The Soft Consensus Model in the Multidistance Framework

Silvia Bortot, Mario Fedrizzi, Michele Fedrizzi, Ricardo Alberto Marques Pereira and Thuy Hong Nguyen

Abstract In the context of the soft consensus model due to (Fedrizzi et al. in Journal international journal of intelligent systems 14:63–77, 1999) [27], (Fedrizzi et al. in New mathematics and natural computation 3:219–237, 2007) [28], (Fedrizzi et al. in Preferences and Decisions: models and applications, studies in fuzziness and soft computing Springer, Heidelberg, pp. 159–182, 2010) [30], we investigate the reformulation of the soft dissensus measure in relation with the notion of multidistance, recently introduced by Martín and Mayor (Information processing and management of uncertainty in knowledge-based systems. Theory and methods, communications in computer and information science, springer, heidelberg, pp. 703–711 2010) [43], Martín and Mayor (Fuzzy sets and systems 167:92–100 2011) [44]. The concept of multidistance is as an extension of the classical concept of binary distance, obtained by means of a generalization of the triangular inequality. The new soft dissensus measure introduced in this paper is a particular form of sum-based multidistance. This multidistance is constructed on the basis of a binary distance defined by means of a subadditive scaling function, whose role is that of emphasizing small distances and attenuating large distances in preferences. We present a detailed study of the subadditive scaling function, which is analogous but not equivalent to the one used in the traditional form of the soft consensus model.

Keywords Multidistances · Dissensus measures · Soft consensus model

1 Introduction

The notion of consensus is central to decision making models involving the aggregation of individual preferences. We can distinguish essentially two complementary readings of the consensus concept. In general terms, it refers to the consensual

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C. Berger-Vachon et al. (eds.), *Complex Systems: Solutions and Challenges in Economics, Management and Engineering*, Studies in Systems, Decision and Control 125, https://doi.org/10.1007/978-3-319-69989-9_10

preference resulting from the aggregation scheme, whether or not preference aggregation is formulated as an iterative consensus reaching process. More specifically, the notion of consensus refers to the construction of consensus (dissensus) measures, which express the level of agreement (disagreement) present in the collective profile of individual preferences.

In general, consensual aggregation models involve some form of explicit or implicit averaging of the individual preferences. In the context of aggregation theory, comprehensive reviews of averaging functions can be found in [2, 7, 21, 33, 34].

In our approach we are primarily interested in the class of aggregation schemes which are based on consensus (dissensus) measures, often constructed on the basis of some binary distance acting pairwise on the individual preferences.

In this respect, the recent literature on the use of penalty functions in aggregation [3–6, 17, 18, 20] provides a suggestive framework in which to describe the interrelation between aggregation functions and consensus (dissensus) measures.

Further interesting investigation on the construction and applications of consensus (dissensus) measures can be found in [1, 8–11, 13–16, 19, 23–26, 46, 47, 49–53].

In the tradition of the fuzzy approach to consensus in the aggregation of individual preferences [31, 32, 35, 36, 40, 41], the soft consensus model was originally proposed in [37–39] and later reformulated in [27–30]. The soft consensus model is based on a dissensus measure constructed from pairwise square differences, composed with a subadditive scaling function (substituting the linguistic quantifiers in the original version of the model), whose role is that of emphasizing small (attenuating large) preference differences by means of a smooth thresholding effect.

In this paper we wish to revisit the soft consensus model and investigate the formulation of the soft dissensus measure in relation with the notion of multidistance, recently introduced in [22, 42–45, 48]. The concept of multidistance is as an extension of the classical concept of binary distance, obtained by means of a generalization of the triangular inequality.

With respect to the traditional soft consensus model, here the idea is to construct a new multidistance dissensus measure directly from the pairwise absolute value differences and the subadditive scaling function, keeping the traditional character of the soft dissensus measure but avoiding the square differences in the functional form.

The paper is organized as follows. In Sect. 2 we briefly review the soft consensus model and the construction of the traditional soft dissensus measure. In Sect. 3 we review the basic notions regarding multidistances and in Sect. 4 we introduce the new multidistance dissensus measure, with a detailed study of the subadditive scaling function. Finally, in Sect. 5 we present some concluding remarks and notes on future research.

2 The Soft Consensus Model

In this section we present a brief review of the traditional soft consensus model in the formulation introduced in [27]. Our point of departure is a set of individual fuzzy preference relations. If $A = \{a_1, \dots, a_m\}$ is a set of decisional alternatives and $I = \{1, \dots, n\}$ is a set of individuals, the fuzzy preference relation R_i of individual i is given by its membership function $R_i : A \times A \to [0, 1]$ with

$$\begin{array}{lll} R_i(a_k, a_l) = & 1 & \text{if } a_k \text{ is definitely preferred over } a_l \\ R_i(a_k, a_l) \in & (0.5, 1) & \text{if } a_k \text{ is preferred over } a_l \\ R_i(a_k, a_l) = & 0.5 & \text{if } a_k \text{ is considered indifferent to } a_l \\ R_i(a_k, a_l) \in & (0, 0.5) & \text{if } a_l \text{ is preferred over } a_k \\ R_i(a_k, a_l) = & 0 & \text{if } a_l \text{ is definitely preferred over } a_k, \end{array}$$

where $i = 1, \dots, n$ and $k, l = 1, \dots, m$. Each individual fuzzy preference relation R_i can be represented by a matrix $[r^i_{kl}]$, $r^i_{kl} = R_i(a_k, a_l)$ which is commonly assumed to be reciprocal, that is $r^i_{kl} + r^i_{lk} = 1$. Clearly, this implies $r^i_{kk} = 0.5$ for all $i = 1, \dots, n$ and $k = 1, \dots, m$.

The general case $A = \{a_1, \dots, a_m\}$ for the set of decisional alternatives is discussed in [27, 28]. Here, for the sake of simplicity, we assume that the alternatives available are only two ($m = 2$), which means that each individual preference relation R_i has only one degree of freedom, denoted by $x_i = r^i_{12}$.

In the framework of the soft consensus model, assuming $m = 2$, the degree of dissensus between individuals i and j as to their preferences between the two alternatives is measured by

$$V_{ij} = g((x_i - x_j)^2) \qquad i, j = 1, \dots, n \tag{1}$$

where $g : [0, 1] \to \mathbb{R}$ is a scaling function defined as

$$g(u) = \frac{1}{\alpha} \ln \left(\frac{1}{1 + e^{-\alpha(u-\beta)}} \right) \qquad u \in [0, 1] \,. \tag{2}$$

In the scaling function formula above, $\beta \in (0, 1)$ is a threshold parameter and $\alpha \in (0, \infty)$ is a free parameter which controls the polarization of the sigmoid function $g' : [0, 1] \to (0, 1)$ given by

$$g'(u) = \frac{1}{1 + e^{\alpha(u-\beta)}} \qquad u \in [0, 1]. \tag{3}$$

In the network representation of the soft consensus model [27], each decision maker $i = 1, \dots, n$ is represented by a pair of connected nodes, a primary node (dynamic) and a secondary node (static). The n primary nodes form a fully connected subnetwork and each of them encodes the individual opinion of a single decision maker. The n secondary nodes, on the other hand, encode the individual opinions originally declared by the decision makers, denoted $s_i \in [0, 1]$, and each of them is connected only with the associated primary node.

The iterative process of preference change corresponds to the gradient descent optimization of a cost function W, depending on both the present and the original network configurations. The value of W combines a measure V of the overall dissensus in the present network configuration with a measure U of the overall change from the original network configuration.

The various interactions involving node i are modulated by interaction coefficients whose role is to quantify the strength of the interaction. The consensual interaction between primary nodes i and j is modulated by the interaction coefficient $v_{ij} \in (0, 1)$, whereas the inertial interaction between primary node i and the associated secondary node is modulated by the interaction coefficient $u_i \in (0, 1)$. In the soft consensus model the values of these interaction coefficients are given by the derivative g' of the scaling function according to

$$v_{ij} = g'((x_i - x_j)^2) \qquad i,j = 1, \dots, n \tag{4}$$

$$v_i = \sum_{j(\neq i)=1}^{n} v_{ij}/(n-1), \quad u_i = g'((x_i - s_i)^2) \qquad i = 1, \dots, n. \tag{5}$$

The average preference $\bar{x}_i$ of the context of individual i is given by

$$\bar{x}_i = \frac{\sum_{j(\neq i)=1}^{n} v_{ij}\, x_j}{\sum_{j(\neq i)=1}^{n} v_{ij}} \qquad i = 1, \dots, n \tag{6}$$

and represents the average preference of the remaining decision makers as seen by decision maker $i = 1, ..., n$.

The construction of the cost function W that drives the dynamics of the soft consensus model is as follows. The individual dissensus cost is given by

$$V_i(\boldsymbol{x}) = \sum_{j(\neq i)=1}^{n} V_{ij}/(n-1) \qquad i = 1, \dots, n \tag{7}$$

and the individual opinion changing cost is

$$U_i(\boldsymbol{x}) = g((x_i - s_i)^2) \qquad i = 1, \dots, n. \tag{8}$$

Summing over the various decision makers we obtain the collective dissensus cost V and inertial cost U,

$$V(\boldsymbol{x}) = \frac{1}{4} \sum_{i=1}^{n} V_i(\boldsymbol{x}), \quad U(\boldsymbol{x}) = \frac{1}{2} \sum_{i=1}^{n} U_i(\boldsymbol{x}) \tag{9}$$

with conventional multiplicative factors 1/4 and 1/2. The full cost function is then

$$W(\boldsymbol{x}) = V(\boldsymbol{x}) + U(\boldsymbol{x}). \tag{10}$$

The consensual network dynamics acts on the individual opinion variables x_i through the iterative process

$$x_i \rightsquigarrow x_i' = x_i - \gamma \frac{\partial W}{\partial x_i} \qquad i = 1, \dots, n. \tag{11}$$

Analyzing the effect of the two dynamical components V and U separately we obtain

$$\frac{\partial V}{\partial x_i} = v_i(x_i - \bar{x}_i) \qquad i = 1, \dots, n \tag{12}$$

where the coefficients v_i were defined in (5) and the average preference $\bar{x}_i$ was defined in (6), and therefore

$$x_i' = (1 - \gamma\, v_i)x_i + \gamma\, v_i \bar{x}_i \qquad i = 1, \dots, n. \tag{13}$$

On the other hand, we obtain

$$\frac{\partial U}{\partial x_i} = u_i(x_i - s_i) \qquad i = 1, \dots, n \tag{14}$$

where the coefficients u_i were defined in (5), and therefore

$$x_i' = (1 - \gamma\, u_i)x_i + \gamma\, u_i s_i \qquad i = 1, \dots, n. \tag{15}$$

The full dynamics associated with the cost function $W = V + U$ acts iteratively according to

$$x_i' = (1 - \gamma\,(v_i + u_i))x_i + +\gamma\, v_i \bar{x}_i + \gamma\, u_i s_i \qquad i = 1, \dots, n \tag{16}$$

and the decision maker i is in dynamical equilibrium, in the sense that $x_i' = x_i$, if the following stability equation holds,

$$x_i = (v_i \bar{x}_i + u_i s_i)/(v_i + u_i) \qquad i = 1, \dots, n \tag{17}$$

that is, if the present opinion x_i coincides with an appropriate weighted average of the original opinion s_i and the average opinion value $\bar{x}_i$ for $i = 1, \dots, n$.

3 The Multidistance Framework

The definition of multidistance has been introduced by Martín and Mayor in [43, 44] as an extension of the classical notion of binary distance to the case of more than two points.

Consider a domain $X \subseteq \mathbb{R}$, with points in X^n being denoted as $\boldsymbol{x} = (x_1, \dots, x_n)$. The multidistance definition given in [44] is as follows.

Definition 1 Given a domain $X \subseteq \mathbb{R}$, a *multidistance* is a function

$$D : \bigcup_{n\geq 2} X^n \to \mathbb{R}$$

with the following properties

(P1) $D(x_1, \dots, x_n) = 0$ if and only if $x_i = x_j$ for all $i, j = 1, \dots, n$
(P2) $D(x_1, \dots, x_n) = D(x_{\pi(1)}, \dots, x_{\pi(n)})$ for any permutation π of $1, \dots, n$
(P2) $D(x_1, \dots, x_n) \leq D(x_1, y) + \cdots + D(x_n, y)$ for all $y \in X$

for all $x_1, \dots, x_n \in X$ and $n \geq 2$. Note that (P1), (P2) and (P3) extend the usual distance axioms. In particular, (P3) generalizes the triangle inequality.

An important class of multidistances, the functionally expressible multidistances, are studied in [45, 48]. Applications of multidistances to the problem of consensus measuring can be found in [16, 22].

Starting from the results obtained in [13, 14] in [16] some connections between m-ary adjacency relations and multidistances were highlighted. It has been shown how m-ary adjacency relations can be modeled on the basis of OWA-based multidistances, and some consensus related optimization problems on m-ary adjacency relations are equivalent to corresponding multidistance minimization problems.

In this paper, a multidistance dissensus measure is introduced as an extension of the relationship between the dissensus measure in the traditional soft consensus model proposed in [27] and the multidistance approach to consensus introduced in [16]. This measure is based on a binary distance defined by means of a subadditive function whose effect is that of emphasizing small distances and attenuating large distances.

There are several methods to construct multidistances. As suggested in [44] given a binary distance $d(x_i, x_j)$, a multidistance may be defined on the basis of the pairwise binary distances, multiplying their sum by a sufficiently small value $\lambda(n)$ depending on n. This type of multidistance is called *sum–based multidistance*.

Proposition 1 (Martín and Mayor [44]) *A function* $D : \bigcup_{n\geq 2} X^n \to \mathbb{R}$ *defined as*

$$D(x_1, ..., x_n) = \lambda(n) \sum_{i,j=1}^{n} d(x_i, x_j) \qquad n \geq 2$$

is a multidistance if and only if the coefficient $\lambda(n)$ *satisfies* $\lambda(2) = 1/2$ *and*

$$0 < \lambda(n) \leq \frac{1}{2(n-1)} \qquad n \geq 3 \tag{18}$$

where $x_1, \dots, x_n \in X$.

In this paper we use the domain $X = [0, 1]$ equipped with the classical distance $d(x, y) = |x - y| \in [0, 1]$, for $x, y \in [0, 1]$, with the usual triangular inequalities $|x + y| \leq |x| + |y|$ and $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in [0, 1]$.

Moreover, we consider the particular coefficient choice

$$\lambda(n) = \frac{1}{n(n-1)} \tag{19}$$

which corresponds to constructing the sum-based multidistance by averaging pairwise binary distances.

Consider now an increasing and subadditive function $f : [0, 1] \to \mathbb{R}$. Due to the subadditivity of the function f, the composition of the distance d with the function f yields a new distance denoted $d_f(x, y) = f(d(x, y))$, which satisfies the triangle inequality $d_f(x, y) \leq d_f(x, z) + d_f(y, z)$. This is obtained as follows,

$$d(x, y) \leq d(x, z) + d(y, z) \tag{20}$$

$$f(d(x, y)) \leq f(d(x, z) + d(y, z)) \leq f(d(x, z)) + f(d(y, z)) \tag{21}$$

where the first inequality is due to the increasingness of f and the second inequality is due to the subadditivity of f. Finally, we obtain

$$d_f(x, y) \leq d_f(x, z) + d_f(y, z). \tag{22}$$

We consider the construction of multidistances based on the binary distance d_f, in particular by averaging pairwise binary distances. In this way we define a multidistance $D_f : \bigcup_{n\geq 2}[0, 1]^n \to \mathbb{R}$ as

$$D_f(x_1, ..., x_n) = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} d_f(x_i, x_j)\,. \tag{23}$$

Consider for instance the case $n = 3$. The multidistance D_f is given by

$$D_f(\boldsymbol{x}) = \frac{1}{6} \sum_{i,j=1}^{3} d_f(x_i, x_j) = \tag{24}$$

$$= \frac{1}{3}\Big(d_f(x_1, x_2) + d_f(x_1, x_3) + d_f(x_2, x_3)\Big) =$$

$$= \frac{1}{3}\Big(f(d(x_1, x_2)) + f(d(x_1, x_3)) + f(d(x_2, x_3))\Big).$$

Notice that each binary distance term is of the form

$$d_f(x_i, x_j) = c_f(x_i, x_j) \cdot d(x_i, x_j) \tag{25}$$

where each classical binary distance $d(x_i, x_j)$ is multiplied by a coefficient

$$c_f(x_i, x_j) = f(d(x_i, x_j)/d(x_i, x_j) \tag{26}$$

depending on the choice of the function f.

The multidistance D_f corresponds to a linear combination of the classical binary distances d, with non negative coefficients c_f. However, notice that these coefficients do not have unit sum and therefore the multidistance D_f does not correspond to a weighted mean of of the classical binary distances d.

4 The Soft Dissensus Measure in the Multidistance Framework

The traditional soft consensus model in group decision making [27, 28] is based on a non linear dissensus measure whose role is that of emphasizing small distances and attenuating large distances in the preference domain.

In this section we reformulate the soft dissensus measure as a sum-based multidistance, in the approach introduced in [12]. In the new multidistance framework, the usual binary distance is composed with a non linear subadditive function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(u) = \frac{2}{\alpha} \ln\left(\frac{1 + e^{\alpha\beta}}{1 + e^{-\alpha(u-\beta)}}\right) \qquad \alpha \in (0, \infty) \quad \beta \in [0, 1] \tag{27}$$

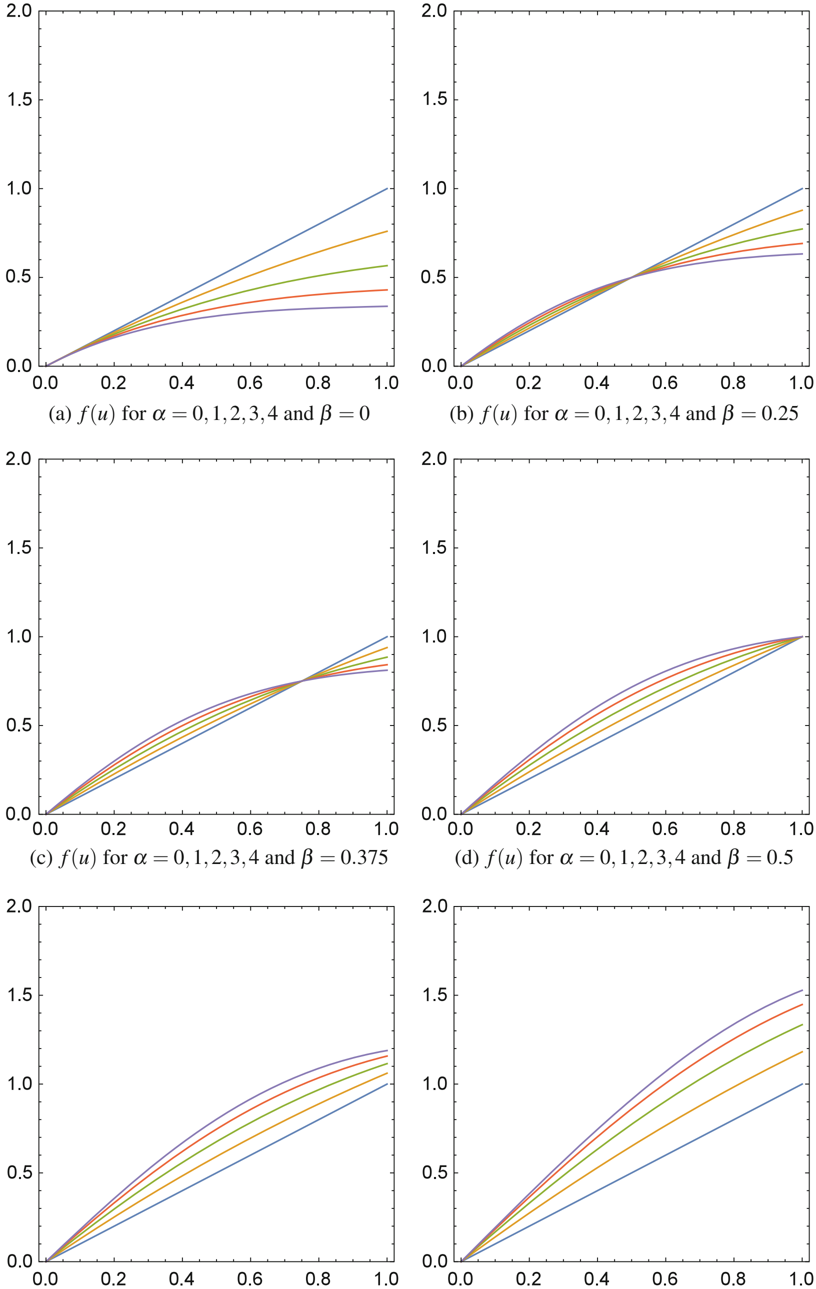
for all $u \in [0, 1]$. The two parameters are $\alpha \in (0, \infty)$ and $\beta \in [0, 1]$, but we can extend the domain of the former by defining $f(u) = u$ for $\alpha = 0$, which in fact corresponds to the asymptotic form of definition (27) at $\alpha = 0^+$.

In Fig. 1 we plot the function $f(u)$ with $u \in [0, 1]$ for various choices of the parameters α, β. In each plot the diagonal line $f(u) = u \in [0, 1]$ is associated with the case $\alpha = 0$.

In the following result we determine the values of the function f at the boundaries of its domain $[0, 1]$, for any choice of the parameters α, β.

Proposition 2 *In relation with the function f defined in (27), we obtain $f(0) = 0$ for any choice of the parameters $\alpha \in (0, \infty)$ and $\beta \in [0, 1]$, and*

- $f(1) < 1$ *for all* $\beta \in [0, 1/2)$



(a) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0$

(b) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.25$

(c) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.375$

(d) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.5$

(e) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.625$

(f) $f(u)$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 1$

Fig. 1 The function $f(u)$ for $u \in [0, 1]$. Each plot is associated with a choice of the parameter β and shows the graph of $f(u)$ for various choices of the parameter α

- $f(1) = 1$ *for* $\beta = 1/2$
- $f(1) > 1$ *for all* $\beta \in (1/2, 1]$

for any choice of the parameter $\alpha \in (0, \infty)$. *In particular, the values of* $f(1)$ *for* $\beta = 0, 1$ *are*

$$f(1) = \frac{2}{\alpha} \ln \left(\frac{2}{1 + e^{-\alpha}} \right), \ \beta = 0 \qquad f(1) = \frac{2}{\alpha} \ln \left(\frac{1 + e^{\alpha}}{2} \right), \ \beta = 1 \,.$$

The limit at $\alpha = 0^+$ *is* $f(1) = 1$ *in both cases* $\beta = 0, 1$, *whereas the limit at* $\alpha = \infty$ *is* $f(1) = 0$ *for* $\beta = 0$ *and* $f(1) = 2$ *for* $\beta = 1$.

Proof From definition (27) we obtain immediately that $f(0) = 0$ for any choice of the parameters α, β, plus also

$$f(1) = 1 + \frac{2}{\alpha} \ln \left(\frac{1 + e^{\alpha\beta}}{e^{\alpha/2} + e^{\alpha\beta} e^{-\alpha/2}} \right) \tag{28}$$

which leads immediately to $f(1) = 1$ for $\beta = 1/2$. Otherwise, writing $N = 1 + e^{\alpha\beta}$ for the numerator and $D = e^{\alpha/2} + e^{\alpha\beta} e^{-\alpha/2}$ for the denominator of the logarithm, it follows that

$$N - D = (1 - e^{-\alpha/2})(e^{\alpha\beta} - e^{\alpha/2}) \,. \tag{29}$$

Considering the second factor in the product, we conclude that the logarithmic term in (29) is negative ($N < D$) for $\beta \in [0, 1/2)$ and is positive ($N > D$) for $\beta \in (1/2, 1]$. The asymptotic limits of $f(1)$ with respect to the parameter α for $\beta = 0, 1$ can be obtained straightforwardly by means of l'Hospital's rule. □

The function f is continuous, strictly increasing and strictly concave in $u \in [0, 1]$ for any choice of the parameters α, β. Continuity is clear from definition (27) and the other properties follow directly from the first and second derivatives of f,

$$f'(u) = \frac{2}{1 + e^{\alpha(u-\beta)}} \qquad f'(u) \in (0, 2) \tag{30}$$

$$f''(u) = -\frac{2\alpha e^{\alpha(u+\beta)}}{(e^{\alpha\beta} + e^{\alpha u})^2} \qquad f''(u) \in (-\alpha/2, 0) \tag{31}$$

Notice that $f'(u = \beta) = 1$ and $f''(u = \beta) = -\alpha/2$. Moreover, we can show that $f''(u) = -\alpha f'(u)(2 - f'(u))/2$ for any choice of the parameters α, β. In the case $\alpha = 0$, for any choice of β, we have the linear form $f(u) = u$ for all $u \in [0, 1]$.

Given that f is (strictly) increasing and $f(0) = 0$ for any choice of the parameters α, β, we have that $f(u) \geq 0$ for all $u \in [0, 1]$. Moreover, we can write

$$f(u) = 2u + \frac{2}{\alpha} \ln \left(\frac{1 + e^{\alpha\beta}}{e^{\alpha u} + e^{\alpha\beta}} \right) \tag{32}$$

for all $u \in [0, 1]$, where the logarithmic term is always non positive. Therefore, we obtain $0 \leq f(u) \leq 2u$ for all $u \in [0, 1]$ and any choice of the parameters α, β.

The function f is subadditive, in the sense that $f(u+v) \leq f(u) + f(v)$. The proof is as follows: assuming $u,\ v \in [0, 1]$ and $u + v \neq 0$, concavity of f implies

$$f(u) \geq \frac{v}{u+v} f(0) + \frac{u}{u+v} f(u+v) = \frac{u}{u+v} f(u+v) \tag{33}$$

$$f(v) \geq \frac{u}{u+v} f(0) + \frac{v}{u+v} f(u+v) = \frac{v}{u+v} f(u+v) \tag{34}$$

and therefore we obtain $f(u) + f(v) \geq f(u+v)$ for $u,\ v \in [0, 1]$.

The composition of the distance d with the subadditive function f yields a new distance denoted

$$d_f(x, y) = f(d(x, y)) \tag{35}$$

satisfying the triangle inequality $d_f(x, y) \leq d_f(x, z) + d_f(y, z)$ as in (20)–(22).

We define the multidistance D_f by averaging pairwise binary distances d_f,

$$D_f(\boldsymbol{x}) = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} d_f(x_i, x_j) = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} f(d(x_i, x_j))\,. \tag{36}$$

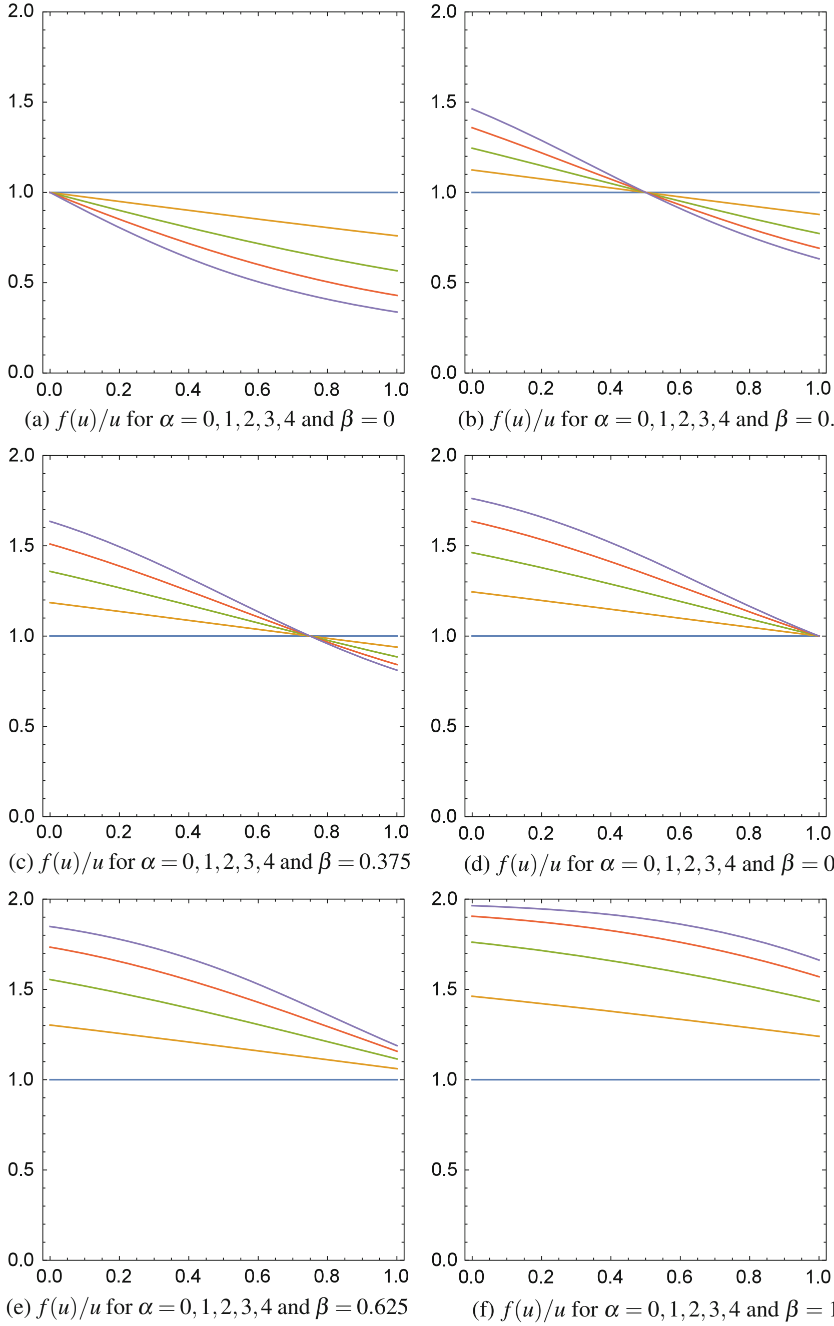
This sum–based multidistance is a natural nonlinear measure of dissensus, analogous but not equivalent to the traditional soft dissensus measure V in (9). The new soft dissensus measure, however, has a more appealing geometrical interpretation as a multidistance.

Finally, recall that each term in (36) is of the form $d_f(x_i, x_j) = c_f(x_i, x_j) \cdot d(x_i, x_j)$, where each binary distance $d(x_i, x_j)$ is multiplied by a coefficient $c_f(x_i, x_j) = f(d(x_i, x_j)/d(x_i, x_j)$ depending on the choice of the function f.

In other words, each term is of the form $f(u) = (f(u)/u) \cdot u$, where each single binary distance u is multiplied by a coefficient $f(u)/u$ which is decreasing with respect to the distance u.

In Fig. 2 we plot the function $f(u)/u$ with various choices of the parameters α and β. In each plot the horizontal line is associated with $\alpha = 0$ and the remaining lines are associated with $\alpha = 1, 2, 3, 4$. In the case $\alpha = 0$ all pairwise distances have the same weight $1/6$ and thus the multidistance corresponds to the weighted average. In the case $\alpha = 1, 2, 3, 4$ the function $f(u)/u$ is monotonically decreasing with respect to pairwise distances, in which a larger weight is assigned to a small distance and a smaller weight is given to a large distance.

The multidistance is defined as the weighted sum of pairwise distances. In this approach, the sum-based multidistance is closely related with the disensus measure in the soft consensus model. There is essentially a single difference: the basic pairwise distance $d_f(x, y)$ involves $|x - y|$ and not $(x - y)^2$, which is not a binary distance.



(a) $f(u)/u$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0$

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(d) $f(u)/u$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.5$

(e) $f(u)/u$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 0.625$

(f) $f(u)/u$ for $\alpha = 0, 1, 2, 3, 4$ and $\beta = 1$

Fig. 2 The function $f(u)/u$ for $u \in [0, 1]$. Each plot is associated with a choice of the parameter β and shows the graph of $f(u)/u$ for various choices of the parameter α

5 Conclusions

We introduce a multidistance measure of dissensus within the framework of the soft consensus model of group decision making. The multidistance dissensus measure is based on a fundamental binary distance d_f associated with a subadditive function f over the domain $X = [0, 1]$, with $d_f(x, y) = f(|x - y|)$. This subadditive function has the effect of emphasizing small distances and attenuating large distances, in analogy with the subadditive scaling function g which plays a central role in the traditional soft consensus model [27, 28, 30].

References

1. Alonso, S., Pérez, I.J., Cabrerizo, F.J., Herrera-Viedma, E.: A linguistic consensus model for Web 2.0 communities. Appl. Soft Comput. **13**, 149–157 (2013)
2. Beliakov, G., Bustince Sola, H., Calvo, T.: A practical guide to averaging functions, studies in fuzziness and soft computing, vol. 329. Springer, Heidelberg (2016)
3. Beliakov, G., Calvo, T., James, S.: Consensus measures constructed from aggregation functions and fuzzy implications. Knowl.-Based Syst. **55**, 1–8 (2014)
4. Beliakov, G., Gagolewski, M., James, S.: Penalty-based and other representations of economic inequality. Int. J. Uncertainty Fuzziness Knowl.-Based Syst. **24**(1), 1–23 (2016)
5. Beliakov, G., James, S.: Unifying approaches to consensus across different preference representations. Appl. Soft Comput. **35**, 888–897 (2015)
6. Beliakov, G., James, S., Wilkin, T.: Aggregation and consensus for preference relations based on fuzzy partial orders. Fuzzy Opti. Decis. Making (online since 23 November 2016)
7. Beliakov, G., Pradera, A., Calvo, T.: Aggregation functions: a guide for practitioners, studies in fuzziness and soft computing, vol. 221. Springer, Heidelberg (2007)
8. Ben-Arieh, D., Chen, Z.: Linguistic group decision-making: opinion aggregation and measures of consensus. Fuzzy Optim. Decis. Making **5**(4), 371–386 (2006)
9. Ben-Arieh, D., Chen, Z.: Linguistic-labels aggregation and consensus measure for autocratic decision making using group recommendations. IEEE Trans. Syst. Man Cybern. **36**(3), 558–568 (2006)
10. Ben-Arieh, D., Easton, T.: Multi-criteria group consensus under linear cost opinion elasticity. Decis. Support Syst. **43**(3), 713–721 (2007)
11. Ben-Arieh, D., Easton, T., Evans, B.: Minimum cost consensus with quadratic cost functions. IEEE Trans. Syst. Man Cybern. **39**(1), 210–217 (2008)
12. Bortot, S., Fedrizzi, M., Fedrizzi, M., Marques Pereira, R.A.: A multidistance approach to consensus modeling. In: Collan, M., Fedrizzi, M., Kacprzyk, J. (eds.) Fuzzy Technol. Stud. Fuzziness Soft Comput. vol. 335, pp. 103–114. Springer, Heidelberg (2016)
13. Brunelli, M., Fedrizzi, M.: A fuzzy approach to social network analysis. In: 2009 International Conference on Advances in Social Network Analysis and Mining, pp. 225–230. IEEE Computer Society (2009)
14. Brunelli, M., Fedrizzi, M., Fedrizzi, M.: OWA-based fuzzy m-ary adjacency relations in social network analysis. In: Yager, R.R., Kacprzyk, J., Beliakov, G. (eds.) Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, Studies in Fuzziness and Soft Computing, vol. 265, pp. 255–267. Springer, Heidelberg (2011)
15. Brunelli, M., Fedrizzi, M., Fedrizzi, M.: Fuzzy m-ary adjacency relations in social network analysis: optimization and consensus evaluation. Inf. Fusion **17**, 36–45 (2014)
16. Brunelli, M., Fedrizzi, M., Fedrizzi, M., Molinari, F.: On some connections between multidistances and valued m-ary adjacency relations. In: Greco, S., Bouchon-Meunier, B.,

Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) Advances in Computational Intelligence, Communications in Computer and Information Science, vol. 297, pp. 201–207. Springer, Heidelberg (2012)
17. Bustince, H., Barrenechea, E., Calvo, T., James, S., Beliakov, G.: Consensus in multi-expert decision making problems using penalty functions defined over a Cartesian product of lattices. Inf. Fusion **17**, 56–64 (2014)
18. Bustince, H., Beliakov, G., Pereira Dimuro, G., Bedregal, B., Mesiar, R.: On the definition of penalty functions in data aggregation. Fuzzy Sets and Syst. (online since 21 September 2016)
19. Cabrerizo, F.J., Moreno, J.M., Pérez, I.J., Herrera-Viedma, E.: Analyzing consensus approaches in fuzzy group decision making: advantages and drawbacks. Soft Comput. **14**(5), 451–463 (2010)
20. Calvo, T., Beliakov, G.: Aggregation functions based on penalties. Fuzzy sets and Syst. **161**(10), 1420–1436 (2010)
21. Calvo, T., Kolesárová, A., Komorníková, M., Mesiar, R.: Aggregation operators: properties, classes and construction methods. In: Calvo, T., Mayor, M., Mesiar, R. (eds.) Aggregation operators: new trends and applications, studies in fuzziness and soft computing, vol. 97, pp. 3–104. Springer, Heidelberg (2002)
22. Calvo, T., Martín, J., Mayor, G.: Measures of disagreement and aggregation of preferences based on multidistances. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) Advances in Computational Intelligence, Communications in Computer and Information Science, vol. 300, pp. 549–558. Springer, Heidelberg (2012)
23. Chen, Z., Ben-Arieh, D.: On the fusion of multi-granularity linguistic label sets in group decision making. Comput. Ind. Eng. **51**, 526–541 (2006)
24. Chiclana, F., Tapia García, J.M., del Moral, M., Herrera-Viedma, E.: A statistical comparative study of different similarity measures of consensus in group decision making. Inf. Sci. **221**, 110–123 (2013)
25. Contreras, I.: A distance-based consensus model with flexible choice of rank-position weights. Group Decis. Negot. **19**(5), 441–456 (2010)
26. Cook, W.D.: Distance-based and ad hoc consensus models in ordinal preference ranking. Eur. J. Oper. Res. **172**(2), 369–385 (2006)
27. Fedrizzi, M., Fedrizzi, M., Marques Pereira, R.A.: Soft consensus and network dynamics in group decision making. Int. J. Intel. Syst. **14**(1), 63–77 (1999)
28. Fedrizzi, M., Fedrizzi, M., Marques Pereira, R.A.: Consensus modelling in group decision making: dynamical approach based on fuzzy preferences. New Math. Nat. Comput. **3**(2), 219–237 (2007)
29. Fedrizzi, M., Fedrizzi, M., Marques Pereira, R.A., Brunelli, M.: Consensual dynamics in group decision making with triangular fuzzy numbers. In: 41st Annual Hawaii International Conference on System Sciences, pp. 70–78. IEEE Computer Society (2008)
30. Fedrizzi, M., Fedrizzi, M., Marques Pereira, R.A., Brunelli, M.: The dynamics of consensus in group decision making: investigating the pairwise interactions between fuzzy preferences. In: Greco, S., Marques Pereira, R.A., Squillante, M., Yager, R.R., Kacprzyk, J (eds.) Preferences and Decisions: Models and Applications, Studies in Fuzziness and Soft Computing, vol. 257, pp. 159–182. Springer, Heidelberg (2010)
31. Fedrizzi, M., Kacprzyk, J., Nurmi, H.: Consensus degrees under fuzzy majorities and fuzzy preferences using OWA (ordered weighted average) operators. Control Cybern. **22**(4), 71–80 (1993)
32. Fedrizzi, M., Pasi, G.: Fuzzy logic approaches to consensus modelling in group decision making. In: Ruan, D., Hardeman, F., van der Meer, K. (eds.) Intelligent Decision and Policy Making Support Systems, Studies in Computational Intelligence, vol. 117, pp. 19–37. Springer, Heidelberg (2008)
33. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support, Theory and Decision Library, series D, vol. 14. Kluwer Academic Publishers, Dordrecht (1994)
34. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation functions, encyclopedia of mathematics and its applications, vol. 127. Cambridge University Press (2009)

35. Herrera-Viedma, E., Cabrerizo, F.J., Kacprzyk, J., Pedrycz, W.: A review of soft consensus models in a fuzzy environment. Inf. Fusion **17**, 4–13 (2014)
36. Herrera-Viedma, E., García-Lapresta, J.L., Kacprzyk, J., Fedrizzi, M., Nurmi, H., Zadrożny, S.: Consensual Processes, Studies in Fuzziness and Soft Computing, vol. 267. Springer, Heidelberg (2011)
37. Kacprzyk, J., Fedrizzi, M.: Soft consensus measures for monitoring real consensus reaching processes under fuzzy preferences. Control Cybern. **15**(3–4), 309–323 (1986)
38. Kacprzyk, J., Fedrizzi, M.: A soft measure of consensus in the setting of partial (fuzzy) preferences. Eur. J. Oper. Res. **34**(3), 316–325 (1988)
39. Kacprzyk, J., Fedrizzi, M.: A human-consistent degree of consensus based on fuzzy login with linguistic quantifiers. Math. Soc. Sci. **18**(3), 275–290 (1989)
40. Kacprzyk, J., Fedrizzi, M., Nurmi, H.: Group decision making and consensus under fuzzy preferences and fuzzy majority. Fuzzy Sets Syst. **49**(1), 21–31 (1992)
41. Kacprzyk, J., Nurmi, H., Fedrizzi, M.: Consensus Under Fuzziness, International Series in Intelligent Technologies, vol. 10. Kluwer Academic Publishers, Dordrecht (1997)
42. Martın, J., Mayor, G.: How separated Palma, Inca and Manacor are? In: Proceedings of the 5th International Summer School of Aggregation Operators AGOP 2009, pp. 195–200. Palma de Mallorca, Spain (2009)
43. Martín, J., Mayor, G.: Some properties of multi-argument distances and fermat multidistance. In: Hllermeier, E., Kruse, R., Hoffmann, F. (eds.) Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Methods, Communications in Computer and Information Science, vol. 80, pp. 703–711. Springer, Heidelberg (2010)
44. Martín, J., Mayor, G.: Multi-argument distances. Fuzzy Sets Syst. **167**(1), 92–100 (2011)
45. Martín, J., Mayor, G., Valero, O.: Functionally expressible multidistances. In: S. Galichet, J.M.G. Mauris (eds.) Proceedings of the 7th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2011) and LFA-2011, Advances in Intelligent Systems Research, pp. 41–46. Atlantis Press, Amsterdam (2011)
46. Merigó, J.M., Casanovas, M.: Decision-making with distance measures and induced aggregation operators. Comput. Ind. Eng. **60**, 66–76 (2011)
47. Meskanen, T., Nurmi, H.: Distance from consensus: a theme and variations. In: Simeone, B., Pukelsheim, F. (eds.) Mathematics and Democracy, Studies in Choice and Welfare, pp. 117–132. Springer, Heidelberg (2006)
48. Molinari, F.: About a new family of multidistances. Fuzzy Sets Syst. **195**, 118–122 (2012)
49. Palomares, I., Martínez, L., Herrera, F.: A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. IEEE Trans. Fuzzy Syst. **22**(3), 516–530 (2014)
50. Parreiras, R.O., Ekel, P.Y., Martini, J.S.C., Palhares, R.M.: A flexible consensus scheme for multicriteria group decision making under linguistic assessments. Inf. Sci. **180**, 1075–1089 (2010)
51. Xu, Z.: Fuzzy ordered distance measures. Fuzzy Optim. Decis Making **11**(1), 73–97 (2012)
52. Xu, Z., Chen, J.: Ordered weighted distance measure. J. Syst. Sci. Syst. Eng. **17**(4), 432–445 (2008)
53. Yu, L., Lai, K.K.: A distance-based group decision-making methodology for multi-person multi-criteria emergency decision support. Decis. Support Syst. **51**, 307–315 (2011)