Elementary Introduction to *p*-Adic Siegel Modular Forms

Siegfried Böcherer

Abstract We give an introduction to the theory of Siegel modular forms mod p and their p-adic refinement from an elementary point of view, following the lines of Serre's presentation (J.-P. Serre, Formes modulaires et fonctions zeta p-adiques. In: Modular Functions of One Variable III. Lecture Notes in Mathematics, vol. 350. Springer, New York, 1973) of the case $SL(2)$.

1 Introduction

In the late sixties of the last century Serre [18] and Swinnerton-Dyer [22] created a theory of p-adic modular forms, which was soon reformulated and refined by Katz [12] in a geometric language. Later on S. Nagaoka and others started to generalize that theory (in the classical language) to Siegel modular forms. In these notes we give a naive introduction, emphasizing level changes and generalizations of Ramanujan's theta operator (i.e. derivatives). Compared with the theory for elliptic modular forms at some points new techniques are necessary. Also some aspects do not appear at all in the degree one case, in particular mod p singular modular forms and also vector-valued modular forms. We will focus on the scalar-valued modular forms, but the vector-valued case will arise naturally in the context of derivatives. We will not enter into the intrinsic theory for the vector-valued case (see e.g. [11] and other papers by the same author); all vector-valued modular forms which appear in our notes arise from scalar-valued ones.

Our naive point of view is that p-adic modular forms encode number theoretic properties (congruences) of Fourier coefficents of Siegel modular forms. We understand that there is a much more sophisticated geometric point of view; in these notes we completely ignore the geometric theory (see e.g. [11, 24, 25]).

S. Böcherer (✉)
Kunzenhof 4B, 79117 Freiburg, Germany
e-mail: boecherer@math.uni-mannheim.de

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2 Basics on Siegel Modular Forms

Mainly to fix notation, we summarize basic facts on Siegel modular forms here. The reader should consult [1, 9, 14] for details.

The symplectic group

$$Sp(n,\mathbb{R}) := \{M \in GL(2n,\mathbb{R}) \mid J_n[M] = J_n\}$$

acts on the Siegel upper half space

$$\mathbb{H}_n := \{Z = Z^t = X + iY \in \mathbb{C}^{(n,n)} \mid Y > 0\}$$

by

$$(M,Z) \longmapsto M < Z > := (AZ+B)(CZ+D)^{-1}.$$

Here J_n denotes the alternating form given by the $2n \times 2n$ matrix $J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and for matrices U, V we put $U[V] := V^t UV$ whenever it makes sense; we decompose the matrix M into block matrices of size n by $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

There are good reasons to look at vector-valued automorphy factors:

For a finite-dimensional polynomial representation $\rho : GL(n,\mathbb{C}) \longrightarrow GL(V_\rho)$ we consider $V = V_\rho$-valued functions $F : \mathbb{H}_n \longrightarrow V$; the group $Sp(n,\mathbb{R})$ acts on such functions from the right via

$$\left(F \mid_\rho M\right)(Z) := \rho(CZ+D)^{-1} F(M < Z >).$$

As usual, we write $|_k\, M$ instead of $F \mid_\rho M$ if $\rho = \det^k$.

We write $\Gamma^n = Sp(n,\mathbb{Z})$ for the full modular group and for $N \geq 1$ we define the principle congruence subgroup of level N by

$$\Gamma(N) := \{M \in \Gamma^n \mid M \equiv 1_{2n} \bmod N\}.$$

We will denote by Γ any group which contains some $\Gamma(N)$ as a subgroup of finite index; typically we will consider the groups

$$\Gamma_0(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv 0 \bmod N\}$$

and

$$\Gamma_1(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv 0 \bmod N, \det(A) \equiv \det(D) \equiv 1 \bmod N\}.$$

The space $M^n_\rho(\Gamma)$ of Siegel modular forms of degree n for ρ consists of all holomorphic functions $F : \mathbb{H}_n \longrightarrow V$, which satisfy the transformation properties $F \mid_\rho \gamma = F$ for all $\gamma \in \Gamma$; only for $n = 1$ we need additional conditions in the cusps, for $n > 1$ such conditions are automatically satisfied ("Koecher principle").

The functions $F \in M^n_\rho(\Gamma)$ are periodic, i.e. $F(Z+S) = F(Z)$ for all $S \in N \cdot \mathbb{Z}^{(n,n)}_{sym}$, their Fourier expansion is then conveniently written in the form

$$F(Z) = \sum_T a_F(T) e^{2\pi i \frac{1}{N} trace(TZ)}. \tag{1}$$

Here T runs over the set $\Lambda^n_{\geq}$ of all symmetric half-integral matrices of size n, which are positive semidefinite.

If we want to emphasize the formal aspects of such a Fourier expansion, then we can view (1) as a formal series as follows:

With $Z = (z_{ij}) \in \mathbb{H}_n$ we put $q_{i,j} = e^{2\pi\sqrt{-1}z_{ij}}$ and we write for $T \in \Lambda^n_{\geq}$

$$q^T := \prod_{i<j} q_{ij}^{2t_{ij}} \prod_j q_{jj}^{t_{jj}}.$$

We consider the q_{ij} as formal variables and we may then view

$$\sum_T a_F(T) q^T$$

as an element of

$$\mathbb{C}[q_{ij}, q_{ij}^{-1}[[q_1, \dots, q_n]] \qquad \text{with} \quad q_j := q_{jj}.$$

We mention two typical examples of number-theoretic interest:

Example 1 (Siegel Eisenstein Series) We consider $\rho = \det^k$ with an even integer $k > n+1$ and

$$E^n_k(Z) := \sum_M 1 \mid_k M = \sum_M \det(CZ+D)^{-k};$$

here M runs over $Sp(n,\mathbb{Z})_\infty \backslash Sp(n,\mathbb{Z})$, where $Sp(n,\mathbb{Z})_\infty$ is defined by the condition $C = 0$.

This defines an element of $M^n_k(\Gamma^n)$ with rational Fourier coefficients with bounded denominators (this is not obvious!).

Example 2 (Theta Series) Let $S \in 2 \cdot \Lambda_{>}^{m}$ be a positive definite even integral matrix of size $m = 2k$ and of level N (i.e. N is the smallest positive integer such that $N \cdot S^{-1} \in \Lambda_{>}^{m}$). Then

$$\vartheta_S^n(Z) := \sum_{R \in \mathbb{Z}^{(m,n)}} e^{\pi i trace(X^t SXZ)}$$

defines an element of

$$M_k^n(\Gamma_0(N), \epsilon_S) := \{F \in M_k^n(\Gamma_1(N)) \mid F \mid_k \gamma = \epsilon_S(\det(D)) \cdot F \quad \forall \gamma \in \Gamma_0(N)\}$$

with the quadratic character

$$\epsilon_S(.) = \left(\frac{(-1)^k \det(S)}{.} \right).$$

It is obvious that such theta series have integral Fourier coefficients.

For a subring $\mathcal{R}$ of $\mathbb{C}$ we denote by $\overline{M_k^n(\Gamma)(\mathcal{R})}$ the $\mathcal{R}$ submodule of all modular forms with all their Fourier coefficients in $\mathcal{R}$. This notion can be extended in an obvious way to the vector-valued case after fixing a basis of the representation space of ρ.

Let ξ_N denote a primitive root of unity and denote by $\mathcal{O}_{\xi_N}$ the ring of integral elements in the N-th cyclotomic field. Then we have the following

Fundamental Property

$$M_k(\Gamma(N)) = M_k^n(\Gamma(N))(\mathcal{O}_{\xi_N}) \otimes \mathbb{C},$$

in particular, the field of Fourier coefficients of a modular form is finitely generated and all modular forms and the Fourier coefficients of a modular form in $M_k^n(\Gamma(N))(\overline{\mathbb{Q}})$ *have bounded denominators.*

The property above will be crucial at several points below (sometimes implicitly). We take this for granted and refer to the literature [20]. In some cases (squarefree levels and large weights) elementary proofs are available, using the solution of the basis problem ("all modular forms are linear combinations of the theta series introduced above", see [3]).

Remark We note here two *important differences* between elliptic modular forms and Siegel modular forms of higher degree:

No Obvious Normalization For $n > 1$ there is no good notion of "first Fourier coefficient" and (even for Hecke eigenforms) we cannot normalize modular forms in a reasonable arithmetic way (note that a normalization by requesting the Petersson product to be one is not an arithmetic normalization!).

Hecke Eigenvalues and Fourier Coefficients Fourier coefficients and Hecke eigenvalues are different worlds for $n > 1$. We briefly explain the reason in the simplest case (scalar-valued modular forms of level one): For $g \in GSp^+(n, \mathbb{Q})$ with $g^t J_n g = \lambda \cdot J_n$ we consider for $\Gamma^n = Sp(n, \mathbb{Z})$ the double coset $\Gamma^n \cdot g \cdot \Gamma^n = \bigcup \Gamma^n \cdot g_i$

with representatives $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}$ with $A_i^t \cdot D_i = \lambda$. Then we define a Hecke operator acting on $F \in M_k^n(\Gamma^n)$ by

$$F \longmapsto G := F \mid \Gamma^n \cdot g \cdot \Gamma^n := \sum_i \det(D_i)^{-k} F((A_i \cdot Z + B_i) \cdot D_i^{-1}.)$$

We may plug in the Fourier expansion $F = \sum a_F(T)q^T$ and we get for the Fourier coefficients of $a_G(S)$ a formula of type

$$a_G(S) = \text{a linear combination of } a_F(T) \text{ with } D_i^{-1}TA_i = S,$$

in particular, S and T are rationally equivalent up to a similitude factor.

The conclusion is that Hecke operators give relations between Fourier coefficients only within a rational similitude class of positive definite matrices $T \in \Lambda_>^n$. For $n \geq 2$, the set $\Lambda_>^n$ however decomposes into *infinitely* many such rational similitude classes. In some sense this is a situation similar to the perhaps more familiar case of degree one modular forms of half-integral weight.

Our aim here will be to study congruences among Fourier coefficients of Siegel modular forms (not congruences among eigenvalues!).

The reader interested in congruences for eigenvalues should consult the work of Katsurada [13], who studies congruences between eigenvalues of different types of automorphic forms (lifts and non-lifts); in a different direction (connection to Galois representations) one may look at the work of Weissauer [23].

3 Congruences

3.1 The Notion of Congruences of Modular Forms

For a prime p we denote by ν_p the (additive) p-adic evaluation $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$, normalized by $\nu_p(p^t) = t$. For a modular form $F = \sum_T a_F(T)q^T \in M_k^n(\Gamma_1(N))(\mathbb{Q})$ we put

$$\nu_p(F) := \inf\{\nu_p(a_F(T)) \mid T \in \Lambda^n\}.$$

By the boundedness of denominators, this number is $> -\infty$.

We defined this notion only for scalar-valued modular forms with Fourier coefficients in $\mathbb{Q}$, but we can easily generalize it to modular forms with Fourier coefficients in $\mathbb{C}$ by extending ν_p to the field generated by the Fourier coefficients. Furthermore, we can define it also for vector-valued modular forms after fixing coordinates and taking the minimum of ν_p on the coordinates (this depends on the choice of coordinates!).

Definition For $F, G \in M_k^n(\Gamma_1(N))(\mathbb{Q})$ we define

$$F \equiv G \bmod p^m \iff \nu_p(F - G) > \nu(F) + m.$$

Note that this definition avoids trivial congruences.

Remark In case of Hecke eigenforms, such congruences for modular forms imply congruences for eigenvalues (but not the other way around!).

3.2 Congruences and Weights

A first observation is that such congruences cannot occur among modular forms of arbitrary weights:

Theorem I *For a prime p and a positive integer N coprime to p we consider $\Gamma = \Gamma_1(N) \cap \Gamma_0(p^l)$. Then for $F_i \in M_{k_i}^n(\Gamma)($ with $i = 1, 2$ a congruence $F_1 \equiv F_2 \bmod p^m$ implies a congruence among the weights:*

$$k_1 \equiv k_2 \bmod \begin{cases} (p-1)p^{m-1} & \text{if } p \neq 2 \\ 2^{m-2} & \text{if } p = 2, \quad m \geq 2. \end{cases}$$

For $n = 1$ this is a result of Katz [12, Corollary 4.4.2]. The case of general degree can be deduced from that by associating to F and G suitable elliptic modular forms f and g with the same weights (possibly with larger level) and satisfying the same congruence (see [6] for details).

As a special case, we mention

Corollary *For an odd prime p a modular form $F \in M_k^n(\Gamma)(\mathbb{Q})$ with Γ as above, can be congruent mod p^m to a constant only if $(p-1) \cdot p^{m-1} \mid k$ holds.*

3.3 Mod p Singular Modular Forms

Singular modular forms are a topic which is specific for higher degree, see [9]; there is an analogue mod p:

Definition We call a modular form $F = \sum a_F(T)q^T \in M_k^n(\Gamma)(\mathbb{Q})$ with $\nu_p(F) = 0$ a mod p-singular modular form of rank r, $0 \leq r \leq n-1$ iff $a_F(T) \equiv 0 \bmod p$ for all $T \in \Lambda^n$ with $rank(T) > r$ and if there exists $T_0 \in \Lambda^n$ with $rank(T_0) = r$ such that $a_F(T_0) \not\equiv 0 \bmod p$.

Theorem II *If $F \in M_k^n(\Gamma_0(N))$ is mod p singular of rank r, then*

$$2k - r \equiv 0 \bmod (p-1)p^{m-1}$$

if p is odd.

The proof is inspired by the method used to prove a similar statement for true singular modular forms [9]: One considers a Fourier-Jacobi-expansion $F(Z) = \sum_{S\in\Lambda^r_{\geq}} \phi_S(z_1, z_2)e^{2\pi i trace(Sz_4)}$ with

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix}, \quad z_1 \in \mathbb{H}_{n-r}, \quad z_4 \in \mathbb{H}_r.$$

We choose $T_0 \in \Lambda^n$ with rank r such that $a_F(T_0) \not\equiv 0 \bmod p$; without loss of generality we may assume that T_0 equals $\begin{pmatrix} 0 & 0 \\ 0 & S_o \end{pmatrix}$ with $S_o \in \Lambda^r_{>}$. The "theta expansion" of the special Fourier-Jacobi coefficient ϕ_{S_0} allows us to arrive at a modular form h of degree r and weight $k - \frac{r}{2}$ which is constant mod p. We may then apply the corollary to h^2.

Example Let S be a positive definite even integral quadratic form in m variables. We assume that S has an integral automorphism σ of order p (the existence of such quadratic forms will be considered below). Let l be the maximal number of linearly independent fixed points of σ. Then ϑ^n_S is mod p singular of rank l.

Other types of examples can be constructed using Siegel Eisenstein series; here divisibility properties of certain Bernoulli numbers play an important role, see [4].

3.4 Existence Theorem

In degree 1 the Clausen-von Staudt property of Bernoulli numbers B_{p-1} implies that the Eisenstein series of weight $p - 1$

$$E_{p-1}(z) = 1 - \frac{2p-2}{B_{p-1}} \sum_{n=1} \left(\sum_{d|n} d^{p-2} \right) e^{2\pi i n z}$$

is congruent 1 mod p for $p \geq 5$. In higher degree the situation is more complicated, the Siegel Eisenstein series of weight $p - 1$ is not necessarily congruent 1 modulo p for irregular primes, see [16].

Before stating a general existence theorem we introduce the "zero dimensional cusps" for a group $\Gamma_0(p)$. It is a consequence of the Bruhat decomposition for the symplectic group over a finite field that a complete set of representatives for the double cosets

$$\Gamma_0(p)\backslash Sp(n, \mathbb{Z})/Sp(n, \mathbb{Z})_\infty$$

is given by the $n+1$ elements

$$\omega_i := \begin{pmatrix} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{pmatrix} \qquad (0 \le i \le n). \tag{2}$$

The theorem below assures the existence of level p modular forms congruent to 1 mod p and with nice behaviour mod p in the other cusps. This is a very usefull technical tool. The proof will be based on the existence of certain quadratic forms with automorphisms of order p. The advantage of theta series (when compared with Eisenstein series) is that the Fourier expansions in *all* cusps are accessible. This point of view is new even for degree one.

We briefly recall the theta transformation formula relevant for us: Let S be an even integral symmetric matrix, positive definite, $\det(S) = p^{2r}$ of size $m = 2k$ and $0 \le j \le n$. Then

$$\vartheta_S^n \mid_k \omega_j = w(S)^j \cdot p^{-jr} \sum_X e^{\pi i S[X]Z}.$$

Here $w(S) = \pm 1$ is the Hasse-Witt invariant of S and X runs over

$$\underbrace{S^{-1}\cdot\mathbb{Z}^m \times \cdots \times S^{-1}\cdot\mathbb{Z}^m}_{j} \times \underbrace{\mathbb{Z}^m \ldots \mathbb{Z}^m}_{n-j}.$$

Theorem III

a) p *odd:* $\exists F \in M^n_{p-1}(\Gamma_0(p)) : F \equiv 1 \bmod p$

b) $p \ge n+3 : \exists F_{p-1} \in M^n_{p-1}(Sp(n,\mathbb{Z})) : F_{p-1} \equiv 1 \bmod p$

c) $p \ge n+3 : \exists k_p : \exists \mathcal{F} \in M^n_{k_p}(\Gamma_0(p)) :$

$$\mathcal{F} \equiv 1 \bmod p \quad \textit{and} \quad \mathcal{F} \mid_{k_p} \omega_i \equiv 0 \bmod p \quad (1 \le i \le n).$$

Proof (sketch)

a) We consider the root lattice

$$A_{p-1} := \{(x_1, \ldots, x_p) \in \mathbb{Z}^p \mid \sum_i x_i = 0\}$$

inside the standard euclidean space $\mathbb{R}^p$. We can act on this lattice by the symmetric group S_p; the only lattice point fixed by a $\sigma \in S_p$ of order p is $\mathbf{0}$.

In particular, the orthogonal sum $A_{p-1} \perp A_{p-1}$ corresponds to an even integral positive definite symmetric matrix S of determinant p^2 with an (integral) automorphism of order p without nontrivial fixed point. The theta series ϑ_S^n has the requested properties [5].

b) We put $T := p \cdot S^{-1}$ with S from above, then

$$F_{p-1} := \pm p^{(p-2)n-\frac{n(n+1)}{2}} \sum_{\gamma \in \Gamma_0(p) \in \backslash Sp(n,\mathbb{Z})} \vartheta_T^n \mid_{p-1} \gamma.$$

The sign depends on the Hasse invariant of the underlying quadratic space.

c) This is more complicated: One has to use not only the lattice $A_{p-1} \perp A_{-1}$ but several lattices $\mathcal{L}_1 \ldots \mathcal{L}_{n+1}$ with determinants $p^2, \ldots, p^{2n+2}$ (all with automorphisms of order p without nonzero fixed points). One can construct such lattices from certain ideals in the cyclotomic field generated by p-th roots of unity. In a first step one may then use linear combinations of theta series for such lattices to construct modular forms $G_i \in M_{p-1}^n(\Gamma_0(p))$ such that

$$G_i \mid_{p-1} \omega_j \equiv 1 \bmod p \quad (0 \leq j \leq i)$$
$$G_i \mid_{p-1} \omega_{i+1} \equiv 0 \bmod p.$$

Typically, the G_i have high powers of p in the denominators of their Fourier coefficients in the cusps ω_j with $j > i + 1$. We may then construct $\mathcal{F}_{k_p}$ by taking suitable products of powers of the G_i.

3.5 *The Ring of Modular Forms Mod p d'après Raum-Richter*

The existence theorem above is an ingredient in the following beautifull recent result (the proof goes beyond our elementary approach).

We define the ring $\tilde{M}^{n,p}$ of modular forms mod p as the image of the ring $\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)})$ under the reduction map˜mod p

$$F = \sum a_F(T) q^T \longmapsto \sum_T \widetilde{a_F(T)} q^T.$$

After Faltings/Chai the ring $\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)})$ of modular forms with coefficients in $\mathbb{Z}_{(p)}$ is finitely generated:

$$\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)}) \simeq \mathbb{Z}_{(p)}[X_1, \ldots, X_r]/C$$

with some ideal C describing the relations. One may in particular write the modular form F_{p-1} as a polynomial B in the generators $X_1, \ldots X_r$ (or rather their images mod C).

Theorem of Raum-Richter [17]

For $p \geq n+3$ we have

$$\tilde{M}^{n,p} \simeq \mathbb{F}_p[X_1, \ldots, X_r]/\tilde{C} + < \tilde{B} - 1 > .$$

We can rephrase this by saying that by reduction mod p, the only new relation among the generators is the one coming from $F_{p-1} \equiv 1 \bmod p$.

4 p-Adic Modular Forms and Level Changes

Definition A formal series

$$F = \sum_{T \in \Lambda_{\geq}^n} a(T) q^T \qquad (a(T) \in \mathbb{Z}_p)$$

is called p-adic modular form if there is a sequence F_j of level one modular forms $F_j \in M_{k_j}^n(Sp(\Gamma^n))(\mathbb{Z}_{(p)})$ such that the sequence (F_j) converges p-adically to F, i.e. $\nu_p(F - F_j) \longrightarrow \infty$, which means that all the sequences $a_{F_j}(T)$ converge p-adically to $a(T)$ uniformly in T.

Some Comments

- It follows from our Theorem I that such a p-adic modular form has a weight in $\mathbb{Z}/(p-1) \cdot \mathbb{Z} \times \mathbb{Z}_p$.
- One can generalize the notion of p-adic modular form to the vector-valued case in an obvious way.
- Clearly, all level one Siegel modular forms with Fourier coefficients in $\mathbb{Z}_p$ are p-adic modular forms.
- It can happen, that such a p-adic limit is itself a modular form, possibly with nontrivial level: A nice example is exhibited by Nagaoka [15] following an observation by Serre in the degree one case [18]: the sequence of Eisenstein series $\left(E_{k_m}^n\right)_{m \in \mathbb{N}}$ with $k_m = 1 + \frac{p-1}{2} p^{m-1}$ converges p-adically to a weight one modular form for $\Gamma_0(p)$, if $p \equiv 3 \bmod 4$, more precisely, it is proportional to the genus Eisenstein series for the genus of positive binary quadratic forms of discriminant $-p$.

Proposition *All modular forms $F \in M_k^n(\Gamma_0(p))(\mathbb{Z}_{(p)})$ are p-adic (p any odd prime).*

We give here a proof for $p \geq n+3$ and refer to [7] for a different proof covering the general case.

We use the existence of a modular form $\mathcal{F}_{k_p}$ as in Theorem IIIc) and we consider for $N \in \mathbb{N}$ a "trace function"

$$G_N := \sum_{\gamma \in \Gamma_0(p) \backslash Sp(n,\mathbb{Z})} \left(F \cdot \mathcal{F}_{k_p}^N\right) \mid_{k+Nk_p} \gamma.$$

According to (2), G_N decomposes naturally into $n + 1$ summands

$$G_N = \sum_i G_{N,i} \quad \text{with} \quad G_{N,i} := \sum_{\gamma_i} \left(F \cdot \mathcal{F}_{k_p}^N \right) \mid_{k+Nk_p} (\omega_i \cdot \gamma_i),$$

where the γ_i run over certain elements of $Sp(n, \mathbb{Z})_\infty$.

For $i \geq 1$ we have $\nu_p(\mathcal{F}_{k_p}^N \mid_{Nk_p}) \geq N$ and therefore $G_{N,i}$ will be divisible by a high power of p if N is large (the denominators which possibly appear in the Fourier expansion of $F \mid_k \omega_i$ will be compensated. As for $G_{N,0} = F \cdot \mathcal{F}_{k_p}^N$ we observe that $\mathcal{F}_{k_p}^N$ is congruent one modulo p^m provided that N is chosen as $N = p^{m-1}$.

We therefore get that G_N is a level one form congruent to F modulo a high power of p provided that $N = p^m$ with m sufficiently large.

The proposition can be generalized to prime power levels:

Proposition *A modular form $F \in M_k^n(\Gamma_0(p^m))$ is p-adic (p odd, m arbitrary).*

We can use the $U(p)$-operator, defined on Fourier series by

$$\sum a(T)q \longmapsto \sum a(p \cdot T)q^T.$$

Such an operator maps modular forms for $\Gamma_0(p^m)$ to modular forms for $\Gamma_0(p^{m-1})$, provided that $m \geq 2$. It is sufficient to show that F is congruent to a modular form for $\Gamma_0(p^{m-1})$ modulo high powers of p, $m \geq 2$. One can start from the elementary observation

$$F^p \mid U(p) \equiv F \bmod p$$

and then apply the same procedure (with $\mathcal{F}$ as in Theorem IIIa)) to

$$\frac{1}{p} \left(F \cdot \mathcal{F} - F^p \mid U(p) \right)$$

to get a congruence mod p^2; iteration gives the desired result; this proof is a straightforward generalization of the one by Serre [19] for degree one.

Remark There is a delicate difference between the two propositions: the first one generalizes in an obvious way to vector-valued situations, whereas for the second proposition a substitute for taking a p-th power is necessary. A natural choice is taking the p-th symmetric tensor; one can get results along this line, but the notion of p-adic modular form has to be generalized, because one varies the representation space V_ρ.

5 Derivatives

In general, derivatives of modular forms are not modular (by derivatives we mean here holomorphic derivatives!)

But there are bilinear holomorphic differential operators, usually called "Rankin-Cohen" operators, e.g. for $n = 1$ and integral weights k, l with $l \neq 0$

$$[\ ,\ ]_{k,l} : \begin{cases} M^1_k(\Gamma) \times M^1_l(\Gamma) \longrightarrow M^1_{k+l+2}(\Gamma) \\ (f, g) \qquad\qquad \longmapsto \frac{1}{2\pi i}\left(f' \cdot g - \frac{k}{l} f \cdot g'\right) \end{cases}$$

We explain how one can use such Rankin-Cohen-operators to prove that derivatives of modular forms are p-adic modular forms; our proof is different from the usual one which uses the Eisenstein series of weight 2, see [18]; note that we cannot expect in higher degree to find a function analogous to the weight 2 Eisenstein series. We advertise here that the Rankin-Cohen operators, together with modular forms congruent one mod p are an appropriate substitute, which also works in higher degree.

To get a congruence mod p in degree one, we may use

$$[f, \mathcal{F}]_{k,p-1} \equiv \frac{1}{2\pi i} f' \bmod p$$

with $\mathcal{F}$ as in Theorem IIIa). For congruences mod p^m, this does not work with $\mathcal{F}^{p^{m-1}}$, because of $l = (p-1)p^{m-1}$ in the denominator of the Rankin-Cohen-operator. We can avoid this problem, if we use the operator V, defined by $g \mid V(t)(z) := g(t \cdot z)$ and consider

$$[f, \mathcal{F}^{p^{m-1}} \mid V(p^m)]_{k,(p-1)p^{m-1}} \equiv \frac{1}{2\pi i} f' \bmod p^m.$$

Here we increase the level by the operator $V(p^m)$; this can be avoided by using a modular form $\mathcal{H}$ of level one and some weight h satisfying

$$\mathcal{H} \equiv \mathcal{F}^{p^{m-1}} \mid V(p^m) \bmod p^m.$$

Then $[f, \mathcal{H}]_{k,h} \equiv \frac{1}{2\pi i} f' \bmod p^m$ holds. Note that the existence of $\mathcal{H}$ is assured by our proposition and by Theorem I, the weight of $\mathcal{H}$ is under control. Clearly this line of reasoning also works for higher derivatives. Furthermore, this proof contains all the ingredients for generalization to higher degree:

First we introduce a symmetric $n \times n$ matrix ∂ of partial derivatives on $\mathbb{H}_n$:

$$(\partial)_{i,j} := \begin{cases} \frac{\partial}{\partial z_{ii}} & \text{if } i = j \\ \frac{1}{2}\frac{\partial}{\partial z_{ij}} & \text{if } i \neq j \end{cases}$$

We fix a weight k and a (possibly vector-valued) automorphy factor ρ and $l = (p-1)p^{m-1}$ with suitable m. Let $Hol(\mathbb{H}_n, V_\rho; |_\rho)$ denote the vector space of all holomorphic V_ρ-valued functions on $\mathbb{H}_n$, equipped with the action of $Sp(n,\mathbb{R})$ defined by the automorphy factor ρ; if $\rho = \det^k$, we just write $Hol(\mathbb{H}_n; |_k)$ We consider a bilinear holomorphic differential operator

$$[\ ,\ ]_{k,l} : Hol(\mathbb{H}_n; |_k) \times Hol(\mathbb{H}_n; |_l) \longrightarrow Hol(\mathbb{H}_n, V_\rho, |_{\rho\otimes\det^{k+l}}),$$

which is equivariant for the action of $Sp(n,\mathbb{R})$, i.e.

$$[F\ |_k\ g, G\ |_l\ g]_{k,l} = [F, G]_{k,l}\ |_{\rho\otimes\det^{k+l}}\ g$$

for all holomorphic functions F, G and all $g \in Sp(n,\mathbb{R})$, in particular, it maps $(F, G) \in M_k^n(\Gamma) \times M_l^n(\Gamma)$ to an element of $M^n_{\rho\otimes\det^{k+l}}(\Gamma)$.

We impose the following three conditions

(RC1) $[F, G]_{k,l}$ is a polynomial in the derivatives of F and G, more precisely, there exists a V_ρ-valued polynomial with rational coefficients in two matrix variables $R_1, R_2 \in \mathbb{C}^{n,n}_{sym}$, homogeneous of degree λ, such that

$$[F, G]_{k,l} = (2\pi i)^{-\lambda}\mathcal{P}(\partial_{Z_1}, \partial_{z_2})(F(Z_1)\cdot G(Z_2))_{|Z=Z_1=Z_2}$$

(RC2) We write $\mathcal{P} = \sum_j \mathcal{P}_j$ where the $\mathcal{P}_j$ are homogenous of degree j when viewed as polynomials in the second variable R_2 alone. Then $\mathcal{P}_0$ should be independent of l.

(RC3) The coefficients of $\mathcal{P}$ depend continuously on l (p-adically)

Comment The existence of such bilinear differential operators is not a problem if we stay away from finitely many values of k and l; this is a matter of invariant theory, see [8, 10]. The condition (RC2) however is delicate and has to be checked case by case as far as I can see.

Using such a Rankin-Cohen operator, we can now define analogues of Ramanujan's theta-operator

$$f = \sum a_t q^t \longmapsto \theta(f) = \frac{1}{2\pi i}f' = \sum_t t\cdot a(t)q^t.$$

For a Rankin-Cohen operator $[\ ,\ ]_{k,l}$ and $F \in M_k^n(\Gamma)$ we define a V_ρ-valued operator by

$$\Theta_{k,\rho}(F) := (2\pi i)^{-\lambda}\mathcal{P}_0(F).$$

Exactly by the same reasoning as for degree one we may show now

Theorem IV *For a modular form $F \in M_k^n(Sp(n,\mathbb{Z}))(\mathbb{Z}_{(p)})$ and a Rankin-Cohen operator $[\ ,\ ]_{k,l}$ with properties (RC1). (RC2), (RC3) the theta operator defines a V_ρ-valued p-adic modular form $\Theta_{k,\rho}(F)$.*

To explain our principle examples, we introduce some convenient notation following [9, III.§6]: For $0 \le i \le n$ and a $n \times n$ matrix A let $A^{[i]}$ be the matrix of size $\binom{n}{i} \times \binom{n}{i}$ consisting of the determinants of all submatrices of size i.

Examples For $0 \le i \le n$ and $F = \sum a_F(T)q^T \in M_k^n(\Gamma)$ we put

$$\Theta^{[i]}F := \sum_T a_F(T) \cdot T^{[i]} q^T$$

For $F \in M_k^n(\Gamma_0(p^r))(\mathbb{Z}_{(p)})$ this expression $\Theta^{[i]}(F)$ is congruent mod p^m to a level one modular form with automorphy factor

$$\det{}^{k+(p-1)p^{m'}} \otimes \underbrace{(\overbrace{2,\dots,2}^{r},0,\dots,0)}_{\text{highest weight of } \rho}$$

for a sufficiently large m', in particular, $\Theta^{[i]}F$ is a p-adic (vector-valued) modular form. This is in particular true for

$$\Theta^{[n]}(F) = \sum_T a_F(T)\det(T)q^T$$

and

$$\Theta^{[1]}(F) = \sum_T a_F(T) \cdot Tq^T.$$

In fact, the corresponding Rankin-Cohen bracket for $\Theta[i](F)$ can be constructed completely explicitly: We define polynomials $Q_{i,j}(R,S)$ in variables $R, S \in \mathbb{C}_{sym}^{(n,n)}$ by

$$(R + xS)^{[i]} = \sum_{j=0}^{i} Q_{i,j}(R,S)x^j.$$

Then there is an explicit linear combination of the

$$Q_{i,j}(\partial_{Z_1}, \partial_{z_2})(F(Z_1)) \cdot G(Z_2)_{Z_1=Z_2}$$

with leading term $(\Theta^{[i]}F) \cdot G$.

Remark If $F \in M_k^n(\Gamma)(\mathbb{Z}_{(p)})$ is mod p singular of rank r, then $\Theta^{[r+1]}(F) \equiv 0 \bmod p$ holds, but not only mod p singular modular forms have this property: let S be a positive definite quadratic forms in $m = 2k$ variables with $rank_{\mathbb{F}_p}(S) = n - j < n$;

we assume that S has no nontrivial integral automorphism. The theta series $\vartheta_S^n = \sum_T a(T)q^T$ is not mod p singular, because $a(S) = 2$. On the other hand, one has

$$\Theta^{[n-j+1]}\vartheta_S^n \equiv \cdots \equiv \Theta^{[n]}\vartheta_S^n \equiv 0 \bmod p.$$

6 Outlook: Quasimodular Forms

There is a sophisticated theory of nearly holomorphic modular forms due to Shimura [21]; they behave like modular forms, but they are no longer holomorphic: they are polynomials in the entries of Y^{-1} with holomorphic coefficients. A very famous example is the nonholomorphic Eisenstein series of weight 2:

$$1 - \frac{3}{\pi i y} - 24 \sum \sigma_1(n)q^n.$$

A quasimodular form is then defined as the constant term of such a nearly holomorphic function. Using the calculus of Rankin-Cohen operators and the full theory of nearly holomorphic modular forms, one can then show that such quasimodular forms are also p-adic modular forms [2].

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