Grade 9 Students' Reasoning About Division of Fractions: What Are their Arguments Anchored in?

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Abstract This paper studies secondary school student's mathematical reasoning when solving tasks about fractions. The aim is to explore what the mathematical foundation is replaced with in their reasoning when reasoning is classified as imitative. Two different foundations were found: incorrect mathematical properties not relevant to the task and beliefs about mathematics and mathematics education. The results suggest that a focus on reasoning provides additional information about students' knowledge about fractions beyond standard error analysis.

Keywords Arithmetics • Beliefs • Fractions • Mathematical reasoning • Secondary level

Introduction

One of the challenges for students in school mathematics is to understand rational numbers (Nunes & Bryant, 2009). A possible explanation is the complexity of this concept. You can, for instance, see fractions as a measurement (e.g. half a pie) or as an arithmetic operation, division (Marshall, 1993). Nunes and Bryant (2009) concluded in their review of research about fractions that it is crucial for children to learn it both as a quantity and as a division. Also, their review shows that the relationship between these two views, quantity and division, doesn't come automatically, meaning teaching about fractions also needs to encompass relationship between different representations. A student's view of fractions affects their

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reasoning: children are more successful ordering fractions in magnitude in situations that involve division than seeing fraction as a quantity (Nunes & Bryant, 2009). Hence, how you construct your reasoning and on which mathematical properties, this including relationships, determines how successful you are.

One of the reasons to focus on understanding of fractions is because it is an important step towards algebra (Fenell et al., 2008; Norton & Hackenberg, 2010). Confusion with fractions will thus make development of algebra more difficult. This paper aims to study student's arguments when solving tasks about fractions. The research questions are as follows: (1) When imitative reasoning, what is replacing the mathematical foundation? (2) What beliefs about mathematics and mathematics education are indicated in students' arguments?

Background

The starting point for this paper is that rational numbers are numbers in the domain of quotients (Brousseau, Brousseau, & Warfield, 2007), meaning numbers that could be defined as *a*/*b*. Children's different way of learning various aspects of rational numbers has been in focus for several research studies (Nunes & Bryant, 2009). As mentioned above, a plausible explanation of the problems with learning fractions might be due to the number of different interpretations including representations. Fractions can be interpreted as part of a whole, ratio, measure, relation and operator (Kieren, 1976, 1993; Marshall, 1993). A lot of the previous research stress the importance of learning about fractions from a multidimensional perspective (Nunes & Bryant, 2009) but also the relations between these interpretations (Nunes & Bryant, 2009). Most of the errors students make when solving tasks about fractions, such as the errors presented in Padberg (1989 in Engström, 1997), stem from confusions students experience when constructing understanding about concepts of numbers including fractions (Siegler & Lortie-Forues, 2015; Stacey, Helme, & Steinle, 2001).

Most research about fractions seems to focus on variants of error analysis or confusions such as Padberg's study, and only a few looks on students' reasoning (Norton & Hackenberg, 2010), in particular fraction arithmetic (Siegler & Lortie-Forues, 2015). Students' reasoning about fractions has been identified as one of the areas that needs more research especially in relation to the development of algebraic reasoning (Norton & Hackenberg, 2010). One of the few is Keijzer and Terwel (2001). They describe their study as an analysis of 'growth in reasoning ability' (Keijzer & Terwel 2001, p.53), from informal reasoning to a more formal reasoning. However, reasoning in their study is not clearly defined, something that is not uncommon in research about reasoning (Lithner, 2008; Sumpter, 2013). Reasoning is instead often reduced to some sort of high-quality thinking based on deductive logic, meaning that we can't talk about different types of reasoning, including incorrect ones, and their foundation. There is also a lack of separation between reasoning and argumentation (Sumpter, 2013).

Therefore, in this paper, we use Lithner's (2008) framework where reasoning is seen as the line of thought adopted to produce assertions and reach conclusions in task solving. It doesn't have to be based on formal logic, and it may even be incorrect (Lithner, 2008). The choice is to see reasoning as a product that appears in the form of a sequence, starting with a task (e.g. exercises, tests, etc.) and ending with an answer. Argumentation is considered to be the substantiation, the part of reasoning that fills the purpose of convincing you or someone else that the reasoning is appropriate. To talk about the content of an argument, we look at the relevant mathematical properties of the components in the reasoning. These components are objects (a fundamental entity, e.g. numbers, variables and functions), transformations (a process to an object where a sequence of these transformations is a procedure, e.g. finding a polynomial maxima) and concepts (a central mathematical idea built on a set of objects, transformations and their properties, e.g. infinity concept).

The division between surface and intrinsic properties aims to capture the relevancy of a property depending on the context. This example, provided by Lithner (2008), illustrates this:

> In deciding if 9/15 or 2/3 is larger, the size of the numbers (9,15,2,3) is a *surface* property that is insufficient to consider while the quotient captures the *intrinsic* property. (Lithner, 2008, p.261)

There are two main types of reasoning: imitative and creative mathematical reasoning (Lithner, 2008). Creative mathematical reasoning (CMR) is reasoning that is novel and plausible and has a mathematical foundation, all which imitative reasoning (IR) does not require. IR is a family of different types of reasoning, but in this paper, we will only separate between CMR and IR. Research looking at university students show that most tasks in tests and textbooks can be solved with IR (Bergqvist, 2006; Lithner, 2004). Similar studies have not yet been done at lower secondary level.

In order to talk about affective factors as part of arguments, such as beliefs, we use the notion Beliefs Indications (BI). BI is defined as 'a theoretical concept and part of a model aiming to describe a specific phenomenon', i.e. the type of arguments given by students when solving school tasks in a lab setting (Sumpter, 2013, p.1116), where beliefs are thought of as "an individual's understandings that shape the ways that the individual conceptualises and engages in mathematical behaviour generating and appearing as thoughts in mind" (Sumpter, 2013, p. 1118). In this sense, beliefs are primarily cognitive structures similar to concept images or misconceptions, and what is expressed by the students in their arguments could be interpreted as indication of beliefs.

Method

Data was collected by video recording task-solving sessions that were fully transcribed. The students' written solutions were also part of the data. Three students, two girls and one boy, from a council school (middle-class suburb to a city) participated in the study. They were in grade 9 which is the last year of compulsory school. Their grades were well above average, but not the highest grade, to ensure that they would have the basic knowledge about fractions. The students were asked to solve three tasks in a lab situation. The tasks were designed to have different levels of difficulties and encompass different aspects of division of fractions. Each task was presented one at the time, and the students could stop whenever they wanted. The first task consisted of five subtasks, with the same question posed: Does the answer get bigger or smaller [than the dividend]? Motivate. Then, solve the task.

1. (a) $\frac{3}{4}\Big/\frac{1}{4}=$ (b) $\frac{15}{3}\Big/2=$ (c) $\frac{4}{5}\Big/\frac{1}{10}=$ (d) $\frac{5}{6}\Big/\frac{13}{7}=$ (e) $\frac{2}{3}\Big/\frac{1}{2}=$

In this task, there is a variation between numerators, denominators and ratios (the answers). The reason why the students were asked to estimate whether the answers were going to be bigger or larger than the dividend was to stimulate logical reasoning based on mathematical properties of division with fractions.

2. Decide the fraction that is half of 4/9. Is the answer bigger or smaller [than 4/9]? Motivate. Then, solve the task.

This task aims to check whether the students understand division of a fraction using 'half of'. Similar tasks can be found in textbook from grade 3.

3. Can you divide two fractions and get the result 5? Explain how you are reasoning.

This task aims to generate data to see if the students could work 'backwards' about division of fractions.

The students were asked to think aloud, a set-up that has been used in previous studies (Jäder, Sidenvall, & Sumpter, 2016; Lithner, 2008; Sumpter, 2013). To structure the data, a four-step reasoning sequence was used (Jäder et al., 2016):

1. A (sub-)task is met, which is denoted as task situation (TS).
2. A strategy choice (SC) is made where 'choice' is seen in a wide sense (choose, recall, construct, discover, guess, etc.).
3. The strategy implementation (SI).
4. A conclusion (C) is obtained.

The characterisation of reasoning types is based on analyses of the explicit arguments for strategy choice and implementation, and the reasoning was characterised as CMR or IR following Lithner's (2008) framework. When studying BI, in this study, we look for explicit metacognitive statements in the transcripts of the task-solving session. BI is data carrying information about the person's beliefs, as defined earlier. Data containing traces of the student's argument were marked. They could be local (for instance, a specific strategy choice) or global (e.g. belief about problem solving). Since the data is not triangulated, e.g. with stimulated recall interviews, here we only see it as indicated beliefs. The BIs were gathered in themes using inductive thematic analysis. The themes were checked against each other and back to the original data since the data within the themes had to 'cohere together meaningfully, while there should be clear and identifiable distinctions between themes' (Braun & Clarke, 2006, p. 91).

Results

In total, the three students generated 33 task situations (TS), most of them ($n = 30$) classified as IR and three classified as CMR. Focusing on IR, two types of foundations were used by the students. The first type of foundation that students used in their arguments was a mathematical foundation although wrong and/or not central for the task. The other type of foundation was indicated beliefs (BI). Here, parts of Ida's and Linn's work illustrate the reasoning and the two different types of arguments. 'I' stands for interviewer.

Ida's Work

Ida is trying to solve the following task: Decide the fraction that is half of 4/9. Is the answer bigger or smaller [than 4/9]? Her work is divided in two parts, and both parts will be presented.

In the first part, Ida tries to decide if the answer to the division is bigger or smaller than the dividend. Her conclusion is reached after just a few seconds, and she gives the following supporting arguments:

Ida: Because when you divide it, then Stefan [the teacher] has always nagged about this thing with cakes… Ah if you have… but how should I explain it, if you have a cake and eh then you should divide into four… and then one can't come so then you should divide into three, then the pieces get bigger.

TS 1: $\frac{4}{9}\Big/2 > \text{or} < \frac{4}{9}$?

SC 1: Reference to teacher's description about division of a cake, comparing magnitudes, here ¼ with 1/3.

SI 1: No further implementation.

C 1: Bigger/Larger.

Ida's reasoning, although referring to mathematical ideas about size of fraction, is not based on mathematical properties central to this task.

In the second part of the solutions process, she tries to solve the division. After some time, when Ida has been quiet, the interviewer asks her about the solution:

I: Do you know the solution/how to proceed?

Ida: No, that is what I'm sitting and eh… thinking of… ninths are not something that we have excessively worked with [laughs nervously].

Shortly after, Ida decides to divide the fraction with 4. She soon realises that she can't divide the denominator (9) with 4 without receiving a reminder and stops without any further explanations:

TS 2 How to solve $\frac{4}{9}\Big/2$?
SC 2 Divide the fraction with 4.
SI 2 Starts, but soon discover 4/9 (i.e. 9/4 gives a reminder)
C 2 Must be wrong, ends with no further arguments.

In this part of the solution, there are no arguments verifying and/or supporting the strategy choice or the implementation, but there is some information about Ida's beliefs about mathematics and mathematics education. She says that 'ninths are not something that we have excessively worked with' as if there is a difference regarding mathematical properties between different types of fractions, and her ability to reason is restricted to what fractions that have been dealt with in mathematics class.

Linn's Work

Linn is trying to solve task 1b: $\frac{15}{3}\Big/2$ =. Does the answer get bigger or smaller [than the dividend]? Motivate. Then, solve the task. Her work is divided in two parts. Here, we focus on the first part since the second part was classified as a CMR.

Linn starts by saying that there are five 3s in 15 and then asks the observer if she can take 'like 5 divided with 2 then?'. She gets no answer from the observer. Linn continues by saying it feels like this should be the case since there are five 3s in 15 and then you should take 5 divided by 2. However, when she starts to implement this strategy, she first starts to write down $\frac{15}{3}$ = 5, i.e. with a horizontal line. When she sees this, Linn says 'Whoopsie, now it got a bit wonky'. She explains that $\frac{15}{3}$ means fractions, and when you write it as 15/3, then it means division. Since she was supposed to divide 15 with 3, she has to write it as a division, because the signs, according to her, have different meanings:

TS 1 How to solve $\frac{15}{3}\Big/2$?
SC 1 Since $\frac{15}{3}$, according to Linn, differs from 15/3, she interprets it now as a division.
SI 1 15 is first divided by 3 and then with 2. Writes it down and performs the division in her head.
C 1 2.5.

The reasoning is based on arguments signalling that her understanding of fractions is dependent on her understanding of division or more specifically the signs used in division. Despite that Linn thinks that the fraction $\frac{15}{3}$ can be performed as a division, she needs to rewrite it to 15/3 before it makes sense. This is interpreted as an indicated belief about difference between division and fraction regarding signs.

Discussion

In this study, three students' solutions were analysed focusing on arguments when the reasoning was classified as IR. Even though there were only a small number of students who participated, which of course means that there are limitations with this study, we can see some results that points to various directions.

First and foremost, the results indicate that the students were not used to formulate arguments based on mathematical properties. The number of reasoning sequences that were classified as CMR was 3 out of 33. Instead, the arguments were anchored in what the students think would be the correct algorithm or a 'feeling' what could be a plausible conclusion, most often with a reference to previous experiences. The mathematical foundation needed for CMR was then replaced with two different themes. The first theme was incorrect mathematical foundation. This is similar to the errors that have been reported in previous research, e.g. Padberg (1989 in Engström, 1997). However, we would like to distinguish what is imitative reasoning and what is incorrect mathematical foundation. When studying students' conceptual knowledge about fraction arithmetic, Siegler and Lortie-Forgues (2015) concluded that although demonstrating ability to perform procedures, students showed weak conceptual understanding of multiplication and division of fractions below 1. This could be interpreted that students can solve tasks with imitative reasoning but at the same time not be able to create mathematical reasoning based on central mathematical properties. Therefore, we look at the arguments. In this paper, we see similar patterns that have previously been described as 'confusion' that arises from when students are trying to make sense of number concepts including fractions (Stacey et al., 2001). And, according to Siegler and Lortie-Forgues (2015), we can add algorithms to this mess: the students' produced imitative reasoning based on incorrect mathematical foundation, but importantly they did not know how to proceed when algorithms did not behave as the students thought they would. Such behaviour has been observed in previous research in mathematical reasoning (Jäder et al., 2016; Sumpter, 2013).

The other theme, which was indicated beliefs (BI), might provide more information about this confusion. The BIs were about mathematics such as there is a difference between division and fractions, although it supposed to be interpretations of the same concept, *a*/*b* (c.f. Brousseau et al., 2007; Marshall, 1993). This would imply that the students, although in grade 9 and last year of compulsory schooling with relatively high grades, have not learnt *a*/*b* both as a quantity and as a division, but also as Nunes and Bryant (2009) emphasised the relationships between these two views. If so, this would indeed be a restriction when trying to perform algebraic reasoning (c.f. Norton & Hackenberg, 2010). Imitative reasoning can be considered to be very unproductive even though it could help you to solve a lot of tasks in little time (Lithner, 2008). Bergqvist (2006) explains this further, focusing on algorithmic reasoning which is a part of IR:

> Using algorithmic reasoning is not a sign of lack of understanding, since algorithms are frequently used by professional mathematicians. […] Algorithmic reasoning is however *possible* to perform without any understanding of the intrinsic mathematics. (p.34)

In this way, an algorithm is designed to avoid to create meaning and understanding (Lithner, 2008), meaning that you could use it without knowing what you are doing.

Another type of beliefs that was indicated was about mathematics education. When Ida says that that 'ninths are not something that we have excessively worked with', it could be interpreted that different fractions have different types of mathematical properties, which would be a belief about mathematics, but also that her reasoning is limited to the fractions that have been in focus in mathematics class. This could be a belief about expectations (Sumpter, 2013): you work with tasks that you have seen in class. In combination, these two indicated beliefs appear to be restricting Ida's possibilities to even start a CMR, even though the task is relatively easy.

Since there are few studies about reasoning about fractions (Norton & Hackenberg, 2010), the implications from this study are two. First, it appears that it is not enough to focus on error analysis in research about students' work in fraction since their arguments also consist of other aspects, here called BIs. Siegler and Lortie-Forgues (2015) stress the importance of working with well-chosen problems including follow-up discussions and not just 'standard' textbook tasks when learning fraction arithmetic. This could, perhaps, not just help students to develop their conceptual understanding but also prevent the creation of unproductive and/or incorrect mathematical beliefs.

Second, when teaching about fractions, it seems it is not enough to focus on fraction as a quantity and as a division. Just as Nunes and Bryant (2009) concluded, the relationships between these two views also need to be in focus since the transfer, as illustrated with Linn's work, is by no means straightforward.

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