coherent	This signification pathway involves associations between signifiers that are mathematically justifiable at all points within the episode. Such pathways include instances where a learner provides an answer immediately to a task as a known fact, and the answer is then subsequently worked with in a verification process There is either no shift in the mathematical model worked with, or the shift is acknowledged in the discourse
Category two Coherent with limitations	This signification pathway is coherent in terms of the signs produced. However the signs made are not generally applicable; instead, the signification pathway is localized and limited to the format of at least one of the signifiers presented within the pathway
Category three Coherent with instances of ambiguity	This signification pathway involves instances of ambiguity in a signification pathway which is generally coherent. Ambiguity within signification pathways relates to one of the following: • No possible signifier at some point or an incorrect signifier for a signified or no acknowledgement of an incorrect sign, but subsequently corrected and filled in; or • A signifier that can be associated with more than one sign or signified; • A signification pathway with a shift between signifiers with no acknowledgement of the shift
Category four Mathematically incoherent	This signification pathway involves mathematical incoherence. The incoherence occurs when the teacher's answer is incorrect or when a signification pathway involved no possible sign at some points, with these gaps not subsequently filled in

written division situations that were, mathematically, coherently able to be associated with the signified (Sd), thereby constructing signs (Sg) through these associations. In the particular case of division, a category one signification pathway produces an answer to a division task without disconnection between the signs in the pathway. For example, if the teacher provided a grouping division situation, the signifiers/signifieds were consistent with grouping actions where the divisor signified the number of items in a group.

Category two signification pathways were coherent in that the signs produced did mathematically link the signifier and the signified. However, a limitation occurred in that the answer or explanation was localized. An example of such a limitation is an appearance of the idea that division always makes smaller, as while this is true for division involving positive integer divisors, it does not generalize to all cases of division.

The third signification pathway (category three) involved locally coherent teachers' explanations containing instances of ambiguity. Three kinds of ambiguity were seen: in the first, the teacher made no clear links to a possible signified, or an incorrect or inappropriate signifier for a signified was used or endorsed, or the teacher accepted or did not acknowledge an incorrect sign. However, in the elaboration of the signification pathway the teacher subsequently replaced the erroneous signifier or provided an appropriate signifier for the signified. For example, within a signification pathway, a teacher may state that the answer to twelve divided by three is six, and then later in the pathway, realize the mistake and correct it by providing the answer of four. The second type of ambiguity occurs when a teacher produces a signifier that can be associated with more than one signified and hence more than one sign. For example, in a picture showing two biscuits, the teacher initially treats the two images as signifying biscuits but subsequently suggests that the two images signify people. The third ambiguity occurs when a teacher shifts a signifier to being associated with a different signified, without acknowledging the shift, for example moving from talking of division as sharing to division as grouping.

The last category of signification pathway is seen when the teacher produces an incorrect answer or offers no associations for a signified, without subsequently offering an association or correcting the incorrect answer.

Next, we present empirical episodes and summarized analyses to further exemplify the signification pathways when teaching division.

14.5.1 A Category One Signification Pathway (Lesson 3—Episode 4)

The teacher asked the learners to pack out thirteen counters and place them on their whiteboards. The teacher instructed the learners to group the counters into 'groups of four'. As she walked between the learners the teacher saw a learner placing four counters into three groups with one remainder, the teacher circled the *four counters as a group. The teacher asked the learners to provide a number sentence and some learners stated "Thirteen divided by three". Another learner stated "Thirteen divided by four" and the teacher wrote '13 ÷ 4 =' on the CB. Above the symbolic inscription '4' she wrote the words 'in a group'. She completed the symbolic inscription by inserting '3 rem 1' to 13 ÷ 4 = as the answer. The teacher reaffirmed that the remainder is the 'left over'.*

Analysis:

Early work in this episode proceeded smoothly with learners able to associate the instruction to place the counters into groups of four (Sd 1) with appropriate counter arrangements (Sf 1). When the teacher asked for a number sentence for the arrangement of the counters (Sf 1), learners provided two number sentences for this: "thirteen divided by three" (Sf 2a) and "thirteen divided by four" (Sf 2b). The teacher's acceptance of the latter, seen in her writing up of "13 ÷ 4=" (Sf 3a) on the board, linked both the action and product arrangement of packing "thirteen counters into groups of four" with its verbal (Sf 2b) and symbolic representations (Sf 3a). She then elaborated on the symbol "4" within the number sentence by associating this sub-part signifier with the phrase "in a group" that was drawn for the original signified entity (Sd 1). Returning to the 13 ÷ 4 = number sentence as a whole, "*3 rem 1*" was added in as the signifier (Sf 3b) for the outcome of the process of packing in groups of four. Shifting once again to a part of this signifier—this time, the "rem 1" portion, she reminded the class orally, that this signified what was 'left over' (see Fig. 14.1).

Mathematically, the instruction to pack counters into groups of four was associated with a grouping action. The teacher's acceptance of the verbal expression 'thirteen divided by four' as opposed to 'thirteen divided by three' was consistent with this grouping action. (If instead the teacher had accepted the learners' offer of 'thirteen divided by three' she would have set up a partitive interpretation of division that could have matched the outcome counter arrangement, but would have dissociated from the process of its construction if no supplementary discourse was offered). Of interest here is that while the teacher then emphasised that the divisor stood for the number of items in a group (Tirosh and Graeber 1991) what is lost is any discussion of the fact that the divisor 'may', rather than 'must', stand for the number of groups. While appropriate for this episode, and leading to a coding of the episode in category one, the broader lack of engagement of differences in meaning

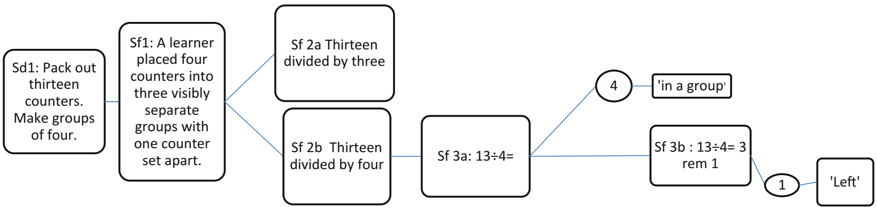

Fig. 14.1 Category one signification pathway

of the divisor across episodes produces a separate working with individual examples that has been noted in broader South African writing (Adler and Venkat 2014; Venkat and Naidoo 2012). However, all associations made or endorsed in instruction within the pathway are mathematically consistent with the signifiers and signifieds in play in this episode.

14.5.2 A Category Two Signification Pathway (Lesson 2—Episode 1)

The teacher said "today we will solve problems and division of whole 1 and 2 digit numbers with solutions". She stated "I'm going to teach you, how to do short division, using a bracket that you may not have seen before and today we will use this bracket". The teacher drew a 'short division bracket'. Thereafter, she drew the division symbol on the chalkboard and asked the learners to identify it. She stated that the "Division symbol and bracket is the same". The teacher revealed five flashcards on the chalkboard reading 'division', 'share', 'smaller', 'less' and 'equal sharing'. She finally stated that "Another name for division" (pointing to the flashcard 'division' on the chalkboard) is "Share" (pointing to the flashcard 'share' on the chalkboard) and then stated that "The answer will always be smaller or less" (see Fig. 14.2).

Analysis:

At the beginning of the signification pathway the teacher mentioned that she was going to teach the learners "short division". After this, she drew a "short division bracket" (Sd 1a) and then a symbolic "division symbol" (Sd 1b) on the chalkboard. She then linked the "short division bracket" and "division symbol" with the utterance "the division symbol and bracket is the same" (Sf 1). Having the two

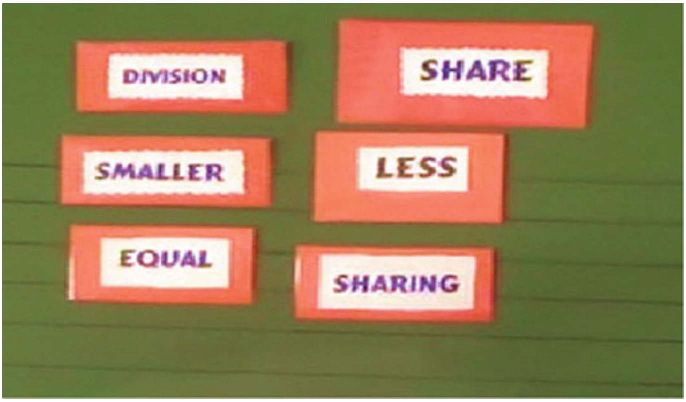

Fig. 14.2 Flashcards on the board

signified inscriptions on the chalkboard, the teacher drew the learners' attention to the idea that they signify division. After placing several flashcards on the chalkboard the teacher pointed to two flashcards with the words "division" (Sd 2a) on the one and "share" (Sd 2b) on the other. She then linked the flashcards with the utterance "another name for division is sharing" (Sf 2). The association the teacher thus established was that division involves sharing. Finally she mentioned that whether you are dividing or sharing, in division problems "the answers will always be smaller or less" (see Fig. 14.3).

The first sign that is produced is that the two signifieds, the 'division bracket' and 'division symbol', both point to the notion of division. The second sign the teacher endorsed was that the use of division should be seen as sharing. The last sign endorsed was that "when dividing or sharing division problems the answers will always be smaller or less." According to Tirosh and Graeber (1991), in the context of division the idea that "division makes smaller" emanates from restrictions in tasks in the early grades to positive integer divisors. This particular episode is coherent in that all the signs that were produced emphasized ways in which division could be expressed. Along the signification pathway the signs are connected to each other in that the teacher shows how the division symbols and actions are interrelated and transformed within the teacher's discourse. However, this episode was categorised within category two due to the signification pathway being localized by being limited by the final signifier to divisors greater than 1 for answers to be 'less and smaller'.

14.5.3 *A Category Three Signification Pathway (Lesson 3—Episode 8)*

The teacher placed a transparency on the overhead projector bearing an image of four squares with two biscuits in each square and one biscuit on the outside of these squares. The following questions were written beneath the image: [Among] how many dogs are the biscuits divided? How many biscuits does each dog get? How many biscuits are there in total? The teacher asked "[Among] how many dogs were the biscuits divided?" One learner responded stating 'four'. The teacher asked the question again and another learner stated 'two'. The teacher pointed towards the

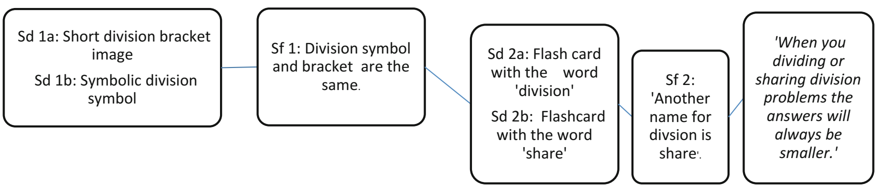

Fig. 14.3 Category two signification pathway

two biscuits in the squares. The teacher accepted the learner's answer of two and asked: 'In how many groups have we put them?' A learner responded 'four groups'. The teacher then asked: "Who can give me a number sentence?" One learner carried out the calculation by counting all the biscuits in the squares and outside the squares. The teacher, together with the learner, counted in twos and added one and established that there were nine biscuits altogether. She asked, "Nine divided by two equals…?", and accepted the learner's answer of 'four, one remainder' without referring to the image.

Analysis:

The teacher placed an image of four squares with two dog biscuits in each square with one dog biscuit on the outside. She posed the question "[Among] how many dogs were the biscuits divided?" At this point, the teacher drew the learner's attention to the idea that the biscuits in the image (Sd 1a) needed to be divided between dogs. The learners provided two offers—of "four" (Sf 1a) and "two" (Sf 1b) dogs. The teacher accepted the latter, seen both in her repeating of the learner's offer of "two" and in pointing to the two biscuits in each square. She thus associated the image of the two biscuits in a square with two dogs. The teacher then emphasized that the four squares signified four groups, before asking for a number sentence. But before the number sentence was provided the teacher asked the learners to count all the biscuits in the squares. The teacher used the image of the squares with biscuits inside and outside (Sd 1a) to make the association that the number sentence was "nine divided by two equals" (Sf 1c). Immediately, the teacher accepted the learners' answer of "four, one remainder" which was added in as a signifier (Sf 1d) for the result of the number sentence. In the latter utterances then, the teacher signified that the images within and outside the squares were nine biscuits (see Fig. 14.4).

Askew (2015) mentions that in solving division situations a "one–many" relationship between a symbolic mathematical operation and situations/actions in division situations becomes visible. In this episode the "one–many" relationship is also visible in having one image that can be solved either through a grouping or sharing interpretation of division. The associations that were produced ensured that the signified in this case (the images within the squares) were linked to two different signifiers, "dogs" at one point and "biscuits" later on. While each association in the

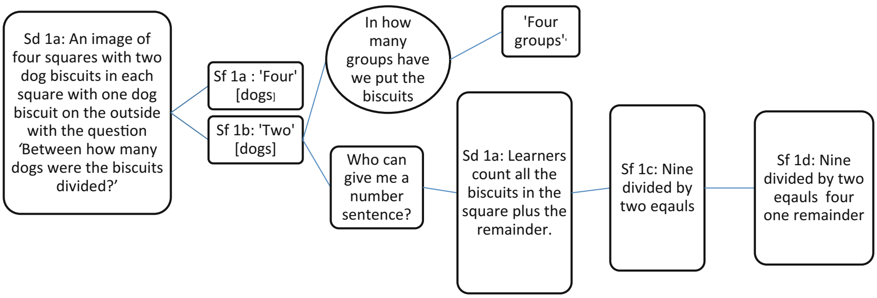

Fig. 14.4 Category three signification pathway

signification pathway could be linked mathematically to the image on the transparency as the outcome of either a sharing or a grouping action, successive signifiers did not cohere with each other in terms of the story setting. The learners' offers of "four" and "two" suggested different division actions. The answer "four" could be interpreted as the result of a sharing action involving four dogs sharing the biscuits, while the answer "two" pointed towards a grouping action (how many dogs can get two biscuits each). The teacher's acceptance of the answer "two" reinforced (as in previous episodes) that the divisor should be understood as the number of items in a group and not the number of groups. The association that the teacher's words and actions endorsed was that a grouping interpretation of division needed to be used in order to produce the spoken number sentence (Ernest 2006). But the image, if linked to a situation of sharing biscuits among dogs, lends itself more easily to signifying a situation involving four dogs sharing the biscuits and getting two biscuits each, with one left over. The teacher's acceptance of two as the 'number of dogs' would require a transformation of the image to 'inscribing' the two dogs and seeing one biscuit in each group as going to each dog. The ambiguity is seen in the teacher producing two signifiers for the images inside the squares—as 'dogs' (signifier) and also as 'biscuits' (signifier). Within the signification pathway of this division task different signifiers for the same signified leads to multiple meanings and associations and possible ambiguity for learners. Therefore, this episode was placed in category three of the signification pathway framework.

14.5.4 A Category Four Signification Pathway (Lesson 4—Episode 9)

The teacher placed a transparency with the image of eleven sweets in two circles with four sweets in each and three sweets on the outside of the circles. The following three questions were written underneath the circles. [Among] how many people are the sweets divided? How many sweets does each person get? How many sweets are there in total? The teacher asked the learners, "Among how many people were the sweets divided?" She then mentioned, "Ok if we share the following, you see all these sweets" (the teacher pointed to the sweets on the transparency). The learners replied, 'Yes madam'. The teacher asked, "How many sweets are there altogether?" The learners responded 'eleven'. The teacher proceeded to ask "How many in a group?" The learners answered 'four'. She asked, "And how many groups are there?" The learners replied 'two'. The teacher asked, "And how many [are] left over?" The learners answered 'four' which she accepted. She then asked the learners to provide a remainder. The learners replied 'three'. The teacher accepted the learners' offer of 'three' as the remainder. She then asked them 'How many sweets are there in total?' The learners answered 'eleven' and she then stated repeated the phrase 'in total' and the learners replied 'eleven' which the teacher accepted.

Analysis:

The teacher placed the transparency on the OHP and proceeded to ask the learners certain questions. The teacher associated the image (Sd 1) with “eleven sweets” (Sf 1a), “four sweets in a group” (Sf 1b) and “two groups” (Sf 1c) and that the sweets outside the three circles stand for the remainder (Sf 1d). She then posed two questions that were drawn from the original signified entity (Sd 1). The associations that the teacher established were that “two” people (Sf 2a) received “four” sweets (Sf 2b). The teacher then re-established the associations which she had created in Sf 1d and Sf 1a (see Fig. 14.5).

From the image provided, the teacher established some signs connected to parts of the image. The initial answer of “two people” each receiving “four sweets” appears to signify an acceptable answer. However, the teacher creating the association that “three is the remainder” brings into question the acceptance of a previous association of sweets being divided between “two people”. In other words, if sweets were shared as mentioned at the beginning of the signification pathway between two people, a complete sharing out action would have produced an image of two circles with five sweets in each and one sweet on the outside as opposed to the image in which each circle has four sweets with three sweets on the outside. The incoherence is seen in that “two” people, as the signifier (Sf 2a) fits poorly with the initial image, coupled with the instruction to ‘share … all the sweets’ (Sd1), and therefore represents an incorrect sign that was established and not subsequently corrected. Literature suggests that incoherence occurs in the teaching of division when disconnects occur between mathematical expressions within teacher’s explanations (Mathews 2014). In this case, the answer does not connect with the image provided and the episode is placed in category four.

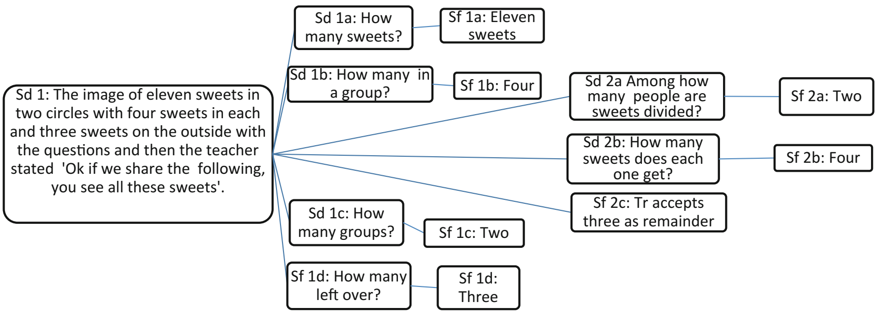

Fig. 14.5 Category four signification pathway

14.6 Implications

The analysis of the episodes highlights the complexity associated with the teaching of division (Anghileri 1995b). These episodes came to challenge some of the traditional notions that 'sharing' as a division interpretation is preferable over 'grouping'. The teachers' attempts at striving for connections among talk, actions, gestures, images and symbolic inscriptions when teaching division are difficult in some instances because of the influence of multiple associations that inhibit the coherent explanation of division (Mathews 2014). In some cases, the explanation of division is localized (Tirosh and Graeber 1991) but treated within the teacher's discourse as generalized. Without careful selection of examples, actions, gestures and other signs, teachers may produce multiple associations that may lead to the endorsement of unintended ways of making sense of division. The introduction of signification pathways within the study of semiotics has introduced the notion that the initial signifieds are not always used to produce new signifiers. Signification pathways are co-produced by the teacher and the learners with the teacher endorsing certain associations and influencing the sequencing of signs (Radford et al. 2008). A signification pathway allows for the unification of the signifier to the signified (Walkerdine 1988) in ways that emphasize the complexity of working within mathematics. However, these signs within the classroom discourse are dependent on the associations that are endorsed (Ernest 2006) by the teacher, which provides new insight into how coherence within division can be reimagined.

References

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction. In H. Venkat, M. Rollnick, J. Loughran, & M. Askew (Eds.), *Exploring mathematics and science teachers' knowledge* (pp. 132–146). London: Routledge.

Anghileri, J. (1995a). Making sense of symbols. In J. Anghileri (Ed.), *Children's mathematical thinking in the primary years* (pp. 74–91). London, UK: Cassell.

Anghileri, J. (1995b). Negotiating meanings for division. *Mathematics in School, 24*(2), 2–4. doi: https://doi.org/10.2307/30215145.

Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME. Revista latinoamericana de investigación en matemática educativa, 9*(1), 267–300.

Askew, M. (2015). *Transforming primary mathematics: Understanding classroom tasks, tools and talk*. London: Routledge.

Askew, M., Venkat, H., & Mathews, C. (2012, July). Coherence and consistency in South African primary mathematics lessons. In *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 27–34). Taipei, Taiwan: PME.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*: ERIC.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner, & Whitson, J. A. (Ed.), *Social, semiotic, and psychological perspectives* (pp. 151–233). Mahwah, NJ: Lawrence Erlbaum.

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics, 61*(1–2), 67–101.
Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3–17.
Gravemeijer, K. (1997). Solving word problems: A case of modelling? *Learning and Instruction, 7*(4), 389–397. https://doi.org/10.1016/S0959-4752(97)00011-X.
Kirshner, D., & Whitson, J. A. (1997). *Situated cognition: Social, semiotic, and psychological perspectives*. Mahwah, NJ: Erlbaum.
Mathews, C. (2014). Teaching division: The importance of coherence in what is made available to learn. In H. Venkat, M. Rollnick, J. Loughran, & M. Askew (Eds.), *Exploring mathematics and science teachers' knowledge* (pp. 84–95). London: Routledge.
Mathews, C. (2016). *The teaching of division across grades 3 and 4—Disruptions to coherence.* Unpublished Ph.D. research report. Johannesburg: University of the Witwatersrand.
Presmeg, N. (1998). Ethnomathematics in teacher education. *Journal of Mathematics Teacher Education, 1*(3), 317–339.
Presmeg, N. (2006). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61*(1–2), 163–182.
Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (Eds.). (2016). *Semiotics in mathematics education*. Hamburg: Springer Open.
Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1–19.
Radford, L., Schubring, G., & Seeger, F. (2008). *Semiotics in mathematics education. Epistemology, history, classroom, and culture*. Rotterdam: Sense Publishers.
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, UK: Cambridge University Press.
Tirosh, D., & Graeber, A. O. (1991). The effect of problem type and common misconceptions on preservice elementary teachers' thinking about division. *School Science and Mathematics, 91*(4), 157–163.
Venkat, H., & Naidoo, D. (2012). Analyzing coherence for conceptual learning in a Grade 2 numeracy lesson. *Education as Change, 16*(1), 21–33. https://doi.org/10.1080/16823206.2012.691686.
Vygotsky, L. (1979). Consciousness as a problem in the psychology of behavior. *Soviet Psychology, 17*(4), 3–35.
Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.
Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: The role of learner generated examples*. Mahwah, NJ: Erlbaum.

Part IV
Semiotic Resources Including Gesturing and Tools

Introduction: Michael Roth

In this part, readers find a collection of chapters concerned with semiotic activity in which gestures and tools are involved and at the center of the analytic process and results. Neither gestures nor tools are particular to humans. They can be found in the non-human animal world not only among our closest relatives but also among birds (the New Caledonian and Hawai'ian crows use tools in multiple steps to get at food) or dolphins (gestural behavior). Even consciousness (conscious awareness), at the core of Chap. 18, is not unique to humans, as studies have shown investigating the behavior of dolphins in front of a mirror and that of chimpanzees in the wild gazing into a still water pool. In humans, all of these forms of activity have come to be amplified and are the core of the life-sustaining process, which thereby has shifted from being dominated by biological and environmental processes to cultural processes. But it is not culture on its own (i.e., "meanings") that determines the human life process. Instead, there is a single substance that manifests itself in biological (body) and cultural ways (thought) in the unity/identity of person and environment (Vygotskij 2001). All chapters in this section can be read as contributing to this unifying, post-Cartesian idea, even though the authors may not express it yet in that way.

In Chap. 15, Abtahi investigates how children and an adult relate through verbal exchange involving the use of common materials to be used in addition of fraction tasks. In the dialectical materialist approach initially articulated by Marx and Engels (1978) and subsequently taken up in the works of L. S. Vygotsky and Leont'ev, these materials therefore do not mediate the activity but constitute the activity as a different one from that in which the materials are not used (Il'enkov 1977). As tools, these thus are not outside of the individuals but are deeply integrated in the new form of activity. They are also integral to what it means to make sense of mathematical entities. Indeed, the mathematical entities, the sum of fractions—as elsewhere the graphs that resulted from scientific research—come to be in synecdochical relation to the activity as a whole, including the materials and tools involved (Roth 2014). Abtahi concludes that tools represent object/meaning and meaning/object relations, which is precisely how de Saussure (1967) defined the sign: relation between a material signifier and an ideal signified. In light of the framework articulated in the introduction, "meaning" is nothing other than relation between people, reflected in the relation between things, attributed to one of the things, here the object.

Krause and Salle place a concept at the heart of Chap. 16, which, though having a well-established foundation in the German literature, has no real equivalent in English: "Grundvorstellung." The term literally refers to basic or foundational ("Grund") Vorstellung, a term that is central to Kant's critique of reason and that philosophers tend to translate as "presentation" or "representation." In fact, Krause and Salle employ the term in contexts and ways that are not unlike the *phenomenological primitives* used in Anglo-Saxon cognitive science to understand how physics students reason in abstract domains (diSessa 1993). Just as the primary Grundvorstellungen in Chap. 16, the phenomenological primitives are based on concrete experiences in the everyday world. Although such phenomenological primitives may be antithetical to mathematical knowing, they are also the condition thereof, the very ground that enables abstract forms of mathematics (Husserl 1939). The authors show how hand movements tracing and symbolizing graphs are part of the sense-making process when an individual is asked to elaborate on a graph. It is important to note that a finger following a graph does not allow us to characterize it as a (symbolic) gesture. Instead, the movement is itself part of the event by means of which the material graph comes to be something perceived and intellected. Hand movements and eye movements in perception are closely related (see introduction).

Much work in mathematics education is concerned with mathematical thought, representations, and mathematical conceptions. Knowledge here is treated as a structure or thing. Recently, there has been a lively debate in *Educational Studies in Mathematics* concerning a variety of dynamic approaches to knowing mathematics (de Freitas and Sinclair 2012; Roth and Maheux 2015). As the verbal forms in the title "Diagramming and Gesturing during Mathematizing" show, Chap. 17 situates itself in that debate. Menz and Sinclair begin their chapter by reminding readers of the critique that has been launched at those semiotic approaches that take signs to be representations of something else. In the introduction, the authors show that underlying such treatment of signs is the cultural practice of taking relations between things and human beings and turning them into attributes of things while forgetting this historical origin. The authors then articulate a dynamic perspective on diagramming and gestures, which, as shown in the introductory chapter, are but two aspects of the human capacity to move. Thus, diagramming may lead to symbolic hand movements that iconically reproduce perceptual forms in diagrams, and the diagrams fix previous movements that articulate the perceptual space (Roth 2012). Whereas the authors use the adjective "embodied" to characterize knowing, their approach actually is consistent with the monist Spinozist-Marxian approach that takes the material form and the related ideal form (thought) as manifestations of the same underlying thinking body. This thinking body itself is inaccessible—perhaps part of the "virtual" that these authors invoke—but is real only in its manifestations that inherently belie the essence of their material origin (Il'enkov 1977).

Chapter 18 situates itself in the cultural–historical tradition generally and in the *theory of objectification* more specifically. This theory explicates the relation between ideal forms, on the one hand, and the ways these are expressed or manifest themselves in concrete entities (Radford 2013). Swidan and Prousak's work thereby can be situated, as outlined in the introductory chapter, in the relation of relations of people and relations of things. In this chapter, the authors investigate how ninth-grade students working in the Virtual Math Team environment, a tool enabling computer-supported cooperative work, accomplish tasks designed to encourage joint engagement. That is, the tool thereby is part of the relation between things and part of the relation between people. A good deal of the chapter is concerned with the emergence of conscious awareness. Consciousness, of course, is nothing other than conscious being, and the being of humans is nothing other than their life process (Marx and Engels 1978). That life process is social, and consciousness but the expression of relations between people and relations between things (see introduction). It is therefore without surprise that Swidan and Prusak notice the emergence of consciousness (awareness) and an objectification of an ideal relation in things (quadrilaterals).

References

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*, 133–152.

de Saussure, F. (1967). *Cours de linguistique générale* [Course in general linguistics]. Paris: Éditions Payot & Rivages.

diSessa, A. (1993). Toward an epistemology of physics. *Cognition and Instruction, 10,* 105–225.

Il'enkov, E. (1977). *Dialectical logic: Essays in its history and theory*. Moscow: Progress.

Husserl, E. (1939). Die Frage nach dem Ursprung der Geometrie als intentional-historisches Problem. *Revue internationale de philosophie, 1*, 203–225.

Marx, K., & Engels, F. (1978). *Werke Band 3* [Works vol. 3]. Berlin: Dietz.

Radford, L. (2013). Three key concepts of the theory of objectification: knowledge, knowing, and learning. *Journal of Research in Mathematics Education, 2*, 7–44.

Roth, W.-M. (2012). Tracking the origins of signs in mathematical activity: A material phenomenological approach. In M. Bockarova, M. Danesi, & R. Núñez (Eds.), *Cognitive science and interdisciplinary approaches to mathematical cognition* (pp. 209–247). Munich: LINCOM EUROPA

Roth, W.-M. (2014). *Graphing and uncertainty in the discovery sciences: With implications for STEM education*. Dordrecht: Springer.

Roth, W.-M., & Maheux, J. F. (2015). The visible and the invisible: Doing mathematics as revelation. *Educational Studies in Mathematics, 88*, 221–238.

Vygotskij, L. S. (2001). *Lekcii po pedologii* [Lectures on pedology]. Izhevsk: Udmurdskij University.

Chapter 15
Gradual Change of Perception: Signs, Tools, and Meaning-Making of Fractions

Yasmine Abtahi

Abstract *We do not see a round object with two moving hands; we see a 'clock'.* In what process do we perceive meanings in our use of tools? In what process do the immediate physical properties of an object—an object with two hands—become subordinate and the meaning of the object—a clock—become dominant? The purpose of this chapter is to engage with these questions through Vygotsky's view of gradual perceptual change in the object/meaning. I use Vygotsky's perspective of gradual change of perception as an instrument to examine the ways in which children attach mathematical meanings to their use of different tools to solve mathematical tasks. I follow my theoretical discussion with a concrete example of two children's interactions with a piece of scotch tape and a ruler, as they attempted to attach fractional meanings to the tools, in order to solve an addition of fractions problem.

Keywords Signs · Tools · Change in perception · Mathematical meaning-making

15.1 Introduction

The basis for all human higher mental functions (memory, will, and so on) evolves socially, entailing interaction (Vygotsky 1986, 1989). Such a view of sociality entails that individual functions stem from forms of collective life. This social collective mediates all our activities—through *tools* and *signs* (Vygotsky 1978). Marx views man working with tools whereby man uses the physical and mechanical properties of objects to reach his goals (Marx and Engels 1865). The Marxian view of tool was then used by Vygotsky as a means of external activity

Y. Abtahi (✉)
University of Ottawa, 145, Jean-Jacques-Lussier Private,
Ottawa, ON K1N 6N5, Canada
e-mail: Yasmine.Abtahi@uottawa.ca

Y. Abtahi
L'Université du Québec à Montréal, Montreal, Canada

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(i.e., labor) with which humans influence the environment. Hammers, nails and chairs are examples of tools. Signs, on the other hand, are means of internal activity that affect humans internally. Languages, various systems for counting, and algebraic symbol systems are examples of signs. With signs and tools being social means themselves, then higher mental functioning becomes the internalized relations of social orders (Vygotsky 1986). That is, a higher mental function first exists as social relation of the child and others, involving signs and tools.

As with any other human actions, children's learning of mathematical concepts is also relational and mediated by signs and tools. They are mediated in the sense that instead of acting directly, in unmediated ways in the mathematical world, children's actions are indirect and mediated by tools and signs. How do signs and tools mediate children's learning? In children's interactions with the tools, how are tools perceived, how are signs tied to the tools and consequently how are tools being used in the solving of mathematical tasks? Here, I outline a view with which I looked at children's interactions with the tools. More specifically, I focus on how children perceived the physical properties of (mathematical) tools, attached signs to their use of tools and gradually made meaning of the mathematical concept in relation to the tools and to the task of the adding two fractions. My rationale for the focus on the concept of fractions is twofold. First, fractions are amongst the most challenging concepts to teach and learn in elementary school mathematics (Steffe and Olive 2010). Lamon (2007) noted that fractions are one of the topics in elementary school mathematics that are among "the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most essential to success in higher mathematics and science" (p. 23). My second reason is that as a learner of mathematics, it was very hard for me to understand the details of the why and how of adding of two fractions. Only later, as a teacher of mathematics, some of the steps made more sense to me.

In the following, I begin with an overview of Vygotsky's view of gradual perceptual change. Then, I turn to a concrete example of two grade 7 children working with a variety of common tools to add two fractions. Then, I relate children's gradual perceptual change to the learning of the mathematical concept of fractions.

15.2 Objects then Meanings

Mathematics reveals itself in the way "painters come to see, when they step back, what they have made visible/revealed in and through their paintings" (Roth and Maheux 2015). In their work Roth and Maheux argued for a mathematics that is not available to be apprehended. For such mathematics to be comprehended it requires what they called "radical creativity" (p. 222). I built on their argument to look at a process within which children bring forth—make visible—the mathematics that is invisible to them as they interact with (mathematical) tools. I describe a mathematical tool as any tool-like object whose mathematical affordances are perceived by the child who is using it to solve a mathematical task, for example, a piece of paper, an apple, or fraction circles.

While a child uses a tool, there is a system of relation between the physical properties and characteristics of the tool and the perception of how the child sees those properties as useful to solve a mathematical task. I refer to the characteristics of the tools as their physical properties such as rigidity, shape, and sharpness. Similarly, I use the term “perception” to refer to the aspects of a child’s thinking that contribute to the kinds of interaction that happen. For example, a paper clip can become a mathematical tool only if a child who is using it, for example, to do a non-standard measurement, perceives its ‘measuring’ affordance. Now, the questions that I ask are as follows: How do children make visible the mathematics that is invisible in the paper clip? And in what process do children’s perception of a paper clip as a “bent piece of wire” that possibly holds paper, change to a “measuring device” that could measure a side of a table?

A particular feature of human perception is what Vygotsky referred to as the *perception of real objects* (Vygotsky 1978). This involves the perception of not only colors and shapes but also of meaning; we do not see a round object with two hands, we see a ‘clock’. The attachment of meaning to an object (not seeing an object with two moving hands but seeing a ‘clock’) is a process that develops through the use of signs and symbols in our interactions with the objects (tools). Vygotsky elaborated the discussion of ‘children’s perception’ through what he referred to as the object/meaning ratio. Using the object/meaning ratio, Vygotsky explained children’s perceptual development.

In short, he argued that at the beginning of a child’s encounter with an object, his/her perception could be expressed figuratively as a ratio in which the numerator is the object and the meaning is denominator—object/meaning. This means that for a young child the object is dominant and the meaning of the object is subordinate. At this stage, the physical properties of things play an important role in the child’s interaction with them. For instance, a stick can be a horse in child’s play, because it looks like a horse, but a box of matches cannot be a horse, because it does not look like a horse. It is only later, when the child can make use of signs and symbols in her/his interaction with the objects, that the meaning becomes the central point and objects are moved from being dominant to being subordinate, thus giving rise to the meaning/object ratio. At this stage Vygotsky noted that, for example, to show a location of a horse on a map a child could put a box of matches down and say: ‘This is a horse’. The perception of the child can now be figuratively expressed as a meaning/object. This figure of perception in which the meaning dominates is the result of tying signs to the tools; the box of matches is a symbol (sign) to present the horse.

Vygotsky borrowed his notion of “ratio” in the object/meaning ratio from others who have used the same notion to express relationships and concepts (e.g., Eco 1986). Although, I believe Vygotsky’ notion of the object/meaning ratio has a direct implication in mathematics teaching and learning with tools, I refine the language of Vygotsky’s object/meaning ratio. I use object/meaning as an intention to represent object-then-meaning, that is when the properties of the tool are dominant to possible mathematical meanings; and similarly I take meaning/object to express meaning-then-object. Such refinement of discourse emphasizes not only the expression of interrelationship but also the signifying process—within which, over time, a child makes visible for himself or herself possible mathematical meaning in tools.

Before moving on, I clarify how I conceptualize the process of meaning making. According to Vygotsky (1986), meaning making is exercised, developed, and enhanced, in the collective. It is formed in a collective in the form of relations among children, adults and tools. That is, meanings become visible in a relation between a child and a tool. In such a relation, the tool not only stands for itself, it also becomes a symbol of something else (a clock, or mathematical symbol); carrying the meaning of a clock or possible mathematical meaning.

Provided that a child is interacting with a (mathematical) tool to solve a mathematical task, at the initial stage of the child's encounter with the mathematical tool, the child may not necessarily perceive any mathematical meaning in his/her working with the tool. As this stage, his/her perception can be presented by the object/meaning; that is, when the physical properties of the object, as a thing that affords some actions, is dominant and the possible mathematical meaning is not yet formed for the child. This is what I call object-then-meaning (object/meaning). This figure of perception expresses how the child perceives the different properties of the tool (i.e., the meaning of the tool as an object in relation to the task at hand) as well as how the child perceives possible mathematics represented by the tool (i.e., the mathematical meaning). At this stage, the tool is dominant and meaning(s)—as an object or its mathematical meaning—is subordinate. Hence, this is the stage at which the physical properties of the mathematical tool play an important role in relation to the child's interaction with them, both to perceive the affordances provided by the tool and to perceive any mathematics represented by the tool. For example, in fraction circles, the relationship between the sizes of the pieces plays an important role in how the child perceives the affordances of the fraction circles and in relation to making sense of the concept of fractions. The physical properties of fraction circles could be perceived as useful for making castles. This implies that meanings (mathematical or otherwise) are brought forth only in relation, by the child, through interactions, while thinking about or working on a mathematical task.

To make mathematical meaning in the interaction with the tool, that is, for a tool to be used as a symbol (a sign) for a mathematical concept, the child needs to increasingly tie signs to their use of the tool. Children do this by talking about what they do, talking about the tasks, drawing, and/or using mathematical symbols. By attaching mathematical signs to their interaction with the tools, the child perceives the tool (object-then-meaning) as a mathematical tool (meaning-then-object). For example, this is when a child calls a half-sized-circle in the fraction circle a "1/2." For this child the wooden (or plastic) piece is not just a cut circle—it means "half," in relation to both the full size of the circle and to the child's perception of such half-ness (Fig. 15.1).

Fig. 15.1 Fraction circle and "half"

It is in the *gradual* process of inverting the object/meaning to a meaning/object, that children grasp the interrelationship between the physical structure of mathematical tools and the meaning of the mathematical concepts that they are intended to represent. I emphasize the word "gradual" because it is often difficult for children to grasp the relationship between tools and the mathematical concepts (e.g., Rabardel and Samucay 2001).

In this chapter, I open up a discussion to show how converting the object/meaning to meaning/object provides the children with the possibility of attaching mathematical meaning to the tools, as they talk, work and act to make the invisible mathematics more visible. Such attachment of mathematical meaning to the tool is a result of the relation and interaction of the child and the tool. A different child might perceive a different mathematical meaning in the same tool. As I explore examples of children's interaction with the tool, I discuss how this attachment, in turn, will afford the children the possibility of becoming acquainted with the newer form of action, reflection and expression, in relation to the tools and to the mathematical concept of the addition of fractions.

Now, to make my points more tangible, I turn to an example. This example is a fragment of a larger data set I collected from a private school in Ottawa, Canada, where 13 children, 12–13 years old, in teams of two, interacted with different tools to solve a series of addition of fractions problems, in different ways. The tools included fractions strips, Cuisenaire Rods, rulers, ribbons, adhesive tape and pieces of paper and cardboard. Children's interactions with the tools were video recorded and transcribed for further analysis. To look at children's interaction with the tools, I made two categories: (1) What children said; and (2) What children did [which were typed in brackets and in italics respectively]. For further clarity, I included screen shots of children's actions. In what follows, I do not display the findings of my study. Instead, through an entire sequence of activities in which two children were intended to use common materials such as tape and ribbons to solve 1/8 + 2/7, I propose to think about how children's perceptions of the objects gradually changed so as to be able to attach mathematical meaning to the tools and to think about or to solve the task. Hence, I discuss a possible application of the theoretical notions of object/meaning.

15.3 J and S's Interaction with a Ruler and a Piece of Tape

In this episode, I gave J and S a wide variety of common materials such as different forms of tape, ribbons, different kinds of paper (graph paper, transparent paper, poster board) from which they needed to select elements and devise what could be perceived as tools for doing the addition tasks provided.

The series of events set out below, shows how S and J perceived the physical properties of a piece of a scotch tape and a ruler, as well as their own previous knowing of fractions, to attach mathematical meaning of addition of fractions to the tools and to solve the task. At first, when J and S started working with the tools, the physical properties of the tools played an important role in their interaction with the tools (Fig. 15.2).

At the beginning stage of this process, the physical properties of the tools played an important role in how J and S perceived the objects. For example, they stated that the red masking tape does not show the marked colors well. Instead they picked a transparent piece of scotch tape. At this stage, I say that the children's perception can be shown figuratively as object/meaning. S and J perceived different physical properties (the colors and their show-ability) of the tools to start attaching mathematical symbols to the tools. In order to think about adding the two fractions of 1/8 and 2/7, they tied mathematical signs to their interactions with the tools, as S explained that they could: "[…] do the eight […] draw sevenths […] then we can color in the eights and the sevenths then we'll see how much that takes." The

J: and we can only use these tools?

Me: yes, these are the tools

S: so what we could do … for example get the piece of tape *[S pickes up the red masking tape]* and do the eight. For example I mean we can draw one color for example the sevenths we will do them in blue and then the eights we will do them in… pink… then we can color in the eighths and the sevenths and then we'll see how much that takes

S: I don't think it would show (the colors) well on red. Do you have any different color [tape]?

I think that would show better if you put it on… *[takes a piece of transparent scotch tape].*

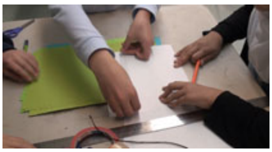

Fig. 15.2 Physical tools and properties

children's interaction with the tools also provided them the possibility of expressing, acting and reflecting about the addition of fractions in a newer form. That is, to color the tape and "see how much that takes."

Over the timespan of a few minutes, the children, unsure of how they could think about adding 1/8 and 2/7, using the piece of scotch tape as a symbol, expressed a series of questions. The following are some of these questions:

01:22	J:	make it a 1 cm each?
01:45	J:	so what should the whole be?
02:27	S:	how long is this?

Using the physical properties of a ruler as well as her own mathematical knowing, S explained that 24 is the right length to represent the whole unit. She spoke as follows (Fig. 15.3):

In this part of the interactions, children used the physical properties and the pieces of tape to form the mathematical meaning of "whole unit" in their interaction with the pieces of tape. At this stage the properties of the tape were subordinate and the meaning of "whole" was becoming dominant as the children talked about and thought about fractions in their interactions with the tool. At this point, for the children, this piece of tape was a sign/symbol for 24. At this stage, the children's perception can be figuratively expressed as meaning/object. That is to them, the piece of tape is a symbol of one whole.

As the interaction unfolded, the children attempted to attach mathematical meanings of 1/8 and 2/7 to the pieces of tape, as they measured, marked, and colored the tape. The following excerpt of transcript shows how they tied signs to the tool, by talking about fractions and acting, as they used the tool to work on the task.

S: we need to cut it up into eight right?... twenty four

J: twenty four

S: so three three three right? Three times eight is twenty four right?

J: yeah… I am pretty sure

[the children picked up the ruler and started marking the tape every 3cm]

Fig. 15.3 Physical properties and mathematical knowing

J:	[*every 3 cm puts a dot on the tape*]
S:	like that… *[she extends the dots to a full line]*
S:	then the seventh… can 24 be divided into seven?
[4 seconds pause]	
S:	I am pretty sure it's decimal …
S:	*[they worked out 24 divided by 7 on a piece of paper and got 3.4 for an answer]* we will put three point four [*S marks with the green marker*] … so this is the seventh like that… oh God this is going to be difficult.

In this part of the interaction the children were attempting to invert the object/meaning to the meaning/object by attaching mathematical meanings of 1/7 and 1/8 to the tape, using the ruler. Making a line of 1/8ths was relatively easy, as they marked the tape every 3 cm.

Yet, the interaction became complex as they attempted to attach the meaning of 1/7 to a whole unit that was 24 cm long, as they asked: "can 24 be divided into seven?" On a separate piece of paper, they calculated 24 divided by seven, to be able to attach a mathematical meaning of 1/7 to the tape. By tying the mathematical symbol of 3.4 to the piece of tape, S created an artifact that meant 1/7 to her, she said: "we will put three point four… so this is the seventh like that." Yet, she found making sevenths to be a difficulty task, if the whole unit is 24 cm; she exclaimed: "oh God this is going to be difficult." And S continued marking the tape, every 3.4 cm.

Again, at this stage, children's interactions with the tools provided them with the means to express fractions in a different form. In her relation with the tape and mathematics, for S, a seventh, was a point that was 3.4 cm away from the beginning of the tape as she said: "we will put three point four [*S marks with the green marker*]".

By marking the tape every 3 cm and every 3.4 cm, S and J made a line segment of 1/7 and a line segment of 1/8. Later, they colored 1 section of 1/8 segment and 2 pieces of 1/7. Now, to the children the artifact that was displayed before them was not mere tape on a piece of paper. By inverting the object/meaning to meaning/object, the children attached the mathematical meaning of a whole, a 1/7 and a 1/8 to the piece of tape (Fig. 15.4).

In the interactions that follow, I show how S and J used the artifact that they created to think about adding 1/8 and 2/7. A tension arose, as S was not sure how to

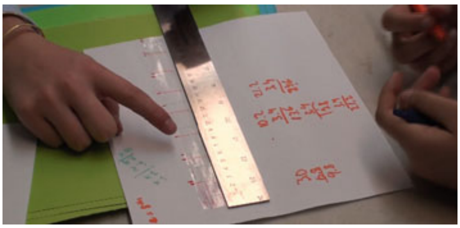

Fig. 15.4 The artifact created meaning for 1/7 and 1/8

proceed, but J was able to perceive the properties of the tool, in relation to adding two fractions. The following excerpt of transcript shows S and J's conversation:

S:	So that we can see how far it is…
J:	Wait
S:	this is not going to work
J:	this is three this is a 3 cm [*pointing to the first colored part*], and this is six point eight
[3 sec pause]	
S:	I think it… might not work this way… I don't
[3 sec pause].	

In this interaction, the physical properties of the created tool were not perceived by S, in such a way as to help her form the mathematical meaning of the addition of fractions to the tool. It was not easy for S to invert the object/meaning to meaning/object, as shown by utterances such as: "this is not going to work" or "might not work this way… I don't.". In contrast, J was able to perceive properties of the created tool to attach mathematical meaning of the addition of fractions to the created tool. She started by attaching mathematical signs to her interaction with the tool, as she said: "this is three… this is a three centimeter *[pointing to the first colored section of the 1/8 line]*, and this is six point eight *[pointing to the first colored section of 1/7 line]*."

After a pause of 3 s, S asked J: "What did you just do?". The interactions that followed show how S and J attached the mathematical meaning of 2/7 + 1/8 to the artifact they created and how they solved the task. The following piece of transcript show this interaction:

S:	what did you just do?
J:	I added these two together [*pointing to the coloured parts of the tape that showed 2/7 and 1/8]* … nine point eight
S:	nine point eight …
[2 sec pause]	
S:	what is it in fractions? that is what we need to know… out of ten…or…
J:	wait.. this is out of twenty four so… nine point eight but you cannot have decimals in fractions
S:	You can you can
J:	oh okay so I guess it is nine point eight out of twenty four.

This part of the interaction shows how J and S's perception of the object gradually changed to perceiving the properties of the artifact to solve 2/7 + 1/8. Again, at the beginning of this interaction, the children used the physical properties of the artifact to think about the task at hand, as J, *pointing to the coloured parts of the tape that showed 2/7 and 1/8,* stated: "I added these two together … nine point eight". Here the interaction reached a point of tension. S's not-so-clear perception made her doubt the usefulness of what J had perceived; using her previous knowing of fractions; S expected a fractional amount as a solution for the addition of two fractions, so she asked: “What is it in fractions? That is what we need to know… out of ten…or…”.

J perceived the physical properties of the object and stated that since the length of the whole tape is 24 cm, so the answer might be nine point eight out of twenty four, as evidenced by her words: “This is out of twenty four so… nine point eight”. The second tension was reached here, as J doubted the possibility of having a decimal in fractions. S considered it to be possible, so their answer was 1/8 + 2/7 = 9.8/24

The interaction of J and S with the created artifact indicated the existence of a complex relationship between the mathematical knowing of the children on the one hand, and their perceptions (and the changes in their perceptions) of the mathematical affordances of the created artifact on the other hand. These knowing(s) and perception(s) were mediated by gradual changes in their perceptions from object/meaning to meaning/object over and over again.

Such changes in the perceptions of the children in relation to a piece of tape, afforded them a newer way of reflecting on and expressing the addition of two fractions. Hence

$$
\begin{aligned}
&2/7 + 1/8 \\
&= 9.8/24
\end{aligned}
$$

It was the interrelationships among the sizes of the parts within the artifact that guided them through this newer form of reflecting, talking and acting. Even though it was not entirely clear for S, J perceived the relationship between the size of the entire tape (24 cm) and the sizes of the colored pieces (3 cm + 3.4 cm + 3.4 cm) as constituting feedback from the created artifact that was useful for solving the task. In this case 2/7 + 1/8 was not 23/56, as it would be according to the children’s previously known steps of adding two fractions, that is:

$$
\begin{aligned}
&1/8 + 2/7 \\
&= 7/56 + 16/56 \\
&= 23/56
\end{aligned}
$$

Instead 1/8 + 2/7 = 9.8/24 (= 23/56), because the children have colored 1 part of 3 cm and two parts of 3.4 cm from an entire tape which was 24 cm. Therefore:

$$
\begin{aligned}
&1/8 = 3\text{cm} \\
&2/7 = 3.4\,\text{cm} + 3.4\,\text{cm} = 6.8\,\text{cm} \\
&\text{The whole tape was } 24\,\text{cm. Hence:} \\
&1/8 + 2/7 = (3 + 3.4 + 3.4)/24 \\
&= 9.8/24
\end{aligned}
$$

This new approach to reflecting and expressing the addition of two fractions emerged as the children participated in interaction with the tools and their created artifacts, along the way.

15.4 Discussion

Through the example of J and S in interaction with the tools, in this chapter I utilized Vygotsky's view of *perception of real objects* (Vygotsky 1978) to systematically examine and make sense of *children's change of perception* and meaning making, as they worked with mathematical tools to think about and/or work on 1/8 + 2/7. As I mentioned, Vygotsky elaborated on the discussion of 'children's perception' through what he referred to as the object/meaning. He also saw meaning-making as being formed in collective relationships. In the step-by-step analysis of S and J's interaction with a piece of tape and a ruler, I showed that the children constantly interacted with the mathematical tools to think about adding two fractions. Following Vygotsky's notion of the object/meaning, I noted that, during the initial stages of their encounter with a mathematical tool, the girls' perception of the tool could be presented using the ratio of object/meaning. This figure of perception represents how the children perceived the physical properties of the tools—that is, the meaning of the tools as objects, respective to the task at hand. At this stage, the tools and their physical properties were dominant and their meanings—as an object or as the mathematical meaning—were subordinate. That is, a tool can be represented by object/meaning.

$$\text{Tool then meaning as} \frac{\text{object}}{\text{meaning}}$$

Hence, this is the stage when the physical properties of the mathematical tools (e.g., the 1 cm-increment of the line segments on the ruler) played an important role in the children's interaction with them, both to perceive the affordances provided by the tools and to form their mathematical meaning(s). In order to invert this ratio—that is, in order for the tools to be used as a symbol (a sign) for a mathematical concept—the children needed increasingly to tie the signs to their use of the tool. I also illustrated that in the analysis the children at times were able to use the physical properties of the tools and their own knowing of fractions to grow and form mathematical meanings for the tools. Hence they accomplished the inversion of object/meaning to meaning/object, where the mathematical meaning was dominant and the physical properties of the tools were subordinate.

$$\text{Meaning then tool as } \frac{\text{meaning}}{\text{object}}$$

Meaning was formed in relation to the girls' interaction with the tool as the children increasingly tied signs to their use of the tool. The children did this by talking about what they did, talking about the tasks, and using mathematical symbols. It is in the gradual relational process of inverting the object/meaning to a meaning/object that children grasp the interrelationship between the affordances of mathematical tools and the meaning of the mathematical concepts that they are intended to represent.

Throughout my analysis of J and S's interaction with the tools, I tried to highlight that the processes of perceiving affordances of the tools and forming meaning in relation to interactions with the tools were complex and gradual, and highly related to the knowing of the child and properties of the tools. What I wrote about in terms of the gradual change of the perception of children as they interacted with tools, and the complicity of such perceptual change in the children's learning of fractions is grounded in the work of many scholars. The effectiveness of mathematical tools has been conceptualized from a variety of angles, including their role in the following: in fostering children's mathematical learning; in reconstruction of mathematical tasks; and in promoting communication and social interactions (Cramer and Henry 2002; Misquitta 2011; Mendiburo and Hasselbring 2011; DeCastro 2008; Mills 2011).

Although mathematical tools are conceptualized as being useful in mathematics classrooms, they have strengths and limitations (Pirie and Kieren 1994; Pimm 2002). At times, in their interactions with the mathematical tools, children encounter difficulties in grasping the relationship between mathematical tools and the mathematical meanings that they are intended to represent (Norman 1993; McNeil and Uttal 2009). McNeil and Fyfe (2012) provided an account of the difficulties children experience in grasping the relationship between mathematical tools and the mathematical concepts that they are intended to represent. They explained that the principle of the dual-representation hypothesis is that any concrete symbol (e.g., a mathematical tool) "can be thought of in two different ways: (a) as an object in its own right and (b) as a representation of something else" (p. 43). That is, the usefulness (or lack of it) of a mathematical tool depends not only on its physical properties and affordances, but also on the child's perceptions while interacting with the tool. A tool that is perceived to be useful for a child in working on/solving a mathematical task might not be perceived as useful for another child or for the same child in another task.

My discussion in this chapter supports and extends the work and findings of other scholars as regards how mathematical meanings are not represented by the tool; rather, different children perceived different mathematical meanings while using the same tool to work on the same task (e.g., Pimm 2002). I further provided a rational basis on which to examine how children form and advance (different) mathematical meanings in their interactions with tools. With respect to the theoretical notion of the inversion of the object/meaning ratio into the meaning/object ratio, I showed how different children perceived the affordances of the tools differently through the use of physical properties of the tools and how, based on their perception of the affordances, they constructed different mathematical meanings in relation to the tools and in relation to the task of adding two fractions.

Another point I pursued here is that in children's interactions with the tools, the children's gradual changes of perception provided the children with newer forms of expression, reflection and action; by uttering statements such as: "See how much that takes", or "We will put three point four… so this is the seventh." In the interactions of the children with the tape, it is important to acknowledge that there is nothing inherently '2/7' about the tape. Rather, it was through a process—doing things with the tape and ruler and thinking about fractions—that the children, at one point, perceived some '2/7-ness' in the tape. As I explained in detail above, at the point at which the children first perceived 1/7 in the tape, that piece of tape ceased to be a mere piece of plastic suitable for sticking things together, but henceforth was '1/7'. Seeing a 1/7-ness of the tape accordingly changed what the children did with the tape (their actions) and what they said (their expressions and reflections). These new forms of expression and action were made possible as the children interacted with the tape and attached mathematical meaning to their interaction with the tape. Now, following Roth and Radford's (2010) conceptualization of learning (i.e., a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection), I could refer to what happened here as 'learning'. Looking at the example above, I say, S and J learnt that 2/7 + 1/8 is not only 16/56 + 7/56 = 23/56 but also (3 + 6.8)/24 = 9.8/24.

What I would like to propose is that, in the process of perceiving, the children also become acquainted with new forms of expression, reflection and action. The process of perceiving could be looked at as one within which a child starts by seeing the physical properties of the artifact (as an object of its own), gradually perceives its affordances in relation to the task, and then attaches a mathematical meaning to the use of the artifact. Hence, much like learning, the process of perceiving is also a complex and gradual process within which newer forms of expression, reflection and action become available to the children.

References

Cramer, K., & Henry, A. (2002). Using manipulative models to build number sense for addition of fractions. In B. Litwiller & G. Bright (Eds.), *National Council of Teachers of Mathematics 2002 Yearbook: Making sense of fractions, ratios, and proportions* (pp. 41–48). Reston, VA: National Council of Teachers of Mathematics.

DeCastro, B. V. (2008). Cognitive models: The missing link to learning fraction multiplication and division. *Asia Pacific Education Review, 9*(2), 101–112.

Eco, U. (1986). *Semiotics and the Philosophy of Language* (Vol. 398). Indiana University Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Reston, VA: NCTM.

Marx, K., & Engels, F. (1865). *The German ideology*. New York: International Publishers.

McNeil, N., & Fyfe, E. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction, 22*(6), 440–448.

McNeil, N. M., & Uttal, D. H. (2009). Rethinking the use of concrete materials in learning: Perspectives from development and education. *Child Development Perspectives, 3*(3), 137–139.

Mendiburo, M., & Hasselbring, T. (2011). *Technology's impact on fraction learning: An experimental comparison of virtual and physical manipulatives*. SREE Conference.

Mills, J. (2011). Body fractions: A physical approach to fraction learning. *APMC, 16*(2), 17–22 (University of Waikato, New Zealand).

Misquitta, R. (2011). A review of the literature: Fraction instruction for struggling learners in mathematics. *Learning Disabilities Research & Practice, 26*(2), 109–119.

Norman, D. A. (1993). *Things that make us smart: Defending human attributes in the age of the machine*. Boston, MA: Addison-Wesley Publishing.

Pimm, D. (2002). *Symbols and meanings in school mathematics*. London: Routledge.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational studies in Mathematics, 26*(2–3), 165–190.

Rabardel, P., & Samucay, R. (2001). *From artifact to instrumented-mediated learning*. New challenges to research on learning. International symposium organized by the Center for Activity Theory and Developmental Work Research, University of Helsinki, March 21–23.

Roth, W.-M., & Maheux, J.-F. (2015). The visible and the invisible: The immanence of doing mathematics and mathematics as revelation. *Educational Studies in Mathematics, 88,* 221–238.

Roth, W. M., & Radford, L. (2010). Re/thinking the zone of proximal development (symmetrically). *Mind, Culture, and Activity*, *17*(4), 292–307 (Scandinavian Press).

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York, NY: Springer.

Vygotsky, L. (1986). *Thought and language*. Cambridge, MA: MIT Press.

Vygotsky, L. L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.

Chapter 16
On the Role of Gestures for the Descriptive Analysis of 'Grundvorstellungen': A Case of Linear Functions

Christina M. Krause and Alexander Salle

Abstract In a German tradition, the concept of 'Grundvorstellungen' (GV) concerns mental models that carry the meaning of mathematical concepts or procedures. While research on these GVs is mostly based on the analysis of verbal utterances or written products, we turn towards an embodied approach and investigate the question of how gestures can take part in revealing the activation of students' GVs, focusing on the specific case of linear functions. In the presented empirical study, we found that gestures are used not only to specify deictic terms, but that they can also reveal a link between a mathematical idea and its non-mathematical grounding as it becomes explicit in speech. How gestures might contribute not only to the descriptive analysis of GVs, but also to the mathematical learning process, is discussed against the background of re-analyzing selected results within two different semiotic approaches.

Keywords Grundvorstellungen · GV · Gestures · Semiotics · Linear functions

16.1 Introduction

'Grundvorstellungen' (GV) are mental models that carry "the meaning of mathematical concepts or procedures" (vom Hofe et al. 2005, p. 68). By activating GVs, students are able to manipulate and translate between various mathematical representations, as well as to model real processes with mathematical objects and operations (vom Hofe and Blum 2016).

C. M. Krause (✉)
Faculty of Mathematics, University of Duisburg-Essen,
Thea-Leymann-Straße 9, 45117 Essen, Germany
e-mail: christina.krause@uni-due.de

A. Salle
Department of Mathematics/Computer Science, Institute of Mathematics,
University of Osnabrück, Albrechtstr. 28a, 49076 Osnabrück, Germany
e-mail: alexander.salle@uni-osnabrueck.de

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Different aspects of GVs make them applicable for normative and descriptive analyses: "The *normative aspect* of GVs works as an *educational guideline* [and] describes possible interpretations of [the] mathematical core" (vom Hofe and Blum 2016, p. 232, emphasis in the original) of concepts or operations, whereas "the *descriptive aspect* allows mental representations of students to be characterized" (vom Hofe and Blum 2016). While students in primary school mainly develop GVs that are "based in concrete actions with real objects" (primary GVs), more abstract and complex mathematical concepts require the formation of GVs that are based on "imagined actions dealing with mathematical objects" (secondary GVs) (vom Hofe and Blum 2016, p. 234). Concerning this distinction, vom Hofe and Blum state more precisely:

> *Primary* GVs are based in concrete actions with real objects. The corresponding concepts can be, so to speak, semi-isomorphically represented by real objects and actions, for instance by joining or dividing real sets of things. Primary GVs for this reason possess a *representational character*.
>
> *Secondary* GVs are based on mathematical operations with symbolic objects. Constituents of the corresponding mathematical structures are not real actions, but rather imagined actions dealing with mathematical objects and means of representing these objects, such as number lines, terms, and function graphs. Secondary GVs for this reason are said to have a *symbolic character*. (p. 234, emphases in the original)

The development of suitable GVs of both kinds is essential for further mathematical understanding and competencies (vom Hofe and Blum 2016).

To reconstruct students' GVs, empirical investigations often use interviews that almost exclusively investigate speech (e.g. Wartha 2007; Stölting 2008; Schüler and Rösken-Winter 2014). Non-verbal modalities are seldom treated explicitly for analysis except for unsystematic clarifications of deictic terms. However, research on thinking and interacting has shown that multiple modalities, especially gestures, have great influence on the development of mental models as well as on the communication and the expression of such models (Edwards et al. 2014).

16.2 Gestures

When cognitive processes are regarded as embodied, "bodily modalities [are] integral components of mathematical thinking, teaching and learning" (Moore-Russo et al. 2014, p. 4). With this understanding of cognition and the assumption that mathematics is originated from physical experience, gesture becomes "a key element in communication and conceptualization" (Radford et al. 2009, p. 93).

Research on gesture has shown that the body and gestures take part in learning mathematics in social interaction (Arzarello et al. 2009; Dreyfus et al. 2014; Krause 2016) as well as in individual learning (Alibali and Goldin-Meadow 1993; Alibali and diRusso 1999; Goldin-Meadow et al. 2009). In social settings, students use gestures to support argumentation and reasoning, to explain mathematical concepts to each other, and for common explorations of problems (e.g., Yoon et al. 2011; Chen and Herbst 2013). In individual learning and instruction, gesture plays an important role for the understanding of mathematical concepts and instructions (Broaders et al. 2007; Goldin-Meadow 2010; Krause and Salle 2016). Important cognitive functions of gesture are specified and synthesized by the Gesture-for-Conceptualization-hypothesis by Kita et al. (2017): "Gesture activates, manipulates, packages and explores spatio-motoric representations for the purposes of speaking and thinking".

For the analysis of empirical data, we consider "idiosyncratic spontaneous movement[s] of the hands and arms accompanying speech" (McNeill 1992, p. 37), which have an expressional rather than a practical purpose (Kendon 2004). The stroke of a gesture, the actual movement, can also be preceded or followed by a holding of the gesture. Stroke and pre- and/or poststroke holds together are called a gesture phrase, the part of a gesture meaningfully assigned to one part of the corresponding spoken utterance (Kendon 2004).

Perceiving a gesture as meaningfully assigned to a spoken utterance comes along with the assumption that gesture and speech are co-expressive, they reveal "different sides of a single underlying mental process" (McNeill 1992, p. 1). Furthermore, McNeill claims that "speech and gesture must cooperate to express the person's meaning" (McNeill 1992, p. 11), "each can include something that the other leaves out" (p. 79). McNeill describes the relationship between gesture (imagery) and (spoken) language as "dialectic of unlike cognitive modes" (McNeill 2002, p. 7), being "key to the evocation, organization, and ultimate execution of meaningful actions shaped to take linguistic form in discourse" (p. 7).

The evocation concerns the idea of the *growth point* (GP) as "the theoretical starting point, in a microgenetic sense, of a speech-gesture combination—"growth" in the sense that it is the seed out of which speech and gesture grows" (McNeill, cited in Montredon et al. 2008, p. 173), or, in more simple terms, "a specific starting point for a thought" (McNeill 2002, p. 20). A growth point can be revealed in the first occurrence of a gesture, referring to a feature that can be traced as recurring in the following discourse. If this feature recurs in other gestures, it gives rise to what McNeill calls a *catchment* (McNeill et al. 2001; McNeill 2002).

A catchment is recognized when one or more gesture features recur in at least two (not necessarily consecutive) gestures. The logic is that the recurrence of an image in the speaker's thinking will generate recurrent gesture features. Recurrent images suggest a common discourse theme. In other words, a discourse theme will produce gestures with recurring features. These gesture features can be detected. Then, working backwards, the recurring features offer clues to the cohesive linkages in the text with which it co-occurs. A catchment is a kind of thread of visuospatial imagery that runs through a discourse to reveal the larger discourse units that emerge out of otherwise separate parts (McNeill 2002, pp. 26–27).

While the growth point marks the emergence of a significant differentiation arising within a certain context, catchments allow the identification of ideas that are related in context and provide "a gesture-based window into discourse-cohesion"

(McNeill 2002, p. 27). Catchments therefore concern the abovementioned organization of meaningful actions within the gesture-language-dialectic. The ultimate execution reveals itself in the verbal manifestation of the idea–or thought–that has been evoked and organized by means of gesture imagery; this process is recognized when the growth point is unpacked, in terms of the dialectic of gesture and imagery.

16.3 Semiotic Perspectives on Gesture

The gestures we are considering can be perceived as signs with respect to two different semiotic traditions—those of Vygotsky and Peirce.

Following *Vygotsky*, gestures as signs may function as psychological tools, "as external or material means of regulation and self-control" (Presmeg et al. 2016, p. 11). In that regard, they can "act as an instrument of psychological activity in a manner analogous to the role of a tool in labor" (Vygotsky 1978/1931, p. 52) and may therefore have influence on the formation and activation of the GVs as mental models. The main difference between signs—and therefore also gestures—as psychological tools and technical tools is described by Vygotsky (1978/1931, p. 55), and also in the following:

> [T]he most essential feature distinguishing the psychological tool from the technical tool is that it directs the mind and behavior whereas the technical tool, which is also inserted as an intermediate link between human activity and the external object, is directed toward producing one or another set of changes in the object itself. The psychological tool changes nothing in the object. It is a means of influencing oneself (or another)—of influencing the mind or behavior; it is not a means of influencing an object. Therefore, an instrumental act results in activation in relation to oneself, not in relation to an object. (Vygotsky 1981, p. 140)

Presmeg et al. (2016) deduce further that "by becoming included in the children's activities, they [the signs] *alter* the way children come to know about the world and about themselves" (p. 11, italics in the original). They can work as semiotic mediators in the process of internalizing, "the internal reconstruction of an external operation" (Vygotsky 1978/1931, p. 56).

Within the framework of Vygotsky, McNeill (2002) considers a growth point to be a psychological unit, "the smallest component that retains the property of being a whole" (p. 14). It needs, however, to be seen against the background of a context from which it differentiates in an instance of instability, provoking "cognitive movement that seeks repose in the form of intuitively complete linguistic form" (p. 21). Within the gesture-speech dialectic model, the growth point as psychological unit already contains the seed for a thought that, however, does not come into existence as such until it is expressed in words (McNeill 2002, p. 14, referring to Vygotsky 1986), in other word, until the growth point is unpacked. A catchment can then contribute to the elaboration of the thought emanating from the initial seed: "A key process is the generation of new meanings during the unpacking process"

(McNeill 2002, p. 33), this new meaning possibly arising from incorporating contexts in a catchment, traceable through the use of recurring gestures.

In contrast to Vygotsky, *Peirce* integrates a representational view of signs. For him, a "sign, or representamen, is something which stands to somebody for something in some respect or capacity" (Peirce CP, 2.228). A sign can thus be considered everything that is somehow interpreted by somebody in a certain way. While being interpreted, it "creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign" (CP, 2.228). Furthermore, "the sign stands for something, its object" (CP, 2.228), represented in the interpretant as interpreting the sign against a certain background and evoking the interpretant thereby. The sign can thereby be seen as bound in a triadic relationship with its object and its interpretant (Fig. 16.1).

This interpretant, as a new sign—being external or internal—stands again for the object and creates a new interpretant, this process—called semiosis—potentially going on ad infinitum. Two kinds of objects can be distinguished; the "immediate object" (Hoffmann 2005, p. 51) represented by a sign in its relationship with the interpretant, and the "dynamic object" (p. 51) as it stands in the hypothetical end of the infinite process of semiosis. Hookway (1985) explains the difference between the immediate and the dynamic object as "two answers to the question: what object does this sign refer to? One [the immediate object] is the answer that could be given when the sign was used; and the other one [the dynamic object] we could give when our scientific knowledge is complete" (p. 139). Obviously, the dynamic object is not reachable, it moreover represents the ideal, "complete" forming of the object. Within this semiotic perspective, we hypothesize on the students' GVs by interpreting the students' utterances as signs, reconstructing the immediate mathematical objects from these utterances.

Based on Bikner-Ahsbahs (2006) work, Krause (2016) used a similar approach to reconstruct the development of mathematical objects within social processes of constructing mathematical knowledge (pp. 58–67). Focusing on the interplay between gesture, speech and inscription in collaborative processes of learning mathematics, she widened the idea of the sign as an utterance to understand it in a multimodal way, as a "semiotic composition", in which different semiotic resources may play together (Fig. 16.2).

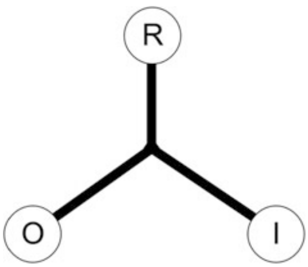

Fig. 16.1 A Peircean understanding of signs as a triadic relation between representamen (R), object (O), and interpretant (I) (Krause 2016, p. 20)

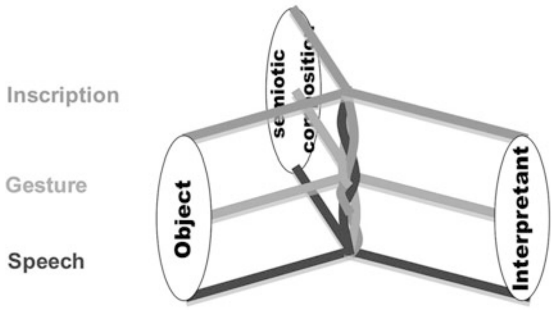

Fig. 16.2 Multimodal sign shaped by the synchronous relationship between the three semiotic sets speech, gesture, and inscription. The "semiotic composition" forms the representamen of the multimodal sign (Krause 2016, p. 50)

16.4 Research Interest

Although the co-expressivity of speech and gesture might be considered implicitly when reconstructing the activation of GV, there has not yet been a systematic integration of gestures in previous analytical approaches. Therefore, this chapter aims at investigating the role of co-speech gestures for descriptive analyses of GVs, focusing on the field of linear functions.

Furthermore, we look beyond the analytical contribution of gestures, aiming at getting more profound insights into the role of gestures for the individual construction of GVs. For this purpose, we discuss the results in the light of the two different semiotic approaches, those of both Vygotsky and Peirce, exploring the influences of gestures on the formation of mental models.

16.5 Methods of the Study

Data were gathered by videotaping German grade 9 students solving the following task, dealing with mobile phone contracts. Further, an interviewer asked questions about their solutions. The first part of the task provides a function in tabular form (Fig. 16.3) and is phrased as follows:

> The following table presents the conditions of two mobile phone contracts. The charges are invoiced per hour. A friend of yours needs advice which contract he should choose. Give him reasons to choose one over the other.

	E-Online	Online-Pro
Monthly basic fee	7 €	3 €
Costs per hour	1 €	2 €

Fig. 16.3 Table with information about mobile phone contracts

For the second part of the task, a graphical representation is given (Fig. 16.4). Based on these data, two offers for contracts should be formulated:

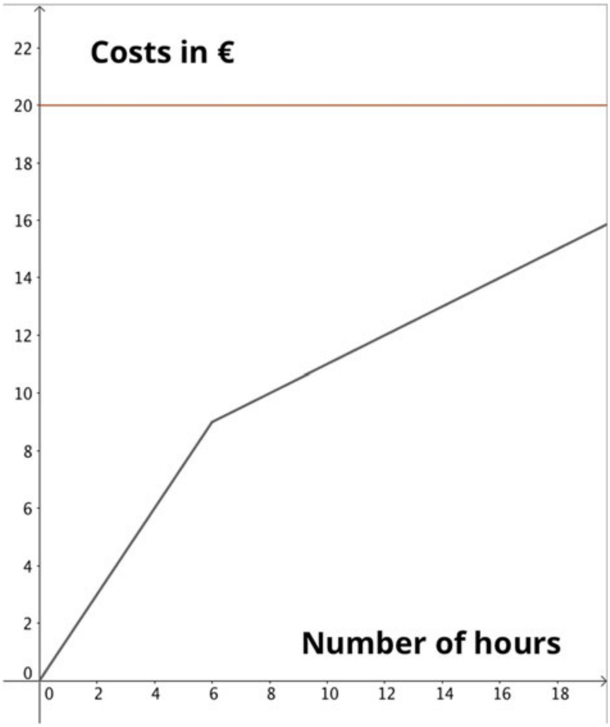

Fig. 16.4 Coordinate system with graphs, given for the second part of the task

Here you see two alternate contracts. How could the corresponding offers be formulated?

The task has been designed to prompt the activation of three GVs that are characteristic for working with real linear (as well as other elementary) functions (Stölting 2008; Vollrath 1989).

> Idea of static mapping: This GV takes into view how x- and y-values are assigned to each other (e.g. well-definedness of a function). It relies on local features of the underlying function.
>
> Idea of varying quantities: This GV describes the dependencies of changes of the ordinate and the abscissa. Basic ideas of different behaviors of functions when "co-varying" their x- and y-values allow learners to describe "what happens, when x increases …". This GV also relies on local features of the underlying function.
>
> Idea of function as an object: This GV regards functions as a whole. Local features from i) and ii) are encapsulated in an object with global features that can be manipulated further. (Dubinsky and Harel 1992)

The analysis was conducted directly from the video data. It was gesture-driven, that is the video was stopped when a gesture occurred. The gestures were interpreted within developing semiotic bundles (Arzarello 2006), taking into account the gestures' relationships to other signs that are used simultaneously or earlier in the interview. Speech, as is the case in most conventionalized modes of expression, provides the context in which the gesture is interpreted, possibly referring to an inscription. Furthermore, the gesture may also be reminiscent of other signs by evoking an iconic similarity. That way, they may become associated with an inscription or a formerly used gesture (Krause 2016).

The interpretation of a gesture within the semiotic bundle is used to decide whether and in which way it may refer to a GV. For the purpose of this study, only those scenes are relevant in which gesture plays a part in revealing the activation of a GV. These scenes were transcribed, taking into account verbal and non-verbal modes of expression. For integrating gestures' co-timing to speech, we use square brackets with parentheses, nested to indicate when several gesture phrases are linked in movement.

16.6 Results: The Case of Victor

In order to solve the first task, Victor created a coordinate system and inserted linear graphs for each of the two contracts. Asked for his final advice, he stated the following.

Scene "In the long run"

01 V That it, well, [pays off, so these are the two equations] (first, points at the first and then to the second graph),

02 so, if one browses less than nine hours, one should rather choose the [second one] (points at the third column of the table),

03 so, chooses the online-pro contract, if one, well, a bit longer, then [the e-online] (points at the second column of the table).

04 because [it then, well, [in the long run]], (first, places his pen at the intersection point of the two graphs, then moves as shown in Fig. 16.5) is more reasonable

In his statement, Victor distinguishes between the two contracts: Up to nine hours, he prefers the "Online pro"-offer (02); when browsing "a bit longer", he would choose "e-online" (03). While formulating his answer, his pointing gestures directed to the graphs and the columns of the table illustrate what he refers to in his speech (01-03). In line 04, he justifies his choice of "e-online" for "the long run" while performing a more complex gesture. His verbal utterances do not hint at activated GVs. Victor formulates his statement in the real-life context in which the task is set, rather than connected to the mathematics he used to solve the task. However, his gesture contributes to the analysis in two ways:

First, he points with his pen to the intersection of the two graphs he has drawn (first stroke). This links his words "it then, well," to the graphical representation of the linear function by indicating the changing condition in the point of intersection. Then, co-timed to the words "in the long run", he moves his pen along an imaginary prolongation of the graph he has labeled with "e-Online" (second stroke). His gestures give hints for two possible interpretations as based on the graphical representation: Victor visually links the contract "e-online" with the object of the graph and identifies the core, global features important for the comparison of the two offers as they become visible in the graphical representation—the linear course of the graphs "in the long run". At the same time, these words and the moving pen give rise to the interpretation that Victor focuses on the interdependence of abscissa and ordinate.

It is the gesture that reveals (a) that he is working out his idea based on the graphical representation, and (b) that his activity is related to the whole graph as object and the variation of quantities "in the long run" revealing possible activations of the idea of function as an object and/or the idea of varying quantities.

Fig. 16.5 Starting point and movement of Victor's gesture in line 4 as embedded within the coordinate system

Scene "Straight line"

Victor is now working on the second task and tries to formulate an advertising offer for a contract depicted by a graph. He concentrates on the lower graph, that consists of two linear parts (in the following we refer to these parts as $f1(x) = 1.5x$, with $x \in [0, 6]$ and $f2(x) = 0.5x + 6$, with $x > 6$) meeting at $P(6, 9)$ (see Fig. 16.4).

11 V And, concerning the second, [one can see that the (moves his index finger along the x-axis between the values $x = 4$ and $x = 10$ back and forth (Fig. 16.6), then points at the value $x = 6$)

12 [first six hours], ehm (positions his hand onto the graph, places his thumb near the origin and his index finger near the kink of the graph)

13 [are more expensive], [but if one, a bit longer, that means after six hours, it always gets more (retained his hand posture unchanged and moves it above the right half of the coordinate system, then positions it onto the graph with this thumb located near the kink and the index finger at the right end of the graph (Fig. 16.7), then moving his hand a little to the right along the graph),

14 [then it gets cheaper]]]

15 I How does one know that?

16 V Ehm, [here, because of this (sets his index finger onto the kink) [kink (moves his index finger to the right along the graph and then back to the kink)

17 I Okay.]

18 V [straight line]] (taps two times on the kink, then moves his index finger to the right along the graph; see Fig. 16.8)

19 Well this means, [[at first, here the first six hours (grabs his triangle ruler, moves one of its tips along the first part of the graph between origin and kink)

20 I Mmh.

21 V The line, it runs steeper here]

22 I Mmh.

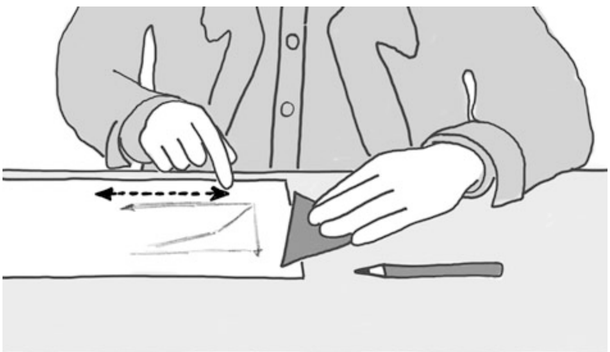

Fig. 16.6 Tracing along the x-axis (line 11)

Fig. 16.7 "Grabbing" the segment with the "Δ-gesture" (Sabena 2007): thumb and index finger positions as if holding a little stick (12–13)

Fig. 16.8 Tracing a "straight line" (18)

23 V (moves the tip of the triangle ruler along the graph between the kink to the right) [than this one]]

In lines 11–14, Victor interprets the lower graph and its characteristics with regard to the given situation, clarifying that by tracing along the x-axis (11, see Fig. 16.6). Following this, he points at the x-axis beneath the kink at P(6, 9) and claims that the "first six hours" (12) "are more expensive" with respect to "after six hours", where "it always gets more, then it gets cheaper" (13–14). There are several ways of interpreting his statement with respect to the observation he expresses in the context of the tasks:

The average costs per hour—with regard to all passed hours—decrease over time (for $x = 8$ the average costs per hour are $f(8)/8 = 1.25$ €).

> In comparison to a contract with costs calculated by f1(x) also for x > 6, the actual costs decrease. That is, the difference f1(x) – f2(x) grows with increasing x starting from x = 6.
>
> The smaller slope of f2(x) means that the costs for every added hour are smaller than the costs for every added hour of the first six hours.

Victor considers two aspects in the course of formulating an answer to the interviewer's question "How does one know that?" (15): first the kink (16), and later the slopes of the two linear segments (18–23). Both reasons support the third interpretation as the most likely interpretation of the first four lines.

The first part of his answer solely consists of "because of this kink" (16), giving no further explanation why this justifies his claim made in lines 11–14. However, he seemingly wants to extend his approach: When the interviewer takes the turn with an "okay," which may be understood as her being satisfied with his answer, Victor interrupts with mentioning a "straight line" (18).

This interjection reveals the value of considering gestures in the analysis not only for the identification of possibly activated GVs, but also for the interpretation of the verbal utterance more generally: In the original video, Victor says "gerade" (18). The translation of this word heavily influences the interpretation of this passage, so we list different possibilities to give the reader a chance to reconstruct the interaction. Possible English translations, that make sense in the context, are: "even" (a property of numbers), "just" (in the temporal sense of now), "just" (in the sense of exactly), "straight" (as an adjective), and "straight line" (as a noun). The interpretation of "gerade" as "straight line" as the favored reading is mainly based on Victor's gestures within the specific context: His index finger moves along the straight linear graph (see Fig. 16.8, right) already while uttering "kink". The movement thus seems to extend the aforementioned reference to the kink visually, reflecting a *growth point* of the idea that started to be formed verbally—the idea of formulating the justification using the slope as characteristic of the function seen as straight line.

This idea becomes more and more elaborated in the following lines (19–23), when Victor reformulates his answer given in lines 11–14 in terms of comparing the slopes of "the lines". Being verbally imprecise, the reference to the graphs of f1 and f2 becomes clarified by his co-timed indication of the linear segments using the tip of the triangle ruler. By this means, he anchors the reference to the two functions in the graphical representation.

An analysis of the verbal statements gives hints of GVs. In lines 13–14, the passage "but if one, a bit longer, that means after six hours, it always gets more, then it gets cheaper" reflects an idea of varying quantities (change of ordinate with a changing abscissa). During the identification of the kink (16), Victor may refer to the idea of static mapping, but in the verbal transcript of this scene, no other hints could be found that confirm this interpretation. His following explanation (19–23) is seen as referring to the idea of function as an object (considering characterizing global features of the function), especially the comparison of the slopes, by which he relates two features of the functions to justify his statement. It seems as if Victor draws on different GVs and does not commit to only one.

Taking into account gestures allowed us to identify (i) additional GVs with respect to those identified from the verbal expression and to identify (ii) GVs prior to being expressed verbally:

(i) In lines 11–14, Victor starts a first approach to interpret the graph. While his verbal utterance refers only to the idea of varying quantities (13/14, see above) his gestures reveal aspects of the idea of static mapping and the idea of function as an object. First, he points at (6, 0) and identifies this abscissa as important—matching his words "first six hours". While the verbal utterance refers to an interval, the gesture indicates the specific point on the x-axis that marks the end of this interval. Right then, Victor "grabs" the segment f1 with thumb and index finger (Fig. 16.7, center), the hand being shaped similarly to what Sabena (2007) called the Δ-gesture, which emerged in the activity on slopes in the context of derivatives (Sabena 2007, pp. 187–188). While he formulates the change of the costs ("[but if one, a bit longer, that means after six hours, [it always gets more, then it gets cheaper]]", 13/14), he retains his hand posture but changes the location, first "grasping" f2 with the Δ-gesture before slightly moving his hand to the right, along and beyond the graphical representation of the straight line.

The first occurrence of the Δ-gesture in line 12 reveals the growth point in which the idea emerges of considering the slope of the function as a novel aspect, important towards the solving of the task. This aspect becomes further elaborated in lines 13 and 14: when "grasping" the two segments, first f1 and then f2, Victor supports his verbally uttered idea visually by comparing the two functions by means of the global property of their slopes. With this, he treats the respective functions as objects. While his verbal expression in lines 12–14 refers merely to the context of the task—using formulations like "more expensive" or "it gets cheaper"—the gestures anticipate the justification as grounded in properties of the functions to the spoken utterance in lines 19–21, now also referring to the steepness of the line in words.

A catchment becomes recognizable in the recurring Δ-gesture in lines 12–14, reflecting the recurrence of the property of slope of the functions. In this, the slope of f1–the graph corresponding to "the first six hours" (12)—becomes then linked to the one of f2 ("then", line 14) by comparing the steepness of the corresponding segments of the graph. The recurrence of the gesture marks the recurring reference to the slope and therefore reveals the graphical comparison of the two-line segments. The global property of the functions, their slopes, and therefore the activation of the idea of function as an object, becomes then explicated in speech; the growth point is unpacked.

In addition, his slight hand movement to the right may support the idea of varying quantities as reflected in speech. It may indicate that he is aware of the function not ending where the graphical representation ends, but that "it gets cheaper" also beyond. This movement could be a reaction to the right open interval and the resulting impossibility of covering the second part of the graph. Consequently, Victor indicates the continuation (and "extends" his grasping) with this movement. Expressing two ideas in one gesture phrase, the gesture reveals the intertwining of the two ideas of function as an object and varying quantities.

(ii) When the interviewer asks for further explanation (15), Victor mentions the kink as reason for his further observations. As has been described before, he traces along the second part of the graph, namely f2, giving a preview of his upcoming verbal reference to a "straight line" (18).

So, grasping the segments by means of the Δ-gesture in lines 12–14 and tracing along the straight line in 16 anticipate Victor's justification in lines 19–23, in which the idea of the function as an object becomes verbalized by specifying his further statements. Speech and gesture together now reveal more precisely that in his justification, he considers the whole graph as consisting of two subgraphs and the different slopes ("the line runs steeper") of the segments as the crucial reason.

From the verbal statements, two kinds of GVs of functions can be perceived, namely, the idea of varying quantities and the idea of function as an object. However, integrating gestures additionally reveals that the idea of function as an object is continually present throughout the whole second scene, as it is manifested in Victor's gestures in anticipation of being expressed in his verbal utterance. Additionally, the idea of function as static mapping can be identified clearly.

16.7 Gestures' Contribution to the Analysis of GVs

Our analyses of GVs show that gestures do not merely complement speech to clarify deictic terms—as we described in the literature review—but enrich verbal expressions with aspects of the mathematical content and ideas substantially.

- Gestures can reveal GVs that cannot be captured verbally at a given time but in which further statements may be grounded: Considering a student's gestures may thus indicate a GV that may still be "under construction" in the student's mind.
- In addition to that, GVs can be identified prior to their verbal expression, which could also be interpreted as a preview of GV under construction.

Thus, gestures can provide significant indications of GVs. A systematic integration of gestures into descriptive analyses of GVs gives a much deeper insight into the ideas of the students. However, beyond the identification and analysis of mental models, gesture may also have influence on the formation of such models.

16.8 Forming Mental Models by Gesturing: Re-analyses of the Data Within Semiotic Perspectives

We now adopt the two semiotic perspectives on gestures discussed in the theoretical background, in order to hypothesize on their contribution to the formation of GVs, reanalyzing the scenes already introduced. In the following, we outline how these two perspectives can be adopted, in the context of analyzing Victor's GVs of linear functions as they manifest themselves in the solving processes. As we show, the questions arising within these two frameworks differ with respect to the functions of gestures focused on—a cognitive one on the Vygotskian side and a representative one on the Peircean side.

16.8.1 The Vygotskian Perspective

Encountering Victor's gesture use from a semiotic perspective in the light of Vygotsky raises the questions of how the gestures he uses while solving the task may "influence his mind or behavior", and how they contribute to the formation of GVs by "altering the way he comes to know about the world." Therefore, we ask, what are the functions of Victor's gestures as semiotic mediators, as "means of regulation and self-control" (Presmeg et al. 2016, p. 11) while solving the task?

Looking at the first scene analyzed above ("In the long run"), we can see how Victor's gestures interact with his verbal utterance, grounding his main argument of the function's behavior "in the long run" in the graphical representation. He coordinates gesture, speech, and inscription to regulate his explanation by altering the graphical representation ephemerally in the gestural extension. Gesture and inscription therefore interact by suggesting the extension of the given graphical representation, which can be seen in light of a GV that is mainly anchored in the graphical representation—the function as an object.

In the second scene ("Straight line"), Victor develops the Δ-gesture as tool to "grasp" the line segments and distinguish the two separate parts of the graph, f1(x) and f2(x) while verbally referring to two different time intervals, the "first six hours" and "after six hours". As in the first example, his gestures ground his explanation—related to the real-world context given by the task—in the graphical representation of the function. The use of the gesture may take part in internalizing the function as an object, constituted of two parts. The fact that the second interval is half-bounded, right-open, may cause him to alter the use of his gesture as a psychological tool, causing the slight movement to the left and suggesting the idea of varying quantities as secondary GV in the imagined action on the representation of the function. Being asked for an explanation (line 15), Victor lets go of the Δ-gesture that might have helped him in regulating the coordination between the real-world context in which two time intervals are distinguished and the line segments as graphical representation that relate these intervals to the effective costs.

His following explanation refers almost exclusively to the graphical representation in words as well as in gestures.

However, this is the moment when the growth point laid out in line 12, identifiable in the first occurrence of the Δ-gesture, is unpacked and the idea of the function as object comes into existence within the Vygotskian perspective. From this growth point, the seed of the idea of function as an object is developed further in the catchment, comparing the idea graphically to combine the two contexts in the gesture-speech dialectic. The graphical representation (imagery/gesture) becomes combined with the everyday formulation in the context of the task in the co-expressive speech. Therefore, the process of unpacking the growth point leads from a justification within the everyday context of the task to grasping the mathematical idea of the function as an object.

In the analysis of both scenes, the gestures act concretely within the coordinate system and thereby suggest a potential operation with the graph as object. The performance of the gesture may reveal a pre-stage of the "imagined actions dealing with mathematical objects" (vom Hofe and Blum 2016, p. 234), still being accomplished as actual action. By this, the gestures bridge between primary and secondary GVs; between the activation of a GV as based in the concrete action and a GV based on mathematical operations with symbolic objects.

16.8.2 The Peircean Perspective

In our case, the social interaction is of a different kind than that described earlier in this chapter, when disclosing the theoretical underpinnings, since it is not primarily an interaction between peers. The knowledge does not develop in the interaction between the participants but in the interaction with oneself and with the signs given by the task. The solving process, as it becomes traceable in the multimodal signs he uses in his explanation, can be seen as an interaction with himself. Assuming this, the signs he used may also be interpreted by him, influencing the shaping of the immediate mathematical object, and may evoke again an interpretant as multimodal sign. Thereby, interpreting his own gestures may give new ideas in an instance of "creating potential mathematics" in an external way (Krause 2016). Reconstructing Victor's solving process within this framework may help identify such instances. Therefore, we ask how Victor's gestures—as interacting with the inscriptions—may take part in his process of solving the task. In the following, we approach this question by interpreting the gestures analyzed in the first part of this chapter in a new light.

The two scenes presented—"In the long run" and "Straight line"—provide very limited access to Victor's process of solving the task, especially as the first scene concerns a fully elaborated explanation rather than the solving process itself. Performing many gestures on level 1—pointing to the graphs while mentioning "these are the two equations" (line 1), to the respective columns of the table while verbally referring to the two contracts (lines 2 and 3)—Victor links the different modes of representing the linear functions. However, these gestures have a communicative function and add to his explanation rather than "creating something new."

In the second scene, Victor uses the Δ-gesture on level 2 above the diagram, creating meaning in the distinction of the two parts (lines 12 and 13) and by slightly extending the second interval to the left (line 13). The gesture might add to providing visual access to the property of the slope in general in a growth point of the idea (12), which might then foster the idea of comparing the slope of the two-line segments. In this graphical comparison he then grounds his argument uttered verbally in the second part of the scene when the growth point is unpacked.

However, these examples give no rich data source for investigating how gestures as signs in the sense of Peirce may be involved in forming mental models in the sense of the concept of GVs, carrying "the meaning of mathematical concepts or procedures" (vom Hofe et al. 2005, p. 68). For the purpose of analyzing the Peircean perspective, one might need to take a closer look at the gestures as they are used in earlier stages of the solving process, when the task and the mathematical representations given within are encountered. This, however, goes beyond the capacity and the data of this paper, in which the semiotic perspective involves the goal of seeing the already presented scenes in a semiotic light.

16.8.3 Summary and Discussion: Semiotic Perspectives on Victor's Learning Process

The two frameworks set the stage for investigating two different functions of the gestures with respect to GVs of linear functions. On the one hand (Vygotsky), the gestures mediate between the task with its representations and the learner and may contribute in establishing GVs. On the other hand (Peirce), the focus is set on gestures' contribution to represent the mathematical object and through this, give new ideas about the object by becoming interpreted as a sign.

This dual semiotic approach is meant to go beyond looking at gestures from an outside perspective, as was adopted in the first part of the chapter where we reconstructed the GVs and the solving process as they become revealed in gestures. It moreover tries to look at gestures' contribution "from the inside" by reconstructing how gestures take part in the process of solving the task and forming GVs in doing so.

The main outcome of seeing gestures as semiotic mediators that may "alter the way one comes to know about the world" (Presmeg et al. 2016, p. 11) concerned the suggestion of a secondary GV by initiating imagined actions with the graphical representation of the mathematical function, such as extending it to the right-hand side in the first scene or grasping the line segments in the second scene. This might help regulate the coordination between the real world context referred to verbally and the graphical representation of the mathematical object.

However, the gestures used by Victor in these two scenes seem not to provide much potential to promote the solving process through representing as signs in a Peircean sense, “standing to Victor for something in some respect or capacity,” The choice of the scenes was gesture-driven so that scenes with communicational character—that is, scenes in which students have been asked to explain their solution—probably involve more gestures. In these explanatory situations the solving process might already be well advanced so that to investigate the question developed within a Peircean approach, scenes from an earlier point in the solving process might be more fruitful.

The analyses also show that the gestures of interest differ depending on the underlying concept of signs, and, respectively, the questions arising within the different frameworks. A look back on the one gesture that was considered in both analyses—the Δ-gesture—provides a starting point for that consideration in the growth point of the thought. The gesture may help the student in coordinating the situation as described in the task with the graphical representation, and in that regard, it may fulfill a cognitive function as semiotic mediator. But it also brings to the fore aspects of the linear function so that it accomplishes a representational function. By interpreting the gesture in its interplay with inscription as sign, new ideas about the linear function—the steepness of the line segments—can be derived.

In a similar context, Hoffmann and Roth (2007) refer to this duality as a “complementarity of the two sign functions” (p. 1), drawing on the claim made by Seeger (2005) to “find and define perspectives, that build on the complementarity of these two paradigms” (p. 73). They develop a model that “can describe how both these functions can be fulfilled, a model which emphasizes that ‘collateral knowledge’ is necessary both to interpret signs and to use signs to generate knowledge, that is to distinguish objects, to structure our experiences, to organize interaction, and so on” (Hoffmann and Roth 2007, p. 20). With collateral knowledge, they refer to “those forms of knowledge that remain hidden though being an essential condition for focusing on something” (p. 7, italics in the original), leaning on Peirce (CP 8.183, 6.338, 8.314). From their analysis of a collaborative learning process they assume that a “‘dialectical’ relation” (p. 20) between the two functions of a sign—representational and epistemological, or cognitive in our case—is required in learning.

In our example, this might concern the use of the gesture to distinguish the two line segments representing different conditions in the contract and the simultaneous representation of the slope as distinguishing feature in the gesture. So in this case, the collateral knowledge concerns an activation of the idea of function as an object. To regard GVs as collateral knowledge and thus to see GVs as “glue” between the different sign functions is one approach for an integration of gestures and GVs with regard to the mathematical learning process.

16.9 Conclusions

In this chapter, we have seen how gestures can take part in researchers gaining a broader picture of the activation of GVs, by integrating gestures in the descriptive analysis. It is also apparent that they contribute to a more thorough understanding of how gestures as signs may be implicated in the formation of GVs when completing a task.

In that regard, this chapter can be seen as subdivided into two parts, providing descriptive and explanatory approaches respectively. Both parts together show the surplus information that can be gained from considering not only what becomes explicated verbally, but what can be learned from considering that "speech and gesture must cooperate to express the person's meaning" (McNeill 1992, p. 11) where "each can include something that the other leaves out" (p. 79). We thereby claim to open not only researchers' ears, but also their eyes, in order to understand better how students conceptualize mathematical ideas, using gestures as a window to apprehend mathematical GVs.

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References

Alibali, M. W., & diRusso, A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development, 14,* 37–56.

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology, 25,* 468–523.

Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME, Numero Especial*, 267–299.

Arzarello, F., Robutti, O., Paola, D., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*, 97–109.

Bikner-Ahsbahs, A. (2006). Semiotic sequence analysis—Constructing epistemic types. In J. Novotná, H. Moraovsá, M. Krátká, & N. Stehliková (Eds.), *Mathematics in the centre. Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 161–168). Prague (Czech Republic): PME.

Broaders, S., Cook, S., Mitchell, Z., & Goldin-Meadow, S. (2007). Making children gesture brings out implicit knowledge and leads to learning. *Journal of Experimental Psychology, 136,* 539–550.

Chen, C., & Herbst, P. (2013). The interplay among gestures, discourse and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics, 83,* 285–307.

Dreyfus, T., Sabena, C., Kidron, I., & Arzarello, F. (2014). The epistemic role of gestures. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 127–152). New York: Springer.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Elements of pedagogy and epistemology* (pp. 85–106). Washington, DC: American Association of America.

Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds.). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, NC: Information Age Publishing.

Goldin-Meadow, S. (2010). When gesture does and does not promote learning. *Language and Cognition, 2*(1), 1–19.
Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological Science, 20*(3), 267–272.
Hoffmann, M. H. (2005). *Erkenntnisentwicklung. Ein semiotisch-pragmatischer Ansatz* [*Knowledge development—A semiotic and pragmatic approach*]. Frankfurt am Main: Klostermann.
Hoffmann, M. H., & Roth, W.-M. (2007). The complementation of a representational and an epistemological function of signs in scientific activity. *Semiotica, 1/4*(164), 1–24.
Hookway, C. J. (1985). *Peirce*. London: Routledge.
Kendon, A. (2004). *Gesture: Visible action as utterance*. Cambridge: Cambridge University Press.
Kita, S., Alibali, M. W., & Chu, M. (2017). How do gestures influence thinking and speaking? The gesture-for-conceptualization hypothesis. *Psychological Review, 124*(3), 245–266.
Krause, C. M. (2016). *The mathematics in our hands: How gestures contribute to constructing mathematical knowledge*. Wiesbaden: Springer Spektrum.
Krause C. M. & Salle, A. (2016): Learners' gestures when learning alone. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 123–130). Szeged, Hungary: PME.
McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.
McNeill, D. (2002). Gesture and language dialectic. *Acta Linguistica Hafniesia, 34*(1), 7–37.
McNeill, D., Quek, F., McCullough, K.-E., Duncan, S., Furuyama, N., Bryll, R., et al. (2001). Catchments, prosody and discourse. *Gesture, 1,* 9–33.
Montredon, J., Amrani, A., Benoit-Barnet, M.-P., Chan You, E., Llorca, R., & Peuteuil, N. (2008). Catchment, growth point, and spatial metaphor: Analyzing Derrida's oral discourse on deconstruction. In A. Cienki & C. Müller (Eds.), *Metaphor and gesture* (pp. 171–194). Amsterdam/Philadelphia: John Benjamins.
Moore-Russo, D., Ferrara, F., & Edwards, L. (2014): Introduction. In L. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 1–23). Charlotte, NC: Information Age Publishing.
Peirce, C. S. (1931–1958). *Collected papers* (CP) (Vol. I–XIII). In P. W. C. Hartshorne & P. Weiss (Eds.). Cambridge, Massachusetts: Harvard University Press.
Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in mathematics education*. Berlin: Springer.
Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics, 70*(2), 91–95.
Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching–learning processes in early Calculus*. PhD. Dissertation. Turin: Turin University.
Schüler, S., & Rösken-Winter, B. (2014). Exploring students' mental models in linear algebra and analytic geometry: Obstacles for understanding basic concepts. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 131–138). Vancouver, Canada: PME.
Seeger, F. (2005). Notes on a semiotically inspired theory of teaching and learning. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign—Grounding mathematics education* (pp. 67–76). New York: Springer.
Stölting, P. (2008). *Die Entwicklung funktionalen Denkens in der Sekundarstufe I*. PhD. Dissertation, Universität Regensburg. Retrieved September 29, 2015, from:http://core.ac.uk/download/pdf/11540300.pdf.
vom Hofe, R., & Blum, W. (2016). 'Grundvorstellungen' as a subject-matter didactics category. *Journal für Mathematikdidaktik, 37*(Supplement 1), 225–254.
vom Hofe, R., Kleine, M., Blum, W., & Pekrun, R. (2005). On the role of 'Grundvorstellungen' for the development of mathematical literacy—First results of the longitudinal study PALMA. *Mediterranean Journal for Research in Mathematics Education, 4*, 67–84.
Vollrath, H.-J. (1989). Funktionales Denken. *Journal für Mathematik-Didaktik, 10*(1), 3–37.

Vygotsky, L. S. (1978/1931). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1981). The instrumental method in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 136–143). Armonk, NY: M.E. Sharpe.

Wartha, S. (2007). *Längsschnittliche Untersuchungen zur Entwicklung des Bruchzahlbegriffs*. Franzbecker: Hildesheim; Berlin.

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*(3), 371–393.

Chapter 17
Diagramming and Gesturing During Mathematizing: Kinesthetic and Haptic Interactions Support Mathematical Ideation

Petra Menz and Nathalie Sinclair

Abstract We focus on the role of diagramming and gesturing in mathematical practice. In most discussions of diagramming in mathematics, including mathematics education research, diagrams are seen as representations of mathematical objects and relations. This view was challenged by the philosopher and historian of mathematics, Gilles Châtelet, who argued that diagramming is a material practice of mathematical invention. His arguments are based on analyses of historical examples of diagrams associated with new mathematical ideas. Guided by his approach, we study the actual diagramming practices of mathematicians, as they work on unsolved problems. By doing so, we aim to identify the various roles that diagrams might play in mathematical invention as well as the material nature of diagramming practice. We hope that a more nuanced understanding of diagramming within mathematical practice will contribute to research on promoting and supporting student diagramming.

Keywords Diagram · Gestures · Materialism · Mathematical practice
Mathematical invention

17.1 Introduction

Diagramming is considered by many to be an essential strategy in mathematical problem solving (Grawemeyer and Cox 2008; Novick 2004; Stylianou and Silver 2004) as well as an essential component of mathematical discourse (O'Halloran

P. Menz (✉)
Department of Mathematics, Simon Fraser University,
8888 University Drive, Burnaby, BC V5A 1S6, Canada
e-mail: pmenz@sfu.ca

N. Sinclair
Faculty of Education, Simon Fraser University, 8888 University Drive,
Burnaby, BC V5A 1S6, Canada
e-mail: nathsinc@sfu.ca

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2005, 2011). In mathematics education research, many have used Peirce's semiotics to study the functioning of diagrams either as a special category of icons or in each of the iconic, indexical and symbolic registers (see Hoffmann 2006).

A critique of the semiotic approach to the study of diagrams relates to their assumed status as representations, either of mathematical objects/relations or as conceptions already formed through mental activity (de Freitas 2012). As representations, they can only be seen as copies of ideal mathematical entities—often flawed and misleading. Rotman (2005) has argued that such a view ignores the material, embodied nature of mathematical diagramming: instead of seeing them as mere representations, we could focus on "the physical activities themselves, the moving around, visualizing, talking, and gesturing involved in learning and communicating the subject" (p. 34).

In other words, the actual drawing of the line on a page that will produce a circle, is not just a representation of a geometric object, but also a mark-producing gesture in which the continuous application of lead on paper is led by the hand and guided by the eyes.

Gestures and diagrams have each received increased attention in mathematics education, especially within the recent emergence of embodied and semiotic perspectives. Châtelet's (2000) work, however, seeks to show, through examples from the history of mathematics, how gestures and diagrams play a pivotal role in mathematical invention. He wants to show how formal mathematics can be seen as continuously emerging from the material mobility of the human body. Diagrams are essential clues for him, since they provide traces of the moving hand, while also enabling—on the surface of the paper—the exploring and creating of new objects and dimensions. Châtelet thus brings together two hitherto-distinct areas of research —on gestures and diagrams, respectively—in pursuing his non-representational, non-dualistic account of mathematical thinking.

Our own material approach to semiotics also draws on Peirce, but with a particular focus on the index. The drawing of a circle can be seen as an icon, which operates according to likeness and resemblance between signifier and signified. Seen as an icon, however, it becomes a representation of a Platonic object (the circle) that is simply being copied, more or less faithfully. But what if we think of the drawing as an index, which has a more material link between signifier and signified? Indeed, unlike icons and symbols, indexical signs are bound to the context in important ways, as they "show something about things, on account of their being physically connected with them" (Peirce 1998, p. 5). The canonical example used by Peirce is that of smoke billowing from a chimney, which indicates that there is a fire in the fireplace so that the smoke indexes the fire. An index "refers to its object not so much because of any similarity or analogy with it, (…) because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other" (Peirce 1932, 2.305). The gesturing hand that leaves a trace of congealed wax can therefore be seen as indexing a circle, producing a temporal and spatial record of circle-making. The circle drawing can thus be seen as an indexical sign that refers to the prior movement of the pencil. As Sinclair and de Freitas (2014) have argued,

> This latter indexical dimension is usually not emphasized in the semiotic study of mathematical meaning making, since we tend to focus on the completed trace and dislocate it from the labour that produced it. Such habits of focus have resulted in our neglect of how the activity of the body and various other material encounters factor in mathematical activity. (p. 356)

Since Châtelet's work was based on historical examples, he only had access to diagrams that have been completed and conserved, and not to actual making of the diagram, a process that would have a particular rhythm, weight, order and tempo that could help provide insight into the inventive process. In this chapter, we study the *live* process of mathematical diagramming. We aim first to corroborate Châtelet's claims in the context of contemporary mathematical practice. We further aim to better understand the embodied, material aspects of mathematical diagramming. In particular, we are interested in how diagrams, if not assumed to be mere representations, can be seen as indexical signs *producing* mathematical concepts.

In the next section, we provide a brief overview of Châtelet's work, which is central to our own approach. We also summarise other work in mathematics education that has adopted this approach. Where relevant, we connect to similar research on diagrams and, especially, gestures.

17.2 Châtelet on Diagramming and Gesturing

Through this flexible view of figuring and defiguring, Châtelet brings new ideas to the study of gestures, diagrams and mathematics. By analyzing historical manuscripts of famous mathematicians such as Oresme, Leibniz and Hamilton and without access to video-recordings of these mathematicians' mathematizing, Châtelet traces their thoughts and actions from their writing and their diagramming. His fundamental insight is that the *virtual* is evoked in historical mathematical inventions through diagramming experiments whose sources can be traced to mobile gestural acts. The virtual is that which is latent in the diagramming, but ontologically new. In this way, Châtelet challenges the disembodied, abstract conceptualization of mathematics. The key ideas of his theory are that (1) the diagram is never really fixed–it is erased, drawn over, reassembled, or redrawn—as it hovers between the actual and the virtual; (2) there is a mutual interaction between gestures and diagram; and (3) it is through the material interaction with the diagram that a person understands or invents mathematics. As Roth and Maheux (2015) point out, "the virtual cannot be grasped but is that which allows grasping to occur" (p. 236).

The crux of Châtelet's study is his revelation of how the virtual is evoked in historical mathematical inventions "through diagramming experiments whose sources Châtelet can trace to mobile gestural acts" (Sinclair and de Freitas 2014, p. 563). This trace of the gesture, which lingers in the creation of new mathematics, provokes and challenges the abstract nature that has come to be associated with mathematics.

Although the main focus of this chapter will be on diagramming, in light of the gestural nature of the production of diagramming coupled with the tight interplay hypothesized by Châtelet, it is worth examining how Châtelet's conception of gesture fits within the existing landscape, which we do in the following subsection.

17.2.1 Situating Châtelet's Conception of Gestures

The last two decades have been fertile ground for the study of gestures in mathematics from the anthropological (Rotman 2012), cognitive scientific (Lakoff and Núñez 2000), educational (Krummheuer 2013; Radford 2001), psycholinguistic (Levelt 1989; McNeill 1992, 2008), sociological (Greiffenhagen 2014), and philosophical (Châtelet 2000) points of view. The predominant line of research in gesture studies, especially within the Anglophone literature, focuses on movements of the body (especially the hand) and their interactions with speech in communication. For example, one of the groundbreaking studies in gesture theory asserts that "in a nutshell, […] *gestures are an integral part of language as much as are words, phrases, and sentences–gesture and language are one system*" (McNeill 1992, p. 2, emphasis in original). Drawing on Peirce's semiotics, this theory identified categories of gestures (iconic, metaphoric, deictic and beat) that distinguish different relationships between gesture and speech.

This type of gesture research emphasizes gesture as part of "the human capacity for language" and the study of gesture as "language in action" (Rossini 2012). In coding gesture only in terms of linguistic potential, such research can overlook the physicality of the hand movements of gestures. As Streeck (2009) indicates, "it is common to treat gesture as a medium of expression, which meets both informational and pragmatic or social-interactional needs, but whose 'manuality' is accidental and irrelevant" (p. 39). To counter this neglect, Streeck defines gesture: "[…] not as a code or symbolic system or (part of) language, but as a constantly evolving set of largely improvised, heterogeneous, partly conventional, partly idiosyncratic, and partly culture-specific, partly universal practices of using the hand to produce situated understanding" (p. 5).

With this definition, Streeck studies gestures for how they are "communicative action of the hands" (p. 4) and thus examines gesture for how it couples with and intervenes in the material world in non-representational ways, which is consistent with the approach of Châtelet. Indeed, for Streeck, distinctions between hand movements 'in the air' and hand movements 'on the page' become fuzzy. And it is in this fuzziness, that Châtelet can dwell on the back and forth interplay between gestures giving rise to diagrams that give rise to gestures.

In line with a French tradition of inquiry into gestures (by philosophers of mathematics such as Cavaillès, Desanti, Longo and Alunni, Châtelet sees the gesture as a tool for non-analytic reasoning (Maddalena 2015). Châtelet not only notices the gesture and its link to the diagram, but he also interprets gesture as even more than a visible, non-verbal, bodily action that carries meaning; indeed, a gesture is the articulation between the virtual and the actual and as such is immediate and embodied. It is in this regard that a gesture is inseparable from where it came from and what it creates. This view led Châtelet to question the boundary between physics and mathematics that Aristotle drew with his interpretation that physics represents applications "that exist in Nature [… and are] mobile" whereas mathematics represents abstraction "which exists only by proxy through the wit" and is "immobile" (Châtelet 2000, p. 17).

Our own research investigates the gesture/diagram coupling in mathematical practice, where it should be possible to study the various hand movements—be they in the air or on the blackboard—made by mathematicians as they work on solving mathematical problems. Our goal is to better understand the material nature of their diagramming practices.

17.3 Methodology

The first author was given the opportunity to participate in a series of research meetings that a group of mathematicians had organized in order to pursue joint work on a particular problem. These meetings struck us as fortuitous settings for studying diagramming practices, since the discussion amongst the mathematicians would help us understand the evolution of their work without having to constantly prod them for explanation, as might be the case for a mathematician working on a problem alone. The research subjects in this study comprise three male, Caucasian, research mathematicians in the field of Topological Graph Theory from two prominent North American universities (see Menz 2015 for details of the full study).

The three mathematicians, referred to as Fred, Colin and Victor, met nine times over a period of three months to study the class of 2-regular directed graphs and how they embed in different surfaces. The mathematicians' goal was to compile a list of obstructions for the projective plane in particular, although other surfaces such as the torus and Klein bottle were also explored at times, and to classify these obstructions for 2-regular directed graphs. All research meetings took place in one university's mathematics seminar room, equipped with blackboards on which all diagrams of this study were drawn. The mathematicians never all sat at the tables in the room together; at least one of them was always standing, close to the blackboard. The first author was present in order to video-record these nine research meetings that varied in length from one to two hours; to capture digital still images of diagrams by the participants using a second camera; and to make field notes during the meeting that attended both to the mathematics that developed during the meeting and to observations regarding diagramming and gesturing.

Based on these observations and multiple viewings of the recorded research meetings, we looked for all diagramming events; an event is regarded to be when a diagram is produced, enhanced, contemplated, referred to, changed or erased. In order to get an overall sense of the frequency and intensity of diagramming, we identified periods of time that contained at least one diagrammatic event. These diagrammatic events were viewed several times in order to describe the mathematics that was occurring, to track who was doing the talking, to identify the overall body language associated with the diagramming, to record spoken phrases, and to label the apparent mood. These descriptions enabled us to identify, more holistically, the different roles that the diagram played during the mathematizing.

Over the course of the 12 h of video-recordings, a total of 122 time intervals were selected for further analysis varying in length from approximately thirty seconds to approximately six minutes, containing a total of 128 distinct diagramming events. For each of these time intervals, we produced a transcript of the event as well as the accompanying gestures (in the air as well as on the blackboard). Our goal was to see whether we could identify different functions that the diagramming played over time, as the mathematicians worked on their problem, and whether some functions were more prevalent than others.

In the first section below, we analyze two different time intervals in order to highlight the material interactions at play, particularly the hand movements that preceded, accompanied and followed diagramming. In the following section, we look across the data set in order to propose different types of diagramming events that occurred, with a view to investigating how the diagramming event enabled invention, comprehension and communication.

17.4 Diagramming Matters

In his analysis, Châtelet focused on particular diagrams that he argued were instances of inventive mathematics and from which he hypothesized the significant role of gestures. In our analysis of the 122 time intervals and the 128 diagrams, we study not only the production of drawings on the blackboard, but also their evolution over the course of time—that is, we focus on the entire life-cycle of diagrams. In doing so, we were able to identify diverse types of diagramming events that occurred—not just the initial production of a diagram through the movement of the chalk on the blackboard, but also subsequent additions and deletions, as well as repetitions of drawings that had been made before. A glimpse into the whole process reveals how diagrams come into being and are engaged with.

Every time a mathematician makes chalk marks on the blackboard that are not writings a diagram emerges. This diagram can consist of anything from just a few strokes, to elaborate, colorful drawings. Once the diagramming mathematician steps or turns away from the diagram, the episode *diagram-is-emerging* is concluded and the episode *diagram-is-present* begins. If a mathematician adds to or erases from the original diagram, the emergence of another diagram is signaled. Five settings are identified when the episode diagram-is-emerging comes about: (1) the mathematician diagrams from scratch during his exploration, which is especially noticeable during sustained high diagramming activity such as in research meetings 3, 5 and 9; (2) the mathematician adds to or erases from an existing diagram, which alters the original diagram; (3) the mathematician re-figures a diagram by tracing the edges and vertices of an existing diagram (usually a graph), which occurred every time the mathematicians tried to find out if a given graph on the blackboard embeds in some surface; (4) the mathematician draws a known diagram by retrieving it from memory; and (5) the mathematician is directed to draw a diagram by another mathematician.

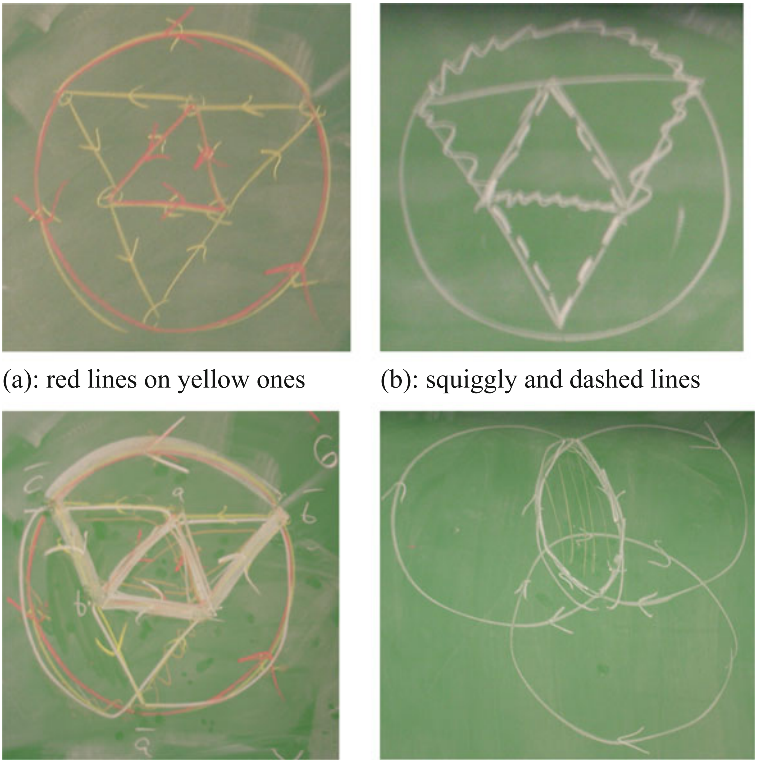
Once a diagram has emerged, it is present on the blackboard, regardless of whether any of the mathematicians engage with or refer to it and no matter for how long a time period. When and how a diagram emerges and becomes present is of specific interest as it reveals how the diagram is interacted with from that moment onwards. In particular, the regularly occurring emergence as described in (3) throughout the nine research meetings provides evidence that the *re*-figuring of diagrams is a crucial element leading to mathematical inventions. In the following example, we describe an instance of (3) in more detail.

17.4.1 Re-figuring Diagrams

Through exploration of the edges and vertices of one 2-regular orientation of C_6^2 (later termed *reversed planar octahedron*), the three mathematicians had discovered an embedding of this particular octahedron in the torus. But they wanted to know if this embedding was unique and how C_6^2 could embed, more generally, in a surface of Euler characteristic zero. For this, they needed to understand how the faces of the embedding were connected to the edges of the original graph, which they referred to as identifying *closed walks*.

The four diagrams in Fig. 17.1 were produced at different times during the research meetings. In some ways, they look ‘the same’ and could thus be considered representations of—or icons of—an embedding of an octahedron on the torus. Such an interpretation would ignore the actions with the hands that produces these diagrams, indexical actions that—as we show—include the mix of improvisation, heterogeneity, idiosyncrasy, and partly conventional, culturally specific, and universal practices evoked earlier by Streeck.

Fred began by drawing the reversed planar octahedron with yellow chalk on the blackboard from memory (they had worked on it in research meeting 2), but seemed to be stuck finding the faces of the embedding. He stood back from the blackboard, looking at it intently. Then Colin joined Fred at the blackboard and asked, “May I please destroy this picture?” to which Fred replied “Oh, sure.” Colin took the red chalk and drew lines on top of some of the yellow edges. He also added some new arrows, some going in directions opposite to the yellow ones. The direction of the lines/arrows corresponded to closed walks (see Fig. 17.1a). The fact that Colin re-drew the lines, right on top of the other lines, shows that the ‘walks’ were just



(a): red lines on yellow ones
(b): squiggly and dashed lines
(c): labels and thickening lines
(d): Victor's incorrect attempt at embedding

Fig. 17.1 **a–d** Different drawings of an octahedron on a torus

that, movements of the hands along the edges of the graphs, movements that started and stopped at the same place—actual closed walks. Had they been mere representations of a closed walk, Colin would have not re-drawn the lines; it would have been enough to change the arrows.

All the while, Fred was several times tracing through the diagram with a cupped right hand (see Fig. 17.2a), which he placed over each of the faces in the diagram as if his hand was mimicking the perimeter of the faces. During the times when he uttered "red, yellow", he alternately moved his index and middle fingers up and down while at the same time moving his hand successively from face to face. Both of these are interesting gestures, as they not only embody the alternation between edge colorings and thereby virtually assemble the faces that the diagram holds, but the gestures also underline the content of Fred's utterances, which is a generalization of the process of finding the faces.

Colin returned to his seat and a discussion about the closed walks ensued, followed by silence. During this time, Fred labelled the vertices of the red/yellow

(a): Fred cupping his hand over faces

(b): Mathematicians stare at diagram shown in Figure 17.1a for several seconds

Fig. 17.2 **a**, **b** Mathematicians interacting with diagram

diagram and softly traced through the edges with red chalk, which can be seen in Fig. 17.1c. It was not enough for him to see what was on the blackboard. Instead, he drew the lines again, his hand performing a walk along the graph. The repetition, by different mathematicians and then by the same mathematician, turns one idiosyncratic choreography of the hand into a convention and forges the closed walk on the imagined surface into an object. This is not done through abstraction, but through ritual action.

In the meantime, at the very beginning of Colin and Fred's exploration at the blackboard, Victor got up saying "I can't see it", but instead of joining Fred and Colin, he quietly drew his own version of the embedding on the blackboard (see Fig. 17.1d) until both Colin and Fred had stepped far away from their diagram. Next, Fred uttered "So that's (.) what am I doing? How did I get such funny numbers? (.) No, but that's perfect. That's, that's an embedding on the torus", during which time he stepped closer to the diagram and touch-pointed it. This statement raised Victor's interest, who stopped drawing and turned to Colin and Fred's diagram and stepped closer. Then, all the mathematicians stared at this diagram for several seconds (see Fig. 17.2b). Fred emphatically repeated "That's an embedding on the torus." Colin replied "Uh, so you have six vertices, twelve" and paused, so Fred finished with "Yah! Uh, uh, six vertices, twelve edges and six faces." Then Colin responded "Six faces is the torus, yah, yah", and Victor chimed in with "Yah, yah, that's what I was doing" and walked to his diagram. The next few minutes, the three mathematicians discussed their findings. Victor wanted to know which polygonal faces Colin and Fred used, both of whom responded by describing and counting the types of polygonal faces while touch-pointing them in the diagram.

During this episode, the diagramming, gesturing, positioning of the bodies, and verbalizing were entwined acts, which culminated in the three expert mathematicians forming a shared understanding of their discovery of the funny octahedron's embedding in the torus.

Next, Fred stated "I am curious how much symmetry of this you can keep when you take this funny embedding of the thing on the torus" and began to redraw the embedding of the funny octahedron in the torus, which took him about two minutes to do. He used Fig. 17.1a to trace along the edges, which guided him in drawing the faces in his new version of the torus embedding using the same yellow and red chalk coloring (see Fig. 17.3). These diagrams were drawn by the mathematicians in order to understand the mathematical structure of the funny octahedron on the torus. It was necessary for Fred to trace along the edges and pause at the vertices of the diagram shown in Fig. 17.1a in order to produce a new version of the same graph. In Châtelet's terms, the constant back and forth between Figs. 17.1a and 17.3 helps actualize the virtual structure through this re-figuring.

Now that the mathematicians had discovered the embedding of C_6^2 in the torus, they wanted to know if this embedding is unique and how C_6^2 can embed generally in a surface of Euler characteristic zero. For this, they needed to understand how the faces of the embedding are connected to the edges of the original graph, which they referred to as closed walks. Fred noticed that "also around each, around each red edge you see a directed four cycle just taking those two faces", and half a minute later he stated "So if we say we have only quadrangles [sic], maybe we can work out, uh, what the possibilities are." Colin responded with "So we can try using no triangles, uh, uh, and then there would be only squares." During this utterance, Victor walked to the blackboard and joined Fred there tracing and touch-pointing at the diagram in Fig. 17.1a while exploring the connection between the edges and faces. After about a minute, Fred exclaimed: "Wait! I don't understand. Are you taking, are you taking these two?" A silence of 11 s ensued, then Colin again came up to the blackboard and picked up the white chalk. He covered some of the edges of the reversed planar octahedron with squiggly and dashed lines (see Fig. 17.1b). Again, Fred walked up to the diagram and this time used the index finger of his right hand to trace over Colin's added lines (see Fig. 17.4a). This is a literal sense in which Colin's gesture/drawing gave rise to a new gesture by Fred, which re-produced the closed path, thus re-figuring the diagram yet again.

Another 10 min passed, during which time Fred and Colin explored the parity of the closed walks in the reversed planar octahedron. They did this by referring to their two diagrams (Fig. 17.1a, b), pointing to them, tracing over them and touching

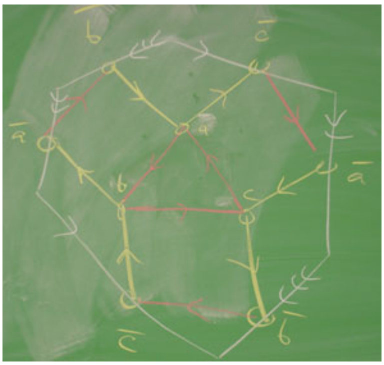


Fig. 17.3 Re-figuring Fig. 17.1a

Fig. 17.4 **a–c** Examples of diagram-is-emerging

the edges and vertices. Then Fred indicated that he finally understood how to argue that the embedding is unique and turned to the diagram drawn in red and yellow. First he traced through the edges (see Fig. 17.4b) and then added thin and thick white lines. He then exclaimed, “Let me make a total mess of this” (producing Fig. 17.1c). Afterwards, Colin traced through the edges of this diagram with his index finger to identify closed walks (see Fig. 17.4c), while both he and Fred verbalized their satisfaction with their argument.

During all this time, the third mathematician, Victor, joined the discussion at times, but also explored the graph on his own on paper. However, his diagramming was ignored by the other two mathematicians. At the very end, Victor was asked by Colin “Is that clear (.) so (.) Victor?’, to which Victor readily responded “yeah” and nodded his head in agreement.

Far from being fixed representations of mathematical objects or relations, the diagrams shown in Fig. 17.1—despite their static nature on the page—unfolded over time, changing and growing with added marks, traces and touches. The continuous lines made in both red and yellow chalk involved quite different hand movements than the dashed, squiggly and thick lines that were piled on top of the original, thin lines—resulting in a layering of new closed walks that were sometimes tentative, and sometimes needing to be insisted upon. The smudges of chalk and finger prints on the blackboard within Fred’s diagram (see Fig. 17.1c) were evidence of the erasing and touching of edges and faces, thereby speaking of the intimate engagement with the objects and relations that were coming into being.

Interestingly, all three diagrams by each of the mathematicians (see Fig. 17.1a, b, d) were drawn with the same orientation, which was not the case in previous and a subsequent explorations of the mathematicians. Their diagrams also display rotational symmetry with a 120-degree angle of rotation. While Fred and Colin’s diagrams are based on an existing representation of the funny octahedron, albeit in the projective plane, Victor’s diagram does not resemble any previous representation of the funny octahedron and employs only one color, although two edges stand out more because he draws over them so often when he is explaining his method to the mathematicians. At a fleeting glance, one can argue that these are still the same diagrams, by just morphing the curved edges of Fig. 17.1a into straight lines and rotating the entire diagram by a 180-degree angle of rotation through the center. Yet, Victor's explanations, which he offers a little while later, elucidate that he is using the idea of Venn diagrams to represent the inner triangle, outer triangle, hexagon and the three quadrilaterals of the embedding. When Fred and Colin are curious how the faces align, they almost immediately discover that Victor's diagram is actually an embedding of the funny octahedron in the Klein bottle rather than the torus. Furthermore, Fred's diagram had vertices labelled and edges oriented, Colin's diagram was void of any labels and orientation, and Victor's diagram had only the orientation of edges. Therefore, the diagrams by the three mathematicians provided further evidence of how uniquely virtual gestures are actualized. However, these personal actualizations are also indicative of a common ground among the mathematicians to realize the possible mathematics, as was discovered here.

In the above example, which occurred over a period of about thirty minutes, the diagram is being brought into being, which Menz (2015) refers to as the *manufacturing phase*. The diagram is present, available both to the mathematician who has drawn it, as well as to his colleagues. This manufacturing phase is one of intense material interaction, but once the diagram is on the blackboard, as when Fig. 17.1a was made, additional material interactions can occur, as in the re-drawing, tracing and touching that was seen above. These interactions serve the purpose of bringing relations and objects into being; providing a site for repeated gestures that can eventually become captured into, in this case, a closed walk; highlighting, bracketing or connecting certain objects and relations for further attention. Each of these actions further shapes the diagram, as it shapes the mathematicians' reasoning.

In the next example, we analyze a similar situation in terms of the material interaction, but where the diagram plays a different function than the ones described above.

17.4.2 Example 2: The Diagram Reveals

During meeting 2, the mathematicians were exploring the embedding of an unknown graph and were expecting the faces of the embedding to consist of four 3-cycles and two 4-cycles. Fred was working at the blackboard and attempting to find these particular cycles. Fred spent about four minutes standing in front of the blackboard, most often with his arms crossed, but several times stepping forward to draw or gesture. The first time, he approached the board, saying "I can go here (tapping on top left vertex, then following arc to top right vertex), here (tapping second vertex, then following an arc to bottom vertex) here" (see Fig. 17.5a). The three instances of "here" were accompanied by the three sounds of taps on the board. No new lines were drawn, but his arms remade the circuit.

A couple of minutes later, standing back from the board, his right hand made quicker and smaller scale, in-the-air movements, that involved more turning of the hand than translating of the hand along lines. Then he approached the board and

(a) (b) (c)

Fig. 17.5 Mathematician tracing edges (**a**), drawing over edges (**b**), and staring at diagram (**c**)

drew a label on the top line (see Fig. 17.5b). He then made several more small, quick in-the-air hand gestures, paused and then erased several of the arrows on the lines with his fingers. This was followed by yet more small, quick gestures and then he started to draw new arcs wrapping around the vertices of the graph. After drawing the arcs, his right hand, which was holding the chalk, traced out yet another circuit, this time returning to a larger gesture. He then stood back (see Fig. 17.5c) and came forward several times, then approached the board and moved his chalk in the air, drawing virtual lines close to the surface of the blackboard. After drawing a few more arcs on the board, he stepped away from the blackboard and exclaimed, "Oh! It's not three, three, three, three, four, four. It's five threes and a five."

There were three types of gestures that occurred in this four-minute interval: the large arm movements that created a virtual circuit and were close to the board; the smaller hand movements that were further from the board but also created virtual circuits; and, the curved hand movements that left a trace on the blackboard around the vertices of the graph. These all followed, evidently, the creating of the actual graph (drawing) on the blackboard, as well as the re-drawing of the graph that we discussed earlier. The drawing, re-drawing and large gesturing involve the same movements of the arm, all at the scale of the graph, with the large gesturing being evidence of Châtelet's sprouting of gestures from diagrams, gestures that begin to actualize a cycle. In the shift from large to small gestures, from arms to hands, the process of actualization continued, now turning a longer traversing of the graph into a quick, settled cycle. The last gestures, which left their traces on the board, carved out the corners of the cycles, inscribing the turns of the wrist that featured in the small gestures—and in this sense, the resulting drawing captures the gesture, as Châtelet theorized.

It is tempting to say that the diagram, as a representation of the embedding, contains the structure required to reveal everything about its cycles. But this would ignore the extended labor involved in drawing and re-drawing edges, in gesturing across from vertex to vertex, as the embedding takes shape, and embedding that is presumed to have a certain property. It is not a question of inferring something from a given diagram since the diagram is being shaped along with the realization of the cycles.

17.5 Characterizing Diagramming Practice Over Time

There were 179 distinct diagrams recorded with the picture camera during the research meetings; however, this number does not include the erased and drawn-over diagrams that were not captured but that are recorded on the videos of the research meetings. Very rarely, a mathematician would place just a few chalk marks on the blackboard and immediately erase them, which occurred at most once in each research meeting. Furthermore, during sustained high diagramming activity such as in research meetings 3, 5 and 9, for every two diagrams of which a digital photo was taken, another diagram was either drawn over or partly erased. Since there is no clear demarcation between the beginning and ending of diagramming, there was a fine balance between getting the data and not disturbing the meeting. Based on the viewing of the video recordings by the first author, there are an estimated twenty diagrams that are recorded only on video and not as digital photographs. Therefore, there are about 200 diagrams drawn during the nine research meetings.

By looking at each one of the diagramming intervals, we were able to identify different functions of interactions between mathematician and diagram in inventive mathematical processes, which are captured in the life-cycle of Fig. 17.6. We were interested in doing this in part because Châtelet provided little insight into how the diagramming process evolved from suggested movements of the hand to more refined, shared and dependable diagrams that could appear as part of the formal mathematical discourse. We propose that the diagram life-cycle is divided into three phases, which concern themselves with (1) how the mathematician brings the diagram into being–*manufacturing phase*; (2) what the relationship is between the diagram and the mathematician–*communication phase*; and (3) how the mathematician resolves, in simplistic terms, whether the diagram stays or not–*dénouement phase*.

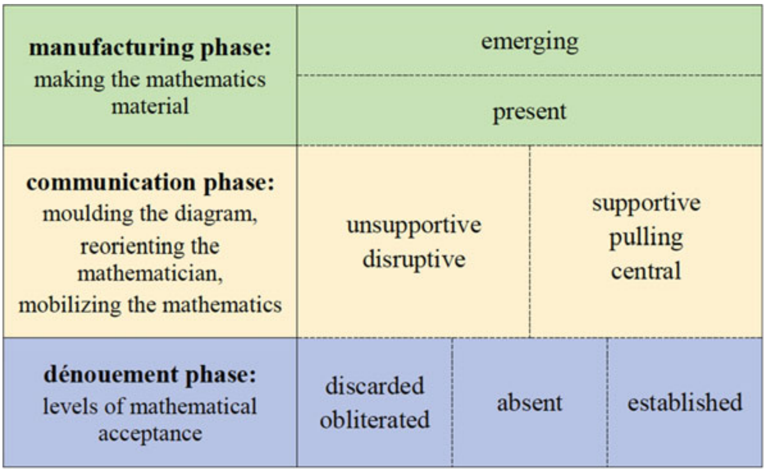


Fig. 17.6 Episodes, phases and their relationships in the life-cycle of a diagram

In each phase, so-called *episodes* describe a particular function of interaction between diagram and mathematician. The term *episode* is chosen because it alludes to both a period of time and a side-story being told. The analysis shows that throughout the life-cycle of a diagram, from its creation to its ‘establishment’ or ‘obliteration’, there are eleven distinct diagram episodes: *emerging*, *present*, *un-supportive*, *disruptive*, *supportive*, *pulling*, *central*, *discarded*, *obliterated*, *absent* or *established* (see Menz 2015 for descriptions and examples of each). It is important to point out that not all of these episodes need to occur in the life-cycle of a particular diagram, except for diagram-is-emerging followed by diagram-is-present, since its occurrence is dependent on the relationship the mathematician has with the diagram.

As indicated in the methodology section, 122 time intervals of interest for further analysis were selected, which contain a total of 128 diagrams. Only the first occurrence of a diagram was counted. In other words, if a diagram in a particular time interval had already been counted in any of the previous time intervals, then this diagram was not counted again. This means that of the estimated total of 200 diagrams, the data analysis contains 64% of the diagrams. Since each diagram was identified with an episode, we were able to conduct a quantitative analysis of the eleven diagram episodes, the result of which is shown in Fig. 17.7. Even though only 94 out of the 128 diagrams are shown to emerge in the selection of 122 time intervals, the episode diagram-is-emerging is still regarded to be 100% of the total number of diagrams, because every one of these diagrams was created by the mathematicians at some point during the nine research meetings. Once a diagram has emerged, however sketchily or incompletely drawn, it is present on the blackboard, no matter for how short or long a time period. Therefore, the episode diagram-is-present also occurs as 100% of the total number of diagrams. The remaining nine types of diagram episodes were counted and their percentage was calculated compared to the 128 total diagrams. Every so often, more than one episode in the life-span of a diagram was identified, and therefore, the percentages in Fig. 17.7 for the bottom nine types of episodes do not add up to 100%.

The two most common of the nine episode types from the communication and dénouement phases are diagram-is-supportive and diagram-is-central at 37 and 38%, respectively. This does not come as a surprise, because the diagram is chosen by the mathematicians in order to explore the mathematics of their research. In other words, a diagram is the quintessential playground, where a mathematician’s intuition and experience shape the mathematics that is under exploration, since “one is infused with the gesture before knowing it” (Châtelet 2000, p. 10). This becomes even more evident in the mathematicians who gestured a diagram during speech before the chalk marks created it on the blackboard. It is in the same sense that de Freitas and Sinclair (2012) emphasize that “[the potential] marks that which is latent or ready in a body. In the case of the diagram, the potential is the virtual motion or mobility that is presupposed in an apparently static figure—and that was central to its creation in the first place” (p. 139). Perhaps this also explains why the episode

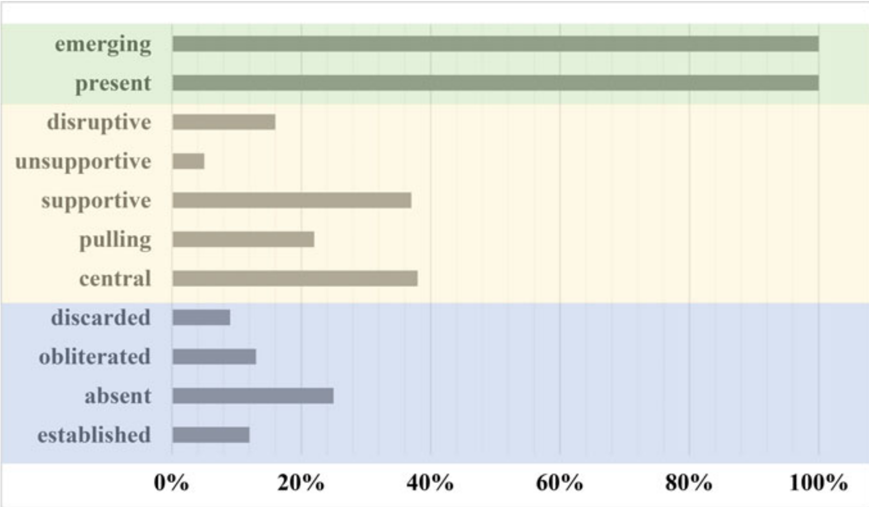


Fig. 17.7 Quantitative analysis of diagram episodes

diagram-is-unsupportive is the least common type of episode with only a 5% occurrence. The mathematician's physical and mathematical intuition shapes his gestures that actualize the diagram, thereby realizing the mathematics that is possible rather than the mathematics that is not.

17.6 Discussion

One might see the discovery of uniqueness (for the embedding of the reversed planar octahedron in the torus using a parity argument of closed walks) as some kind of mental process, and as such an abstraction that does not partake of the physical world. However, the data we have presented and analyzed contrast this point of view and provide evidence that the mathematical result derives from an embodied, material engagement with diagrams. The diagrams involve material objects (e.g., edges, vertices, graphs, embeddings) and relationships (e.g., closed walks) that are made by the hand in time, manipulated and experimented with, and actualized through gestures and drawings. The two mathematicians' abundant production of material, indexical signs on the blackboard, their subsequent direct and intimate engagement with these marks, and the prolific gestures that accompany speech and hint at the mathematical entities that are being explored through the diagram are all evidence that the diagram is not merely a representation of a known object, nor merely a fixed platform on which to experiment. The diagram is the mathematical becoming.

Furthermore, we argue that the diagram is given a voice through the virtual and physical gestures of the mathematician, which speaks not only to the gesturing mathematician but all of its participating onlookers such as the third mathematician Victor. Considering all the episodes that are possible in the life-cycle of a diagram, the diagram is more than what it depicts: it constantly reassembles itself for the mathematician as he engages with the diagram virtually through gestures (e.g., points, traces, stares) or physically (e.g., adds, erases, draws over). Through the lenses of anthropology, archaeology, art and architecture, Ingold (2013) offers the following insight from his study of drawings: "The drawing that tells is not an image, nor is it the expression of an image; it is the trace of a gesture. […] Thus the drawing is not the visible shadow of a mental event; *it is a process of thinking, not the projection of thought*" (p. 128, emphasis in original). Ingold's conclusion equally serves this study of mathematical drawings, because ultimately, the diagram tells the story of how innovative mathematics comes into being.

Researchers such as Kita (2000), Krummheuer (2013), Lakoff and Núñez (2000), McNeill (2008) and Radford (2001) agree that gestures make thought visible. What is new in recent studies on gestures and thought "is the effort to identify causal and measureable relations and interactions between bodily behavior and hypothesized internal processes and to explain these within embracing and detailed theories of kinetic, communicative, cognitive, and symbolic systems" (Streeck 2009, p. 172). As exemplified above, in the culture of mathematical research in Topological Graph Theory, gestures play a vital role in that they not only support communication among the mathematicians, but also kinesthetic and haptic experience with the diagram and the mathematical meaning that the diagram holds; and thereby they support each mathematician in his ideational realm.

Aside from Châtelet's diagrammatical studies that cover Algebraic Manifolds, Calculus, Complex Numbers, Coordinate Geometry, Geometry, Infinite Sequences and Series, Ordinary Differential Equations, Partial Differential Equations, Symmetry, Classical Mechanics, Electricity, Magnetism, Quantum Mechanics and Relativity Theory, there are other studies that similarly investigate diagramming, gesturing and mathematizing. The sociologist Christian Greiffenhagen (2014) uses one mathematical lecture in Mathematical Logic as a paradigmatic example "to extend the 'material turn' to an instance of abstract thought" (p. 21). In his analysis, he similarly observes gestures of pointing and of mathematical entities during utterances and notes that while logicians are "predominately [sic] concerned with symbolic writing [diagrams] are nevertheless a crucial aspect of some parts of their practice" (p. 23). Barany and MacKenzie (2014) studied the activities of mathematicians engaged in partial differential equations and related topics while working on the blackboard. They draw awareness to the gesture space between the blackboard and the mathematician that lends itself to exhibit intuition, invention and discovery: "In the pregnant space between chalk and slate there reposes a germ of the bursts of inspiration, triumphs of logic, and leaps of intuition that dominate mind centered accounts of mathematics" (pp. 10–11). We cannot say with certainty that every mathematical field is prone to so much diagramming and gesturing as witnessed in Topological Graph Theory through the research meetings.

Nonetheless, the findings suggest that awareness needs to be raised to the acts of diagramming in particular as indexical signs of engaging with and creating mathematics.

Lastly, we want to discuss some possible implications of these findings for the teaching and learning at all levels of acquiring mathematics. In their creative phase the mathematicians needed to *see* and *feel* objects and their relationships in order to gain an understanding about them or to explore how they could be altered or newly positioned to extract new insights in their field. How do we, as teachers in general and at all levels, allow our students to *see* and *feel* in the classroom or the office? How prevalent is the use of regurgitated visual products in order to explain, for example, what a parabola looks like instead of what a parabola is? How often do we create an environment that places each student in the position of a mathematician, where the student explores the relationship between input x and output $x^2 + c$, perhaps dynamically using geometry software, or drawing by hand the locus of points such that the distance to a given focus equals the distance to a given directrix? These are all environments, where the student is bound to point, touch, hold, trace, add and delete, which may lead to further material engagement with the curve and the curving that is the parabola.

17.7 Conclusion

While Châtelet only hints at the whole process of diagramming during mathematical invention, the analysis of the 122 time intervals and the 128 diagrams allows a glimpse into the whole process and reveals how diagrams—seen from a material semiotic perspective—come into being and are engaged with. These findings enrich Châtelet's theories by being able to identify the different functions of interactions between mathematician and diagram in inventive mathematical processes. The explications of the two episodes above reveal that diagramming during mathematizing is a ritual kind of practice (done over and over again) that offers a physical arena for experimentation. We suggest that the life-cycle functions may also offer fruitful avenues for helping students incorporate diagramming into their own mathematical problem solving, if only by seeing how drawing lines on paper might initiate new gestures and new ideas.

References

Barany, M., & MacKenzie, D. (2014). Chalk: Materials and concepts in mathematics research. In C. Coopmans, J. Vertesi, M. Lynch, & S. Woolgar (Eds.), *Representation in scientific practice revisited* (pp. 107–130). Cambridge, MA: The MIT Press.

Châtelet, G. (2000). *Figuring space: Philosophy, mathematics and physics* (R. Shore & M. Zagha, Trans.). Dordrecht, The Netherlands: Kluwer Academic Publishers. (Original work published 1993).

de Freitas, E. (2012). The diagram as story: Unfolding the event-structure of the mathematical diagram. *For the Learning of Mathematics, 32*(2), 27–33.

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1), 133–152.

Greiffenhagen, C. (2014). The materiality of mathematics: Presenting mathematics at the blackboard. *British Journal of Sociology, 65*(3), 502–528.

Grawemeyer, B., & Cox, R. (2008, September). The effects of users' background diagram knowledge and task characteristics upon information display selection. In *International Conference on Theory and Application of Diagrams* (pp. 321–334). Berlin and Heidelberg: Springer.

Hoffmann, M. H. (2006). What is a "semiotic perspective", and what could it be? Some comments on the contributions to this special issue. *Educational Studies in Mathematics, 61*(1–2), 279–291.

Ingold, T. (2013). *Making: Anthropology, archaeology, art and architecture*. London, UK: Routledge.

Kita, S. (2000). How representational gestures help speaking. In D. McNeill (Ed.), *Language and gesture* (pp. 162–185). Cambridge: Cambridge University Press.

Krummheuer, G. (2013). The relationship between diagrammatic argumentation and narrative argumentation in the context of the development of mathematical thinking in the early years. *Educational Studies in Mathematics, 84*(2), 1–17.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Levelt, W. (1989). *Speaking. From intention to articulation*. Cambridge, MA: The MIT Press.

Maddalena, G. (2015). *The philosophy of gesture: Completing pragmatists' incomplete revolution*. Montreal & Kingston: McGill-Queen's University Press.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago, IL: University of Chicago Press.

McNeill, D. (2008). *Gesture and thought*. Chicago, IL: University of Chicago Press.

Menz, P. (2015). *Unfolding of diagramming and gesturing between mathematics graduate student and supervisor during research meetings*. Ph.D dissertation, Department of Mathematics, Simon Fraser University. Available from https://theses.lib.sfu.ca/thesis/etd9092.

Novick, L. R. (2004). Diagram literacy in preservice math teachers, computer science majors, and typical undergraduates: The case of matrices, networks, and hierarchies. *Mathematical Thinking and Learning, 6*(3), 307–342.

O'Halloran, K. L. (2005). *Mathematical discourse: Language, symbolism and visual images*. London and New York: Continuum (reprinted 2008).

O'Halloran, K. L. (2011). Multimodal discourse analysis. In K. Hyland & B. Paltridge (Eds.), *Companion to discourse analysis* (pp. 120–137). London, UK: Continuum.

Peirce, C. S. (1894/1998). What is a sign? In Peirce edition Project (Eds.) *The essential Peirce: Selected philosophical writings* (vol. 2, pp. 4–10). Bloomington, IN: Indiana University Press.

Peirce, C. S. (1932). Elements of logic. In C. Hartshorne & P. Weiss (Eds.), *Collected papers of Charles Sanders Peirce* (Vol. 2). Cambridge, UK: Harvard University Press.

Radford, L. (2001, July). On the relevance of semiotics in mathematics education. Paper Presented to the Discussion Group on Semiotics in Mathematics Education at the 25th PME International Conference, July 12–17. (2001). *University of Utrecht*. The Netherlands: Utrecht.

Rossini, N. (2012). *Reinterpreting gesture as language. Language "in action"*. Amsterdam, The Netherlands: IOS Press.

Roth, W.-M., & Maheux, J. F. (2015). The visible and the invisible: Doing mathematics as revelation. *Educational Studies in Mathematics, 88*(2), 221–238.

Rotman, B. (2005, July). *Gesture in the head: Mathematics and mobility*. In Mathematics and Narrative Conference, Mykonos, Greece.

Rotman, B. (2012). Topology, algebra, diagrams. *Theory, Culture & Society, 29*(4–5), 247–260.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture, 14*(3), 351–374.
Stylianou, D., & Silver, E. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning, 6*(4), 353–387.
Streeck, J. (2009). *Gesturecraft: The manufacture of meaning*. Amsterdam, The Netherlands: John Benjamins Publishing Company.

Chapter 18
Objectifying the Inclusion Relationship of Quadrilaterals in a Synchronic-Interactive and Collaborative Computer Supported Environment

Osama Swidan and Naomi Prusak

Abstract This study aims at understanding the role of synchronous collaborative computer supported environments in objectifying the inclusion relationships of quadrilaterals. We choose the inclusion relationships of quadrilaterals since, on one hand, this topic has the potential for promoting the development of geometrical thinking, and on the other hand, several studies have clearly shown that a great number of students have problems with learning this complicated concept. The study reports on six ninth grade students working with the Virtual Math Team environment (VMT) while coping with three tasks that were meticulously designed to motivate collaboration among them. The study is guided by the theory of knowledge objectification and the data were analyzed using the semiotic tools of attention and awareness. The analysis of the data suggests that the students objectified the inclusion relationships of quadrilaterals. The study shed light on the nature of the social interactions between the students and the technological artifact during the objectification processes. Insights for re-designing the learning tasks and for further research were also drawn.

Keywords Objectification · Geometry · Inclusion relationship · Collaborative learning · Argumentation

O. Swidan (✉)
Ben-Gurion University of the Negev, Beer-Sheva 653, Israel
e-mail: osamasw@gmail.com

N. Prusak
School of Education, The Hebrew University of Jerusalem,
5, Rozen st., Jerusalem 9692521, Israel
e-mail: inlrap12@netvision.net.il

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18.1 Introduction

The promotion of collaborative learning has become a central educational goal in the last decade. It was found that problem solving and concept learning may be enhanced by the use of cooperative groups. Collaborative settings in which argumentation is fostered may promote conceptual learning (Schwarz 2009). However, this potentiality is rarely realized. As noted by Webb (1995), arranging students in small groups rarely leads them to collaborate, even when students are given scripts or instructions in advance for collaborating. Indeed, teachers also need to structure tasks to support collaboration (e.g., Mercer 1996). Design principles such as the creation of socio-cognitive conflict, or the administration of complex tasks that demand coordination of action may encourage collaboration in certain cases. Yet, even when appropriate conditions are created to trigger collaboration, students have difficulties in maintaining it in the end. In this situation, scientists have created a new field—Computer-Supported Collaborative Learning (CSCL), in which technological tools are designed for affording productive collaboration in group-work (Stahl 2013). The present chapter presents a study in which a CSCL environment was used to foster the learning of a challenging idea in geometry, namely, the inclusion relationships of quadrilaterals.

Several studies have clearly shown that many students have problems with the inclusion relationships of quadrilaterals (e.g., Fujita and Jones 2007). The term inclusion relationship means here the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts. The inclusion relationship of quadrilaterals helps in promoting the development of geometrical thinking (Fujita and Jones 2007). According to De Villiers (1994) there are some important functions of this inclusion relationship: (a) It simplifies the deductive systematization and derivation of the properties of more special concepts; (b) It often provides a useful conceptual schema during problem solving; and (c) It sometimes suggests alternative definitions and new propositions. For example, to justify why a square is a kite, learners need to be able to examine its properties. The fact that a square has more properties than a kite should not impinge on the right answer, but in everyday reasoning, it does: children find it difficult to distinguish between critical and non-critical properties (Erez and Yerushalmy 2006). Several researchers observed that dynamic geometry software offers great potential for conceptually enabling many children to see and accept the possibility of hierarchical inclusions; for example, letting them drag the vertices of a dynamic parallelogram to transform it into a rectangle, a rhombus or a square (Fujita and Jones 2007). Dynamic geometry tools enable the testing of hypotheses and the manipulating of geometrical objects. Indeed, through the dragging action, students often switch back and forth from examples of figures to concepts and this move might push students from material manipulations to high-level reasoning (Hanna 2000).

Our general assumption was that collaboration supported by dedicated tools and peer interaction would entail a shift from reasoning based on perceptual objects that appear on the screen to reasoning moved by logical necessity, ultimately leading to proving, thereby providing the possibility of facilitating the objectification of the inclusion relationship of quadrilaterals. The aim of the research was to better understand the role of synchronous collaborative computer supported environments in students becoming fully aware of the inclusion relationship of quadrilaterals. To shed light on the effectiveness of the suggested learning environment, we used semiotic lenses in analyzing the development of students' understanding of the inclusion relationship through peer interactions. By this means, in this chapter we address the question of how middle school students become aware of the inclusion relationship of quadrilaterals in a synchronous collaborative computer supported environments.

18.2 Theoretical Perspective

The theory of knowledge objectification, which guided the present study, considers learning as a process of objectification (Radford 2003). In other words, learning is a matter of actively endowing with meaning the conceptual objects (e.g., inclusion relationship of quadrilaterals) made available by artifacts. Objectification as a theoretical tool enables a thorough analysis of the interaction between conceptual objects deployed in the artifact and the subjective meanings of the students.

The theory of objectification considers thinking to be a mediated reflection in accordance with the form or mode of activity of individuals (Radford 2008). Radford uses the term "mediating" to refer to the role played by artifacts in carrying out social practices and considered artifacts as constitutive and consubstantial parts of thinking. He maintained that we think with and through cultural artifacts, which are essential sources of learning, and shape our thinking. Artifacts are carriers of the historical knowledge that has been produced through the activities of the last generations. Therefore, social practices are not limited to the interactions between people in a learning setting but also include the interactions conducted with the cultural artifact. For Radford, reflection means a dialectical movement between a historically and culturally constituted reality and individuals who refract and modify it according to their subjective interpretations, actions, and feelings. In the mediated reflection process of learning, the semiotic means are considered fundamental constituents of thinking because they mediate the social activity, and bind the individual to the historical and cultural dimension of the mathematical object. Bartolini Bussi and Mariotti (2008) argued that the relation between the artifact and the accomplishment of a task is expressed by signs such as gestures, speech, and drawings. Arzarello et al. (2009) broadened the range of signs relevant in the teaching-learning process to include words, gestures, glances, drawings, extra-linguistic modes of expression, and technological artifacts. In agreement with Radford (2008), Arzarello et al. (2009) and Mariotti (2009) considered these semiotic means to be crucial in learning processes because of the role they play in the meaning coming to life. Learning as objectification requires making use of different semiotic means such as words, symbols, actions with the artifact, rhythmic speech, and gestures available in the universe of the discourse (Radford 2003). Radford referred to this process as a "semiotic means of objectification."

From a pragmatic point of view, Radford suggests a semiotic tool to analyze educational mathematical activities in the classroom. The basic components of the semiotic tool are the students' *attention* and *awareness* of the mathematical object. Varieties of semiotic means of objectification that have a representational function attract the students' *attention* to mathematical objects. Furthermore, the properties of the artifact can help students to attend to the mathematical objects related to the activity under consideration. Through *paying attention* to the necessary aspects of the mathematical phenomenon and using various semiotic means of objectification, students become aware of the attributes of mathematical objects within that phenomenon. By being aware, students attain objectification of the mathematical objects, which then become apparent to them through various devices and signs (Roth 2015).

It has been suggested that in order to boost learning processes through technological tools, a series of learning tasks should be given to students with the aim of achieving a specific learning objective. In the theory of knowledge objectification, learning tasks are means for achieving the type of praxis cogitans[1] or cultural reflection (Radford 2008). The character of the tasks can encourage students to actively engage with and explore the mathematical relationships contained in the artifact and to become aware of these relationships. In addition, tasks have a mediating role between the artifacts, the signs that emerge as a result of using the artifact, and the mathematics culture.

Being in a problem solving situation and thinking mathematically require the students to reflect on the task. The process of reflection facilitates the students' *awareness* of their own solution processes and methods. Students who are encouraged to offer explanations for their reasoning and are motivated to reflect on it, will eventually come to understand that such processes are intrinsic to learning by doing mathematics (Schoenfeld 1994). Tasks that are good candidates for considering and learning about mathematical relationships provide challenges to students, encourage argumentation processes, and provide social interactions between learners (Andriessen and Schwarz 2009). To design tasks that encourage productive argumentation and rich social interaction, scholars (e.g., Prusak et al. 2012) proposed several factors to be considered: (a) the creation of collaborative situations; (b) the design of activities that allow for the establishment of socio-cognitive conflict; and (c) the provision of tools for checking hypotheses.

Based on the aforementioned principles, we designed the research experiment activity presented in the next section. We used the task design principles to design the three tasks that invite the students to engage in the process of objectifying the hierarchical inclusion relationship of quadrilaterals.

[1]The Latin term, *cogitans*, means thinking. Radford uses the term *praxis cogitans* to stress the idea that the way we think is a form of cultural practice.

18.3 The Research Design Experiment

18.3.1 The Synchronic-Interactive and Collaborative Artifact

To fulfill our research aim we used a Computer-Supported Collaborative learning (CSCL) environment that used technological tools designed for affording productive collaboration in group-work. We designed the research experiment in a way that allows students to engage in productive dyadic argumentation. The CSCL tool we chose was Virtual Math Teams (VMT). The Virtual Math Teams is a mathematics forum that includes a GeoGebra applet (Fig. 18.1) shared by all participants, and offers them the opportunity to collaborate on geometrical tasks (Stahl 2013).

The VMT interface includes two spaces:

(a) A drawing space in which small groups of students can share their mathematical explorations and co-construct geometric figures online. When one of the participants drags or constructs a geometrical figure, all the others can see the changes of the figure. Only one student can interact with the figure at a given moment, by taking control of manipulations.
(b) A text chat window, where students write their ideas and share them with their peer. In this way, students can scroll up and down to return to previous conversations (Fig. 18.1).

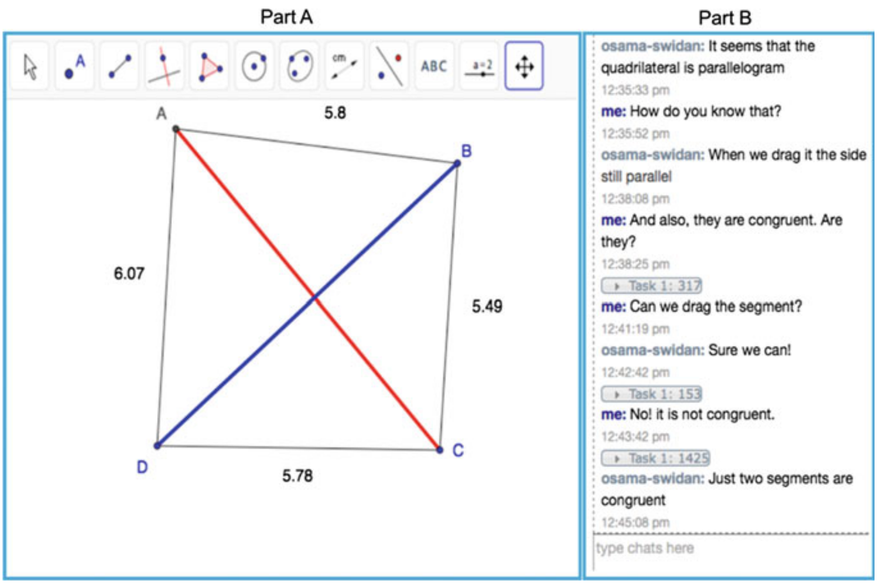


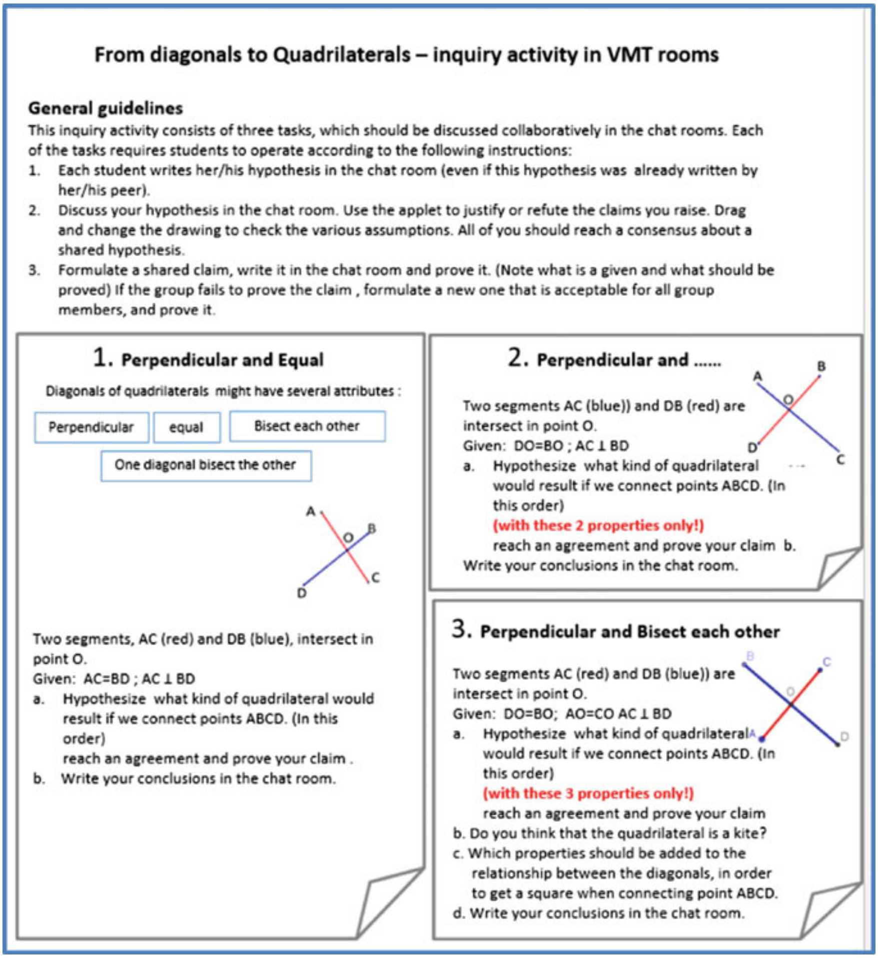
Fig. 18.1 VMT interface

18.3.2 The Participants

The participants in this research were six ninth-grade students. All participants were high achievers in mathematics and studied mathematics at the highest level in their schools. At the time the meetings took place, the students had already learned geometrical basic concepts, such as special lines of triangles, congruency of triangles and the properties of different quadrilaterals. The students were familiar with the dynamic geometry environment (DGE), which was part of their previous study of geometry within the framework of the formal school curriculum. They were familiar with using congruence theorems and were able to write and prove elementary theorems, based on the use of congruence theorems. The students studied Euclidian geometry from a textbook that is available for middle school students in Israel. Generally, the textbooks contain a brief theoretical discussion of the concept and provide exercises to apply the concepts and strategies to solve geometrical problems.

18.3.3 The Tasks

The design of the tasks relied on the theory of knowledge objectification and the design principles aforementioned. Accordingly, we designed an inquiry activity that consists of a sequence of three tasks. We designed the tasks to invite the students to engage in a problem-solving situation and to explore the mathematical structure of the inclusion relationship of quadrilaterals contained in the artifact. The aim in the first task was to provide a situation in which two properties of the diagonals are given and yet these properties are not sufficient for generating a specific type of quadrilateral. We assumed that the student would hypothesize that the quadrilateral should be either a kite or a rhombus, and the dragging option of the dynamic geometry (DGE) software would generate a situation of conflict between their expectation and the outcomes. We assumed that this situation of conflict would stimulate the need for proving. In each of the three tasks, students were asked to go through similar phases: (a) to find individually a solution and write it in the chat room; (b) to work collaboratively in the VMT room in dyads or triads and to try to reach a consensus concerning the solution; (c) to check conjectures through the DGE software. In the cases in which a wrong conjecture was drawn, students were asked to raise a new conjecture and to recheck it with the DG tools; (d) to prove and draw conclusions. In the second and the third parts of each task, the properties of the diagonals were changed in a way illustrating the hierarchical inclusion relationship of quadrilaterals. The aims in the second and third tasks were as follows: (i) to realize that the properties of the diagonals are sufficient for generating a specific quadrilateral; (ii) to realize that when dragging and exploring, the shape of the quadrilateral might change, but some of its properties remain constant the very properties that characterize the type of quadrilateral; (iii) to construct hierarchical



From diagonals to Quadrilaterals – inquiry activity in VMT rooms

General guidelines

This inquiry activity consists of three tasks, which should be discussed collaboratively in the chat rooms. Each of the tasks requires students to operate according to the following instructions:

1. Each student writes her/his hypothesis in the chat room (even if this hypothesis was already written by her/his peer).
2. Discuss your hypothesis in the chat room. Use the applet to justify or refute the claims you raise. Drag and change the drawing to check the various assumptions. All of you should reach a consensus about a shared hypothesis.
3. Formulate a shared claim, write it in the chat room and prove it. (Note what is a given and what should be proved) If the group fails to prove the claim , formulate a new one that is acceptable for all group members, and prove it.

1. Perpendicular and Equal

Diagonals of quadrilaterals might have several attributes :

Perpendicular | equal | Bisect each other | One diagonal bisect the other

Two segments, AC (red) and DB (blue), intersect in point O.
Given: AC=BD ; AC ⊥ BD

a. Hypothesize what kind of quadrilateral would result if we connect points ABCD. (In this order)
reach an agreement and prove your claim .
b. Write your conclusions in the chat room.

2. Perpendicular and ......

Two segments AC (blue)) and DB (red) are intersect in point O.
Given: DO=BO ; AC ⊥ BD

a. Hypothesize what kind of quadrilateral would result if we connect points ABCD. (In this order)
(with these 2 properties only!)
reach an agreement and prove your claim b.
Write your conclusions in the chat room.

3. Perpendicular and Bisect each other

Two segments AC (red) and DB (blue)) are intersect in point O.
Given: DO=BO; AO=CO AC ⊥ BD

a. Hypothesize what kind of quadrilateral would result if we connect points ABCD. (In this order)
(with these 3 properties only!)
reach an agreement and prove your claim
b. Do you think that the quadrilateral is a kite?
c. Which properties should be added to the relationship between the diagonals, in order to get a square when connecting point ABCD.
d. Write your conclusions in the chat room.

Fig. 18.2 The tasks and instructions given to the students

inclusion relations in the family of quadrilaterals (see Fig. 18.2). For each task, we designed an applet in such a way that the two segments given fulfill the requirements of the task.

18.3.4 Data Analysis

We analyzed the data using the pragmatic point of view of the theory of knowledge objectification in order to fulfill the research aim. The theoretical assumption of the theory of knowledge objectification is that through the process of *becoming aware* of mathematical relationships, students mobilize multi-modal actions (e.g. gesturing, speaking, writing, drawing, etc.), through which mathematical meanings are formed. We examined the ways in which dyads and triads used synchronic-interactive and collaborative artifacts in the process of objectification of the inclusion relationship of quadrilaterals. We used the basic components of the semiotic tool, namely, the students' progressive attention and different stages of awareness of the mathematical object, as suggested by Radford (2003) and Roth (2015), to analyze the dynamic development of the inclusion relationship.

We analyzed the data in two phases. In the first phase, we read the transcripts to detect the *focuses of objectification* of the inclusion relationship of quadrilaterals. Focuses of objectification were defined as segments of discourse in which the students sought to discover possible relationships between the quadrilaterals and to come up with conjectures about the inclusion relationship of quadrilaterals. In the second phase, we wrote descriptions of each student's work, drawing special attention to the semiotic means used by the students and the interactions between the students themselves and the artifacts. In this phase, we also analysed the becoming aware movement of the inclusion relationship. Doing so, we used the three dimensions of becoming aware suggested by (Roth 2015), namely, being unaware, latent awareness, and full awareness. We said that the students are being unaware if the students did not pay attention to the conceptual object that is displayed in the artifact. We said that the students *have latent awareness* of the inclusion relationship when they declare the existence of an inclusion relationship of quadrilaterals. Declarations about the existence of inclusion relationships of the quadrilaterals tended to be based on visual and experimental considerations. For example, when students observed a square while dragging a family of rectangles and declared that a square is a special case of a rectangle, we said that the students had *latent awareness* of an inclusion relationship. Justifications and interpretations based on taking into account the common properties of the shapes and considering the deductive reasoning of what students had noticed were defined as *full awareness*. For example, we said that students became fully aware of an inclusion relationship between a square and a rectangle when they justified this phenomenon by explaining that the diagonals of squares have the same properties as rectangles, i.e., diagonals are congruent and they bisect each other.

18.4 Findings

In the first task, the students were asked to explore for which quadrilaterals, diagonals are equal and perpendicular.

18.4.1 Latent Awareness of the Relationship Between the Diagonals and the Quadrilaterals

1.	Neta		Drags point D
2.	Avia	I think it is a kite or a rhombus, but I am not sure about the difference between them.	
3.	Neta	It seems to be a rhombus, since the diagonals are perpendicular and congruent.	
4.	Avia		Drags point D
5.	Neta	In a kite the diagonals are not congruent	

This excerpt (Fig. 18.3) illustrates the being unaware phase of the ***inclusion relationship between quadrilaterals***. Although the excerpt does not include an indication of using an inclusion relationship, it still considers an important phase in objectifying the inclusion relationship. This excerpt demonstrates the process of latent awareness of the ***relationship between the diagonals and the quadrilateral.*** Neta's utterances [2] and [3] suggest that they are not yet fully aware of the relationship between the diagonals and the quadrilaterals. Their subjective meaning is partially fitting the mathematical meaning of the relationship between the diagonals and quadrilaterals. In other words, the rhombus's diagonals are not always congruent, and the kite's diagonals might be congruent.

Furthermore, the dragging action afforded the mediation and reflection processes in the course of becoming aware of relationships between the diagonals and the quadrilaterals. The social practices resulting from the dragging tool were characterized by Avia raising a conjecture [2] regarding the type of quadrilateral, and her partner elaborating the conjecture [3]. Dragging the quadrilateral allowed the students to see different objects on the screen. This aspect encouraged discussion between the students, which afforded the students latent awareness of the inclusion relationship, as the next excerpt illustrates.

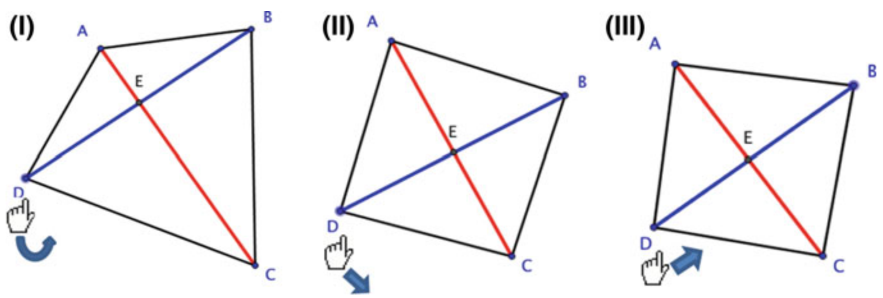


Fig. 18.3 **I** Neta drags point D and produces a kite; **II** Neta drags point D and produces a rhombus; **III** Avia drags point D and maintains a rhombus

18.4.2 Latent Awareness of the Inclusion Relationship Between Kite and Rhombus

It seems that Avia was not fully convinced by Neta's determination that the quadrilateral is a rhombus. So Avia dragged point D once, and then point B. Her action dragging point B suggests that Avia was looking for new type of quadrilaterals, other than rhombuses and kites (Fig. 18.4).

6.	Avia		Drags point D and then point B.
7.	Avia	It is a rhombus …	
8.	Avia	But in a rhombus, the sides must be congruent. Are they?	
9.	Neta	Yeah, so we must prove it.	
10.	Avia		Drags point B
11.	Neta	It may be a kite.	
12.	Avia		Drags point B
13.	Avia	It also could be a square. Although sometimes when you drag it. It does not maintain as a square …	
14.	Neta	So, it is not a square.	
15.	Avia	It is a kite or a rhombus.	
16.	Neta	It is a kite since rhombus is a special case of kite	

This except illustrates the processes that allowed the students to achieve latent awareness of the inclusion relationship between rhombus and kite, as suggested by utterance [16]. At this phase the students just became latently aware of the inclusion relation but not fully aware of it, as evidenced by the students mentioning the inclusion relationship without discussing the connections between the two quadrilaterals and their diagonals.

The process of becoming latently aware of the inclusion relationship was a complex one and included iterative engagement with the artifact, raising conjectures, refuting them, and becoming aware of the invariant aspect within the artifact. The dragging action played a central role in the dialectical movement between the

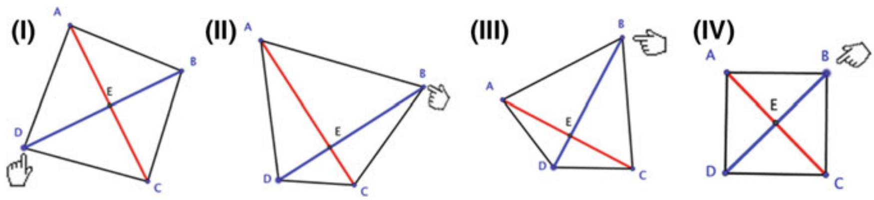


Fig. 18.4 **I** Avia drags point D and produces rhombus; **II** then Avia drags point B and produces a quadrilateral which is not rhombus; **III** Avia drags point B and produces a kite; **IV** Avia drags point B and produces a square

subjective meaning of the students and endowing it with the mathematical meaning of the inclusion relationship. This dialectical movement afforded the students to become aware latently of the inclusion relationship. In addition, the social practices that emerged throughout this phase of learning also afforded them to attain this latent awareness. The social practices are characterized by a chain of processes: dragging the quadrilateral; raising a conjecture; challenging the conjecture; raising a new conjecture. Looking for the quadrilateral whose diagonals are perpendicular and congruent through this chain of processes, engaged the students in producing various shapes of kites. One of these kites was a rhombus. All of these processes led the students to the latent awareness of the inclusion relationship between kites and rhombuses.

To summarize this phase of the learning, the dragging tools and the shared screen allowed the students to become latently aware of the inclusion relationship in the case of kites and rhombuses. The interaction between the students in the chat window allowed them to become aware of the congruence between the quadrilateral's properties and the invariant aspects that were kept in the figure. Yet, at this phase, the students did not become fully aware of the inclusion relationship between the quadrilaterals. It is worth noticing that the correct answer for this task is: a "no-name" quadrilateral.

18.4.3 Deductive Reasoning as a Bridge for Becoming Fully Aware of the Inclusion Relationship

In the second task, the students were asked to explore the nature of the quadrilateral whose diagonals are perpendicular and one of them bisects the other (Fig. 18.5). The expected answer is a kite. The students conjectured that the quadrilateral that satisfies the task requirements is a kite (the right answer).

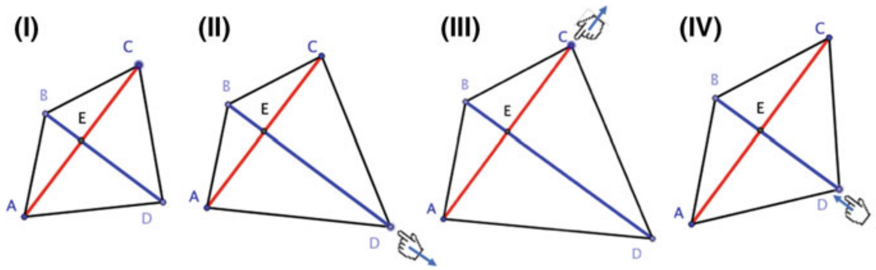


Fig. 18.5 **I** The figure that appears on the screen; **II** Avia Drags point D; **III** Avia drags point C; **IV** Avia drags point D

17	Avia	I think it is a kite	
18.	Neta	I also think it is a kite.	
19.	Avia		Drags point D, then Point C, and finally again point D (Fig. 18.5)
20.	Neta	Okay! Avia we need to prove it.	
21.	Avia	It is because the two diagonals are perpendicular, and two sides are congruent.	
22.	Neta	The sides' length is not given in the task. You must prove that they are congruent.	
23.	Avia	But it is given that one diagonal bisect the other.	
24.	Neta	So…	
25.	Neta	BE is a height in the triangle ABC	
26.	Avia	We must prove that there are pairs of congruent triangles	
27.	Neta	BE is a median since it split AC into two equal segments. Therefore, triangle ABC is an isosceles triangle because the height coincides with the median.	

This excerpt illustrates one aspect in the process of becoming fully aware of the inclusion relationship between quadrilaterals, namely, deductive justification of the observed phenomenon. The social practices identified in this excerpt are generally similar to those revealed in the excerpt aforementioned. However the chain of the social practices in this excerpt is characterized by raising the same conjecture by the two students, verifying the conjecture perceptually by producing various shapes of quadrilaterals, and most importantly—the transition from the experimental aspect to deductive justification. Also in this excerpt, the dragging actions, in regard to the social practices, played a central role in the process of verifying the conjecture, namely the dragging action allowed the students to produce several figures that satisfied the quadrilateral they conjectured.

Being aware of the necessity for a deductive proof was invited by the task's design, which explicitly asked for that. However, the transition towards the deductive justification was not smooth. It continued to include perceptual aspects observed on the screen. Students seemed to depend on these in the deductive processes. Due to the awareness of one of the students and the collaboration created between them, the students shifted their reasoning from what was given as data in the task to what should be proved deductively.

To summarize this stage of learning, indeed the students were not engaged in an inclusion relationship between quadrilaterals in this task but the current task allowed them to become aware of the need to prove a statement in a deductive way as a vehicle for being convinced for its truth. It seems that this stage of learning bridged the objectification of the inclusion relationships, as we show in the next excerpt.

18.4.4 Objectifying the Inclusion Relation in Quadrilaterals

In task 3, the students were asked to explore the nature of the quadrilateral whose diagonals bisect each other and are perpendicular (Fig. 18.6). It is noteworthy that the conditions in task 3 include the properties given in task 2, and one more condition was added—the diagonals bisect each other. We expected that the students would notice this fact, and would assume that the quadrilateral is a 'special' kite with an additional condition.

28.	Avia	The two segments are bisecting each other, so it is a parallelogram.	
29.	Neta	It is a rhombus. In the parallelogram, the diagonals are not perpendicular.	
30.	Avia		Drags point B, then drags point A, adds the sides of the quadrilateral, and again drags point A (Fig. 18.6)
31.	Avia	It is a rhombus. It is a specific kind of kite. Kite which is a rhombus. We need to prove that all the triangles are congruent.	
32.	Neta	We can prove that the triangles BEC, BEA, AED, and DEC are congruent. By using the theorem S.A.S. All of them are given.	

By adding one more property (bisect each other) to the two properties given in task 2 (one diagonal bisects the other and they are perpendicular), we intended to invite students to pay attention to the fact that the new obtained quadrilateral first of all should be a kite, and then as a result of having one more property it should become a regular kind of a kite. In fact, one of the students claimed that the quadrilateral is a parallelogram [28]. This claim might be the result of the great significance given to the theorem in traditional geometry lessons, in which they did not use dynamic technology or inquiry based learning. Although students partially

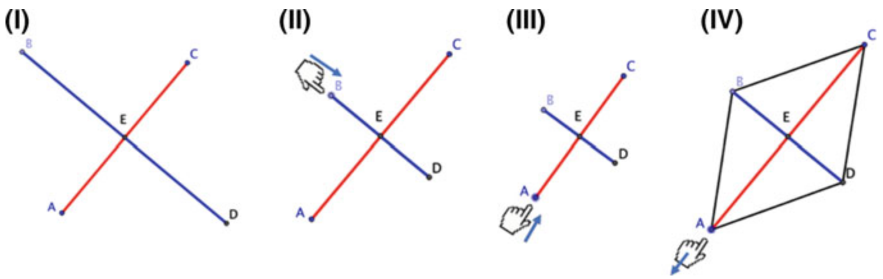


Fig. 18.6 **I** The figure that appears on the screen; **II** Avia Drags point B; **III** Avia drags point A; **IV** Avia adds the segments of the quadrilateral and drags point D

accepted that the solution to task 3 was a parallelogram, this answer caused a conflict between the students. This conflict suggested that the students, at this phase of the learning process, had not fully become aware of the inclusion relationship between quadrilaterals [28–29].

At the end of the activity the researcher, who was virtually present in the VMT with the students, asked the students to summarize their learning activity.

33.	Mediator	Could you explain how the properties change in the tasks and how this change is reflected in the quadrilateral's type?
34.	Avia	If the diagonals bisect each other and are perpendicular, the quadrilateral will be a rhombus.
35.	Neta	She is asking about all the tasks and not only about the last one. Anyway, in each task, one new property is added, thus the new quadrilateral we get is a specific kind of the latter.
36	Mediator	Neta! Could you please explain in more detail what you mean?
37.	Avia	In the second task, the diagonals are perpendicular and one of them bisects the other. In this case, the quadrilateral was a kite. In the following task, one property was added. It was still a kite but a specific kind of kite, rhombus. Because the properties of the kite were kept.

Later the mediator again asked the students: What are the properties of the diagonals you should add to the last task (in which the answer was a rhombus) in order to get a square?

38.	Avia	The diagonals should be congruent.
39.	Neta	Yes, I agree with Avia, because a quadrilateral whose diagonals bisect each other and are perpendicular, is a rhombus. When we add a new property to these properties, it will remain a rhombus but of special kind. A rhombus that has one of its angles 90° is a square.
40.	Mediator	How do you know that the angle is 90°?
41.	Avia	If a median of a triangle is equal to half of the base, the triangle must be a right triangle.

This excerpt illustrates the processes of becoming fully aware of the inclusion relationship between kites, rhombuses, and squares [37, 39]. In [33] the intervention of the mediator introduced a new social practice when she asked the students how the change in the diagonals' properties may affect the shape of the quadrilateral. The mediator's question was creative in a manner that boosted the students' awareness of the inclusion relationship of quadrilaterals. Furthermore, the mediator's questions not only prompted the becoming aware process but these questions also promoted deductive reasoning, which is a central issue in learning geometry. However, from the mediator's intervention and the students' responses, we could learn that the students were in the zone of proximal development (Vygotsky 1980). Therefore, we assume that, in order to reach a better objectification of the inclusion relationship, students should engage in additional tasks, that emphasis various inclusion relationships. This could be done, for example, starting with diagonals bisecting each other (a parallelogram), then diagonals congruent and bisecting each other (a rectangle), and finally bisecting each other, congruent, and perpendicular (a square).

18.5 Final Remarks

The aim of this study was to explore the role of synchronous collaborative computer supported environments and the role of task design for affording collaboration and leading to the objectification of the inclusion relationships of quadrilaterals. The findings of the study confirm that the design partially supported the objectification of inclusion relationships of quadrilaterals. Indications of this objectification can be found in the first and the third tasks. It seems that looking for a quadrilateral with the properties given in task 1 lines [6–16]: the diagonals are perpendicular and congruent), afforded the students *paying attention* to the inclusion relationship. Possibly through the dragging actions, students were prompted to examine special cases (such as a kite, a rhombus, a square) of a general case (the 'no name' quadrilateral), emphasizing the inclusion relationships of quadrilaterals. These findings are consistent with the conclusion drawn by Fujita and Jones (2007), who observed that dynamic geometry software enables many students to see and accept the possibility of inclusion relationships among different types of quadrilaterals. Additional indications for objectifying the inclusion relationships were found in task 3 (in lines [35–37] and [38–41]). In this case the students *became fully aware* of the inclusion relationships by considering how the addition of one property affected the newly obtained quadrilateral, especially when the students explicitly articulated that the rhombus is a special case of a kite. Moreover, they summarized their insights into the inclusion relationship when they said: "In the second task the quadrilateral was a kite. In the third task, one property was added. It remained a kite but a specific kind of a kite, a rhombus. Because the properties of the kite were kept."

Although in the course of performing the second task, the students' discourse did not refer directly to the inclusion relationship, it facilitated them in *becoming aware* of the need of using deductive reasoning. Through mediated reflected use of the artifact, the students became aware of the distinction between data displayed on the screen and data given in the task [20–22]. Following Radford (2008) who claimed that learning is more than endowing a conceptual object with meaning, we suggest that learning is a matter of becoming part of the mini-culture we helped to establish. We conclude that engaging in task 2 contributed to the process of introducing deductive reasoning as a part of the students' culture in this environment.

As the data analysis suggests, the objectification of the inclusion relationship among quadrilaterals was achieved in the context of rich social interactions between the students themselves as well as the interaction with the VMT software. Each student took into consideration the actions and the claims of her peer. In addition, we observed dynamic changes in students' insights that emerged in the course of the social interaction. These interactions were both dialogical and dialectical. They were dialogical when each student co-elaborated the other's articulation towards the emergence of new ideas. They were dialectical when students challenged each other. These two types of interaction contributed to the objectification of the inclusion relationships. It appears that the Virtual Math Team environment and the designed tasks afforded these two kinds of interactions.

The analysis of becoming aware allows the exploration of the role of the environment and the tasks' design in suggesting alternative definitions and propositions concerning quadrilaterals. The analysis reveals the use of words that indicate the utilization of terms that are typical in inclusion relationships. The use of these terms indicates a shift in thinking from a quadrilateral to its properties, to thinking from properties to types of quadrilaterals. This shift evidences a higher level of thinking.

In this chapter we demonstrate the use of the objectification theory as a methodology for examining the meaning making processes of a complicated geometrical concept in a computer supported collaborative learning environment. Using the objectification theory as a methodology required the elaboration of the term awareness to include levels of awareness: being unaware, latent awareness and full awareness. In addition we presented in the paper a practical definition for these levels. The use of the three levels and their definition is a contribution for applying the objectification theory analytically.

Using a semiotic framework, this study also contributes to better understanding the role of collaborative and dynamic geometrical environments in learning complex mathematical concepts. Compared with the traditional use of DGE, this environment offers new affordances as well as some constraints. For example the shared drawing space in which small groups of students can share their mathematical explorations and co-construct geometric figures online, and the text chat window, where students *write* their ideas and share them with their peers, afford the students a collaborative situation. This environment influences the type of semiotic means that were generated during the experiment. It encourages the student to argue and develop shared ideas using the applet as a tool for checking assumptions. The fact that students were not able to communicate verbally or to gesture freely are limitations that we took upon ourselves in choosing this environment. We assumed that these limitations would allow us to concentrate on precise semiotic analysis.

The present case study provides empirical evidence for the efficiency of synchronous collaborative computer supported environments in objectifying the inclusion relationship of quadrilaterals. Further studies should be conducted to better understand the role of CSCL environments in objectifying the inclusion relationship of quadrilaterals, and to report on new challenges that emerge as a result of utilizing this environment. However, since VMT relies on GeoGebra, we anticipate that the structure of this research might serve other mathematics education researchers in the investigation of the learning of other mathematical concepts.

References

Andriessen, J. E. B., & Schwarz, B. B. (2009). Argumentative design. In N. Muller-Mirza & A-N. Perret-Clermont (Eds.), *Argumentation and education—Theoretical foundations and practices* (pp. 145–174). New York: Springer.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109. https://doi.org/10.1007/s10649-008-9163-z.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York: Routledge.

De Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics, 14*(1), 11–18.

Erez, M., & Yerushalmy, M. (2006). "If you can turn a rectangle into a square, you can turn a square into a rectangle": Young students' experience the dragging tool. *International Journal of Computers for Mathematical Learning, 11*(3), 271–299.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education, 9*(1&2), 3–20.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44,* 5–23.

Mariotti, M. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM—The International Journal on Mathematics Education*, *41*(4), 427–440. https://doi.org/10.1007/s11858-009-0199-z.

Mercer, N. (1996). The quality of talk in children's collaborative activity in the classroom. *Learning and Instruction, 6*(4), 359–377.

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics, 79,* 19–40.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70. https://doi.org/10.1207/s15327833mtl0501_02.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishers.

Roth, W. M. (2015). Becoming aware: Towards a post-constructivist theory of learning. *Learning: Research and Practice, 1*(1), 38–50.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53–69). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schwarz, B. B. (2009). Argumentation and learning. In M. Mirza & A-N. Perret-Clermont (Eds.), *Argumentation and education—Theoretical foundations and practices* (pp. 91–126). New York: Springer.

Stahl, G. (2013). Translating Euclid: Designing a human-centered mathematics. San Rafael, CA: Morgan & Claypool Publishers. http://GerryStahl.net/elibrary/euclid.

Webb, N. M. (1995). Group collaboration in assessment: Multiple objectives, processes, and outcomes. *Educational Evaluation and Policy Analysis, 17*(2), 239–261.

Vygotsky, L. S. (1980). *Mind in society: The development of higher psychological processes*. Boston, MA: Harvard University Press.

Chapter 19
Discussion and Conclusions

Norma Presmeg, Luis Radford, Wolff-Michael Roth and Gert Kadunz

Abstract After an introductory section that addresses the nature of semiotics, the editors discuss themes that highlight issues that have arisen from and that illustrate what has been accomplished in the varied chapters of this monograph. The final section provides some suggestions, based on these issues, for further research on the various threads that pertain to the potential significance of semiotics in mathematics education. The editors believe that there is room for both theoretical development and further empirical studies designed in resonance with these theories, in order to address the full potential of semiotics in areas of research that have not yet received widespread attention.

Keywords Semiotics · Signification · Iconic and indexical relationships Qualitative methodologies · Interactive technology

N. Presmeg (✉)
Mathematics Department, Illinois State University,
5160, Candlewood Court, Maryville, TN 37804, USA
e-mail: npresmeg@msn.com

L. Radford
École des sciences de l'éducation, Université Laurentienne, Sudbury
ON P3E 2C6, Canada
e-mail: lradford@laurentian.ca

W.-M. Roth
Applied Cognitive Science, University of Victoria, MacLaurin Building A567,
Victoria, BC V8P 5C2, Canada
e-mail: mroth@uvic.ca

G. Kadunz
Department of Mathematics, Alpen-Adria
Universitaet Klagenfurt, Universitaetsstrasse 65, 9020 Klagenfurt, Austria
e-mail: gert.kadunz@aau.at

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19.1 The Nature of Semiotics Revisited

Dictionaries—because they provide definitions according to common usage—do not always address the latest theoretical developments in thinking about terms, such as those relevant to semiotics, that are the concern of researchers in fields that address human knowledge and its implications for education. This is so particularly in mathematics education, which is the field of this monograph. Nevertheless, some aspects of dictionary definitions address the core of certain words. For example, here are some definitions that are pertinent:

sign (noun): any indicative or significant object or event; a symbol; token (Funk and Wagnall).
represent (transitive verb): to bring before the mind (Webster); to serve as the symbol, expression, or designation of; to set forth a likeness or image of (Funk and Wagnall).
representation (noun): the act of representing, or the state of being represented; the stage or process of mental conservation that consists in the presenting to itself by the mind of objects previously known (Funk and Wagnall).
mediate (transitive verb): to be the means of (Webster); to serve as the medium for effecting (a result) or conveying (an object, information, etc.); to occur or be in an intermediate relation or position (Funk and Wagnall).

Semiotics is the science of signs; it is clear that the nature of a *sign* is that it *stands for* some entity, and that this representation is the essence of the term.

Now, based on the writings of Peirce (1931–1958), the sign frequently is described as the triadic relationship among sign vehicle, object, and interpretant (e.g., Presmeg 2002; Sáenz-Ludlow and Kadunz 2016a; Sáenz-Ludlow and Presmeg 2006). However, there are also powerful voices that express different opinions. Some philosophers suggest that Peirce never thought about the sign in the triangle that Ogden and Richards later used, charging it with introducing a dualism of mind and world (Dewey and Bentley 1949/1999). Some semioticians, too, consider the triangular diagram over-simplified, though giving it some credit when theorizing the production of signs (Eco 1976). The triad could imply an interpreted *iconic* likeness of sign vehicle to its object, in which case the relationship could involve an image, a diagram, or a metaphor (Kinach, Chap. 13). But the triad could also imply something of the order of indexicality. As Menz and Sinclair point out (Chap. 17), during *creative* mathematizing with diagrams the signifying relationship appears to be indexical, something of a pointing or indicative nature. Although philosophers hoped that objective expressions—all those built on "the principles, theorems, proofs, and theories of the 'abstract sciences'" (Husserl 1913b, p. 81)—and subjective–indexical expressions could be distinguished, some social scientists later noted the indexical properties of *all* forms of signs (Garfinkel 1967). Pragmatic philosophers thus encourage us to investigate signs (language) in actual use rather than trying to figure out some unchanging structural relation, such as that between sign (signifier) and meaning (Dewey and Bentley 1999/1949; Wittgenstein 1997/1953).

Along this pragmatic line of thought, it is worth recalling that Peirce (1992, 1998) was one of the first semioticians to insist on the advantage of thinking of semiotics as a *process*. Although his work starts with the triadic definition of a sign, the force of his approach is the continuous relationship between a sign and a new sign, a process that goes from signs to signs. What this view means is that a sign is not a static entity; it is rather an entity that is part of a chain. If the defining feature of a *sign* is that it *stands for* some entity, and *representation* is the essence of the term, we need to understand this representation not in terms of a mapping but in a fluid and dynamic manner. The focus shifts from object-representation to representing, something that using a more modern terminology we may call *signifying*. In this view signs are dynamic entities arising according to specific forms of signifying.

As some of the dictionary definitions hint, the something or entity indicated by a sign vehicle may have a variety of characteristics, including those of a purely ideal nature. The nature of the object to which a sign refers often changes from one semiotic approach to another. For example, in semiology, the signifier–signified relationship such as between the word "tree" and the image of a tree is psychological (de Saussure 1967). In other approaches, this entity, which is not *necessarily* a physical object, is assumed to have ontological presence irrespective of human observation. Other semioticians and philosopher-anthropologists have defined the sign as a relation between two parts of the *material* continuum, each being shaped in its own way (Eco 1984; Latour 1993). All in all, the important aspect for understanding human behavior through semiotic lenses is the manner in which the different parts come to be made to correspond and thus how such things as diverse as a mathematical equation or function come to be related to the functioning of a winch, pulley, or table of paired data. And as we have seen, the question of how signs signify as well as the question of the nature of the object the sign stands for are two questions that receive different answers in the varied range of semiotic approaches.

Several of the chapters in this book grapple with the ontological question of how and what signs signify. In line with other authors, Morgan (Chap. 8) and Otte (Chap. 9) eschew the Platonic approach to mathematical objects. In his chapter, Otte provides useful instances of the *complementarity* of sense and reference, and of structure and function. He points out that man is a symbolic being, and that every thought is a sign. Similarly, Kadunz (Chap. 7) points out that the translational approach to semiotics that he espouses does not need the assumption of "an objectively existing mathematical instance." In Chap. 12, Salazar reminds the reader that semiosis—an action that Peirce understands as the irreducible relation of three subjects, such as sign vehicle, object, and interpretant (Eco 1976)—and noesis—intentionally oriented thinking (Husserl 1913a)—are inseparable. The representational correlate of semiosis is *production*, the purpose of noesis (e.g. judging) is the conceptual *apprehension* of an object and its correlate (outcome) is the noema (what is judged).

It is a welcome development that non-representational theories that take into account the sociocultural nature of human signification (see especially the chapters in Parts I and IV of the book), or the richness of linguistics in this regard (Part II), are being promulgated in many of the chapters. They shift the attention from a static view of representation to a dynamic view of sign production and meaning-making. Based on work in mathematics education and the social studies of sciences, some educational scholars have suggested conceptualizing what tend to be called mathematical representations in terms of the concept of *inscriptions*, which orients investigators to the use of diagrams, graphs, equations, and other forms of mathematical expression (Roth and McGinn 1998; Presmeg 2006a). The shift goes hence from the diagram per se and what it may signify to the *use* of the diagram and the broad spectrum of signifying possibilities that arise in using it in activity.

A contentious point is the role in semiotic processes of artifacts and cultural devices, which traditionally have been conceptualized as *mediators* of teachers' and students' activity (see, e.g., Bartolini-Bussi and Mariotti 2008). Roth and Jornet (2017) contend that there is no necessity to invoke the concept of mediation, particularly in cultural historical approaches to cognition and learning. Resorting to a materialist viewpoint, Radford (2015) has suggested that signs and artifacts be understood as *part* of the teachers' and students' activity and as *the material fabric of thinking*. In this view, signs and artifacts do not mediate activity: they are key components of it in that they contribute to configure, organize and re-organize classroom activity and teachers' and students' thinking.

The various chapters in this book suggest that, in the complexity of human teaching and learning, there are a variety of questions that arise, and thus it is appropriate to endorse theories that are appropriate to the questions and issues: no single theory can take into account the need for different grain sizes in data analysis (see Chap. 16, in which Vygotskian and Peircean theories are applied in analysis of the same data set), or the affordances offered by different conceptual frameworks that are suitable for the multitude of empirical research questions that concern mathematics education (e.g., Chap. 10, Semiotic Conceptual Analysis; or Chap. 12, Dynamic Figural Register).

In the following section, we select some of the specific threads that arise in the chapters of this monograph, and discuss these with a view to highlighting what has been accomplished to date. The final section takes the discussion further, pointing out possible future directions for theory and empirical research, and identifying lacunae in current research foci that might profitably be addressed using semiotics as a lens.

19.2 What Has Been Accomplished?

The following subsections address specific threads from the chapters.

19.2.1 Theoretical Developments

In recent years, semiotics has undergone a series of developments that resonate with developments in other areas of inquiry in the social sciences. Such developments have led to revisiting the concept of knowledge and knowing and the role that signs play therein. It is in this context that there has been a shift of paradigms. In the first paradigm individuals were conceived of as producing ideas in their minds as they acted in the world. Signs in this paradigm were the conduits in which thought was cast. Actions, words and semiotic activity in general were explained in terms of ideas that justified the actions, the words, and the symbols to which individuals resorted in their endeavors.

In the second paradigm, ideas are conceived of as not necessarily prior to action and semiotic activity. Actions and semiotic activity are fused in the processes out of which individuals achieve deeper forms of reflective awareness, and their ideas become clearer through more and more sophisticated expression. In this paradigm, signs, tools, and thought are entangled. As a result, the semiotic lenses to understand and provide accounts of teaching and learning include signs and tools. The tool is not independent of cognition and learning, just as cognition and learning are not independent of the tool. Both are integral and irreducible parts of the same category. Any behavior we may analyze, therefore, has aspects of both. As the research on aptitude has shown, understanding the role of personal characteristics in the behavior requires knowing environmental characteristics, and understanding the role of the environmental characteristics in performance requires knowing the relevant personal characteristics (Snow 1992). In this paradigm, in considering the role of technology, we may be wise not to give special status to computing or other more recent forms (e.g. virtual reality) to be theorized differently from other things that are part of, determining and being determined by, current activity. Just as GeoGebra (Salazar, Chap. 12) or Virtual Math Teams (Swidan and Prusak, Chap. 18) do, pencils, diagrams (Menz and Sinclair, Chap. 17), rulers, slide rules, Dienes blocks, and natural or formal languages (e.g. C++) may (but do not have to) have a function in activity and change the task from activities in which they are not present. In this context, human behavior appears as the result of the irreducible characteristics of the person–environment unity/identity. These remarks are not intended to diminish the investigation of tools and technology. On the contrary, we believe that it is useful to investigate cognitive and semiotic systems in which different types of technology are an integral part. For one thing, this investigation provides our community with materials for theorizing these systems. But also it can help us understand the deep relationships between tools, signs, and thought.

In the second paradigm, the question of the social is reformulated in new terms. In the first paradigm—a paradigm that was informed to a great extent by idealist and subjectivist understandings of knowledge and knowing—the social is not foregrounded; a discrete set of individuals coordinate actions in order to accomplish something. In the second paradigm, an individual's action is *transindividual* as it occurs with others (present or virtually present through language or social context). Knowing appears here as something that is social in nature through and through. In this line of thought, cultural-historical perspectives have emphasized the relation between teacher and student as something deeply social and as an extremely important root of cognition. For cultural-historical theorists, the relation between the teacher and the student is cognitively important because it *is* the higher psychological function that can later be found in the behavior of the student (Vygotsky 1989; Vygotskij 2001). The relation is important here because, as Vygotsky describes, what has become one in the behavior of the teacher now is unfolded across the relation and the two persons involved. It is the "experimental unfolding of a higher process (voluntary attention) into a *small drama*" (p. 58). Thus, when Stott (Chap. 5) writes about space of joint attention, then this "space" includes the relation between student and teacher. This relation is not something abstract. Indeed, the word or any other form of sign involved in the teacher-student interaction *is* an essential part of the relation. The same happens in the mathematical game that kindergarten students play (Radford, Chap. 2). The space of joint action is a space where an understanding of the mathematical rule is disclosed through verbal and embodied forms of social relations.

The social relations that are encompassed in classroom interaction have been conceptualized in the theory of objectification as one of the two main lines or axes around which classroom activity is organized. Social relations are defined as part of the forms of *human cooperation* that underlie all activity. In the aforementioned first paradigm (the idealist-subjectivist one), forms of human cooperation often appear through utilitarian lenses: individuals exchange ideas in accordance with a logic of self-interest. "Classroom communication is good if the student learns more than he or she would if working alone. This is still the 'me-perspective': I am willing to transact with you if, at the end, my wealth (here knowledge) increases" (Radford 2012, p. 109). The second paradigm makes room to envision social relations and, more generally, classroom forms of human cooperation in a different way. The theory of objectification makes here a contribution in terms of delineating a communitarian ethic that, through pedagogical action, emphasizes a sense of the social as embodied in solidarity, answerability, and responsibility, moving away from individualist forms of human cooperation fostered by neoliberal conceptions of the school (Radford 2014). The question, however, is not merely to come and be able to participate in certain forms of non-alienating activity. The question is that the activity that individuals produce is the same one that produces them. The result of this dialectic movement, which appears in particular in Chap. 2, is a function of the activity's forms of human cooperation to which the pedagogical project resorts (either implicitly or explicitly).

The other line or axis around which classroom activity is organized in the theory of objectification is the one of the *modes of knowledge production*. They include the technological or material dimension (tools, signs, diagrams, etc.) and the various *savoir-faire* through which individuals produce their means of subsistence and fulfill their needs. Strictly speaking, the forms of human cooperation and the modes of knowledge production do not emerge in situ, that is in the classroom, as something out of the blue. They have been produced and refined in the course of historical and cultural developments. It is these historical and cultural dimensions that the concept of *semiotic systems of cultural signification* try to capture. Human activity unfolds, indeed, (often implicitly) organized by a fluid and ever-changing super-symbolic structure from which individuals draw ideas of right and wrong both in their dealings with what is to be known and in their dealings with others.

Since both dealings involve language, symbols, and material culture, the result is that signs (and semiotics) are not the repository and conveyers of ideas (as idealism suggested), nor are they the mere channels out of which ideas and knowledge are formed. Inescapably, signs and semiotics appear as imbued in an ethical and political view. There is always an ethical and a political stance behind knowing and becoming. In this line of thought, the question is not about representing knowledge, but how we come to signify through signs. That is, how we signify things in the world and how we come to signify ourselves and are signified by others through the activities in which we engage.

19.2.2 Methodological Concerns

The question of *grain size* needed in different research designs is one that is salient in some of the chapters but latently present in all. Krause and Salle (Chap. 16), in their investigation of gestures and their role in the formation of *Grundvorstellungen* (GV, "mental models that carry the meaning of mathematical concepts or procedures"), use lenses of both Vygotskian and Peircean semiotics in their analysis. Vygotskian theory proved to be more suitable for the grain size of their research data: in this theory, gestures bridged between *primary* and *secondary* GVs, whereas in Peircean theory it appeared that signs led to the shaping of the immediate mathematical task, but the given data were insufficient to answer the question of *how* gestures led to the solving of the problem. According to the authors, a broader view encompassing scenes from an earlier point in the research might have been helpful in the Peircean approach.

The issue of grain size appears also in Morgan's use of systemic functional linguistics (SFL, Chap. 8) to explore social semiotic issues. However, the theoretical stance of SFL and associated methodology, adapted to different grain sizes, proved to be suitable both for the institutional assumptions involved in the fine-grained analysis of how an examination question is written, and in Morgan's second, society-level, focus involving a thematic analysis of official documents.

Another issue addressed in some chapters is methodology used in research that aims at the *constitution of typologies*. In the chapters of both Salazar (Chap. 12, constitution of the dynamic figural register, using Duval's theory) and Mathews et al. (Chap. 14, signification pathways, based on Peirce's semiotics), a fine-grained analysis of theoretical positions (Chap. 12) or qualitative empirical data in a naturalistic study (Chap. 14) resulted in typologies that move the field forward. However, in the case of the study by Mathews et al., the semiotic chaining framework that was a starting point for the methodology required modification and greater specification before it became useful for the research purpose. In the study by Kinach (Chap. 13) involving primary school teacher preparation, the semiotic chaining model broke down, in the sense that it could be applied only to the first three of five phases in an interactive computer game developed for the research purpose.

It is clear in all the chapters of this monograph that research purposes determine the nature of the methodologies used, based on appropriate theoretical considerations. However, it is noteworthy that with the exception of Chap. 17 (Menz and Sinclair), none of the chapters use quantitative analysis of data sets, or psychometric designs that require the use of statistics. It remains an open question whether research designs that require such methods are useful in research on semiotics in mathematics education. It is possible that qualitative research designs are more suited to the issues pertinent to semiosis in mathematics education, in which *paradigm cases* (of individuals or groups) are likely to shed light on questions of teaching and learning, even when sociocultural lenses are applied. Even Menz and Sinclair's chapter is based largely on the paradigm case of three research mathematicians. It is clear that issues of "joint attention" (Shvarts, Chap. 5), "togethering" and "moments of poēsis" (Stott, Chap. 5) and many of the issues investigated and reported in other chapters, are more amenable to qualitative rather than quantitative research methodologies, notwithstanding the current trend for mixed methods to be used in mathematics education research (Bikner-Ahsbahs et al. 2015).

The traditional question of grain size obtains—or should obtain—a new sense within cultural-historical theories (e.g., Chaps. 2–6). This is so because the minimum unit of analysis, as made explicit in cultural-historical activity theory, is *societal* activity. This unit is based on the recognition that what characterizes humans as distinct from other animals are the specific relations that constitute society. Cultural-historical psychologists thus consider personality to be the ensemble of societal relations (Leont'ev 1978; Vygotsky 1989). As a result, every analysis of a relation between two people also makes for an investigation of society; every analysis of societal relations also constitutes an investigation in available psychological functions and personality traits (Roth 2012a). The issue is particularly important in the case of the birth of signs (Chap. 3, Roth). Whereas signs may be created by individuals, they *inherently* have *social* character. But they may not be universal but be characteristic only of a group; but universality is a suprasensible (ideal) potential of anything material exchanged (Marx and Engels 1962, 1978), a point taken up in Chap. 3 (Roth). Thus, if investigators adhere to the cultural-historical approach, they (have to) accept the inherently social, thus shared and societal nature of the investigated phenomena—though, admittedly, investigators may fail to point out or exhibit this feature. For example, Salinas-Hernández and Miranda (Chap. 4) conduct their investigation in a 12th-grade physics course. But what the students generally say and do, or any particular student specifically, is not an individual construction. This is so because their talk is designed *for* others, using a language that they have learned *from* others, and which, in their talk, returns *to* the other. Whatever students say, thus, is a reflection of the culture, a mirror of society, a microcosm thereof (e.g., Zeyer and Roth 2009).

A very different approach to method arises when research focuses on the ways in which phenomena come about—i.e. the work in which the birth of a sign is founded (see Roth, Chap. 3)—rather than the outcomes of this work. The distinction maps on to that made above between noesis, an event, and noema, its outcome. Chapter 3 exhibits the work children do to assist each other in making some aspect visible (perceivable), whether this is an aspect of the hidden object or the different shapes they make from plasticine. It has been pointed out that research using formal methods (such as psychometric analysis)—specified in the methods sections of research reports—focuses on the identification of noema, social or material things, whereas ethnomethodology focuses on the work that real people do to constitute and make visible the situationally relevant social and material things (Garfinkel 1996). Formal methodologies include quantitative (mathematical, experimental) studies as well as some qualitative (interpretive) studies (Bikner-Ahsbahs et al. 2015), and these differ incommensurably from ethnomethodology and other naturalistic methodologies that investigate the methods the people themselves employ to make and make visible social and material *things* (i.e. the noema). In this light, naturalistic studies appear to be particularly suited to empirical research on semiotic issues—and indeed, many of the empirical studies described in chapters in this book fall into this genre.

19.2.3 The Role of the Teacher

The role of the teacher is one of the elements that has been recently under intense discussion in mathematics education research. There seems to be a need to reconceptualize this role in teaching and learning. In the chapters included in this book, the role of the teacher takes various forms.

In Chap. 2 (Radford), it is the sensitive intervention of the teacher that breaks the impasse between Carl and Jack in the dice game and allows them to move forward. Rather than facilitator, the teacher actively participates in the creation of a social space where the rule of the game regains focus. Carl's and Jack's interaction becomes again *reorganized* by the rule. This reorganization makes room for the unfolding of a cultural form of intersubjectivity in which individuals take positions and answer, in a responsible manner, to each other. The teacher plays a preponderant role here in order to make sure that the students' answers are coherent with the operating semiotic systems of cultural signification (Radford 2008; see also Chap. 2) that convey ideas (mostly implicitly) about how individuals should behave. And as we saw, the joint creation of the social space by the teacher and the students, where the rule can be again attended to, occurs in difficult moments that are full of tensions and emotions—tensions and emotions of which the teacher is, like the students, a part. In Chap. 3 (Roth), the teacher points out to Melissa, Jane, and Sylvia that there is *one* object in the shoebox, and therefore the group should decide on one plasticine model for this object: Melissa's cube model *dies* in favor of the flatter rectangular model of Jane and Sylvia, in an example of the gap-crossing needed to reconcile two structured portions of the continuum.

In Chap. 4 (Salinas and Miranda), there is a hint from the researcher, but it is the interactive technology that stands in lieu of the teacher in the students' processes of objectification dealing with the meanings of Cartesian graphs. The interaction with the technology enables the students to check their predictions using the software. The students' description of the motion of a tennis ball down an inclined plane was *preceded* by their understanding of the mathematical relations of physical variables of space and time in the graphs; this understanding evolved as the students formulated new hypotheses and interpreted the software outcomes. The teacher appears here as a kind of virtual entity whose contours change as the students ask questions and the software provides answers.

In the dual eye-tracking synchronized research of Shvarts (Chap. 5), the role of the parent is crucial in establishing the joint attention with the child that enables the child to apprehend the *cultural way* of using coordinates on a Cartesian plane. Indeed, phylogenetically speaking, the higher psychological function of attention is the result of a lengthy culturally and historically evolved way of attending things, which is materialized here through the concrete relations with other people, generally a parent. Ontogenetically speaking, the birth of the capacity to orient to the same objects and to signal whether joint attention has been achieved lies early in the life of a child (Vygotsky 1997), exemplified in a recent analysis of reading multimodal texts involving a mother and her one-year-old boy (Roth 2016). Shvarts gives examples both of the ambiguity of visual and gestural presentations (and verbal terms) that cause misunderstanding, and in a nice contrast, the seamless joint attention that results from the adult's sensitivity to the child's anticipations, as the parent *followed* the child's attention, in an outcome that was satisfying to both of them. Finally, resonating with Shvarts's study, in Chap. 5 Stott gives examples in an after-school program, of the importance of the role of the facilitator in establishing the *space of joint action*, and *togethering*, that are necessary for cultural meanings. In Stott's first example, this space of joint action with Anathi and Thembela leads to the cultural meanings that are the goal of the activity; in the second example, with Akhona and Kuhle, the facilitator does not succeed in intervening in *attention catching*, and the absence of togethering prevents the moment of poēsis that is apparent in the first example.

All five of these chapters highlight the indispensable role of the teacher or surrogate teacher and the various forms this role takes, depending on the activity, the object of the activity and the tools and artifacts involved in the classroom activity.

19.2.4 The Affordances of Interactive Technology

In several of the chapters, interactive computer software plays a pivotal role in accomplishing the goals of the research. For instance, GeoGebra is an essential tool in both Chap. 12 (Salazar) and Chap. 18 (Swidan and Prusak). It has been shown empirically for some time now that the affordances of such software (compared with paper-and-pencil methods) change the processes of learning mathematics quite radically (e.g., Yu 2004, using a conceptual framework based on semiotics).

In the case of Salazar's research, which employs the theoretical formulations of Duval concerning formation, treatment and conversion of semiotic registers, the use of a *dynamic representation environment* such as that provided by GeoGebra is essential. Duval did not take into account the differences that the use of computer software introduced in examining formation, treatment, and conversion. Thus Salazar's examination of the *dynamic figural register* in such an environment, highlighting the reconfiguration of quadrilaterals to study their area, moves this field forward both theoretically and empirically. Indeed, from a cultural-historical perspective, Salazar and Duval are investigating different phenomena, different forms of activity, for as soon as a tool is introduced into a person–environment unit, then cognition, consciousness, sense, and the related affects change. As research on the relation between cognition, technology, and work shows, even a minor variation in a tool—such as a 5-centimeter displacement of a dial in an airline cockpit—changes the cognitive system as a whole (e.g., Soo et al. 2016).

Swidan and Prusak's research (Chap. 18) is also concerned with the study of quadrilaterals, but from the viewpoint of the inherent inclusion relationships in a hierarchy of quadrilaterals. The new field of Computer-Supported Collaborative Learning (CSCL) is central in their research methods, because the aim of their study was to investigate the role of CSCL in the processes of students becoming *fully* aware (as the final stage of a triad of processes, starting with unawareness, followed by latent awareness) of selected inclusion relationships of quadrilaterals. They concluded that the research design (partially) supported the objectification of these relationships. The authors note that the computer environment was limiting in that it did not provide access to certain kinds of data: the students could not gesture or communicate verbally (or at least the researchers did not have access to these data sources) in this environment, in which communication was in a chat room with written explanations.

In the case of Kinach's research (Chap. 13), an "area game" designed for prospective elementary school teachers also invoked the affordances of technology: the design was intended to provide the prospective teachers with an introduction to semiotic chaining in the Peircean framework, as a basis for their pedagogy, as they worked through five levels of the children's game. In practice, the chaining broke down after level three; but nevertheless the research provided access to Peirce's three principles of diagrammatic reasoning, and highlighted some of Krutetskii's (1976) problem types.

Technology was also an essential component of Shvarts's (Chap. 5) methodology. In this case, two synchronized eye-trackers and an external video camera enabled the researcher to capture the gazes, gestures (with a thin stick pointed at the computer screen), and verbal interactions of mother and child as both looked at the same computer screen and engaged with tasks involving Cartesian coordinates of points. The power of this methodology was illustrated in the micro-mechanisms that were identified in the processes of joint attention, and objectification of the canonical way of working with such coordinates.

19.2.5 Research on Diagramming

Although diagrams enter into the mathematical components of data sources of several chapters in this monograph, there are two chapters (Chap. 11, Sáenz-Ludlow; and Chap. 17, Menz and Sinclair) that engage with processes of diagramming specifically, using different theoretical lenses and investigating different issues. Sáenz-Ludlow's chapter, on *Iconicity and diagrammatic reasoning in meaning making*, is largely theoretical in nature, based on Peirce's (1931–1958) semiotics, and provides two convincing examples of the power of diagrammatic reasoning. Sáenz-Ludlow highlights Peirce's definition of a diagram as an "icon of possible relationships." For Peirce, *all* knowledge is the product of the self-corrective activity of the human mind, and diagrammatic reasoning is an inferential process that addresses the relationships among sign vehicles, their interpretants, and the interpreted objects in question. Sáenz-Ludlow explores the three triads at the root of the ten-fold classification of sign vehicles and their relationships according to Peirce. Mathematical diagrams are considered to be epistemological tools for inferential thinking: dynamic interpretants give way to logical interpretants, which in turn lead to chains of inferences. Cultural-historical scholars point out that the diagrams are not merely tools in but constitutive of thinking in the same way as words and thinking are integral parts of the same communicative activity, thinking and speaking being two dynamic processes that are flexibly related (Vygotsky 1987). Thought completes itself in the word, which, as part of the material reality, determines further thinking. The same is the case for all forms of signs, including the relationship between diagrams and reasoning: reasoning *becomes* what it is in the diagram, and diagrams determine the situationally appropriate reasoning. This general relationship between signs and reasoning, paired with Sáenz-Ludlow's concrete analyses, help us better understand how initial *transformand* diagrams give way to more meaningful and useful *transformate* diagrams in the processes of diagrammatic reasoning. Sáenz-Ludlow's analyses also show specific ways in which visualization is indispensable in the process. In this complex theoretical formulation, diagrams have the potential to have both iconic and indexical properties (and symbolic, metaphoric and metonymic ones too, Presmeg 1992).

In a different theoretical formulation, in their chapter on *Diagramming and gesturing during mathematizing*, Menz and Sinclair challenge the traditional formulation of diagrams as representations of mathematical objects and relations. Peirce (1992, 1998) was one of the first semioticians to conceive of semiotics in a dynamic way. He called attention to the role of diagrams in *processes of inference and abduction*. For Peirce a diagram is made up of signs (indexical and others) that serve to emphasize relations that come to mind in the investigation of a problem. For Peirce a diagram is first of all a kind of iconic sign, not in virtue of a resemblance with something but in virtue of the relationships that it brings to the fore—

e.g., he argued that algebraic formulas are icons. Focusing on the context of mathematical invention, Menz and Sinclair emphasize the indexical nature of diagrams. They base their arguments on the writings of Châtelet, who, as a result of his deep investigation of the thinking of great mathematicians according to the records that they left, described diagramming as a *material* practice of mathematical invention. Metz and Sinclair illustrate this theoretical formulation convincingly in the case of three research mathematicians, Fred, Colin, and Victor, working collaboratively on problems of Topological Graph Theory. About 200 diagrams were identified during nine meetings of the mathematicians. The life-cycles of the diagrams were analyzed according to three phases, namely, *manufacturing*, *communicating*, and a final phase of *dénouement*. From their data, the authors identified five elements in the life-cycles of the mathematicians' diagrams: firstly, the diagram is emerging ("is born," as Roth, Chap. 3, might say); then a mathematician adds to or erases from an existing diagram; next, the diagram is refigured by tracing edges and vertices; the fourth element is a mathematician drawing a known diagram by retrieving it from memory; finally, a mathematician might be directed to draw a diagram by his colleague.[1] Although the indexical nature of these diagrams is clear from this data set, the diagrams might also be considered to have iconic properties. For instance, in the case of the diagram drawn from memory, memory must not be thought of in terms of representations, but in terms of the movements or actions themselves that have become habitual. The movements producing mathematical diagrams and those gesturing over and about the diagrams produced before or as a result are the same—as shown in a semiotic study of lecturing with graphs in university physics courses (Roth 2012b). Iconicity appears here not as a property of the sign itself but rather as a property of its production: "as the projection of an earlier experience onto a new one" (Radford 2008, p. 94)—something that makes this experience at the same time old and new, "similar and different" (p. 94), as in the various hand movements and the ensuing figures inscribed on the blackboard.

More generally, because the emergent and evolving meaning of signs changes continuously as semiotic activity unfolds, the nature of signs moves constantly around from being indexes, to being icons, and to being symbols; in fact, signs can be more than indexes, or icons, or symbols only; signs can be one, two or three things all at the same time. However, it is worth remembering that there are many roles of diagrams in the teaching and learning of mathematics. One open question is the influence of *prototypical images* on students' and mathematicians' thinking in all areas of mathematics (e.g., prototypical images of octahedron and torus in Menz and Sinclair's study, or triangles and pentagons in Sáenz-Ludlow's), both as affordances and as constraints (Presmeg 1992, 2006a).

[1]See also Roth (2015) where the continued changes in thinking and writing of mathematical graphs and equations, associated with continued erasure and rewriting, are theorized in the field as the birth of understanding arising from the excess of graphical movements.

19.2.6 Types of Signification

In view of the rich variety of theories of signification offered in the chapters in this monograph, in this section we honor the view that there are different types of ways that signs may signify. Just as mental imagery may signify in different ways in mathematical activity (Presmeg 1997, 1998, 2006b), the signs identified in the chapters of this book have different contexts, purposes, and roles in mathematics education. As in Shakespeare's words, "A tale told by an idiot, Full of sound and fury, Signifying nothing" (Macbeth, Act 5, scene 5), for *some* students mathematics may have no personal signification, despite its sound and fury. Signification implies *meaning*.

The meaning of activities for the learning of mathematics is a relevant topic in mathematics education at least since Jean Piaget (e.g., Piaget and Inhelder 2013/1958): numerous researchers have presented texts on research questions in this area (Boaler 2002; Wittmann 1995). Semiotics offers several other views on mathematical activities. In the Peircean tradition, besides diagrammatic reasoning (see Sáenz-Ludlow, Chap. 11), which focuses on Peirce's use of signs, the pragmatic maxim introduced by Peirce in (1878) and slightly changed by William James (1907) is a relevant theoretical principle to explain the meaning of concepts in common usage and in mathematics in particular (Sáenz-Ludlow and Kadunz 2016b). Concentrating on the pragmatic maxim also builds a bridge to the ideas of Ludwig Wittgenstein (see Kadunz, Chap. 7), another philosopher widely known but rarely used in mathematics education (Dörfler 2016). On the one hand Wittgenstein cannot be seen as a semiotician. On the other hand we can find numerous hints in his posthumously published book *On Certainty* (1975/1969) very similar to the pragmatic maxim. Boncompagni (2016) stated the pragmatic maxim as follows:

> Roughly, it is the connection between the meaning of a conception and its practical effects, thanks to which we can claim to know something completely only insofar as we know its effects in factual and/or behavioural terms. (p. 140)

Peirce sees *action* not as a concept to be studied in detail, but as a tool to interpret various different situations. For him, action is a part of a life practice, a behavior (habit), which is controlled by action and shows itself at the same time. What is the meaning of this maxim for learning mathematics? If we follow Peirce, it is a tool to achieve clarity about ideas and concepts, including those that determine mathematics. It is remarkable that, in Peirce's view, "One of the goals is to illustrate that certain hypotheses or concepts do not bear any cognitive content at all" (Hookway 2012, in Boncampagni 2016, p. 43). The pragmatic view focuses mainly on the *outcome* of activities. Peirce's definition of meaning is "translation of one sign into another system of signs" (CP4.127). This definition resonates with Eco's (1984) characterization of semiosis as one segmentation of the material continuum in relation to another segmentation.

Although they espouse different traditions, Kadunz (Chap. 7) and Roth (Chap. 3) both recognize the principle of a field transverse to the flow of ideas, as in the metaphor of a river. Roth, in the tradition of the later Vygotsky, calls it a communicative field. Kadunz points out that the German word for translation, *Übersetzung*, implies a leaping across: a good translator "leaps to the other river bank" (Chap. 7). As in the pragmatic maxim, *use* is another focus in the worldview of Kadunz that has resonance in Roth's chapter, in which sign vehicles are adumbrated in terms of the buying and selling of commodities: the buyer is interested in the *use-value*; the seller in the *exchange value*. A human relation is involved. The transaction is future-oriented (as in the pragmatic maxim): *use* matters. Roth writes of "lines of becoming" (Chap. 3). Two questions are involved in the foregoing, as it relates to mathematics education: (a) What is signified? (b) How is it signified?

Some indication of an answer to these questions is given by Krause and Salle (Chap. 16). In their use of a framework of *Grundvorstellungen*, "mental models that carry the meaning of mathematical concepts and procedures," they invoke theories of both Vygotsky and Peirce. *What* is signified by Victor's coordination of gestures, speech, and inscriptions is indicated in his mental models that proceed from an initial "growth point" to a "catchment" (a confluence of growth points) that leads him to the idea of *function as object*, in the context of linear equations and mobile telephone contracts. Thus function as object is what is signified by this catchment. *How* it is signified, using a Peircean frame, was not able to be identified with the dataset that they presented. It would require data from an earlier point in time.

Thus there are many aspects that concern signification in the teaching and learning of mathematics. The use-value may be identified in mental models. But in the learning processes involved, "individuals come to position themselves in differential, polyglossic, and ideological ways" (Radford, Chap. 2). Otte, too, writes that "man is a symbolic being" (Chap. 9), characterizing semiosis as meaningfulness, not objective reference. What is signified, according to Radford, is uniqueness, being in flux, shaped by a common cultural ground of ideology, leading to a student as a "continuous, moving sign in the making" (Chap. 2). Then the student *is* the sign. Radford's (2008) theory of objectification, involving *semiotic systems of cultural signification* (SSCS), is used in the empirical research reported in several of the chapters of this monograph (Salinas-Hernandez and Miranda, Chap. 4; Shvarts, Chap. 5; and others more implicitly). SSCS and the identity of a student in flux are not incompatible with Sáenz-Ludlow's Peircean analysis (Chap. 11) of dynamic, continuous sign-building in diagrammatic reasoning. The focus is on Peircean chains that "stretch out in both directions" (Otte, Chap. 9) without limit. The Peircean analysis may be better suited to hinting at the details of *how* the signification proceeds, but the SSCS view is better able to capture the mathematical *identity in flux*, of a student learning in a cultural context, with all the emotions and affect that belong to this process. It is not only a question of the grain of the research; Shvarts's study, for example, is extremely fine-grained—and provides a detailed account of *how* a case of objectification of a cultural view of Cartesian coordinates takes place through joint attention (see also Stott, Chap. 6). It is also a question of the *focus*, in view of the conceptual framework of the research, which enables and constrains the research questions. Perhaps Otte's (Chap. 9) principle of complementarity is a useful one to adopt in the context of theories that inform a particular research question. Salazar (Chap. 12) successfully integrates Peirce's qualisign, sinsign, and logisign with Duval's registers of formation, treatment, and conversion respectively, in her configuration of the *dynamic figural register* in a GeoGebra environment. European researchers have been leaders in the "networking" of theories that together can provide a broader picture of a phenomenon (Bikner-Ahsbahs et al. 2015).

19.3 Suggestions for Future Research on Topics Concerning Semiotics in Mathematics Education

Whether or not a principle of complementarity of theoretical positions is embraced, the foregoing issues, and the rich variety of theoretical positions on semiosis adopted in the chapters of this monograph, suggest further research in a broad array of different areas.

Firstly, is there scope for more *system-wide* research involving semiotics in mathematics education? Morgan's research (Chap. 8) indicates that indeed that is the case. Using social semiotics and systemic functional linguistics, she addresses three questions: (a) What is mathematics? (b) Who or what does mathematics? and (c) What do teachers and students do? Both her examples are fine-grained in their analysis, but they manifest the versatility of ways that system-wide issues may be addresses through these lenses. Examination of the wording of past national mathematics examination questions highlights the subtlety of messages signified, in characterizing what mathematics is and who should do it (the first two research questions). In the second example (question 3), official government documents are analyzed to decode what is considered "good practice" in teaching mathematics—agency, performance, support, etc. are all key words that signify a view of teaching that is rewarded by the system. This kind of study addresses both kinds of research, namely, *descriptive-analytic* (what *is*), and *normative* (what could be different).

With a different focus, there is much scope for further investigation of Radford's question (Chap. 2), What is the nature of the human activity that, at school, produces teachers and students? His research, based on the classroom constitution of mathematical subjectivities, suggests that *what* is learned, and *how* it is learned, are the "threads out of which subjectivities are made," involving *togethering* (Radford and Roth 2011). Specifically, Roth (Chap. 3) indicates three possible areas of investigations of signification in this regard: (a) creation of signs; (b) transparent reading of signs; and (c) interpretation of signs. Roth has addressed all three of these foci in his writings, but Chap. 3 is devoted to the first, the birth of signs. Chapters 4, 5, and 6 are all involved in examining the micro-mechanisms that address the mathematical subjectivities of students in processes of learning, and the field is ripe for more research in this area. There is also scope for further research on attention and awareness in progressive objectification as characterized by Swidan and Prusak (Chap. 18), who introduced in their research Roth's (2015) three-fold distinction of being unaware, being latently aware, and culminating in full awareness.

There are several theoretical formulations in the chapters of this monograph that could fruitfully be put to the test as lenses in contexts other than those described. One such formulation is that of Menz and Sinclair (Chap. 17), using Châtelet's theory in research on diagrammatic reasoning in the creation of new mathematics by three research mathematicians. Since Châtelet's theory was founded on the written work of famous research mathematicians, it would be interesting to investigate the scope of his theory in the classroom teaching and learning of mathematics at all levels. Are the five steps in the life-cycle of diagrams, which Menz and Sinclair identified, replicable in other contexts? Sáenz-Ludlow's research (Chap. 11) also concerns diagrammatic reasoning, but within a Peircean framework: there is much scope for the further investigation of the usefulness of *transformate* diagrams in teaching and learning geometry. Further theoretical formulations that might be tested as lenses in different contexts are as follows: Peirce's three-step plan for chaining diagrams (Kinach, Chap. 13), the Dynamic Figural Register (Salazar, Chap. 12), the typology of Signification Pathways, co-produced by teachers and learners (Mathews, Venkat, and Askew, Chap. 14), and Vygotsky's view of gradual perceptual change in the Object-Meaning Ratio (Abtahi, Chap. 15).

Finally, because linguistics addresses a large area of semiotic expression, there is need for further research that involves elements of translation in mathematics education (Kadunz, Chap. 7). Kadunz asks what conditions must be met in order that key terms can be translated? This is still a focal question that might be addressed using Semiotic-Conceptual Analysis (Priss, Chap. 10), the framework that Priss used in investigating why formal language is difficult for students to learn. The conceptual lens of SCA introduces the fine grain of four structural aspects in questions of language: incompleteness, polysemy, synonymy, and iconicity.

It is clear from the rich diversity of theoretical formulations, and the numerous issues still pertinent to semiosis in mathematics education, that signification will remain an important aspect of research in this field.

References

Bartolini Bussi, M., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York: Routledge, Taylor and Francis.

Bikner-Ahsbahs, A., Knipping, C., & Presmeg, N. (Eds.). (2015). *Approaches to qualitative research in mathematics education: Examples of methodology and methods*. Dordrecht, The Netherlands: Springer.

Boaler, J. (2002). Exploring the nature of mathematical activity: Using theory, research and 'working hypotheses' to broaden conceptions of mathematics knowing. *Education Studies in Mathematics, 51*(1–2), 3–21.

Boncompagni, A. (2016). *Wittgenstein and pragmatism (History of analytic philosophy)*. London: Macmillan.

de Saussure, F. (1967). *Cours de linguistique générale [Course in general linguistics]*. Paris: Éditions Payot & Rivages.

Dewey, J., & Bentley, A. F. (1999). Knowing and the known. In R. Handy & E. E. Hardwood, *Useful procedures of inquiry* (pp. 97–209). Great Barrington, MA: Behavioral Research Council (First published in 1949).

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In W. Dörfler, A. Bikner-Ahsbahs, A. Vohns, & R. Bruder (Eds.), *Theories in and of mathematics education* (pp. 21–31). Berlin, New York: Springer.

Eco, U. (1976). *A theory of semiotics*. Bloomington: Indiana University Press.

Eco, U. (1984). *Semiotics and the philosophy of language*. Bloomington: Indiana University Press.

Garfinkel, H. (1967). *Studies in ethnomethodology*. Englewood Cliffs, NJ: Prentice Hall.

Garfinkel, H. (1996). Ethnomethodology's program. *Social Psychology Quarterly, 59,* 5–21.

Hookway, C. (2012). *The pragmatic maxim. Essays on Peirce and pragmatism*. Oxford: Oxford University Press.

Husserl, E. (1913a). *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Erstes Buch: Allgemeine Einführung in die reine Phänomenologie [Ideas to a pure phenomenology and phenomenological philosophy vol. 1: General introduction to a pure phenomenology]*. Halle a.d.S.: Max Niemeyer.

Husserl, E. (1913b). *Logische Untersuchungen. Zweiter Band. Untersuchungen zur Phänomenologie und Theorie der Erkenntnis [Logical investigations vol. 2. Investigations of phenomenology and theory of knowledge]*. Halle a.d.S: Max Niemeyer.

James, W. (1907). *Pragmatism: a new name for some old ways of thinking*. London, New York: Longmans, Green & Co.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Latour, B. (1993). *La clef de Berlin et autres leçons d'un amateur de sciences [The key to Berlin and other lessons of a science lover]*. Paris: Éditions la Découverte.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs, NJ: Prentice-Hall.

Marx, K., & Engels, F. (1962). *Werke Band 23 [Works vol. 23]*. Berlin: Dietz.

Marx, K., & Engels, F. (1978). *Werke Band 3 [Works vol. 3]*. Berlin: Dietz.

Peirce, C. S. (1878). How to make our ideas clear. *Popular Science Monthly, 12,* 286–302.

Peirce, C. S. (1931–1958). *Collected papers* (CP, Vols. 1–8). Cambridge: Harvard University Press.

Peirce, C. S. (1992). In N. Houser & C. Kloesel (Eds.), *The essential Peirce* (Vol. 1). Bloomington, IN: Indiana University Press.

Peirce, C. S. (1998). In The Peirce Edition Project (Eds.), *The essential Peirce* (Vol. 2). Bloomington, IN: Indiana University Press.

Piaget, J., & Inhelder, B. (2013/1958). *The growth of logical thinking from childhood to adolescence* (A. Parsons & S. Milgram, Trans.). New York: Routledge.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23*(6), 595–610.

Presmeg, N. C. (1997). A semiotic framework for linking cultural practice and classroom mathematics. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey (Eds.), *Proceedings of the 19th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 151–156). Columbus, Ohio.

Presmeg, N. C. (1998). Ethnomathematics in teacher education. *Journal of Mathematics Teacher Education, 1*(3), 317–339.

Presmeg, N. C. (2002). A triadic nested lens for viewing teachers' representations of semiotic chaining. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 263–276). Mexico City: Cinvestav IPN.

Presmeg, N. C. (2006a). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19–34). Prague: PME.

Presmeg, N. C. (2006b). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics, 61,* 163–182.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishers.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics, 80*(1), 101–118.

Radford, L. (2014). On teachers and students. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the American Chapter* (Vol. 1, pp. 1–20). Vancouver, Canada: PME.

Radford, L. (2015). Early algebraic thinking: Epistemological, semiotic, and developmental issues. In S. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 209–227). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-12688-3_15.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*(2–3), 227–245.

Roth, W.-M. (2012a). Societal mediation of mathematical cognition and learning. *Orbis Scholae, 6* (2), 7–22.

Roth, W.-M. (2012b). Tracking the origins of signs in mathematical activity: A material phenomenological approach. In M. Bockarova, M. Danesi, & R. Núñez (Eds.), *Cognitive science and interdisciplinary approaches to mathematical cognition* (pp. 209–247). Munich: LINCOM EUROPA.

Roth, W.-M. (2015). Excess of graphical thinking: Movement, mathematics and flow. *For the Learning of Mathematics, 35*(1), 2–7.

Roth, W.-M. (2016). *Concrete human psychology*. New York: Routledge.

Roth, W.-M., & Jornet, A. (2017). Theorizing without "mediators". *Integrative Psychological and Behavioral Science*. https://doi.org/10.1007/s12124-016-9376-0.

Roth, W.-M., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of Educational Research, 68,* 35–59.

Sáenz-Ludlow, A., & Kadunz, G. (2016a). *Semiotics as a tool for learning mathematics: How to describe the construction, visualisation, and communication of mathematics concepts*. Rotterdam: Sense Publishers.

Sáenz-Ludlow, A., & Kadunz, G. (2016b). Constructing knowledge as a semiotic activity. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics* (pp. 1–21). Rotterdam: Sense Publishers.

Sáenz-Ludlow, A., & Presmeg, N. (2006). Semiotic perspectives in mathematics education. *Educational Studies in Mathematics. Special Issue*, *61*(1–2).

Snow, R. E. (1992). Aptitude theory: Yesterday, today, and tomorrow. *Educational Psychologist, 27,* 5–32.

Soo, K., Mavin, T. J., & Roth, W.-M. (2016). Mixed-fleet flying in commercial aviation: A joint cognitive systems perspective. *Cognition, Technology & Work, 18,* 449–463.

Vygotskij, L. S. (2001). *Lekcii po pedologii [Lectures on pedology]*. Izhevsk: Udmurdskij University.

Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky, vol. 1: Problems of general psychology*. New York: Springer.

Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.

Vygotsky, L. S. (1997). *The collected works of L. S. Vygotsky, vol. 4: The history of the development of higher mental functions.* New York: Springer.

Wittgenstein, L. (1975/1969). *On certainty* (revised ed.). New Jersey: Wiley.

Wittgenstein, L. (1997). *Philosophical investigations/Philosophische Untersuchungen* (2nd ed.). Oxford: Blackwell (First published in 1953).

Wittmann, E. C. (1995). Mathematics education as a "design science". *Educational Studies in Mathematics, 29*(4), 355–374.

Yu, P. W. (2004). *Prototype development and discourse among middle school students in a dynamic geometry environment.* Unpublished Ph.D. dissertation, Illinois State University.

Zeyer, A., & Roth, W.-M. (2009). A mirror of society: a discourse analytic study of 14–15-year-old Swiss students' talk about environment and environmental protection. *Cultural Studies of Science Education, 4,* 961–998.