Bounds for Fourier-Jacobi Coefficients of Siegel Cusp Forms of Degree Two

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Abstract We discuss and prove several estimates involving Peterrson norms of Fourier-Jacobi coefficients of Siegel cusp forms of degree two.

1 Introduction

Let f be an elliptic cusp form of integral weight k for the Hecke congruence subgroup $\Gamma_0(M) \subset SL_2(\mathbf{Z})$ of level M and write $a(n)$ $(n \geq 1)$ for its Fourier coefficients. Then Deligne's bound (previously the Ramanujan-Petersson conjecture) says that

$$a(n) \ll_{f,\epsilon} n^{\frac{k-1}{2}+\epsilon} \quad (\epsilon > 0). \tag{1}$$

While (1) is deep, various bounds for sums related to the $a(n)$ can be derived in a rather elementary way. For example, using Parseval's formula one can easily show that

$$\sum_{n \leq N} |a(n)|^2 \ll_f N^k \tag{2}$$

and from this—using the Cauchy-Schwarz inequality—that

$$\sum_{n \leq N} |a(n)| \ll_f N^{\frac{k+1}{2}} \tag{3}$$

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(cf. e.g. [6, Thm. 5.1, Cor. 5.2]). We note that vice versa (up to the occurrence of the ϵ), the bound (1) directly implies (2) and (3) and so (2) and (3), respectively can be viewed as the Deligne bound on average.

On the other hand, it was proved in [6, Thm. 5.3] that for any real α one has

$$\sum_{n\leq N} a(n)e^{2\pi i\alpha n} \ll_f N^{\frac{k}{2}}\log(2N) \tag{4}$$

where the implied constant depends only on f and not on α. Note that (4) saves $\frac{1}{2}-\epsilon$ ($\epsilon > 0$) in the power of N in comparison to using the triangle inequality and (3) and so there must be many cancellations in (4).

In this paper we would like to discuss and prove similar estimates as above in the case of a Siegel cusp form F of degree two, where the Fourier coefficients of f in the classical setting are replaced by the Fourier-Jacobi coefficients of F and we work with the Petersson norm. When using Fourier-Jacobi coefficients rather than usual Fourier coefficients, the situation seems to become a bit more uniform, as will be demonstrated. For example, while a generalized Ramanujan-Petersson conjecture is known to fail for the Fourier coefficients of a form F in the Maass subspace [2, sect. 2], an analogous conjecture can be proved in the setting of Fourier-Jacobi coefficients, cf. Sect. 3.

2 Jacobi Forms and Norms

We denote by $\mathcal{H}_2$ the Siegel upper half-space of degree two consisting of symmetric complex $(2,2)$-matrices with positive definite imaginary part. For $M \in \mathbf{N}$ we let

$$\Gamma_0^{(2)}(M) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbf{Z}) \mid C \equiv 0 \pmod{M}\}$$

the Hecke congruence subgroup of level M and degree two.

If $F : \mathcal{H}_2 \to \mathbf{C}$ is a Siegel cusp form of weight k for $\Gamma_0^{(2)}(M)$, we write its Fourier-Jacobi expansion as

$$F(Z) = \sum_{m\geq 1} \phi_m(\tau, z)e^{2\pi i m\tau'} \quad (Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2).$$

Then $\phi_m \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$, the space of Jacobi cusp forms of weight k and index m for $\Gamma_0(M)_J := \Gamma_0(M) \rhd \mathbf{Z}^2$ [4, 5].

For $\phi \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$ put

$$\Phi(\tau, z) := \phi(\tau, z)e^{-2\pi m y^2/v} v^{\frac{k}{2}} \quad (\tau = u + iv, z = x + iy). \tag{5}$$

Then $|\Phi(\tau, z)|$ is invariant under $\Gamma_0(M)_J$ and $\Phi(\tau, z)$ is bounded on $\mathcal{H} \times \mathbf{C}$.

For $\phi, \psi \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$ we denote their Petersson scalar product by

$$\langle \phi, \psi \rangle = \int_{\mathcal{F}} \Phi(\tau, z)\overline{\Psi(\tau, z)}d\mu,$$

where Ψ is defined in an analogous way as Φ, and $\mathcal{F}$ is any fundamental domain for the action of $\Gamma_0(M)_J$ on $\mathcal{H} \times \mathbf{C}$. Also

$$d\mu = \frac{dxdydudv}{v^3}$$

is the invariant measure.

Note that by definition the Petersson norm $||\phi||$ of ϕ is equal to the L^2-norm $||\Phi||$ of the corresponding function Φ restricted to $\mathcal{F}$.

We want to extend the L^2-norm on the space of functions as above to the space $B(\mathcal{H} \times \mathbf{C})$ of continuous and bounded functions on $\mathcal{H} \times \mathbf{C}$ (not necessarily satisfying any invariance properties under $\Gamma_0(M)_J$). For any choice of fundamental domain $\mathcal{F}$ for $\Gamma_0(M)_J$, and any $\Phi \in B(\mathcal{H} \times \mathbf{C})$ we have the L^2-norm

$$||\Phi||_{\mathcal{F}} := \left(\int_{\mathcal{F}} |\Phi(\tau, z)|^2 d\mu\right)^{1/2}.$$

We put

$$||\Phi|| := \sup_{\mathcal{F}} ||\Phi||_{\mathcal{F}}. \tag{6}$$

Then $||\,.\,||$ is a norm on $B(\mathcal{H} \times \mathbf{C})$ and if Φ is obtained from a Jacobi form ϕ as in (5), then (6) coincides with the L^2-norm $||\Phi||$ as above, i.e. with the Petersson norm $||\phi||$.

The norm (6) will come into play later in Sect. 5.

3 A Generalized Ramanujan-Petersson Conjecture

We will first show

Theorem 1 *Let F be a cusp form of weight k for $\Gamma_0^{(2)}(M)$ and let $\phi_m (m \geq 1)$ be its Fourier-Jacobi coefficients. Then*

$$\sum_{m \leq N} ||\phi_m||^2 \ll_F N^k. \tag{7}$$

Proof The proof works in a similar way as in the case of an elliptic modular form, mutatis mutandis, cf. [6, Thm. 5.1]. By Parseval's formula

$$\sum_{m\geq 1}|\phi_m(\tau,z)|^2e^{-4\pi mv'}=\int_0^1|F(Z)|^2du' \quad (Z=\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}, \tau'=u'+iv').$$

Since

$$(\det Y)^{k/2}|F(Z)| \quad (Y=\Im(Z))$$

is bounded on $\mathcal{H}_2$, F being a cusp form, we find that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2e^{-4\pi mv'}\leq\sum_{m\geq 1}|\phi_m(\tau,z)|^2e^{-4\pi mv'}$$

$$\ll_F (\det Y)^{-k}.$$

We choose

$$v'=\frac{y^2}{v}+\frac{1}{N}$$

and note that with this choice

$$\det Y = vv'-y^2$$

$$=\frac{v}{N}.$$

We then infer that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2e^{-4\pi my^2/v}\cdot e^{-4\pi m/N}\ll_F N^kv^{-k}.$$

Since

$$e^{-4\pi}\leq e^{-4\pi m/N}$$

for $m\leq N$ we obtain that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2e^{-4\pi my^2/v}\cdot v^k\ll_F N^k. \tag{8}$$

Integrating (8) over a fundamental domain $\mathcal{F}$ with respect to the measure $d\mu$ we finally conclude that

$$\sum_{m\le N} ||\phi_m||^2 \ll_F N^k$$

as claimed.

Writing

$$||\phi_m|| = 1 \cdot ||\phi_m||$$

and using the Cauchy-Schwarz inequality we obtain from Theorem 1

Corollary *One has*

$$\sum_{m\le N} ||\phi_m|| \ll_F N^{\frac{k+1}{2}}. \tag{9}$$

Remark We believe that the bound of Theorem 1 is best possible so that we have a similar situation as in the case of elliptic modular forms. Indeed, at least if $M = 1$, i.e. we work with the full Siegel modular group, one can prove an asymptotic formula

$$\sum_{m\le N} ||\phi_m||^2 \asymp c_F N^k \quad (N \to \infty)$$

where $c_F > 0$ is a constant depending only on F. This follows from the analytic properties of the Dirichlet series

$$D_{F,F}(s) = \zeta(2s - 2k + 4) \sum_{m\ge 1} ||\phi_m||^2 m^{-s} \quad (\sigma := \Re(s) \gg 1)$$

proved in [10] in conjunction with a usual Tauberian theorem. Note that these properties are more difficult to prove, while the proof of (7) was quite straightforward.

In an analogous way as in the case of elliptic modular forms, based on (7) and (9) one is tempted to make the following

Conjecture (Ramanujan-Petersson) One has

$$||\phi_m|| \ll_{F,\epsilon} m^{\frac{k-1}{2}+\epsilon} \quad (\epsilon > 0). \tag{10}$$

Remarks

i) Note that the potential bound (10) was also addressed in [9, p. 718].
ii) The best general bound in the direction of (10) known so far seems to be

$$||\phi_m|| \ll_{F,\epsilon} m^{k/2-2/9+\epsilon} \quad (\epsilon > 0)$$

(cf. [7]). One also knows that there are infinitely many m such that $||\phi_m|| \ll_F m^{(k-1)/2}$ and infinitely many m such that $||\phi_m|| \gg_F m^{(k-1)/2}$ (if $F \neq 0$), cf. [9].

iii) Note that in the literature there is also a conjectured bound for the usual Fourier coefficients of a Siegel cusp form which is due to Resnikoff and Saldaña and which also could be viewed as a generalization of the Ramanujan-Petersson conjecture for classical cusp forms [11]. In the case of degree two this conjecture says that

$$a(T) \ll_{F,\epsilon} (\det T)^{k/2-3/4+\epsilon} \quad (\epsilon > 0), \tag{11}$$

for any positive definite symmetric half-integral matrix T of size 2, where $a(T)$ denote the Fourier coefficients of F. The estimate (11) can be motivated "on average" using the analytic properties of the Rankin-Selberg zeta function attached to F, cf. [8]. While one believes that (11) should be true "generically", there are well-known "exceptional" cases where it fails to hold, e.g. when F is a Hecke eigenform in the Maass space [2, loc. cit.]. Contrary to the above situation, we will prove estimate (10) for F in the Maass space in the next section.

4 Hecke Eigenforms in the Maass Space

Recall that the space of Siegel cusp forms of even weight k for $Sp_2(\mathbf{Z})$ has a special subspace, the so-called Maass space. It has a basis of Hecke eigenforms F whose spinor zeta function $Z_F(s)$ factors as

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s) \quad (\sigma \gg 1) \tag{12}$$

where f is a normalized cuspidal Hecke eigenform of weight $2k-2$ for $SL_2(\mathbf{Z})$ and $L(f,s)$ $(\sigma \gg 1)$ is its Hecke L-series [4].

Theorem 2 *Suppose that F is a cuspidal Hecke eigenform of even weight k for $Sp_2(\mathbf{Z})$ in the Maass subspace. Then conjecture (10) is true.*

Proof Under the given hypothesis one has

$$||\phi_m||^2 = \lambda_m ||\phi_1||^2 \quad (m \geq 1)$$

where λ_m is the m-th eigenvalue of F under the usual Hecke operator $T(m)$, cf. [3, Remark on p. 530] and [10]. Hence one only has to show that

$$\lambda_m \ll_\epsilon m^{k-1+\epsilon} \quad (\epsilon > 0).$$

Recall that the eigenvalues λ_m and the spinor zeta function of F are related by the identity

$$\sum_{m\geq 1} \lambda_m m^{-s} = \frac{Z_F(s)}{\zeta(2s-2k+4)} \quad (\sigma \gg 1)$$

as is well-known [1]. In particular, for F in the Maass subspace, using (12) we get

$$\sum_{m\geq 1} \lambda_m m^{-s} = \frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-2k+4)} \cdot L(f,s) \quad (\sigma \gg 1). \tag{13}$$

We note that the quotient of Riemann zeta functions on the right-hand side of (13) equals

$$\frac{\zeta(w-1)\zeta(w)}{\zeta(2w)},$$

where $w = s - k + 2$. Since

$$\frac{\zeta(w)}{\zeta(2w)} = \prod_p [1 + p^{-w})$$

$$= \sum_{m\geq 1} |\mu(m)| m^{-w} \quad (\Re(w) \gg 1)$$

where μ is the Möbius function, the general coefficient of the above quotient is equal to

$$\alpha(m) = m^{k-2} \sum_{d|m} |\mu(\frac{m}{d})| d.$$

Clearly we have

$$\alpha(m) \leq m^{k-1}\sigma_0(m)$$

$$\ll_\epsilon m^{k-1+\epsilon} \quad (\epsilon > 0).$$

Here $\sigma_0(m)$ denotes the number of positive divisors of m.

Hence denoting by $\beta(m)$ the Hecke eigenvalues of f and observing that

$$\beta(m) \ll_\epsilon m^{k-3/2+\epsilon} \quad (\epsilon > 0)$$

by Deligne's bound we find that

$$\lambda(m) = \sum_{d|m} \alpha(d)\beta(\frac{m}{d})$$

$$\ll_\epsilon \sum_{d|m} d^{k-1+\epsilon} \cdot (\frac{m}{d})^{k-3/2+\epsilon}$$

$$= m^{k-3/2+\epsilon} \sum_{d|m} d^{1/2}$$

$$\leq m^{k-3/2+\epsilon} \cdot m^{1/2} \sigma_0(m)$$

$$\ll_\epsilon m^{k-1+2\epsilon}.$$

This proves our assertion.

5 Bounds for Twisted Sums

In this section we will prove an estimate analogous to the bound (4) in the classical case. Let again F be a Siegel cusp form of weight k for $\Gamma_0(M)_J$ and let ϕ_m be its m-th Fourier-Jacobi coefficient.

Following Sect. 2 we put

$$\Phi_m(\tau, z) := \phi_m(\tau, z) e^{-2\pi m y^2/v} v^{k/2} \quad (\tau = u + iv,\ z = x + iy).$$

Let $\alpha \in \mathbf{R}$. Using Cauchy-Schwarz we see that

$$|\sum_{m \leq N} \Phi_m(\tau, z) e^{2\pi i m \alpha}|$$

$$\leq N^{1/2} \cdot \sqrt{\sum_{m \leq N} |\Phi_m(\tau, z)|^2}$$

$$\ll_F N^{\frac{k+1}{2}}$$

where in the last line we have used (8). Thus the function

$$\sum_{m \leq N} \Phi_m e^{2\pi i m \alpha}$$

is bounded on $\mathcal{H} \times \mathbf{C}$ and we can talk about its norm as defined in Sect. 2.

Theorem 3 *With the above notations we have*

$$|| \sum_{m \leq N} \Phi_m e^{2\pi i m\alpha} || \ll_F N^{k/2} \log(2N). \tag{14}$$

Remark Note that if we estimate the left-hand side of (14) by brute force, using the triangle inequality and the Corollary to Theorem 1 we only get the bound $N^{\frac{k+1}{2}}$.

Proof The proof follows a similar pattern as that of inequality (4) for elliptic modular forms, again mutatis mutandis.

We will use the notation

$$e(z) := e^{2\pi i z} \quad (z \in \mathbf{C}).$$

Since ϕ_m is the m-th Fourier-Jacobi coefficient of F, we have

$$S_{N,\alpha}(\tau, z) := \sum_{m \leq N} \Phi_m(\tau, z) e^{2\pi i m\alpha}$$

$$= v^{k/2} \sum_{m \leq N} e^{-2\pi m y^2/v} \int_0^1 F(\begin{pmatrix} \tau & z \\ z & \tau' + \alpha \end{pmatrix}) e(-m\tau') du' \quad (\tau' = u' + iv').$$

We put

$$v' = \frac{y^2}{v} + \frac{1}{N} \tag{15}$$

and obtain

$$S_{N,\alpha}(\tau, z) = v^{k/2} \int_0^1 \Big(\sum_{m \leq N} e(-m(u' + \frac{i}{N})) \Big) F(\begin{pmatrix} \tau & z \\ z & u' + \alpha + i(\frac{y^2}{v} + \frac{1}{N}) \end{pmatrix}) du'. \tag{16}$$

Summing the geometric series now gives

$$\sum_{1 \leq m \leq N} e(-m(u' + \frac{i}{N})) = \frac{e(-N(u' + \frac{i}{N})) - 1}{1 - e(u' + \frac{i}{N})}$$

$$\ll \frac{1}{|1 - e(u' + \frac{i}{N})|}.$$

According to [6, p. 71] one has

$$\int_0^1 \frac{du'}{|1 - e(\tau')|} \ll \log(2 + \frac{1}{v'}). \tag{17}$$

Applying (17) with $v' = \frac{1}{N}$ we see that

$$\int_0^1 \Big(\sum_{m \le N} e(-m(u' + \frac{i}{N}))\Big) du' \ll \log(2+N)$$

$$\ll \log(2N).$$

Finally, since

$$F(Z) \ll_F (\det Y)^{-k/2}$$

and by (15) we have

$$\det Y = \frac{v}{N},$$

we obtain altogether from (16) that

$$S_{N,\alpha}(\tau, z) \ll_F N^{k/2} \log(2N). \qquad (18)$$

Now (18) implies that

$$||S_{N,\alpha}||_{\mathcal{F}} \ll_F N^{k/2} \log(2N)$$

for any fundamental domain $\mathcal{F}$ (where the implied constant depends only on F and not on $\mathcal{F}$) and hence that

$$||S_{N,\alpha}|| \ll_F N^{k/2} \log(2N).$$

This proves Theorem 3.

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