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Jan Hendrik Bruinier
Winfried Kohnen *Editors*

L-Functions and Automorphic Forms

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Editors
Jan Hendrik Bruinier
Fachbereich Mathematik
Technische Universität Darmstadt
Darmstadt, Germany

Winfried Kohnen
Mathematisches Institut
Universität Heidelberg
Heidelberg, Germany

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Preface

This volume presents the proceedings of the conference *L-Functions and Automorphic Forms* that took place from February 22 to 26, 2016 at the University of Heidelberg, Germany. The theory of automorphic forms and their associated L-functions is one of the central research areas in modern number theory, linking number theory, arithmetic algebraic geometry, representation theory, and complex analysis in many profound ways. It is an area of enduring interest to a wide class of mathematicians. The present volume contains carefully refereed articles by leading experts in the field, giving new and original results. The topics include automorphic L-functions and their special values, p-adic modular forms, Eisenstein series, Borcherds products, automorphic periods, and many more.

In addition, four preparatory courses took place in the week before the conference, with the aim to introduce PhD students to basic and advanced concepts of the theory of automorphic forms. Lecture notes of three of those courses are also included.

The conference took place in the *Internationales Wissenschaftsforum Heidelberg*, which is a center for scholarly exchange in all areas of science and academic research located in the old town of Heidelberg. We are grateful for the hospitality and that we had the opportunity to use this excellent venue.

We thank Claudia Alfes-Neumann and Eric Hofmann for their help in preparing the conference program, the webpage, and the preparatory courses. Special thanks are also due to Nicole Umlas and David Obermayr for their support in preparing and running the conference. We are grateful for the generous financial support from the DFG-Forschergruppe 1920 Heidelberg/Darmstadt *Symmetry, Geometry, and Arithmetic* and the Mathematics Center Heidelberg (MATCH). Finally, we wish to extend our sincere thanks to all contributors to this volume and all speakers of the conference.

Darmstadt, Germany — Jan Hendrik Bruinier
Heidelberg, Germany — Winfried Kohnen
December 2017

Group picture of the conference

Contents

Sturm-Like Bound for Square-Free Fourier Coefficients

Pramath Anamby and Soumya Das

Abstract In this short article, we show the existence of an analogue of the classical Sturm's bound in the context of the *square-free* Fourier coefficients for cusp forms of square-free levels. This number is a cut-off to determine a cusp form from its initial few *square-free* Fourier coefficients. We also mention some questions in this regard.

1 Introduction

The theory of modular forms by now occupies a central place in number theory, and its wide ranging applications in various branches of mathematics is well known. One pleasant, and computationally important feature of these objects is that if f is such a form in $M_k(\Gamma)$ ($\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup and $k \geq 0$) with a Fourier expansion, say

$$f(\tau) = \sum_{n=0}^{\infty} a(f,n)e^{2\pi i n\tau}, \qquad (\tau \in \mathscr{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}) \tag{1.1}$$

then there exists a number $A > 0$ depending on the space such that if $a(f,n) = 0$ for all $n \leq A$, then $f = 0$. The smallest such bound in general is known as Sturm's bound in the literature. Let us denote it by $\mu(k,\Gamma)$ and recall that

$$\mu(k,\Gamma) := \frac{k\,[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]}{12}. \tag{1.2}$$

In fact Sturm's bound is known for various kinds of modular forms, e.g., half-integer weight forms, Siegel modular forms etc. In this paper we will discuss the following

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P. Anamby • S. Das (✉)
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
e-mail: pramatha@iisc.ac.in; pramath.anamby@gmail.com; soumya@iisc.ac.in; soumya.u2k@gmail.com

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question. Suppose that $\Gamma = \Gamma_0(N)$ and consider the spaces $S_k(N, \chi)$ where χ is a Dirichlet character modulo N with conductor m_χ and k is an integer.

Question Let $f \in S_k(N, \chi)$ have the Fourier expansion (1.1). Suppose N/m_χ is square-free. Does there exist a number $B > 0$, depending only on k, N such that if $a(f, n) = 0$ for all $n \leq B$ and *n square-free*, then $f = 0$?

We stress that the question has a trivial solution if f is a newform, in which case one can take $B = \mu(k, \Gamma_0(N))$ with $n \leq B$ to be prime. The above question was motivated by recent work of the authors, who proved the following result.

Theorem 1 ([1]) *Let χ be a Dirichlet character of conductor m_χ and let N be a positive integer with $m_\chi | N$ such that N/m_χ is square-free. Suppose that $f \in S_k(N, \chi)$ and that $a(f, n) = 0$ for all but finitely many square-free integers n. Then we have $f = 0$.*

The above mentioned **Question** is now a natural, finite counterpart to the above theorem. Note that the condition on the ratio of the level and conductor is necessary, this can be seen by taking the example of a non-zero form $g(\tau) \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ and considering $g(m^2\tau)$ for some $m > 1$.

We would like to mention that before Theorem 1, the possibility of the existence of such a constant B as above, didn't cross our minds. In this paper, we show the existence of B, and prove a crude bound for it in terms of k, N. We believe, with some efforts these bounds can be improved, and indicate some avenues for improvements.

Let us define $\mu_{\mathsf{sf}}(k, N)$ to be the smallest integer such that whenever $f \in S_k(N, \chi)$ with N/m_χ square-free and $a(f, n) = 0$ for all square-free $n \leq \mu_{\mathsf{sf}}(k, N)$, then $f = 0$.

Theorem 2 *Let N be square-free and $k \geq 2$. Then $\mu_{sf}(k, N)$ exists and satisfies the bound*

$$\mu_{sf}(k, N) \leq a_0 \cdot N \cdot 2^{r(r-1)/2} e^{4r \log^2(7k^2 N)},$$

where a_0 is an absolute constant and $r = \frac{(k-1)N}{2}$.

The plan of the paper is as follows. In Sect. 2 we prove the main result of the paper, and also include some conditional results. The proof follows an argument of Balog and Ono from [2] and it uses the existence of primes p in suitable intervals such that $a'(f, p) \neq 0$, where we put $a'(f, n) := a(f, n)n^{-\frac{k-1}{2}}$. Moreover we also need such primes which distinguish between two newforms, say $f_1 \neq f_2$; i.e., $a(f_1, p) \neq a(f_2, p)$ with the p's distinct and as small as possible.

In Sect. 2.1, we mention how to improve upon the various bounds by using the prime number theorem in short intervals for automorphic representations or by assuming a form of Maeda's conjecture in the case of $\mathrm{SL}_2(\mathbf{Z})$. In the last section, several related questions are discussed.

2 Proof of the Main Result

The following lemma follows immediately from the Prime Number Theorem, but we need to keep track of the dependence on the analytic conductors.

Lemma 2.1 *There exists $C = C(N, k) > N$ such that for all $X \geq C$, there exists a prime $p \in (X, 2X]$ for which*

(i) given any newform f of any level $M \mid N$ contained in $S_k(N, \chi)$, one has $a(f, p) \neq 0$;
(ii) given any two distinct newforms f, g as in (i), one has $a(f, p) \neq a(g, p)$.

Proof For $X \geq 1$, define $\psi_{f\times g}(X) = \sum_{n\leq x} \Lambda_{f\times g}(n)$, where we put

$$-L'(f \otimes g, s)/L(f \otimes g, s) = \sum_{n=1}^{\infty} \Lambda_{f\times g}(n)n^{-s}.$$

By the prime number theorem for $f \otimes \overline{g}$, one has (see [5, Theorem 5.13]) that

$$|\psi_{f\times g}(X) - r_{f,g}X| \leq c_2\, \mathfrak{q}_{f,g}^{1/2}\, X \exp\{-c_1(\log(X)^{1/2})\} \tag{2.1}$$

where $c_1, c_2 > 0$ are absolute constants, $\mathfrak{q}_{f,g}$ is the analytic conductor of the automorphic representation attached to $f \otimes \overline{g}$ (see [5, § 5.1]), and $r_{f,g}$ is the order of the possible pole (or zero) at $s = 1$. For future reference, let us note here that

$$\mathfrak{q}_{f,g} \leq 3^4 \mathfrak{q}_f^2 \cdot \mathfrak{q}_g^2. \tag{2.2}$$

and that $\mathfrak{q}_h = M(\frac{k-1}{2} + 3)(\frac{k+1}{2} + 3)$, if h is a newform of level M.

Moreover since N is square-free, the Euler-factors of $L(f \otimes g, s)$ behave nicely, and an easy estimate (see [5, (5.49)]) shows that for an absolute constant c_3,

$$|\psi_{f\times g}(X) - \sum_{p\leq X} a_f'(p)a_g'(p)\log(p)| \leq c_3\, X^{1/2} \log^2(X\mathfrak{q}_{f,g}). \tag{2.3}$$

Let us call the quantities on the right-hand side of (2.1) and (2.3) to be $R_1(X)$ and $R_2(X)$ respectively and put $R(X) := |R_1(X)| + |R_2(X)|$ and $F := f - \kappa g$, with f, g as in the theorem and $\kappa \in \{0, 1\}$. Taking into account that $r_{f,g} \leq 1$, and equality holds if and only if $g = \overline{f}$. It is immediate from the above that

$$\begin{aligned}\sum_{X<p\leq 2X} |a_F'(p)|^2 \log(p) &= |a_f'(p)|^2 \log(p) + \kappa|a_g'(p)|^2 \log(p) - 2\kappa\Re\{a_f'(p)a_g'(p)\log(p)\}\\ &\geq X - 4(R(X) + R(2X)).\end{aligned} \tag{2.4}$$

A simple calculation shows that (2.4) is positive provided $X > c_4 \exp\{4\log^2(k^2N)\}$, where c_4 is an absolute constant. So the lemma follows by taking $C = c_5 \exp\{4\log^2(7k^2N)\}$ with c_5 chosen so that $C > N$. □

The rest of the section is devoted to a proof of Theorem 2.

If $f \in S_k(N, \chi)$ is a newform then the result follows from the multiplicativity of the Fourier coefficients and the fact that for every prime p, the Hecke operators $T(p^\nu)$ are polynomials in $T(p)$ and clearly for this f, $B = \mu(k, \Gamma_0(N))$ works. The following is adapted from Balog-Ono [2].

So let $f \in S_k(N, \chi)$ be non-zero. Consider the set $\{f_1, f_2, \ldots\ldots f_s\}$ of all newforms of weight k and level dividing N contained in $S_k(N, \chi)$. Let their Fourier expansions be given by $f_i(z) = \sum_{n=1}^{\infty} b_i(n)q^n$. Then for all primes p, one has $T_p f_i = b_i(p) f_i$. By "multiplicity-one", if $i \neq j$, we can find infinitely many primes $p > N$ such that $b_i(p) \neq b_j(p)$. Now by the theory of newforms, there exists $\alpha_{i,\delta} \in \mathbf{C}$ such that $f(z)$ has can be written uniquely in the form

$$f(z) = \sum_{i=1}^{s} \sum_{\delta | N} \alpha_{i,\delta} f_i(\delta z). \tag{2.5}$$

Since $f \neq 0$, we may, after renumbering the indices, assume $\alpha_{1,\delta} \neq 0$ for some $\delta | N$. Let C be as in Lemma 2.1. Let p_1 be any prime such that $p_1 \in (2C, 2^2C]$ and $b_1(p_1) \neq b_2(p_1)$. Note that $(p_1, N) = 1$. Then consider the form $g_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n := T_{p_1} f(z) - b_2(p_1) f(z)$ so that

$$g_1(z) = \sum_{i=1}^{s} (b_i(p_1) - b_2(p_1)) \sum_{\delta | N} \alpha_{i,\delta} f_i(\delta z).$$

The cusp forms $f_2(\delta z)$ for any $\delta \mid N$, do not appear in the decomposition of $g_1(z)$ but $f_1(\delta z)$ does for some $\delta | N$. Also it is easy to see that $a_1(n) = a(f, p_1 n) + \chi(p_1) p_1^{k-1} a(f, n/p_1) - b_2(p_1) a(f, n)$. Proceeding inductively in this way, and choosing the primes $p_i \in (2^i C, 2^{i+1} C]$ $(2 \leq i \leq s-1)$, we can remove all the non-zero newform components $f_i(\delta z)$ for all $i = 2, \ldots, s$, to obtain a cusp form $F(z)$ in $S_k(N, \chi)$. After dividing by a suitable non-zero complex number we get

$$F(z) = \sum_{n=1}^{\infty} A(n) q^n := \sum_{\delta | N} \alpha_{1,\delta} f_1(\delta z).$$

Now by repeating the above steps we get finitely many algebraic numbers β_j and positive rational numbers γ_j such that for every n

$$A(n) = \sum_{\delta | N} \alpha_{1,\delta} b_1(n/\delta) = \sum_j \beta_j a(f, \gamma_j n). \tag{2.6}$$

Let δ_1 be the smallest divisor of N such that $\alpha_{1,\delta_1} \neq 0$ in (2.5). Define $S = \{p\colon p$ prime, $p|N\} \cup \{p\colon p$ prime, $b_1(p) = 0\}$.

Let us choose $p \in (C, 2C]$ as in Lemma 2.1 (i). Then $p \notin S$, and $A(\delta_1 p) = \alpha_{1,\delta_1} b_1(p) \neq 0$. Note that δ_1 is square-free and $(\delta_1, \gamma_j p) = 1$ when $\gamma_j p \in \mathbf{N}$. From (2.6), we get a $\gamma_j \in \mathbf{N}$ such that $a(f, \delta_1 \gamma_j p) \neq 0$. Now combining the estimates

$$\delta_1 \leq N, \quad \gamma_j \leq 2^{\frac{s(s-1)}{2}-1} C^{s-1}, \quad p \leq 2C,$$

we obtain $n \leq B$ such that $a(f, n) \neq 0$. Moreover B satisfies

$$\mu_{\mathsf{sf}}(k, N) \leq B \leq 2^{\frac{s(s-1)}{2}} N C^s.$$

Since N is square-free, $s \leq (k-1)N/2$ (see [6]). The theorem follows by substituting the expressions for s and C. This completes the proof of Theorem 2. □

2.1 Remarks

In this section, we discuss the number $\mu_{\mathsf{sf}}(N, k)$ and remark on a few ways of improving its bound, mostly based on some conjectures and numerical evidence.

(i) *An inequality.* When $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, by employing the so-called Miller's basis, we can easily show that $\mu_{\mathsf{sf}}(k, 1) \geq \mu(k, 1)$.

(ii) *A conjecture.* Based on standard heuristics about Fourier coefficients of cusp forms and that the square-free integers have a positive natural density, we are led to believe that for all $N \geq 1$,

$$\mu(k, N) \leq \mu_{\mathsf{sf}}(k, N) \leq a \cdot \mu(k, N)$$

for some absolute constant $a > 1$. Numerical experiments with SAGE supports this.

(iii) *Application of a form of Maeda's Conjecture.* Let us recall a result due to P. Bengoechea [3] which states that if $k \equiv 0 \bmod 4$, and if for some $n \geq 1$ the characteristic polynomial $T_{n,k}(X)$ of the n-th Hecke operator T_n on $\mathrm{SL}_2(\mathbf{Z})$ is irreducible over $\mathbf{Z}$ and has full Galois group, then so are $T_{p,k}(X)$ for all primes p. So, if we assume the aforementioned condition for some n, an inspection of the proof of Theorem 2 shows that we can choose $p, p_1, p_2, \ldots, p_{s-1}$ to be consecutive primes in increasing order. We easily get that $\mu_{\mathsf{sf}}(k, N) \ll \exp\left(\frac{\mu(k,1)}{\log \mu(k,1)}\right)$, where the implied constant is absolute. In particular, due to computations by Ghitza and Mc Andrew [4], this holds for all such $k \leq 12{,}000$.

(iv) *Primes in short intervals.* If instead of the prime number theorem in long intervals that we used, one uses a short interval version for the PNT for the Rankin-Selberg convolution $\pi \otimes \widetilde{\pi}$ where π is the irreducible unitary representation obtained from a newform using [8], there is a possibility of improving the bound in Theorem 2, but we do not know the dependence of the implied constants on π in the above mentioned result.

2.2 Further Questions

Throughout this section we assume N/m_χ is square-free, $k \geq 2$.

(i) *Eisenstein series.* Even though we have stated the result only for cusp forms, it should be true for the space of Eisenstein series as well.

(ii) *Different method of proof.* In order to prove something like (ii) in Sect. 2.1, one should come up with a more natural method of proof, which does not involve choosing primes distinguishing between modular forms. Let us mention here that in [1], apart from Theorem 1, a certain asymptotic formula was proved for the second moment of Fourier co-efficients of any $f \in S_K(N, \chi)$ supported over square-free indices; where the implied constants depend on k, N and f. We do not see immediately how one can deduce from there an answer to the above **Question** uniformly for all $f \in S_k(N, \chi)$.

(iii) *Finite version.* Let $M_k(N, \mathscr{O}_F)$ denote the space of modular forms in $M_k(N)$ whose Fourier expansion at ∞ lies in $\mathscr{O}_F$, the ring of integers of a number field F. Fix a prime ideal $\mathscr{P} \subset \mathscr{O}_F$.

In analogy with Sturm's bound for finite primes (see e.g., [7]), is it true (with standard notations) that if $f, g \in M_k(N, \mathscr{O}_F)$ such that $a(f, n) \equiv a(g, n) \bmod \mathscr{P}$ for all *square-free* $n \leq \mu_{\mathsf{sf}}(k, N)$, then $f \equiv g \bmod \mathscr{P}$?

(iv) *Half-integral weight forms.* Given a result of Saha [9] (which holds e.g., in Kohnen's plus space $S^+_{k+1/2}(4N)$ of level $4N$ with N odd, square-free) that $f \in S^+_{k+1/2}(4N)$ are determined by Fourier coefficients which are indexed by fundamental discriminants, one can ask for the existence of a finite 'fundamental' version of our result.

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Images of Maass-Poincaré Series in the Lower Half-Plane

Nickolas Andersen, Kathrin Bringmann, and Larry Rolen

Abstract In this note we extend integral weight harmonic Maass forms to functions defined on the upper and lower half-planes using the method of Poincaré series. This relates to Rademacher's "expansion of zero" principle, which was recently employed by Rhoades to link mock theta functions and partial theta functions.

1 Introduction and Statement of Results

In [12], Rhoades found a method to uniformly describe partial theta functions and mock theta functions as manifestations of a single function. He showed that Ramanujan's mock theta function $f(q)$ (defined below), with $q := e^{2\pi i\tau}$ and τ in the upper half-plane $\mathbb{H}$, in some sense "leaks" through the real line to a partial theta function $\psi(q^{-1})$ (given below) on the lower half-plane $-\mathbb{H}$. His construction follows the "expansion of zero" principle of Rademacher (see [4], [6], [7, Chapter IX], and [10]). Rademacher showed, using his exact formula for the partition function, that the partition generating function can be extended to the lower half-plane, and he later proved [11] that this extension is identically zero in the lower half plane. We note that there are other relations of partial theta functions and mock theta functions. For example one, which is due to Zagier and Zwegers, passes through asymptotic expansions (see for example [5]).

Let us now say a few more words concerning mock theta functions. Originally introduced by Ramanujan in his last letter to Hardy, mock theta functions have since

N. Andersen
Mathematics Department, UCLA, Box 951555, Los Angeles, CA 90095, USA
e-mail: nandersen@math.ucla.edu

K. Bringmann (✉)
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: kbringma@math.uni-koeln.de

L. Rolen
Hamilton Mathematics Institute & School of Mathematics, Trinity College, Dublin 2, Ireland

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160
e-mail: lrolen@maths.tcd.ie

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found applications in many areas of mathematics. We now understand that they fit into the larger framework of harmonic Maass forms, as shown by Zwegers [14] (see also [9, 13]). That is, the mock theta functions are examples of mock modular forms, which are the holomorphic parts of harmonic Maass forms (see Sect. 2 for definitions). Thus, it is natural to ask whether Rhoades' construction applies to the non-holomorphic completion of $f(q)$ and, if so, what is the image of that function in the lower half-plane? One hope, which has not yet been realized, is that this might shed some light on the problem of finding a completion of the partial theta functions to non-holomorphic modular forms. General *partial theta functions* have the shape

$$\sum_{n\geq 0} \psi(n) n^{\nu} q^{n^2}$$

with ψ a primitive Dirichlet character and $\nu \in \mathbb{Z}$ such that $\psi(-1) = (-1)^{\nu+1}$. In particular these functions are not modular forms.

We begin by more closely recalling Rhoades' results. The partial theta function which Rhoades studied is given by

$$\psi(q) := \sum_{n\geq 1} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}},$$

and the associated mock theta function is Ramanujan's third order function

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(1+q)^2 \cdot \ \ldots \ \cdot (1+q^n)^2}.$$

Now set

$$\alpha_c(s) := \sum_{m\geq 0} \left(\frac{\pi}{12c}\right)^{2m+\frac{1}{2}} \frac{1}{\Gamma\left(m+\frac{3}{2}\right)} \frac{1}{s^{m+1}}$$

and (with $\zeta_a^b := e^{\frac{2\pi i b}{a}}$)

$$\Phi_{c,d}(\tau) := \frac{1}{2\pi i} \int_{|s|=r} \frac{\alpha_c(s) e^{23s}}{1 - \zeta_{2c}^d \, q \, e^{24s}} ds,$$

where r is taken sufficiently small such that $|\operatorname{Log}\left(\zeta_{2c}^d q\right)| \gg r$ and such that the integral converges. Moreover let $\omega_{h,c}$ be the multiplier of the Dedekind η-function (which can be given explicitly in terms of Dedekind sums, see [11]). Then define the function

$$F(\tau) := 1 + \pi \sum_{c \geq 1} \frac{(-1)^{\lfloor \frac{c+1}{2} \rfloor}}{c}$$
$$\times \sum_{d \pmod{2c}^*} \omega_{-d,2c} \exp\left(2\pi i \left(-\frac{d}{8}(1+(-1)^c) + \frac{d}{2c} + \tau\right)\right) \Phi_{c,d}(\tau),$$

where $d \pmod{2c}^*$ indicates that the sum ranges over those d modulo $2c$ with $\gcd(d, 2c) = 1$. This function converges in both the upper and lower half-planes, i.e., for $\tau \in \mathbb{H} \cup (-\mathbb{H})$. Moreover, Rhoades' main result states that

$$F(\tau) = \begin{cases} f(q) & \text{if } \tau \in \mathbb{H}, \\ 2\psi\left(q^{-1}\right) & \text{if } \tau \in -\mathbb{H}. \end{cases}$$

As discussed above, we describe a similar phenomenon for both the holomorphic and non-holomorphic parts of Maass–Poincaré series. To state our results, we first require some notation. Throughout, let $k \in 2\mathbb{Z}$, and let $M_k^!(\Gamma_0(N))$ denote the space of weakly holomorphic modular forms of weight k on $\Gamma_0(N)$. Let $S_k^!(\Gamma_0(N))$ denote the subspace of $M_k^!(\Gamma_0(N))$ consisting of forms whose Fourier expansion at $i\infty$ has constant term equal to zero. For $f(\tau) =: \sum_n c_f(n)q^n \in S_k^!(\Gamma_0(N))$, we define the (holomorphic) *Eichler integral*

$$\mathcal{E}_f(\tau) := \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{c_f(n)}{n^{k-1}} q^n$$

and the *non-holomorphic Eichler integral* ($\tau = u + iv$)

$$f^*(\tau) := -(4\pi)^{1-k} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\overline{c_f(n)}}{n^{k-1}} \Gamma(k-1, 4\pi n v) q^{-n}.$$

Here $\Gamma(s, y)$ denotes the incomplete gamma function defined in (2.1). Note that with

$$D^{k-1} := \left(q\frac{d}{dq}\right)^{k-1} \qquad \text{and} \qquad \xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}},$$

we have

$$D^{k-1}(\mathcal{E}_f) = f \qquad \text{and} \qquad \xi_{2-k}(f^*) = f.$$

For even $k > 2$ and $m \in \mathbb{Z}$, let $P_{k,m}$ denote the holomorphic Poincaré series (defined in Sect. 2 below). If $m < 0$, these functions are weakly holomorphic forms, while for $m > 0$, they are cusp forms. For $k \in -2\mathbb{N}$ and $m > 0$, let $F_{k,-m}$ be the

Maass-Poincaré series of Sect. 2. They are harmonic Maass forms with exponential growth in their holomorphic part.

Theorem 1.1 *Let* $k \in -2\mathbb{N}$ *and* $m \in \mathbb{N}$. *Then the function* $H_{k,m} := H^{+}_{k,m} + H^{-}_{k,m}$ (*defined in* (3.3) *and* (3.6) *below*) *converges for all* $\tau \in \mathbb{H} \cup (-\mathbb{H})$. *Furthermore, if* $\tau \in \mathbb{H}$ *we have*

$$H_{k,m}(\tau) = F_{k,-m}(\tau),$$

and if $\tau \in -\mathbb{H}$ *we have*

$$H_{k,m}(\tau) = m^{1-k}\left(\mathcal{E}_{P_{2-k,m}}(-\tau) - \frac{(4\pi)^{1-k}}{(-k)!}P^{*}_{2-k,-m}(-\tau)\right).$$

To prove Theorem 1.1, we determine the extension of the holomorphic and non-holomorphic parts of $F_{k,-m}$ separately, in Sects. 3.1 and 3.2, respectively. The computation involving the holomorphic part closely follows [12], and the extension is provided by the simple fact that

$$\frac{1}{1-q} = \begin{cases} \sum\limits_{n\geq 0} q^n & \text{if } |q| < 1, \\ -\sum\limits_{n\geq 1} q^{-n} & \text{if } |q| > 1. \end{cases}$$

For the non-holomorphic part, the situation is similar, but somewhat more complicated, and the extension is provided by the functional equation of the polylogarithm $\mathrm{Li}_{1-k}(q)$ (defined in (3.5) below), namely

$$\mathrm{Li}_{k-1}(q) = \mathrm{Li}_{k-1}\left(q^{-1}\right) \qquad \text{for } k \in -2\mathbb{N}. \tag{1.1}$$

Remark If one tries to mimic the computations of Sect. 3.2 in the case of half-integral weight, the situation is complicated by the analogue of (1.1) for $k \notin \mathbb{Z}$, namely

$$i^{1-k}\,\mathrm{Li}_{k-1}\left(e^{2\pi iu}\right) + i^{k-1}\,\mathrm{Li}_{k-1}\left(e^{-2\pi iu}\right) = \frac{(2\pi)^{k-1}}{\Gamma(k-1)}\zeta(2-k,u),$$

where $\zeta(2-k,u)$ denotes the Hurwitz zeta function. It is unclear whether the resulting function in the lower-half plane has any relation to a known modular-type object.

The paper is organized as follows. In Sect. 2 we recall the definitions and some basic properties of harmonic Maass forms and Poincaré series. In Sect. 3 we prove Theorem 1.1.

2 Preliminaries

2.1 Harmonic Maass Forms

In this section we recall basic facts of harmonic Maass forms, first introduced by Bruinier and Funke in [2]. We begin with their definition.

Definition For $k \in 2\mathbb{N}$, a *weight k harmonic Maass form for* $\Gamma_0(N)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(1) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

(2) We have $\Delta_k(f) = 0$ where Δ_k is the *weight k hyperbolic Laplacian*

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

(3) There exists a polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$f(\tau) - P_f(\tau) = O(e^{\varepsilon v})$$

as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Denote the space of such harmonic Maass forms by $H_k(\Gamma_0(N))$. Every $f \in H_k(\Gamma_0(N))$ has an expansion of the form

$$f(\tau) = f^+(\tau) + f^-(\tau)$$

with the *holomorphic part* having a q-expansion

$$f^+(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n$$

and the *non-holomorphic part* having an expansion of the form

$$f^-(\tau) = \sum_{n>0} c_f^-(n) \Gamma(1-k, 4\pi n v) q^{-n}.$$

Here $\Gamma(s, v)$ is the *incomplete gamma function* defined, for $v > 0$, as the integral

$$\Gamma(s, v) := \int_v^\infty t^{s-1} e^{-t}\, dt. \tag{2.1}$$

2.2 Poincaré Series

In this section, we recall the definitions and properties of various Poincaré series. The general construction is as follows. Let φ be any translation-invariant function, which we call the *seed* of the Poincaré series in question. Then, in the case of absolute convergence, we can define a function satisfying weight k modularity by forming the sum

$$\mathbb{P}_k(\varphi;\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \varphi|_k\gamma(\tau),$$

where $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n\in\mathbb{Z}\}$ is the group of translations. Convergence is, in particular, satisfied by functions φ satisfying $\varphi(\tau) = O(v^{2-k+\varepsilon})$ as $v\to 0$.

A natural choice for φ is a typical Fourier coefficient in the space of automorphic functions one is interested in. For example, in the case of weakly holomorphic modular forms one may choose, for $m\in\mathbb{Z}$,

$$\varphi(\tau) = \varphi_m(\tau) := q^m.$$

Define for $k\in 2\mathbb{N}$ with $k>2$ and $m\in\mathbb{Z}$ the *Poincaré series of exponential type* by

$$P_{k,m}(\tau) := \mathbb{P}_k(\varphi_m;\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \varphi_m|_k\gamma(\tau).$$

To give their Fourier expansion, we require the *Kloosterman sums*

$$K(m,n;c) := \sum_{d \pmod{c}^*} e\left(\frac{m\overline{d}+nd}{c}\right), \tag{2.2}$$

where $e(x) := e^{2\pi i x}$. A direct calculation yields the following duality:

$$K(-m,-n;c) = K(m,n;c).$$

A very useful property of the Poincaré series is that they have explicit Fourier expansions, as given in the following theorem.

Theorem 2.1 *Suppose that $k>2$ is even.*

i) *If $m\in\mathbb{N}$, the Poincaré series $P_{k,m}$ are in $S_k(\Gamma_0(N))$. We have the Fourier expansion $P_{k,m}(\tau) = \sum_{n=1}^{\infty} b_{k,m}(n)q^n$, where*

$$b_{k,m}(n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}}\left(\delta_{m,n} + 2\pi(-1)^{\frac{k}{2}}\sum_{\substack{c>0\\ N|c}}\frac{K(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right).$$

Here $\delta_{m,n}$ is the Kronecker delta-function and J_s denotes the usual J-Bessel function.

ii) *For $m \in -\mathbb{N}$, the Poincaré series $P_{k,m}$ are elements of $M_k^!(\Gamma_0(N))$. We have the Fourier expansion $P_{k,m}(\tau) = q^m + \sum_{n=1}^{\infty} b_{k,m}(n)q^n$, where*

$$b_{k,m}(n) = 2\pi(-1)^{\frac{k}{2}} \left|\frac{n}{m}\right|^{\frac{k-1}{2}} \sum_{\substack{c>0\\ N|c}} \frac{K(m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Here I_s denotes the usual I-Bessel function. Moreover, $P_{k,m}$ is holomorphic at the cusps of $\Gamma_0(N)$ other than $i\infty$.

We next turn to the construction of harmonic Maass forms via Poincaré series. Such series have appeared in many places in the literature, indeed in the works of Niebur [8] and Fay [3] in the 1970's, long before the recent advent of harmonic Maass forms. Define

$$F_{k,m} := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \phi_{k,m}|_k\gamma,$$

where the seed $\phi_{k,m}$ is given by

$$\phi_{k,m}(\tau) := \big(1 - \Gamma^*(1-k, 4\pi|m|v)\big)q^m.$$

Here Γ^* is the *normalized incomplete gamma function*

$$\Gamma^*(s,v) := \frac{\Gamma(s,v)}{\Gamma(s)}.$$

The analogous exact formula for coefficients of these Poincaré series is then given in the following theorem (see, e.g., [3] or [1] for a proof).

Theorem 2.2 *If $k < 0$ is even and $m \in -\mathbb{N}$, then $F_{k,m} \in H_k(\Gamma_0(N))$. We have*

$$\xi_k\left(F_{k,m}\right) = -\frac{(4\pi m)^{1-k}}{(-k)!} P_{2-k,-m}$$

and

$$D^{1-k}\left(F_{k,m}\right) = m^{1-k} P_{2-k,m}.$$

We have the Fourier expansion

$$F_{k,m}(\tau) = \big(1 - \Gamma^*(1-k, -4\pi m v)\big)\, q^m + \sum_{n=0}^{\infty} a^+_{k,m}(n) q^n + \sum_{n=1}^{\infty} a^-_{k,m}(n) \Gamma^*(1-k, 4\pi n v) q^{-n}$$

with

$$a^+_{k,m}(0) = \frac{(2\pi)^{2-k}(-1)^{\frac{k}{2}+1} m^{1-k}}{(1-k)!} \sum_{\substack{c>0\\ N|c}} \frac{K(m,0;c)}{c^{2-k}}.$$

Moreover, for $n \geq 1$ and $\varepsilon \in \{+,-\}$, we have

$$a^{\varepsilon}_{k,m}(n) \;=\; 2\pi(-1)^{\frac{k}{2}} \left|\frac{m}{n}\right|^{\frac{1-k}{2}} \sum_{\substack{c>0\\ N|c}} \frac{K(m,\varepsilon n;c)}{c} \times \begin{cases} I_{1-k}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \varepsilon n > 0,\\ J_{1-k}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \varepsilon n < 0. \end{cases}$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, we consider the holomorphic and non-holomorphic parts of $F_{k,-m}$ separately.

3.1 The Holomorphic Part

We first extend the holomorphic part $F^+_{k,-m}$ of $F_{k,-m}$ to a function defined for $|q| \neq 1$, closely following [12]. Using Theorem 2.2, we have, for $|q| < 1$,

$$F^+_{k,-m}(\tau) = q^{-m} + a^+_{k,-m}(0) + 2\pi(-1)^{\frac{k}{2}} m^{\frac{1-k}{2}} \sum_{\substack{c>0\\ N|c}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}}{c}\right) A^+_m(c,d),$$

where

$$A^+_m(c,d) = A^+_m(c,d;\tau) := \sum_{n\geq 1} n^{\frac{k-1}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) \zeta_c^{nd} q^n. \tag{3.1}$$

Using the series expansion of the I-Bessel function

$$I_\alpha(x) = \sum_{j\geq 1} \frac{1}{j!\Gamma(j+\alpha+1)} \left(\frac{x}{2}\right)^{2j+\alpha},$$

we obtain

$$n^{\frac{k-1}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{j\geq 0} \beta^+_{m,c}(j) \frac{n^j}{j!},$$

where

$$\beta^+_{m,c}(j) := \frac{\left(\frac{2\pi\sqrt{m}}{c}\right)^{2j+1-k}}{(j+1-k)!}.$$

We insert the integral representation (for $r > 0$)

$$\frac{n^j}{j!} = \frac{1}{2\pi i} \int_{|s|=r} \frac{e^{ns}}{s^{j+1}}\, ds, \tag{3.2}$$

and we conclude that

$$n^{\frac{k-1}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \frac{1}{2\pi i} \int_{|s|=r} \alpha^+_{m,c}(s) e^{ns}\, ds,$$

where $\alpha^+_{m,c}(s)$ is the series

$$\alpha^+_{m,c}(s) := \sum_{j\geq 0} \frac{\beta^+_{m,c}(j)}{s^{j+1}},$$

which is absolutely convergent for all s. Equation (3.1) then becomes

$$A^+_m(c,d) = \frac{1}{2\pi i} \int_{|s|=r} \alpha^+_{m,c}(s) \sum_{n\geq 1} \left(e^s \zeta_c^d q\right)^n\, ds = \frac{1}{2\pi i} \zeta_c^d q \int_{|s|=r} \frac{\alpha^+_{m,c}(s) e^s}{1 - e^s \zeta_c^d q}\, ds.$$

Here we take r sufficiently small so that $|e^s \zeta_c^d q| < 1$. Define

$$\phi^+_{k,m}(c,d;\tau) := \frac{1}{2\pi i} \int_{|s|=r} \frac{\alpha^+_{m,c}(s) e^s}{1 - e^s \zeta_c^d q}\, ds.$$

We can now define the function which exists away from the real line. To be more precise, since $\phi^{+}_{k,m}$ is regular for all $v \neq 0$, the function

$$H^{+}_{k,m}(\tau) := q^{-m} + a^{+}_{k,-m}(0) + 2\pi(-1)^{\frac{k}{2}} m^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ N|c}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}+d}{c} + \tau\right) \phi^{+}_{k,m}(c,d;\tau) \tag{3.3}$$

is defined for $\tau \in \mathbb{H} \cup (-\mathbb{H})$.

We now consider the Fourier expansion of the function $H^{+}_{k,m}(\tau)$ for τ in the lower half-plane, so suppose for the remainder of the proof that $v < 0$. In this case, we have

$$\begin{aligned} \zeta_c^d q\, \phi^{+}_{k,m}(c,d;\tau) &= \frac{1}{2\pi i} \int_{|s|=r} \alpha^{+}_{m,c}(s) \frac{e^s \zeta_c^d q}{1 - e^s \zeta_c^d q} ds \\ &= -\frac{1}{2\pi i} \int_{|s|=r} \alpha^{+}_{m,c}(s) \sum_{n \geq 0} \left(e^s \zeta_c^d q\right)^{-n} ds, \end{aligned}$$

where r is chosen so that $|e^{-s}q^{-1}| < 1$. By reversing the calculation which led to (3.3), making the change of variables $s \mapsto -s$, and using that $I_{1-k}(-ix) = i^{1-k} J_{1-k}(x)$, we find that

$$\begin{aligned} \zeta_c^d q\, \phi^{+}_{k,m}(c,d;\tau) &= -\beta^{+}_{m,c}(0) - \sum_{n \geq 1} \left(\sum_{j \geq 0} (-1)^j \beta^{+}_{m,c}(j) \frac{n^j}{j!} \right) e\left(\frac{-nd}{c}\right) q^{-n} \\ &= -\beta^{+}_{m,c}(0) - \sum_{n \geq 1} e\left(\frac{-nd}{c}\right) n^{\frac{k-1}{2}} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) q^{-n}. \end{aligned}$$

Thus we have

$$\begin{aligned} &H^{+}_{k,m}(\tau) \\ &= q^{-m} + a^{+}_{k,-m}(0) - 2\pi(-1)^{\frac{k}{2}} m^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}}{c}\right) \beta^{+}_{m,c}(0) \\ &\quad - 2\pi(-1)^{\frac{k}{2}} \sum_{n \geq 1} \left(\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ N|c}} \frac{K(-m,-n;c)}{c} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) \right) q^{-n}. \end{aligned} \tag{3.4}$$

Note that the second and the third terms on the right-hand side of (3.4) cancel, since

$$-2\pi(-1)^{\frac{k}{2}} m^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ N|c}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}}{c}\right) \beta^+_{m,c}(0)$$
$$= \frac{(2\pi)^{2-k}(-1)^{\frac{k}{2}+1} m^{1-k}}{(1-k)!} \sum_{\substack{c>0 \\ N|c}} \frac{K(-m,0;c)}{c^{2-k}} = -a^+_{k,-m}(0).$$

By (2.2), we conclude that

$$H^+_{k,m}(\tau) = q^{-m} - \sum_{n\geq 1} \left(\frac{m}{n}\right)^{1-k}$$
$$\times \left(2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ N|c}} \frac{K(m,n;c)}{c} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right) q^{-n}$$
$$= m^{1-k}\left(m^{k-1}q^{-m} + \sum_{n\geq 1} n^{k-1} b_{2-k,m}(n) q^{-n}\right) = m^{1-k}\mathcal{E}_{P_{2-k,m}}(-\tau)$$

if τ is in the lower half-plane.

3.2 The Non-holomorphic Part

Next we extend the non-holomorphic part $F^-_{k,-m}(\tau)$ to a function $H^-_{k,m}(\tau)$, which is defined for $|q| \neq 1$. We have, by Theorem 2.2,

$$F^-_{k,-m}(\tau) = -\Gamma^*(1-k, 4\pi m v) q^{-m}$$
$$+ 2\pi(-1)^{\frac{k}{2}} m^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ N|c}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}}{c}\right) A^-_m(c,d;\tau),$$

where

$$A^-_m(c,d) = A^-_m(c,d;\tau)$$
$$:= \sum_{n\geq 1} n^{\frac{k-1}{2}} e\left(\frac{-nd}{c}\right) J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) \Gamma^*(1-k, 4\pi n v) q^{-n}.$$

Using the integral representation of the incomplete Gamma function and making a change of variables, we find that

$$\Gamma(1-k, y) = y^{1-k} \int_1^\infty t^{-k} e^{-yt} dt,$$

thus

$$A_m^-(c,d) = \frac{(4\pi v)^{1-k}}{(-k)!} \int_1^\infty t^{-k} \sum_{n\geq 1} n^{\frac{1-k}{2}} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-4\pi n v t} \left(\zeta_c^d q\right)^{-n} dt.$$

As above, we use the series expansion of the J-Bessel function

$$J_\alpha(x) = \sum_{j\geq 1} \frac{(-1)^j}{j!\Gamma(j+\alpha+1)} \left(\frac{x}{2}\right)^{2j+\alpha}$$

to obtain

$$n^{\frac{1-k}{2}} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) = n^{1-k} \sum_{j\geq 0} \beta_{m,c}^-(j) \frac{n^j}{j!},$$

where

$$\beta_{m,c}^-(j) := \frac{(-1)^j}{(j+1-k)!} \left(\frac{2\pi\sqrt{m}}{c}\right)^{2j+1-k}.$$

Thus we have, again using (3.2),

$$n^{\frac{1-k}{2}} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \frac{1}{2\pi i} \int_{|s|=r} n^{1-k} e^{ns} \alpha_{m,c}^-(s) ds,$$

where

$$\alpha_{m,c}^-(s) := \sum_{j\geq 0} \frac{\beta_{m,c}^-(j)}{s^{j+1}}.$$

Here r is chosen so that $r < 2\pi v$. Thus

$$\begin{aligned} A_m^-(c,d) &= \frac{(4\pi v)^{1-k}}{(-k)!} \int_1^\infty t^{-k} \left(\frac{1}{2\pi i} \int_{|s|=r} \alpha_{m,c}^-(s) \sum_{n\geq 1} n^{1-k} \left(e^{s-4\pi v t} \zeta_c^{-d} q^{-1}\right)^n ds\right) dt \\ &= \frac{(4\pi v)^{1-k}}{(-k)!} \int_1^\infty t^{-k} \left(\frac{1}{2\pi i} \int_{|s|=r} \alpha_{m,c}^-(s) \, \mathrm{Li}_{k-1}\left(e^{s-4\pi v t} \zeta_c^{-d} q^{-1}\right) ds\right) dt. \end{aligned}$$

Here $\mathrm{Li}_s(w)$ is the *polylogarithm*, defined for $s \in \mathbb{C}$ and $|w| < 1$ by

$$\mathrm{Li}_s(w) := \sum_{n=1}^{\infty} \frac{w^n}{n^s}. \tag{3.5}$$

We now again introduce a function defined away from the real line. Define

$$\phi_{k,m}^{-}(c,d;\tau) := \frac{1}{2\pi i}\int_1^{\infty} t^{-k} \int_{|s|=r} \alpha_{m,c}^{-}(s)\,\mathrm{Li}_{k-1}\left(e^{s-4\pi v t}\zeta_c^{-d}q^{-1}\right)\,ds\,dt$$

and

$$\begin{aligned} H_{k,m}^{-}(\tau) &:= -\Gamma^*(1-k,4\pi m v)q^{-m} \\ &+ \frac{2\pi(-1)^{\frac{k}{2}}(4\pi v)^{1-k}}{(-k)!} m^{\frac{1-k}{2}} \sum_{\substack{c>0\\ N|c}} \frac{1}{c} \sum_{d \pmod{c}^*} e\left(\frac{-m\overline{d}}{c}\right)\phi_{k,m}^{-}(c,d;\tau). \end{aligned} \tag{3.6}$$

Using the functional equation

$$\mathrm{Li}_{-n}(z) = (-1)^{n+1}\mathrm{Li}_{-n}\left(\frac{1}{z}\right),$$

we obtain, for $v < 0$, and using that k is even,

$$A_m^{-}(c,d) = \frac{(4\pi v)^{1-k}}{(-k)!}\int_1^{\infty} t^{-k}\left(\frac{1}{2\pi i}\int_{|s|=r}\alpha_{m,c}^{-}(s)\,\mathrm{Li}_{k-1}\left(e^{-s+4\pi v t}\zeta_c^{d}q\right)\,ds\right)dt.$$

Since $v < 0$ we can now use the series representation of Li_{k-1}. This yields

$$A_m^{-}(c,d) = \frac{(4\pi v)^{1-k}}{(-k)!}\int_1^{\infty} t^{-k}\left(\sum_{n\ge 1} n^{1-k}e^{4\pi n v t}\left(\zeta_c^{d}q\right)^n \frac{1}{2\pi i}\int_{|s|=r}\alpha_{m,c}^{-}(s)e^{-ns}\,ds\right)dt.$$

The innermost integral is (inserting the definition of $\alpha_{m,c}^{-}(s)$, making the change of variables $s \mapsto -s$, and using (3.2))

$$\begin{aligned} \frac{1}{2\pi i}\int_{|s|=r}\alpha_{m,c}^{-}(s)e^{-ns}ds &= \sum_{j\ge 0}\frac{1}{(j+1-k)!}\left(\frac{2\pi\sqrt{m}}{c}\right)^{2j+1-k}\frac{n^j}{j!} \\ &= n^{\frac{k-1}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned}$$

Thus

$$A_m^-(c,d) = \frac{(4\pi v)^{1-k}}{(-k)!}\sum_{n\geq 1} n^{\frac{1-k}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right)\left(\int_1^\infty t^{-k}e^{4\pi nvt}dt\right)\left(\zeta_c^d q\right)^n$$
$$= \sum_{n\geq 1} n^{\frac{k-1}{2}} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Gamma^*(1-k, 4\pi n|v|)\left(\zeta_c^d q\right)^n.$$

Therefore, using that $K(m,n;c)$ is real, we conclude that

$$H_{k,m}^-(\tau) = -\Gamma^*(1-k, 4\pi mv)q^{-m} + 2\pi(-1)^{\frac{k}{2}}\sum_{n\geq 1}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}$$
$$\times \sum_{\substack{c>0\\ N|c}} \frac{K(-m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Gamma^*(1-k, 4\pi n|v|)q^n$$
$$= -\frac{(4\pi m)^{1-k}}{(-k)!}P^*_{2-k,-m}(-\tau).$$

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On Denominators of Values of Certain L-Functions When Twisted by Characters

Siegfried Böcherer

Abstract We try to bound the denominators of standard L-functions attached to Siegel modular forms when we twist them by Dirichlet characters. Main tools are our modification of the doubling method (Böcherer et al., Ann. Inst. Fourier 50:1375–1443, 2000) together with its application to congruences by the method of Katsurada (Math. Z. 259:97–111, 2008) and integrality properties of Bernoulli numbers with characters.

1 Introduction

Katsurada [15] has shown that the primes appearing in the denominators of critical values for standard L-functions for Siegel modular forms are congruence primes. We emphasize that this is a notion concerning *eigenvalues of Hecke operators*. In the first part of the present paper we show that his method can be combined with some calculations from Böcherer and Schmidt [8] to get a similar result after twisting the L-function in question by a primitive Dirichlet character. This may be of independent interest; a similar statement appears in Katsurada [17]. As a consequence we get as a main result of this work a finiteness statement concerning the set of primes appearing in such denominators, when we vary over all Dirichlet characters. We are not aware of such a result for other L-functions except for the Riemann zeta function, where it follows from well known integrality properties of generalized Bernoulli numbers [9, 19]. We emphasize that the main method in our proof is to give an interpretation of the denominators in terms of congruence primes. From this, our finiteness statement is (almost) evident. It is not clear to us whether this is a special property of standard L-functions or whether one should expect such finiteness properties in general.

Katsurada's method seems not directly applicable to congruences for powers of prime ideals. To get also a result concerning the powers of the primes in

S. Böcherer (✉)
Kunzenhof 4B, 79117 Freiburg, Germany
e-mail: boecherer@math.uni-mannheim.de

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such denominators we switch from congruences for Hecke eigenvalues (this was Katsurada's main concern) to congruences for modular forms (i.e. simultaneous congruences for the Fourier coefficients of modular forms).

In the final section we indicate how our method can also be applied for other types of L-functions. In our exposition, we do not aim at greatest generality or at the strongest possible result, e.g. we focus on level one and on scalar-valued modular forms. Also, we refrain from any consideration concerning p-integrality statements for primes dividing the conductor of the Dirichlet character in question. This more delicate question may possibly be handled by within our framework by a careful bookkeeping concerning primes, which appear in the conductor of the character at hand.

The origin of this work is a discussion with H. Katsurada and T. Chida during my visit to Muroran in 2014. I want to thank Katsurada for several exchanges about this subject, starting from the Hakuba seminar 2012 on congruences; the seminar program organized by him was highly inspiring. Also discussions with S. Takemori and A. Raghuram were very helpful.

2 Preliminaries

2.1 General Notation

For a rational prime p we denote by ν_p the usual additive valuation on $\mathbb{Q}$, normalized by $\nu_p(p) = 1$. For a number field K let $\mathcal{O}$ be its ring of integers. We extend ν_p to a valuation $\nu_\mathfrak{p}$ on K, given by a prime ideal $\mathfrak{p}$ dividing the prime p. Note that such a valuation also makes sense when applied to a fractional ideal of K. We say that an element x of K is p-integral, if $\nu_\mathfrak{p}(x) \geq 0$ for all prime ideals $\mathfrak{p}$ dividing p.

2.2 Siegel Modular Forms

We refer to [1] and [10] for basic facts on Siegel modular forms. The proper symplectic similitude group $G^+Sp(n, \mathbb{R})$ acts on Siegel's upper half space $\mathbb{H}_n$ in the usual way. For an integer l we get an action of $G^+Sp(n, \mathbb{R})$ on functions g on $\mathbb{H}_n$ by

$$(g, M) \longmapsto g \mid_l M(Z) := \det(M)^{\frac{l}{2}} \det(CZ + D)^{-l} g(M < Z >)$$

with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

For a congruence subgroup Γ of $Sp(n, \mathbb{Z})$ the space $M^n_l(\Gamma)$ of Siegel modular forms of weight l for Γ consists of all holomorphic functions g on $\mathbb{H}_n$ which satisfy $f \mid_l \gamma = f$ for all $\gamma \in \Gamma$, with the usual additional condition in the cusps if $n = 1$. We write $S^n_l(\Gamma)$ for the subspace of cusp forms. The most convenient case for us will be "modular forms of nebentypus": Let N be a positive integer and ψ a Dirichlet character mod N; then we denote by $M_l^n(N, \psi)$ the space of all modular forms g satisfying $g \mid_l M = \psi(\det(D)) \cdot g$ for all

$$M \in \Gamma_0(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z} \mid C \equiv 0 \bmod N\}.$$

Such modular forms g have a Fourier expansion of type

$$g(Z) = \sum_{T \in \Lambda_{\geq}^n} a_g(T) q^T,$$

where $\Lambda_{\geq}^n$ denotes the set of all symmetric semiintegral positive semidefinite matrices of size n and q^T stands for $exp(2\pi i tr(TZ))$. Similarly, $\Lambda_{>}^n$ denotes the subset of all positive definite elements in $\Lambda_{\geq}^n$.

It is well known ([22], see also [11] for a different approach) that such a space has a basis with Fourier coefficients in the cyclotomic field generated by ψ and that the Fourier coefficients have bounded denominators. Furthermore, for any $\sigma \in Aut(\mathbb{C})$ and $g \in M_l^n(N, \psi)$ we have $g^\sigma \in M_l^n(N, \psi^\sigma)$ where $g^\sigma := \sum a_g^\sigma q^T$.

2.3 Eisenstein Series

We will use Siegel type Eisenstein series of even degree $2n$ with nebentypus: Let ψ be a primitive character mod N, $N > 1$ and k a positive integer with $\psi(-1) = (-1)^k$. We put

$$E_k^{2n}(Z, \psi, s) := \sum_{C,D} \psi(det(C)) det(CZ + D)^{-k} \mid det(CZ + D) \mid^{-2s} \det(Y)^s,$$

where (C, D) runs over second rows of matrices in $Sp(2n, \mathbb{Z})$ up to the action of $GL(2n, \mathbb{Z})$ from the left. This series is known to converge for $k + 2\Re(s) > 2n + 1$ and to have meromorphic continuation to $\mathbb{C}$ as a function of s. Moreover, for $k \geq n + 2$

$$E_k^{2n}(Z, \psi) := E_k^{2n}(Z, \psi, s)_{s=0}$$

defines a holomorphic modular form with Fourier coefficients in a cyclotomic number field.[1]

[1]Under additional conditions on n and ψ this is also true for $k = n + 1$.

2.4 Petersson Product

It is important to note that we always use an *unnormalized* version of the Petersson inner product, i.e. for $f, g \in S_l^n(\Gamma)$ we denote by $< f, g >_\Gamma$ the integral of $f \cdot \overline{g} \cdot \det(Y)^l$ over a fundamental domain for Γ; in the special case $\Gamma = \Gamma_0(N)$ we just write $< f, g >_N$; we omit the index $N = 1$.

2.5 Hecke Operators and L-Functions

For $f \in S_l^n(\Gamma_0(N))$ and $M \in GSp^+(\mathbb{Q}) \cap \mathbb{Z}^{2n,2n}$ with $\det(M)$ coprime to N we define the Hecke operator $\mathbb{T}(M)$ by

$$f \mid \mathbb{T}(M)(Z) := \det(M)^{\frac{l}{2}-\frac{n+1}{2}} \sum_\gamma f \mid_l \gamma,$$

where γ runs over representatives of $\Gamma_0(N)\backslash\Gamma_0(N) \cdot M \cdot \Gamma_0(N)$. We may choose γ in upper triangular form, i.e. $\gamma = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ and in this normalization, $\mathcal{T}(M)$ defines an endomorphism of $S_l^n(\Gamma_0(N))(\mathbb{Z})$ if $l > n$, see [20]. In this normalization, Hecke eigenvalues are algebraic integers [15].

If $f \in M_l^n(\Gamma_0(N))$ is an eigenform of all Hecke operators $\mathcal{T}(M)$ with $\det(M)$ a square and coprime to N, we can associate to it the standard-L-function

$$D^{(N)}(f, s) := \prod_{p \nmid N} \left(\frac{1}{1 - p^{-s}} \prod_{i=1}^{n} \frac{1}{1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s})} \right)$$

where the $\alpha_{i,p}$ are the Satake parameters attached to f. The analytic and number theoretic properties of such L-functions can be studied by the Rankin-Selberg-method [2, 24] or the doubling method [4, 23].

3 The Main Construction

In this section we describe a twisted version of Katsurada's setting in [15].

We start from the following data

- an integer $k \geq n + 2$
- a positive integer ν
- a primitive Dirichlet character χ modulo N

We put $l = k + \nu$ and we impose the condition $\chi(-1) = (-1)^k$.

The doubling method in its twisted version as in [8] allows us to construct explicitly a function $g^n_{k,\nu}(\chi) \in S^n_l \otimes S^n_l$ with

$$g^n_{k,\nu}(\chi)(z,w) = \sum_i \Lambda(f_i, k-n, \chi) f_i(z) \cdot f_i(w), \tag{1}$$

where the f_i run over an orthogonal basis of Hecke eigenforms in S^n_l and

$$\Lambda(f, k-n, \chi) := \Gamma(k-n) \prod_{i=1}^{n} \Gamma(2l-n-i) \frac{D(f, \chi, k-n)}{\pi^d < f, f >} \tag{2}$$

with

$$d = \frac{3}{2}n(n+1) - 2nk - n \cdot \nu - k.$$

This normalization coincides with the one in [15] for trivial character and—up to the Γ-factor and Gauss sums—with the one in [8, A11], where it is also shown that (2) behaves smoothly under the action of Aut($\mathbb{C}$) provided that we choose our eigenforms such that all their Fourier coefficients lie in the field generated by their Hecke eigenvalues. Such a choice is always possible and we will (sometimes tactitly) assume throughout this paper that all our eigenforms have this property.

For a given weight l, decomposed as $l = k - \nu$ with $\chi(-1) = (-1)^k$ we then cover all the positive critical weights of $D(f, s, \chi)$ except the smallest one (corresponding to $k = n+1$) and the largest one (corresponding to $l = k$), see [8].

We always keep n, k, ν fixed and study the variation of the values of these L-functions with χ.

We will show that $g^n_{k,\nu}(\chi)$ has Fourier coefficients in a cyclotomic field K_N and these Fourier coefficients are p-integral for all primes coprime to N and coprime to $(2k-n-1)!$.

3.1 The Construction of $g^n_{k,\nu}(\chi)$

We get $g^n_{k,\nu}(\chi)$ from the Eisenstein series $E^{2n}_k(\chi)$ in several step; our main source is [8].

First Step: Exterior Twist

We consider (following [8])

$$G^{2n}_k(Z, \chi) := \sum_X \chi(\det(X)) E^{2n}_k(Z, \chi) \mid \begin{pmatrix} 1_{2n} & S(\frac{X}{N}) \\ 0_{2n} & 1_{2n} \end{pmatrix},$$

where X runs over $\mathbb{Z}^{(n,n)}$ mod N and

$$S(X) := \begin{pmatrix} 0_n & X \\ X^t & 0_n \end{pmatrix} \in \mathbb{Z}^{(2n,2n)}.$$

The function G is then a modular form for a congruence subgroup Γ of level N^2 which contains $\Gamma_0(N^2) \times \Gamma_0(N^2)$ (no nebentypus anymore!).

Second Step: Differential Operators

We decompose $Z \in \mathbb{H}_{2n}$ into block matrices of size n as

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \qquad (z_3 = z_2^t)$$

The differential operators $\overset{o\ \nu}{\mathfrak{D}}_{n,k}$ in question are polynomials in the holomorphic derivatives, evaluated in $z_2 = 0$. They are considered in the work of Ibukiyama [13] or from a different point of view in [3]. They map C^∞-functions on $\mathbb{H}_{2n}$ to C^∞-functions on $\mathbb{H}_n \times \mathbb{H}_n$ and they satisfy

$$\overset{o\ \nu}{\mathfrak{D}}_{n,k}(F \mid_k \iota(g,h)) = \left(\overset{o\ \nu}{\mathfrak{D}}_{n,k} F\right) \mid^z_{k+\nu} g \mid^w_{k+\nu} h$$

for $g, h \in Sp(n, \mathbb{R})$. Here ι is the natural embedding of $Sp(n) \times Sp(n) \hookrightarrow Sp(2n)$, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ c & 0 & d & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

The upper indices z and w at the slash-operators indicate which variable is considered. We also note here, that these differential operators map modular forms to cusp forms, i.e. for $\nu > 0$ we get

$$\overset{o\ \nu}{\mathfrak{D}}_{n,k}(F) \in S^n_{k+\nu}(N) \otimes S^n_{k+\nu}(N)$$

for modular forms $F \in M^{2n}_k(N)$.

The differential operators are unique up to normalization, we normalize them as in [8, 15].

Third Step: Spectral Decomposition

We may compute the pullback of such Siegel Eisenstein series after applying "superior twists" and differential operators of the type above. For the calculation we refer to [8, section 3]. For a Dirichlet character χ mod N and a positive integer $k > n+1$ with $\chi(-1) = (-1)^k \chi(-1)$ and $l = k + \nu$ with $\nu > 0$ we have

$$\overset{o\ \nu}{\mathfrak{D}}_{n,k}(G_k^{2n}(*, \chi)(z, w)) =$$

$$\frac{\Omega_{l,\nu}}{\mathfrak{L}(k, \bar{\chi})} N^{n(2k+\nu-n-1)} N^{n(n+1)-nl}$$

$$\sum_g D(g \mid_l \begin{pmatrix} 0 & -1 \\ N^2 & 0 \end{pmatrix}, k-n, \chi) \frac{g \mid_l \begin{pmatrix} 0 & -1 \\ N^2 & 0 \end{pmatrix}(z) \cdot g(w)}{< g, g >_{N^2}}$$

where g runs over a basis of Hecke eigenforms in $S^l(N^2)$ and

$$\mathfrak{L}(s, \psi) := L(s, \psi) \prod_{i=1}^{n} L(2s - 2i, \psi^2).$$

The number $\Omega_{l,\nu}$ is a product of a power of π and certain values of the Γ-function.

3.2 *Forth Step: Level Change*

We decompose the space of cusp forms into a direct orthogonal sum:

$$S_l^n(N^2) = S_l^n \perp S_l^n(N^2)^o$$

and we use that $S_l^n(N^2)^o$ can be characterized as the kernel of the trace map

$$tr^{N^2} : \begin{cases} S_l^n(N^2) \longrightarrow & S_l^n \\ \quad g \quad \longmapsto & \sum_\gamma g \mid_l \gamma \end{cases},$$

where γ runs over $\Gamma_0(N^2)\backslash Sp(n, \mathbb{Z})$.
Then

$$tr_z^{N^2} \overset{o\ \nu}{\mathfrak{D}}_{n,k} G^{2n}(*, \chi)(z, w) =$$

$$\frac{\Omega_{l,\nu}}{\mathfrak{L}(k, \bar{\chi})} N^{n(2k+\nu-n-1)} N^{n(n+1)-nl} \sum_g D(g, k-n, \chi) \frac{g(z) g(N^2 \cdot w)}{< g, g >}.$$

The summation goes now over an orthogonal basis of S_l^n consisting of Hecke eigenforms.

Our candidate for the function $g_{k,\nu}^n(\chi) \in S_l^n \otimes S_l^n$ is then—up to normalization

$$tr_z^{N^2} \overset{o\ \nu}{\mathfrak{D}}_{n,k} G^{2n}(*, \chi)(\frac{1}{N^2} \cdot z, w).$$

Fifth Step: Normalization

We follow Katsurada [15] in normalization the function $g_{k,\nu}^n$ appropriately such that Eq. (1) holds. In the case of trivial character Katsurada showed that with this normalization the function $g_{k,\nu}^n(id)$ has rational Fourier coefficients and it is p-integral for all primes $p > (2k-n-1)!$.

An analogous statement for of this is

Proposition *The Fourier coefficients of $g_{k,\nu}^n(\chi)$ are in a cyclotomic field K_N generated by the values of χ and the Nth roots of unity. The Fourier coefficients are p-integral for p coprime to N and $p > (2k-n-1)!$.*

One can follow the lines of Katsurada's reasoning; we should use integrality properties of Bernoulli numbers with character instead of the usual Clausen-von Staudt property (see [9, 19]). Furthermore one has to observe that in step 4 the integrality properties outside primes dividing N are preserved when taking the trace. This follows from the q-expansion principal (see e.g. [14]) or by a careful analysis of Fourier expansions of Eisenstein series in all cusps, see [18].

4 Congruence Primes and Denominators

Let $f \in S_l^n$ be a Hecke eigenform and $\mathcal{M}$ a subspace of S_l^n stable under the Hecke algebra. We assume that $\mathcal{M}$ is contained in the orthogonal complement $(\mathbb{C} \cdot f)^\perp$ of $\mathbb{C} \cdot f$ with respect to the Petersson product. We denote by $\mathbb{Q}(f)$ the algebraic number field generated by the Hecke eigenvalues of f. A prime ideal $\mathfrak{p}$ in $\mathcal{O}_{\mathbb{Q}(f)}$ is called a congruence prime of f with respect to $\mathcal{M}$ if there exists a Hecke eigenform g in $\mathcal{M}$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \mathfrak{P}$$

for all integral linear combinations of Hecke operators $\mathbb{T}(M)$, where $\mathfrak{P}$ is some prime ideal in $\mathcal{O}_{\mathbb{Q}(f)\cdot\mathbb{Q}(g)}$. If $\mathcal{M} = (\mathbb{C}f)^\perp$ we call $\mathfrak{p}$ a congruence prime for f.

With the function $g_{k,\nu}^n(\chi)$ and their properties at hand, we may now follow Katsurada's procedure [15, 16] line by line.

Theorem (Version 1) *Let $f \in S_l^n$ be among the eigenforms above with Fourier coefficients in the field $\mathbb{Q}(f)$ generated by the Hecke eigenvalues. We define the fractional ideal $\mathfrak{I}_f$ in $\mathbb{Q}(f)$ generated by all the Fourier coefficients of f. Let χ be* *a character mod N and let $\mathfrak{P}$ be a prime ideal in the composite field generated by $\mathbb{Q}(f)$ and K_N and assume that it does not divide $(2k-n-1)!$ and does not divide N. Assume that*

$$\nu_{\mathfrak{P}}(\Lambda(f,k-n,\chi)\mathfrak{I}_f^2) < 0$$

then $\mathfrak{P}$ is a congruence prime for f.

This version appears also in [17]. It is not sensitive for multiplicity problems. Katsurada [16] refined his procedure by starting from a system λ of Hecke eigenvalues and considering an orthogonal basis $(f_1,\ldots,f_t)$ of the corresponding eigenspace $S_l^n(\lambda)$. We denote the Fourier expansions by

$$f_i(z) = \sum_{T\in\Lambda_>^n} a_i(T)q^T$$

and we define

$$I_{T,S} := \sum_{i=1}^{t} \frac{a_i(S)\overline{a_i(T)}\tilde{\Lambda}(f_i,k-n,\chi)}{<f_i,f_i>},$$

where

$$\Lambda(f_i,k-n,\chi) = \frac{\tilde{\Lambda}(f_i,k-n,\chi}{<f_i,f_i>}.$$

We consider the fractional ideal $\mathfrak{I}$ in the composite field $\mathbb{Q}(f_1,N)$ of $\mathbb{Q}(f_1)$ and K_N generated by all the $I_{S,T}$. This only depends on the system λ and on χ.

Theorem (Version 2) *For a system λ of Hecke eigenvalues and $f\in S_l^n(\lambda)$ and a character χ* mod *N we define the fractional ideal $\mathfrak{I}$ as above. Let $\mathfrak{P}$ be a prime ideal in the field $\mathbb{Q}(f,N)$ and assume that $\mathfrak{P}$ is coprime to N and coprime to* $(2k-n-1)!$. *If*

$$\nu_{\mathfrak{P}}(\mathfrak{I}) < 0$$

then $\mathfrak{P}$ is a congruence prime with respect to $S_l^n(\lambda)^{\perp}$.

Clearly, the number of such congruence primes has to be finite. As the main result of this paper we get a finiteness statement concerning the primes which may appear in the denominators of critical values of standard-L-functions, when twisted by Dirichlet characters:

Finiteness Theorem For an eigenspace $S_l^n(\lambda)$ and a decomposition $l = k+\nu$ the set

$$\{p \mid \exists\chi : p \nmid \mathrm{cond}(\chi)\,\exists\mathfrak{P} \mid p : \nu_{\mathfrak{P}}(\mathfrak{I}) < 0\}$$

is a finite set of primes.

Remark We should mention that for elliptic modular forms Hida [12] gives a characterization of congruence primes using the numerator of a value of the standard L-function. We emphasize that the value considered by Hida is *not* among the values considered in our work.

5 Congruences for Modular Forms

In sequel, it will be sufficient to use a "naive" notion of congruences: Let K be a number field and $\mathcal{A}$ an integral ideal in K. We call two modular forms $f = \sum a_f(T)q^T$ and $g = \sum a_g(T)q^T$ with Fourier coefficients in K congruent mod $\mathcal{A}$ if $a_f(T) - a_g(T) \in \mathcal{A}$ for all T. This is a naive notion, because (unlike the more sophisticated notion introduced in [21]) it considers modular forms to be congruent mod $\mathcal{A}$ even if all the Fourier coefficients of f and g are in $\mathcal{A}$.

A modular form should not satisfy "too many" nontrivial congruences; a precise version of this somewhat vague statement, suitable for our situation, is given below: It will be our main technical tool in proving a stronger finiteness statement.

Proposition *Let $f, g_1, \ldots, g_t$ be $t+1$ fixed, linearly independent Siegel modular forms with integral Fourier coefficients in an algebraic number field K. There exists a nonzero integral element $a \in K$ such that for all finite extensions L/K and all integral ideals $\mathfrak{A}$ in L the following property holds:*

Any congruence

$$f \equiv \sum \alpha_i g_i \bmod \mathfrak{A}$$

with arbitrary coefficients $\alpha_i \in L$ implies

$$a \in \mathfrak{A}, \quad i.e. \quad a \in \mathfrak{a} := \mathfrak{A} \cap \mathcal{O}_K$$

Proof Whenever convenient we use g_0 for f; we also write $a_i(T)$ for $a_{g_i}(T)$. For $T \in \Lambda^n_>$ we denote by ϵ_T the linear form on S^n_l, which maps $g \in S^n_l$ to its Fourier coefficient $a_g(T)$. By linear algebra we may choose $T_0, \ldots, T_t \in \Lambda^n_>$ such that the linear forms $\epsilon_{T_1}, \ldots, \epsilon_{T_t}$ provide an isomorphism between $\oplus_{i=1}^t \mathbb{C} \cdot g_i$ and $\mathbb{C}^t$ and $\epsilon_{T_0}, \ldots, \epsilon_{T_t}$ provide an isomorphism from $\oplus_{i=0}^t \mathbb{C} \cdot g_i$ and $\mathbb{C}^{t+1}$. In particular, the matrices $R := \left(a_i(T_j)\right)_{1 \le i,j \le t}$ and $\tilde{R} := \left(a_i(T_j)\right)_{0 \le i,j \le t}$ are then invertible matrices in $GL(t, K) \cap M_t(\mathcal{O}_K)$ and in $GL(t+1, K) \cap M_{t+1}(\mathcal{O}_K)$ respectively.

The simple formula

$$\det(\begin{pmatrix} x & \mathfrak{x} \\ \mathfrak{y} & R \end{pmatrix}) = \det(R) \cdot \left(x - \mathfrak{x} \cdot R^{-1} \cdot \mathfrak{y}^t\right),$$

valid for R as above and any $x \in K, \mathfrak{y}, \mathfrak{x}^t \in K^t$ implies that

$$a' := \frac{\det(\tilde{R})}{\det(R)} = a_f(T_0) - (a_1(T_0), \dots a_t(T_0)) \cdot R^{-1} \cdot \begin{pmatrix} a_f(T_1) \\ \vdots \\ a_f(T_t) \end{pmatrix} \neq 0 \tag{3}$$

Starting from a congruence $f \equiv \sum_{i=1}^{t} \alpha_i g_i \bmod \mathfrak{A}$ we get t+1 equations

$$a_f(T_0) = \sum_{i=1}^{t} \alpha_i a_i(T_0) + x_0 \tag{4}$$

$$a_f(T_j) = \sum_{i=1}^{t} \alpha_i a_i(T_j) + x_j \qquad (1 \leq j \leq t) \tag{5}$$

with certain elements $x_0 \dots, x_t \in \mathfrak{A}$

We choose a nonzero element $c \in \mathcal{O}_K$ such that

$$c \cdot R^{-1} \in M_t(\mathcal{O}_K).$$

Then the system (5) implies

$$c \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix} = c \cdot R^{-1} \cdot \begin{pmatrix} a_f(T_1) \\ \vdots \\ a_f(T_t) \end{pmatrix} + cR^{-1} \cdot \mathfrak{x}$$

with $\mathfrak{x} = (x_1, \dots, x_t)^t$.

We may then rewrite (4) as

$$c \cdot a_f(T_0) = c \cdot \sum_{i=1}^{t} \alpha_i \cdot a_i(T_0) + c \cdot x_0$$

$$= \big(a_(T_0), \dots, a_t(T_0)\big) \cdot \left(c \cdot R^{-1} \cdot \begin{pmatrix} a_f(T_1) \\ \vdots \\ a_f(T_t) \end{pmatrix} + c \cdot R^{-1} \cdot \mathfrak{x} \right) + c \cdot x_0$$

This implies

$$c \cdot a' \in \mathfrak{A} \cap K$$

and $a := c \cdot a'$ has the requested property.

Remark It should be clear that in the proposition above, we may weaken the integrality conditions to "integrality outside a finite set of prime ideals".

Remark As was pointed out to me by Takemori, the special case $L = K$ and $\mathfrak{A} = \mathfrak{p}^r$ with a prime ideal $\mathfrak{p}$ in K should follow from a $\mathfrak{p}$-adic completeness statement for subspaces in a finite-dimensional $\mathfrak{p}$-adic vector space.

6 Denominators Again

How to use the function $g^n_{k,\nu}(\chi)$ as a source of congruences as in the proposition? We use the same normalization as Katsurada involving $\mathfrak{I}_f^2$. For a fixed $S \in \Lambda^n_>$ we get a function $g^n_{k,\nu}(\chi)_S$ by taking the S-Fourier coefficient of $g^n_{k,\nu}(\chi)$ w.r.t. w; this gives

$$g^n_{k,\nu}(\chi)(Z) = \sum_i a_{f_i}(S)\Lambda(f_i, k-n, \chi) \cdot f_i(z).$$

We are interested in the values of the L-function attached to $f = f_1$.

We define $\mathbb{Q}(l)$ to be the number filed generated by all the Hecke eigenvalues appearing in S^n_l and we define $\mathbb{Q}(N, l)$ to be the composite of $\mathbb{Q}(l)$ and K_N. We fix a prime ideal $\mathfrak{P}$ in $\mathbb{Q}(N, l)$ and we choose $S_0 \in \Lambda^n_>$ such that

$$\nu_{\mathfrak{P}}(a_f(S_0)) = \mathrm{Min}(\{\nu_{\mathfrak{P}}(a_f(S)) \mid S \in \Lambda^n_>\}).$$

Then

$$\frac{f}{a_f(S_0)} + \sum_{i=1}^{t} \frac{a_{f_i}(S_0) \cdot \Lambda(f_i, k-n, \chi)}{a_f(S_0)^2 \cdot \Lambda(f, k-n, \chi)} \cdot f_i = \frac{g^n_{k,\nu}(\chi)_S}{a_f(S_0)^2\Lambda(f, k-n, \chi)}$$

If $\nu_{\mathfrak{P}}(\mathfrak{I}_f^2 \cdot \Lambda(f, k-n, \chi)) = -r < 0$ for some prime ideal $\mathfrak{P}$ in $\mathbb{Q}(N, l)$, then we get a congruence for f as in the proposition (in view of the integrality properties of the numerator $g^n_{k,\nu}(\chi)$ on the right side), at least if the underlying rational prime p does not divide $(2k-n-1)!$. Even stronger, taking into account that $\nu_{\mathfrak{P}}(\frac{f}{a_f(S_0)}) = 0$, we get a congruence mod $\mathfrak{P}^r$ for $\frac{f}{a_f(S_0)}$ in the strong sense.

Our proposition tells us that this can happen only for finitely many prime ideals $\mathfrak{p}$ in $\mathbb{Q}(l)$ (with nontrivial power r), more precisely, we get

Theorem *For $l \in \mathbb{N}$ with $l = k + \nu$, $k \geq n+2$ and a fixed Hecke eigenform $f \in S^n_l$ the set*

$$\{p \mid \exists \chi : p \nmid cond(\chi) \, \exists \mathfrak{P} \mid p : \nu_{\mathfrak{P}}(\mathfrak{I}_f^2 \Lambda(f, k-n, \chi)) < 0\}$$

is a finite set and there exists $r > 0$: For all $p, \chi, \mathfrak{P}$ from above: we have

$$\nu_{\mathfrak{P}}(\mathfrak{I}_f^2 \Lambda(f, k-n, \chi)) \geq -r.$$

7 Outlook

Remark One may ask about the primes dividing the conductor of χ. The most delicate point is then the fourth step in our construction of $g_{k,\nu}^n$, in particular the p-integrality. By a variant of our strategy it is possible to bound the power of p (or rather of prime ideals dividing p, which can then appear in the denominator of the twisted L-function; it remains open whether that power of p can be bounded independently of p.

Remark The method exhibited here works in a similar style for triple product L-functions following the lines of [6, 7], using the observation that denominators of (suitably normalized) critical values of triple product L-functions give rise to congruence primes, see [5]; this will be worked out in detail in future work. More generally, our approach should work in all situations, where integral representations of L-functions are obtained by restricting Siegel type Eisenstein series, e.g. Hermitian modular forms or Siegel modular forms of half-integral weight, see e.g. [23].

Remark It is possible to include the smallest positive critical value in our procedure, if we exclude quadratic characters. The largest critical value is somewhat delicate because here there is no differential operator to kill the noncuspidal contribution in this situation (the second step in our construction is trivial. i.e. $\nu = 0$.)

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First Order p-Adic Deformations of Weight One Newforms

Henri Darmon, Alan Lauder, and Victor Rotger

Abstract This article studies the first-order p-adic deformations of classical weight one newforms, relating their fourier coefficients to the p-adic logarithms of algebraic numbers in the field cut out by the associated projective Galois representation.

Introduction

Let g be a classical cuspidal newform of weight one, level N and nebentypus character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}_p^\times$, with fourier expansion

$$g(q) = \sum_{n=1}^{\infty} a_n q^n.$$

The *p-stabilisations* of g attached to a rational prime $p \nmid N$ are the eigenforms of level Np defined by

$$g_\alpha(q) := g(q) - \beta \cdot g(q^p), \qquad g_\beta(q) := g(q) - \alpha \cdot g(q^p), \tag{1}$$

where α and β are the (not necessarily distinct) roots of the Hecke polynomial

$$x^2 - a_p x + \chi(p) =: (x - \alpha)(x - \beta).$$

H. Darmon (✉)
McGill University, Montreal, QC, Canada
e-mail: darmon@math.mcgill.ca

A. Lauder
University of Oxford, Oxford, UK
e-mail: lauder@maths.ox.ac.uk

V. Rotger
Universitat Politècnica de Catalunya, Barcelona, Spain
e-mail: victor.rotger@upc.edu

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The forms g_α and g_β are eigenvectors for the Atkin U_p operator, with eigenvalues α and β respectively. Since α and β are roots of unity, these eigenforms are both *ordinary* at p.

An important feature of classical weight one forms is that they are associated to odd, irreducible, two-dimensional Artin representations, via a construction of Deligne-Serre. Let $\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(\mathbb{C})$ denote this Galois representation, and write V_g for the underlying representation space.

A fundamental result of Hida asserts the existence of a *p-adic family* of ordinary eigenforms specialising to g_α (or to g_β) in weight one. Bellaiche and Dimitrov [1] later established the uniqueness of this Hida family, under the hypothesis that g is *regular at* p, i.e., that $\alpha \neq \beta$, or equivalently, that the frobenius element at p acts on V_g with distinct eigenvalues. In the intriguing special case where g is the theta series of a character of a real quadratic field F in which the prime p is split, the result of Bellaiche-Dimitrov further asserts that the unique ordinary first-order infinitesimal p-adic deformation of g is an overconvergent (but not classical) modular form of weight one. In [3], the Fourier coefficients of this non-classical form were expressed as p-adic logarithms of algebraic numbers in a ring class field of F, suggesting that a closer examination of such deformations could have some relevance to explicit class field theory for real quadratic fields.

The primary purpose of this note is extend the results of [3] to general weight one eigenforms.

Part A considers the regular setting where $\alpha \neq \beta$, in which the results exhibit a close analogy to those of [3].

Part B takes up the case where g is irregular at p. Here the results are more fragmentary and less definitive. Let $S_1^{(p)}(N, \chi)$ denote the space of p-adic overconvergent modular forms of weight 1, level N, and character χ, and let $S_1^{(p)}(N, \chi)[[g]]$ denote the generalised eigenspace attached to the system of Hecke eigenvalues of an irregular weight one form $g \in S_1(N, \chi)$. The main conjecture of the second part asserts that $S_1^{(p)}(N, \chi)[[g]]$ is always four dimensional, with a two-dimensional subspace consisting of classical forms. Under this conjecture, an explicit description of the elements of the generalised eigenspace in terms of their q-expansions is provided. The resulting concrete description of the generalised eigenspace that emerges from Part B is an indispensable ingredient in the extension of the "elliptic Stark conjectures" of [4] to the irregular setting that will be presented in [5].

Part A: The Regular Setting

Let $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ denote the Iwasawa algebra, and let

$$\mathcal{W} := \mathrm{Hom}_{\mathrm{cts}}(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, \mathbb{C}_p)$$

denote the associated *weight space*. For each $k \in \mathbb{Z}_p$, write $\nu_k \in \mathcal{W}$ for the "weight k" homomorphism sending the group-like element $a \in 1 + p\mathbb{Z}_p$ to a^{k-1}. The rule $\lambda(k) := \nu_k(\lambda)$ realises elements of Λ as analytic functions on $\mathbb{Z}_p$. The spectrum $\tilde{\mathcal{W}} := \mathrm{Hom}_{\mathrm{alg}}(\tilde{\Lambda}, \mathbb{C}_p)$ of a finite flat extension $\tilde{\Lambda}$ of Λ is equipped with a "weight map"

$$\mathrm{w} : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}$$

of finite degree. A $\mathbb{Q}_p$-valued point $x \in \tilde{\mathcal{W}}$ is said to be *of weight k* if $\mathrm{w}(x) = \nu_k$, and is said to be étale over $\mathcal{W}$ if the inclusion $\Lambda \subset \tilde{\Lambda}$ induces an isomorphism between Λ and the completion of $\tilde{\Lambda}$ at the kernel of x, denoted $\tilde{\Lambda}_x$. An element of this completion thus gives rise to an analytic function of $k \in \mathbb{Z}_p$ in a natural way.

A *Hida family* is a formal q-series

$$\mathbf{g} := \sum a_n q^n \in \tilde{\Lambda}[[q]]$$

with coefficients in a finite flat extension $\tilde{\Lambda}$ of Λ, specialising to a classical ordinary eigenform of weight k at almost all points x of $\tilde{\mathcal{W}}$ of weight $k \in \mathbb{Z}^{\geq 2}$. Two Hida families $\mathbf{g}_1 \in \tilde{\Lambda}_1[[q]]$ and $\mathbf{g}_2 \in \tilde{\Lambda}_2[[q]]$ are regarded as equal if the Λ-algebras $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ can be embedded in a common extension $\tilde{\Lambda}$ in such a way that $\mathbf{g}_1$ and $\mathbf{g}_2$ are identified. A well known theorem of Hida and Wiles asserts the existence of a Hida family specialising to g_α in weight one. The following uniqueness result for this Hida family plays an important role in our study.

Theorem (Bellaiche, Dimitrov) *Assume that the weight one form g is regular at p, and let x_α and x_β denote the distinct points on $\tilde{\mathcal{W}}$ attached to g_α and g_β respectively. Then*

(a) *The curve $\tilde{\mathcal{W}}$ is smooth at x_α and x_β, and in particular there are unique Hida families $\mathbf{g}_\alpha, \mathbf{g}_\beta \in \tilde{\Lambda}[[q]]$ specialising to g_α and g_β at x_α and x_β respectively.*
(b) *The weight map $\mathrm{w} : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}$ is furthermore étale at x_α if any only if*

(†) *g is not the theta series of a character of a real quadratic K in which p splits.*

The setting where g is regular at p but w is not étale at x_α has been treated in [3], and the remainder of Part A will therefore focus on the scenarios where (†) is satisfied. In that case, after viewing elements of the completion $\tilde{\Lambda}_{x_\alpha}$ of $\tilde{\Lambda}$ at x_α as analytic functions of the "weight variable" k, one may consider the *canonical* q-series

$$g'_\alpha := \left(\frac{d}{dk} \mathbf{g}_\alpha \right)_{k=1}$$

representing the first-order infinitesimal ordinary deformation of $\mathbf{g}$ at the weight one point x_α, along this canonical "weight direction". The q-series g'_α is analogous to the overconvergent generalised eigenform considered in [3], with the following differences:

(a) While the overconvergent generalised eigenform of [3] is a (non-classical, but overconvergent) modular form of weight one, such an interpretation is not available for the q-series g'_α, which should rather be viewed as the first order term of a "modular form of weight $1 + \varepsilon$".
(b) In the non-étale setting of [3], the absence of a natural local coordinate with respect to which the derivative would be computed meant that the overconvergent generalised eigenform of loc.cit. could only be meaningfully defined up to scaling by a non-zero multiplicative factor. This ambiguity is not present in the definition of g'_α, whose fourier coefficients are therefore entirely well-defined.

The main results of Part A give explicit formulae for these fourier coefficients: they are stated in Theorems 10, 14, 16, and 19 below.

1 The General Case

The goal of this section is to describe a general formula for the fourier coefficients of g'_α.

The Artin representation V_g can be realised as a two-dimensional L-vector space, where L is a finite extension of $\mathbb{Q}$, contained in a cyclotomic field. Let $W_g = \hom(V_g, V_g)$ denote the adjoint equipped with its usual conjugation action of $G_\mathbb{Q}$, denoted

$$\sigma \cdot w := \varrho_g(\sigma) w \varrho_g(\sigma)^{-1}, \qquad \sigma \in G_\mathbb{Q}, \quad w \in W_g.$$

Let $H \subset H_g$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by the representations W_g and V_g respectively, and write $G := \mathrm{Gal}\,(H/\mathbb{Q})$.

For notational simplicity, the following assumption is made in the rest of this paper:

Assumption 1 *The prime p splits completely in the field L of coefficients of the Artin representation V_g.*

This assumption amounts to a simple congruence condition on p. The choice of an embedding of L into $\mathbb{Q}_p$, which is fixed henceforth, will allow us, when it is convenient, to view V_g and W_g as representations of $G_\mathbb{Q}$ with coefficients in $\mathbb{Q}_p$, and the weight one form g as a modular form with fourier coefficients in $\mathbb{Q}_p$ rather than in L. The $\mathbb{Q}_p$-vector spaces V_g and W_g are thus equipped with natural $G_\mathbb{Q}$-stable L-rational structures, denoted V_g^L and W_g^L respectively.

An embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ is fixed once and for all, determining a prime $\wp$ of H and of H_g above p, and an associated frobenius element $\tau_\wp$ in Gal $(H_g/\mathbb{Q})$ and in G. Let $G_\wp \subset G$ be the decomposition subgroup generated by $\tau_\wp$.

The representations V_g and W_g admit the following decompositions as $\tau_\wp$-modules:

$$V_g = V_g^\alpha \oplus V_g^\beta, \qquad W_g = W_g^{\alpha\alpha} \oplus W_g^{\alpha\beta} \oplus W_g^{\beta\alpha} \oplus W_g^{\beta\beta},$$

where V_g^α and $\mathcal{V}_g^\beta$ denote the α and β-eigenspaces for the action of $\tau_\wp$ on V_g, and

$$W_g^{\xi\eta} := \hom(V_g^\xi, V_g^\eta), \qquad \text{for } \xi, \eta \in \{\alpha, \beta\}$$

is a $G_\wp$-stable line, on which $\tau_\wp$ acts with eigenvalue η/ξ. Let

$$W_g^{\text{ord}} := \hom(V_g/V_g^\alpha, V_g) = W_g^{\beta\alpha} \oplus W_g^{\beta\beta}.$$

Of course, W_g^{ord} is stable under the action of $G_\wp$ but not under the action of G.

We propose to give a general formula for the ℓth fourier coefficient of g'_α as the trace of a certain explicit endomorphism of V_g, which is constructed via a series of lemmas. In the lemma below, we let G act on $\mathcal{O}_H^\times \otimes W_g$ diagonally on both factors in the tensor product.

Lemma 2 *The $\mathbb{Q}_p$-vector space $(\mathcal{O}_H^\times \otimes W_g)^G$ of G-invariant vectors is one-dimensional.*

Proof Let G_∞ be the subgroup of G generated by a complex conjugation c, which has order two, since V_g is odd. By Dirichlet's unit theorem, the global unit group $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ is isomorphic to $\mathrm{Ind}_{G_\infty}^G(\mathbb{Q}_p) - \mathbb{Q}_p$ as a $\mathbb{Q}_p[G]$-module. Let W_g^0 denote the three-dimensional representation of G consisting of trace zero endomorphisms of V_g. As a representation of G, we have $W_g = W_g^0 \oplus \mathbb{Q}_p$, and W_g^0 does not contain the trivial representation as a constituent. By Frobenius reciprocity,

$$\dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W_g)^G) = \dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W_g^0)^G) = \dim_{\mathbb{Q}_p}((W_g^0)^{c=1}) = 1.$$

The result follows. □

Assume that the field L of coefficients is large enough so that the semisimple ring $L[G]$ becomes isomorphic to a direct sum of matrix algebras over L. The $L[G]$-module $\mathcal{O}_H^\times \otimes L$ decomposes as a direct sum of V-isotypic components,

$$\mathcal{O}_H^\times \otimes L = \oplus_V \mathcal{O}_H^\times[V],$$

where V runs over the irreducible representations of G, and $\mathcal{O}_H^\times[V]$ denotes the largest subrepresentation of $\mathcal{O}_H^\times \otimes L$ which is isomorphic to a direct sum of copies of V as an $L[G]$-module. For a general, not necessarily irreducible, representation W, the module $\mathcal{O}_H^\times[W]$ is defined as the direct sum of the $\mathcal{O}_H^\times[V]$ as V ranges over the irreducible constituents of W. Because W_g (viewed, for now, as a representation with coefficients in L) is self-dual, Lemma 2 can be recast as the assertion that $\mathcal{O}_H^\times[W_g]$ is isomorphic to a single irreducible subrepresentation of W_g. More precisely:

- In the case of "exotic weight one forms" where ϱ_g has non-dihedral projective image (isomorphic to A_4, S_4 or A_5), then
$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[W_g^0] \simeq W_g^0, \tag{2}$$
and hence is three-dimensional.
- If ϱ_g is induced from a character ψ_g of an imaginary quadratic field K, then
$$W_g = L \oplus L(\chi_K) \oplus V_\psi,$$
where χ_K is the odd quadratic Dirichlet character associated to K and V_ψ is the two-dimensional representation of G induced from the ring class character $\psi = \psi_g/\psi_g'$ which cuts out the abelian extension H of K. The representation V_ψ is irreducible if and only if ψ is non-quadratic, and in that case,
$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[V_\psi] \simeq V_\psi. \tag{3}$$
In the special case where ψ is quadratic, the representation V_ψ further decomposes as the direct sum of one-dimensional representations attached to an even and an odd quadratic Dirichlet character, denoted χ_1 and χ_2 respectively. That special case, in which V_g is also induced from a character of the real quadratic field cut out by χ_1, is thus subsumed under (4) below.
- If ϱ_g is induced from a character ψ_g of a real quadratic field F, then
$$W_g = L \oplus L(\chi_F) \oplus V_\psi, \qquad V_\psi := \mathrm{Ind}_F^{\mathbb{Q}}(\psi), \quad \psi := \psi_g/\psi_g',$$
and one always has
$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[\chi_F] \simeq L(\chi_F), \tag{4}$$
i.e., $\mathcal{O}_H^\times[W_g]$ is generated by a fundamental unit of F.

Let $U_g^\times$ be any generator of the one-dimensional $\mathbb{Q}_p$-vector space $(\mathcal{O}_H^\times \otimes W_g)^G$ and let
$$U_g := (\log_\wp \otimes \mathrm{id})(U_g^\times) \in H_\wp \otimes W_g \tag{5}$$
be the image of this vector under the linear map
$$\log_\wp \otimes \mathrm{id} : \mathcal{O}_H^\times \otimes W_g \longrightarrow H_\wp \otimes W_g,$$
where $\log_\wp$ is the p-adic logarithm on the $\wp$-adic completion $H_\wp$ of H at $\wp$.

Lemma 3 *There exists a non-zero endomorphism $A \in H_\wp \otimes W_g$ satisfying the following conditions:*

(a) $\mathrm{Trace}(AU_g) = 0$.
(b) *A belongs to $H_\wp \otimes W_g^{\mathrm{ord}}$, i.e., $A(V_g^\alpha) = 0$.*

This endomorphism is unique up to scaling.

Proof The space $H_\wp \otimes W_g$ is four-dimensional over $H_\wp$ and the conditions in Lemma 3 amount to three linear conditions on A. More precisely, choose a $\tau_\wp$-eigenbasis (v_α, v_β) for V_g for which

$$\tau_\wp v_\alpha = \alpha v_\alpha, \qquad \tau_\wp v_\beta = \beta v_\beta.$$

Relative to this basis, the endomorphism U_g is represented by a matrix of the form

$$U_g : \begin{pmatrix} \log_\wp(u_1) & \log_\wp(u_{\beta/\alpha}) \\ \log_\wp(u_{\alpha/\beta}) & -\log_\wp(u_1) \end{pmatrix},$$

where $u_1, u_{\alpha/\beta}$, and $u_{\beta/\alpha}$ are generators of $\mathcal{O}_H^\times[W_g]$ which (when non-zero) are eigenvectors for $\tau_\wp$, satisfying

$$\tau_\wp(u_1) = u_1, \qquad \tau_\wp(u_{\beta/\alpha}) = (\beta/\alpha)u_{\beta/\alpha}, \qquad \tau_\wp(u_{\alpha/\beta}) = (\alpha/\beta)u_{\alpha/\beta}.$$

The endomorphism A satisfies condition (b) above if and only if the matrix representing it in the basis (v_α, v_β) is of the form

$$A : \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}, \qquad x, y \in H_\wp,$$

and condition (a) implies the further linear relation

$$\log_\wp(u_{\alpha/\beta}) \cdot x - \log_\wp(u_1) \cdot y = 0. \tag{6}$$

The injectivity of $\log_\wp : \mathcal{O}_H^\times \otimes L \longrightarrow H_\wp$, which follows from the linear independence over $\bar{\mathbb{Q}}$ of logarithms of algebraic numbers, implies that the coefficients $\log_\wp(u_{\alpha/\beta})$ and $\log_\wp(u_1)$ in (6) vanish simultaneously if and only if

$$u_{\alpha/\beta} = u_1 = 0$$

in $\mathcal{O}_H^\times \otimes L$, i.e., if and only if $\mathcal{O}_H^\times[W_g]$ is one-dimensional over L and generated by $u_{\beta/\alpha}$. This immediately rules out (2) and (3) as scenarios for the structure of $\mathcal{O}_H^\times[W_g]$, leaving only (4). Hence, V_g is induced from a character of a real quadratic field F. In that case, the lines spanned by $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ are interchanged under the action of any reflection in G, and hence the condition $u_{\alpha/\beta} = 0$ implies that $u_{\beta/\alpha} = 0$ as well, thus forcing the vanishing of the full $\mathcal{O}_H^\times[W_g]$. This contradiction to Lemma 2 shows that (6) imposes a non-trivial linear condition on x and y, and therefore that A is unique up to scaling. □

Lemma 4 *Let A be any element of $H_\wp \otimes W_g$ satisfying the conditions in Lemma 3. Then the following are equivalent:*

(a) $\mathrm{Trace}(A) \neq 0$*;*
(b) *The representation ϱ_g is not induced from a character of a real quadratic field in which the prime p splits.*

Proof The vanishing of $\mathrm{Trace}(A)$ is equivalent to the vanishing of the entry y in (6), and hence to the vanishing of $\log_\wp(u_{\alpha/\beta})$, and therefore of $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ as well. This implies that $\mathcal{O}_H^\times[W_g]$ is one-dimensional and generated by u_1. As in the proof of Lemma 3, this rules out (2) and (3), leaving only (4) as a possibility, i.e., V_g is necessarily induced from a character of a real quadratic field F. Furthermore, $\tau_\wp$ fixes the group $\mathcal{O}_H^\times[W_g]$ generated by the fundamental unit of F, which occurs precisely when p splits in F. The lemma follows. □

Assume from now on that the equivalent conditions of Lemma 4 hold. One can then define $A_g \in H_\wp \otimes W_g$ to be the unique $H_\wp^\times$-multiple of A satisfying

$$\mathrm{Trace}(A_g) = 1.$$

As in Lemma 2, $H_\wp \otimes W_g$ is endowed with the diagonal action of $G_\wp$ which acts on both $H_\wp$ and on W_g in a natural way. Given $A \in H_\wp \otimes W_g$ and $\sigma \in G_\wp$, let us write ${}^\sigma A$ for the image of A by the action of σ on the first factor $H_\wp$, and $\sigma \cdot A_g$ for the image of A by the action of σ by conjugation on the second factor W_g.

Lemma 5 *The endomorphism A_g belongs to the space $(H_\wp \otimes W_g)^{G_\wp}$ of $G_\wp$-invariants for the diagonal action of $G_\wp$ on $H_\wp \otimes W_g$, i.e.,*

$${}^{\tau_\wp}A_g = \tau_\wp^{-1} \cdot A_g.$$

Proof Relative to the $\mathbb{Q}_p$-basis for V_g used in the proof of Lemma 3, the endomorphism A_g is represented by a matrix of the form

$$\begin{pmatrix} 0 & \frac{\log_\wp(u_1)}{\log_\wp(u_{\alpha/\beta})} \\ 0 & 1 \end{pmatrix}.$$

The lemma follows immediately from this in light of the fact that conjugation by $\varrho_g(\tau_\wp)$ preserves the diagonal entries in such a matrix representation while multiplying its upper right hand entry by α/β, whereas $\tau_\wp$ acts on the upper right-hand entry of the above matrix as multiplication by β/α. □

The matrix A_g gives rise to a G-equivariant homomorphism $\Phi_g : H^\times \longrightarrow H_\wp \otimes W_g$ by setting

$$\Phi_g(x) = \sum_{\sigma \in G} \log_\wp({}^\sigma x) \cdot (\sigma^{-1} \cdot A_g), \tag{7}$$

where, just as above, the group G acts on $H_\wp \otimes W_g$ trivially on the first factor and through the usual conjugation action induced by ρ_g on the second factor.

Lemma 6 *The homomorphism Φ_g takes values in W_g.*

Proof For any $x \in (H \otimes \mathbb{Q}_p)^\times$ we have

$$\begin{aligned} {}^{\tau_\wp}\Phi_g(x) &= \sum_{\sigma \in G} \log_\wp({}^{\tau_\wp \sigma} x) \cdot (\sigma^{-1} \cdot {}^{\tau_\wp} A_g) \\ &= \sum_{\sigma \in G} \log_\wp({}^{\tau_\wp \sigma} x) \cdot (\sigma^{-1} \cdot \tau_\wp{}^{-1} \cdot A_g) \\ &= \sum_{\sigma \in G} \log_\wp({}^{\tau_\wp \sigma} x) \cdot ((\tau_\wp \sigma)^{-1} \cdot A_g) \\ &= \Phi_g(x), \end{aligned}$$

where Lemma 5 has been used to derive the second equation.

By a slight abuse of notation, we shall continue to denote with the same symbol the homomorphism

$$\Phi_g : (H \otimes \mathbb{Q}_p)^\times \longrightarrow H_\wp \otimes W_g$$

obtained from (7) by extending scalars. Note that $H_\wp^\times$ embeds naturally in $(H \otimes \mathbb{Q}_p)^\times$.

Lemma 7 *The homomorphism Φ_g vanishes on $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ and $\Phi_g(H_\wp^\times) \subseteq H_\wp \otimes W_g^{\mathrm{ord}}$.*

Proof Picking $u \in \mathcal{O}_H^\times$ and an arbitrary $B \in W_g$, set

$$U_g^\times := \sum_{\sigma \in G} {}^\sigma u \otimes (\sigma \cdot B) \in (\mathcal{O}_H^\times \otimes W_g)^G, \qquad U_g := (\log_\wp \otimes \mathrm{id})(U_g)$$

as in the statement of Lemma 3. Note that $U_g^\times$ is either trivial or a generator of the one-dimensional space $(\mathcal{O}_H^\times \otimes W_g)^G$. We have

$$\begin{aligned} \mathrm{Trace}(\Phi_g(u) \cdot B) &= \mathrm{Trace}\left(\left(\sum_{\sigma \in G} \log_\wp({}^\sigma u) \cdot (\sigma^{-1} \cdot A_g)\right) \cdot B\right) \\ &= \mathrm{Trace}\left(A_g \cdot \left(\sum_{\sigma \in G} \log_\wp({}^\sigma u) \cdot (\sigma \cdot B)\right)\right) \\ &= \mathrm{Trace}\left(A_g \cdot (\log_\wp \otimes \mathrm{Id})(U_g^\times)\right) = \mathrm{Trace}\left(A_g \cdot U_g\right). \end{aligned}$$

It follows from Property (a) satisfied by A (and hence A_g in particular) in Lemma 3 that

$$\mathrm{Trace}\left(\Phi_g(u)\cdot B\right)=0, \qquad \text{for all } B\in H_\wp\otimes W_g.$$

The first assertion in the lemma follows from the non-degeneracy of the $H_\wp$-valued trace pairing on $H_\wp\otimes W_g$. The second assertion follows from Property (b) satisfied by A and by A_g in Lemma 3. □

Let now $\ell\nmid Np$ be a rational prime, and let λ be a prime of H above ℓ. Let $u(\lambda)\in\mathcal{O}_H[1/\lambda]^\times\otimes\mathbb{Q}$ be a λ-unit of H satisfying

$$\mathrm{Norm}^H_{\mathbb{Q}}(u(\lambda))=\ell. \tag{8}$$

This condition makes $u(\lambda)$ well-defined up to the addition of elements in $\mathcal{O}_H^\times\otimes\mathbb{Q}$, and hence the element

$$A_g(\lambda):=\Phi_g(u(\lambda))=\sum_{\sigma\in G}\log_\wp({}^\sigma u(\lambda))\cdot(\sigma^{-1}\cdot A_g)$$

is well-defined, by Lemma 7. Although $A_g(\lambda)$ only belongs to $H_\wp\otimes W_g$ a priori, we have:

Lemma 8 *The trace of the endomorphism* $A_g(\lambda)$ *is equal to* $\log_p(\ell)$.

Proof Since the trace of A_g and its conjugates are all equal to 1, we have

$$\begin{aligned}\mathrm{Trace}(A_g(\lambda)) &= \sum_{\sigma\in G}\log_\wp({}^\sigma u(\lambda))\cdot\mathrm{Trace}(\sigma^{-1}\cdot A_g)\\ &= \sum_{\sigma\in G}\log_\wp({}^\sigma u(\lambda))\\ &= \log_\wp\left(\mathrm{Norm}^H_{\mathbb{Q}}(u(\lambda))\right).\end{aligned}$$

The latter expression is equal to $\log_p(\ell)$, by (8). □

Remark 9 Although $A_g(\lambda)$ belongs to W_g by Lemma 6, the entries of the matrix representing $A_g(\lambda)$ relative to an L-basis for V_g^L are L-linear combinations of products of $\wp$-adic logarithms of units and ℓ-units in H, and in particular $A_g(\lambda)$ need not lie in W_g^L. (In fact, it never does, since its trace is not algebraic.)

In addition to the invariant $A_g(\lambda)$, the choice of the prime λ of H above ℓ also determines a well-defined Frobenius element τ_λ in $G=\mathrm{Gal}\,(H/\mathbb{Q})$, and even in $\mathrm{Gal}\,(H_g/\mathbb{Q})$, since $\mathrm{Gal}\,(H_g/H)$ lies in the center of this group.

We are now ready to state the main theorem of this section:

Theorem 10 *For all rational primes* $\ell\nmid Np$,

$$a_\ell(g'_\alpha)=\mathrm{Trace}(\varrho_g(\tau_\lambda)A_g(\lambda)).$$

Remark 11 This invariant does not depend on the choice of a prime λ of H above ℓ, since replacing λ by another such prime has the effect of conjugating the endomorphisms $\varrho_g(\tau_\lambda)$ and $A_g(\lambda)$ by the same element of $\mathrm{Aut}(V_g)$.

Proof of Theorem 10 Let $\mathbb{Q}[\varepsilon]$ denote the ring of dual numbers over $\mathbb{Q}_p$, with $\varepsilon^2 = 0$, and let

$$\tilde{\varrho}_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(\mathbb{Q}_p[\varepsilon])$$

be the unique first order α-ordinary deformation of ϱ_g satisfying

$$\det \tilde{\varrho}_g = \chi_g(1 + \log_p \chi_{\mathrm{cyc}} \cdot \varepsilon).$$

This representation may be written as

$$\tilde{\varrho}_g = (1 + \varepsilon \cdot \kappa_g) \cdot \varrho_g \quad \text{for some} \quad \kappa_g : G_{\mathbb{Q}} \longrightarrow W_g. \tag{9}$$

The multiplicativity of $\tilde{\varrho}_g$ implies that the function κ_g is a 1-cocycle on $G_{\mathbb{Q}}$ with values in W_g, whose class in $H^1(\mathbb{Q}, W_g)$ (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of $\tilde{\varrho}_g$. Furthermore,

$$a_\ell(g_\alpha) + \varepsilon \cdot a_\ell(g'_\alpha) = \mathrm{Trace}(\tilde{\varrho}_g(\tau_\lambda)) = a_\ell(g) + \varepsilon \cdot \mathrm{Trace}(\kappa_g(\tau_\lambda)\varrho_g(\tau_\lambda)),$$

and hence

$$a_\ell(g'_\alpha) = \mathrm{Trace}(\varrho_g(\tau_\lambda)\kappa_g(\tau_\lambda)). \tag{10}$$

To make $\kappa_g(\tau_\lambda)$ explicit, observe that the inflation-restriction sequence combined with global class field theory for H gives rise to a series of identifications

$$\begin{aligned} H^1(\mathbb{Q}, W_g) &\xrightarrow{\mathrm{res}_H} \hom(G_H, W_g)^G \\ &= \hom_G\left(\frac{(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Q}_p}, W_g\right). \end{aligned}$$

Under this identification, the class κ_g can be viewed as an element of the space

$$H^1_{\mathrm{ord}}(\mathbb{Q}, W_g) = \left\{ \Phi \in \hom_G\left(\frac{(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Q}_p}, W_g\right) \text{ such that } \Phi(H_{\wp}^\times) \subset W_g^{\mathrm{ord}} \right\}.$$

But the homomorphism Φ_g of (7) belongs to the same one-dimensional space, by Lemmas 6 and 7. By global class field theory, the endomorphism $\kappa_g(\tau_\lambda)$ is therefore a $\mathbb{Q}_p^\times$-multiple of $\Phi_g(u_g(\lambda)) = A_g(\lambda)$. The fact that these endomorphisms are actually equal now follows by comparing their traces and noting that

$$\mathrm{Trace}(\kappa_g(\tau_\lambda)) = \log_p \chi_{\mathrm{cyc}}(\ell) = \log_p(\ell),$$

while

$$\mathrm{Trace}(A_g(\lambda)) = \log_p(\ell),$$

by Lemma 8. Theorem 10 follows. □

Corollary 12 *If the rational prime $\ell \nmid Np$ splits completely in $H/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

Proof The hypothesis implies that $\varrho_g(\tau_\lambda)$ is a scalar, and hence that $\varrho_g(\tau_\lambda) = \frac{1}{2}a_\ell(g)$. It follows that

$$\mathrm{Trace}(\varrho_g(\tau_\lambda)A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \mathrm{Trace}(A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

The corollary now follows from Theorem 10. □

Example 13 Let χ be a Dirichlet character of conductor 171 with order 3 at 9 and 2 at 19. Then $S_1(171, \chi)$ is a $\mathbb{Q}(\chi)$-vector space of dimension 2. It is spanned by an eigenform

$$g = q + \zeta q^2 + \zeta^3 q^3 - \zeta^2 q^5 + (\zeta^2 - 1)q^6 + \cdots$$

defined over $L := \mathbb{Q}(\zeta)$, with ζ a primitive 12th root of unity, and its Galois conjugate. (See [2] for all weight one eigenforms of level at most 1500.) The associated projective representation ϱ_g has A_4-image and factors through the field

$$H = \mathbb{Q}(a),\ a^4 + 10a^3 + 45a^2 + 81a + 81 = 0.$$

Let $p = 13$, which splits completely in L. The representation ϱ_g is regular at 13, with eigenvalues $\alpha = \zeta$ and $\beta = -\zeta^3$. We computed the first order deformations through each of g_α and g_β to precision 13^{10}, and q-adic precision $q^{37{,}000}$, using methods based upon the algorithms in [7].

The predictions made from Theorem 10 for $a_\ell(g'_\alpha)$ depend upon the conjugacy class of the Frobenius at ℓ in $\mathrm{Gal}\,(H/\mathbb{Q})$. For all primes $\ell < 37{,}000$ which split completely in H, such as $\ell = 109, 179, 449, 467, 521, \ldots$, we verified that

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_{13}(\ell) \pmod{13^{10}},$$

as asserted by Corollary 12.

2 CM Forms

This section focuses on the case where $g = \theta_{\psi_g}$ is the CM theta series attached to a character

$$\psi_g : G_K \longrightarrow L^\times$$

of a quadratic imaginary field K. The main theorems are Theorems 14 and 16 below, which will be derived in two independent ways, both "from first principles" and by specialising Theorem 10.

As in the previous section, the choice of an embedding of L into $\mathbb{Q}_p$ allows us to view ψ_g as a $\mathbb{Q}_p^\times$-valued character, and the weight one form g as a modular form with coefficients in $\mathbb{Q}_p$.

For a character $\psi : G_K \longrightarrow L^\times$, the notation ψ' will be used to designate the composition of ψ with conjugation by the non-trivial element in Gal $(K/\mathbb{Q})$:

$$\psi'(\sigma) = \psi(\tau\sigma\tau^{-1}),$$

where τ is any element of $G_\mathbb{Q}$ which acts non-trivially on K.

The Artin representation ϱ_g is induced from ψ_g and its restriction to G_K is the direct sum $\psi_g \oplus \psi_g'$ of two characters of K, which are *distinct* by the irreducibility of ϱ_g resulting from the fact that g is a cusp form. In this case, the field H is the ring class field of K which is cut out by the non-trivial ring class character $\psi := \psi_g/\psi_g{}'$. The Galois group $G :=$ Gal $(H/\mathbb{Q})$ is a generalised dihedral group containing $Z :=$ Gal (H/K) as its abelian normal subgroup of index two.

The case of CM forms can be further subdivided into two sub-cases, depending on whether p is split or inert in K.

2.1 *The Case Where p Splits in K*

Write $p\mathcal{O}_K = \mathfrak{p}\,\mathfrak{p}'$, and fix a prime $\wp$ of $\bar{\mathbb{Q}}$ above $\mathfrak{p}$. The roots of the pth Hecke polynomial of g are

$$\alpha = \psi_g(\mathfrak{p}), \qquad \beta = \psi_g(\mathfrak{p}').$$

This case is notable in that the Hida family $\mathbf{g}$ passing through g_α can be written down explicitly as a family of theta series. Its weight k specialisation $\mathbf{g}_k$ is the theta-series attached to the character $\psi_g\Psi^{k-1}$, where Ψ is a CM Hecke character of weight $(1, 0)$ which is unramified at $\mathfrak{p}$. For all rational primes $\ell \nmid Np$, the ℓth fourier coefficient of $\mathbf{g}_k$ is given by

$$a_\ell(\mathbf{g}_k) = \begin{cases} \psi_g(\lambda)\Psi^{k-1}(\lambda') + \psi_g(\lambda')\Psi^{k-1}(\lambda) & \text{if } \ell = \lambda\lambda' \text{ splits in } K; \\ 0 & \text{if } \ell \text{ is inert in } K. \end{cases} \qquad (11)$$

Letting h be the class number of K and t the cardinality of the unit group $\mathcal{O}_K^\times$, the character Ψ satisfies

$$\Psi(\lambda)^{ht} = u_\lambda^t, \qquad \text{where } (u_\lambda) := \lambda^h,$$

for any prime ideal λ of $\mathcal{O}_K$ whose norm is the rational prime $\ell = \lambda\lambda'$. Let u'_λ denote the conjugate of u_λ in $K/\mathbb{Q}$. It follows that

$$\frac{d}{dk}\Psi^{k-1}(\lambda)_{k=1} = \log_{\mathfrak{p}}(u(\lambda)), \qquad \text{where } u(\lambda) := u_\lambda \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q},$$

and likewise that

$$\frac{d}{dk}\Psi^{k-1}(\lambda')_{k=1} = \log_{\mathfrak{p}}(u(\lambda)'), \qquad \text{where } u(\lambda)' := u'_\lambda \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q}.$$

In light of (11), we have obtained:

Theorem 14 *For all rational primes ℓ that do not divide Np,*

$$a_\ell(g'_\alpha) = \begin{cases} \left(\psi_g(\lambda)\log_{\mathfrak{p}}(u(\lambda')) + \psi_g(\lambda')\log_{\mathfrak{p}}(u(\lambda))\right) & \text{if } \ell = \lambda\lambda' \text{ splits in } K; \\ 0 & \text{if } \ell \text{ is inert in } K. \end{cases} \tag{12}$$

Thus, the prime fourier coefficients of g'_α are supported at the primes ℓ which are split in K, where they are (algebraic multiples of) the $\mathfrak{p}$-adic logarithms of ℓ-units in this quadratic field. This general pattern will persist in the other settings to be described below, with the notable feature that the fourier coefficients of g'_α will be more complicated expressions involving, in general, the p-adic logarithms of units and ℓ-units in the full ring class field H.

The reader will note Theorem 14 is consistent with Theorem 10, and could also have been deduced from it. More precisely, choose a basis of V_g consisting of eigenvectors for the action of G_K (and hence also, of $\tau_\wp$) which are interchanged by some element $\tau \in G_\mathbb{Q} - G_K$. Relative to such a basis, the endomorphisms U_g and A_g are represented by the following matrices, in which u_ψ and τu_ψ are generators of the spaces of ψ and ψ^{-1}-isotypic vectors in the group of elliptic units in $\mathcal{O}_H^\times \otimes L$:

$$U_g : \begin{pmatrix} 0 & \log_\wp(u_\psi) \\ \log_\wp(u'_\psi) & 0 \end{pmatrix}, \qquad A_g : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that, if $\ell = \lambda\lambda'$ is split in K, then $A_g(\lambda)$ is represented by the matrix

$$A_g(\lambda) : \begin{pmatrix} \log_\wp(u(\lambda')) & 0 \\ 0 & \log_\wp(u(\lambda)) \end{pmatrix},$$

while $A_g(\lambda) = \frac{1}{2}\log_p(\ell)$ is the scalar matrix with trace equal to $\log_p(\ell)$ if ℓ is inert in K.

2.2 The Case Where p Is Inert in K

We now turn to the more interesting case where p is inert in K. Let $\sigma_\wp := \tau_\wp^2$ denote the frobenius element in G_K attached to the prime $\wp$ of H (which is well-defined modulo the inertia subgroup at $\wp$). Note that the prime p splits completely in H/K, since the image of $\tau_\wp$ in G is a reflection in this generalised dihedral group. The image of $\sigma_\wp$ in $\mathrm{Gal}\,(H_g/K)$ therefore belongs to the subgroup $\mathrm{Gal}\,(H_g/H)$ whose image under ϱ_g consists of scalar matrices. Similar notations and remarks apply to any rational prime ℓ which is inert in $K/\mathbb{Q}$.

Relative to an eigenbasis (e_1, e_2) for the action of G_K on V_g, the Galois representation ϱ_g takes the form

$$\varrho_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi'_g(\sigma) \end{pmatrix} \quad \text{for } \sigma \in G_K. \tag{13}$$

The homomorphisms $\psi_g, \psi'_g : G_K \longrightarrow \mathbb{Q}_p^\times$ factor through $\mathrm{Gal}\,(H_g/K)$ and satisfy

$$\psi_g(\tau\sigma\tau^{-1}) = \psi'_g(\sigma), \qquad \text{for all } \tau \in G_\mathbb{Q} - G_K, \quad \sigma \in G_K.$$

It follows that $\varrho_g(\tau)$ interchanges the lines spanned by e_1 and e_2, for any element $\tau \in G_\mathbb{Q} - G_K$. The restriction of ϱ_g to $G_\mathbb{Q} - G_K$ can therefore be described in matrix form by

$$\varrho_g(\tau) = \begin{pmatrix} 0 & \eta_g(\tau) \\ \eta'_g(\tau) & 0 \end{pmatrix} \quad \text{for } \tau \in G_\mathbb{Q} - G_K, \tag{14}$$

where η_g and η'_g are L-valued functions on $G_\mathbb{Q} - G_K$ that satisfy

$$\eta_g(\tau_1)\eta'_g(\tau_2) = \psi_g(\tau_1\tau_2) = \psi'_g(\tau_2\tau_1), \qquad \text{for all } \tau_1, \tau_2 \in G_\mathbb{Q} - G_K, \tag{15}$$

as well as the relations

$$\begin{array}{ll} \eta_g(\sigma\tau) = \psi_g(\sigma)\eta_g(\tau), & \eta_g(\tau\sigma) = \psi'_g(\sigma)\eta_g(\tau), \\ \eta'_g(\sigma\tau) = \psi'_g(\sigma)\eta'_g(\tau), & \eta'_g(\tau\sigma) = \psi_g(\sigma)\eta'_g(\tau), \end{array} \quad \text{for all } \sigma \in G_K, \ \tau \in G_\mathbb{Q} - G_K. \tag{16}$$

After re-scaling e_1 and e_2 if necessary, we may assume that $\tau_\wp \in G_\mathbb{Q} - G_K$ is sent to the matrix

$$\varrho_g(\tau_\wp) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}, \qquad \text{with } \zeta^2 = -\chi_g(p). \tag{17}$$

The eigenvalues of $\varrho_g(\tau_\wp)$ are equal to $\alpha := \zeta$ and $\beta := -\zeta$, and hence g is *always regular* in this setting.

Let

$$\tilde{\varrho}_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}(\tilde{V}_g)$$

denote the first-order infinitesimal deformation of ϱ_g attached to the Hida family $\mathbf{g}$ passing through a choice of p-stabilization g_α of g, where $\alpha \in \{\zeta, -\zeta\}$. The module $\tilde{V}_g$ is free of rank two over the ring $\mathbb{Q}_p[\varepsilon]/(\varepsilon^2) = \mathbb{Q}_p[[T]]/(T^2)$ arising from the mod T^2 reduction of the representation $\varrho_{\mathbf{g}}$ attached to $\mathbf{g}$. Choose any $\mathbb{Q}_p[\varepsilon]$-basis $(\tilde{e}_1, \tilde{e}_2)$ of $\tilde{V}_g$ lifting (e_1, e_2), and note that the restriction of $\tilde{\varrho}_g$ to G_K is given by:

$$\tilde{\varrho}_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) \cdot (1 + \kappa(\sigma) \cdot \varepsilon) & \psi_g'(\sigma) \cdot \kappa_\psi(\sigma) \cdot \varepsilon \\ \psi_g(\sigma) \cdot \kappa_\psi'(\sigma) \cdot \varepsilon & \psi_g'(\sigma) \cdot (1 + \kappa'(\sigma) \cdot \varepsilon) \end{pmatrix}, \quad \text{for all } \sigma \in G_K. \tag{18}$$

In this expression,

(a) The functions κ and κ' are continuous homomorphisms from G_K to $\mathbb{Q}_p$, i.e., elements of $H^1(K, \mathbb{Q}_p)$, which are interchanged by conjugation by the involution in Gal $(K/\mathbb{Q})$:

$$\kappa(\tau\sigma\tau^{-1}) = \kappa'(\sigma), \qquad \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.$$

(b) The functions $\kappa_\psi, \kappa_\psi' : G_K \longrightarrow \mathbb{Q}_p$ are one-cocycles with values in $\mathbb{Q}_p(\psi)$, and give rise to well defined classes

$$\kappa_\psi \in H^1(K, \mathbb{Q}_p(\psi)), \qquad \kappa_\psi' \in H^1(K, \mathbb{Q}_p(\psi^{-1})),$$

which also satisfy

$$\kappa_\psi(\tau\sigma\tau^{-1}) = \kappa_\psi'(\sigma), \qquad \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.$$

For each rational prime $\ell \nmid Np$, the ℓth fourier coefficient $a_\ell(g_\alpha')$ is given by

$$a_\ell(g_\alpha') = \frac{d}{dk}\mathrm{Trace}(\varrho_{\mathbf{g}}(\tau_\lambda))_{k=1} \tag{19}$$

Observe that the spaces $H^1(K, \mathbb{Q}_p)$ and $H^1(K, \mathbb{Q}_p(\psi))$ are of dimensions two and one respectively over $\mathbb{Q}_p$, since $\psi \neq 1$. More precisely, restriction to the inertia group at p combined with local class field theory induces an isomorphism

$$H^1(K, \mathbb{Q}_p) = \hom(\mathcal{O}_{K_p}^\times, \mathbb{Q}_p) = \mathbb{Q}_p \log_p(z) \oplus \mathbb{Q}_p \log_p(z'). \tag{20}$$

Let $\mathcal{O}_H^{\times,\psi}$ denote the (one-dimensional) ψ-isotypic component of $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ on which Gal (H/K) acts through the character ψ, and denote by $\wp$ the prime of H above p arising from our chosen embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Restriction to the inertia group at $\wp$ in G_H likewise gives rise to an identification

$$H^1(K, \mathbb{Q}_p(\psi)) = \hom(\mathcal{O}_{H_\wp}^\times / \mathcal{O}_H^{\times,\psi}, \mathbb{Q}_p) = \mathbb{Q}_p \cdot (\log_\wp(u'_\psi)\log_\wp(z) - \log_\wp(u_\psi)\log_\wp(z')). \tag{21}$$

In the above equation, u_ψ is to be understood as the natural image in $\mathcal{O}_{H_\wp}^\times = \mathcal{O}_{K_p}^\times$ of an element of the form

$$\sum_{\sigma \in Z} \psi^{-1}(\sigma) u^\sigma \in (\mathcal{O}_H^\times \otimes L)^\psi,$$

where u is an $L[G]$-module generator of $\mathcal{O}_H^\times \otimes L$, and u'_ψ is the image of u_ψ under the conjugation action $K_p \longrightarrow K_p$. Note that replacing u by λu for some $\lambda \in L[G]$ has the effect of multiplying both u_ψ and u'_ψ by $\psi(\lambda) \in \mathbb{Q}_p$, so that the $\mathbb{Q}_p$-line spanned by the right-hand side of (21) is independent of the choice of $u \in \mathcal{O}_H^\times$.

It follows from (20) and (21) that the total deformation space of ϱ_g (before imposing any ordinarity hypotheses, or restrictions on the determinant) is three dimensional.

Let $v_g^+ := e_1 + e_2$ and $v_g^- := e_1 - e_2$ be the eigenvectors for $\tau_\wp$ acting on V_g, with eigenvalues ζ and $-\zeta$ respectively. Let κ_p and $\kappa_{\psi,\wp}$ denote the restrictions κ and κ_ψ to the inertia groups at p and $\wp$ in G_H and G_K respectively. Both can be viewed as characters of $K_p^\times = H_\wp^\times$ after identifying the abelianisations of G_{K_p} and $G_{H_\wp}$ with a quotient of $K_p^\times$ via local class field theory.

Lemma 15 *The following are equivalent:*

(a) *The inertia group at $\wp$ acts as the identity on some lift $\tilde{v}_g^+$ of v_g^+ to $\tilde{V}_g$;*
(b) *The inertia group at $\wp$ acts as the identity on all lifts $\tilde{v}_g^+$ of v_g^+ to $\tilde{V}_g$;*
(c) *The restrictions κ_p and $\kappa_{\psi,\wp}$ satisfy*

$$\kappa_p(x) = -\kappa_{\psi,\wp}(x), \qquad \text{for all } x \in \mathcal{O}_{K_p}^\times.$$

Similar statements hold when v_g^+ is replaced by v_g^-, where the conclusion is that $\kappa_p = \kappa_{\psi,\wp}$.

Proof The equivalence of the first two conditions follows from the fact that $\varepsilon\tilde{V}_g \simeq V_g$ is unramified at p and hence that inertia acts as the identity on the kernel of the natural map $\tilde{V}_g \longrightarrow V_g$. To check the third, note that the inertia group I_p at p is contained in G_K, since K is unramified at p, and observe that any $\sigma \in I_p$ sends $\tilde{e}_1 + \tilde{e}_2$ to

$$\begin{aligned}\tilde{\varrho}_g(\sigma)(\tilde{e}_1 + \tilde{e}_2) &= \tilde{e}_1 + \tilde{e}_2 + \varepsilon \cdot (\kappa(\sigma)\tilde{e}_1 + \kappa'_\psi(\sigma)\tilde{e}_2 + \kappa_\psi(\sigma)\tilde{e}_1 + \kappa'(\sigma)\tilde{e}_2) \\ &= \tilde{e}_1 + \tilde{e}_2 + \varepsilon \cdot ((\kappa(\sigma) + \kappa_\psi(\sigma))\tilde{e}_1 + (\kappa'(\sigma) + \kappa'_\psi(\sigma))\tilde{e}_2).\end{aligned}$$

The lemma follows. □

A lift $\tilde{\varrho}_g$ of ϱ_g is ordinary relative to the space spanned by v_g^+ if and only if it satisfies the equivalent conditions of Lemma 15. This lemma merely spells out the proof of the Bellaiche-Dimitrov theorem on the one-dimensionality of the tangent space of the eigencurve at the point associated to g_α. More precisely, the general ordinary first-order deformation of ϱ_g is completely determined by the pair $(\kappa_p, \kappa_{\psi,\wp})$, which depends on a single linear parameter $\mu \in \bar{\mathbb{Q}}_p$ and is given by the rule

$$\kappa_p(z) = \mu(\log_\wp(u'_\psi) \cdot \log_\wp(z) - \log_\wp u_\psi \cdot \log_\wp(z')), \tag{22}$$

$$\kappa_{\psi,p}(z) = \pm\mu(\log_\wp(u'_\psi) \cdot \log_\wp(z) - \log_\wp u_\psi \cdot \log_\wp(z')), \tag{23}$$

where the sign in the second formula depends on whether one is working with the ordinary deformation of g_α or g_β.

Let us now make use of the fact that

$$\det(\tilde{\varrho}_g) = 1 + \varepsilon \log_p \chi_{\text{cyc}} = 1 + \varepsilon \log_p(zz').$$

Since $\det(\tilde{\varrho}_g) = 1 + \varepsilon(\kappa + \kappa')$, this condition implies that

$$\mu = \frac{1}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)},$$

and hence that κ_p and $\kappa_{\psi,\wp}$ are given by

$$\kappa_p(z) = \frac{\log_\wp(u'_\psi) \cdot \log_\wp(z) - \log_\wp(u_\psi) \cdot \log_\wp(z')}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)}, \tag{24}$$

$$\kappa_{\psi,\wp}(z) = \pm\frac{\log_\wp(u'_\psi) \cdot \log_\wp(z) - \log_\wp(u_\psi) \cdot \log_\wp(z')}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)}. \tag{25}$$

Equations (24) and (25) give a completely explicit description of the first order deformation $\tilde{\varrho}_{g_\alpha}$ and $\tilde{\varrho}_{g_\beta}$, from which the fourier coefficients of g'_α and g'_β shall be readily calculated.

The formula for the ℓth fourier coefficient of g'_α involves the unit u_ψ above as well as certain ℓ-units in $\mathcal{O}_H[1/\ell]^\times \otimes L$ whose definition depends on whether or not the prime ℓ is split or inert in $K/\mathbb{Q}$.

If $\ell = \lambda\lambda'$ splits in $K/\mathbb{Q}$, let $u(\lambda)$ and $u(\lambda')$ denote, as before, the ℓ-units in $\mathcal{O}_K[1/\ell]^\times \otimes \mathbb{Q}$ of norm ℓ with prime factorisation λ and λ' respectively. Set

$$u_g(\lambda) := u(\lambda) \otimes \psi_g(\lambda) + u(\lambda') \otimes \psi_g(\lambda'), \qquad u_g(\lambda') := u(\lambda') \otimes \psi_g(\lambda) + u(\lambda) \otimes \psi_g(\lambda').$$

In other words, $u_g(\lambda)$ is the unique element of $\mathcal{O}_K[1/\ell]^\times \otimes L$ whose prime factorisation is equal to $\psi_g(\lambda) \cdot \lambda + \psi_g(\lambda') \cdot \lambda'$. Note that, if ℓ splits completely in $H/\mathbb{Q}$, i.e., if $\varrho_g(\tau_\lambda)$ is equal to a scalar ζ, then $u_g(\lambda) = u_g(\lambda') = \ell \otimes \zeta$, but that otherwise $u_g(\lambda)$ and ℓ generate the L-vector space $\mathcal{O}_K[1/\ell]^\times \otimes L$ of ℓ-units of K (tensored with L).

If ℓ is inert in $K/\mathbb{Q}$, choose a prime λ of H lying above ℓ, and let $u(\lambda) \in \mathcal{O}_H[1/\lambda]^\times \otimes \mathbb{Q}$ be any λ-unit of H satisfying $\mathrm{ord}_\lambda(u(\lambda)) = 1$, which is well defined up to units in $\mathcal{O}_H^\times$. Define the elements

$$u_\psi(\lambda) = \sum_{\sigma\in Z} \psi^{-1}(\sigma) \otimes {}^\sigma u(\lambda) \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times,$$
$$u'_\psi(\lambda) = \tau_\wp u_\psi(\lambda) \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times.$$

Thus $u_\psi(\lambda)$ lies in the ψ-component $\mathcal{O}_H[1/\ell]^\times[\psi]$ and is well-defined up to the addition of multiples of u_ψ, where

$$u_\psi := \sum_{\sigma\in Z} \psi^{-1}(\sigma) \otimes {}^\sigma u \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times,$$

for any unit $u \in \mathcal{O}_H^\times$, while $u'_\psi(\lambda)$ lies in the ψ^{-1} component and is well-defined up to the addition of multiples of u'_ψ, where

$$u'_\psi = \tau_\wp u_\psi.$$

Recall the function $\eta'_g : G_\mathbb{Q} \setminus G_K$ introduced in (14), with values in the roots of unity of $L^\times$. The main result of this section is:

Theorem 16 *Let $\ell \nmid Np$ be a rational prime.*

(a) *If $\ell = \lambda\lambda'$ splits in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = a_\ell(g'_\beta) = \frac{\log_\wp(u'_\psi)\cdot\log_\wp(u_g(\lambda)) - \log_\wp(u_\psi)\cdot\log_\wp(u_g(\lambda'))}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)}. \tag{26}$$

(b) *If ℓ remains inert in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = \eta'_g(\tau_\lambda)\frac{\log_\wp(u'_\psi)\log_\wp(u_\psi(\lambda)) - \log_\wp(u_\psi)\log_\wp(u'_\psi(\lambda))}{\log_\wp(u'_\psi) - \log_\wp(u_\psi))}.$$

Proof Let us first compute first the fourier coefficients at primes $\ell \nmid Np$ that split as $\ell = \lambda\lambda'$ in K. Let σ_λ and $\sigma_{\lambda'}$ be the frobenius elements associated to λ and λ' respectively. They are well-defined elements in the Galois group of any abelian extension of K in which ℓ is unramified.

It follows from (19) and the matrix expression for $\tilde{\varrho}_{g|G_K}$ given in (18) that

$$\begin{aligned} a_\ell(g'_\alpha) &= \psi_g(\lambda)\kappa(\lambda) + \psi_g(\lambda')\kappa(\lambda') \\ &= \psi_g(\lambda)\kappa_p(u(\lambda)) + \psi_g(\lambda')\kappa_p(u'(\lambda)). \end{aligned}$$

Equation (26) then follows from the formula for $\kappa_p(z)$ given in (24).

We now turn now to the computation of the fourier coefficients of g'_α at primes $\ell \nmid Np$ that remain inert in K. Let τ_λ denote the Frobenius element in Gal $(M/\mathbb{Q})$ associated to the choice of a prime ideal λ above ℓ in $\bar{\mathbb{Q}}$, and let $\sigma_\lambda := \tau_\lambda^2$ denote the associated frobenius element in Gal (M/K).

Since τ_λ belongs to $G_\mathbb{Q} - G_K$, it follows from (14) that the matrix $\tilde{\varrho}_g(\tau_\lambda)$ is of the form

$$\tilde{\varrho}_g(\tau_\lambda) = \begin{pmatrix} r_\ell \cdot \varepsilon & \eta_g(\tau_\lambda)(1 + s_\ell \cdot \varepsilon) \\ \eta'_g(\tau_\lambda)(1 + t_\ell \cdot \varepsilon) & u_\ell \cdot \varepsilon \end{pmatrix},$$

for suitable scalars r_ℓ, s_ℓ, t_ℓ, and $u_\ell \in \bar{\mathbb{Q}}_p$. Since

$$a_\ell(g_\alpha) + a_\ell(g'_\alpha)\varepsilon = \mathrm{Trace}(\tilde{\varrho}_g(\tau_\lambda)) = (r_\ell + u_\ell) \cdot \varepsilon,$$

it follows that

$$a_\ell(g'_\alpha) = r_\ell + u_\ell. \tag{27}$$

In order to compute this trace, we observe that it arises in the upper right-hand and lower left-hand entries of the matrix

$$\tilde{\varrho}_g(\sigma_\lambda) = \tilde{\varrho}_g(\tau_\lambda)^2 = \begin{pmatrix} \psi_g(\sigma_\lambda)(1 + (s_\ell + t_\ell) \cdot \varepsilon) & \eta_g(\tau_\lambda)(r_\ell + u_\ell) \cdot \varepsilon \\ \eta'_g(\tau_\lambda)(r_\ell + u_\ell) \cdot \varepsilon & \psi_g(\sigma_\lambda)(1 + (s_\ell + t_\ell) \cdot \varepsilon) \end{pmatrix}. \tag{28}$$

On the other hand, since σ_λ belongs to G_K it follows from (18) that

$$\tilde{\varrho}_g(\sigma_\lambda) = \begin{pmatrix} \psi_g(\sigma_\lambda) \cdot (1 + \kappa(\sigma_\lambda) \cdot \varepsilon) & \psi'_g(\sigma_\lambda) \cdot \kappa_\psi(\sigma_\lambda) \cdot \varepsilon \\ \psi_g(\sigma_\lambda) \cdot \kappa'_\psi(\sigma_\lambda) \cdot \varepsilon & \psi'_g(\sigma_\lambda) \cdot (1 + \kappa'(\sigma_\lambda) \cdot \varepsilon) \end{pmatrix}. \tag{29}$$

By comparing upper-right entries in the matrices in (28) and (29) and invoking (27) together with the relation $\psi'_g(\sigma_\lambda)\eta_g(\tau_\lambda)^{-1} = \eta'_g(\tau_\lambda)$ arising from (15), we deduce that

$$a_\ell(g'_\alpha) = \eta'_g(\tau_\lambda)\kappa_\psi(\sigma_\lambda).$$

It is worth noting that each of the expressions $\eta'_g(\tau_\lambda)$ and $\kappa_\psi(\sigma_\lambda)$ depend on the choice of a prime λ of H above ℓ that was made to define τ_λ and σ_λ, since changing this prime replaces τ_λ and σ_λ by their conjugates $\sigma\tau_\lambda\sigma^{-1}$ and $\sigma\sigma_\lambda\sigma^{-1}$ by some $\sigma \in G_K$. More precisely, by (16) and the cocycle property of κ_ψ,

$$\eta'_g(\sigma\tau_\lambda\sigma^{-1}) = \psi^{-1}(\sigma)\eta'_g(\tau_\lambda), \qquad \kappa_\psi(\sigma\sigma_\lambda\sigma^{-1}) = \psi(\sigma)\kappa_\psi(\sigma_\lambda).$$

In particular, the product $\eta_g'(\tau_\lambda)\kappa_\psi(\sigma_\lambda)$ is independent of the choice of a prime above ℓ, as it should be. Note that $\eta_g'(\tau_\lambda)$ is a simple root of unity belonging to the image of ψ_g, while $\kappa_\psi(\sigma_\lambda)$ represents the interesting "transcendental" contribution to the fourier coefficient $a_\ell(g_\alpha')$.

By the description of $\kappa_\psi(\sigma_\lambda)$ arising from local and global class field theory, we conclude from (25) that

$$a_\ell(g_\alpha') = \eta_g'(\tau_\lambda)\frac{\log_\wp(u_\psi')\log_\wp(u_\psi(\lambda)) - \log_\wp(u_\psi)\log_\wp(u_\psi'(\lambda))}{\log_\wp(u_\psi') - \log_\wp(u_\psi)}, \tag{30}$$

as was to be shown. □

A more efficient (but somewhat less transparent) route to the proof of Theorem 16 is to specialise Theorem 10 to this setting. Relative to a basis of the form $(v, \tau_\wp v)$ for V_g, where v spans a G_K-stable subspace of V_g on which G_K acts via ψ_g, the matrix for U_g is proportional to one of the form

$$U_g : \begin{pmatrix} 0 & \log_\wp(u_\psi) \\ \log_\wp(\tau_\wp u_\psi) & 0 \end{pmatrix},$$

and the ordinarity condition implies that the matrix representing A_g is proportional to a matrix of the form

$$A : \begin{pmatrix} x & -x \\ y & -y \end{pmatrix}.$$

The relations $\mathrm{Trace}(A_g U_g) = 0$ and $\mathrm{Trace}(A_g) = 1$ show that A_g is represented by the matrix

$$A_g : \frac{1}{\log_\wp(u_\psi) - \log_\wp(\tau_\wp u_\psi)} \cdot \begin{pmatrix} \log_\wp(u_\psi) & -\log_\wp(u_\psi) \\ \log_\wp(\tau_\wp u_\psi) & -\log_\wp(\tau_\wp u_\psi) \end{pmatrix},$$

and Theorem 16 is readily deduced from the general formula for the fourier coefficients of g_α' given in Theorem 10. The details are left to the reader.

2.3 Numerical Examples

We begin with an illustration of Theorem 16 in which the image of ϱ_g is isomorphic to the symmetric group S_3.

Example 17 Let χ be the quadratic character of conductor 23 and

$$g = q - q^2 - q^3 + q^6 + q^8 + \cdots \in S_1(23, \chi)$$

be the theta series attached to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-23})$. The Hilbert class field H of K is

$$H = \mathbb{Q}(\alpha) \quad \text{where} \quad \alpha^6 - 6\alpha^4 + 9\alpha^2 + 23 = 0.$$

Write $\mathrm{Gal}\,(H/K) = \langle\sigma\rangle$. The smallest prime which is inert in K is $p = 5$. The deformations g_1' and g_{-1}' were computed to a 5-adic precision of 5^{40} (and q-adic precision q^{600}).

Consider the inert prime $\ell = 7$ in K. Let $u(7) = (2\alpha^4 - 7\alpha^2 + 5)/9$, a root of $x^3 - x^2 + 2x - 7 = 0$. Taking ω a primitive cube root of unity we have

$$\begin{aligned}\log_5(u_\psi(7)) &= \log_5(u(7)) + \omega \log_5(u(7)^\sigma) + \omega^2 \log_5(u(7)^{\sigma^2})\\ \log_5(u'_\psi(7)) &= \log_5(u(7)) + \omega^2 \log_5(u(7)^\sigma) + \omega \log_5(u(7)^{\sigma^2}).\end{aligned}$$

Let $u = (\alpha^2 - 1)/3$ be the elliptic unit in H, a root of $x^3 - x^2 + 1 = 0$. Then likewise we have

$$\begin{aligned}\log_5(u_\psi) &= \log_5(u) + \omega \log_5(u^\sigma) + \omega^2 \log_5(u^{\sigma^2})\\ \log_5(u'_\psi) &= \log_5(u) + \omega^2 \log_5(u^\sigma) + \omega \log_5(u^{\sigma^2}).\end{aligned}$$

Now

$$a_7(g_1') = -a_7(g_{-1}') = 408307984761015709227253754 8 \cdot 5 \bmod 5^{40}$$

and one checks to 40 digits of 5-adic precision that

$$a_7(g_1') = \frac{\log_5(u_\psi(7)) \log_5(u'_\psi) - \log_5(u'_\psi(7)) \log_5(u_\psi)}{\log_5(u_\psi) - \log_5(u'_\psi)}$$

as predicted by part (b) of Theorem 16.

Consider next the prime $\ell = 13$, which splits in K and factors as $(l) = \lambda\lambda'$, where $\lambda^3 = (u_\lambda)$ is a principal ideal generated by $u_\lambda = -6\alpha^3 + 18\alpha - 37$, a root of $x^2 + 74x + 2197$. After setting $u(\lambda) = u_\lambda \otimes \frac{1}{3}$, we let

$$\begin{aligned}\log_5(u_g(\lambda)) &= \left(\omega \log_5(u(\lambda)) + \omega^2 \log_5(u'(\lambda))\right)\\ \log_5(u'_g(\lambda)) &= \tfrac{1}{3}\left(\omega^2 \log_5(u(\lambda)) + \omega \log_5(u'(\lambda))\right).\end{aligned}$$

We have

$$a_{13}(g_1') = a_{13}(g_{-1}') = -6388941316808301988520085 92 \cdot 5 \bmod 5^{40}$$

and one sees that

$$a_{13}(g_{\pm 1}') = \frac{\log_5(u'(\psi)) \log_5(u_g(\lambda)) - \log_5(u(\psi)) \log_5(u'_g(\lambda))}{\log_5(u'_\psi) - \log_5(u_\psi)}$$

to 40 digits of 5-adic precision, confirming Part (a) of Theorem 16.

The experiment below focuses on the case where ψ_g is a quartic ring class character, so that ϱ_g has image isomorphic to the dihedral group of order 8. The associated ring class character $\psi = \psi_g/\psi_g' = \psi_g{}^2$ of K is quadratic, i.e., a genus character which cuts out a biquadratic extension H of $\mathbb{Q}$ containing K. Let F denote the unique real quadratic subfield of H, and let K' the unique imaginary quadratic subfield of H which is distinct from K. The unit u_ψ is a power of the fundamental unit of F. Observe that the prime p is necessarily inert in $K'/\mathbb{Q}$, since otherwise ϱ_g would be induced from a character of the real quadratic field F in which p splits. It follows that $u'_\psi = u_\psi^{-1}$, so that by Theorem 16,

$$a_\ell(g'_\alpha) = \frac{-\log_p u_\psi \cdot (\log_p(u_g(\ell)) + \log_p(u'_g(\ell)))}{-2\log_p(u_\psi)} = \frac{1}{2}\cdot(\log_p(u_g(\ell))+\log_p(u'_g(\ell))).$$

It follows from the definition of $u_g(\ell)$ that

$$a_\ell(g'_\alpha) = \mathrm{trace}(\varrho_g(\lambda)) \cdot \log_p(\ell).$$

In particular, we obtain

$$a_\ell(g'_\alpha) = \begin{cases} \log_p(\ell) & \text{if } \psi_g(\lambda) = 1, \\ 0 & \text{if } \psi_g(\lambda) = \pm i, \\ -\log_p(\ell) & \text{if } \psi_g(\lambda) = -1, \end{cases} \tag{31}$$

in perfect agreement with the experiments below.

Example 18 Let $\chi = \chi_3\chi_{13}$ where χ_3 and χ_{13} are the quadratic characters of conductors 3 and 13, respectively. The space $S_1(39, \chi)$ is one dimensional and spanned by the form $g = q - q^3 - q^4 + q^9 + \cdots$. The representation ρ_g has projective image D_4 and is induced from characters of two imaginary quadratic fields and one real quadratic field. In particular, it is induced from the quadratic character ψ_g of the Hilbert class field

$$H = \mathbb{Q}(\sqrt{-39}, a),\ a^4 + 4a^2 - 48 = 0$$

of $\mathbb{Q}(\sqrt{-39})$ (and also ramified characters of ray class fields of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{13})$). Let $p = 7$, which is inert in $\mathbb{Q}(\sqrt{-39})$. We computed $g'_{\pm 1}$ to 20-digits of 7-adic precision (and to q-adic precision q^{900}).

First consider the case of $\ell = \lambda\lambda'$ split in $\mathbb{Q}(\sqrt{-39})$. Then one observes to 20-digits of 7-adic precision and all such $\ell < 900$ that both $a_\ell(g'_1)$ and $a_\ell(g'_{-1})$ satisfy (31). Next we consider the case that ℓ is inert in $\mathbb{Q}(\sqrt{-39})$. Here one observes numerically that the Fourier coefficients are zero when ℓ is inert in $\mathbb{Q}(\sqrt{-3})$. When ℓ is split in $\mathbb{Q}(\sqrt{-3})$ the Fourier coefficients of the two stabilisations are opposite in sign and both equal to the p-adic logarithm of a fundamental ℓ unit of norm 1 in $\mathbb{Q}(\sqrt{-3})$. (Observe that p is split in $\mathbb{Q}(\sqrt{-3})$ and our numerical observations are consistent in this example with both our theorems for p split and p inert in the CM case.)

3 RM Forms

Consider now the case where g is the theta series attached to a character

$$\psi_g : G_K \longrightarrow L^\times$$

of mixed signature of a real quadratic field K. As before, assume for simplicity that the field L may be embedded into $\mathbb{Q}_p$ and fix one such embedding. We also continue to denote H_g the abelian extensions of K which is cut out by ϱ_g, and let H be the ring class field of K cut out by the non-trivial ring class character $\psi := \psi_g/\psi_g{}'$. Since ψ_g has mixed signature, it follows that ψ is totally odd and thus H is totally imaginary. As before, write $G := \mathrm{Gal}\,(H/\mathbb{Q})$ and $Z := \mathrm{Gal}\,(H/K)$.

As explained in the introduction, the case where p splits in K was already dealt with in [3], so in this section we only consider the case where p is inert in $K/\mathbb{Q}$. The prime p then splits completely in H/K and we fix a prime $\wp$ of H above p. This choice determines an embedding $H \longrightarrow H_\wp = K_p$ and we write $z \mapsto z'$ for the conjugation action of $\mathrm{Gal}\,(K_p/\mathbb{Q}_p)$. Let $u_K \in \mathcal{O}_K^\times$ denote the fundamental unit of $\mathcal{O}_K$ of norm 1, which we regard as an element of $K_p^\times = H_\wp^\times$ through the above embedding, and let $u'_K = u_K^{-1}$ denote its algebraic conjugate.

Let (v_1, v_2) be a basis for V_g consisting of eigenvectors for the action of G_K, and which are interchanged by the frobenius element $\tau_\wp$. Just as in the previous section, relative to this basis the Galois representation ϱ_g takes the form

$$\varrho_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi'_g(\sigma) \end{pmatrix} \text{ for } \sigma \in G_K, \qquad \varrho_g(\tau) = \begin{pmatrix} 0 & \eta_g(\tau) \\ \eta'_g(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_\mathbb{Q} - G_K, \tag{32}$$

where η_g and η'_g are functions taking values in the group of roots of unity in $L^\times$. The element $U_g \in (H_\wp \otimes W_g)$ of (5) is thus represented the matrix

$$U_g : \begin{pmatrix} \log_\wp(u_K) & 0 \\ 0 & \log_\wp(u'_K) \end{pmatrix},$$

and hence the endomorphism A_g of Lemma 5 is represented by the particularly simple matrix

$$A_g : \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It follows that, if $\ell = \lambda\lambda'$ is split in $K/\mathbb{Q}$, we have

$$A_g(\lambda) : \frac{1}{2}\begin{pmatrix} \log_p(\ell) & 0 \\ 0 & \log_p(\ell) \end{pmatrix}, \tag{33}$$

while if ℓ is inert in $K/\mathbb{Q}$ and λ is a prime of H lying above ℓ,

$$A_g(\lambda) : \frac{1}{2}\begin{pmatrix} \log_p(\ell) & -\log_\wp(u_\psi(\lambda)) \\ -\log_\wp(u'_\psi(\lambda)) & \log_p(\ell) \end{pmatrix}. \tag{34}$$

Theorem 19 *For all rational primes $\ell \nmid Np$,*

(a) *If ℓ is split in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = \frac{1}{2}a_\ell(g) \cdot \log_p(\ell).$$

(b) *If ℓ is inert in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = -\left(\eta_g(\lambda)\log_\wp(u'_\psi(\lambda)) + \eta_g(\lambda')\log_\wp(u_\psi(\lambda))\right).$$

Proof This follows directly from Theorem 10 in light of (33) and (34). □

Remark 20 As already remarked in [3], Theorem 19 above (and also Theorem 29 of Part B) display a striking analogy with Theorem 1.1. of [6] concerning the fourier expansions of mock modular forms whose shadows are weight one theta series attached to characters of imaginary quadratic fields. The underlying philosophy is that the p-adic deformations considered in this paper behave somewhat like mock modular forms of weight one, “with ∞ replaced by p”. This explains why the analogy remains compelling when the quadratic imaginary fields of [6] are replaced by real quadratic fields in which p is inert (these fields being “imaginary” from a p-adic perspective).

We illustrate Theorem 19 on the form of smallest level whose associated Artin representation is induced from a character of a real quadratic field, but of no imaginary quadratic field. The projective image in this example is the dihedral group D_8 of order 8:

Example 21 Let $\chi = \chi_5\chi_{29}$ where χ_5 and χ_{29} are quadratic and quartic characters of conductor 5 and 29, respectively. Then $S_1(145, \chi)$ is one dimensional and spanned by the eigenform

$$g = q + iq^4 - iq^5 + iq^9 + (-i-1)q^{11} - q^{16} + (-i-1)q^{19} + \cdots.$$

The form g is induced from a quartic character of a ray class group of $K = \mathbb{Q}(\sqrt{5})$ (see [4, Example 4.1] for a further discussion on this form). The relevant ring class field H is

$$H = \mathbb{Q}(\alpha) \text{ where } \alpha^8 - 2\alpha^7 + 4\alpha^6 - 26\alpha^5 + 94\alpha^4 - 212\alpha^3 + 761\alpha^2 - 700\alpha + 980 = 0.$$

Write $\mathrm{Gal}\,(H/K) = \langle \psi \rangle$. Take $p = 13$, and note that $\chi(p) = 1$ and so $\alpha = i$ and $\beta = -i$. We compute $g'_{\pm i}$ to 10 digits of 13-adic precision (and q-adic precision $q^{28,000}$).

Consider first the prime $\ell = 7$ which is inert in K. We take the 7-unit $u(7) \in H$ to satisfy $x^4 + 13x^3 + 38x^2 + 5x + 343 = 0$ and define

$$\log_{13}(u(7, \pm i)) := \log_{13}(u(7)) \mp i \log_{13}(u(7)^{\psi}) \mp \log_{13}(u(7)^{\psi^2}) \pm i \log_{13}(u(7)^{\psi^3})$$

and so $\log_{13}(u(7, i)) + \log_{13}(u(7, -i)) = \log_{13}(v)$ where $v = u(7)/u(7)^{\psi^2} \in H$. Then one checks that to 10 digits of 13-adic precision

$$a_7(g'_i) = -\frac{1}{6} \cdot \log_{13}(v)$$

which is in line with Theorem 19. Next we take the prime $\ell = 11$ which is split in K. Then to 10-digits of 13-adic precision

$$a_{11}(g'_i) = -\frac{i+1}{2} \cdot \log_{13}(11)$$

exactly as predicted by Theorem 19.

Part B: The Irregular Setting

Denote by $S_k(Np, \chi)$ (resp. by $S_k^{(p)}(N, \chi)$) the space of classical (resp. p-adic overconvergent) modular forms of weight k, level Np (resp. tame level N) and character χ, with coefficients in $\mathbb{Q}_p$. The Hecke algebra $\mathbb{T}$ of level Np generated over $\mathbb{Q}$ by the operators T_ℓ with $\ell \nmid Np$ and U_ℓ with $\ell \mid Np$ acts naturally on the spaces $S_k(Np, \chi)$ and $S_k^{(p)}(N, \chi)$.

As in the introduction, let $g \in S_1(N, \chi)$ be a newform and let $g_\alpha \in S_1(Np, \chi)$ be a p-stabilisation of g. The eigenform g_α gives rise to a ring homomorphism $\varphi_{g_\alpha} : \mathbb{T} \longrightarrow L$ to the field L generated by the fourier coefficients of g_α, satisfying

$$\varphi_{g_\alpha}(T_\ell) = a_\ell(g_\alpha) \text{ if } \ell \nmid Np, \qquad \varphi_{g_\alpha}(U_\ell) = \begin{cases} a_\ell(g_\alpha) & \text{if } \ell \mid N; \\ \alpha & \text{if } \ell = p. \end{cases} \tag{35}$$

For any ideal I of a ring R and any R-module M, denote by $M[I]$ the I-torsion in M. Let $I_{g_\alpha} \lhd \mathbb{T}$ be the kernel of φ_{g_α}, and set

$$S_1(Np, \chi)[g_\alpha] := S_1(Np, \chi)[I_{g_\alpha}], \qquad S_1(Np, \chi)[[g_\alpha]] := S_1(Np, \chi)[I_{g_\alpha}^2].$$

Our main object of study is the subspace

$$S_1^{(p)}(N,\chi)[[g_\alpha]] := S_1^{(p)}(N,\chi)[I_{g_\alpha}^2].$$

of the space of overconvergent p-adic modular forms of weight one, which is contained in the generalised eigenspace attached to I_{g_α}. An element of $S_1^{(p)}(N,\chi)[[g_\alpha]]$ is called an *overconvergent generalised eigenform* attached to g_α, and it is said to be *classical* if it belongs to $S_1(Np,\chi)[[g_\alpha]]$. The theorem of Bellaiche and Dimitrov stated in the opening paragraphs of Part A implies that the natural inclusion

$$S_1(Np,\chi)[[g_\alpha]] \hookrightarrow S_1^{(p)}(N,\chi)[[g_\alpha]]$$

is an isomorphism, i.e., every overconvergent generalised eigenform is classical, except possibly in the following cases:

(a) g is the theta series attached to a finite order character of a real quadratic field in which the prime p splits, or
(b) g is *irregular* at p, i.e., $\alpha = \beta$.

The study of $S_1^{(p)}(N,\chi)[[g_\alpha]]$ in scenario (a) was carried out in [3] when $\alpha \neq \beta$. The main result of loc.cit. is the description of a basis $(g_\alpha, g_\alpha^\flat)$ for $S_1^{(p)}(N,\chi)[[g_\alpha]]$ which is *canonical up to scaling*, and an expression for the fourier coefficients of the non-classical $g_\alpha^\flat$ (or rather, of their ratios) in terms of p-adic logarithms of certain algebraic numbers.

Assume henceforth that g is not regular at p, i.e., that $\alpha = \beta$. In that case, the form g admits a unique p-stabilisation $g_\alpha = g_\beta$. The Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid N$ act semisimply (i.e., as scalars) on the two-dimensional vector space

$$S_1(Np,\chi)[[g_\alpha]] = \mathbb{Q}_p g_\alpha \oplus \mathbb{Q}_p g', \qquad g'(q) := g(q^p),$$

but the Hecke operator U_p acts non-semisimply via the formulae

$$U_p g_\alpha = \alpha g_\alpha, \qquad U_p g' = g_\alpha + \alpha g'.$$

Because

$$(a_1(g_\alpha), a_p(g_\alpha)) = (1,\alpha), \qquad (a_1(g'), a_p(g')) = (0,1),$$

the classical subspace $S_1(Np,\chi)[[g_\alpha]]$ has a natural linear complement in $S_1^{(p)}(N,\chi)[[g_\alpha]]$, consisting of the generalised eigenforms $\tilde{g}$ whose q-expansions satisfy

$$a_1(\tilde{g}) = a_p(\tilde{g}) = 0. \tag{36}$$

A modular form satisfying (36) is said to be *normalised*, and the space of normalised generalised eigenforms is denoted $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$. The main goal of Part B is to study this space and give an explicit description of its elements in terms of their fourier expansions. The idoneous fourier coefficients will be expressed as determinants of 2×2 matrices whose entries are p-adic logarithms of algebraic numbers the number field H cut out by the projective Galois representation attached to g (cf. Theorems 26, 27 and 29).

4 Generalised Eigenspaces

We begin by recalling some of the notations that were already introduced in Part A. Let

$$\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V_g) \simeq \mathbf{GL}_2(\mathbb{Q}_p)$$

be the odd, two-dimensional Artin representation associated to g by Deligne and Serre (but viewed as having p-adic rather than complex coefficients; as in Part A, we assume for simplicity that the image of ϱ_g can be embedded in $\mathbf{GL}_2(\mathbb{Q}_p)$ and not just in $\mathbf{GL}_2(\bar{\mathbb{Q}}_p)$).

The four-dimensional $\mathbb{Q}_p$-vector space $W_g := \mathrm{Ad}(V_g) := \mathrm{End}(V_g)$ of endomorphisms of V_g is endowed with the conjugation action of $G_{\mathbb{Q}}$,

$$\sigma \cdot M := \varrho_g(\sigma) \circ M \circ \varrho_g(\sigma)^{-1}, \qquad \text{for any } \sigma \in G_{\mathbb{Q}}, \quad M \in W_g.$$

Let H be the field cut out by this Artin representation. The action of $G_{\mathbb{Q}}$ on W_g factors through a faithful action of the finite quotient $G := \mathrm{Gal}\,(H/\mathbb{Q})$. Let $W_g^\circ := \mathrm{Ad}^0(V_g)$ denote the three-dimensional $G_{\mathbb{Q}}$-submodule of W_g consisting of trace zero endomorphisms. The exact sequence

$$0 \longrightarrow W_g^\circ \longrightarrow W_g \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

of G-modules admits a canonical G-equivariant splitting

$$p : W_g \longrightarrow W_g^\circ, \qquad p(A) := A - 1/2 \cdot \mathrm{Tr}(A).$$

Because the action of $G_{\mathbb{Q}}$ on V_g also factors through a finite quotient, the field $L \subset \mathbb{Q}_p$ generated by the traces of ϱ_g is a finite extension of $\mathbb{Q}$, and ϱ_g maps the semisimple algebra $L[G_{\mathbb{Q}}]$ to a central simple algebra of rank 4 over L. By eventually enlarging L, it can be assumed that $\varrho_g(L[G_{\mathbb{Q}}]) \simeq M_2(L)$, and therefore that ϱ_g is realised on a two-dimensional L-vector space V_g^L equipped with an identification $\iota : V_g^L \otimes_L \mathbb{Q}_p \longrightarrow V_g$. The spaces

$$W_g^L := \mathrm{Ad}(V_g^L), \qquad W_g^{\circ L} := \mathrm{Ad}^0(V_g^L)$$

likewise correspond to G-stable L-rational structures on W_g and W_g° respectively, equipped with identifications

$$\iota : W_g^L \otimes_L \mathbb{Q}_p \longrightarrow W_g, \qquad \iota : W_g^{\circ L} \otimes_L \mathbb{Q}_p \longrightarrow W_g^\circ.$$

The spaces W_g and W_g° (as well as W_g^L and $W_g^{\circ L}$) are equipped with the Lie bracket $[\ ,\ ]$ and with a symmetric non-degenerate pairing $\langle\ ,\ \rangle$ defined by the usual rules

$$[A, B] := AB - BA, \qquad \langle A, B\rangle := \mathrm{Tr}(AB),$$

which are compatible with the G-action in the sense that

$$[\sigma \cdot A, \sigma \cdot B] = \sigma \cdot [A, B], \qquad \langle \sigma \cdot A, \sigma \cdot B\rangle = \langle A, B\rangle, \qquad \text{for all } \sigma \in G.$$

These operations can be combined to define a G-invariant determinant function—i.e., a non-zero, alternating trilinear form—on W_g° and on $W_g^{\circ L}$ by setting

$$\det(A, B, C) := \langle [A, B], C\rangle.$$

The rule described in (35) gives rise to natural identifications

$$S_1(Np, \chi)[g_\alpha] \simeq \mathrm{Hom}(\mathbb{T}/I_{g_\alpha}, \mathbb{Q}_p), \quad S_1^{(p)}(N, \chi)[[g_\alpha]] \simeq \mathrm{Hom}(\mathbb{T}/I_{g_\alpha}^2, \mathbb{Q}_p),$$

and hence the dual of the short exact sequence

$$0 \to I_{g_\alpha}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha} \to 0$$

can be identified with

$$0 \longrightarrow S_1(Np, \chi)[g_\alpha] \longrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]] \longrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \longrightarrow 0.$$

In particular, one has the isomorphism

$$S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \simeq \mathrm{Hom}(I_{g_\alpha}/I_{g_\alpha}^2, \mathbb{Q}_p). \tag{37}$$

Let $\mathbb{Q}_p[\varepsilon] = \mathbb{Q}_p[x]/(x^2)$ denote the ring of dual numbers. Given $g^\flat \in S_1^{(p)}(N, \chi)[[g_\alpha]]_0$, the modular form $\tilde{g} := g_\alpha + \varepsilon \cdot g^\flat$ is an eigenform for $\mathbb{T}$ with coefficients in $\mathbb{Q}_p[\varepsilon]$. Its associated Galois representation

$$\varrho_{\tilde{g}} : G_\mathbb{Q} \longrightarrow \mathbf{GL}_2(\mathbb{Q}_p[\varepsilon])$$

satisfies

(i) $\varrho_{\tilde{g}} = \varrho_g \pmod{\varepsilon}$ and $\det(\varrho_{\tilde{g}}) = \chi$,
(ii) for every prime number $\ell \nmid Np$, the trace of an arithmetic Frobenius τ_ℓ at ℓ is

$$\mathrm{Tr}(\varrho_{\tilde{g}}(\tau_\ell)) = a_\ell(g_\alpha) + \varepsilon \cdot a_\ell(g^\flat). \tag{38}$$

Conjecture 22 Assume that g is irregular at p. Then the assignment $g^\flat \mapsto \varrho_{\tilde{g}}$ gives rise to a canonical isomorphism between $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ and the space $\mathrm{Def}^0(\varrho_g)$ of isomorphism classes of deformations of ϱ_g to the ring of dual numbers, with constant determinant.
We now derive some consequences of this conjecture.

Proposition 23 *Assume Conjecture 22. If g is irregular at p, then the space $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ is two-dimensional over $\mathbb{Q}_p$.*

Proof Since any $\tilde{\varrho} \in \mathrm{Def}^0(\varrho_g)$ has constant determinant, it may be written as

$$\tilde{\varrho} = (1 + \varepsilon \cdot c) \cdot \varrho_g \quad \text{for some} \quad c = c(\tilde{\varrho}) : G_{\mathbb{Q}} \longrightarrow W_g^\circ. \tag{39}$$

The multiplicativity of $\tilde{\varrho}$ implies that the function c is a 1-cocycle of $G_{\mathbb{Q}}$ with values in W_g°, whose class in $H^1(\mathbb{Q}, W_g^\circ)$ (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of $\tilde{\varrho}$. The assignment $\tilde{\varrho} \mapsto c(\tilde{\varrho})$ realises an isomorphism (cf. for instance [8, §1.2])

$$\mathrm{Def}^0(\varrho_g) \longrightarrow H^1(\mathbb{Q}, W_g^\circ).$$

Under Conjecture 22, this yields an isomorphism

$$S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \xrightarrow{\sim} H^1(\mathbb{Q}, W_g^\circ), \quad g^\flat \mapsto c_{g^\flat}. \tag{40}$$

The inflation-restriction sequence combined with global class field theory for H now gives rise to a series of identifications

$$\begin{aligned} H^1(\mathbb{Q}, W_g^\circ) &\xrightarrow{\mathrm{res}_H} \hom(G_H, W_g^\circ)^G \\ &= \hom_G\left(\frac{(\mathcal{O}_H \otimes \mathbb{Z}_p)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Z}_p}, W_g^\circ\right) \\ &= \hom_G\left(\frac{H_p}{U}, W_g^\circ\right) \\ &= \ker\left(\hom_G(H_p, W_g^\circ) \xrightarrow{\mathrm{res}_U} \hom_G(U, W_g^\circ)\right), \end{aligned} \tag{41}$$

where U denotes the natural image of $\mathcal{O}_H^\times \otimes \mathbb{Z}_p$ in $H_p := H \otimes \mathbb{Q}_p$ under the p-adic logarithm map

$$\log_p : H_p^\times \longrightarrow H_p.$$

As representations for G, the space H_p is isomorphic to the regular representation

$$H_p \simeq \mathrm{Ind}_1^G \mathbb{Q}_p,$$

while U, by the Dirichlet unit theorem, is induced from the trivial representation of the subgroup $G_\infty \subset G$ generated by a complex conjugation:

$$U \simeq \mathrm{Ind}_{G_\infty}^G \mathbb{Q}_p.$$

Complex conjugation acts on W_g° with eigenvalues 1, -1 and-1, and hence by Frobenius reciprocity,

$$\dim_{\mathbb{Q}_p} \hom_G(H_p, W_g^\circ) = 3, \qquad \dim_{\mathbb{Q}_p} \hom_G(U, W_g^\circ) = 1. \tag{42}$$

It follows from (41) that $H^1(\mathbb{Q}, W_g^\circ)$ is two-dimensional over $\mathbb{Q}_p$. Proposition 23 follows. □

For any $\ell \nmid Np$, the ℓth fourier coefficient of $g^\flat$ is given in terms of the associated cocycle $c_{g^\flat}$ by the rule

$$a_\ell(g^\flat) = \mathrm{Tr}(c_{g^\flat}(\sigma_\lambda)\varrho_g(\sigma_\lambda)) \tag{43}$$

where $\lambda|\ell$ is any prime above ℓ and σ_λ denotes the arithmetic Frobenius associated to it. Note that the right-hand side of (43) does not depend on the choice of λ.

Our next goal is to parametrise the elements of (41) explicitly, and then to derive concrete formulae for the fourier expansions of the associated modular forms in $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ via (40) and (43). After treating the general case in Sect. 5, Sects. 6 and 7 focus on the special features of the scenarios where W_g° is reducible, i.e.,

(i) the CM case where V_g is induced from a character of an imaginary quadratic field;
(ii) the RM case where V_g is induced from a character of a real quadratic field.

5 The General Case

The Galois representation W_g° is irreducible if and only if $G := \mathrm{Gal}(H/\mathbb{Q})$ is isomorphic to A_4, S_4, or A_5. Otherwise, the representation ϱ_g has dihedral projective image and G is isomorphic to a dihedral group.

The irregularity assumption implies that the prime p splits completely in H, and H can therefore be viewed as a subfield of $\mathbb{Q}_p$ after fixing an embedding $H \hookrightarrow \mathbb{Q}_p$ once and for all. This amounts to choosing a prime $\wp$ of H above p. Let $\log_\wp : H_p^\times \longrightarrow \mathbb{Q}_p$ denote the associated $\wp$-adic logarithm map, which factors through $\log_p$.

The Dirichlet unit theorem implies (via the second Eq. in (42)) that

$$\dim_L(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G = 1.$$

In particular, for all $\mathfrak{u} \in \mathcal{O}_H^\times$ and all $w \in W_g^{\circ L}$, the element

$$\xi(\mathfrak{u}, w) := \frac{1}{\#G} \times \sum_{\sigma \in G} (\sigma \mathfrak{u}) \otimes (\sigma \cdot w) \in (\mathcal{O}_H^\times \otimes W_g^{\circ L})^G \tag{44}$$

only depends on the choices of $\mathfrak{u}$ and w up to scaling by a (possibly zero) factor in L. As $\mathfrak{u}$ varies over $\mathcal{O}_H^\times$ and w over $W_g^{\circ L}$, the elements

$$\xi_\wp(\mathfrak{u}, w) := (\log_\wp \otimes \mathrm{id})\xi(\mathfrak{u}, w) = \frac{1}{\#G} \times \sum_{\sigma \in G} \log_\wp(\sigma \mathfrak{u}) \cdot (\sigma \cdot w) \in W_g^\circ \tag{45}$$

therefore lie in a one-dimensional L-vector subspace of W_g°. Choose a generator $w(1)$ for this space. The coordinates of $w(1)$ relative to a basis (e_1, e_2, e_3) for $W_g^{\circ L}$ are $\wp$-adic logarithms of units in $\mathcal{O}_H$, namely, we can write

$$w(1) = \log_\wp(\mathfrak{u}_1)e_1 + \log_\wp(\mathfrak{u}_2)e_2 + \log_\wp(\mathfrak{u}_3)e_3, \tag{46}$$

for appropriate $\mathfrak{u}_i \in (\mathcal{O}_H^\times) \otimes_{\mathbb{Z}} L$.

Let $\ell \nmid Np$ be a rational prime. For any prime λ of H above ℓ, let $\tilde{\mathfrak{u}}_\lambda$ be a generator of the principal ideal λ^h, where h is the class number of H, and set

$$\mathfrak{u}_\lambda := \tilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in (\mathcal{O}_H[1/\ell]^\times) \otimes_{\mathbb{Z}} L.$$

Let

$$\tilde{w}_\lambda := \varrho_g(\sigma_\lambda) \in W_g^L, \qquad w_\lambda := p(\tilde{w}_\lambda) \in W_g^{\circ L} \tag{47}$$

be the endomorphisms of V_g arising from the image of σ_λ under ϱ_g. The element $\mathfrak{u}_\lambda$ is well-defined up to multiplication by elements of $\mathcal{O}_H^\times$, and hence the elements

$$\xi(\mathfrak{u}_\lambda, w_\lambda) := \frac{1}{\#G} \times \sum_{\sigma \in G} (\sigma \mathfrak{u}_\lambda) \otimes (\sigma \cdot w_\lambda) \in (\mathcal{O}_H[1/\ell]^\times \otimes W_g^{\circ L})^G,$$

$$w(\ell) = \xi_\wp(\mathfrak{u}_\lambda, w_\lambda) := \frac{1}{\#G} \times \sum_{\sigma \in G} \log_\wp(\sigma \mathfrak{u}_\lambda) \cdot (\sigma \cdot w_\lambda) \in W_g^\circ \tag{48}$$

are defined up to translation by elements of the one-dimensional L-vector spaces $(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G$ and $L \cdot w(1)$ respectively. Furthermore, the image of $w(\ell)$ in the quotient $W_g^\circ/(L \cdot w(1))$ does not depend on the choice of the prime λ of H above ℓ

that was made to define it. The Lie bracket

$$\mathfrak{W}(\ell) := [w(1), w(\ell)] \in W_g^\circ$$

is thus independent of the choices that were made in defining $w(\ell)$.

Remark 24 The coordinates of $w(\ell)$ relative to a basis (e_1, e_2, e_3) for $W_g^{\circ L}$ are $\wp$-adic logarithms of ℓ-units in H, i.e., one can write

$$w(\ell) = \log_\wp(\mathfrak{v}_1)e_1 + \log_\wp(\mathfrak{v}_2)e_2 + \log_\wp(\mathfrak{v}_3)e_3, \tag{49}$$

with $\mathfrak{v}_i \in (\mathcal{O}_H[1/\ell]^\times)_L$ for $i = 1, 2, 3$. A direct computation shows that

$$\dim_{\mathbb{Q}_p}(W_g^\circ)^{\sigma_\lambda=1} = \begin{cases} 1 & \text{if } g \text{ is regular at } \ell; \\ 3 & \text{if } g \text{ is irregular at } \ell. \end{cases}$$

It follows that for all regular primes ℓ,

$$\dim_L(\mathcal{O}_H[1/\ell]^\times \otimes W_g^{\circ L})^G = 2,$$

and therefore that the element $\xi(\mathfrak{v}, w)$ attached to any pair $(\mathfrak{v}, w) \in \mathcal{O}[1/\ell]^\times \times W_g^{\circ L}$ as in (48) is well-defined up to scaling by L and up to translation by elements of the one-dimensional space $(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G$. In particular, the associated vector $\mathfrak{W}(\ell)$ lies in a canonical one-dimensional subspace of W_g°, namely, the orthogonal complement in W_g° of

$$(\log_\wp \otimes \mathrm{Id})(\mathcal{O}_H[1/\ell]^\times \otimes W_g^\circ)^G \subset W_g^\circ.$$

If the basis $(e_1 e_2, e_3)$ for $W_g^{\circ L}$ in (46) and (49) is taken to be the standard basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$\mathfrak{W}(\ell) \;=\; \det\begin{pmatrix} \mathfrak{u}_2 & \mathfrak{u}_3 \\ \mathfrak{v}_2 & \mathfrak{v}_3 \end{pmatrix} \cdot e_1 \;+\; 2\det\begin{pmatrix} \mathfrak{u}_1 & \mathfrak{u}_2 \\ \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot e_2 \;-\; 2\det\begin{pmatrix} \mathfrak{u}_1 & \mathfrak{u}_3 \\ \mathfrak{v}_1 & \mathfrak{v}_3 \end{pmatrix} \cdot e_3.$$

Remark 25 Observe that if the prime ℓ is irregular for g, the vector $\tilde{w}_\lambda$ is a scalar endomorphism in W_g and hence $w_\lambda = w(\ell) = \mathfrak{W}(\ell) = 0$.

Our main result is

Theorem 26 *Assume Conjecture 22. For all $w \in W_g^\circ$, there exists an overconvergent generalised eigenform $g_w^\flat \in S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ satisfying*

$$a_\ell(g_w^\flat) = \langle w, \mathfrak{W}(\ell)\rangle = \det(w, w(1), w(\ell)),$$

for all primes $\ell \nmid Np$. The assignment $w \mapsto g_w^\flat$ induces an isomorphism between W_g°/U and $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$.

Proof The semi-local field $H_p = H \otimes_{\mathbb{Q}} \mathbb{Q}_p = \oplus_{\wp|p}\mathbb{Q}_p$ is naturally identified with the set of vectors $h = (h_\wp)_{\wp|p}$ with entries $h_\wp \in \mathbb{Q}_p$, indexed by the primes of H above p. The function which to $w \in W_g^\circ$ associates the linear transformation

$$\tilde{\varphi}_w : H_p \longrightarrow W_g^\circ, \qquad \tilde{\varphi}_w(h) = \frac{1}{\#G} \times \sum_{\sigma\in G} (\sigma^{-1}h)_\wp \cdot (\sigma \cdot w)$$

identifies W_g° with $\hom_G(H_p, W_g^\circ)$. The linear function $\tilde{\varphi}_w$ is trivial on $U := \log_p(\mathcal{O}_H^\times) \subset H_p$ if and only if, for all $\mathrm{u} \in \mathcal{O}_H^\times$ and all $w' \in W_g^\circ$,

$$\langle \tilde{\varphi}_w(\log_p(\mathrm{u})), w'\rangle = 0.$$

But

$$\begin{aligned}
\langle \tilde{\varphi}_w(\log_p(\mathrm{u})), w'\rangle &= \frac{1}{\#G} \times \left\langle \sum_{\sigma\in G} \log_\wp(\sigma^{-1}(\mathrm{u})) \cdot (\sigma \cdot w),\ w' \right\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma^{-1}(\mathrm{u})) \cdot \langle \sigma \cdot w, w'\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma^{-1}(\mathrm{u})) \cdot \langle w, \sigma^{-1} \cdot w'\rangle \\
&= \langle w, \xi_\wp(\mathrm{u}, w')\rangle,
\end{aligned}$$

and hence $\tilde{\varphi}_w$ is trivial on $U = \log_p(\mathcal{O}_H^\times)$ if and only if w is orthogonal in W_g° to the line spanned by $w(1)$. It follows that the G-equivariant linear function

$$\varphi_w := \tilde{\varphi}_{[w,w(1)]} : H_p \longrightarrow W_g^\circ$$

factors through H_p/U. The assignment $w \mapsto \varphi_w$ identifies $W_g^\circ/(L \cdot w(1))$ with $\hom_G(H_p/U, W_g^\circ)$, and gives an explicit description of the latter space.

Let $\tilde{g}_w = g + \varepsilon g_w^\flat$ be the eigenform with coefficients in $\mathbb{Q}_p[\varepsilon]$ which is attached to the cocycle $\varphi_w \in \hom_G(H_p/U, W_g^\circ) = H^1(\mathbb{Q}, W_g^\circ)$. Equation (43) with $g^\flat = g_w^\flat$ (and hence $c_{g^\flat} = \varphi_w$) combined with (47) shows that the ℓth the fourier coefficient of $g_w^\flat$ at a prime $\ell \nmid Np$ is equal to

$$a_\ell(g_w^\flat) = \mathrm{Tr}(\varphi_w(\sigma_\lambda) \cdot \varrho_g(\sigma_\lambda)) = \langle \varphi_w(\sigma_\lambda), \tilde{w}_\lambda\rangle = \langle \varphi_w(\sigma_\lambda), w_\lambda\rangle. \tag{50}$$

Class field theory for H implies that

$$\varphi_w(\sigma_\lambda) = \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma^{-1}\mathrm{u}_\lambda) \cdot \sigma \cdot [w, w(1)].$$

Hence

$$\begin{aligned}
a_\ell(g_w^\flat) &= \frac{1}{\#G} \times \left\langle \sum_{\sigma\in G} \log_\wp(\sigma^{-1}\mathrm{u}_\lambda)\cdot\sigma\cdot[w,w(1)],\ w_\lambda \right\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma^{-1}\mathrm{u}_\lambda)\cdot\langle\sigma\cdot[w,w(1)],\ w_\lambda\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma^{-1}\mathrm{u}_\lambda)\cdot\langle[w,w(1)],\ \sigma^{-1}\cdot w_\lambda\rangle \\
&= \langle[w,w(1)],w(\ell)\rangle = \det(w,w(1),w(\ell)) = \langle w,\mathfrak{W}(\ell)\rangle.
\end{aligned}$$

The theorem follows. □

If w in a vector in $W_g^{\circ L}$, Theorem 26 shows that the associated overconvergent generalised eigenform $g_w^\flat$ has fourier coefficients which are L-rational linear combinations of determinants of 2×2 matrices whose entries are the $\wp$-adic logarithms of algebraic numbers in H. In the CM and RM cases to be discussed below, the representation W_g° is reducible and decomposes further into non-trivial irreducible representations. In that case the choice of an L-basis for $W_g^{\circ L}$ which is compatible with this decomposition leads to canonical elements of $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ which can sometimes be re-scaled so that their fourier expansions admit even simpler expressions, as will be described in the next two sections.

6 CM Forms

Assume that g is the theta series attached to a character of a quadratic imaginary field K, i.e., that

$$V_g^L = \mathrm{Ind}_K^{\mathbb{Q}}\,\psi_g,$$

where $\psi_g : \mathrm{Gal}\,(\bar K/K) \longrightarrow L^\times$ is a finite order character. Let ψ_g' denote the character deduced from ψ_g by composing it with the involution in $\mathrm{Gal}\,(K/\mathbb{Q})$. The irreducibility assumption on V_g^L implies that the characters ψ_g and ψ_g' are distinct, and therefore the representations V_g^L and V_g decompose canonically as a direct sum of two G_K-stable one-dimensional subspaces

$$V_g^L = \mathcal{L}_{\psi_g}^L \oplus \mathcal{L}_{\psi_g'}^L, \qquad V_g = \mathcal{L}_{\psi_g} \oplus \mathcal{L}_{\psi_g'}$$

on which G_K acts via the characters ψ_g and ψ_g' respectively. The representations W_g^L and W_g also decompose as direct sums of four G_K-stable lines

$$W_g^L = \left(\hom(\mathcal{L}^L_{\psi_g}, \mathcal{L}^L_{\psi_g}) \oplus \hom(\mathcal{L}^L_{\psi_g'}, \mathcal{L}^L_{\psi_g'})\right) \oplus \left(\hom(\mathcal{L}^L_{\psi_g'}, \mathcal{L}^L_{\psi_g}) \oplus \hom(\mathcal{L}^L_{\psi_g}, \mathcal{L}^L_{\psi_g'})\right),$$

$$W_g = \left(\hom(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi_g}) \oplus \hom(\mathcal{L}_{\psi_g'}, \mathcal{L}_{\psi_g'})\right) \oplus \left(\hom(\mathcal{L}_{\psi_g'}, \mathcal{L}_{\psi_g}) \oplus \hom(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi_g'})\right).$$

The direct summands in parentheses are also stable under $G_{\mathbb{Q}}$ and are isomorphic to the induced representations $\mathrm{Ind}_K^{\mathbb{Q}} 1$ and $\mathrm{Ind}_K^{\mathbb{Q}} \psi$ respectively, where $\psi := \psi_g/\psi_g'$, is the *ring class character* of K associated to ψ_g. It follows that

$$W_g^{\circ L} = L(\chi_K) \oplus Y_g^L, \qquad W_g^{\circ} = \mathbb{Q}_p(\chi_K) \oplus Y_g, \qquad Y_g^L := \mathrm{Ind}_K^{\mathbb{Q}} \psi, \quad Y_g := Y_g^L \otimes_L \mathbb{Q}_p.$$

It will be convenient to choose a basis $(e_1, e_2) \in \mathcal{L}^L_{\psi_g} \times \mathcal{L}^L_{\psi_g'}$ for V_g^L, and to denote by $e_{11}, e_{12}, e_{21}, e_{22}$ the resulting basis of W_g^L, where e_{ij} is the elementary matrix whose (i', j')-entry is $\delta i = i'\delta_{j=j'}$. Relative to the identification of $W_g^{\circ L}$ with the space of 2×2 matrices of trace zero with entries in L via this basis, the representation $L(\chi_K) = L \cdot (e_{11} - e_{22})$ is identified with the space of diagonal matrices of trace 0, while $Y_g^L = L \cdot e_{12} \oplus L \cdot e_{21}$ is identified with the space of off-diagonal matrices in $M_2(L)$. Fix an element $\tau \in G_{\mathbb{Q}} = G_K$ once and for all. By eventually re-scaling e_1 and e_2, it can (and shall, henceforth) be assumed that $\varrho_g(\tau)$ is represented by the matrix $\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ in this basis, where $-t^2 := \chi(\tau)$.

Let $Z := \mathrm{Gal}\,(H/K)$ be the maximal abelian normal subgroup of the dihedral group $G = \mathrm{Gal}\,(H/\mathbb{Q})$. Note that every element in $G - Z$ (such as the image of τ in G) is an involution, and that Z operates transitively on $G - Z$ by either left or right multiplication.

The field H through which W_g° factors is the ring class field of K attached to the character ψ. The group $\mathcal{O}_H^\times \otimes \mathbb{Q}$ of units of H is isomorphic to the regular representation of Z minus the trivial representation, and a finite index subgroup of $\mathcal{O}_H^\times$ can be constructed explicitly from the elliptic units arising in the theory of complex multiplication. Let

$$e_\psi := \frac{1}{\#Z} \sum_{\sigma \in Z} \psi^{-1}(\sigma)\sigma$$

be the idempotent in the group ring of Z giving rise to the projection onto the ψ-isotypic component for the action of Z. Choose a unit $\mathfrak{u} \in \mathcal{O}_H^\times$ and let

$$\mathfrak{u}_\psi := e_\psi \mathfrak{u}, \qquad \tau \mathfrak{u}_\psi = e_{\psi'}(\tau \mathfrak{u})$$

be elements of $\mathcal{O}_H^\times \otimes L$ on which Z acts via the characters ψ and $\psi' = \psi^{-1}$ respectively. With these choices, we can let

$$w(1) = \begin{pmatrix} 0 & \log_\wp(\mathrm{u}_\psi) \\ \log_\wp(\tau \mathrm{u}_\psi) & 0 \end{pmatrix}. \tag{51}$$

The description of the canonical vectors $w(\ell), \mathfrak{W}(\ell) \in W_g^\circ$ attached to a rational prime $\ell \nmid Np$ depends in an essential way on whether ℓ is split or inert in $K/\mathbb{Q}$.

If $\ell = \lambda\lambda'$ is split in K and ℓ is regular for g, i.e., $\psi_g(\sigma_\lambda) \neq \psi_g(\lambda')$, then the natural map

$$(\mathcal{O}_K[1/\ell]^\times \otimes W_g^{\circ L})^G \subset \left(\frac{\mathcal{O}_H[1/\ell]^\times}{\mathcal{O}_H^\times} \otimes W_g^{\circ L}\right)^G$$

is an isomorphism of L-vector spaces.

Let $\tilde{\mathrm{u}}_\lambda$ be a generator of λ^h where h is the class number of K, and set

$$\mathrm{u}_\lambda := \tilde{\mathrm{u}}_\lambda \otimes h^{-1}.$$

Since

$$\tilde{w}_\lambda = \begin{pmatrix} \psi_g(\lambda) & 0 \\ 0 & \psi_g(\lambda') \end{pmatrix}, \qquad w_\lambda = \frac{\psi_g(\lambda) - \psi_g(\lambda')}{2} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

a direct calculation shows that

$$w(\ell) = \log_\wp(\mathrm{u}_\lambda/\mathrm{u}'_\lambda) \times \frac{(\psi_g(\lambda) - \psi_g(\lambda'))}{2} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \log_\wp(\mathrm{u}_\lambda/\mathrm{u}'_\lambda) \times (\psi_g(\lambda) - \psi_g(\lambda')) \times \begin{pmatrix} 0 & -\log_\wp(\mathrm{u}_\psi) \\ \log_\wp(\mathrm{u}'_\psi) & 0 \end{pmatrix}. \tag{52}$$

If ℓ is inert in K then ℓ is always regular for g since $\varrho_g(\tau_\ell)$ has trace 0 and hence has distinct eigenvalues. The prime ℓ splits completely in H/K, and hence the group $(\mathcal{O}_H[1/\ell]^\times) \otimes L$ is isomorphic to two copies of the regular representation of Z minus a trivial representation. The choice of a prime λ of H_g above ℓ determines a matrix (and not just a conjugacy class)

$$\tilde{w}_\lambda = w_\lambda = \varrho(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}$$

with entries in L. Let $\mathfrak{u}_\lambda$ be an element of $(\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times)\otimes L$ whose prime factorisation is given by

$$(\mathfrak{u}_\lambda) = b_\lambda \lambda + c_\lambda(\tau\lambda). \quad (53)$$

This ℓ-unit is only well defined by (53) up to translation by $\mathcal{O}_H^\times \otimes L$, and the defining Eq. (53) of course depends crucially on the choice of the prime λ above ℓ. However, the ψ-isotypic projection

$$\mathfrak{u}_\psi(\ell) := e_\psi \mathfrak{u}_\lambda \quad (54)$$

is independent of this choice. A direct calculation shows that

$$w(\ell) = \frac{1}{2} \times \begin{pmatrix} 0 & \log_\wp(\mathfrak{u}_\psi(\ell)) \\ \log_\wp(\tau\mathfrak{u}_\psi(\ell)) & 0 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \begin{pmatrix} R_\psi(\ell) & 0 \\ 0 & -R_\psi(\ell) \end{pmatrix}, \quad (55)$$

where

$$R_\psi(\ell) = \det \begin{pmatrix} \log_\wp(\mathfrak{u}_\psi) & \log_\wp(\tau\mathfrak{u}_\psi) \\ \log_\wp(\mathfrak{u}_\psi(\ell)) & \log_\wp(\tau\mathfrak{u}_\psi(\ell)) \end{pmatrix}$$

is an ℓ-unit regulator attached to ψ, which is independent of the choice of prime λ of H above ℓ. The function $\ell \mapsto R_\psi(\ell)$ does depend on the choice of the unit $\mathfrak{u}$, but only up to scaling by $L^\times$.

Theorem 27 *Assume Conjecture 22. The space $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ has a canonical basis $(g_1^\flat, g_2^\flat)$ which is characterised by the properties:*

(i) *The fourier coefficients $a_\ell(g_1^\flat)$ are 0 for all primes $\ell \nmid Np$ that are inert in K. If $\ell = \lambda\lambda'$ is split in K, then*

$$a_\ell(g_1^\flat) = (\psi_g(\lambda) - \psi_g(\lambda')) \times \log_\wp(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda)$$

is a simple algebraic multiple of the p-adic logarithm of the fundamental ℓ-unit of norm 1 in K.

(ii) *The fourier coefficients of $g_2^\flat$ are 0 at all the primes $\ell \nmid Np$ that are split in K. If ℓ is inert in K, then*

$$a_\ell(g_s^\flat) = R_\psi(\ell).$$

Proof This follows directly from the calculation of the matrices $\mathfrak{W}(\ell)$ in (52) and (55) in light of Theorem 26. □

Example 28 Let χ be the quadratic character of conductor 59. The space $S(59, \chi)$ is one dimensional and spanned by the theta series

$$g = q - q^3 + q^4 - q^5 - q^7 - q^{12} + q^{15} + q^{16} + 2q^{17} - \cdots .$$

Here $K = \mathbb{Q}(\sqrt{-59})$ and the ring class field attached to ψ is

$$H = K(\alpha) \text{ where } \alpha^3 - 3\alpha + 46\sqrt{-59} = 0.$$

The inert primes ℓ in K are $2, 3, 13, 23, \cdots$ and the unit and first few ℓ-units are

$$u = \tfrac{1}{612}\left(13\alpha^2 - 7\sqrt{-59}\alpha - 26\right), \qquad u_2 = \tfrac{1}{612}\left(-5\alpha^2 - 13\sqrt{-59}\alpha - 194\right)$$

$$u_{11} = \tfrac{1}{306}\left(5\alpha^2 + 13\sqrt{-59}\alpha - 112\right), \ u_{13} = \tfrac{1}{612}\left(13\alpha^2 - 7\sqrt{-59}\alpha - 1250\right)$$

$$u_{23} = \tfrac{1}{204}\left(-\alpha^2 + 11\sqrt{-59}\alpha + 138\right).$$

Let $p = 17$, an irregular prime for g. We computed a basis of q-expansions for the generalised eigenspace modulo p^{20} and $q^{30,000}$. One observes that it contains the classical space spanned by the forms $g_\alpha(q)$ and $g(q^p)$ and in addition a complementary space of dimension two. This space is canonically spanned by two normalised generalised eigenforms

$$\tilde{g}_1^\flat = q^3 + \cdots + 0 \cdot q^p + \cdots \quad \text{and} \quad \tilde{g}_2^\flat = q^2 + 0 \cdot q^3 + \cdots + \cdots + 0 \cdot q^p + \cdots .$$

Note that the natural scaling of the forms output by our algorithm is with leading Fourier coefficients equal to 1. By Theorem 27 one expects that for ℓ inert in K, or ℓ split in K but irregular, we have $a_\ell(\tilde{g}_1^\flat) = 0$; and for ℓ split in K we have that

$$a_\ell(\tilde{g}_1^\flat) = \frac{\log_p(u_\ell)}{\log_p(u_3)}$$

where u_ℓ is a fundamental ℓ-unit in K (the logarithm of this is well-defined up to sign). We checked this to 20-digits of 17-adic precision for primes $\ell < 1000$. Further, one expects that

$$a_\ell(\tilde{g}_2^\flat) = \frac{R_\psi(\ell)}{R_\psi(2)} \text{ for } \ell \text{ inert in } K, \text{ and } a_\ell(\tilde{g}_2^\flat) = 0 \text{ for } \ell \text{ split in } K.$$

We checked this for all split primes $\ell < 30{,}000$ and for the inert primes $\ell = 2, 3, 11$ and 23, constructing $R_{\psi}(\ell)$ using the unit u and ℓ-unit u_ℓ above.

7 RM Forms

We now turn to the RM setting where F is a real quadratic field and

$$V_g = \mathrm{Ind}_F^{\mathbb{Q}} \psi_g,$$

where $\psi_g : \mathrm{Gal}\,(\bar{F}/F) \longrightarrow L^\times$ is a finite order character of mixed signature. Letting ψ_g' denote the character deduced from ψ_g by composing it with the involution in $\mathrm{Gal}\,(F/\mathbb{Q})$, the ratio $\psi := \psi_g/\psi_g'$ is a totally odd L-valued ring class character of F.

As before, let H denote the ring class field of F which is fixed by the kernel of ψ, and set $Z := \mathrm{Gal}\,(H/F)$ and $G := \mathrm{Gal}\,(H/\mathbb{Q})$. Just as in the previous section,

$$W_g^\circ = \chi_K \oplus Y_g, \qquad Y_g := \mathrm{Ind}_K^{\mathbb{Q}} \psi,$$

and we can set

$$w(1) = \begin{pmatrix} \log_\wp(\mathrm{u}_F) & 0 \\ 0 & -\log_\wp(\mathrm{u}_F) \end{pmatrix},$$

where u_F is a fundamental unit of F.

If ℓ is split in $K/\mathbb{Q}$, it is easy to see that the vector $w(\ell)$ is proportional to $w(1)$, and hence that

$$\mathfrak{W}(\ell) = 0. \tag{56}$$

If ℓ is inert in K, let U_g and $U_g(\ell)$ denote the subspaces $(\mathcal{O}_H^\times \otimes Y_g)^{G_\mathbb{Q}}$ and $(\mathcal{O}_H[1/\ell]^\times \otimes Y_g)^{G_\mathbb{Q}}$. The the dimensions of these spaces are 0 and 1 respectively. Choose a prime λ of H above ℓ, and let u_λ and $\mathrm{u}_\psi(\ell)$ be the elements of $\mathcal{O}_H[1/\ell]^\times$ determined by the relations

$$(\mathrm{u}_\lambda) = b_\lambda \lambda + c_\lambda \tau\lambda, \qquad \mathrm{u}_\psi(\ell) = e_\psi(\mathrm{u}_\lambda), \qquad \mathrm{u}_\psi'(\ell) = \tau \mathrm{u}_\psi(\ell),$$

where

$$\varrho_g(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}.$$

The $\wp$-adic logarithms

$$\log_{\wp}(\mathrm{u}_{\psi}(\ell)), \qquad \log_{\wp}(\mathrm{u}'_{\psi}(\ell))$$

are well-defined invariants of ℓ and ϱ which do not depend on the choice of a prime λ lying above ℓ, and

$$w(\ell) = \frac{1}{2} \times \begin{pmatrix} 0 & \log_{\wp}(\mathrm{u}_{\psi}(\ell)) \\ \log_{\wp}(\mathrm{u}'_{\psi}(\ell)) & 0 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \log_{\wp}(\mathrm{u}_F) \times \begin{pmatrix} 0 & \log_{\wp}(\mathrm{u}_{\psi}(\ell)) \\ -\log_{\wp}(\mathrm{u}'_{\psi}(\ell)) & 0 \end{pmatrix}. \tag{57}$$

Theorem 29 *Assume Conjecture 22. The space $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ has a canonical basis $(g_1^\flat, g_2^\flat)$ which is characterised by the properties:*

(i) *The fourier coefficients of $g_1^\flat$ and $g_2^\flat$ are 0 at all primes $\ell \nmid Np$ that are split in F.*
(ii) *If ℓ is inert in F, then*

$$a_\ell(g_1^\flat) = \log_{\wp}(\mathrm{u}_{\psi}(\ell)), \qquad a_\ell(g_2^\flat) = \log_{\wp}(\mathrm{u}'_{\psi}(\ell)).$$

Proof This follows directly from Theorem 26 in light of Eqs. (56) and (57). □

Example 30 Let χ_8 and χ_7 denote the quadratic characters of conductors 8 and 7, respectively, and define $\chi := \chi_8\chi_7$. Then $S_1(56, \chi)$ is one-dimensional and spanned by the form

$$g = q - q^2 + q^4 - q^7 - q^8 - q^9 + q^{14} + q^{16} + q^{18} + 2q^{23} - \cdots.$$

We take $p = 23$, an irregular prime for g, and compute a basis for the generalised eigenspace modulo (p^{15}, q^{3000}). The two dimensional space complementary to the classical space has a natural basis

$$\tilde{g}_1^\flat = q^3 + \cdots + 0 \cdot q^p + \cdots \quad \text{and} \quad \tilde{g}_2^\flat = q^2 + 0 \cdot q^3 + \cdots + \cdots + 0 \cdot q^p + \cdots.$$

Take

$$g_1^\flat := \frac{1}{2} \cdot \log_p(u_2) \cdot \tilde{g}_1^\flat \quad \text{and} \quad g_2^\flat := \log_p(u_3) \cdot \tilde{g}_2^\flat.$$

Here u_ℓ, $\ell = 2$ and 3, denotes a fundamental ℓ-unit of norm 1 in $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-56})$, respectively. One finds that the coefficients at primes ℓ which are split in $\mathbb{Q}(\sqrt{8})$ of both forms $g_1^\flat$ and $g_2^\flat$ are zero. At inert primes the coefficients of $g_1^\flat$ are the logarithms of fundamental ℓ-units of norm 1 in $\mathbb{Q}(\sqrt{-7})$, and those of $g_2^\flat$ are the logarithms of fundamental ℓ-units of norm 1 in $\mathbb{Q}(\sqrt{-56})$ (such logarithms are well-defined up to sign; interestingly, the forms $g_j^\flat$ single out a consistent choice of signs).

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Computing Invariants of the Weil Representation

Stephan Ehlen and Nils-Peter Skoruppa

Abstract We propose an algorithm for computing bases and dimensions of spaces of invariants of Weil representations of $\mathrm{SL}_2(\mathbb{Z})$ associated to finite quadratic modules. We prove that these spaces are defined over $\mathbb{Z}$, and that their dimension remains stable if we replace the base field by suitable finite prime fields.

1 Introduction

The Weil representations associated to a finite abelian groups A equipped with a non-degenerate quadratic form Q provides a fundamental tool in the theory of automorphic forms. They are at the basis of the theory of automorphic products, the theory of Jacobi forms or Siegel modular forms of singular and critical weight, and they also find applications in other disciplines like coding theory or quantum field theory. Of particular interest for the mentioned applications is the space $\mathbb{C}[A]^G$ of invariants of the Weil representations of $G = \mathrm{SL}_2(\mathbb{Z})$ associated to a given finite quadratic module (A, Q). Despite the importance of $\mathbb{C}[A]^G$ for the indicated applications neither any explicit closed formula is known for the dimension of $\mathbb{C}[A]^G$ nor any useful description[1] of its elements.

[1]However, if (A, Q) possesses a self-dual isotropic subgroup U (i.e. a subgroup which equals its dual with respect to the bilinear form associated to Q and such that $Q(x) = 0$ for all x in U) then the characteristic function of U is quickly checked to be an invariant. Moreover, one can show that in this case the characteristic functions of the self-dual isotropic subgroups span in fact the space $\mathbb{C}[A]^G$ (A proof of this will be given in [10]). An arbitrary finite quadratic module does not necessarily possess self-dual isotropic subgroups and still admits nonzero invariants if its order is big enough.

S. Ehlen (✉)
Mathematisches Institut, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: stephan.ehlen@math.uni-koeln.de

N.-P. Skoruppa
Fachbereich Mathematik, Universität Siegen, Walter-Flex-Str. 3, 57072 Siegen, Germany
e-mail: nils.skoruppa@gmail.de

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The purpose of the present note is to discuss questions related to the computation of the dimension and a basis of $\mathbb{C}[A]^G$ for a given finite quadratic module (A, Q). In particular, we develop an algorithm (Algorithm 4.2) for computing a basis of $\mathbb{C}[A]^G$ which we also implemented and ran successfully in various examples.[2] We mention two results of this article which might be of independent interest. First, we prove that $\mathbb{C}[A]^G$ always possesses a basis whose elements are in $\mathbb{Z}[A]$ (Theorem 3.3). Second, if a finite prime field $\mathbb{F}_\ell$ contains the Nth roots of unity, where N is the level of (A, Q), then the Weil representation can also be defined on $\mathbb{F}_\ell[A]$. We prove that then $\dim \mathbb{C}[A]^G = \dim \mathbb{F}_\ell[A]^G$ (except for possibly $(N, \ell) = (2, 3)$). Our algorithm has already been used successfully to compute the dimension of spaces of vector valued cusp forms of weight 2 and $3/2$ in [3], where a classification of all lattices of signature $(2, n)$ without obstructions to the existence of weakly holomorphic modular forms of weight $1 - \frac{n}{2}$ for the associated Weil representation was given.

The plan of this note is as follows. In Sect. 2 we recall the basic definitions and facts from the theory of finite quadratic modules and its associated Weil representations. In Sect. 3 we prove some basic facts about the space of invariants $\mathbb{C}[A]^G$. Most of the material of this section is probably known to specialists. However, since it is often difficult to find suitable references we decided to include this section. To our knowledge Theorem 3.3 is new, which shows that the space of invariants $\mathbb{C}[A]^G$ is in fact defined over $\mathbb{Z}$. In Sect. 4 we explain our algorithm for computing a basis for $\mathbb{C}[A]^G$, and we discuss some improvements. In Sect. 5 we consider the reduction of Weil representations modulo suitable primes ℓ and prove that the dimension of the space of invariants remains stable under reduction. This interesting fact can be used to improve the run-time of our algorithm. Finally, in Sect. 6 we provide tables of dimensions for quadratic modules of small order.

2 Finite Quadratic Modules and Weil Representations

A *finite quadratic module* (also called a finite quadratic form or discriminant form in the literature) is a pair $\mathfrak{A} = (A, Q)$ consisting of a finite abelian group A together with a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form Q on A. The bilinear form corresponding to Q is defined as

$$Q(x, y) := Q(x + y) - Q(x) - Q(y).$$

The quadratic form Q is called non-degenerate if $Q(\cdot, \cdot)$ is non-degenerate, i.e. if there exists no $x \in A \setminus \{0\}$, such that $Q(x, y) = 0$ for all $y \in A$. Two finite quadratic modules $\mathfrak{A} = (A, Q)$ and $\mathfrak{B} = (B, R)$ are called isomorphic if there exists an isomorphism of groups $f : A \to B$ such that $Q = R \circ f$. The theory of finite quadratic modules has a long history; see e.g. [14], [15], [6] and the upcoming [10].

[2] An implementation is available at [4].

If $\mathfrak{L} = (L, \beta)$ is an even lattice, the quadratic form β on L induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form Q on the discriminant group L'/L of $\mathfrak{L}$. The pair $\mathrm{D}_{\mathfrak{L}} := (L'/L, Q)$ defines a finite quadratic module, which we call the *discriminant module* of $\mathfrak{L}$. According to [14, Thm. (6)], any finite quadratic module can be obtained as the discriminant module of an even lattice $\mathfrak{L}$. If $\mathfrak{A} = (A, Q)$ is a finite quadratic module and $\mathfrak{L}$ a lattice whose discriminant module is isomorphic to $\mathfrak{A}$, then the difference $b^+ - b^-$ of the real signature (b^+, b^-) of $\mathfrak{L}$ is already determined modulo 8 by $\mathfrak{A}$. Namely, by Milgram's formula [5, p. 127] one has

$$\frac{1}{\sqrt{\mathrm{card}\,(A)}} \sum_{x \in A} e\,(Q(x)) = e((b^+ - b^-)/8),$$

where we use $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. We call

$$\mathrm{sig}(\mathfrak{A}) := b^+ - b^- \bmod 8 \in \mathbb{Z}/8\mathbb{Z}$$

the *signature of* $\mathfrak{A}$. The number

$$N = \min\{n \in \mathbb{Z}_{>0} \mid nQ(x) \in \mathbb{Z} \text{ for all } x \in A\}$$

is called the *level of* $\mathfrak{A}$.

The metaplectic extension $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ (i.e. the nontrivial twofold central extension of $\mathrm{SL}_2(\mathbb{Z})$) can be realized as the group of pairs $(M, \phi(\tau))$, where $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ and ϕ is a holomorphic function on the complex upper half plane $\mathbb{H}$ with $\phi(\tau)^2 = c\tau + d$ (see e.g. [9]). The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \qquad \text{and} \qquad S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right],$$

and the group $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $T^* := (T, 1)$ and $S^* = (S, \sqrt{\tau})$ with relations $S^{*2} = (S^*T^*)^3 = \zeta$, where $\zeta = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

The Weil representation $\rho_{\mathfrak{A}}$ associated to $\mathfrak{A}$ is a representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[A]$. Here, and throughout, we denote the standard basis of $\mathbb{C}[A]$ by $(\mathfrak{e}_x)_{x\in A}$. (Recall that $\mathbb{C}[A]$ is the complex vector space of maps from A into $\mathbb{C}$ and $\mathfrak{e}_x$ the function which maps x to 1 and any $y \neq x$ to 0.) The action of $\rho_{\mathfrak{A}}$ can then be given in terms of the generators $S^*, T^* \in \mathrm{Mp}_2(\mathbb{Z})$ as follows:

$$\rho_{\mathfrak{A}}(T^*)\mathfrak{e}_x = e(Q(x))\mathfrak{e}_x,$$
$$\rho_{\mathfrak{A}}(S^*)\mathfrak{e}_x = \frac{e(-\,\mathrm{sig}(\mathfrak{A})/8)}{\sqrt{\mathrm{card}\,(A)}} \sum_{y \in A} e\,(-Q(x, y))\,\mathfrak{e}_y.$$

We shall sometimes simply write $\alpha.v$ for $\rho_{\mathfrak{A}}(\alpha)v$, i.e. we consider $\mathbb{C}[A]$ as $\mathrm{Mp}_2(\mathbb{Z})$-module via the action $(\alpha, v) \mapsto \rho_{\mathfrak{A}}(\alpha)v$. For details of the theory of Weil representations attached to finite quadratic modules we refer the reader to [1], [7], [8], [10], [13].

The kernel of the projection of $\mathrm{Mp}_2(\mathbb{Z})$ onto its first coordinate is the subgroup generated by $(1,-1)$. It is easily checked that $\rho_{\mathfrak{A}}((1,-1)) = \rho_{\mathfrak{A}}(S^*)^4$ acts as multiplication by $e(\mathrm{sig}(\mathfrak{A})/2)$. This simple observation has two immediate consequences. First of all, the space of invariants $\mathbb{C}[A]^{\mathrm{Mp}_2(\mathbb{Z})}$, i.e. the subspace of elements v in $\mathbb{C}[A]$ fixed by $\mathrm{Mp}_2(\mathbb{Z})$, reduces to $\{0\}$ unless $\mathrm{sig}(\mathfrak{A})$ is even. Secondly, $\rho_{\mathfrak{A}}$ descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$ if and only $\mathrm{sig}(\mathfrak{A})$ is even. Note, that $\mathrm{sig}(\mathfrak{A})$ is always even if the level of $\mathfrak{A}$ is odd as follows from Milgram's formula. (Namely, since $e\,(Q(x))$, for x in A, is an Nth root of unity, we conclude that the square of the left hand side of Milgram's formula is contained in the Nth cyclotomic field K_N and hence $e\,(\mathrm{sig}(\mathfrak{A})/4)$ is a root of unity in K_N. If N is odd, this implies that $e\,(\mathrm{sig}(\mathfrak{A})/4)$ is a $2N$th root of unity and thus $\mathrm{sig}(\mathfrak{A})$ is even in this case.)

3 Invariants

Let $\mathfrak{A} = (A, Q)$ be a finite quadratic module of level N. We shall assume in this section that $\mathrm{sig}(\mathfrak{A})$ is even. As we saw at the end of the last section the space of invariants is otherwise zero. The representation $\rho_{\mathfrak{A}}$ then descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$ and, even more, factors through a representation of the finite group $\Gamma(N)\backslash\,\mathrm{SL}_2(\mathbb{Z})$, i.e. of the group

$$G_N := \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

We will denote this representation also by $\rho_{\mathfrak{A}}$.

An easy closed and explicit formula for the dimension of $\mathbb{C}[A]^{G_N}$ is not known for general $\mathfrak{A}$. Of course, orthogonality of group characters yields

$$\dim \mathbb{C}[A]^{G_N} = \frac{1}{\mathrm{card}\,(G_N)} \sum_{g \in G_N} \mathrm{tr}(\rho_{\mathfrak{A}}(g)).$$

While it is therefore in principle possible to compute the dimension of $\mathbb{C}[A]^{G_N}$, there are two obstructions in practice . First of all, the size of the sum on the right can become very large. More precisely, the number of conjugacy classes of G_N is asymptotically equal to N for increasing N (see [7, Tabelle 2]). Secondly, the straight-forward formulas for $\mathrm{tr}(\rho_{\mathfrak{A}}(g))$ which follow directly from the explicit well-known formulas for the matrix coefficients of $\rho_{\mathfrak{A}}(g)$ (see e.g. [13, Theorem 6.4]) involve trigonometric sums with about $\mathrm{card}\,(A)^2$ many terms.[3]

[3] However, in [1] a different and simpler formula is given, which expresses the traces of the Weil representations in terms of the natural invariants for the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$.

The following proposition implies that we can compute the invariants or the dimension of the space of invariants “locally”, i.e. for every p-component of A separately. For a given prime p, denote the p-subgroup of A by A_p. It is quickly verified that $\mathfrak{A}_p := (A_p, Q|_{A_p})$ is again a finite quadratic module. Moreover, the decomposition $A = \bigoplus_{p|\mathrm{card}(A)} A_p$ of A as sum over its p-subgroups A_p induces an orthogonal direct sum decomposition of $\mathfrak{A}$. We also decompose G_N as a product

$$G_N \cong \prod_{p^\nu \| N} G_{p^\nu}$$

with $G_{p^\nu} := \mathrm{SL}_2(\mathbb{Z}/p^\nu\mathbb{Z})$ via the Chinese remainder theorem. In this way $\bigotimes_p \mathbb{C}[A_p]$ becomes a G_N-module in the obvious way. For this, we note that the set of primes dividing N is equal to the set of primes dividing card (A).

Proposition 3.1 *Let $A = \bigoplus_{p|N} A_p$ be the decomposition of A as sum over its p-subgroups A_p. Then $\mathfrak{e}_{\oplus_p a_p} \mapsto \otimes_p \mathfrak{e}_{a_p}$ defines via linear extension an isomorphism of G-modules*

$$\mathbb{C}[A] \xrightarrow{\cong} \bigotimes_{p|N} \mathbb{C}[A_p].$$

Under this isomorphism we have

$$\mathbb{C}[A]^{G_N} \cong \bigotimes_{p^\nu \| N} \mathbb{C}[A_p]^{G_{p^\nu}}.$$

Remark 3.2 The proposition implies in particular

$$\dim_{\mathbb{C}} \mathbb{C}[A]^{G_N} = \prod_{p^\nu \| N} \dim_{\mathbb{C}} \mathbb{C}[A_p]^{G_{p^\nu}}.$$

Proof of Proposition 3.1 The given map clearly defines an isomorphism of complex vector spaces. That this map commutes with the action of G_N, where G_N acts component-wise on the right-hand side, as described above, is easily checked using the formulas for the S and T-action. It follows that

$$\mathrm{tr}(g, \mathbb{C}[A]) = \prod_p \mathrm{tr}(g_p, \mathbb{C}[A_p])$$

for all $g = \otimes_p g_p$ in G_N, which implies, in particular, the second statement of the theorem via orthogonality of group characters. □

A natural problem is to determine the field or ring of definition[4] of the space $\mathbb{C}[A]^{G_N}$. From the formulas defining $\rho_{\mathfrak{A}}$, it is clear that $\mathbb{C}[A]^{G_N}$ is defined over the cyclotomic field K_N.[5] However, it turns out that the invariants are in fact defined over the field of rational numbers, as we shall see in a moment. This will allow us in Sect. 5 to compute a basis for $\mathbb{C}[A]^{G_N}$ by doing the computations in $\mathbb{F}_\ell[A]$ for suitable sufficiently large primes ℓ.

Theorem 3.3 *The space $\mathbb{C}[A]^{G_N}$ is defined over $\mathbb{Z}$.*

For the proof we need some preparations. For any d in $(\mathbb{Z}/N\mathbb{Z})^\times$, let σ_d denote the automorphism of K_N which sends each Nth root of unity z to z^d.

Lemma 3.4 *For any $g = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and x in A, one has*

$$\rho_{\mathfrak{A}}(g)\mathfrak{e}_x = \chi_{\mathfrak{A}}(d)\, e\left(bdQ(x)\right) \mathfrak{e}_{dx},$$

where $\chi_{\mathfrak{A}}(d) = \sigma_d(w)/w$ with $w = \sum_{x\in A} e\left(Q(x)\right)$.

A careful analysis of $\chi_{\mathfrak{A}}$ yields

$$\chi_{\mathfrak{A}}(d) = \begin{cases} \left(\dfrac{d}{\mathrm{card}\,(A)}\right) & \text{if } \mathrm{card}\,(A) \text{ is odd,} \\ \left(\dfrac{d}{\mathrm{card}\,(A)}\right)\left(\dfrac{-4}{d}\right)^s & \text{if } \mathrm{card}\,(A) \text{ is even,} \end{cases}$$

where $s = \mathrm{sig}(\mathfrak{A}_2)/2$ (see e.g. [13, Lemma 3.9]). However, we shall not need this formula.

Proof of Lemma 3.4 Since $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & bd \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & bd \\ 0 & 1 \end{smallmatrix}\right]\mathfrak{e}_x = e(bdQ(x))\mathfrak{e}_x$ it suffices to consider the action of $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. For this we write

$$\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] = S^{-1}\left[\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right] S \left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right] S \left[\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right]$$

and apply the formulas for the action of S and T to obtain after a standard computation

$$\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]\mathfrak{e}_x = \gamma\, \mathfrak{e}_{dx},$$

[4]We say that *a subspace V of $\mathbb{C}[A]$ is defined over the ring R* if it possesses a basis whose elements are in $R[A]$.

[5]For this one needs that $e(-\,\mathrm{sig}(\mathfrak{A})/8)/\sqrt{\mathrm{card}\,(A)}$ is in K_N, which can be read off from Milgram's formula.

where

$$\gamma = e\left(-\operatorname{sig}(\mathfrak{A})/8\right)\sqrt{\operatorname{card}(A)}\sum_{x\in A} e\left(dQ(x)\right).$$

The product of the first two factors equals $1/w$ by Milgram's formula. The lemma is now obvious. □

For any endomorphism f of $\mathbb{C}[A]$, which leaves $K_N[A]$ invariant, say $f\mathfrak{e}_x = \sum_{y\in A} f(x,y)\mathfrak{e}_y$ with $f(x,y)$ in K_N, we use $\sigma_s(f)$ for the endomorphism of $\mathbb{C}[A]$ such that

$$\sigma_s(f)\mathfrak{e}_x = \sum_{y\in A}\sigma_s\left(f(x,y)\right)\mathfrak{e}_y.$$

Note that $f \mapsto \sigma_s(f)$ defines an automorphism of the ring of endomorphisms of $\mathbb{C}[A]$ which leave $K_N[A]$ invariant.

Lemma 3.5 *For any* $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ *in* G_N, *one has*

$$\sigma_s(\rho_{\mathfrak{A}}\left(\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\right)) = \rho_{\mathfrak{A}}\left(\left(\begin{smallmatrix} a & sb\\ s^{-1}c & d\end{smallmatrix}\right)\right).$$

Proof Both sides of the claimed identity are multiplicative in $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ (for this note that the map $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a & sb\\ s^{-1}c & d\end{smallmatrix}\right]$ defines a automorphism of G_N). It suffices therefore to prove the claimed formula for the generators T and S of G_N. For T the formula can be read off immediately from the formula for the action of T. For S one has on the one hand for any x in A

$$\sigma_s(\rho_{\mathfrak{A}}(S))\mathfrak{e}_x = \sigma_s(w)\sum_{y\in A} e\left(-sQ(x,y)\right)\mathfrak{e}_y,$$

where $w = e(-\operatorname{sig}(\mathfrak{A})/8)/\sqrt{\operatorname{card}(A)} = \sum_{x\in A} e(-Q(x))/\operatorname{card}(A)$. On the other hand, $\left[\begin{smallmatrix} 0 & -s\\ s^{-1} & 0\end{smallmatrix}\right] = S\left[\begin{smallmatrix} s^{-1} & 0\\ 0 & s\end{smallmatrix}\right]$, and hence, using Lemma 3.4,

$$\rho_{\mathfrak{A}}\left(\left[\begin{smallmatrix} 0 & s\\ -s^{-1} & 0\end{smallmatrix}\right]\right)\mathfrak{e}_x = \chi_{\mathfrak{A}}(s)w\sum_{y\in A} e\left(-Q(sx,y)\right)\mathfrak{e}_y.$$

But $\sigma_s(w)/w = \chi_{\mathfrak{A}}(s)$, which implies the claimed formula. □

Proof of Theorem 3.3 The G_N-invariant projection $\mathcal{P} : \mathbb{C}[A] \to \mathbb{C}[A]^{G_N}$ is given by the formula, in other words, that we have $\sigma_s(\mathcal{P}) = \mathcal{P}$ for any s in $(\mathbb{Z}/N\mathbb{Z})^\times$.

$$\mathcal{P} = \frac{1}{\operatorname{card}(G_N)}\sum_{g\in G_N}\rho_{\mathfrak{A}}(g).$$

It suffices to show that, for any x in A, we have $\mathcal{P}\mathfrak{e}_x = \sum_y \mathcal{P}(x,y)\mathfrak{e}_y$ with rational numbers $\mathcal{P}(x,y)$, in other words, that we have, for any s in $(\mathbb{Z}/N\mathbb{Z})^\times$ the identity $\sigma_s(\mathcal{P}) = \mathcal{P}$. But this follows from Lemma 3.5 and the fact that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a & sb \\ s^{-1}c & d \end{smallmatrix}\right]$ permutes the elements of G_N. This proves the theorem. □

4 The Algorithm

In this section we explain our algorithm for computing a basis for the space of invariants. We then discuss various easy and natural improvements. We fix a finite quadratic module $\mathfrak{A} = (A, Q)$ of level N, and assume that $\operatorname{sig}(\mathfrak{A})$ is even (since otherwise the space of invariants of the associated Weil representation is trivial). The Weil representation $\rho_{\mathfrak{A}}$ is then a representation of $G = \mathrm{SL}_2(\mathbb{Z})$, which factors even through a representation of $G_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Define

$$\operatorname{Iso}(\mathfrak{A}) := \{x \in A : Q(x) = 0\},$$

and, for $v \in \mathbb{C}[A]$,

$$\operatorname{supp}(v) := \{x \in A \ : \ v(x) \neq 0\}.$$

Note that, for any G-submodule M of $\mathbb{C}[A]$, we have

$$M^T := \{v \in M : \rho_{\mathfrak{A}}(T)v = v\} = \{v \in M : \operatorname{supp}(v) \subseteq \operatorname{Iso}(\mathfrak{A})\}$$

as follows immediately from the formula for the action of T in Sect. 2. Our algorithm is based on the following observation.

Proposition 4.1 *Let M be a G-submodule of $\mathbb{C}[A]$. Then*

$$M^G = \left(1 + \rho_{\mathfrak{A}}(S) + \rho_{\mathfrak{A}}(S)^2 + \rho_{\mathfrak{A}}(S)^3\right)(M^T) \cap M^T.$$

Proof An element v of M is invariant under all of G if it is invariant under the generators T and S of G, i.e. if it is contained in M^T and the set M^S of vectors invariant under $\rho_{\mathfrak{A}}(S)$. Since $S^4 = 1$ we have $M^S = \mathrm{Tr}_S(M)$, where

$$\mathrm{Tr}_S = 1 + \rho_{\mathfrak{A}}(S) + \rho_{\mathfrak{A}}(S)^2 + \rho_{\mathfrak{A}}(S)^3.$$

But $M^G \subseteq M^T$, hence $M^G = \mathrm{Tr}_S(M^G) \subseteq \mathrm{Tr}_S(M^T)$, and therefore

$$M^G = M^G \cap M^T \subseteq \mathrm{Tr}_S(M^T) \cap M^T.$$

The proposition is now obvious. □

The proposition is quickly converted into a first version of our algorithm:

Algorithm 4.1 *(Computing a Basis for the Space $\mathbb{C}[A]^G$ of Invariants)*

1. *Find the isotropic elements $a_1, \ldots, a_m$ and the non-isotropic elements $b_1, \ldots, b_n$ in A.*
2. *Compute the $(m+n) \times m$ matrix H such that*

$$(L\mathfrak{e}_{a_1}, \ldots, L\mathfrak{e}_{a_m}) = (\mathfrak{e}_{a_1}, \ldots, \mathfrak{e}_{a_m}, \mathfrak{e}_{b_1}, \ldots, \mathfrak{e}_{b_n})H,$$

   where $L = 1 + \rho_{\mathfrak{A}}(S) + \rho_{\mathfrak{A}}(S)^2 + \rho_{\mathfrak{A}}(S)^3$.
3. *Let U and V be the matrices obtained by extraction the first m and the last n rows of H, respectively.*
4. *Compute a basis $\mathfrak{V}$ for the space of vectors x such that $Vx = 0$.*
5. *Return a basis for the space of all Ux, where x runs through the basis $\mathfrak{V}$.*

For implementing this algorithm we need, first of all, to decide over which field K we would like to do the computations. One possibility is to use floating point numbers to do a literal implementation using the field of complex numbers. However, the matrix coefficients of $\rho_{\mathfrak{A}}(S)$ with respect to the natural basis of $\mathbb{C}[A]$ are elements of the Nth cyclotomic field K_N. Hence it is reasonable to perform the calculations over $K_N = \mathbb{Q}[x]/(\phi_N)$, where ϕ_N is the Nth cyclotomic polynomial. Another choice for K will be discussed in Sect. 5.

There are two easy improvements which can help to reduce the computing time. The first one is due to the following observation.

Proposition 4.2 *The subspaces $\mathbb{C}[A]^+$ and $\mathbb{C}[A]^-$ of even and odd functions are G-submodules of $\mathbb{C}[A]$. Let $\epsilon = (-1)^{\mathrm{sig}(\mathfrak{A})/2}$. Then $\mathbb{C}[A]^G = (\mathbb{C}[A]^\epsilon)^G$ and $(\mathbb{C}[A]^{-\epsilon})^G = \{0\}$.*

Proof The first statement follows immediately from the observation that the map $\mathfrak{e}_a \mapsto \mathfrak{e}_{-a}$ intertwines with the action of S and T, and hence with the action of G, as is obvious from the formulas for the action of S and T.

For the proof of the second statement we note that $S^2\mathfrak{e}_a = \epsilon\mathfrak{e}_{-a}$ which is again an immediate consequence of the formula for the action of S. In other words, any invariant v satisfies $v(a) = (S^2v)(a) = \epsilon v(-a)$ for all a in A. □

Let $\rho_{\mathfrak{A}}^{\pm} : G \to \mathrm{GL}\left(\mathbb{C}[A]^{\pm}\right)$ afforded by the G-modules $\mathbb{C}[A]^{\pm}$. As we saw in the proof of the preceding proposition S^2 acts on $\mathbb{C}[A]^\epsilon$ ($\epsilon = (-1)^{\mathrm{sig}(\mathfrak{A})/2}$) as identity, i.e. $\rho_{\mathfrak{A}}^{\epsilon}(S^2) = 1$. Using this Propositions 4.1, 4.2 imply

$$\mathbb{C}[A]^G = (\mathbb{C}[A]^\epsilon)^G = \left\{v \in \left(1 + \rho_{\mathfrak{A}}^{\epsilon}(S)\right)(\mathbb{C}[A]^\epsilon) : \mathrm{supp}(v) \subseteq \mathrm{Iso}(\mathfrak{A})\right\}.$$

A basis for $\mathbb{C}[A]^\epsilon$ is obtained by replacing in the standard basis $\mathfrak{e}_a$ by $\mathfrak{e}_a^\epsilon = \frac{1}{2}(\mathfrak{e}_a + \epsilon\mathfrak{e}_{-a})$ and omitting all zeroes and all duplicated vectors. This leads to the following modified algorithm.

Algorithm 4.2 *(Modified Algorithm for Computing a Basis for the Space of Invariants)*

1. *As in Algorithm 4.1.*
2.a *Construct the basis* $\mathfrak{e}^{\epsilon}_{a_i}$, $\mathfrak{e}^{\epsilon}_{b_j}$ $(1 \le i \le m', 1 \le j \le n')$ *of* $\mathbb{C}[A]^{\epsilon}$ *obtained from the standard basis* $\mathfrak{e}_{a_i}$, $\mathfrak{e}_{b_j}$ *by (anti-)symmetrizing, suppressing zeroes and duplicates, and possibly renumbering the* a_i *and* b_j.
2.b *Compute the* $(m' + n') \times n'$ *matrix* H' *such that*

$$(L\mathfrak{e}^{\epsilon}_{a_1}, \ldots, L\mathfrak{e}^{\epsilon}_{a_{m'}}) = (\mathfrak{e}^{\epsilon}_{a_1}, \ldots, \mathfrak{e}^{\epsilon}_{a_{m'}}, \mathfrak{e}^{\epsilon}_{b_1}, \ldots, \mathfrak{e}^{\epsilon}_{b_{n'}})H',$$

where $L = 1 + \rho^{\epsilon}_{\mathfrak{A}}(S)$.
3.–5. *As in Algorithm 4.1 with* H, m, n *replaced by* H', m', n'.

The dimension of $\mathbb{C}[A]^{\pm}$ equals $\frac{1}{2}\,(\mathrm{card}\,(A) + \mathrm{card}\,(A[2]))$, where $A[2]$ denotes the subgroup of elements annihilated by "multiplication by 2". Note that $A[2] = \{0\}$ if $\mathrm{card}\,(A)$ is odd. Therefore the size of H' is about half of the size of H in Algorithm 4.1. Also note that H' has entries in the totally real subfield K_N^+ of K_N (see the subsequent formula for the entries h_{ij} of H'). This implies that $\mathbb{C}[A]^G$ is in fact defined over K_N^+ and we can perform our computations over K_N^+ instead of K_N.

To implement the algorithm, we still need an explicit formula for the entries of the matrix $H' = (h_{ij})$, where $1 \le i, j \le m' + n'$. We just write $x_i = a_i$ for $1 \le i \le m'$ and $x_i = b_{i-m'}$ for $m' < i \le m' + n'$ for the elements of A. By a straightforward calculation, we obtain

$$\begin{aligned} h_{ij} &= f_i^{-1} \left\langle \rho_{\mathfrak{A}}(S)\mathfrak{e}^{\epsilon}_{x_j} + \mathfrak{e}^{\epsilon}_{x_j}, \mathfrak{e}^{\epsilon}_{x_i} \right\rangle \\ &= \frac{e(-\operatorname{sig}(\mathfrak{A})/8)}{2f_i\sqrt{\mathrm{card}\,(A)}}\big(e(-Q(x_j, x_i)) + \epsilon e(Q(x_j, x_i))\big) + \delta_{i,j}, \end{aligned}$$

where $f_i = \langle \mathfrak{e}^{\epsilon}_{x_i}, \mathfrak{e}^{\epsilon}_{x_i} \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}[A]$ (conjugate-linear in the second component), such that $\langle \mathfrak{e}_x, \mathfrak{e}_y \rangle = \delta_{x,y}$. Note that $f_i = \frac{1}{2}$ if $x_i \ne -x_i$ and $f_i = 1$, otherwise.

Given a finite quadratic module the exact value of quantity $\operatorname{sig}(\mathfrak{A})$ is not immediately clear. For finding the ϵ of the preceding proposition the following is helpful.

Proposition 4.3 *For odd* card (A) *one has*

$$(-1)^{\operatorname{sig}(\mathfrak{A})/2} = \left(\frac{-1}{\mathrm{card}\,(A)}\right).$$

Proof Indeed, directly from the formula for the S-action we obtain $S^2\mathfrak{e}_x = (-1)^{\operatorname{sig}(\mathfrak{A})/2}\mathfrak{e}_{-x}$. On the other hand $S^2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and therefore we obtain by Lemma 3.4 that $S^2\mathfrak{e}_x = \chi_{\mathfrak{A}}(-1)\mathfrak{e}_{-x}$. For odd card (A) it is then easy to deduce from the formula of the lemma for $\chi_{\mathfrak{A}}$ that $\chi_{\mathfrak{A}}(-1) = \left(\frac{-1}{\mathrm{card}(A)}\right)$ (see also the remark after Lemma 3.4). □

The second possible improvement is the factorization into local components as explained in Proposition 3.1. We first compute the local components $\mathfrak{A}_p := (A_p, Q|_{A_p})$, and then apply Algorithm 4.2 to the finite quadratic modules $\mathfrak{A}_p$. If the number of different primes in card (A) is large this reduces the run-time of our algorithm enormously. Indeed, the two bottle necks of our algorithm are the search for the isotropic elements in A and the computation of the kernel of a matrix of size $n \times m$, where m is the number of isotropic and anisotropic elements of A, respectively. If card (A) contains more than two different primes, say card $(A) = p_1^{k_1} \cdots p_r^{k_r}$ with $r \geq 2$, then it takes $p_1^{k_1} \cdots p_r^{k_r}$ many search steps to find all isotropic elements in A, whereas an application of Proposition 3.1 allows us to dispense with $p_1^{k_1} + \cdots + p_r^{k_r}$ many search steps to eventually find all invariants of A. A similar comparison applies to the size of the matrices in our algorithm when run either directly on A or else separately on the p-parts A_{p_j}.

5 Reduction Mod ℓ

In this section we fix again a finite quadratic module $\mathfrak{A} = (A, Q)$ of level N. Let ℓ denote a prime such that $\ell \equiv 1 \bmod N$. Then $\mathbb{Q}_\ell$ contains the Nth roots of unity, hence the Nth cyclotomic field. Accordingly, we can consider $\rho_{\mathfrak{A}}$ as a representation of $G_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ taking values in $\mathrm{GL}(\mathbb{Q}_\ell[A])$, and $\mathbb{Q}_\ell[A]$ as G_N-module. From the formulas for the action of S and T on $\mathbb{Q}_\ell[A]$ it is clear that $\mathbb{Z}_\ell[A]$ is invariant under G_N, and that the $\mathbb{Z}_\ell$-rank of $\mathbb{Z}_\ell[A]^{G_N}$ equals the dimension of $\mathbb{C}[A]^{G_N}$.

For computing the rank of $\mathbb{Z}_\ell[A]^{G_N}$ it is natural to consider the reduction modulo ℓ of $\mathbb{Z}_\ell[A]$. More precisely, note that $\ell\mathbb{Z}_\ell[A]$ is a G_N-submodule of $\mathbb{Z}_\ell[A]$, so that we have the exact sequence of G_N-modules

$$0 \longrightarrow \ell\mathbb{Z}_\ell[A] \longrightarrow \mathbb{Z}_\ell[A] \xrightarrow{r} \mathbb{F}_\ell[A] \longrightarrow 0,$$

where r denotes the reduction map $r(f) : a \mapsto f(a) + \ell\mathbb{Z}_\ell$ and the second arrow the inclusion map. Here the action of G_N on $\mathbb{F}_\ell[A] \cong \mathbb{Z}_\ell[A]/\ell\mathbb{Z}_\ell[A]$ is the one induced by the action on $\mathbb{Z}_\ell[A]$.

Theorem 5.1 *Suppose that* $(N, \ell) \neq (2, 3)$. *Then*

$$\dim_{\mathbb{Q}_\ell} \mathbb{Q}_\ell[A]^{G_N} = \dim_{\mathbb{F}_\ell} \mathbb{F}_\ell[A]^{G_N}.$$

Remark 5.2 Numerical computed examples suggest that the theorem is also true for $N = 2$ and $\ell = 3$. However, we did not try to pursue this further.

Proof of Theorem 5.1 From the short exact sequence preceding the theorem we obtain the long exact sequence in cohomology

$$0 \longrightarrow \ell\mathbb{Z}_\ell[A]^{G_N} \longrightarrow \mathbb{Z}_\ell[A]^{G_N} \xrightarrow{r} \mathbb{F}_\ell[A]^{G_N} \longrightarrow \mathrm{H}^1(G_N, \ell\mathbb{Z}_\ell[A]) \longrightarrow \dots .$$

We shall show in a moment that the order of G_N is a unit of $\mathbb{Z}_\ell$. Hence, the cohomology group $\mathrm{H}^1(G_N, \ell\mathbb{Z}_\ell[A])$ is trivial [2, Corollary 10.2]. It follows then that $\mathbb{F}_\ell[A]^{G_N} \cong \mathbb{Z}_\ell[A]^{G_N}/\ell\mathbb{Z}_\ell[A]^{G_N}$. Since $\mathbb{Z}_\ell[A]^{G_N}$ is free we conclude that $\dim_{\mathbb{F}_\ell} \mathbb{F}_\ell[A]^{G_N}$ equals the $\mathbb{Z}_\ell$-rank of $\mathbb{Z}_\ell[A]^{G_N}$. It is quickly checked that every $\mathbb{Z}_\ell$-basis of $\mathbb{Z}_\ell[A]^{G_N}$ yields a basis of the $\mathbb{Q}_l$-vector space $\mathbb{Q}_l[A]^{G_N}$, which implies then the theorem.

For proving that $\mathrm{card}\,(G_N)$ is not divisible by ℓ, first note that $\ell \equiv 1 \bmod N$ implies that $\ell > N$. Then, recall that the order of $G_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is given by

$$\mathrm{card}\,(G) = N^3 \prod_{p|N} \frac{p^2-1}{p^2}.$$

Hence, if $\ell \mid \mathrm{card}\,(G)$, we conclude that there is a prime $p \mid N$, such that $\ell \mid p+1$ or $\ell \mid p-1$. However, $p-1 < N < \ell$ and thus the only possibility is $\ell = p+1$ and $N = p$. Since ℓ and p are primes we conclude $N = 2$ and $\ell = 3$, which we excluded in the statement of the theorem. □

The results on reduction modulo ℓ are not only interesting from a theoretical point of view. Our implementation profits tremendously from reduction modulo a suitable prime ℓ as it speeds up the calculation in practice. The reason is that there are highly optimized libraries for computation with matrices over finite fields (and/or over the integers) available. For instance, in `sage` (which uses the linbox library default), computing the nullity of a random 200×200 matrix with entries in a cyclotomic field $\mathbb{Q}(\zeta_{11})$ takes about 4 s on our test machine, whereas computing the nullity of a 2000×2000 matrix over $\mathbb{F}_{23}$ takes about 600 ms. This immediately speeds up the computation of the dimension of $\mathbb{C}[A]^G$ although it does not give a basis for $\mathbb{C}[A]^G$.

6 Tables

Tables 1, 2, 3, 4, 5, and 6 list the values $s = \mathrm{sig}(\mathfrak{A})$ and dimension $d = \dim \mathbb{C}[A]^{\mathrm{SL}_2(\mathbb{Z})}$ for various p-modules $\mathfrak{A} = (A, Q)$, where $p = 2, 3, 5$. We use *genus symbols* for denoting isomorphism classes of finite quadratic modules (see [3, 10]). In short, for a power q of an odd prime p and a nonzero integer d the symbol q^d stands for the quadratic module

$$\left((\mathbb{Z}/q\mathbb{Z})^k, \frac{x_1^2 + \cdots + x_{k-1}^2 + ax_k^2}{q}\right),$$

where $k = |d|$ and a is an integer such that $\left(\frac{2a}{p}\right) = \mathrm{sign}(d)$. For a 2-power $q = 2^e$, we have the following symbols: We write q_a^d for the module

$$\left((\mathbb{Z}/q\mathbb{Z})^k, \frac{x_1^2 + \cdots + x_{k-1}^2 + ax_k^2}{2q}\right),$$

with $k = |d|$ and $\left(\frac{a}{2}\right) = \mathrm{sign}(d)$. We normalize a to be contained in the set $\{1, 3, 5, 7\}$ and if $q = 2$, we take $a \in \{1, 7\}$. Finally, we write q^{+2k} for

$$\left((\mathbb{Z}/q\mathbb{Z})^{2k}, \frac{x_1x_2 + \cdots + x_{k-1}x_k}{q}\right),$$

and q^{-2k} for

$$\left((\mathbb{Z}/q\mathbb{Z})^{2k}, \frac{x_1x_2 + \cdots + x_{k-3}x_{k-2} + x_{k-1}^2 + 2x_{k-1}x_k + x_k^2}{q}\right).$$

The concatenation of such symbols stands for the direct sum of the corresponding modules. For instance, $3^{-1}9^{+1}27^{-2}$ denotes the finite quadratic module

$$\left(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/27\mathbb{Z})^2, \frac{x^2}{3} - \frac{y^2}{9} + \frac{z^2 - w^2}{27}\right).$$

It can be shown that every finite quadratic p-module is isomorphic to a module which can be described by such symbols, and that this description is essentially unique (up to some ambiguities for $p = 2$). For details of this we refer to [10].

For the computations we used [12], the additional package [11] and our implementation of Algorithm 4.2 (available at [4]).

Table 1 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 2-modules of even signature s

$\mathfrak{A}$ $s=0$	d	$\mathfrak{A}$ $s=4$	d	$\mathfrak{A}$ $s=0$	d	$\mathfrak{A}$	d	s
2^{+2}	2	2^{-2}	0	4^{+2}	3	4^{-8}	1191	0
2^{+4}	5	2^{-4}	1	4^{+4}	16	2_0^{+2}	1	0
2^{+6}	15	2^{-6}	7	4^{+6}	141	2_2^{+2}	0	2
2^{+8}	51	2^{-8}	35	4^{+8}	1711	2_0^{+4}	2	0
2^{+10}	187	2^{-10}	155	4^{-2}	1	2_4^{+4}	0	4
2^{+12}	715	2^{-12}	651	4^{-4}	6	2_6^{+6}	0	6
2^{+14}	2795	2^{-14}	2667	4^{-6}	73	2_0^{+6}	5	0

Table 2 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 2-modules of even signature s

$\mathfrak{A}$ $s=0$	d	$\mathfrak{A}$ $s=4$	d	$\mathfrak{A}$ $s=2$	d
$2^{+2}4^{+2}$	8	$2^{+2}8^{-2}$	1	$2^{+4}4_2^{+2}$	0
$2_0^{+2}4^{+2}$	4	$2_0^{+2}8^{-2}$	0	$2_0^{+4}4_2^{+2}$	0
$2^{+4}4^{+2}$	25	$2^{+4}8^{-2}$	7	4_2^{+4}	1
$2^{+2}8^{+2}$	11	$2_0^{+4}8^{-2}$	1	$2^{+2}4_2^{+4}$	4
$2_0^{+4}4^{+2}$	11	$4^{+2}8^{-2}$	2	$2_0^{+2}4_2^{+4}$	4
$2_7^{+1}4^{+2}8_1^{+1}$	4	$4_0^{+2}8^{-2}$	4	$4_2^{+2}8^{+2}$	3
$2_5^{-1}4^{+2}8_3^{-1}$	4	$2^{+2}4^{-2}8^{-2}$	1	$4_1^{+3}16_1^{+1}$	1
$2_0^{+2}8^{+2}$	6	$2_0^{+2}4^{-2}8^{-2}$	3	$4_7^{-3}16_3^{-1}$	1

Table 3 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 3-modules of signature s

$\mathfrak{A}$ $s=6$	d	$\mathfrak{A}$ $s=2$	d	$\mathfrak{A}$ $s=0$	d	$\mathfrak{A}$	d	s
3^{+1}	0	3^{-1}	0	9^{+1}	1	27^{+1}	0	6
3^{-2}	2	3^{+2}	0	9^{+2}	1	27^{+2}	0	4
3^{+3}	1	3^{-3}	1	9^{+3}	5	27^{+3}	5	2
3^{-4}	1	3^{+4}	7	9^{+4}	33	27^{-1}	0	2
3^{+5}	10	3^{-5}	10	9^{+5}	121	27^{-2}	4	0
3^{-6}	40	3^{+6}	22	9^{-1}	1	27^{-3}	5	6
3^{+7}	91	3^{-7}	91	9^{-2}	3	81^{+1}	1	0
3^{-8}	247	3^{+8}	301	9^{-3}	5	81^{+2}	1	0
3^{+9}	820	3^{-9}	820	9^{-4}	11	81^{-1}	1	0
3^{-10}	2542	3^{+10}	2380	9^{-5}	121	81^{-2}	5	0

Table 4 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 3-modules of signature s

$\mathfrak{A}$ $s = 6$	d	$\mathfrak{A}$ $s = 2$	d	$\mathfrak{A}$ $s = 6$	d	$\mathfrak{A}$ $s = 2$	d
$3^{+1}27^{-1}$	2	$3^{+1}27^{+1}$	0	$3^{+1}243^{-1}$	2	$3^{+1}243^{+1}$	0
$3^{-2}27^{-1}$	1	$3^{-2}27^{+1}$	1	$3^{-2}243^{-1}$	1	$3^{-2}243^{+1}$	1
$3^{+3}27^{-1}$	1	$3^{+3}27^{+1}$	7	$3^{+3}243^{-1}$	1	$3^{+3}243^{+1}$	7
$3^{-4}27^{-1}$	10	$3^{-4}27^{+1}$	10	$3^{-4}243^{-1}$	10	$3^{-4}243^{+1}$	10
$3^{+5}27^{-1}$	40	$3^{+5}27^{+1}$	22	$3^{+5}243^{-1}$	40	$3^{+5}243^{+1}$	22
$3^{-1}27^{+1}$	2	$3^{-1}27^{-1}$	0	$3^{-1}243^{+1}$	2	$3^{-1}243^{-1}$	0
$3^{+2}27^{+1}$	1	$3^{+2}27^{-1}$	1	$3^{+2}243^{+1}$	1	$3^{+2}243^{-1}$	1
$3^{-3}27^{+1}$	1	$3^{-3}27^{-1}$	7	$3^{-3}243^{+1}$	1	$3^{-3}243^{-1}$	7
$3^{+4}27^{+1}$	10	$3^{+4}27^{-1}$	10	$3^{+4}243^{+1}$	10	$3^{+4}243^{-1}$	10
$3^{-5}27^{+1}$	40	$3^{-5}27^{-1}$	22	$3^{-5}243^{+1}$	40	$3^{-5}243^{-1}$	22

Table 5 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 5-modules of signature s

$\mathfrak{A}$ $s = 4$	d	$\mathfrak{A}$ $s = 0$	d	$\mathfrak{A}$ $s = 4$	d	$\mathfrak{A}$ $s = 0$	d
5^{+1}	0	5^{-1}	0	25^{+1}	1	125^{+1}	0
5^{-2}	0	5^{+2}	2	25^{+2}	3	125^{+2}	4
5^{+3}	1	5^{-3}	1	25^{+3}	7	125^{-1}	0
5^{-4}	1	5^{+4}	11	25^{-1}	1	125^{-2}	0
5^{+5}	26	5^{-5}	26	25^{-2}	1		
5^{-6}	106	5^{+6}	156	25^{-3}	7		
5^{+7}	651	5^{-7}	651				

Table 6 Dimension $d = \dim_{\mathbb{C}} \mathbb{C}[A]^G$ for some 5-modules of signature s

$\mathfrak{A}$ $s = 4$	d	$\mathfrak{A}$ $s = 0$	d	$\mathfrak{A}$ $s = 4$	d	$\mathfrak{A}$ $s = 0$	d
$5^{+1}125^{-1}$	0	$5^{-1}125^{+1}$	0	$5^{+1}125^{+1}$	2	$5^{-1}125^{-1}$	2
$5^{-2}125^{-1}$	1	$5^{+2}125^{+1}$	1	$5^{-2}125^{+1}$	1	$5^{+2}125^{-1}$	1
$5^{+3}125^{-1}$	1	$5^{-3}125^{+1}$	1	$5^{+3}125^{+1}$	11	$5^{-3}125^{-1}$	11
$5^{-4}125^{-1}$	26	$5^{+4}125^{+1}$	26	$5^{-4}125^{+1}$	26	$5^{+4}125^{-1}$	26

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The Metaplectic Tensor Product as an Instance of Langlands Functoriality

Wee Teck Gan

Abstract We interpret the metaplectic tensor product construction of Mezo for the genuine representations of the Kazhdan-Patterson covering groups in terms of the L-group formalism of Weissman.

1 Kazhdan-Patterson Coverings and Metaplectic Tensor Product

Let F be a characteristic 0 local field which contains all n-th roots of unity (for a fixed $n \in \mathbb{N}$). The goal of this note is to interpret the metaplectic tensor product construction of Mezo [5] for the Kazhdan-Patterson covering groups of $\mathrm{GL}_r(F)$ in the framework of Langlands functoriality for Brylinski-Deligne extensions.

1.1 Kazhdan-Patterson Covering

We shall be working with Brylinski-Deligne covers of the group $G_r = \mathrm{GL}_r$ over F. Let $T_r \subset B_r$ be the diagonal torus of GL_r contained in the upper triangular Borel subgroup; this defines a based root datum $(X(T_r), \Delta_r, Y(T_r), \Delta_r^\vee)$ for GL_r, and we may consider the standard pinning. Let us write

$$Y = Y(T_r) = \oplus_{i=1}^r \mathbb{Z} \cdot e_i$$

and let Y_{sc} be the sublattice spanned by the simple coroots $\Delta_r^\vee = \{e_i - e_{i+1} : i = 1, \dots r-1\}$.

W.T. Gan (✉)
Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore
e-mail: matgwt@nus.edu.sg

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For $c \in \mathbb{Z}$, let Q_c be the Weyl-invariant quadratic form, whose associated symmetric bilinear form B_c is given by

$$B_c(e_i, e_j) = \begin{cases} 2c \text{ if } i = j; \\ 2c + 1 \text{ if } i \neq j. \end{cases}$$

Note that for each $\alpha^\vee \in \Delta_r^\vee$,

$$Q_c(\alpha^\vee) = -1.$$

One has the (non symmetric) bilinear form D_c given by

$$D_c(e_i, e_j) = \begin{cases} c \text{ if } i = j; \\ 2c + 1, \text{ if } i < j; \\ 0, \text{ if } i > j. \end{cases}$$

Hence we have $B_c(x, y) = D_c(x, y) + D_c(y, x)$, so that D_c is a bisector of B_c in the sense of [2, §2.6]. If $\eta_0 : Y_{sc} \longrightarrow F^\times$ is the trivial map (sending every element to 1), then the pair (D_c, η_0) is an object in the category $\mathrm{Bis}_{\mathrm{GL}_r}$ in [2, §2.6] and gives rise to a Brylinski-Deligne extension $\overline{G}_{r,c}$ of GL_r:

$$1 \longrightarrow K_2 \longrightarrow \overline{G}_{r,c} \longrightarrow \mathrm{GL}_r \longrightarrow 1.$$

Taking F-points and pushing out by the n-th Hilbert symbol $(-,-)_n$, we obtain a topological central extension

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G}_{r,c} \stackrel{p}{\longrightarrow} \mathrm{GL}_r(F) \longrightarrow 1$$

The covering group $\overline{G}_{r,c}$ is none other than the degree n Kazhdan-Patterson cover of $\mathrm{GL}_r(F)$ associated to the twisting parameter c studied in [3].

The bisector D_c is basically providing a cocycle for the maximal torus T_r. More precisely, one may realise

$$\overline{T}_{r,c} := p^{-1}(T_r) = T_r(F) \times \mu_n(F) \quad \text{as a set}$$

with group law given by

$$(e_i(a_i), \epsilon_i) \cdot (e_j(a_j), \epsilon_j) = \left(e_i(a_i)e_j(a_j), \epsilon_i\epsilon_j \cdot (a_i, a_j)_n^{D_c(e_i,e_j)}\right)$$

for $a_i, a_j \in F^\times$. An important observation to make here is that, with

$$T_r^n = \{t^n : t \in T_r(F)\},$$

the subset

$$T_r^n \times \mu_n(F) \subset T_r(F) \times \mu_n(F)$$

is a subgroup. In particular, one has a natural splitting of the subgroup T_r^n into $\overline{T}_{r,c}$, given by embedding into the first coordinate in the above presentation. Henceforth, we shall regard T_r^n as a subgroup of $\overline{T}_{r,c}$ in this way.

We shall be considering irreducible genuine representations of $\overline{G}_{r,c}$. More precisely, let us fix an embedding

$$\epsilon : \mu_n(F) \hookrightarrow \mathbb{C}^\times,$$

and let $\mathrm{Irr}_\epsilon(\overline{G}_{r,c})$ denote the set of isomorphism classes of ϵ-genuine irreducible representations of $\overline{G}_{r,c}$.

1.2 Covers of Levi Subgroups

Now suppose that $M_r \subset \mathrm{GL}_r$ is a Levi subgroup, with

$$M_r = \mathrm{GL}_{r_1} \times \ldots . \times \mathrm{GL}_{r_k}.$$

Note that one such M_r is the split torus T_r. Restricting the cover $\overline{G}_{r,c}$ to M_r gives a cover $\overline{M}_{r,c}$. On the other hand, for each $G_{r_i} = \mathrm{GL}_{r_i}$ in M_r, the restriction of the cover to G_{r_i} is none other than the (degree n) Kazhdan-Patterson cover $\overline{G}_{r_i,c}$, i.e.

$$p^{-1}(G_{r_i}) \cong \overline{G}_{r_i,c}.$$

While G_{r_i} and G_{r_j} commute with each other, it is no longer true in general that $p^{-1}(G_{r_i})$ and $p^{-1}(G_{r_j})$ commute. Hence , in general, there is no direct way of relating the covering groups $\overline{M}_{r,c}$ and the almost direct product

$$\overline{G}_{r_1,c} \times_{\mu_n} \ldots .. \times_{\mu_n} \overline{G}_{r_k,c}.$$

In particular, an irreducible genuine representation of $\overline{M}_{r,c}$ is not obtained as a tensor product of irreducible genuine representations of the $\overline{G}_{r_i,c}$.

1.3 Metaplectic Tensor Product

However, in [5], Mezo described a construction which constructs an irreducible genuine representation of $\overline{M}_{r,c}$ out of irreducible genuine representations of $\overline{G}_{r_i,c}$, for $1 \leq i \leq k$, and one extra piece of data. Let us recall his construction of this "metaplectic tensor product" briefly.

Let π_i be irreducible genuine representations of $\overline{G}_{r_i,c}$. Let

$$G^{(n)}_{r_i,c} = \{g \in G_{r_i,c} : \det(g) \in F^{\times n}\}$$

and set

$$\overline{G}^{(n)}_{r_i,c} = p^{-1}(G^{(n)}_{r_i,c}).$$

For $i \neq j$, $p^{-1}(G^{(n)}_{r_i,c})$ and $p^{-1}(G^{(n)}_{r_j,c})$ commute with each other. Hence,

$$\overline{M}^{(n)}_{r,c} := p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right) = \overline{G}^{(n)}_{r_1,c} \times_{\mu_n} \times \ldots\ldots. \times_{\mu_n} \overline{G}^{(n)}_{r_i,c}.$$

Now consider the restriction of π_i to $\overline{G}^{(n)}_{r_i,c}$ (this restriction is semisimple of finite length since $Z(\overline{G}_{r_i,c}) \cdot \overline{G}^{(n)}_{r_i,c}$ is a finite index subgroup of $\overline{G}_{r_i,c}$) and let $\sigma_i \subset \pi_i$ be an irreducible summand in this restriction. One then has an irreducible representation

$$\sigma_1 \boxtimes \ldots. \boxtimes \sigma_k \quad \text{of } \overline{M}^{(n)}_{r,c} = p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right).$$

Next, one picks an irreducible genuine character χ of $Z(\overline{G}_{r,c})$ such that

$$\chi = \otimes_{i=1}^{k} \omega_{\pi_i} \quad \text{on } Z(\overline{G}_{r,c}) \cap p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right). \tag{1.1}$$

One then obtains an irreducible representation

$$\chi \boxtimes \sigma_i \boxtimes \ldots. \boxtimes \sigma_k \text{ of } Z(\overline{G}_{r,c}) \cdot p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right) \subset \overline{M}_{r,c}.$$

One extends this irreducible representation as much as possible to a subgroup $\overline{M}'_{r,c}$ of $\overline{M}_{r,c}$ and sets

$$\Pi = \mathrm{ind}_{\overline{M}'_{r,c}}^{\overline{M}_{r,c}} \chi \boxtimes (\boxtimes_{i=1}^{k} \sigma_i).$$

It was shown in [5] that the above construction gives a well-defined surjective map

$$\tilde{\otimes} : \left(\mathrm{Irr}_\epsilon(\overline{G}_{r_1,c}) \times \ldots . \times \mathrm{Irr}_\epsilon(\overline{G}_{r_k,c}) \times \mathrm{Irr}_\epsilon(Z(\overline{G}_{r,c}))\right)^{\heartsuit} \longrightarrow \mathrm{Irr}_\epsilon(\overline{M}_{r,c}). \tag{1.2}$$

Here the superscript in $(\ldots.)^{\heartsuit}$ indicates that one is considering tuples $(\pi_1, \ldots, \pi_k, \chi)$ satisfying (1.1). This map is the so-called metaplectic tensor product. It is not injective: replacing each π_i by $\pi_i \otimes (\chi_i \circ \det)$ where χ_i is a character of $F^\times$ such that $\chi_i^n = 1$ would give the same output, but this is the only reason for the non-injectivity.

There is a global analog of the metaplectic tensor product for automorphic representations which has been developed by Takeda; see [6, 7].

2 L-Group Formalism

The goal of this note is to give an interpretation of this construction of Mezo in the framework of Langlands functoriality, as developed in [8] and [2]. To do this, we shall need to recall briefly the theory of dual groups and L-groups for Brylinski-Deligne extensions.

2.1 Dual Group

We first describe the dual group of a Brylinski-Deligne extension following McNamara [4, §11]. Set

$$Y^\# = \{y \in Y : B_c(y, z) \in n\mathbb{Z} \text{ for all } z \in Y\} \subset Y \otimes_{\mathbb{Z}} \mathbb{Q},$$

and let $X^\#$ be its dual lattice in $X(T_r) \otimes_{\mathbb{Z}} \mathbb{Q}$. For $\alpha \in \Delta_r$, put

$$n_\alpha = n/(n, Q_c(\alpha^\vee)) = n \quad (\text{since } Q_c(\alpha^\vee) = -1),$$

and set

$$\alpha_\#^\vee = n \cdot \alpha^\vee, \quad \text{and} \quad \alpha_\# = n^{-1} \cdot \alpha.$$

Denote by $\Delta_\#^\vee$ and $\Delta_\#$ the sets of these modified coroots and roots. Then $(Y^\#, \Delta_\#^\vee, X^\#, \Delta_\#)$ is a based root datum and the associated connected reductive group $\overline{G}_{r,c}^\vee$ over $\mathbb{C}$ is the Langlands dual group of $\overline{G}_{r,c}$. It is explicitly given by [2, §16.2]

$$\overline{G}_{r,c}^\vee \cong \{(g, \lambda) \in \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^d\} \subset \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

with

$$d = \mathrm{GCD}(n, (2c+1)r - 1).$$

2.2 Structural Facts

Let $H_{r,c}$ be the split linear algebraic group (pinned) whose dual group is $\overline{G}_{r,c}^{\vee}$. Then

$$H_{r,c} = (\mathrm{GL}_r \times \mathrm{GL}_1)/\{(\lambda, \lambda^{-d}) : \lambda \in \mathrm{GL}_1\}$$

contains the (diagonal) maximal split torus

$$A_{r,c} = Y^{\#} \otimes \mathbb{G}_m \cong (T_r \times \mathrm{GL}_1)/\{(\lambda, \lambda^{-d}) : \lambda \in \mathrm{GL}_1\}.$$

Since $Y^{\#} \subset Y$ and $Y^{\#}_{sc} = \mathbb{Z} \cdot \Delta^{\vee}_{\#} \subset Y_{sc}$, one may restrict the bisector D_c to $Y^{\#}$ and η_0 to $Y^{\#}_{sc}$. The data $(D_c|_{Y^{\#}}, \eta_0)$ then gives rise to a Brylinski-Deligne cover $\overline{H}_{r,c}$ of $H_{r,c}$ whose dual group is $\overline{H}_{r,c}^{\vee} = \overline{G}_{r,c}^{\vee}$.

The inclusion $Y^{\#} \hookrightarrow Y$ induces an isogeny

$$i : A_{r,c} \longrightarrow T_r$$

which is explicitly given by

$$i(t, \lambda) = \lambda^{n/d} \cdot t^n.$$

This isogeny plays a crucial role in the structure theory and representation theory of $\overline{G}_{r,c}$.

For example, one has

$$i(A_{r,c}(F)) = p(Z(\overline{T}_{r,c})),$$

where $Z(\overline{T}_{r,c})$ denotes the center of $\overline{T}_{r,c}$. Alternatively, one may pullback the cover $\overline{T}_{r,c}$ to $A_{r,c}$ via i, yielding a cover $\overline{A}_{r,c} \subset \overline{H}_{r,c}$. Then one has

$$i(\overline{A}_{r,c}) = Z(\overline{T}_{r,c}).$$

Observe that

$$Z(\overline{T}_{r,c}) \supset T_r^n.$$

On the other hand, let $Z(\overline{G}_{r,c})$ be the center of $\overline{G}_{r,c}$. Then one has

$$i(Z(H_{r,c})) = p(Z(\overline{G}_{r,c})) = Z(G_r) \cap p(Z(\overline{T}_{r,c})).$$

While the second equality is true in general, the first is a special feature of Kazhdan-Patterson covers. In any case, we have

$$Z(\overline{T}_{r,c}) = Z(\overline{G}_{r,c}) \cdot T_r^n.$$

Because of the above, the central character of an irreducible genuine representation of $\overline{G}_{r,c}$ is a genuine character of $Z(\overline{H}_{r,c})$ which is trivial on $\mathrm{Ker}(i)$. Note that

$$p(Z(\overline{H}_{r,c})) = \{1\} \times \mathrm{GL}_1(F) \subset A_{r,c}(F).$$

and

$$Z(\overline{H}_{r,c}) = \overline{Z(H_{r,c})}.$$

In particular, $Z(\overline{H}_{r,c})$ is an example of a Brylinski-Deligne cover of GL_1, and its associated dual group is

$$\overline{Z(H_{r,c})}^{\vee} = \overline{H}_{r,c}^{\vee}/[\overline{H}_{r,c}^{\vee}, \overline{H}_{r,c}^{\vee}] = \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}].$$

Thus, its L-group is the pushout of ${}^L\overline{G}_{r,c}$ (introduced below) by the natural map $\overline{G}_{r,c}^{\vee} \longrightarrow \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$.

2.3 L-Group and LLC

In a foundational paper [8] of Weissman, the dual group $\overline{G}_{r,c}^{\vee}$ is enhanced to give an L-group extension ${}^L\overline{G}_{r,c}$:

$$1 \longrightarrow \overline{G}_{r,c}^{\vee} \longrightarrow {}^L\overline{G}_{r,c} \longrightarrow W_F \longrightarrow 1$$

where W_F is the Weil group of F. A more down-to-earth construction of ${}^L\overline{G}_{r,c}$, also due to Weissman, is described in [2, §5], where it is shown that this L-group extension is split. The set $Spl({}^L\overline{G}_{r,c})$ of splittings over the Weil-Deligne group $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$, modulo the conjugation action of $\overline{G}_{r,c}^{\vee}$, is the set of L-parameters for $\overline{G}_{r,c}$. These L-parameters are expected to classify the irreducible genuine representations of $\overline{G}_{r,c}$.

More precisely, the local Langlands correspondence (LLC) predicts that there is a natural map

$$\mathcal{L} : \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c}) \longrightarrow Spl({}^L\overline{G}_{r,c}).$$

Unlike the case of linear reductive groups, this map is not expected to be surjective, as a consequence of the fact that the isogeny $i : A_{r,c} \longrightarrow T_r$ is not an isomorphism. It is however expected to be injective for the groups $\overline{G}_{r,c}$.

Likewise, if we consider the cover $\overline{M}_{r,c}$ of the Levi subgroup M_r, then

$$\overline{M}_{r,c}^{\vee} \hookrightarrow \overline{G}_{r,c}^{\vee}$$

is the Levi subgroup of type $(r_1, \ldots, r_k)$ and one has a commutative diagram of short exact sequences [2, Lemma 5.3]

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{M}_{r,c}^{\vee} & \longrightarrow & {}^L\overline{M}_{r,c} & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{G}_{r,c}^{\vee} & \longrightarrow & {}^L\overline{G}_{r,c} & \longrightarrow & W_F & \longrightarrow & 1. \end{array} \qquad (2.1)$$

The LLC predicts a natural map

$$\mathcal{L}_M : \mathrm{Irr}_{\epsilon}(\overline{M}_{r,c}) \longrightarrow Spl({}^L\overline{M}_{r,c}).$$

2.4 Desiderata

We highlight some expected properties of the LLC which will be used later on.

- (Central characters) If $\pi \in \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c})$ has central character ω_{π}, regarded as a genuine character of $Z(\overline{H}_{r,c}) = \overline{Z(H_{r,c})}$, then the L-parameter of ω_{π} is deduced from that of π by the pushout via $\overline{G}_{r,c}^{\vee} \longrightarrow \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$. One way of expressing this is that one has commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c}) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{G}_{r,c}) \\ \downarrow & & \downarrow \\ \mathrm{Irr}_{\epsilon}(Z(\overline{G}_{r,c})) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{Z(H_{r,c})}) \end{array}$$

where the first vertical arrow is the central character map and the second vertical arrow is induced by the natural map from $\overline{G}_{r,c}^{\vee}$ to its cocenter $\overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$.
- (Twisting) If $\pi \in \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c})$ and $\chi : F^{\times} \longrightarrow \mathbb{C}^{\times}$ is a 1-dimensional character, then $\pi \otimes (\chi \circ \det) \in \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c})$ also. If the L-parameter of π is $\phi : WD_F \longrightarrow {}^L\overline{G}_{r,c}$ and that of χ is

$$\phi_{\chi} : W_F \longrightarrow Z(G_r^{\vee}) \cong \mathbb{C}^{\times} \subset G_r^{\vee} \cong \mathrm{GL}_r(\mathbb{C}),$$

then the L-parameter of $\pi \otimes (\chi \circ \det)$ should be given by [2, §12.2]. Specializing to the case of interest here, we have a natural map

$$\delta : Z(G_r^\vee) = \mathbb{C}^\times \longrightarrow Z(\overline{G}_{r,c}^\vee) = \{(a \cdot I_r, b) \in \mathbb{C}^\times \times \mathbb{C}^\times : a^r = b^d\}$$

given by

$$\delta(z) = (z^n, z^{r \cdot \frac{n}{d}}).$$

Then the L-parameter of $\pi \otimes (\chi \circ \det)$ is given by $\phi \otimes (\delta \circ \phi_\chi)$, and for $w \in W_F$, one has

$$\delta \circ \phi_\chi(w) = (\chi(w)^n I_r, \chi(w)^{rn/d}) \in \overline{G}_{r,c}^\vee.$$

2.5 LLC for Covering Tori

We now specialize to the case when $M_r = T_r$. In this case, the LLC has been shown, i.e. the map $\mathcal{L}_T$ has been constructed. More precisely, since $\overline{T}_{r,c}$ is a Heisenberg group, an irreducible genuine representation is determined by its central character. Hence we have natural maps:

$$\mathrm{Irr}_\epsilon(\overline{T}_{r,c}) \longleftrightarrow \mathrm{Irr}_\epsilon(Z(\overline{T}_{r,c})) \hookrightarrow \mathrm{Irr}_\epsilon(\overline{A}_{r,c})$$

where the inclusion is induced by $i : \overline{A}_{r,c} \longrightarrow \overline{T}_{r,c}$. It was shown in [2, §8] that one has a map

$$\mathrm{Irr}_\epsilon(\overline{A}_{r,c}) \longleftrightarrow Spl({}^L\overline{A}_{r,c}) = Spl({}^L\overline{T}_{r,c}).$$

The composite of these maps gives the desired

$$\mathcal{L}_T : \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) \longrightarrow Spl({}^L\overline{T}_{r,c}).$$

The above construction of the LLC for $\overline{T}_{r,c}$ does not care that T_r is a maximal split torus of G_r. Let us take that into account now. In this case, the Weyl group $W = N_{G_r}(T_r)/T_r$ acts naturally on $\mathrm{Irr}_\epsilon(\overline{T}_{r,c})$ and $Spl({}^L\overline{T}_{r,c})$. It was shown in [2, §9.3 and Prop. 9.5] that the LLC map $\mathcal{L}_T$ is W-equivariant.

2.6 LLC for Principal Series

The above properties of the LLC for $\overline{T}_{r,c}$ allows us to define the LLC map $\mathcal{L}$ for principal series representations of $\overline{G}_{r,c}$. Namely, one expects a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) & \xrightarrow{\mathcal{L}_T} & Spl({}^L\overline{T}_{r,c}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Irr}_\epsilon(\overline{G}_{r,c}) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{G}_{r,c}) \end{array} \tag{2.2}$$

Here the first vertical arrow is via parabolic induction and taking Langlands quotient whereas the second is by the natural inclusion of L-groups. Because the LLC map $\mathcal{L}_T$ is W-equivariant and the two vertical arrows are W-invariant, this commutative diagram serves to define the map $\mathcal{L}$ on the set $\mathrm{Irr}_{\epsilon,ps}(\overline{G}_{r,c})$ of those genuine representations of $\overline{G}_{r,c}$ which are Langlands quotient of standard modules induced from the Borel subgroup B_r.

Explicitly, a principal series representation of $\overline{G}_{r,c}$ is of the form

$$I(\chi) = \mathrm{Ind}_{\overline{B}_r}^{\overline{G}_{r,c}} \tau(\chi)$$

where $\tau(\chi)$ is the irreducible representation of $\overline{T}_{r,c}$ with central character χ on $Z(\overline{T}_{r,c})$, or equivalently χ is a character of $\overline{A}_{r,c}$ which is trivial on $\mathrm{Ker}(i)$. By replacing χ by a W-translate, we may assume $I(\chi)$ is a standard module and denote its unique irreducible quotient by $J(\chi)$. If the L-parameter of χ is

$$\phi_\chi : W_F \longrightarrow {}^L A_{r,c} = {}^L\overline{T}_{r,c},$$

then the L-parameter of $J(\chi)$ is

$$\mathcal{L}(J(\chi)) : W_F \xrightarrow{\phi_\chi} {}^L T_{r,c} \longrightarrow {}^L\overline{G}_{r,c}.$$

Likewise, one has a classification of the set $\mathrm{Irr}_{\epsilon,ps}(\overline{M}_{r,c})$ of (Langlands quotients of) principal series representations of $\overline{M}_{r,c}$, since T_r is a maximal split torus of M_r. In other words, one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) & \longrightarrow & Spl({}^L\overline{T}_{r,c}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Irr}_{\epsilon,ps}(\overline{M}_{r,c}) & \xrightarrow{\mathcal{L}_M} & Spl({}^L\overline{M}_{r,c}) \end{array} \tag{2.3}$$

where the first vertical arrow is parabolic induction (and taking Langlands quotient) and is W_M-invariant.

2.7 Distinguished Splittings

In [2, §6 and §7], we have defined, constructed and classified a set of so-called distinguished splittings of the L-group extension ${}^L\overline{G}_{r,c}$. It was shown that a distinguished splitting of ${}^L\overline{G}_{r,c}$ takes value in ${}^L\overline{T}_{r,c}$ and gives rise to the following:

- it gives an isomorphism

$$ {}^L\overline{G}_{r,c} \cong \overline{G}_{r,c}^{\vee} \times W_F = {}^L H_{r,c}, $$

  and hence a bijective map (depending on the distinguished splitting).

$$ \{L\text{-parameters of } \overline{G}_{r,c}\} \longleftrightarrow \{L\text{-parameters of } H_{r,c}\}. $$

- it gives a distinguished W-invariant genuine character χ of $Z(\overline{T}_{r,c})$, or equivalently a genuine character of $\overline{A}_{r,c}$, which is trivial on the kernel of i. One can restrict such a distinguished W-invariant character of $Z(\overline{T}_{r,c})$ to the center $Z(\overline{G}_{r,c})$.

We highlight a key property of these distinguished characters in the context of the Kazhdan-Patterson covers, which follows from their definition and construction; see [2, §7 and §16.2]:

Lemma 2.1 *The distinguished characters of $Z(\overline{T}_{r,c})$ are trivial on the subgroup $T_r^n \subset Z(\overline{T}_{r,c})$. Any two distinguished characters differ from each other by twisting by a character of $p(Z(\overline{T}_{r,c}))/T_r^n \cong (F^\times)^{n/d}/F^{\times n}$. Pulled back to $A_{r,c}$ via i, this gives a character of $Z(H_{r,c})/Z(H_{r,c})^d$ (which is a quotient of $A_{r,c}$ by the second projection).*

Moreover, it was shown in [2, §7 and §16.2] that given an additive character ψ of F, one can construct an associated distinguished splitting and hence a W-invariant genuine character χ_ψ of $Z(\overline{T}_{r,c})$. Using this, one has an associated bijection (depending on ψ)

$$ Spl({}^L\overline{G}_{r,c}) \longleftrightarrow \mathrm{Hom}(W_F, \overline{G}_{r,c}^{\vee})/\overline{G}_{r,c}^{\vee} - \text{conjugacy}. $$

The analogous statement holds for any of the Levi covers $\overline{M}_{r,c}$. Thus, the use of a distinguished splitting (or equivalently a distinguished W-invariant genuine character of $\overline{T}_{r,c}$) is to serve as a base-point and thus allow one to work with the dual group instead of the L-group extensions.

3 L-Group Interpretation of Metaplectic Tensor Product

With the above preparation, we are now ready to formulate an interpretation of the metaplectic tensor product using the L-group.

3.1 Setup

Recall that $M \subset \mathrm{GL}_r$ is a Levi subgroup, with

$$M = \mathrm{GL}_{r_1} \times \ldots . \times \mathrm{GL}_{r_k}.$$

Restricting the cover $\overline{G}_{r,c}$ to M gives a cover $\overline{M}_{r,c}$, whose dual group is

$$\overline{M}_{r,c}^{\vee} = \{(g_1, \ldots, g_k, \lambda) \in \prod_{i=1}^{k} \mathrm{GL}_{r_i}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \prod_{i=1}^{k} \det(g_i) = \lambda^d\}.$$

On the other hand, for each $G_{r_i} = \mathrm{GL}_{r_i}$ in M, we have the (degree n) Kazhdan-Patterson cover $p^{-1}(G_{r_i,c}) \cong \overline{G}_{r_i,c}$, with its own dual group $\overline{G}_{r_i,c}^{\vee}$. Setting

$$d_i = \mathrm{GCD}(n, (2c+1)r_i - 1),$$

one has

$$\overline{G}_{r_i,c}^{\vee} = \{(g, \lambda) : \mathrm{GL}_{r_i}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^{d_i}\}.$$

We shall write det for the character of $\overline{G}_{r_i,c}$ given by the composite of the first projection to $\mathrm{GL}_{r_i}(\mathbb{C})$ and the determinant map of $\mathrm{GL}_{r_i}(\mathbb{C})$.

In the metaplectic tensor product, one starts with a tuple $(\pi_1, \ldots, \pi_k, \chi)$ satisfying the compatibility condition (1.1). Let us imagine for a moment that LLC holds and we have fixed a nontrivial additive character ψ of F, which determines distinguished splittings of each ${}^L\overline{G}_{r_i,c}$, ${}^L\overline{Z(H_{r,c})}$ and ${}^L\overline{M}_{r,c}$. Let

$$\phi_i : WD_F \longrightarrow \overline{G}_{r_i,c}^{\vee} \quad \text{and} \quad \phi_\chi : W_F \longrightarrow \mathbb{C}^\times$$

be the associated L-parameters of π_i and χ. Hence, we have

$$\phi_1 \times \ldots \times \phi_k \times \phi_\chi : WD_F \longrightarrow \left(\prod_{i=1}^{k} \overline{G}_{r_i,c}^{\vee} \right) \times \mathbb{C}^\times.$$

How is the compatibility condition (1.1) expressed in terms of L-parameters?

Lemma 3.2 *The compatibility condition (1.1) is equivalent to*

$$\prod_{i=1}^{k} \det \phi_i = \phi_\chi^d.$$

Proof We need to work out

$$Z(\overline{G}_{r,c}) \cap p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right).$$

The projection of this to $\mathrm{GL}_r(F)$ consists of scalar matrices $a^{n/d} \cdot I_r$ satisfying

$$a^{nr_i/d} \in F^{\times n} \quad \text{for each } i = 1, \ldots, k.$$

This is equivalent to

$$a \in (F^{\times})^{d/(d,r_i)} \quad \text{for each } i. \tag{3.1}$$

Now observe that

$$(d, r_1, r_2, \ldots, r_k) = (n, (2c+1) \cdot (\sum_{i=1}^{k} r_i) - 1, r_1, \ldots, r_k) = 1.$$

Hence

$$\mathrm{LCM}(d/(d, r_1), \ldots.., d/(d, r_k)) = d.$$

Hence the condition in (3.1) is equivalent to $a \in F^{\times d}$. In particular,

$$p\left(Z(\overline{G}_{r,c}) \cap p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right)\right) = \{a^n I_r : a \in F^{\times}\} = Z(G_r)^n.$$

Since

$$i(A^{d_i}_{r_i,c}) = T^n_{r_i} \quad \text{and} \quad i(A^{d}_{r,c}) = T^n_{r},$$

it follows that (1.1) is equivalent to the identity of L-parameters in the lemma. □

The above lemma implies that the parameter $\phi_1 \times \ldots \times \phi_k \times \phi_\chi$ factors through the subgroup

$$\mathcal{M}^{\heartsuit} \subset \left(\prod_{i=1}^{k} \overline{G}^{\vee}_{r_i,c}\right) \times \mathrm{GL}_1(\mathbb{C})$$

consisting of those elements

$$\left(\prod_{i=1}^{k} (g_i, \lambda_i), \lambda\right)$$

satisfying

$$\lambda^d = \prod_{i=1}^{k} \det(g_i) = \prod_{i=1}^{k} \lambda_i^{d_i}.$$

3.2 The Conjecture

Observe that one may define a map

$$f : \mathcal{M}^{\heartsuit} \longrightarrow \overline{M}_{r,c}^{\vee}$$

by

$$\left(\prod_{i=1}^{k}(g_i, \lambda_i), \lambda\right) \mapsto (g_1, \ldots, g_k, \lambda).$$

Note that the kernel of f is

$$\mu_{d_i} \times \ldots\ldots \times \mu_{d_k},$$

consisting of elements $\left(\prod_{i=1}^{k}(g_i, \lambda_i), \lambda\right)$ with $g_i = 1$, $\lambda = 1$ and $\lambda_i \in \mu_{d_i}$. The above discussion motivates the following conjecture:

Conjecture The metaplectic tensor product $\tilde{\otimes}$ defined in (1.2) is the Langlands functorial lift associated to the map $f : \mathcal{M}^{\heartsuit} \longrightarrow \overline{M}_{r,c}^{\vee}$ defined above.

The above conjecture is not a statement which can be proved at this moment, since it is conditional upon the LLC for covering groups. We make a couple of remarks as a sort of consistency check:

- the metaplectic tensor product construction does not depend on the choice of distinguished characters of $Z(\overline{T}_{r,c})$, $Z(\overline{G}_{r,c})$ or $Z(\overline{T}_{r_i,c})$, but the map f only induces a lifting of L-parameters if one fixes distinguished characters on these groups. So it will be pertinent to check that in fact, the induced lifting of L-parameters is independent of the choice of such distinguished characters.

  To see this, note that for each i, it follows by Lemma 2.1 that two distinguished characters of $Z(\overline{T}_{r_i,c})$, regarded as characters of $\overline{A}_{r_i,c}$, differ by twisting by a character μ of $Z(H_{r_i,c})$ with $\mu^{d_i} = 1$. Their L-parameters differ by a homomorphism $\phi_\mu : W_F \hookrightarrow Z(H_{r_i,c})^\vee = \mathbb{C}^\times$ with $\phi_\mu^{d_i} = 1$. Hence $f \circ \phi_\mu$ is trivial, so that the choice of the distinguished character of $Z(\overline{T}_{r_i,c})$ is not important.

  On the other hand, having chosen and fixed a distinguished character χ on $Z(\overline{T}_{r,c})$, we inherit one on $Z(\overline{G}_{r,c})$ by restriction and hence one on $\overline{Z(H_{r,c})}$ by pullback. One checks that as long as one uses distinguished splittings of ${}^L\overline{G}_{r,c}$

and ${}^L\overline{Z(H_{r,c})}$ related in this way, the lifting of L-parameters induced by f is independent of the choice of distinguished splittings.

- the lifting of L-parameters induced by f is not injective: $(\phi_i, \ldots, \phi_k, \chi)$ and $(\phi'_1, \ldots, \phi'_k, \chi')$ have same image if and only if $\chi = \chi'$ and for each i, ϕ_i and ϕ'_i differs by a homomorphism

$$\mu_i : W_F \longrightarrow \{1\} \times \mu_{d_i} \subset \{1\} \times \mathrm{GL}_1(\mathbb{C}).$$

This agrees exactly with the non-injectivity of the metaplectic tensor product construction, as we now explain. By the discussion at the end of Sect. 1.3, the metaplectic tensor product does not change if and only if we replace the representation π_i by $\pi_i \otimes (\chi_i \circ \det)$ with $\chi_i^n = 1$. By the desiderata (Twisting) in Sect. 2.4, this replaces the L-parameter ϕ_i by $\phi_i \otimes (\delta \circ \phi_{\chi_i})$, where ϕ_{χ_i} is the L-parameter of χ_i and δ is defined in Sect. 2.4. But for $w \in W_F$,

$$\delta \circ \phi_{\chi_i}(w) = (\chi_i(w)^n, \chi_i(w)^{r_i \cdot \frac{n}{d_i}}) = (1, \chi_i^{r_i n/d_i}(w)).$$

The character $\mu_i := \chi_i^{r_i n/d_i}$ satisfies $\mu_i^{d_i} = 1$; its order is the same as that of χ_i^{n/d_i} since $(r_i, d_i) = 1$.

3.3 Case of Principal Series

As we have explained in the previous section, the LLC is known for principal series representations induced from a Borel subgroup. The main result of this note is the demonstration of the above conjecture for such principal series representations.

Proposition 3.2 *The above conjecture holds when each π_i belongs to* $\mathrm{Irr}_{\epsilon,ps}(\overline{G}_{r_i,c})$.

Proof The metaplectic tensor product of principal series representations was determined by Cai [1, Theorem 3.26] and the point is to interpret the result on the dual side. We give an independent treatment here.

We first consider the case when $M_r = T_r$ is the maximal split torus; this is the key case to understand. Thus, we are assuming that $r_i = 1$ for all i and $k = r$, so that

$$d_0 := d_i = \mathrm{GCD}(n, 2c).$$

Then

$$H_{1,c} = A_{1,c} = (\mathrm{GL}_1(F) \times \mathrm{GL}_1(F))/\{(t, t^{-d_0}) : t \in F^\times\} \cong \mathrm{GL}_1(F)$$

by the second projection, so that

$$H_{1,c}^\vee = A_{1,c}^\vee \cong \{(g, \lambda) \in \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : g = \lambda^{d_0}\} \cong \mathbb{C}^\times$$

via the second projection. The isogeny $i : A_{1,c} \longrightarrow T_1$ is the map $\lambda \mapsto \lambda^{n/d_0}$.

The input into the metaplectic tensor product is then a tuple $(\pi_1, \ldots, \pi_r, \chi)$ where each π_i is a genuine representation $\tau(\chi_\psi \chi_i)$ of $\overline{T}_{1,c}$, where χ_ψ is a distinguished character of $Z(\overline{T}_{1,c}) = p^{-1}(T_1^{n/d_0})$ and χ_i is a character of $A_{1,c} \cong F^\times$ which is trivial on $\mu_{n/d_0}(F)$. The compatibility condition (1.1) is given by

$$(\prod_{i=1}^{k} \chi_i)^{d_0} = \chi^d.$$

Moreover, one has

$$\phi_{\chi_1} \times \ldots \times \phi_{\chi_r} \times \phi_\chi : W_F \longrightarrow \mathcal{M}^\heartsuit \subset \mathrm{GL}_1(\mathbb{C})^r \times \mathrm{GL}_1(\mathbb{C}),$$

so that

$$f \circ (\phi_{\chi_1} \times \ldots \times \phi_{\chi_r} \times \phi_\chi) = (\phi_{\chi_1}^{d_0} \times \ldots \times \phi_{\chi_r}^{d_0}) \times \phi_\chi : W_F \longrightarrow \overline{T}_{r,c}^\vee \subset \mathrm{GL}_1(\mathbb{C})^r \times \mathrm{GL}_1(\mathbb{C}).$$

Consider now the construction of the metaplectic tensor product. We first restrict each π_i to

$$T_1^n \times \mu_n(F) = \overline{T}_{1,c}^{(n)} = i\left(\overline{A}_{1,c}^{d_0}\right).$$

Since T_1^n is contained in the center of $\overline{T}_{1,c}$ (this center is $\overline{T}_{1,v}^{(n/d_0)}$), the restriction of π_i to T_1^n is simply the isotypic sum of its central character χ_i restricted to T_1^n (here we have used Lemma 2.1 which says that the distinguished character χ_ψ is trivial on T_1^n). We then consider the character

$$\chi_\psi \cdot \left(\chi \boxtimes (\boxtimes_{i=1}^r \chi_i)\right) \text{ on the subgroup } Z(\overline{G}_{r,c}))(T_1^n \times \ldots \times T_1^n),$$

where now χ_ψ denotes a distinguished character of $Z(\overline{T}_{r,c})$ restricted to $Z(\overline{G}_{r,c})$. But this subgroup is precisely the center $Z(\overline{T}_{r,c})$ of $\overline{T}_{r,c}$, and so this character determines an irreducible genuine representation of $\overline{T}_{r,c}$. Explicitly, the character

$$\chi \boxtimes (\boxtimes_{i=1}^r \chi_i) \text{ of } p(Z(\overline{G}_{r,c}))(T_1^n \times \ldots \times T_1^n) = p(Z(\overline{T}_{r,c}))$$

is given by:

$$\begin{pmatrix} a_1^n \lambda^{n/d} & & & \\ & a_2^n \lambda^{n/d} & & \\ & & \cdots & \\ & & & a_r^n \lambda^{n/d} \end{pmatrix} \mapsto \chi_1(a_1)^{d_0} \cdot \ldots \cdot \chi_r(a_r)^{d_0} \cdot \chi(\lambda).$$

By our construction of the LLC for $\overline{T}_{r,c}$, we see that the L-parameter of the genuine representation of $\overline{T}_{r,c}$ with this central character is precisely

$$(\phi_{\chi_1}^{d_0} \times \ldots \times \phi_{\chi_r}^{d_0}) \times \phi_\chi : W_F \longrightarrow T_r^\vee \times \mathrm{GL}_1(\mathbb{C}) \longrightarrow \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

which is equal to $f \circ (\phi_{\chi_1} \times \ldots . \times \phi_{\chi_r} \times \phi_\chi)$, as desired.

Now we consider the case of general $M_r = \mathrm{GL}_{r_1} \times \ldots \times \mathrm{GL}_{r_k}$. Given irreducible genuine principal series representations π_i of $\overline{G}_{r_i,c}$ associated to genuine characters χ_i of $p(Z(\overline{T}_{r_i,c}))$, we are to restrict them to $\overline{G}_{r_i,c}^{(n)}$, take an irreducible summand σ_i and then consider the representation

$$\boxtimes_{i=1}^k \sigma_i \boxtimes \chi \text{ on } Z(\overline{G}_{r,c}) \cdot \left(\overline{G}_{r_1,c}^{(n)} \times_{\mu_n} \times \ldots .. \times_{\mu_n} \overline{G}_{r_k,c}^{(n)} \right).$$

The resulting metaplectic tensor product representation is undoubtedly a principal series representation of $\overline{M}_{r,c}$ (as shown in [1, Theorem 3.26]), and hence is determined by a character of $p(Z(\overline{T}_{r,c}))$. Now the main point is that

$$Z(\overline{T}_{r,c}) = Z(\overline{G}_{r,c}) \cdot T_r^n = Z(\overline{G}_{r,c}) \cdot (T_{r_1}^n \times \ldots \times T_{r_k}^n) \subset Z(\overline{G}_{r,c}) \cdot \left(\overline{G}_{r_1,c}^{(n)} \times_{\mu_n} \times \ldots .. \times_{\mu_n} \overline{G}_{r_k,c}^{(n)} \right).$$

Hence, the resulting metaplectic tensor product representation is determined by the behaviour of $(\boxtimes_{i=1}^k \pi_i) \boxtimes \chi$ on $Z(\overline{T}_{r,c})$. Because of the commutativity in (2.2) and (2.3), we are basically reduced to a question on covering tori, which is a slight generalization of the case when $M_r = T_r$ treated above. Arguing as in that special case, one sees that the metaplectic tensor product on $\overline{G}_{r,c}$ is constructed from the character

$$\chi \boxtimes (\boxtimes_{i=1}^k \chi_i) \quad \text{ of } p(Z(\overline{G}_{r,c})) \cdot T_r^n,$$

and this gives the parameter

$$\phi_{\chi_1}^{d_1} \times \ldots \times \phi_{\chi_k}^{d_k} \times \phi_\chi = f \circ \left(\phi_{\chi_1} \times \ldots \times \phi_{\chi_k} \times \phi_\chi \right)$$

as desired. □

As we mentioned earlier, Takeda [6, 7] has developed the notion of metaplectic tensor product in the global setting of automorphic representations. The proposition thus allows one to interpret his construction as an instance of weak Langlands functorial lifting relative to the homomorphism $f : \mathcal{M}^\heartsuit \longrightarrow \overline{M}_{r,c}^\vee$ of dual groups.

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On Scattering Constants of Congruence Subgroups

Miguel Grados and Anna-Maria von Pippich

Abstract Let Γ be a congruence subgroup of level N. The scattering constant of Γ at two cusps is given by the constant term at $s = 1$ in the Laurent expansion of the scattering function of Γ at these cusps. Scattering constants arise in Arakelov theory when establishing asymptotics for Arakelov invariants of the modular curve associated to Γ, as the level N tends to infinity. More precisely, in the known cases, scattering constants essentially contribute to the leading term of the asymptotics for the self-intersection of the relative dualizing sheaf. In this article, we prove an identity relating the scattering constants of Γ to certain scattering constants of the principal congruence subgroup $\overline{\Gamma}(N)$. Providing an explicit formula for the latter, in case that $N = 2$ or $N \geq 3$ is odd and square-free, we thereby present a systematic way of computing the scattering constants of Γ in these cases.

1 Introduction

1.1 Scattering Constants

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ be a congruence subgroup which acts on the hyperbolic upper half-plane $\mathbb{H}$ by fractional linear transformations. The quotient space $\Gamma\backslash\mathbb{H}$ admits the structure of a hyperbolic Riemann surface of finite hyperbolic volume v_Γ, having $p_\Gamma > 0$ cusps and possibly finitely many elliptic fixed points. Associated to any cusp q, there is a non-holomorphic Eisenstein series $E_q^\Gamma(z, s)$, defined, for $z \in \mathbb{H}$

M. Grados
Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
e-mail: grados@mathematik.hu-berlin.de

A.-M. von Pippich (✉)
Fachbereich Mathematik, Technische Universität Darmstadt, Schloßgartenstr. 7, 64289 Darmstadt, Germany
e-mail: pippich@mathematik.tu-darmstadt.de

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and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, by

$$E_q^\Gamma(z,s) := \sum_{\gamma \in \Gamma_q \backslash \Gamma} \mathrm{Im}(\sigma_q^{-1}\gamma z)^s,$$

where $\Gamma_q := \mathrm{Stab}_\Gamma(q)$ is the stabilizer subgroup of q in Γ and $\sigma_q \in \mathrm{PSL}_2(\mathbb{R})$ is a scaling matrix of q. The Eisenstein series $E_q^\Gamma(z,s)$ is an automorphic form with respect to Γ, which is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and admits a meromorphic continuation to the whole s-plane. Furthermore, one has the functional equation

$$E^\Gamma(z,s) = \Phi^\Gamma(s)\, E^\Gamma(z,1-s),$$

where $E^\Gamma(z,s) = (E_{q_1}^\Gamma(z,s) \, \ldots \, E_{q_{p_\Gamma}}^\Gamma(z,s))^t$ denotes the vector of all Eisenstein series and

$$\Phi^\Gamma(s) := \left(\varphi_{q_j q_k}^\Gamma(s)\right)_{j,k=1,\ldots,p_\Gamma}$$

is the so-called scattering matrix. The scattering matrix plays an important role in the spectral theory of the hyperbolic Laplacian. The entries $\varphi_{q_j q_k}^\Gamma(s)$ of the scattering matrix are called scattering functions. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, they are given by a Dirichlet series and arise in the constant term of the Fourier expansion of $E_{q_j}^\Gamma(z,s)$ with respect to the cusp q_k (see, e.g., [8, 10], or [15]). For the precise definition, we refer the reader to Sect. 2.3. The scattering function $\varphi_{q_j q_k}^\Gamma(s)$ admits a meromorphic continuation to the whole s-plane with a simple pole at $s = 1$ of residue equal to v_Γ^{-1}. The scattering constant

$$\mathscr{C}_{q_j q_k}^\Gamma := \lim_{s \to 1}\left(\varphi_{q_j q_k}^\Gamma(s) - \frac{v_\Gamma^{-1}}{s-1}\right)$$

at the cusps q_j and q_k is then defined to be the constant term in the Laurent expansion of the scattering function $\varphi_{q_j q_k}^\Gamma(s)$ at $s = 1$.

For the congruence subgroups $\overline{\Gamma}(N)$, $\overline{\Gamma}_0(N)$, and $\overline{\Gamma}_1(N)$, the determinant of the scattering matrix can be expressed in terms of known functions from analytic number theory (see [7]) and Hejhal [6] provides explicit formulas for the scattering functions, under certain restrictions on the level N. Further formulas are available in the literature, for example, in the thesis [9], identities for the determinant of the scattering matrix for subgroups of finite index of the modular group are studied. Moreover, in the thesis [14], formulas for the scattering functions of the subgroup $\overline{\Gamma}_0(N)$ are stated in the framework of Kronecker limit formulas, whereas in the thesis [13], which establishes relations between dessins d'enfants and non-holomorphic Eisenstein series, the case $\overline{\Gamma}(2)$ as well as its subgroups of finite index are considered.

Our main motivation to study scattering constants of congruence subgroups arises from questions in Arakelov theory, when considering bounds for Arakelov invariants, such as the Faltings's delta function and, more importantly for us, the self-intersection of the relative dualizing sheaf. Let us give a few more details. By its very definition, the self-intersection of the relative dualizing sheaf on a modular curve is the sum of a geometric contribution, that encodes the finite intersection of divisors coming from the cusps, and an analytic contribution, which is given in terms of the Arakelov Green's function evaluated at these cusps. In their influential work [1], A. Abbes and E. Ullmo in particular developed a method to explicitly compute this analytic contribution. Their method relies on a crucial identity relating the Arakelov Green's function $g_{\mathrm{Ar}}(q_1, q_2)$ at two cusps q_1, q_2 of a modular curve X to certain fundamental constants, among them the scattering constant $\mathscr{C}^{\Gamma}_{q_1 q_2}$. Namely, one has (see [1, Proposition E])

$$-\frac{1}{2\pi}\, g_{\mathrm{Ar}}(q_1, q_2) = \mathscr{C}^{\Gamma}_{q_1 q_2} + v_{\Gamma}^{-1} - \mathcal{G}^{\Gamma} - \mathcal{R}^{\Gamma}_{q_1} - \mathcal{R}^{\Gamma}_{q_2},$$

where $\mathcal{G}^{\Gamma}$ is a constant involving the automorphic Green's function on X and $\mathcal{R}^{\Gamma}_{q_j}$ ($j = 1, 2$) is the constant term in the Laurent expansion at $s = 1$ of the Rankin–Selberg transform associated to q_j of the Arakelov metric. For a precise definition of these constants, we refer the reader to [1].

In [1] and [12] A. Abbes–E. Ullmo and P. Michel–E. Ullmo considered the compactification $X_0(N)$ of the Riemann surface $\overline{\Gamma}_0(N)\backslash\mathbb{H}$ for the congruence subgroup $\overline{\Gamma}_0(N)$ recalled in Sect. 2.1. To state their result, let $\mathcal{X}_0(N)/\mathbb{Z}$ denote the minimal regular model of $X_0(N)/\mathbb{Q}$ and let $\overline{\omega}^2_{\mathcal{X}_0(N)/\mathbb{Z}}$ denote the self-intersection of the relative dualizing sheaf. Then, the following asymptotics holds (see [1] and [12, Théorème 1.1])

$$\overline{\omega}^2_{\mathcal{X}_0(N)/\mathbb{Z}} \sim 3 g_{\overline{\Gamma}_0(N)} \log(N), \tag{1}$$

as $N \to \infty$, where N is assumed to be square-free and such that $2, 3 \nmid N$, and $g_{\overline{\Gamma}_0(N)}$ denotes the genus of $X_0(N)$. Here, the geometric contribution to the leading term of the asymptotics (1) is given by $g_{\overline{\Gamma}_0(N)} \log(N)$, whereas the analytic contribution equals $2g_{\overline{\Gamma}_0(N)} \log(N)$. A remarkable fact is that the leading term $2g_{\overline{\Gamma}_0(N)} \log(N)$ from the analytic contribution essentially comes from the scattering constants at the corresponding cusps. The same phenomenon also occurs in the article [11], where an analogous asymptotics for the modular curves $X_1(N)$ associated to the congruence subgroup $\overline{\Gamma}_1(N)$ is established, as well as in the upcoming article [5], see also the dissertation [4], where an asymptotics in the case of the modular curves $X(N)$ associated to the principal congruence subgroup $\overline{\Gamma}(N)$ is proved. Summing up, in all three known cases, the leading term of the analytic contribution essentially comes from the involved scattering constants, and one might expect this to be true for other congruence subgroups. Therefore, explicit formulas for the scattering constants of arbitrary congruence subgroups play a crucial role for the purpose of proving similar asymptotics in Arakelov theory.

1.2 Purpose of this Article

In this article, we provide a formula for the scattering constants of an arbitrary congruence subgroup Γ of level N, where $N = 2$ or $N \geq 3$ is odd and square-free. This formula expresses the scattering constants of Γ in terms of scattering constants of the principal congruence subgroup $\overline{\Gamma}(N)$, which in turn can be explicitly derived from the work of Hejhal [6]. For instance, this formula can be used to obtain expressions for scattering constants arising in Arakelov geometry when establishing bounds for the self-intersection of the relative dualizing sheaf on modular curves. In this article, we compute the relevant scattering constants of the groups $\overline{\Gamma}_0(N)$, $\overline{\Gamma}_1(N)$, and $\overline{\Gamma}(N)$. We thus give another proof for the scattering constants obtained in [1] and [11], in a uniform way. Moreover, we provide the scattering constants that will be needed in the upcoming article [5]. We also indicate that there is a missing term in [11, Lemma 3.6]. We discuss this in Remark 9 of Sect. 6 below. We emphasize that this does not affect the asymptotics proven in [11].

The authors would like to point out, that the results and methods of proofs presented here are likely known to the experts. However, to the best knowledge of the authors, the explicit formulas for the scattering constants $\overline{\Gamma}(N)$ and the uniform formula given in Theorem 8 have not been stated elsewhere.

1.3 Outline of the Article

The paper is organized as follows. In Sect. 2, we recall basic notations and review facts on cusps, non-holomorphic Eisenstein series, scattering functions, and scattering constants for congruence subgroups. In Sect. 3, we turn our attention to the principal congruence subgroup $\overline{\Gamma}(N)$. In particular, we recall the parametrization of the cusps given by Hejhal and in Theorem 1 we state Hejhal's explicit formula for the scattering functions of $\overline{\Gamma}(N)$. In Sect. 4, Proposition 5, we provide a relation expressing the non-holomorphic Eisenstein series of a congruence subgroup Γ as a sum of certain non-holomorphic Eisenstein series of the principal congruence subgroup $\overline{\Gamma}(N)$. In Sect. 5, we establish the main results of this article. In Proposition 6, we first prove an explicit formula for the scattering constants of $\overline{\Gamma}(N)$, for $N = 2$ or $N \geq 3$ odd and square-free. In Theorem 8, using the relation between Eisenstein series, we express the scattering constants of an arbitrary congruence subgroup Γ of level N in terms of certain scattering constants of the principal congruence subgroup. Finally, in Sect. 6, we combine Theorem 8 with Proposition 6 to obtain closed formulas for several scattering constants of the subgroups $\overline{\Gamma}(N)$, $\overline{\Gamma}_0(N)$, and $\overline{\Gamma}_1(N)$.

2 Background Material

2.1 Congruence Subgroups

Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}; y > 0\}$ denote the hyperbolic upper half-plane and $\mathbb{H}^* := \mathbb{H} \sqcup \mathbb{P}^1(\mathbb{R})$ the union of $\mathbb{H}$ with its topological boundary. The hyperbolic volume form is given by $\mu_{\mathrm{hyp}}(z) = \mathrm{d}x\,\mathrm{d}y/y^2$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^*$ by fractional linear transformations. This action is transitive on $\mathbb{H}$, since $z = x + iy = n(x)a(y)i$ with

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad a(y) := \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}.$$

By abuse of notation, we represent an element of $\mathrm{PSL}_2(\mathbb{R})$ by a matrix. We set $I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$.

As mentioned in the introduction, we let Γ denote a congruence subgroup, that is a subgroup of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ containing the principal congruence subgroup

$$\overline{\Gamma}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv \pm 1 \bmod N,\, b \equiv c \equiv 0 \bmod N \right\}$$

for some N. Observe that $\overline{\Gamma}(1) = \mathrm{PSL}_2(\mathbb{Z})$. The level of a congruence subgroup Γ is the lowest positive integer N such that $\overline{\Gamma}(N) \subseteq \Gamma$. The subgroups $\overline{\Gamma}_0(N)$ and $\overline{\Gamma}_1(N)$ given by

$$\overline{\Gamma}_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$

$$\overline{\Gamma}_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv \pm 1 \bmod N,\; c \equiv 0 \bmod N \right\},$$

respectively, are instances of congruence subgroups of level N.

The quotient space $X := \Gamma\backslash\mathbb{H}^*$ admits the structure of a compact Riemann surface of genus g_Γ. The hyperbolic volume of X is given by $v_\Gamma := \int_X \mu_{\mathrm{hyp}}(z)$ and it is finite. By abuse of notation, we will at times identify points of X with their preimages in $\mathbb{H}^*$. By $\Gamma_z = \{\gamma \in \Gamma \mid \gamma z = z\}$ we denote the stabilizer subgroup of a point $z \in \mathbb{H}^*$ with respect to Γ.

2.2 Cusps

A cusp of X is the Γ-orbit of a parabolic fixed point of Γ in $\mathbb{H}^*$. The number of cusps of X is finite and will be denoted by p_Γ. By $P_\Gamma \subseteq \mathbb{P}^1(\mathbb{Q})$ we denote a complete set of representatives for the cusps of X. We will always identify a cusp of X with its representative in P_Γ. Hereby, identifying $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$, we write elements of $\mathbb{P}^1(\mathbb{Q})$ as α/β for $\alpha, \beta \in \mathbb{Z}$, not both equal to 0, and we always assume that $\gcd(\alpha, \beta) = 1$; we set $1/0 := \infty$.

The width of a cusp $q \in P_\Gamma$ is given by $w_q := [\mathrm{PSL}_2(\mathbb{Z})_q : \Gamma_q]$. We define the subgroup

$$B(w_q) := \left\{ n(bw_q) \;\middle|\; b \in \mathbb{Z} \right\} \subset \mathrm{PSL}_2(\mathbb{Z}) \tag{2}$$

and we set $B := B(1)$. For each cusp $q \in P_\Gamma$, we choose an element $g_q \in \mathrm{PSL}_2(\mathbb{Z})$ such that $g_q\infty = q$, namely, if $q = \alpha/\beta \in P_\Gamma$, we choose

$$g_q = \begin{pmatrix} \alpha & * \\ \beta & * \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}). \tag{3}$$

Then, we have

$$g_q^{-1}\Gamma_q g_q = B(w_q). \tag{4}$$

Furthermore, the element $\sigma_q := g_q\, a(w_q) \in \mathrm{PSL}_2(\mathbb{R})$ is a scaling matrix for the cusp q, since it satisfies $\sigma_q\infty = q$ and $\sigma_q^{-1}\Gamma_q\sigma_q = B$. Unless otherwise stated, we choose all scaling matrices to be of this form.

2.3 Non-holomorphic Eisenstein Series

Let $q \in P_\Gamma$ be a cusp with scaling matrix $\sigma_q \in \mathrm{PSL}_2(\mathbb{R})$. For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the non-holomorphic Eisenstein series associated to the cusp q is given by

$$E_q^\Gamma(z, s) := \sum_{\gamma \in \Gamma_q \backslash \Gamma} \mathrm{Im}(\sigma_q^{-1}\gamma z)^s. \tag{5}$$

Note that this definition does not depend on the choice of the representative for the cusp q. The series in (5) converges absolutely and locally uniformly for any $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Thus, $E_q^\Gamma(z, s)$ defines a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, which is Γ-invariant in z. Furthermore, the Eisenstein series admits a meromorphic continuation to the complex s-plane. At $s = 1$ there is always a simple pole of $E_q^\Gamma(z, s)$ with residue equal to v_Γ^{-1}.

Let $q_1, q_2 \in P_\Gamma$ be two cusps, not necessarily distinct, with scaling matrices $\sigma_{q_1}, \sigma_{q_2} \in \mathrm{PSL}_2(\mathbb{R})$, respectively. Then, the Fourier expansion of $E^{\Gamma}_{q_1}(z, s)$ with respect to the cusp q_2 is given by (see, e.g., [8, Theorem 3.4])

$$E^{\Gamma}_{q_1}(\sigma_{q_2} z, s) = \delta_{q_1 q_2} y^s + \varphi^{\Gamma}_{q_1 q_2}(s) y^{1-s} + \sum_{n \neq 0} \varphi^{\Gamma}_{q_1 q_2}(n; s) y^{1/2} K_{s-1/2}(2\pi |n| y) e^{2\pi i n x},$$

where $\delta_{q_1 q_2}$ is the Dirac delta function, $K_\mu(Z)$ denotes the modified Bessel function of the second kind (see, e.g., [2, pp. 374–377]), and we have set

$$\varphi^{\Gamma}_{q_1 q_2}(s) := \sqrt{\pi}\, \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \sum_{c > 0} c^{-2s} \Bigg(\sum_{\substack{d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_{q_1}^{-1} \Gamma \sigma_{q_2}}} 1 \Bigg), \tag{6}$$

$$\varphi^{\Gamma}_{q_1 q_2}(n; s) := \frac{2\pi^s}{\Gamma(s)} |n|^{s-1/2} \sum_{c > 0} c^{-2s} \Bigg(\sum_{\substack{d \bmod c \\ \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_{q_1}^{-1} \Gamma \sigma_{q_2}}} e^{2\pi i (dm + an)/c} \Bigg).$$

For further details on non-holomorphic Eisenstein series, we refer the reader to the vast literature, e.g., [6, 8], or [10].

2.4 Scattering Functions and Scattering Constants

Let $q_1, q_2 \in P_\Gamma$ be two cusps, not necessarily distinct, with scaling matrices $\sigma_{q_1}, \sigma_{q_2} \in \mathrm{PSL}_2(\mathbb{R})$, respectively. As mentioned in the introduction, the function $\varphi^{\Gamma}_{q_1 q_2}(s)$, defined in (6), is called scattering function of Γ at the cusps q_1 and q_2. Note that the definition of $\varphi^{\Gamma}_{q_1 q_2}(s)$ does not depend on the choice of the representatives for the cusps q_1 and q_2 nor on the choice of the scaling matrices. The scattering function $\varphi^{\Gamma}_{q_1 q_2}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and admits a meromorphic continuation to the complex s-plane. At $s = 1$ there is always a simple pole of $\varphi^{\Gamma}_{q_1 q_2}(s)$ with residue equal to v_Γ^{-1}. Furthermore, we have

$$\varphi^{\Gamma}_{q_1 q_2}(s) = \varphi^{\Gamma}_{q_2 q_1}(s).$$

For further properties of the scattering functions, we refer the reader to the literature, e.g., [8]. For the modular group $\Gamma = \overline{\Gamma}(1)$, we have $p_{\overline{\Gamma}(1)} = 1$ and we choose $P_{\overline{\Gamma}(1)} = \{\infty\}$. Then, the scattering function is explicitly given by

$$\varphi^{\overline{\Gamma}(1)}_{\infty\infty}(s) = \sqrt{\pi}\, \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}, \tag{7}$$

where $\zeta(s)$ denotes the Riemann zeta function. We will also make use of formulas for the scattering functions of the principal congruence subgroup, which have been established by Hejhal in [6]. These will be recalled in Sect. 3.2.

The scattering constant $\mathscr{C}^{\Gamma}_{q_1q_2}$ of Γ at the cusps q_1 and q_2 is defined by

$$\mathscr{C}^{\Gamma}_{q_1q_2} := \lim_{s\to 1}\left(\varphi^{\Gamma}_{q_1q_2}(s) - \frac{v_{\Gamma}^{-1}}{s-1}\right).$$

Again, the definition of $\mathscr{C}^{\Gamma}_{q_1q_2}$ does not depend on the choice of the representatives for the cusps q_1 and q_2. We note the identity

$$\mathscr{C}^{\Gamma}_{q_1q_2} = \mathscr{C}^{\Gamma}_{q_2q_1}.$$

For $\Gamma = \overline{\Gamma}(1)$, we have $v_{\overline{\Gamma}(1)} = \pi/3$ and we derive from (7) the Laurent expansion

$$\varphi^{\overline{\Gamma}(1)}_{\infty\infty}(s) = \frac{3/\pi}{s-1} + \frac{6}{\pi}\mathscr{C} + O(s-1), \tag{8}$$

at $s = 1$, where the constant $\mathscr{C}$ is given by

$$\mathscr{C} := 1 - \log(4\pi) + \frac{\zeta'(-1)}{\zeta(-1)}. \tag{9}$$

Therefore, we have $\mathscr{C}^{\overline{\Gamma}(1)}_{\infty\infty} = (6/\pi)\mathscr{C}$.

3 The Principal Congruence Subgroup

In this section, we review facts concerning the cusps of $X(N) := \overline{\Gamma}(N)\backslash\mathbb{H}^*$ and we state Hejhal's formula for the scattering functions of $\overline{\Gamma}(N)$.

3.1 Cusps

Choosing for each $(u, v) \in (\mathbb{Z}/N\mathbb{Z})^2$ with $\gcd(u, v, N) = 1$ a lift $(\alpha, \beta) \in \mathbb{Z}^2$ with $\gcd(\alpha, \beta) = 1$, then the set of all of these $\alpha/\beta \in \mathbb{P}^1(\mathbb{Q})$ constitutes a possible choice for $P_{\overline{\Gamma}(N)}$ (see, e.g., [3, pp. 99–101]). From this, the number of cusps of $X(N)$ can be easily computed. In particular, we have $p_{\overline{\Gamma}(2)} = 3$ and, for $N \geq 3$ square-free, we have the formula

$$p_{\overline{\Gamma}(N)} = \frac{1}{2}\prod_{p|N}(p^2 - 1) = \frac{1}{2}\sigma(N)\varphi(N),$$

where $\sigma(N)$ is the sum of all positive divisors of N and $\varphi(N)$ denotes the Euler's totient function. Since $\overline{\Gamma}(N)$ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, all the cusps $q \in P_{\overline{\Gamma}(N)}$ have the same width $w_q = N$. The hyperbolic volume of $X(N)$ is therefore given by $v_{\overline{\Gamma}(N)} = v_{\overline{\Gamma}(1)} N p_{\overline{\Gamma}(N)} = \pi N p_{\overline{\Gamma}(N)}/3$. In particular, we have $v_{\overline{\Gamma}(2)} = 2\pi$ and, for $N \geq 3$ square-free, we obtain

$$v_{\overline{\Gamma}(N)} = \frac{\pi}{6} N \prod_{p|N} (p^2 - 1). \tag{10}$$

Let now $N \geq 3$ be an integer that is odd and square-free. We state another possible choice for $P_{\overline{\Gamma}(N)}$, which was employed by Hejhal in [6] when establishing explicit formulas for the scattering functions. For an integer d satisfying $d|N$ and $1 < d < N$, let $\gamma_d \in \mathrm{PSL}_2(\mathbb{Z})$ be such that

$$\gamma_d \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod d \quad \text{and} \quad \gamma_d \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N/d;$$

we refer to [6, Lemma 4.1] for the existence of such γ_d. We further set $\gamma_1 := I$ and $\gamma_N := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Next, let $U \subset \mathbb{Z}$ be a set of representatives for $(\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$, which contains 1. For $\xi \in U$, we define the matrix

$$h(\xi) := \begin{pmatrix} \xi & 1 \\ kN & \tilde{\xi} \end{pmatrix} \in \overline{\Gamma}_0(N)$$

with $\tilde{\xi}, k \in \mathbb{Z}$ chosen such that $\xi\tilde{\xi} - kN = 1$. With these data one can choose (see [6, Lemma 5.1])

$$P_{\overline{\Gamma}(N)} = \left\{ n(t)h(\xi)\gamma_d \infty \;\middle|\; (d, \xi, t) \in Q_{\overline{\Gamma}(N)} \right\}$$

with parameter set given by

$$Q_{\overline{\Gamma}(N)} := \left\{ (d, \xi, t) \in \mathbb{Z}^3 \;\middle|\; d|N \text{ with } 1 \leq d \leq N,\ \xi \in U,\ 0 \leq t < d \right\}.$$

We say that a cusp $q_j \in P_{\overline{\Gamma}(N)}$ is parametrized by $(d_j, \xi_j, t_j) \in Q_{\overline{\Gamma}(N)}$, if $\overline{\Gamma}(N)q_j = \overline{\Gamma}(N)n(t_j)h(\xi_j)\gamma_{d_j}\infty$.

3.2 Scattering Functions

Let $N \geq 3$ be an integer that is odd and square-free. Consider $q_1, q_2 \in P_{\overline{\Gamma}(N)}$ two cusps, not necessarily distinct, parametrized by $(d_j, \xi_j, t_j) \in Q_{\overline{\Gamma}(N)}$ with $j = 1, 2$, and let $\tilde{\xi}_j$ be an inverse of ξ_j modulo N. With these data, set $e_j := N/d_j$ and $d := \gcd(\xi_1 +$

$t_1 - \xi_2 - t_2, \gcd(d_1, d_2))$. Then, the level N can be decomposed as $N = K_{q_1q_2}M_{q_1q_2}$ with

$$K_{q_1q_2} := \gcd(e_1, e_2)d, \quad M_{q_1q_2} := \frac{\gcd(d_1, d_2)}{d}\gcd(d_1, e_2)\gcd(d_2, e_1).$$

Note that $\gcd(K_{q_1q_2}, M_{q_1q_2}) = 1$, since N is square-free. In case that $d_1 = d_2 = 1$, one has $K_{q_1q_2} = N$, hence $M_{q_1q_2} = 1$. Finally, let $\nu = \nu(K_{q_1q_2}, M_{q_1q_2})$ be an integer such that $\nu \bmod M_{q_1q_2}$ belongs to $(\mathbb{Z}/M_{q_1q_2}\mathbb{Z})^\times$ and ν satisfies the congruences

$$\begin{aligned} K_{q_1q_2}\nu &\equiv -\tilde{\xi}_1\xi_2 \bmod \gcd(d_1, e_2), \\ K_{q_1q_2}\nu &\equiv \xi_1\tilde{\xi}_2 \bmod \gcd(d_2, e_1), \\ \gcd(e_1, e_2)\nu &\equiv \tilde{\xi}_1\tilde{\xi}_2\left(\frac{\xi_1 + t_1 - \xi_2 - t_2}{d}\right) \bmod \frac{\gcd(d_1, d_2)}{d}. \end{aligned}$$

We now state Hejhal's formula for the scattering functions $\varphi_{q_1q_2}^{\overline{\Gamma}(N)}(s)$. To do this, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the function

$$\phi_{q_1q_2}^{\overline{\Gamma}(N)}(s) := \frac{K_{q_1q_2}^{1-2s}}{\varphi(M_{q_1q_2})} \sum_{\substack{\chi \bmod M_{q_1q_2} \\ \text{even}}} \overline{\chi}(\nu)\, \ell_{q_1q_2}(s, \chi), \tag{11}$$

where the sum runs over all Dirichlet characters modulo $M_{q_1q_2}$ with $\chi(-1) = 1$, if $M_{q_1q_2} \neq 1$, and, if $M_{q_1q_2} = 1$, then $\chi \bmod 1$ denotes the function $\chi_{\mathrm{triv}} : \mathbb{Z} \longrightarrow \mathbb{C}^\times$, which is identically 1 on $\mathbb{Z}$. Furthermore, we have set

$$\ell_{q_1q_2}(s, \chi) := \frac{L(2s-1, \chi)}{L(2s, \chi)} \prod_{p | K_{q_1q_2}} \left(1 - \frac{\chi(p)}{p^{2s}}\right)^{-1} \tag{12}$$

with $L(s, \chi)$ denoting the Dirichlet L-function associated to the character χ and the product runs over all p prime numbers dividing $K_{q_1q_2}$.

Theorem 1 (Hejhal) *Let $N \geq 3$ be an integer that is odd and square-free. Suppose that $q_1, q_2 \in P_{\overline{\Gamma}(N)}$ are two cusps, not necessarily distinct. Then, for $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$, *the following identity holds*

$$\varphi_{q_1q_2}^{\overline{\Gamma}(N)}(s) = 2\sqrt{\pi}\, \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} N^{-2s} \phi_{q_1q_2}^{\overline{\Gamma}(N)}(s),$$

where $\phi_{q_1q_2}^{\overline{\Gamma}(N)}(s)$ is the function given by (11)*. Furthermore, this identity is valid for all $s \in \mathbb{C}$ by meromorphic continuation.*

Proof See [6, pp. 540–549]. □

We note that there is also a formula for the scattering function $\varphi_{q_1 q_2}^{\overline{\Gamma}(2)}(s)$ (see, e.g., [13]). Namely, we have

$$\varphi_{q_1 q_2}^{\overline{\Gamma}(2)}(s) = \begin{cases} \sqrt{\pi}\,\dfrac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,\dfrac{\zeta(2s-1)}{\zeta(2s)}\,\dfrac{2^{-2s}}{2^{2s}-1}, & \text{if } q_1 = q_2; \\[2ex] \sqrt{\pi}\,\dfrac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,\dfrac{\zeta(2s-1)}{\zeta(2s)}\,\dfrac{1-2^{1-2s}}{2^{2s}-1}, & \text{if } q_1 \neq q_2. \end{cases}$$

This formula is obtained by substituting [13, (C.0.4.1)] into [13, Proposition C.0.5] and multiplying the resulting expression by $2^{-2s}\varphi_{\infty\infty}^{\overline{\Gamma}(1)}(s)$.

4 Relation for Non-holomorphic Eisenstein Series

In this section, we prove a relation between the non-holomorphic Eisenstein series of an arbitrary congruence subgroup Γ of level N and the sum of certain non-holomorphic Eisenstein series of the principal subgroup $\overline{\Gamma}(N)$. We first prove the following useful lemma.

Lemma 2 *Let Γ be a congruence subgroup of level N and let $q = \alpha/\beta \in P_\Gamma$ be a cusp with $g_q \in \mathrm{PSL}_2(\mathbb{Z})$ chosen as in (3). Then, we have the following bijection*

$$g_q^{-1}\left(\Gamma_q\backslash\Gamma\right) \xrightarrow{\simeq} \left\{(m,n) \in \mathbb{Z}^2 \;\middle|\; \gcd(m,n) = 1,\ \Gamma\,\frac{n}{-m} = \Gamma\,\frac{\alpha}{\beta}\right\}. \tag{13}$$

Proof We start by noting that an element $g_q^{-1}\Gamma_q\gamma \in g_q^{-1}\left(\Gamma_q\backslash\Gamma\right)$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ can be written as

$$g_q^{-1}\Gamma_q\gamma = B(w_q)g_q^{-1}\gamma = B(w_q)\begin{pmatrix} * & * \\ \alpha c - \beta a & \alpha d - \beta b \end{pmatrix}$$

with $B(w_q)$ given by (2), where for the first equality we used the identity (4). Thus, the assignment

$$B(w_q)\begin{pmatrix} * & * \\ m & n \end{pmatrix} \longmapsto (m,n)$$

induces a map $g_q^{-1}\left(\Gamma_q\backslash\Gamma\right) \longrightarrow \left\{(m,n) \in \mathbb{Z}^2 \mid \gcd(m,n) = 1,\ \Gamma\,(-n/m) = \Gamma\,\alpha/\beta\right\}$. It is straightforward to show that this map is well-defined. In particular, the element $g_q^{-1}\Gamma_q \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in g_q^{-1}\left(\Gamma_q\backslash\Gamma\right)$ maps to $(\alpha c - \beta a, \alpha d - \beta b) \in \mathbb{Z}^2$ and we have

$$\frac{\alpha d - \beta b}{-\alpha c + \beta a} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\frac{\alpha}{\beta} \in \Gamma\,\frac{\alpha}{\beta}.$$

To prove the injectivity of this map, let $g_q^{-1}\Gamma_q\gamma_1, g_q^{-1}\Gamma_q\gamma_2 \in g_q^{-1}(\Gamma_q\backslash\Gamma)$ with $\gamma_1, \gamma_2 \in \Gamma$ be such that $g_q^{-1}\gamma_1 = \left(\begin{smallmatrix} r_1 & s_1 \\ m & n \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ and $g_q^{-1}\gamma_2 = \left(\begin{smallmatrix} r_2 & s_2 \\ m & n \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$. Then, we have

$$\gamma_1\gamma_2^{-1} = g_q \begin{pmatrix} r_1 & s_1 \\ m & n \end{pmatrix} \begin{pmatrix} r_2 & s_2 \\ m & n \end{pmatrix}^{-1} g_q^{-1} = g_q \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} g_q^{-1}.$$

This implies that $\gamma_1\gamma_2^{-1} \in \Gamma_q$, hence $\Gamma_q\gamma_1 = \Gamma_q\gamma_2$, as desired. Finally, to prove the surjectivity, let $(m, n) \in \mathbb{Z}^2$ with $\gcd(m, n) = 1$ and $\Gamma\,(-n/m) = \Gamma\,\alpha/\beta$. Then, there exists an element $\gamma \in \Gamma$ such that $\gamma\,(-n/m) = \alpha/\beta$. Hence, $g_q^{-1}\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $g_q^{-1}\gamma\,(-n/m) = \infty$. Therefore, $g_q^{-1}\gamma\,\left(\begin{smallmatrix} n & * \\ -m & * \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})_\infty = B$, which yields $g_q^{-1}\gamma = \left(\begin{smallmatrix} * & * \\ m & n \end{smallmatrix}\right)$, as desired. This completes the proof of the lemma.

Lemma 3 *Let Γ be a congruence subgroup of level N and let $q = \alpha/\beta \in P_\Gamma$ be a cusp. Then, for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the following identity holds*

$$E_q^\Gamma(z, s) = w_q^{-s} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1 \\ \Gamma\frac{n}{-m}=\Gamma\frac{\alpha}{\beta}}} \frac{y^s}{|mz+n|^{2s}}.$$

Proof Indeed, from the definition (5) of the Eisenstein series $E_q^\Gamma(z, w)$, we obtain

$$E_q^\Gamma(z, s) = \sum_{\gamma\in\Gamma_q\backslash\Gamma} \mathrm{Im}(\sigma_q^{-1}\gamma z)^s = w_q^{-s} \sum_{\gamma\in\Gamma_q\backslash\Gamma} \mathrm{Im}(g_q^{-1}\gamma z)^s,$$

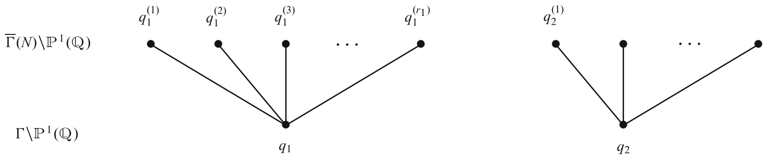
where for the second equality we used the identity $\sigma_q = g_q\,a(w_q)$. Now, by means of the bijection (13), we deduce

$$E_q^\Gamma(z, s) = w_q^{-s} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1 \\ \Gamma\frac{n}{-m}=\Gamma\frac{\alpha}{\beta}}} \frac{y^s}{|mz+n|^{2s}},$$

as asserted. □

In the remaining part of this article, we will often consider the following setting.

Setting (*) *Let $q_1, q_2 \in P_\Gamma$ be two cusps, not necessarily distinct, and write $q_1^{(1)}$ resp. $q_2^{(1)}$ for q_1 and q_2 regarded as elements of $P_{\overline{\Gamma}(N)}$, respectively. Let $q_1^{(1)}, \ldots, q_1^{(r_1)} \in P_{\overline{\Gamma}(N)}$ be a complete list of all the cusps that are Γ-equivalent to q_1. Here, $r_1 = w_{q_1}[\Gamma : \overline{\Gamma}(N)]/N$. The next picture illustrates this setting.*



Remark 4 In setting (*), the width w_{q_1} of the cusp $q_1 \in P_\Gamma$ can be interpreted as the ramification index of q_1 with respect to the map $X \longrightarrow X(1)$. Similarly, the width N of the cusp $q_1^{(j)} \in P_{\overline{\Gamma}(N)}$ ($j = 1, \ldots, r_1$) equals the ramification index of $q_1^{(j)}$ with respect to the map $X(N) \longrightarrow X(1)$. Thus, $q_1^{(j)}$ has ramification index N/w_{q_1} with respect to the map $X(N) \longrightarrow X$. Therefore, we can deduce the equality $r_1 = [\Gamma : \overline{\Gamma}(N)]/(N/w_{q_1})$, as asserted.

Proposition 5 *Let Γ be a congruence subgroup of level N and let $q_1 \in P_\Gamma$ be a cusp. In the notation of setting (*), the following relation holds*

$$E^{\Gamma}_{q_1}(z,s) = \left(\frac{N}{w_{q_1}}\right)^s \sum_{j=1}^{r_1} E^{\overline{\Gamma}(N)}_{q_1^{(j)}}(z,s), \tag{14}$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Proof Let $q_1 = \alpha_1/\beta_1 \in P_\Gamma$. By Lemma 3, we have the representation

$$E^{\Gamma}_{q_1}(z,s) = w_{q_1}^{-s} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1 \\ \Gamma\,\frac{n}{-m} = \Gamma\,\frac{\alpha_1}{\beta_1}}} \frac{y^s}{|mz+n|^{2s}}.$$

Using the decomposition $\Gamma\,\alpha_1/\beta_1 = \bigsqcup_{j=1}^{r_1} \overline{\Gamma}(N)\, q_1^{(j)}$ and writing $q_1^{(j)} = \alpha_j/\beta_j$, we deduce

$$E^{\Gamma}_{q_1}(z,s) = w_{q_1}^{-s} \sum_{j=1}^{r_1} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1 \\ \overline{\Gamma}(N)\,\frac{n}{-m} = \overline{\Gamma}(N)\,\frac{\alpha_j}{\beta_j}}} \frac{y^s}{|mz+n|^{2s}}.$$

Again, by Lemma 3, the innermost sum on the right hand side of this identity is equal to $N^s E^{\overline{\Gamma}(N)}_{q_1^{(j)}}(z,s)$, since $w_{q_1^{(j)}} = N$, for all $j = 1, \ldots, r_1$. This proves the asserted relation. □

5 Formulas for Scattering Constants

In this section, we first employ Theorem 1 to prove a formula for the scattering constants of $\overline{\Gamma}(N)$, where N has the same restrictions as in Sect. 3.2. Then, we relate the scattering constants of a given congruence subgroup Γ of arbitrary level N to scattering constants of the principal congruence subgroup $\overline{\Gamma}(N)$.

Let $q_1, q_2 \in P_{\overline{\Gamma}(N)}$. For $M_{q_1q_2} \neq 1$, we define the constant

$$\kappa(\nu; K_{q_1q_2}, M_{q_1q_2}) := \frac{2\pi}{N^2 K_{q_1q_2}\varphi(M_{q_1q_2})} \sum_{\substack{\chi \bmod M_{q_1q_2} \\ \text{even} \\ \chi \neq \chi_0}} \overline{\chi}(\nu)\, \ell_{q_1q_2}(1, \chi), \tag{15}$$

where χ_0 denotes the principal Dirichlet character modulo $M_{q_1q_2}$. Note that in this case, the function $\ell_{q_1q_2}(s, \chi)$, defined in (12), is well-defined at $s = 1$, since $L(s, \chi)$ is holomorphic at $s = 1$, provided that $\chi \neq \chi_0$, and $L(2, \chi) \neq 0$.

Proposition 6 *Let $N \geq 3$ be an integer that is odd and square-free. Suppose that $q_1, q_2 \in P_{\overline{\Gamma}(N)}$ are two cusps, not necessarily distinct. Then the following identity holds*

$$\mathscr{C}_{q_1q_2}^{\overline{\Gamma}(N)} = 2v_{\overline{\Gamma}(N)}^{-1}\left(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right) + C_{q_1q_2}^{\overline{\Gamma}(N)},$$

where $C_{q_1q_2}^{\overline{\Gamma}(N)} := 0$, if $M_{q_1q_2} = 1$, and

$$C_{q_1q_2}^{\overline{\Gamma}(N)} := 2v_{\overline{\Gamma}(N)}^{-1} \sum_{p|M_{q_1q_2}} \frac{p}{p-1}\log(p) + \kappa(\nu; K_{q_1q_2}, M_{q_1q_2}),$$

if $M_{q_1q_2} \neq 1$. Here, $\mathscr{C}$ is the constant given by (9) and $\kappa(\nu; K_{q_1q_2}, M_{q_1q_2})$ is given by (15).

Proof To simplify the notation, we write $M = M_{q_1q_2}$ and $K = K_{q_1q_2}$ throughout the proof.

We first deduce from Theorem 1, for $M = 1$ and hence $K = N$, the equality

$$\varphi_{q_1q_2}^{\overline{\Gamma}(N)}(s) = 2\sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} N^{-2s} \phi_{q_1q_2}^{\overline{\Gamma}(N)}(s)$$

with

$$\phi_{q_1q_2}^{\overline{\Gamma}(N)}(s) = N^{1-2s}\ell_{q_1q_2}(s, \chi_{\mathrm{triv}}) = N^{1-2s}\frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p|N}\left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

Therefore, by (7) and using that N is square-free, we get

$$\varphi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) = 2N^{1-2s} f(s)\, \varphi_{\infty\infty}^{\overline{\Gamma}(1)}(s),$$

where we have set

$$f(s) := \prod_{p|N} \frac{1}{p^{2s}-1}.$$

To obtain the Laurent expansion at $s = 1$ of the function $f(s)$, we compute its logarithmic derivative

$$\frac{f'(s)}{f(s)} = -\sum_{p|N} \frac{2p^{2s}}{p^{2s}-1} \log(p).$$

Thus, at $s = 1$, we obtain

$$f(s) = f(1)\bigg(1 - 2\Big(\sum_{p|N} \frac{p^2}{p^2-1} \log(p)\Big)(s-1) + O\big((s-1)^2\big)\bigg).$$

Multiplying this expansion with

$$2N^{1-2s} = \frac{2}{N}\Big(1 - 2\log(N)(s-1) + O\big((s-1)^2\big)\Big),$$

we obtain

$$2N^{1-2s} f(s) = \frac{2}{N} f(1)\bigg(1 - 2\Big(\log(N) + \sum_{p|N} \frac{p^2}{p^2-1} \log(p)\Big)(s-1) + O\big((s-1)^2\big)\bigg).$$

Consequently, employing the expansion (8) and observing that (10) gives

$$\frac{3}{\pi}\frac{2}{N} f(1) = \frac{6}{\pi}\frac{1}{N} \prod_{p|N} \frac{1}{p^2-1} = v_{\overline{\Gamma}(N)}^{-1},$$

we finally get

$$\lim_{s\to 1}\left(\varphi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) - \frac{v_{\overline{\Gamma}(N)}^{-1}}{s-1}\right) = 2v_{\overline{\Gamma}(N)}^{-1}\bigg(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1} \log(p)\bigg).$$

Next, we deduce from Theorem 1, for $M \neq 1$, the equality

$$\varphi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) = 2\sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} N^{-2s} \phi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) \tag{16}$$

with

$$\phi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) = \frac{K^{1-2s}}{\varphi(M)}\, \ell_{q_1 q_2}(s, \chi_0) + \frac{K^{1-2s}}{\varphi(M)} \sum_{\substack{\chi \bmod M \\ \text{even} \\ \chi \neq \chi_0}} \overline{\chi}(\nu)\, \ell_{q_1 q_2}(s, \chi),$$

where χ_0 denotes the principal character modulo M. Using that $N = KM$ is square-free and employing the well-known identity

$$L(s, \chi_0) = \zeta(s) \prod_{p|M} \left(1 - \frac{1}{p^s}\right),$$

we deduce from the definition (12) of $\ell_{q_1 q_2}(s, \chi_0)$ the identity

$$\begin{aligned} \ell_{q_1 q_2}(s, \chi_0) &= N^{2s} f(s) \frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p|M} \left(1 - \frac{1}{p^{2s-1}}\right) \\ &= N^{2s} f(s) \frac{\zeta(2s-1)}{\zeta(2s)}\, M^{1-2s} \prod_{p|M} (p^{2s-1} - 1). \end{aligned}$$

Substituting this identity into (16) and using (7), we obtain

$$\begin{aligned} \varphi_{q_1 q_2}^{\overline{\Gamma}(N)}(s) &= 2N^{1-2s} f(s)\, \varphi_{\infty\infty}^{\overline{\Gamma}(1)}(s) \varphi(M)^{-1} \prod_{p|M} (p^{2s-1} - 1) \\ &\quad + 2\sqrt{\pi}\, \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} N^{-2s} \frac{K^{1-2s}}{\varphi(M)} \sum_{\substack{\chi \bmod M \\ \text{even} \\ \chi \neq \chi_0}} \overline{\chi}(\nu) \ell_{q_1 q_2}(s, \chi). \end{aligned}$$

To obtain the Laurent expansion at $s = 1$ of the function

$$g(s) := \varphi(M)^{-1} \prod_{p|M} (p^{2s-1} - 1),$$

we compute its logarithmic derivative

$$\frac{g'(s)}{g(s)} = \sum_{p|M} \frac{2p^{2s-1}}{p^{2s-1} - 1} \log(p).$$

Observing that $g(1) = 1$, since M is square-free, we get

$$g(s) = 1 + 2\biggl(\sum_{p|M} \frac{p}{p-1} \log(p)\biggr)(s-1) + O\bigl((s-1)^2\bigr),$$

at $s = 1$. Employing, at $s = 1$, the Laurent expansion

$$2N^{1-2s} f(s)\, \varphi_{\infty\infty}^{\overline{\Gamma}(1)}(s) = \frac{v_{\overline{\Gamma}(N)}^{-1}}{s-1} + 2v_{\overline{\Gamma}(N)}^{-1}\biggl(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1} \log(p)\biggr) + O(s-1),$$

we thus obtain

$$\lim_{s\to 1}\biggl(\varphi_{q_1q_2}^{\overline{\Gamma}(N)}(s) - \frac{v_{\overline{\Gamma}(N)}^{-1}}{s-1}\biggr) = 2v_{\overline{\Gamma}(N)}^{-1}\biggl(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1} \log(p) + \sum_{p|M} \frac{p}{p-1} \log(p)\biggr)$$
$$+ 2\pi N^{-2} \frac{K^{-1}}{\varphi(M)} \sum_{\substack{\chi \bmod M \\ \text{even} \\ \chi \neq \chi_0}} \overline{\chi}(\nu)\ell_{q_1q_2}(1,\chi).$$

This completes the proof. $\square$

Remark 7 There is also a formula for the scattering constant $\mathscr{C}_{q_1q_2}^{\overline{\Gamma}(2)}$ (see, e.g., [13, Remark 4.4.3, p. 86]). Namely, we have

$$\mathscr{C}_{q_1q_2}^{\overline{\Gamma}(2)} = \begin{cases} 2v_{\overline{\Gamma}(2)}^{-1}\Bigl(\mathscr{C} - \dfrac{7}{3}\log(2)\Bigr), & \text{if } q_1 = q_2; \\[2ex] 2v_{\overline{\Gamma}(2)}^{-1}\Bigl(\mathscr{C} - \dfrac{1}{3}\log(2)\Bigr), & \text{if } q_1 \neq q_2. \end{cases}$$

We now consider setting (*) given in Sect. 4.

Theorem 8 *Let Γ be a congruence subgroup of level N and let $q_1, q_2 \in P_\Gamma$ be two cusps, not necessarily distinct. In the notation of setting (*), the following identity holds*

$$\mathscr{C}_{q_1q_2}^{\Gamma} = v_\Gamma^{-1} \log\biggl(\frac{N^2}{w_{q_1}w_{q_2}}\biggr) + \frac{N}{w_{q_1}} \sum_{j=1}^{r_1} \mathscr{C}_{q_1^{(j)}q_2^{(1)}}^{\overline{\Gamma}(N)}.$$

Proof We start by proving the following relation for scattering functions

$$\varphi_{q_1q_2}^{\Gamma}(s) = \frac{N^{2s-1}}{w_{q_1}^{s} w_{q_2}^{s-1}} \sum_{j=1}^{r_1} \varphi_{q_1^{(j)}q_2^{(1)}}^{\overline{\Gamma}(N)}(s), \tag{17}$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. To simplify the notation, we write $\widehat{\sigma}_2 = \widehat{g}_2\, a(N)$ for the scaling matrix of $q_2^{(1)}$ in $\overline{\Gamma}(N)$. Furthermore, recall that we write $q_2^{(1)}$ for the cusp $q_2 \in P_\Gamma$ regarded as an element of $P_{\overline{\Gamma}(N)}$. The relation (14) states the identity

$$E_{q_1}^{\Gamma}(\widehat{\sigma}_2 z, s) = \left(\frac{N}{w_{q_1}}\right)^s \sum_{j=1}^{r_1} E_{q_1^{(j)}}^{\overline{\Gamma}(N)}(\widehat{\sigma}_2 z, s). \tag{18}$$

Substituting the Fourier expansion of $E_{q_1^{(j)}}^{\overline{\Gamma}(N)}(z,s)$ with respect to the cusp $q_2^{(1)} \in P_{\overline{\Gamma}(N)}$ into (18) yields

$$E_{q_1}^{\Gamma}(\widehat{\sigma}_2 z, s) = \left(\frac{N}{w_{q_1}}\right)^s \delta_{q_1 q_2} y^s + \left(\frac{N}{w_{q_1}}\right)^s y^{1-s} \sum_{j=1}^{r_1} \varphi_{q_1^{(j)} q_2^{(1)}}^{\overline{\Gamma}(N)}(s) \tag{19}$$

$$+ \left(\frac{N}{w_{q_1}}\right)^s \sum_{n \neq 0} y^{1/2} K_{s-1/2}(2\pi |n| y) e^{2\pi i n x} \sum_{j=1}^{r_1} \varphi_{q_1^{(j)} q_2^{(1)}}^{\overline{\Gamma}(N)}(n; s).$$

We next consider the matrix

$$\sigma_2 := \widehat{\sigma}_2\, a\big(w_{q_2}/N\big) \in \mathrm{PSL}_2(\mathbb{R}). \tag{20}$$

Note that σ_2 is a scaling matrix of q_2 in Γ. Indeed, we have $\sigma_2 \infty = \widehat{\sigma}_2 \infty = q_2^{(1)} = q_2$ and

$$\sigma_2^{-1} \Gamma_{q_2} \sigma_2 = a(N/w_{q_2}) \widehat{\sigma}_2^{-1} \Gamma_{q_2} \widehat{\sigma}_2\, a(w_{q_2}/N) = a(w_{q_2}^{-1}) B(w_{q_2}) a(w_{q_2}) = B,$$

where we used (4) for the second equality. Now, by (20), we have

$$E_{q_1}^{\Gamma}(\widehat{\sigma}_2 z, s) = E_{q_1}^{\Gamma}\left(\sigma_2\left(\frac{N}{w_{q_2}} z\right), s\right),$$

and employing the Fourier expansion of $E_{q_1}^{\Gamma}(w,s)$ with respect to the cusp q_2, letting $w := Nz/w_{q_2}$, we obtain

$$E_{q_1}^{\Gamma}(\widehat{\sigma}_2 z, s) = \delta_{q_1 q_2}\left(\frac{Ny}{w_{q_2}}\right)^s + \varphi_{q_1 q_2}^{\Gamma}(s)\left(\frac{Ny}{w_{q_2}}\right)^{1-s} \tag{21}$$

$$+ \sum_{m \neq 0} \varphi_{q_1 q_2}^{\Gamma}(m; s)\left(\frac{Ny}{w_{q_2}}\right)^{1/2} K_{s-1/2}(2\pi |m| Ny/w_{q_2}) e^{2\pi i m N x/w_{q_2}}.$$

Comparing the coefficients in the asymptotics of the two expansions (19) and (21) of $E^{\Gamma}_{q_1}(\widehat{\sigma}_2 z, s)$ for large y and taken into account the exponential decay of the modified Bessel function of the second kind, we particularly derive the identity

$$\left(\frac{N}{w_{q_1}}\right)^s \sum_{j=1}^{r_1} \varphi^{\overline{\Gamma}(N)}_{q_1^{(j)} q_2^{(1)}}(s) = \varphi^{\Gamma}_{q_1 q_2}(s) \left(\frac{N}{w_{q_2}}\right)^{1-s}.$$

This proves relation (17).

Now, using $r_1 = [\Gamma : \overline{\Gamma}(N)] w_{q_1}/N$, we first observe that

$$v_{\Gamma}^{-1} = \frac{N}{w_{q_1}} r_1 v^{-1}_{\overline{\Gamma}(N)}.$$

Furthermore, at $s = 1$, we have the Laurent expansion

$$\frac{N^{2s-1}}{w^s_{q_1} w^{s-1}_{q_2}} = \frac{N}{w_{q_1}} + \frac{N}{w_{q_1}} \log\left(\frac{N^2}{w_{q_1} w_{q_2}}\right)(s-1) + O\big((s-1)^2\big).$$

Therefore, employing relation (17) and recalling the definition of the scattering constants in question, we get

$$\begin{aligned}\lim_{s\to 1}\left(\varphi^{\Gamma}_{q_1 q_2}(s) - \frac{v_{\Gamma}^{-1}}{s-1}\right) &= \lim_{s\to 1}\left(\frac{N^{2s-1}}{w^s_{q_1} w^{s-1}_{q_2}} \sum_{j=1}^{r_1} \varphi^{\overline{\Gamma}(N)}_{q_1^{(j)} q_2^{(1)}}(s) - \frac{N}{w_{q_1}} \frac{r_1 v^{-1}_{\overline{\Gamma}(N)}}{s-1}\right) \\ &= \frac{N}{w_{q_1}} \sum_{j=1}^{r_1} \mathscr{C}^{\overline{\Gamma}(N)}_{q_1^{(j)} q_2^{(1)}} + \frac{N}{w_{q_1}} \log\left(\frac{N^2}{w_{q_1} w_{q_2}}\right) r_1 v^{-1}_{\overline{\Gamma}(N)}.\end{aligned}$$

This completes the proof of the theorem. □

6 Examples

In this section, we first compute several scattering constants of $\overline{\Gamma}(N)$ for particular cusps using Proposition 6 and Remark 7. For this, we assume that $N = 2$ or $N \geq 3$ is odd and square-free. In particular, we compute the scattering constants, that are needed for the asymptotics in the upcoming article [5]. We then illustrate how to combine Theorem 8 with Proposition 6 in order to obtain closed formulas for scattering constants of other congruence subgroups, by considering examples for $\overline{\Gamma}_0(N)$ and $\overline{\Gamma}_1(N)$. In particular, we compute in a uniform way the scattering constants that arise in [1] and [11].

Example 1 In the notation of Sect. 3, recall that U denotes a set of integers containing 1 representing $(\mathbb{Z}/N\mathbb{Z})^{\times}/\{\pm 1\}$. For $\xi \in U$, we denote by ∞_ξ the cusp parametrized by the triple $(1, \xi, 0) \in \mathcal{Q}_{\overline{\Gamma}(N)}$ and by 0_ξ the cusp parametrized by the triple $(N, \xi, N-1) \in \mathcal{Q}_{\overline{\Gamma}(N)}$. In particular, if $\xi = 1$, then $\infty_1 = \infty$, since $n(0)h(1)\gamma_1\infty = \infty$, and $0_1 = 0$, since $n(N-1)h(1)\gamma_N\infty = \left(\begin{smallmatrix} N & -1 \\ 1 & 0 \end{smallmatrix}\right)\infty = N/1$ and $\overline{\Gamma}(N)\, N/1 = \overline{\Gamma}(N)\, 0/1$. In the notation of Sect. 3.2, we now compute the following scattering constants of $\overline{\Gamma}(N)$.

(i) Let us consider the cusps $q_1 := \infty_\xi$ and $q_2 := \infty$. Then, we have that $d_1 = d_2 = 1$, $\xi_1 = \xi$, $\xi_2 = 1$, and $t_1 = t_2 = 0$. These values give $e_1 = e_2 = N$, $d = 1$, $K_{q_1 q_2} = N$, and $M_{q_1 q_2} = 1$. From Proposition 6, we obtain

$$\mathscr{C}^{\overline{\Gamma}(N)}_{\infty_\xi \infty} = 2v^{-1}_{\overline{\Gamma}(N)}\left(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right).$$

In particular, letting $\xi = 1$, this yields one of the scattering constants used in [5].

(ii) Let us consider the cusps $q_1 := \infty_\xi$ and $q_2 := 0$. Then, we have that $d_1 = 1$, $d_2 = N$, $\xi_1 = \xi$, $\xi_2 = 1$, $t_1 = 0$, and $t_2 = N-1$. These values give $e_1 = N$, $e_2 = 1$, $d = 1$, $K_{q_1 q_2} = 1$, and $M_{q_1 q_2} = N$. Take $\nu = \xi$. From Proposition 6, we obtain

$$\begin{aligned}\mathscr{C}^{\overline{\Gamma}(N)}_{\infty_\xi 0} &= 2v^{-1}_{\overline{\Gamma}(N)}\left(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right) \\ &\quad + 2v^{-1}_{\overline{\Gamma}(N)}\sum_{p|N}\frac{p}{p-1}\log(p) + \kappa(\xi; 1, N) \\ &= 2v^{-1}_{\overline{\Gamma}(N)}\left(\mathscr{C} - \sum_{p|N}\frac{p^2-p-1}{p^2-1}\log(p)\right) + \kappa(\xi; 1, N).\end{aligned}$$

(iii) Let us consider the cusps $q_1 := \infty$ and $q_2 := 0_\xi$. Then, we have that $d_1 = 1$, $d_2 = N$, $\xi_1 = 1$, $\xi_2 = \xi$, $t_1 = 0$, and $t_2 = N-1$. These values give $e_1 = N$, $e_2 = 1$, $d = 1$, $K_{q_1 q_2} = 1$, and $M_{q_1 q_2} = N$. Take $\nu = \tilde{\xi}$. From Proposition 6, we obtain

$$\begin{aligned}\mathscr{C}^{\overline{\Gamma}(N)}_{\infty 0_\xi} &= 2v^{-1}_{\overline{\Gamma}(N)}\left(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right) \\ &\quad + 2v^{-1}_{\overline{\Gamma}(N)}\sum_{p|N}\frac{p}{p-1}\log(p) + \kappa(\tilde{\xi}; 1, N) \\ &= 2v^{-1}_{\overline{\Gamma}(N)}\left(\mathscr{C} - \sum_{p|N}\frac{p^2-p-1}{p^2-1}\log(p)\right) + \kappa(\tilde{\xi}; 1, N).\end{aligned}$$

This gives the other scattering constants used in [5].

Example 2 In this example, the following coset decomposition will be useful

$$\overline{\Gamma}_0(N) = \bigsqcup_{\xi \in U} \bigsqcup_{t=1}^{N} \overline{\Gamma}(N) n(t) h(\xi). \tag{22}$$

In particular, this shows that $[\overline{\Gamma}_0(N) : \overline{\Gamma}(N)] = N\varphi(N)/2$. In the notation of setting (*), we now compute several scattering constants of $\Gamma = \overline{\Gamma}_0(N)$.

(i) Let us consider the cusps $q_1 := \infty, q_2 := \infty \in P_{\overline{\Gamma}_0(N)}$. Then, we have $w_{q_1} = w_{q_2} = 1$ and $r_1 = [\overline{\Gamma}_0(N) : \overline{\Gamma}(N)] w_{q_1}/N = \varphi(N)/2$. From the coset decomposition (22), we can deduce that each cusp $q_1^{(j)} \in P_{\overline{\Gamma}(N)}$ ($j = 1, \dots, \varphi(N)/2$) corresponds to exactly one of the cusps ∞_ξ given in Example 1 (i). Therefore, by Theorem 8, we have the identity

$$\mathscr{C}_{\infty\infty}^{\overline{\Gamma}_0(N)} = v_{\overline{\Gamma}_0(N)}^{-1} \log(N^2) + N \sum_{\xi \in U} \mathscr{C}_{\infty_\xi \infty}^{\overline{\Gamma}(N)}.$$

Using the formula given in Example 1 (i), we obtain

$$\begin{aligned} \mathscr{C}_{\infty\infty}^{\overline{\Gamma}_0(N)} &= 2v_{\overline{\Gamma}_0(N)}^{-1} \log(N) + N \sum_{\xi \in U} 2v_{\overline{\Gamma}(N)}^{-1} \left(\mathscr{C} - \log(N) - \sum_{p \mid N} \frac{p^2}{p^2 - 1} \log(p) \right) \\ &= 2v_{\overline{\Gamma}_0(N)}^{-1} \left(\mathscr{C} - \sum_{p \mid N} \frac{p^2}{p^2 - 1} \log(p) \right). \end{aligned}$$

This recovers the scattering constant given in [1, p. 59].

(ii) Let us consider the cusps $q_1 := \infty, q_2 := 0 \in P_{\overline{\Gamma}_0(N)}$. Then, we have $w_{q_1} = 1$, $w_{q_2} = N$, and $r_1 = [\overline{\Gamma}_0(N) : \overline{\Gamma}(N)] w_{q_1}/N = \varphi(N)/2$. Therefore, from Theorem 8, we have the identity

$$\mathscr{C}_{\infty 0}^{\overline{\Gamma}_0(N)} = v_{\overline{\Gamma}_0(N)}^{-1} \log(N) + N \sum_{\xi \in U} \mathscr{C}_{\infty_\xi 0}^{\overline{\Gamma}(N)}.$$

Using the formula given in Example 1 (ii), we obtain

$$\begin{aligned} \mathscr{C}_{\infty 0}^{\overline{\Gamma}_0(N)} &= v_{\overline{\Gamma}_0(N)}^{-1} \log(N) + 2v_{\overline{\Gamma}_0(N)}^{-1} \left(\mathscr{C} - \sum_{p \mid N} \frac{p^2 - p - 1}{p^2 - 1} \log(p) \right) + N \sum_{\xi \in U} \kappa(\xi; 1, N) \\ &= 2v_{\overline{\Gamma}_0(N)}^{-1} \left(\mathscr{C} - \frac{1}{2} \sum_{p \mid N} \frac{p^2 - 2p - 1}{p^2 - 1} \log(p) \right). \end{aligned}$$

Note that the sum of $\kappa(\xi; 1, N)$ over $\xi \in U$ vanishes by virtue of the orthogonality relations for Dirichlet characters. This recovers the scattering constant given in [1, p. 67].

Example 3 Note that the coset decomposition

$$\overline{\Gamma}_1(N) = \bigsqcup_{t=1}^{N} \overline{\Gamma}(N)n(t)$$

in particular shows that $[\overline{\Gamma}_1(N) : \overline{\Gamma}(N)] = N$. In the notation of setting (*), we now compute several scattering constants of $\Gamma = \overline{\Gamma}_1(N)$.

(i) Let us consider the cusps $q_1 := \infty, q_2 := \infty \in P_{\overline{\Gamma}_1(N)}$. Then, we have $w_{q_1} = w_{q_2} = 1$ and $r_1 = [\overline{\Gamma}_1(N) : \overline{\Gamma}(N)]w_{q_1}/N = 1$. Therefore, from Theorem 8, we have the identity

$$\mathscr{C}_{\infty\infty}^{\overline{\Gamma}_1(N)} = 2v_{\overline{\Gamma}_1(N)}^{-1} \log(N) + N\mathscr{C}_{\infty\infty}^{\overline{\Gamma}(N)}.$$

Using the formula given in Example 1 (i), we obtain

$$\mathscr{C}_{\infty\infty}^{\overline{\Gamma}_1(N)} = 2v_{\overline{\Gamma}_1(N)}^{-1} \log(N) + 2Nv_{\overline{\Gamma}(N)}^{-1}\left(\mathscr{C} - \log(N) - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right)$$

$$= 2v_{\overline{\Gamma}_1(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2}{p^2-1}\log(p)\right).$$

This recovers the scattering constant given in [11, (5.9), p. 140].

(ii) Let us consider the cusps $q_1 := \infty, q_2 := 0_\xi \in P_{\overline{\Gamma}_1(N)}$ with $\xi \in U$. Then, we have $w_{q_1} = 1$, $w_{q_2} = N$, and $r_1 = [\overline{\Gamma}_1(N) : \overline{\Gamma}(N)]w_{q_1}/N = 1$. Therefore, from Theorem 8, we have the identity

$$\mathscr{C}_{\infty 0_\xi}^{\overline{\Gamma}_1(N)} = v_{\overline{\Gamma}_1(N)}^{-1} \log(N) + N\mathscr{C}_{\infty 0_\xi}^{\overline{\Gamma}(N)}.$$

Using the formula given in Example 1 (iii), we obtain

$$\mathscr{C}_{\infty 0_\xi}^{\overline{\Gamma}_1(N)} = v_{\overline{\Gamma}_1(N)}^{-1} \log(N) + 2v_{\overline{\Gamma}_1(N)}^{-1}\left(\mathscr{C} - \sum_{p|N} \frac{p^2-p-1}{p^2-1}\log(p)\right) + N\kappa(\tilde{\xi}; 1, N)$$

$$= 2v_{\overline{\Gamma}_1(N)}^{-1}\left(\mathscr{C} - \frac{1}{2}\sum_{p|N} \frac{p^2-2p-1}{p^2-1}\log(p)\right) + N\kappa(\tilde{\xi}; 1, N).$$

Remark 9 Letting $\xi = 1$ in Example 3 (ii), we obtain a formula for $\mathscr{C}_{\infty 0}^{\overline{\Gamma}_1(N)}$. It turns out that this expression differs from [11, Lemma 6.1, p. 143] in the additive term $N\kappa(1; 1, N)$. To be more precise, the scattering function $\varphi_{0\infty}^{\overline{\Gamma}_1(N)}(s) = \varphi_{\infty 0}^{\overline{\Gamma}_1(N)}(s)$ can be explicitly computed using relation (17), which yields $\varphi_{\infty 0}^{\overline{\Gamma}_1(N)}(s) = N^s \varphi_{\infty 0}^{\overline{\Gamma}(N)}(s)$, and then applying Hejhal's formula stated in Theorem 1. In this special case, a straightforward computation then yields the identity

$$\varphi_{0\infty}^{\overline{\Gamma}_1(N)}(s) = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,\frac{1}{N^s} \sum_{\substack{c\in\mathbb{Z}\\ c\neq 0\\ c\equiv -1 \bmod N}} \frac{\varphi(c)}{c^{2s}}.$$

The congruence condition in the above series is missing in [11].

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The Bruinier–Funke Pairing and the Orthogonal Complement of Unary Theta Functions

Ben Kane and Siu Hang Man

Abstract We describe an algorithm for computing the inner product between a holomorphic modular form and a unary theta function, in order to determine whether the form is orthogonal to unary theta functions without needing a basis of the entire space of modular forms and without needing to use linear algebra to decompose this space completely.

1 Introduction

In this paper, we are interested in the decomposition of holomorphic modular forms. Suppose that f is a weight $3/2$ holomorphic modular form on some congruence subgroup Γ. One can decompose f into an Eisenstein series component E, a sum Ψ of (cuspidal) unary theta functions (see (2.6) for the definition), and a cusp form g in the orthogonal complement of unary theta functions. This is an orthogonal splitting with respect to the usual Petersson inner product, since the Eisenstein series is orthogonal to cusp forms. It is thus natural to try to compute the individual pieces. The Eisenstein series component may be computed by determining the growth of f towards the cusps. Furthermore, its Fourier coefficients may be explicitly computed, and these generally constitute the main asymptotic term of the Fourier coefficients of f. In a number of combinatorial applications, this is quite useful in determining the overall growth of the coefficients of f. For example, if f is the generating function for the number of representations by a ternary quadratic form Q, then the coefficients of the Eisenstein series count the number of local representations, and the fact that this is (usually) the main asymptotic term implies an equidistribution result about the representations of integers in the *genus* of Q (i.e., those quadratic forms which are locally equivalent to Q). This equidistribution result does not always hold, however; the coefficients of Ψ grow as fast as the coefficients of E within their support, although they are only supported in finitely many square classes (known on the algebraic side of the theory of quadratic forms as spinor exceptional square classes).

B. Kane (✉) • S.H. Man
Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong
e-mail: bkane@hku.hk; bkane@maths.hku.hk; der.gordox@gmail.com

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Using an upper bound of Duke [6] for the coefficients of g, Duke and Schulze-Pillot [7] combined these ideas to conclude an equidistribution result for the primitive representations by every element of the genus away from these spinor exceptional square classes.

It is natural to ask whether similar results hold true when the quadratic form is replaced with a *totally positive quadratic polynomial* (i.e., a form constructed as a linear combination of a positive-definite integral quadratic form, linear terms, and the unique constant such that the quadratic polynomial only represents non-negative integers and represents zero). One such example is sums of polygonal numbers. For $n \in \mathbb{Z}$, the nth *generalized m-gonal number* is

$$p_m(n) := \frac{(m-2)n^2 - (m-4)n}{2},$$

and for $a, b, c \in \mathbb{N}$ we investigate sums of the type

$$P(x, y, z) = P_{a,b,c}(x, y, z) := ap_m(x) + bp_m(y) + cp_m(z),$$

where $x, y, z \in \mathbb{Z}$. We consider a, b, and c to be fixed and vary x, y, and z. We package P into a generating function

$$\sum_{x,y,z\in\mathbb{Z}} e^{2\pi i P(x,y,z)\tau}$$

with $\tau \in \mathbb{H} := \{\alpha \in \mathbb{C} : \operatorname{Im}(\alpha) > 0\}$; this is known as the *theta function* for P. We may then investigate the Fourier coefficients of this theta function in order to attempt to understand which integers are represented by P. It is actually more natural to complete the square to rewrite

$$p_m(x) = \frac{(2(m-2)x - (m-4))^2}{8(m-2)} - \frac{(m-4)^2}{8(m-2)}$$

Adding an appropriate constant, we obtain a theta function for a shifted lattice $L + \nu$, where $\nu \in \mathbb{Q}L$ inside a quadratic space with associated quadratic norm Q; quadratic forms are simply the case when $\nu = 0$ (or equivalently, $\nu \in L$). These theta functions are again modular forms and the unary theta functions govern whether the local-to-global principle fails finitely or infinitely often.

Theorem 1.1 *Suppose that L is a ternary positive-definite lattice and ν is a vector in the associated quadratic space over $\mathbb{Q}$. Suppose further that the congruence class $(M\mathbb{Z} + r) \cap \mathbb{N}_0$ is primitively represented locally by the associated quadratic form Q on $L + \nu$ and denote by $a_{L+\nu}(Mn + r)$ the number of vectors of length $Mn + r$ in $L + \nu$ (i.e., the number of $\mu \in L + \nu$ for which $Q(\mu) = Mn + r$). If*

$$\Theta_{L+\nu}(\tau) := \sum_{\mu\in L+\nu} e^{2\pi i Q(\mu)\tau}$$

is orthogonal to unary theta functions, then

$$\{n \in \mathbb{Z} : \nexists \mu \in L+\nu,\ Q(\mu) = Mn + r\}$$

is finite.

Remark If $\Theta_{L+\nu}$ is orthogonal to unary theta functions for every $L+\nu$ in a given genus, then one obtains an equidistribution result for representations of $Mn+r$ (for n sufficiently large, but with an ineffective bound) across the entire genus in the same manner as for the case of quadratic forms.

There are a number of cases where Theorem 1.1 has been employed to show that certain quadratic polynomials P are *almost universal* (i.e., they represent all but finitely many integers). In the case of triangular numbers (that is to say, $m = 3$), the first author and Sun [10] obtained a near-classification which was later fully resolved by Chan–Oh [5]; further classification results about sums of triangular numbers and squares were completed by Chan–Haensch [4]. More recently, the case $a = b = c = 1$ with arbitrary m was considered by Haensch and the first author [8]. In [8], a number of almost universality results are obtained by taking advantage of the fact that the structure of modular forms may be used to determine that certain congruence classes are not in the support of the coefficients of all of the unary theta functions in the same space, and hence directly obtaining the orthogonality needed for Theorem 1.1. This was generalized by the second author and Mehta [11] to include many more cases of a, b, c where the same phenomenon implies orthogonality. We next consider a case which does not immediately follow from this approach.

Proposition 1.2 *Every sufficiently large positive integer may be written in the form* $p_8(x) + 3p_8(y) + 3p_8(z)$ *with* $x, y, z \in \mathbb{Z}$. *In other words,* $p_8(x) + 3p_8(y) + 3p_8(z)$ *is almost universal.*

In order to show Proposition 1.2, we use Theorem 1.1 and show that the theta function $\Theta_{L+\nu}$ associated to $p_8(x) + 3p_8(y) + 3p_8(z)$ is orthogonal to all unary theta functions. One can numerically compute the inner product with unary theta functions directly from the definition as an integral over a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ or use a method called unfolding to write it as infinite sums involving products of the Fourier coefficients of $\Theta_{L+\nu}$ and those of the unary theta functions. However, this is not sufficient for our purposes, since we need to algebraically verify that the inner product is indeed zero and the first method is only a numerical approximation while the second method yields an infinite sum. Since the associated space of modular forms is finite-dimensional and there is a natural orthogonal basis of Hecke eigenforms, one can decompose the space explicitly to determine whether this orthogonality holds, but the linear algebra involved is usually computationally expensive and is not feasible in many cases. We hence use a pairing of Bruinier and Funke [3] to rewrite the inner product as a finite sum. The basic idea is to use Stokes' Theorem to rewrite the inner product as a (finite) linear combination of products of the Fourier coefficients of $\Theta_{L+\nu}$ and coefficients of certain "pre-images" of the unary theta functions under a natural differential operator. In order to find these pre-images, we employ work of Zwegers [17], who showed that these pre-images are related to the mock theta functions of Ramanujan.

The paper is organized as follows. In Sect. 2, we give some preliminary information about modular forms and harmonic Maass forms. In Sect. 3, we describe how to compute the inner product using the Bruinier–Funke pairing and construct explicit pre-images of unary theta functions using [17] (see Theorem 3.4). Finally, in Sect. 4, we prove Theorem 1.1 and Proposition 1.2.

2 Preliminaries

We recall some results about modular forms and harmonic Maass forms.

2.1 *Basic Definitions*

Let $\mathbb{H}$ denote the *upper half-plane*, i.e., those $\tau = u + iv \in \mathbb{C}$ with $u \in \mathbb{R}$ and $v > 0$. The matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ (the space of two-by-two integral matrices with integer coefficients and determinant 1) act on $\mathbb{H}$ via *fractional linear transformations* $\gamma\tau := \frac{a\tau+b}{c\tau+d}$. For

$$j(\gamma, \tau) := c\tau + d,$$

a *multiplier system* for a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and *weight* $r \in \mathbb{R}$ is a function $\nu : \Gamma \mapsto \mathbb{C}$ such that for all $\gamma, M \in \Gamma$ (cf. [12, (2a.4)])

$$\nu(M\gamma)j(M\gamma, \tau)^r = \nu(M)j(M, \gamma\tau)^r \nu(\gamma)j(\gamma, \tau)^r.$$

The *slash operator* $|_{r,\nu}$ of weight r and multiplier system ν is then

$$f|_{r,\nu}\gamma(\tau) := \nu(\gamma)^{-1} j(\gamma, \tau)^{-r} f(\gamma\tau).$$

A *harmonic Maass form* of weight $r \in \mathbb{R}$ and multiplier system ν for Γ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following criteria:

1. The function f is annihilated by the weight r hyperbolic Laplacian

$$\Delta_r := -\xi_{2-r} \circ \xi_r,$$

where

$$\xi_r := 2iv^r \overline{\frac{\partial}{\partial \overline{\tau}}}. \tag{2.1}$$

2. For every $\gamma \in \Gamma$, we have

$$f|_{r,\nu}\gamma = f. \tag{2.2}$$

3. The function f exhibits at most linear exponential growth towards every *cusp* (i.e., those elements of $\Gamma\backslash(\mathbb{Q} \cup \{i\infty\})$). This means that at each cusp ϱ of $\Gamma\backslash\mathbb{H}$, the Fourier expansion of the function $f_\varrho(\tau) := f|_{r,\nu}\gamma_\varrho(\tau)$ has at most finitely many terms which grow, where $\gamma_\varrho \in \mathrm{SL}_2(\mathbb{Z})$ sends $i\infty$ to ϱ.

If f is holomorphic and the Fourier expansion at each cusp is bounded, then we call f a holomorphic modular form. Furthermore, if f is a holomorphic modular form and vanishes at every cusp (i.e., the limit $\lim_{\tau\to i\infty} f_\varrho(\tau) = 0$), then we call f a *cusp form*.

2.2 *Half-Integral Weight Forms*

We are particularly interested in the case where $r = k + 1/2$ with $k \in \mathbb{N}_0$ and, in the example given in Theorem 1.1 that motivates this study we may choose Γ to be an intersection between the groups

$$\Gamma_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : M \mid c \right\},$$

$$\Gamma_1(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : M \mid c, a \equiv d \equiv 1 \pmod{M} \right\}$$

for some $M \in \mathbb{N}$ divisible by 4. The multiplier system we are particularly interested in is given in [13, Proposition 2.1], although we do not need the explicit form of the multiplier for this paper.

If $T^N \in \Gamma$ with $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, then by (2.2) we have $f(\tau + N) = f(\tau)$, and hence f has a Fourier expansion ($c_f(v; n) \in \mathbb{C}$)

$$f(\tau) = \sum_{n \gg -\infty} c_f(v; n) e^{\frac{2\pi i n \tau}{N}}. \tag{2.3}$$

Moreover, f is meromorphic if and only if $c_f(v; n) = c_f(n)$ is independent of v. For holomorphic modular forms, an additional restriction $n \geq 0$ follows from the fact that f is bounded as $\tau \to i\infty$. There are similar expansions at the other cusps. One commonly sets $q := e^{2\pi i \tau}$ and associates the above expansion with the corresponding formal power series, using them interchangeably unless explicit analytic properties of the function f are required.

2.3 Theta Functions for Quadratic Polynomials

In [13, (2.0)], Shimura defined theta functions associated to lattice cosets $L+\nu$ (for a lattice L of rank n) and polynomials P on lattice points. Namely, he defined

$$\Theta_{L+\nu,P}(\tau) := \sum_{\boldsymbol{x}\in L+\nu} P(\boldsymbol{x})q^{Q(\boldsymbol{x})},$$

where Q is the quadratic map in the associated quadratic space. We omit P when it is trivial. In this case, we may write $r_{L+\nu}(\ell)$ for the number of elements in $L+\nu$ of norm ℓ and we get

$$\Theta_{L+\nu}(\tau) = \sum_{\ell\geq 0} r_{L+\nu}(\ell)q^{\ell}. \tag{2.4}$$

Shimura then showed (see [13, Proposition 2.1]) that $\Theta_{L+\nu}$ is a modular form of weight $n/2$ for $\Gamma = \Gamma_0(4N^2)\cap\Gamma_1(2N)$ (for some N which depends on L and ν) and a particular multiplier. Note that we have taken $\tau \mapsto 2N\tau$ in Shimura's definition. To show the modularity properties, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$, we compute

$$2N\gamma(\tau) = 2N\frac{a\tau+b}{c\tau+d} = \frac{a(2N\tau)+2Nb}{\frac{c}{2N}(2N\tau)+d} = \begin{pmatrix} a & 2Nb \\ \frac{c}{2N} & d\end{pmatrix}(2N\tau). \tag{2.5}$$

Since $\gamma\in\Gamma$, we have

$$\begin{pmatrix} a & 2Nb \\ \frac{c}{2N} & d\end{pmatrix} \in \Gamma(2N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I_2 \pmod{N}\right\} \subset \Gamma_1(2N),$$

so we may then use [13, Proposition 2.1]. Specifically, the multiplier is the same multiplier as Θ^3, where $\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$ is the classical Jacobi theta function.

We only require the associated polynomial in one case. Namely, for $n=1$ and $P(x) = x$, we require the *unary theta functions* (see [13, (2.0)] with $N \mapsto N/t$, $P(m) = m$, $A = (N/t)$, and $\tau\mapsto 2N\tau$)

$$\vartheta_{h,t}(\tau) = \vartheta_{h,t,N}(\tau) := \sum_{\substack{r\in\mathbb{Z}\\ r\equiv h \pmod{\frac{2N}{t}}}} rq^{tr^2}, \tag{2.6}$$

where h may be chosen modulo $2N/t$ and t is a squarefree divisor of $2N$. These are weight $3/2$ modular forms on $\Gamma_0(4N^2)\cap\Gamma_1(2N)$ with the same multiplier system as $\Theta_{L+\nu}$.

3 The Bruinier–Funke Pairing

In this section, we describe how to compute the inner product with unary theta functions. We again begin by noting the decomposition of a weight $3/2$ modular form f as

$$f = E + \Psi + g,$$

where E is an Eisenstein series, Ψ is a linear combination of unary theta functions, and g is a cusp form in the orthogonal complement of unary theta functions. Since the decomposition above is an orthogonal splitting with respect to the Petersson inner product, one may instead compute the inner product

$$\langle f, \Theta_j \rangle$$

for each unary theta function Θ_j. Recall that Petersson's classical definition of the inner product between two holomorphic modular forms f and h (for which fh is cuspidal) is (here and throughout $\tau = u + iv$)

$$\langle f, h \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma\backslash\mathbb{H}} f(\tau)\overline{h(\tau)} v^{\frac{3}{2}} \frac{dudv}{v^2},$$

where $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ denotes the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$. While one may be able to approximate the integral well numerically, we are interested in obtaining a precise (algebraic) formula for the inner product (and hence an explicit formula for Ψ). In order to do so, we rely on a formula of Bruinier and Funke (see [3, Theorem 1.1 and Proposition 3.5]) known as the *Bruinier–Funke pairing*. The basic premise is to use Stokes' Theorem in order to compute the inner product in a different way. Suppose that we have a preimage $\mathcal{H}$ under the operator $\xi_{1/2}$, where

$$\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial\overline{\tau}}}$$

is a differential operator which sends functions satisfying weight κ modularity to functions satisfying weight $2-\kappa$ modularity. Note that since h is holomorphic and $\xi_{1/2}(\mathcal{H}) = h$, the fact that the kernel of $\xi_{2-\kappa}$ is holomorphic functions implies that the function $\mathcal{H}$ is necessarily annihilated by the weight κ hyperbolic Laplacian (for $\kappa = 1/2$)

$$\Delta_\kappa = -\xi_{2-\kappa} \circ \xi_\kappa.$$

If we further impose that $\mathcal{H}$ is modular of weight κ on Γ and has certain restrictions on its singularities in $\Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\})$ (see Sect. 2 for further details), then we obtain a harmonic Maass form. Due to the fact that Γ is a congruence subgroup, it contains T^N for some N, where $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Similarly, if $\gamma_\varrho \in \mathrm{SL}_2(\mathbb{Z})$ sends $i\infty$ to a cusp ϱ, then T^{N_ϱ} is contained in $\gamma_\varrho^{-1}\Gamma\gamma_\varrho$ for some $N_\varrho \in \mathbb{N}$; here N_ϱ is known as the cusp width at ϱ. Using this, one can show that it has a Fourier expansion around each cusp ϱ of Γ of the shape

$$\mathcal{H}_\varrho(\tau) = \sum_{n\in\mathbb{Z}} c_{\mathcal{H},\varrho}(v; n) e^{\frac{2\pi i n\tau}{N_\varrho}},$$

for some $c_{\mathcal{H},\varrho}(y; n) \in \mathbb{C}$, and where $\mathcal{H}_\varrho := \mathcal{H}|_\kappa\gamma_\varrho$ is the expansion around ϱ. Note however, that since $\mathcal{H}$ is not holomorphic, the Fourier coefficients may depend on v. Solving the differential equation $\Delta_\kappa(\mathcal{H}) = 0$ termwise yields a natural splitting of the Fourier expansion into holomorphic and non-holomorphic parts, namely

$$\mathcal{H}_\varrho(\tau) = \mathcal{H}_\varrho^+(\tau) + \mathcal{H}_\varrho^-(\tau)$$

with

$$\mathcal{H}_\varrho^+(\tau) = \sum_{n\gg-\infty} c_{\mathcal{H},\varrho}^+(n) e^{\frac{2\pi i n\tau}{N_\varrho}}$$

$$\mathcal{H}_\varrho^-(\tau) = c_{\mathcal{H},\varrho}^-(0) v^{2-\kappa} + \sum_{\substack{n\ll\infty \\ n\neq 0}} c_{\mathcal{H},\varrho}^-(n) \Gamma\left(2-\kappa, -\frac{4\pi n v}{N_\varrho}\right) e^{\frac{2\pi i n\tau}{N_\varrho}},$$

where now the coefficients are independent of v. It is these Fourier coefficients which are used by Bruinier and Funke to compute the inner product explicitly in [3, Proposition 3.5]. To state their formula, let $\mathcal{S}_\Gamma$ denote the set of cusps and write

$$f_\varrho(\tau) = \sum_{n\geq 0} c_{f,\varrho}(n) e^{\frac{2\pi i n\tau}{N_\varrho}}.$$

Theorem 3.1 (Bruinier–Funke) *We have*

$$\langle f, h\rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \sum_{\varrho\in\mathcal{S}_\Gamma} \sum_{n\geq 0} c_{f,\varrho}(n) c_{\mathcal{H},\varrho}^+(-n).$$

Theorem 3.1 is algebraic, precise, and is actually a finite sum since there are only finitely many n for which $c_{\mathcal{H},\varrho}^+(-n) \neq 0$, allowing one to explicitly compute the inner product. We will assume that sufficiently many Fourier coefficients of f are known, or in other words the input to our algorithm will be the Fourier coefficients $c_{f,\varrho}(n)$ and the function h, which in our case will be a unary theta function. The assumption that the expansions are known at every cusp may at first seem to be a somewhat strong assumption, since in combinatorial applications we often only know the expansion at one cusp. However, when $f = \Theta_{L+\nu}$ is the theta function for a shifted lattice, Shimura [13] has computed the modularity properties for all of $\mathrm{SL}_2(\mathbb{Z})$ and one obtains modularity for $\mathrm{SL}_2(\mathbb{Z})$ in a vector-valued sense, where the components of the vector are the functions f_ϱ. In other words, given just the theta function f, one can determine the functions f_ϱ as long as one can write γ_ϱ explicitly in terms of the generators $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and T of $\mathrm{SL}_2(\mathbb{Z})$. Although this rewriting is well-known, we provide the details for the convenience of the reader.

Lemma 3.2 *Given $\varrho = a/c$, there is an algorithm to determine $\gamma_\varrho \in \mathrm{SL}_2(\mathbb{Z})$ explicitly in terms of S and T.*

Proof First, we need to construct γ_ϱ for which $\gamma_\varrho(i\infty) = a/c$. In other words, we want a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Since $ad - bc = 1$ and a and c are necessarily prime, we see that b and $-c$ are precisely the coefficients from Bezout's theorem. We next construct the sequence of S and T recursively as follows.

Let $\gamma_0 := \gamma_\varrho$. At step $j+1$ (with $j \in \mathbb{N}_0$) we will construct γ_{j+1} inductively/recursively from γ_j by multiplying either by S or by ST^m for some $m \in \mathbb{Z}$, and eventually obtain $\gamma_\ell = \pm T^m$ for some step ℓ and $m \in \mathbb{Z}$. Suppose that

$$\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

If $c_j = 0$, then $a_j = d_j = \pm 1$ and $\ell = j$ with $\gamma_j = \pm T^{\pm b_j}$, and reversing back through the recursion gives the expansion of γ_0 in terms of S and T, so we are done.

If $c_j \neq 0$, then we choose $r \in \mathbb{Z}$ such that $|a_j + rc_j|$ is minimal (if there are two choices, i.e, if $a_j + rc_j = c_j/2$ for some r, then we take this choice of r). We then set

$$\gamma_{j+1} := ST^r\gamma_{j-1} = S\begin{pmatrix} a_j + rc_j & b_j + rd_j \\ c_j & d_j \end{pmatrix} = \begin{pmatrix} -c_j & -d_j \\ a_j + rc_j & b_j + rd_j \end{pmatrix}.$$

Note that $|a_j + rc_j| \leq |c_j|/2$ by construction, so the entry in the lower-left corner is necessarily smaller at step $j+1$ than it was at step j. Therefore the algorithm will halt after a finite number of steps. □

In order to determine the inner product $\langle f, h\rangle$, it remains to compute the preimage $\mathcal{H}$ and compute its Fourier expansion. Luckily, motivated by Ramanujan's mock theta functions, Zwegers [17] constructed pre-images of the unary theta functions using a holomorphic function μ which he "completed" to obtain a harmonic Maass form (actually, he is even able to view his completed object as a non-holomorphic Jacobi form, and one obtains the pre-images of unary theta functions by plugging in elements of $\mathbb{Q} + \mathbb{Q}\tau$ for the elliptic variable z). Choosing z to be an appropriate element of $\mathbb{Q} + \mathbb{Q}\tau$, one may compute the expansions at all cusps by viewing Zwegers's function as a component of a vector-valued modular form. As a first example, Zwegers himself computed the corresponding vector when the unary theta function is given by

$$\Theta_0(\tau) := \sum_{n\in\mathbb{Z}} \left(n + \frac{1}{6}\right) e^{3\pi i\left(n+\frac{1}{6}\right)^2\tau}.$$

This is related to the third order mock theta function $f(q)$, and played an important role in Bringmann and Ono's [1] proof of the Andrews–Dragonette conjecture. One may find the full transformation properties listed in [1, Theorem 2.1]. Specifically, let

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

and

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}.$$

Setting ($q := e^{2\pi i\tau}$)

$$F(\tau) = (F_0(\tau), F_1(\tau), F_2(\tau))^T := \left(q^{-\frac{1}{24}}f(q), 2q^{\frac{1}{3}}\omega\left(q^{\frac{1}{2}}\right), 2q^{\frac{1}{3}}\omega\left(-q^{\frac{1}{2}}\right)\right)^T,$$

we have the following.

Theorem 3.3 (Zwegers [17]) *There is a vector-valued harmonic Maass form $\mathcal{H} = (\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2)^T$ whose meromorphic part is F (component-wise). The harmonic Maass form satisfies*

$$\xi_{\frac{1}{2}}(\mathcal{H}_0) = \Theta_0$$

and the modularity properties for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\mathcal{H}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \mathcal{H}(\tau),$$

$$\mathcal{H}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathcal{H}(\tau),$$

where $\zeta_n := e^{2\pi i/n}$.

Pre-images of a more general family of unary theta functions were investigated by Bringmann and Ono in [2]; these are connected to Dyson's rank for the partition function, and the modularity of the relevant functions is given in [2, Theorem 1.2], with the full vector-valued transformation properties given in [2, Theorem 2.3].

Theorem 3.3 is the first case of a much more general theorem which follows by combining the results in Zwegers's thesis [17]. To describe this result, for $a, b \in \mathbb{C}$ and $\tau \in \mathbb{H}$, define the holomorphic function

$$\mu(a, b; \tau) := \frac{e^{\pi i a}}{\theta(b; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i \left(n^2+n\right)\tau + 2\pi i n b}}{1 - e^{2\pi i n \tau + 2\pi i a}},$$

and also define the real-analytic function

$$R(a; \tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} \left(\operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im}(a)}{v}\right)\sqrt{2v}\right)\right)(-1)^{\nu-\frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i a \nu},$$

where $\operatorname{sgn}(x)$ is the usual sign function,

$$\theta(z; \tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu \left(z + \frac{1}{2}\right)},$$

and

$$E(z) := \operatorname{sgn}(z)\left(1 - \beta\left(z^2\right)\right)$$

with (for $x \in \mathbb{R}_{\geq 0}$)

$$\beta(x) := \int_x^\infty t^{-\frac{1}{2}} e^{-\pi t} dt.$$

One then defines

$$\widetilde{\mu}(a, b; \tau) := \mu(a, b; \tau) + \frac{i}{2} R(a - b; \tau). \tag{3.1}$$

The function $\widetilde{\mu}$ is essentially a weight $1/2$ harmonic Maass form.

Theorem 3.4 *For $h, t, N \in \mathbb{N}$ with $t \mid 2N$, the function*

$$\mathcal{F}_{h,t,N}(\tau) := -e^{-2\pi i \left(h - \frac{N}{t}\right)^2 \tau} \widetilde{\mu}\left(\frac{ht - N}{2N} \frac{8N^2\tau}{t^2}, -\frac{1}{2}; \frac{8N^2\tau}{t^2}\right)$$

is a weight $1/2$ harmonic Maass form on $\Gamma := \Gamma_1(4N/t) \cap \Gamma_0(16N^2/t^2)$ with some multiplier system. Furthermore, it satisfies

$$\xi_{\frac{1}{2}}\left(\mathcal{F}_{h,t,N}\right) = \vartheta_{h,t,N}(\tau).$$

Proof The modularity properties of $\mathcal{F}_{h,t,N}$ follow by Zwegers [17, Theorem 1.11]. In particular, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\gamma' = \left(\begin{smallmatrix} a & 8N^2b/t^2 \\ ct^2/(8N^2) & d \end{smallmatrix}\right)$, a change of variables in [17, Theorem 1.11 (2)] together with (2.5) implies that (with $v(\gamma') := \eta(\gamma'\tau)/(j(\gamma',\tau)\eta(\tau))$ denoting the multiplier system of the Dedekind η-function $\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n)$)

$$\begin{aligned}
&\mathcal{F}_{h,t,N}\left(\frac{a\tau+b}{c\tau+d}\right)\\
&= -e^{-2\pi i\left(h-\frac{N}{t}\right)^2\frac{a\tau+b}{c\tau+d}}\widetilde{\mu}\left(\frac{ht-N}{2N}\frac{a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}}{\frac{ct^2}{8N^2}\left(\frac{8N^2}{t^2}\tau\right)+d}, -\frac{1}{2}; \frac{a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}}{\frac{ct^2}{8N^2}\left(\frac{8N^2}{t^2}\tau\right)+d}\right)\\
&= -e^{-2\pi i\left(h-\frac{N}{t}\right)^2\left(\frac{a\tau+b}{c\tau+d}\right)}v(\gamma')^{-3}(c\tau+d)^{\frac{1}{2}}e^{-\pi i\frac{\frac{ct^2}{8N^2}\left(\frac{ht-N}{2N}\left(a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}\right)+\frac{c\tau+d}{2}\right)^2}{c\tau+d}}\\
&\quad\times\widetilde{\mu}\left(\left(\frac{ht}{2N}-\frac{1}{2}\right)\left(a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}\right), -\frac{c\tau+d}{2}; \frac{8N^2\tau}{t^2}\right). \qquad (3.2)
\end{aligned}$$

We next use the fact that $a \equiv 1 \pmod{4N/t}$ to obtain

$$\frac{ht-N}{2N}\left(a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}\right) \equiv \frac{ht-N}{2N}\frac{8N^2\tau}{t^2} \pmod{\mathbb{Z}\frac{8N^2\tau}{t^2}+\mathbb{Z}},$$

while $16N^2/t^2 \mid c$ and $d \equiv 1 \pmod{4N/t}$ imply that

$$\frac{c\tau+d}{2} \equiv \frac{1}{2} \pmod{\mathbb{Z}\frac{8N^2\tau}{t^2}+\mathbb{Z}},$$

Hence by Zwegers [17, Theorem 1.11 (1)], we have

$$\begin{aligned}
&\widetilde{\mu}\left(\frac{ht-N}{2N}\left(a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}\right), -\frac{c\tau+d}{2}; \frac{8N^2\tau}{t^2}\right)\\
&= (-1)^{(a-1)\frac{ht-N}{2N}+\left(h-\frac{N}{t}\right)\frac{4Nb}{t}-\frac{ct^2}{16N^2}-\frac{d-1}{2}}\\
&\quad e^{\pi i\left((a-1)\frac{ht-N}{2N}+\frac{ct^2}{16N^2}\right)^2\frac{8N^2\tau}{t^2}+2\pi i\left((a-1)\frac{ht-N}{2N}+\frac{ct^2}{16N^2}\right)\left(\frac{ht-N}{2N}\frac{8N^2\tau}{t^2}+\frac{1}{2}\right)}\\
&\quad\times\widetilde{\mu}\left(\frac{ht-N}{2N}\frac{8N^2\tau}{t^2}, -\frac{1}{2}; \frac{8N^2\tau}{t^2}\right). \qquad (3.3)
\end{aligned}$$

The power of -1 modifies the multiplier system accordingly. Plugging back into (3.2), we see that it remains to simplify the exponentials to match the power of τ.

The parameter of the exponential (or rather, the part which involves τ) is $\frac{2\pi i}{c\tau+d}$ times

$$\begin{aligned}
&-\left(h-\frac{N}{t}\right)^2(a\tau+b)-\frac{ct^2}{16N^2}\left(\frac{ht-N}{2N}\left(a\left(\frac{8N^2}{t^2}\tau\right)+\frac{8N^2b}{t^2}\right)+\frac{c\tau+d}{2}\right)^2\\
&\quad+\frac{1}{2}\left((a-1)\frac{ht-N}{2N}+\frac{ct^2}{16N^2}\right)^2\frac{8N^2\tau}{t^2}(c\tau+d)\\
&\quad+\left((a-1)\frac{ht-N}{2N}+\frac{ct^2}{16N^2}\right)\frac{ht-N}{2N}\frac{8N^2\tau}{t^2}(c\tau+d)\\
&\quad=-\left(h-\frac{N}{t}\right)^2(a\tau+b)-c(a\tau+b)^2\left(h-\frac{N}{t}\right)^2\\
&\quad-\frac{ct}{4N}\left(h-\frac{N}{t}\right)(a\tau+b)(c\tau+d)-\frac{ct^2}{64N^2}(c\tau+d)^2\\
&\quad+(a-1)^2\left(h-\frac{N}{t}\right)^2\tau(c\tau+d)+\frac{t}{4N}(a-1)\left(h-\frac{N}{t}\right)c\tau(c\tau+d)\\
&\quad+\frac{t^2}{64N^2}c^2\tau(c\tau+d)+2(a-1)\left(h-\frac{N}{t}\right)^2\tau(c\tau+d)\\
&\quad+\frac{t}{4N}\left(h-\frac{N}{t}\right)c\tau(c\tau+d).
\end{aligned}\tag{3.4}$$

We consider (3.4) as a polynomial in $h-N/t$ and simplify the coefficients of each power of $h-N/t$. We first combine and simplify the terms in (3.4) with $(h-N/t)^2$. Using $ad-bc=1$, these are $(h-N/t)^2$ times

$$\begin{aligned}
&-(a\tau+b)-c(a\tau+b)^2+(a-1)^2\tau(c\tau+d)+2(a-1)\tau(c\tau+d)\\
&\quad=-a\tau-b-2abc\tau-b^2c+a^2d\tau+d\tau-c\tau^2-2d\tau\\
&\quad=-a\tau-b-2abc\tau-b(ad-1)+a(1+bc)\tau+d\tau-c\tau^2-2d\tau\\
&\quad=-abc\tau-abd+d\tau-c\tau^2-2d\tau=-(c\tau+d)(\tau+ab).
\end{aligned}$$

Thus the exponential corresponding to the terms with $(h-N/t)^2$ is

$$e^{\frac{2\pi i}{c\tau+d}\left(h-\frac{N}{t}\right)^2(c\tau+d)(-\tau-ab)}=e^{-2\pi i\left(h-\frac{N}{t}\right)^2\tau}e^{-2\pi i\left(h-\frac{N}{t}\right)^2ab}.$$

The first factor is precisely the factor in front of $\mathcal{F}_{h,t,N}$ and the second contributes to the multiplier system.

We next simplify the terms in (3.4) with $h - N/t$. These give

$$\left(h - \frac{N}{t}\right)(c\tau + d)\frac{ct}{4N}\left(-(a\tau + b) + (a-1)\tau + \tau\right) = -b\left(h - \frac{N}{t}\right)(c\tau + d)\frac{ct}{4N}.$$

The resulting exponential contributes to the multiplier system since the factor $c\tau + d$ cancels.

Finally, we see directly that the terms in (3.4) which are constant when considered as a polynomial in $h - N/t$ cancel. Therefore, the simplification of (3.4) yields that the exponential is

$$e^{-2\pi i\left(h-\frac{N}{t}\right)^2\tau} e^{-2\pi i\left(h-\frac{N}{t}\right)^2 ab} e^{-2\pi i b\left(h-\frac{N}{t}\right)\frac{ct}{4N}}. \tag{3.5}$$

Altogether, plugging (3.3) and (3.5) into (3.2) (note that in the simplification we left out one exponential term in (3.3) because it was independent of τ) yields

$$\begin{aligned}\mathcal{F}_{h,t,N}\left(\frac{a\tau + b}{c\tau + d}\right) &= \upsilon(\gamma')^{-3}(c\tau + d)^{\frac{1}{2}}(-1)^{(a-1)\frac{ht-N}{2N} + \left(h-\frac{N}{t}\right)\frac{4Nb}{t} - \frac{ct^2}{16N^2} - \frac{d-1}{2}} \\ &\quad \times e^{-2\pi i\left(h-\frac{N}{t}\right)^2 ab} e^{-2\pi i b\left(h-\frac{N}{t}\right)\frac{ct}{4N}} e^{\pi i\left((a-1)\frac{ht-N}{2N} + \frac{ct^2}{16N^2}\right)} \mathcal{F}_{h,t,N}(\tau). \end{aligned} \tag{3.6}$$

We see from (3.6) that $\mathcal{F}_{h,t,N}$ has the desired modularity properties.

We next compute the image under $\xi_{1/2}$. Since the μ-function is holomorphic on the upper half-plane, it is annihilated by $\xi_{1/2}$. Therefore, plugging in the definition (3.1) of $\widetilde{\mu}$, we have

$$\xi_{\frac{1}{2}}\left(\mathcal{F}_{h,t,N}(\tau)\right) = -\frac{1}{2i}\xi_{\frac{1}{2}}\left(e^{-2\pi i\left(h-\frac{N}{t}\right)^2\tau} R\left(\frac{8N^2}{t^2}\left(\frac{ht}{2N} - \frac{1}{2}\right)\tau + \frac{1}{2}; \frac{8N^2\tau}{t^2}\right)\right).$$

Noting that we have

$$\frac{\operatorname{Im}\left(\frac{8N^2}{t^2}\left(\frac{ht}{2N} - \frac{1}{2}\right)\tau + \frac{1}{2}\right)}{\operatorname{Im}\left(\frac{8N^2\tau}{t^2}\right)} = \frac{ht}{2N} - \frac{1}{2},$$

we then employ [17, Theorem 1.16] to rewrite this as

$$\xi_{\frac{1}{2}}\left(\mathcal{F}_{h,t,N}(\tau)\right) = -\frac{1}{2i}\xi_{\frac{1}{2}}\left(\int_{-\frac{8N^2\overline{\tau}}{t^2}}^{i\infty} \frac{g_{\frac{ht}{2N},0}(z)}{\sqrt{-i\left(z + \frac{8N^2\tau}{t^2}\right)}}dz\right), \tag{3.7}$$

where

$$g_{a,b}(\tau) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2\tau+2\pi ib\nu}.$$

The remaining integral is what is known as a non-holomorphic Eichler integral, and is easily evaluated by the Fundamental Theorem of Calculus as

$$\xi_{\frac{1}{2}}\left(\int_{-\frac{8N^2\overline{\tau}}{t^2}}^{i\infty} \frac{g_{\frac{ht}{2N},0}(z)}{\sqrt{-i\left(z+\frac{8N^2\tau}{t^2}\right)}}dz\right) = -2iv^{\frac{1}{2}}\frac{8N^2}{t^2}\frac{g_{\frac{ht}{2N},0}\left(\frac{8N^2\tau}{t^2}\right)}{\frac{2N}{t}\sqrt{-2i(\tau-\overline{\tau})}}$$
$$= -2i\frac{2N}{t}g_{\frac{ht}{2N},0}\left(\frac{2N}{t}\tau\right).$$

Therefore (3.7) becomes

$$\xi_{\frac{1}{2}}\left(\mathcal{F}_{h,t,N}(\tau)\right) = \frac{2N}{t}g_{\frac{ht}{2N},0}\left(\frac{8N^2}{t^2}\tau\right).$$

We finally rewrite

$$\frac{2N}{t}g_{\frac{ht}{2N},0}\left(\frac{8N^2}{t^2}\tau\right) = \frac{2N}{t}\sum_{\nu\in\frac{ht}{2N}+\mathbb{Z}} \nu e^{\frac{8\pi iN^2\nu^2\tau}{t^2}} = \sum_{\nu\in h+\frac{2N}{t}\mathbb{Z}} \nu e^{2\pi i\nu^2\tau} = \vartheta_{h,t,N}(\tau).$$

□

In order to prove Proposition 1.2, we are particularly interested in the case of $N = 3$ and $h = 2$. It turns out that congruence conditions immediately rule out all of the possible unary theta functions except for the form

$$\vartheta_{\chi_{-3}}(\tau) := \sum_{n\in\mathbb{Z}} \chi_{-3}(n)ne^{2\pi in^2\tau}, \tag{3.8}$$

where $\chi_d(n) := \left(\frac{d}{n}\right)$ is the usual Kronecker–Jacobi character (also known as the extended Legendre symbol). We rewrite this form in the notation from this paper as follows.

Lemma 3.5 *We have*

$$\vartheta_{\chi_{-3}}(\tau) = \vartheta_{2,1,3}\left(\frac{\tau}{4}\right).$$

Remark By Theorem 3.4, Lemma 3.5 together with the chain rule implies that

$$\xi_{\frac{1}{2}}\left(\mathcal{F}_{2,1,3}\left(\frac{\tau}{4}\right)\right) = \frac{1}{4}\vartheta_{\chi_{-3}}(\tau).$$

Proof We compute

$$\begin{aligned}\vartheta_{\chi_{-3}}(\tau) &= \sum_{n\in\mathbb{Z}}(3n+1)e^{2\pi i(3n+1)^2\tau} - \sum_{n\in\mathbb{Z}}(3n-1)e^{2\pi i(3n-1)^2\tau}\\ &= \sum_{n\in\mathbb{Z}}(3n+1)e^{2\pi i(3n+1)^2\tau} - \sum_{n\in\mathbb{Z}}(-3n-1)e^{2\pi i(-3n-1)^2\tau}\\ &= 2\sum_{n\equiv 1 \pmod 3} ne^{2\pi i n^2\tau} = \sum_{n\equiv 2 \pmod 6} ne^{2\pi i n^2\frac{\tau}{4}} = \vartheta_{2,1,3}\left(\frac{\tau}{4}\right).\end{aligned}$$

□

4 An Application to Lattice Theory

4.1 An Application

To motivate this study, we first prove Theorem 1.1.

Proof of Theorem 1.1 We decompose $\Theta_{L+\nu}$ as an Eisenstein series E, a unary theta function, and a cusp form g which is orthogonal to unary theta functions. Since the unary theta function is trivial by assumption, we have

$$\Theta_{L+\nu} = E + g.$$

We then compare the coefficients of $E + g$. Since every element of $M\mathbb{Z} + r$ is primitively represented locally, the local densities increase as a function of n. The product of the local densities were shown in [16] (and independently in [14]) to be the Fourier coefficients of E, paralleling the famous Siegel–Weil formula. Since $\nu \in \mathbb{Q}L$, there exists $R \in \mathbb{N}$ for which $R\nu \in L$. Note further that (denoting the localization at the prime p by $L_p := L \otimes \mathbb{Q}_p$) for each prime $p \nmid R$, we have $\nu \in L_p$ (because R is invertible in $\mathbb{Q}_p$) Therefore

$$L_p + \nu = L_p.$$

In other words, the local density at p for $L + \nu$ and for L agree. Denoting the local densities for $L + \nu$ by β_p and the local densities of L by α_p, we have

$$\prod_p \beta_p = \frac{\prod_{p|R}\beta_p}{\prod_{p|R}\alpha_p}\prod_p \alpha_p.$$

The product $\prod_p \alpha_p$ is known to be a (Hurwitz) class number for an imaginary quadratic field (see [9, Theorem 86]) and these are known to grow faster than $n^{\frac{1}{2}-\varepsilon}$

by Siegel's [15] famous (ineffective) lower bound for the class numbers. On the other hand, Duke [6] has shown that the coefficients of g grow slower than $n^{3/7+\varepsilon}$. Therefore, the coefficients of E are the main asymptotic term and they are positive. For n sufficiently large the coefficient must be positive, yielding the claim. □

It is worth noting that the Fourier coefficients of the unary theta function grow at the same rate as the coefficients of the Eisenstein series. In other words, when the unary theta function is not trivial, it is often the case that the set investigated in Theorem 1.1 is actually infinite. One such example is worked out in [8, Theorem 1.5] with an applications to sums of polygonal numbers, and a proposed algebraic explanation for this behavior involving the spinor genus of $L+\nu$ is given in [8, Conjecture 1.3].

4.2 An Individual Case

In individual cases, one may combine Theorem 1.1 with Theorem 3.4 to show that certain quadratic polynomials are almost universal. We demonstrate one such example in Proposition 1.2.

Proof of Proposition 1.2 Let $L+\nu$ be the corresponding shifted lattice. By Theorem 1.1, it suffices to show that the inner product of $\Theta_{L+\nu}$ against all theta functions in the same space is trivial. For the diagonal lattice corresponding to the quadratic form $Q(x,y,z) = x^2+3y^2+3z^2$, an inclusion-exclusion argument implies that (recalling that $\Theta(\tau)=\sum_{n\in\mathbb{Z}} q^{n^2}$)

$$\Theta_{L+\nu}(\tau) = (\Theta(\tau)-\Theta(9\tau))\,(\Theta(3\tau)-\Theta(27\tau))\,(\Theta(3\tau)-\Theta(27\tau))\,,$$

from which one sees that $\Theta_{L+\nu}$ is actually a weight $3/2$ modular form on $\Gamma_0(108)$. Specifically, in Shimura's notation, we have

$$\Theta_{L+\nu}(\tau) = \theta\left(6\tau;\begin{pmatrix}3\\9\\9\end{pmatrix},\begin{pmatrix}3&0&0\\0&9&0\\0&0&9\end{pmatrix},9,1\right),$$

where (for A a symmetric $n\times n$ matrix, $h\in\mathbb{Z}^n$ satisfying $Ah\in N\mathbb{Z}^n$, and P a spherical function)

$$\theta(\tau;h,A,N,P) := \sum_{\substack{x\in\mathbb{Z}^n\\ x\equiv h \pmod N}} P(x)e^{\frac{2\pi i\tau}{2N^2}{}^t xAx}. \tag{4.1}$$

Here ${}^t x$ denotes the transpose of x. We write h_L and A_L for the corresponding vector and lattice in our case and omit $P=1$ in the notation in the following.

A straightforward check of congruence conditions implies that the only relevant theta function is $\vartheta_{\chi_{-3}}$ defined in (3.8). By Lemma 3.5, Theorems 3.4, and 3.1, it suffices to show that, for $\Gamma = \Gamma_0(108)$,

$$\sum_{\varrho\in\mathcal{S}_\Gamma}\sum_{n\geq 0} c_{\Theta_{L+\nu},\varrho}(n)c_{\mathcal{F}_{2,1,3}(\tau/4),\varrho}(-n) = 0, \tag{4.2}$$

where we abuse notation to write $c_{\mathcal{F}_{2,1,3}(\tau/4),\varrho}(-n)$ as the $(-n)$th coefficient of $\mathcal{F}_{2,1,3}(\tau/4)$. In order to compute the expansions at other cusps, we apply S and T repeatedly (using Lemma 3.2) and note that [17, Theorem 1.11 (2)] yields the fully modularity properties of $\mathcal{F}_{2,1,3}(\tau/4)$ as a vector-valued modular form, while $\Theta_{L+\nu}$ behaves as a vector-valued modular form on the full modular group by [13, (2.4) and (2.5)]. Specifically, we have (for arbitrary h satisfying $A_L h \in 9\mathbb{Z}^3$)

$$\theta\left(-\frac{1}{z}; h, A_L, 9\right) = \sum_{\substack{k \pmod 9\\ A_L k\equiv 0 \pmod 9}} e^{\frac{2\pi i}{27}(k_1h_1+3k_2h_2+3k_3h_3)}\theta(z;k,A_L,9),$$

$$\theta(z+2;;h,A_L,9) = e^{\frac{2\pi i}{27}\left(h_1^2+3h_2^2+3h_3^2\right)}\theta(z;h,A_L,9).$$

Note that the restriction $A_L h \equiv 0 \pmod 9$ is equivalent to $3 \mid h_1$, so the exponential in the first identity may be simplified as

$$e^{\frac{2\pi i}{9}\left(\frac{k_1h_1}{3}+k_2h_2+k_3h_3\right)}$$

and the exponential in the second identity may be simplified as

$$e^{\frac{2\pi i}{9}\left(\frac{h_1^2}{3}+h_2^2+h_3^2\right)}.$$

Since the only terms contributing to the sum in (4.2) are the principal parts (the terms where the power of q is negative) of the expansions around each cusp of $\mathcal{F}_{2,1,3}$, we only need to compute a few Fourier coefficients for each of the components of the vector-valued modular forms corresponding to $\Theta_{L+\nu}$ and $\mathcal{F}_{2,1,3}$. A computer check then verifies (4.2), yielding the claim in the proposition. □

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Bounds for Fourier-Jacobi Coefficients of Siegel Cusp Forms of Degree Two

Winfried Kohnen and Jyoti Sengupta

Abstract We discuss and prove several estimates involving Peterrson norms of Fourier-Jacobi coefficients of Siegel cusp forms of degree two.

1 Introduction

Let f be an elliptic cusp form of integral weight k for the Hecke congruence subgroup $\Gamma_0(M) \subset SL_2(\mathbf{Z})$ of level M and write $a(n)$ $(n \geq 1)$ for its Fourier coefficients. Then Deligne's bound (previously the Ramanujan-Petersson conjecture) says that

$$a(n) \ll_{f,\epsilon} n^{\frac{k-1}{2}+\epsilon} \quad (\epsilon > 0). \tag{1}$$

While (1) is deep, various bounds for sums related to the $a(n)$ can be derived in a rather elementary way. For example, using Parseval's formula one can easily show that

$$\sum_{n\leq N} |a(n)|^2 \ll_f N^k \tag{2}$$

and from this—using the Cauchy-Schwarz inequality—that

$$\sum_{n\leq N} |a(n)| \ll_f N^{\frac{k+1}{2}} \tag{3}$$

W. Kohnen
Mathematisches Institut, Universität Heidelberg, INF 205, 69120 Heidelberg, Germany
e-mail: winfried@mathi.uni-heidelberg.de

J. Sengupta (✉)
School of Mathematics, T.I.F.R., Homi Bhabha Road, Mumbai 400 005, India
e-mail: sengupta@math.tifr.res.in

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(cf. e.g. [6, Thm. 5.1, Cor. 5.2]). We note that vice versa (up to the occurrence of the ϵ), the bound (1) directly implies (2) and (3) and so (2) and (3), respectively can be viewed as the Deligne bound on average.

On the other hand, it was proved in [6, Thm. 5.3] that for any real α one has

$$\sum_{n\leq N} a(n)e^{2\pi i\alpha n} \ll_f N^{\frac{k}{2}} \log(2N) \tag{4}$$

where the implied constant depends only on f and not on α. Note that (4) saves $\frac{1}{2} - \epsilon$ ($\epsilon > 0$) in the power of N in comparison to using the triangle inequality and (3) and so there must be many cancellations in (4).

In this paper we would like to discuss and prove similar estimates as above in the case of a Siegel cusp form F of degree two, where the Fourier coefficients of f in the classical setting are replaced by the Fourier-Jacobi coefficients of F and we work with the Petersson norm. When using Fourier-Jacobi coefficients rather than usual Fourier coefficients, the situation seems to become a bit more uniform, as will be demonstrated. For example, while a generalized Ramanujan-Petersson conjecture is known to fail for the Fourier coefficients of a form F in the Maass subspace [2, sect. 2], an analogous conjecture can be proved in the setting of Fourier-Jacobi coefficients, cf. Sect. 3.

2 Jacobi Forms and Norms

We denote by $\mathcal{H}_2$ the Siegel upper half-space of degree two consisting of symmetric complex $(2,2)$-matrices with positive definite imaginary part. For $M \in \mathbf{N}$ we let

$$\Gamma_0^{(2)}(M) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbf{Z}) \mid C \equiv 0 \pmod{M}\}$$

the Hecke congruence subgroup of level M and degree two.

If $F : \mathcal{H}_2 \to \mathbf{C}$ is a Siegel cusp form of weight k for $\Gamma_0^{(2)}(M)$, we write its Fourier-Jacobi expansion as

$$F(Z) = \sum_{m\geq 1} \phi_m(\tau, z)e^{2\pi i m\tau'} \quad (Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2).$$

Then $\phi_m \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$, the space of Jacobi cusp forms of weight k and index m for $\Gamma_0(M)_J := \Gamma_0(M) \rhd \mathbf{Z}^2$ [4, 5].

For $\phi \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$ put

$$\Phi(\tau, z) := \phi(\tau, z)e^{-2\pi m y^2/v} v^{\frac{k}{2}} \quad (\tau = u + iv, z = x + iy). \tag{5}$$

Then $|\Phi(\tau, z)|$ is invariant under $\Gamma_0(M)_J$ and $\Phi(\tau, z)$ is bounded on $\mathcal{H} \times \mathbf{C}$.

For $\phi, \psi \in J_{k,m}^{cusp}(\Gamma_0(M)_J)$ we denote their Petersson scalar product by

$$\langle \phi, \psi \rangle = \int_{\mathcal{F}} \Phi(\tau, z)\overline{\Psi(\tau, z)}d\mu,$$

where Ψ is defined in an analogous way as Φ, and $\mathcal{F}$ is any fundamental domain for the action of $\Gamma_0(M)_J$ on $\mathcal{H} \times \mathbf{C}$. Also

$$d\mu = \frac{dxdydudv}{v^3}$$

is the invariant measure.

Note that by definition the Petersson norm $||\phi||$ of ϕ is equal to the L^2-norm $||\Phi||$ of the corresponding function Φ restricted to $\mathcal{F}$.

We want to extend the L^2-norm on the space of functions as above to the space $B(\mathcal{H}\times\mathbf{C})$ of continuous and bounded functions on $\mathcal{H}\times\mathbf{C}$ (not necessarily satisfying any invariance properties under $\Gamma_0(M)_J$). For any choice of fundamental domain $\mathcal{F}$ for $\Gamma_0(M)_J$, and any $\Phi \in B(\mathcal{H} \times \mathbf{C})$ we have the L^2-norm

$$||\Phi||_{\mathcal{F}} := \left(\int_{\mathcal{F}} |\Phi(\tau, z)|^2 d\mu\right)^{1/2}.$$

We put

$$||\Phi|| := \sup_{\mathcal{F}} ||\Phi||_{\mathcal{F}}. \tag{6}$$

Then $||\,.\,||$ is a norm on $B(\mathcal{H} \times \mathbf{C})$ and if Φ is obtained from a Jacobi form ϕ as in (5), then (6) coincides with the L^2-norm $||\Phi||$ as above, i.e. with the Petersson norm $||\phi||$.

The norm (6) will come into play later in Sect. 5.

3 A Generalized Ramanujan-Petersson Conjecture

We will first show

Theorem 1 *Let F be a cusp form of weight k for $\Gamma_0^{(2)}(M)$ and let $\phi_m (m \geq 1)$ be its Fourier-Jacobi coefficients. Then*

$$\sum_{m \leq N} ||\phi_m||^2 \ll_F N^k. \tag{7}$$

Proof The proof works in a similar way as in the case of an elliptic modular form, mutatis mutandis, cf. [6, Thm. 5.1]. By Parseval's formula

$$\sum_{m\geq 1}|\phi_m(\tau,z)|^2 e^{-4\pi m v'} = \int_0^1 |F(Z)|^2 du' \quad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}, \tau' = u' + iv'\right).$$

Since

$$(\det Y)^{k/2}|F(Z)| \quad (Y = \Im(Z))$$

is bounded on $\mathcal{H}_2$, F being a cusp form, we find that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2 e^{-4\pi m v'} \leq \sum_{m\geq 1}|\phi_m(\tau,z)|^2 e^{-4\pi m v'}$$

$$\ll_F (\det Y)^{-k}.$$

We choose

$$v' = \frac{y^2}{v} + \frac{1}{N}$$

and note that with this choice

$$\det Y = v v' - y^2$$

$$= \frac{v}{N}.$$

We then infer that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2 e^{-4\pi m y^2/v}\cdot e^{-4\pi m/N} \ll_F N^k v^{-k}.$$

Since

$$e^{-4\pi} \leq e^{-4\pi m/N}$$

for $m \leq N$ we obtain that

$$\sum_{m\leq N}|\phi_m(\tau,z)|^2 e^{-4\pi m y^2/v}\cdot v^k \ll_F N^k. \tag{8}$$

Integrating (8) over a fundamental domain $\mathcal{F}$ with respect to the measure $d\mu$ we finally conclude that

$$\sum_{m \le N} ||\phi_m||^2 \ll_F N^k$$

as claimed.

Writing

$$||\phi_m|| = 1 \cdot ||\phi_m||$$

and using the Cauchy-Schwarz inequality we obtain from Theorem 1

Corollary *One has*

$$\sum_{m \le N} ||\phi_m|| \ll_F N^{\frac{k+1}{2}}. \tag{9}$$

Remark We believe that the bound of Theorem 1 is best possible so that we have a similar situation as in the case of elliptic modular forms. Indeed, at least if $M = 1$, i.e. we work with the full Siegel modular group, one can prove an asymptotic formula

$$\sum_{m \le N} ||\phi_m||^2 \asymp c_F N^k \quad (N \to \infty)$$

where $c_F > 0$ is a constant depending only on F. This follows from the analytic properties of the Dirichlet series

$$D_{F,F}(s) = \zeta(2s - 2k + 4) \sum_{m \ge 1} ||\phi_m||^2 m^{-s} \quad (\sigma := \Re(s) \gg 1)$$

proved in [10] in conjunction with a usual Tauberian theorem. Note that these properties are more difficult to prove, while the proof of (7) was quite straightforward.

In an analogous way as in the case of elliptic modular forms, based on (7) and (9) one is tempted to make the following

Conjecture (Ramanujan-Petersson) One has

$$||\phi_m|| \ll_{F,\epsilon} m^{\frac{k-1}{2}+\epsilon} \quad (\epsilon > 0). \tag{10}$$

Remarks

i) Note that the potential bound (10) was also addressed in [9, p. 718].
ii) The best general bound in the direction of (10) known so far seems to be

$$||\phi_m|| \ll_{F,\epsilon} m^{k/2-2/9+\epsilon} \quad (\epsilon > 0)$$

(cf. [7]). One also knows that there are infinitely many m such that $||\phi_m|| \ll_F m^{(k-1)/2}$ and infinitely many m such that $||\phi_m|| \gg_F m^{(k-1)/2}$ (if $F \neq 0$), cf. [9].

iii) Note that in the literature there is also a conjectured bound for the usual Fourier coefficients of a Siegel cusp form which is due to Resnikoff and Saldaña and which also could be viewed as a generalization of the Ramanujan-Petersson conjecture for classical cusp forms [11]. In the case of degree two this conjecture says that

$$a(T) \ll_{F,\epsilon} (\det T)^{k/2-3/4+\epsilon} \quad (\epsilon > 0), \tag{11}$$

for any positive definite symmetric half-integral matrix T of size 2, where $a(T)$ denote the Fourier coefficients of F. The estimate (11) can be motivated "on average" using the analytic properties of the Rankin-Selberg zeta function attached to F, cf. [8]. While one believes that (11) should be true "generically", there are well-known "exceptional" cases where it fails to hold, e.g. when F is a Hecke eigenform in the Maass space [2, loc. cit.]. Contrary to the above situation, we will prove estimate (10) for F in the Maass space in the next section.

4 Hecke Eigenforms in the Maass Space

Recall that the space of Siegel cusp forms of even weight k for $Sp_2(\mathbf{Z})$ has a special subspace, the so-called Maass space. It has a basis of Hecke eigenforms F whose spinor zeta function $Z_F(s)$ factors as

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s) \quad (\sigma \gg 1) \tag{12}$$

where f is a normalized cuspidal Hecke eigenform of weight $2k-2$ for $SL_2(\mathbf{Z})$ and $L(f,s)$ $(\sigma \gg 1)$ is its Hecke L-series [4].

Theorem 2 *Suppose that F is a cuspidal Hecke eigenform of even weight k for $Sp_2(\mathbf{Z})$ in the Maass subspace. Then conjecture (10) is true.*

Proof Under the given hypothesis one has

$$||\phi_m||^2 = \lambda_m ||\phi_1||^2 \quad (m \geq 1)$$

where λ_m is the m-th eigenvalue of F under the usual Hecke operator $T(m)$, cf. [3, Remark on p. 530] and [10]. Hence one only has to show that

$$\lambda_m \ll_\epsilon m^{k-1+\epsilon} \quad (\epsilon > 0).$$

Recall that the eigenvalues λ_m and the spinor zeta function of F are related by the identity

$$\sum_{m\geq 1} \lambda_m m^{-s} = \frac{Z_F(s)}{\zeta(2s-2k+4)} \quad (\sigma \gg 1)$$

as is well-known [1]. In particular, for F in the Maass subspace, using (12) we get

$$\sum_{m\geq 1} \lambda_m m^{-s} = \frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-2k+4)} \cdot L(f,s) \quad (\sigma \gg 1). \tag{13}$$

We note that the quotient of Riemann zeta functions on the right-hand side of (13) equals

$$\frac{\zeta(w-1)\zeta(w)}{\zeta(2w)},$$

where $w = s-k+2$. Since

$$\frac{\zeta(w)}{\zeta(2w)} = \prod_p [1 + p^{-w})$$

$$= \sum_{m\geq 1} |\mu(m)| m^{-w} \quad (\Re(w) \gg 1)$$

where μ is the Möbius function, the general coefficient of the above quotient is equal to

$$\alpha(m) = m^{k-2} \sum_{d|m} |\mu(\frac{m}{d})| d.$$

Clearly we have

$$\alpha(m) \leq m^{k-1}\sigma_0(m)$$

$$\ll_\epsilon m^{k-1+\epsilon} \quad (\epsilon > 0).$$

Here $\sigma_0(m)$ denotes the number of positive divisors of m.

Hence denoting by $\beta(m)$ the Hecke eigenvalues of f and observing that

$$\beta(m) \ll_\epsilon m^{k-3/2+\epsilon} \quad (\epsilon > 0)$$

by Deligne's bound we find that

$$\lambda(m) = \sum_{d|m} \alpha(d)\beta(\frac{m}{d})$$

$$\ll_\epsilon \sum_{d|m} d^{k-1+\epsilon} \cdot (\frac{m}{d})^{k-3/2+\epsilon}$$

$$= m^{k-3/2+\epsilon} \sum_{d|m} d^{1/2}$$

$$\leq m^{k-3/2+\epsilon} \cdot m^{1/2}\sigma_0(m)$$

$$\ll_\epsilon m^{k-1+2\epsilon}.$$

This proves our assertion.

5 Bounds for Twisted Sums

In this section we will prove an estimate analogous to the bound (4) in the classical case. Let again F be a Siegel cusp form of weight k for $\Gamma_0(M)_J$ and let ϕ_m be its m-th Fourier-Jacobi coefficient.

Following Sect. 2 we put

$$\Phi_m(\tau, z) := \phi_m(\tau, z)e^{-2\pi m y^2/v} v^{k/2} \quad (\tau = u + iv,\ z = x + iy).$$

Let $\alpha \in \mathbf{R}$. Using Cauchy-Schwarz we see that

$$|\sum_{m\leq N} \Phi_m(\tau, z)e^{2\pi i m\alpha}|$$

$$\leq N^{1/2} \cdot \sqrt{\sum_{m\leq N} |\Phi_m(\tau, z)|^2}$$

$$\ll_F N^{\frac{k+1}{2}}$$

where in the last line we have used (8). Thus the function

$$\sum_{m\leq N} \Phi_m e^{2\pi i m\alpha}$$

is bounded on $\mathcal{H} \times \mathbf{C}$ and we can talk about its norm as defined in Sect. 2.

Theorem 3 *With the above notations we have*

$$|| \sum_{m \le N} \Phi_m e^{2\pi i m\alpha} || \ll_F N^{k/2} \log(2N). \tag{14}$$

Remark Note that if we estimate the left-hand side of (14) by brute force, using the triangle inequality and the Corollary to Theorem 1 we only get the bound $N^{\frac{k+1}{2}}$.

Proof The proof follows a similar pattern as that of inequality (4) for elliptic modular forms, again mutatis mutandis.

We will use the notation

$$e(z) := e^{2\pi i z} \quad (z \in \mathbf{C}).$$

Since ϕ_m is the m-th Fourier-Jacobi coefficient of F, we have

$$S_{N,\alpha}(\tau, z) := \sum_{m \le N} \Phi_m(\tau, z) e^{2\pi i m\alpha}$$

$$= v^{k/2} \sum_{m \le N} e^{-2\pi m y^2/v} \int_0^1 F(\begin{pmatrix} \tau & z \\ z & \tau' + \alpha \end{pmatrix}) e(-m\tau') du' \quad (\tau' = u' + iv').$$

We put

$$v' = \frac{y^2}{v} + \frac{1}{N} \tag{15}$$

and obtain

$$S_{N,\alpha}(\tau, z) = v^{k/2} \int_0^1 \Big(\sum_{m \le N} e(-m(u' + \frac{i}{N})) \Big) F(\begin{pmatrix} \tau & z \\ z & u' + \alpha + i(\frac{y^2}{v} + \frac{1}{N}) \end{pmatrix}) du'. \tag{16}$$

Summing the geometric series now gives

$$\sum_{1 \le m \le N} e(-m(u' + \frac{i}{N})) = \frac{e(-N(u' + \frac{i}{N})) - 1}{1 - e(u' + \frac{i}{N})}$$

$$\ll \frac{1}{|1 - e(u' + \frac{i}{N})|}.$$

According to [6, p. 71] one has

$$\int_0^1 \frac{du'}{|1 - e(\tau')|} \ll \log(2 + \frac{1}{v'}). \tag{17}$$

Applying (17) with $v' = \frac{1}{N}$ we see that

$$\int_0^1 \Big(\sum_{m \le N} e(-m(u' + \frac{i}{N})) \Big) du' \ll \log(2+N)$$

$$\ll \log(2N).$$

Finally, since

$$F(Z) \ll_F (\det Y)^{-k/2}$$

and by (15) we have

$$\det Y = \frac{v}{N},$$

we obtain altogether from (16) that

$$S_{N,\alpha}(\tau, z) \ll_F N^{k/2} \log(2N). \qquad (18)$$

Now (18) implies that

$$||S_{N,\alpha}||_{\mathcal{F}} \ll_F N^{k/2} \log(2N)$$

for any fundamental domain $\mathcal{F}$ (where the implied constant depends only on F and not on $\mathcal{F}$) and hence that

$$||S_{N,\alpha}|| \ll_F N^{k/2} \log(2N).$$

This proves Theorem 3.

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Harmonic Eisenstein Series of Weight One

Yingkun Li

Abstract In this short note, we will construct a harmonic Eisenstein series of weight one, whose image under the ξ-operator is a weight one Eisenstein series studied by Hecke (Math Ann 97(1):210–242, 1927).

1 Introduction

In the theory of automorphic forms, Eisenstein series occupy an important place. Holomorphic Eisenstein series can be explicitly constructed and are usually the first examples of modular forms people encounter. Furthermore, their constant Fourier coefficients are special values of the Riemann zeta function, whereas the non-constant coefficients are the sums of the powers of divisors. Modularity then connects these two types of interesting quantities together.

Holomorphic theta series constructed from positive definite lattices provide another source of modular forms besides Eisenstein series. In [13], Siegel introduced non-holomorphic theta series associated to indefinite lattices, and showed that they can be integrated to produce Eisenstein series. Later in his seminal work [14], Weil studied this phenomenon for algebraic groups, and deduced the famous Siegel-Weil formula.

In the setting of theta correspondence between the orthogonal and sympletic groups, the Siegel-Weil formula is an equality between the integral of a theta function on the orthogonal side and an Eisenstein series on the symplectic side. With the knowledge of the theta kernel, one can then construct various symplectic Eisenstein series. A prototypical example of such a construction was already carried out by Hecke around 1926 [7], where he constructed a theta kernel $\Theta(\tau, t)$ from an indefinite lattice of signature $(1, 1)$ and integrated it to produce a holomorphic modular form $\vartheta(\tau)$ of weight one. This is an Eisenstein series if the lattice is isotropic and a cusp form otherwise. In [8], Kudla extended this construction to

Y. Li (✉)
Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany
e-mail: li@mathematik.tu-darmstadt.de

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produce holomorphic Siegel modular forms of genus g and weight $\frac{g+1}{2}$, as a prelude to the important works by Kudla and Millson later [10, 11].

In this note, we will consider a different theta kernel $\tilde{\Theta}(\tau, t)$ for an isotropic, indefinite lattice of signature $(1, 1)$. Rather than holomorphic, its integral in t is a harmonic function $\tilde{\vartheta}(\tau)$ and related to the holomorphic Eisenstein series $\vartheta(\tau)$ constructed by Hecke via

$$\xi\tilde{\vartheta}(\tau) = \vartheta(\tau),$$

where $\xi = \xi_1$ is the differential operator introduced by Bruinier and Funke [2]. In the notion loc. cit., $\tilde{\vartheta}(\tau)$ is a harmonic Maass form of weight one. For any $k \in \frac{1}{2}\mathbb{Z}$, a harmonic Maass form of weight k is a real analytic functions on the upper half-plane $\mathcal{H} := \{\tau = u + iv : v > 0\}$ that transforms with weight k with respect to a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and is annihilated by the weight k hyperbolic Laplacian

$$\Delta_k := -\xi_{2-k} \circ \xi_k, \quad \xi_k(f) := 2iv^k \overline{\frac{\partial f}{\partial \overline{\tau}}}.$$

Harmonic Maass forms can be written as the sum of a holomorphic part and a non-holomorphic part. The Fourier coefficients of their holomorphic parts are expected to contain interesting arithmetic information concerning the ξ_k-images of the non-holomorphic parts (see e.g. [3, 5]).

In [12], Kudla, Rapoport and Yang considered an Eisenstein series, which is harmonic. The Fourier coefficients of its holomorphic part are logarithms of rational numbers, and can be interpreted as the arithmetic degree of special divisors on an arithmetic curve. In view of their work and the appearance of other weight one harmonic Maass forms in connection with the Kudla program [6, 9], we expect the Fourier coefficients of the harmonic Eisenstein series we construct to have a similar arithmetic interpretation as well.

The idea to construct $\tilde{\vartheta}(\tau)$ is rather straightforward. If we can construct a function $\tilde{\Theta}(\tau, t)$ such that it is modular in τ and satisfies $\xi\tilde{\Theta}(\tau, t) = \Theta(\tau, t)$ for each t, then simply integrating it in t will produce the desirable $\tilde{\vartheta}(\tau)$. This idea has already been used in [4], where $\xi_{1/2}$ connected the theta kernels constructed from the Gaussian and the Kudla-Millson Schwartz form. In our setting, we will introduce an L^∞ function $\tilde{\varphi}_\tau$, which is a ξ-preimage of the Schwartz function used in constructing $\Theta(\tau, t)$ under ξ (see Proposition 3.4). We will then use this function to form a theta kernel $\tilde{\Theta}(\tau, t)$ and integrate it to obtain the harmonic Eisenstein series $\tilde{\vartheta}(\tau)$ in Theorem 4.3 in the last section.

2 Theta Lift from O(1, 1) to SL_2

In this section, we will recall the construction of the Eisenstein series in [7] and [8]. For $N \in \mathbb{N}$, let $L = N\mathbb{Z}^2$ be a lattice with quadratic form $Q(\binom{a}{b}) := \frac{ab}{N}$ and $B(\cdot, \cdot) : L \times L \to \mathbb{Z}$ the associated bilinear form. The dual lattice $L^* \subset V_{\mathbb{Q}} := L \otimes \mathbb{Q}$ is then $\mathbb{Z}^2$ and the discriminant group is $L^*/L = (\mathbb{Z}/N\mathbb{Z})^2$.

Let ρ_L be the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L^*/L]$. As usual, let $\{\mathfrak{e}_h : h \in L^*/L\}$ denote the canonical basis of $\mathbb{C}[L^*/L]$ and $\mathbf{e}(a) := e^{2\pi i a}$ for any $a \in \mathbb{C}$. Then the action of ρ_L on the generators $T, S \in \mathrm{SL}_2(\mathbb{Z})$ is given by (see e.g. [1, §4])

$$\rho_L(T)(\mathfrak{e}_h) = \mathbf{e}(Q(h))\mathfrak{e}_h,\ \rho_L(S)(\mathfrak{e}_h) = \frac{1}{N}\sum_{\delta \in L^*/L} \mathbf{e}(-(\delta, h))\mathfrak{e}_\delta.$$

The symmetric domain attached to $V_{\mathbb{R}} := L \otimes \mathbb{R}$ is given by the hyperbola

$$\mathcal{D} := \{Z \in V_{\mathbb{R}} | B(Z, Z) = -1\}.$$

We denote one of its two connected components by $\mathcal{D}^+$ and parametrize it by

$$\begin{aligned} \Phi : \mathbb{R}_+^\times &\to \mathcal{D}^+ \\ t &\mapsto Z_t := \sqrt{\frac{N}{2}}\begin{pmatrix} t \\ -1/t \end{pmatrix}. \end{aligned}$$

Let $W_t := \sqrt{N/2}\binom{t}{1/t} \in Z_t^\perp$. Then $d\Phi\left(t\frac{d}{dt}\right) = W_t \in V_{\mathbb{R}}$ and $\{W_t, Z_t\}$ is an orthogonal basis of $V_{\mathbb{R}}$. For any $X = \binom{x_1}{x_2} \in V_{\mathbb{R}}$, one can write $X = X_{W_t} + X_{Z_t}$, where

$$X_{W_t} := B(X, W_t)W_t = \frac{t^{-1}x_1 + tx_2}{\sqrt{2N}}W_t,\ X_{Z_t} := -B(X, Z_t)Z_t = \frac{t^{-1}x_1 - tx_2}{\sqrt{2N}}Z_t.$$

Then the majorant of $B(\cdot,\cdot)$ associated to Z_t, denoted by $B(\cdot,\cdot)_t$, is given by the positive definite quadratic form

$$Q(X)_t := Q(X_{W_t}) - Q(X_{Z_t}) = \frac{B(X, W_t)^2 + B(X, Z_t)^2}{2} = \frac{t^{-2}x_1^2 + t^2x_2^2}{2N}.$$

Let $\mathbb{R}^{1,1} = \{(x, y) : x, y \in \mathbb{R}\}$ be a quadratic space of signature $(1, 1)$ with the quadratic form $Q'((x, y)) = \frac{x^2 - y^2}{2}$ with associated bilinear form $B'(\cdot,\cdot)$. Given $\tau = u + iv \in \mathcal{H}$ in the upper half plane, we define the Schwartz function φ_τ on $\mathbb{R}^{1,1}$ by

$$\begin{aligned} \varphi_\tau : \mathbb{R}^{1,1} &\to \mathbb{C} \\ (x, y) &\mapsto \sqrt{2v}\cdot x \cdot \mathbf{e}\left(\frac{x^2}{2}\tau - \frac{y^2}{2}\overline{\tau}\right). \end{aligned} \tag{2.1}$$

Now summing φ_τ over any even, integral lattice $M \subset \mathbb{R}^{1,1}$ of rank 2 would produce a real-analytic theta series of weight 1 that transforms with respect to ρ_M. In our setting, we let M be the image of L under the following isometry

$$\begin{aligned} \iota_t : V_{\mathbb{R}} &\to \mathbb{R}^{1,1} \\ X &\mapsto (B(X, W_t), B(X, Z_t)) \end{aligned} \tag{2.2}$$

for each $t \in \mathbb{R}_+^\times$. Now the vector-valued theta function

$$\Theta(\tau, t) := \sum_{h \in L^*/L} \Theta_h(\tau, t)\mathfrak{e}_h, \;\; \Theta_h(\tau, t) := \sum_{X \in L+h} \varphi_\tau(\iota_t(X)) \tag{2.3}$$

transforms with weight 1 and representation ρ_L in the variable τ by Theorem 4.1 in [1]. For $h = \binom{h_1}{h_2} \in \mathbb{Z}^2$, we have explicitly

$$\Theta_h(\tau, t) = \sqrt{\frac{v}{N}} \sum_{\substack{x_1 \equiv h_1 (N) \\ x_2 \equiv h_2 (N)}} (t^{-1}x_1 + tx_2)\mathbf{e}\left(\frac{x_1x_2}{N}u + \frac{t^{-2}x_1^2 + t^2x_2^2}{2N}iv\right)$$

Integrating over $t \in \mathbb{R}_+^\times$ with respect to the invariant differential $\frac{dt}{t}$ defines

$$\vartheta_h(\tau, s) := \int_0^1 \Theta_h(\tau, t)t^s\frac{dt}{t} + \int_1^\infty \Theta_h(\tau, t)t^{-s}\frac{dt}{t}. \tag{2.4}$$

Here, the integral converges for $\Re s \gg 0$. As a function of s, it has analytic continuation to $s \in \mathbb{C}$. Let $\vartheta_h(\tau)$ be the constant term in the Laurent expansion of $\vartheta_h(\tau, s)$ around $s = 0$. Then $\vartheta_h(\tau)$ is holomorphic and $\vartheta(\tau) := \sum_{h \in L^*/L} \vartheta_h(\tau)\mathfrak{e}_h$ is an Eisenstein series of weight 1 on $\mathrm{SL}_2(\mathbb{Z})$ and transforms with respect to ρ_L. It has the following Fourier expansion (see [8, Theorem 3.2]).

Proposition 2.1 *Write $h = \binom{h_1}{h_2} \in \mathbb{Z}^2$. Then $\vartheta_h(\tau)$ has the Fourier expansion $\vartheta_h(\tau) = \sum_{n \in \mathbb{Q}_{\geq 0}} c_h(n)q^n$ with*

$$\begin{aligned} c_h(0) &:= \begin{cases} \frac{1}{2} - \langle \frac{h_1}{N} \rangle & N \nmid h_1, N \mid h_2, \\ \frac{1}{2} - \langle \frac{h_2}{N} \rangle & N \mid h_1, N \nmid h_2, \\ 0 & \text{otherwise}, \end{cases} \\ c_h(n) &:= \sum_{X = \binom{x_1}{x_2} \in L+h,\ Q(X)=n} \mathrm{sgn}(x_1), \qquad n > 0. \end{aligned} \tag{2.5}$$

Here $\langle x \rangle \in (0, 1]$ is defined by the property $x - \langle x \rangle \in \mathbb{Z}$.

Proof Use the identity $e^{-2\pi y} = \sqrt{y}\int_0^\infty e^{-\pi y(t^2+t^{-2})}(t + t^{-1})\frac{dt}{t}$ and $H(0, x) = \frac{1}{2} - x$, where $H(s, x) := \sum_{n=0}^\infty (x + n)^{-s}$ is the Hurwitz zeta function. □

3 Some Special Functions

In this section, we will introduce a special function $\tilde{\varphi}_\tau$ on $\mathbb{R}^{1,1}$ such that $\xi\tilde{\varphi}_\tau = \varphi_\tau$.

3.1 Non-holomorphic Part

Define the functions $f_\tau^* : \mathbb{R} \to \mathbb{R}$ and $\varphi_\tau^* : \mathbb{R}^{1,1} \to \mathbb{C}$ by

$$f_\tau^*(x) := \operatorname{sgn}(x) - \operatorname{erf}\left(\sqrt{2\pi v}x\right) = \operatorname{sgn}(x)\operatorname{erfc}(\sqrt{2\pi v}|x|),$$
$$\varphi_\tau^*(x, y) := \mathbf{e}\left(\frac{y^2 - x^2}{2}\tau\right) f_\tau^*(x). \tag{3.1}$$

where $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$ and $\operatorname{erfc}(x)$ are the error and complementary error functions. Straightforward calculations show that

$$\xi(\varphi_\tau^*(x, y)) = -\varphi_\tau(x, y) \tag{3.2}$$

for all $(x, y) \in \mathbb{R}^{1,1}$.

For each $y \in \mathbb{R}$, the function $\varphi_\tau^*(x, y)$ decays like a Schwartz function in x. Also, $\varphi_\tau^*(x, y)$ satisfies

$$\lim_{x \to 0^+} \varphi_\tau^*(x, y) - \lim_{x \to 0^-} \varphi_\tau^*(x, y) = 2\mathbf{e}\left(\frac{y^2}{2}\tau\right),$$

hence has a jump discontinuity at $x = 0$. Away from 0, it is smooth. Thus, we can view it as a tempered distribution on $\mathbb{R}^{1,1}$ and calculate its Fourier transform with respect to $-Q'$ as follows.

First, notice that as a distribution, f_τ^* satisfies the differential equation

$$\frac{d}{dx}\left(f_\tau^*(x)\right) = 2 \cdot \delta(x) - 2\sqrt{2v}e^{-2\pi v x^2},$$

where $\delta(x)$ is the Dirac delta function. This follows from $\frac{d}{dx}|x| = \operatorname{sgn}(x)$, $\frac{d}{dx}\operatorname{sgn}(x) = 2\delta(x)$ as tempered distributions. Substituting in the definition of $\varphi_\tau^*(x, y)$, we see that it satisfies

$$\frac{\partial}{\partial x}\left(\varphi_\tau^*(x, y)\right) + 2\pi i\tau x\varphi_\tau^*(x, y) = \left(2\delta(x) - 2\sqrt{2v}\mathbf{e}\left(-\frac{x^2}{2}\overline{\tau}\right)\right)\mathbf{e}\left(\frac{y^2}{2}\tau\right). \tag{3.3}$$

Notice that $\mathbf{e}\left(-\frac{x^2}{2}\tau\right) 2\delta(x) = 2\delta(x)$ as a distribution.

For a Schwartz function ϕ on $\mathbb{R}^{1,1}$, we define its Fourier transform $\mathcal{F}(\phi)$ with respect to the quadratic form $-Q'$ by

$$\mathcal{F}(\phi)(x, y) := \int_{\mathbb{R}^{1,1}} \phi(w, z)\mathbf{e}(-wx + yz)dwdz. \tag{3.4}$$

If ϕ is not a Schwartz function and the integral above converges, we also use it to denote its Fourier transform. Using the standard facts of Fourier transform (see e.g. [1, Lemma 3.1]), we have

$$\begin{aligned} &-\tau \frac{\partial}{\partial x} \mathcal{F}(\varphi^*_\tau)(x, y) + 2\pi i x \mathcal{F}(\varphi^*_\tau)(x, y) \\ &= \left(2 - 2\sqrt{2v}\,(i\overline{\tau})^{-1/2}\, \mathbf{e}\left(-\frac{x^2}{2}\overline{(-1/\tau)}\right)\right) \frac{\mathbf{e}\left(\frac{y^2}{2}(-1/\tau)\right)}{\sqrt{-i\tau}}. \end{aligned}$$

After dividing by $-\tau$ on both sides and making the change of variable $\tau \mapsto -1/\tau$, the equation becomes

$$\begin{aligned} &\frac{\partial}{\partial x} \mathcal{F}(\varphi^*_{-1/\tau})(x, y) + 2\pi i x \tau \mathcal{F}(\varphi^*_{-1/\tau})\,(x, y) \\ &= -2\tau \left(\sqrt{2v}\mathbf{e}\left(-\frac{x^2}{2}\overline{\tau}\right) - \sqrt{-i\tau}\right) \mathbf{e}\left(\frac{y^2}{2}\tau\right), \end{aligned}$$

Now define

$$\mathcal{D}^*_\tau(x, y) := \varphi^*_\tau(x, y) - \frac{\mathcal{F}(\varphi^*_{-1/\tau})(x, y)}{\tau}. \tag{3.5}$$

Then it satisfies the differential equation

$$\mathbf{e}\left(-\frac{x^2}{2}\tau\right) \frac{d}{dx}\left(\mathbf{e}\left(\frac{x^2}{2}\tau\right) \mathcal{D}^*_\tau(x, y)\right) = 2\left(\delta(x) - \sqrt{-i\tau}\right) \mathbf{e}\left(\frac{y^2}{2}\tau\right). \tag{3.6}$$

We have the following result concerning the solutions to this differential equation.

Proposition 3.1 *For fixed $\tau_0 \in \mathcal{H}, y_0 \in \mathbb{R}$, the only jump discontinuity of any piecewise continuous solution to the differential equation (3.6) is at $x = 0$. Suppose further that it is bounded in x. Then the solution agrees with the function $\mathcal{D}_{\tau_0}(x, y_0)$ defined by*

$$\mathcal{D}_{\tau_0}(x, y_0) := \mathbf{e}\left(\frac{y_0^2 - x^2}{2}\tau_0\right) \mathrm{sgn}(x)\mathrm{erfc}(\sqrt{-i\tau_0}|x|) \tag{3.7}$$

whenever the solution is continuous. In particular, $\mathcal{D}^*_\tau(x, y) = \mathcal{D}_\tau(x, y)$ for all $(x, y) \in \mathbb{R}^{1,1}$ and $\tau \in \mathcal{H}$.

Remark 3.2 Here in $\tau_0 = u_0 + iv_0$, the function $\mathrm{erfc}(\sqrt{-i\tau_0}|x|)$ is the unique holomorphic extension of $\mathrm{erfc}(\sqrt{v_0}|x|)$.

Proof The first claim is clear as a jump discontinuity at $x = x_0$ of a piecewise solution would produce a constant times $\delta(x - x_0)$. By the fundamental theorem of calculus, the solution, whenever continuous, would agree with $\mathcal{D}_{\tau_0}(x, y_0)$ up to a constant multiple of $\mathbf{e}\left(-\frac{x^2}{2}\tau\right)$, which is unbounded as $x \to \infty$. Since $\mathcal{D}_{\tau_0}(x, y_0)$ is assumed to be bounded, the second claim follows. Finally, for any fixed $\tau_0 \in \mathcal{H}, y_0 \in \mathbb{R}$, we have $\varphi^*_{\tau_0}(x, y_0) \in L^1(\mathbb{R})$. Thus its Fourier transform is continuous and bounded. That implies $\mathcal{D}^*_\tau(x, y)$ is bounded and has no removable discontinuity on $\mathbb{R}$, hence the third claim. □

3.2 Holomorphic Part

Now, we will define the holomorphic counterpart to φ^*_τ as

$$\varphi^+_\tau(x, y) := \mathbf{e}\left(\frac{y^2 - x^2}{2}\tau\right)\mathrm{sgn}(x)\mathbb{1}_{y^2>x^2}, \tag{3.8}$$

where $\mathbb{1}_{y^2>x^2}$ is the characteristic function of the set $\{(x, y) \in \mathbb{R}^{1,1} : y^2 > x^2\}$. Even though $\varphi^+_\tau(x, y)$ is not a Schwartz function, it decays nicely enough such that we have the following result.

Proposition 3.3 *The following integral*

$$\mathcal{F}(\varphi^+_\tau)(x, y) := \int_{\mathbb{R}^{1,1}} \varphi^+_\tau(w, z)\mathbf{e}(-wx + zy)dwdz$$

converges uniformly on compact subsets of $\{(x, y) \in \mathbb{R}^{1,1} : x^2 \neq y^2\}$. *Furthermore, the function* $\mathcal{F}(\varphi^+_\tau)$ *is bounded and continuously differentiable on* $\{(x, y) \in \mathbb{R}^{1,1} : x^2 \neq y^2\}$.

Proof Let $A = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and make the rotational change of variables

$$\begin{pmatrix} a \\ b \end{pmatrix} := A \cdot \begin{pmatrix} w \\ z \end{pmatrix}, \quad \begin{pmatrix} x' \\ y' \end{pmatrix} := A \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

we can rewrite the integral above as

$$\mathcal{F}(\varphi^+_\tau)(x', y') = \lim_{T,T'\to\infty} \int_{-T'}^{T'} \int_{-T}^{T} \mathbb{1}_{ab>0}\mathbf{e}(ab\tau)\mathrm{sgn}(a - b)\mathbf{e}(ay' + bx')dbda.$$

The integral over $0 < a < T', 0 < b < T$ can be evaluated explicitly as

$$\int_0^{T'} \frac{\mathbf{e}(a^2\tau + a(x'+y'))}{\pi i(a\tau + x')} - \frac{\mathbf{e}((a\tau + x')T + ay')}{2\pi i(a\tau + x')} - \frac{\mathbf{e}(ay')}{2\pi i(a\tau + x')} da.$$

The same can be done for the region $-T' < a < 0, -T < b < 0$. As $T \to \infty$, the middle term vanishes, and we are left with

$$\mathcal{F}(\varphi_\tau^+)(x', y') = \lim_{T' \to \infty} \int_{-T'}^{T'} \frac{\mathbf{e}(a^2\tau + a(x'+y'))}{\pi i(a\tau + x')} - \frac{\mathbf{e}(ay')}{2\pi i(a\tau + x')} da. \tag{3.9}$$

The integral of the first term can be bounded with $\int_{-\infty}^{\infty} \frac{e^{-r^2}}{\sqrt{r^2+(x')^2}} dr \ll |x'|^{-1}$, which implies that the integral converges uniformly and defines a continuously differentiable function away from $x'y' = 0$. Furthermore, it is bounded when $|x'|$ is large. On the other hand when $|x'|$ is close to zero, we can fix an absolute constant $\epsilon > 0$ such that the integral over $|a| \in (\epsilon, \infty)$ converges absolutely independent of x', y'. The rest of the integrand can be written as

$$\begin{aligned} &\frac{\mathbf{e}(a^2\tau + a(x'+y'))}{a\tau + x'} + \frac{\mathbf{e}(a^2\tau - a(x'+y'))}{-a\tau + x'} \\ &= C_1(a, x', \tau)\frac{x'}{(a\tau)^2 - (x')^2} + C_2(a, x', \tau)\frac{\sin(ay')}{a} \end{aligned}$$

with $|C_j(a, x', \tau)|$ bounded above independently of a and x'. Since $|\int_0^\epsilon \frac{x'}{(a\tau)^2-(x')^2} da|$ and $|\int_0^\epsilon \frac{\sin(ay')}{a} da|$ are bounded independent of x' and y', the integral of the first term in Eq. (3.9) defines a bounded and continuously differentiable function on $\{(x, y) \in \mathbb{R}^{1,1} : x^2 \neq y^2\}$.

Away from $x'y' = 0$, the integral of the last term converges uniformly using integration by parts and defines a continuous function. Using standard formula in one dimensional Fourier transform, we can in fact evaluate it explicitly as

$$\int_{-\infty}^{\infty} \frac{\mathbf{e}(ay')}{2\pi i(a\tau + x')} da = \tau^{-1}\mathbf{e}(x'y'(-1/\tau))\mathbb{1}_{x'y'>0}.$$

From this, it is clear that it is bounded. □

Since φ_τ^+ is bounded, integrating against it defines a tempered distribution on $\mathbb{R}^{1,1}$. Thus, we can then study its Fourier transform $\mathcal{F}(\varphi_\tau^+)$ as we have done for φ_τ^*. The analogue to Eq. (3.3) is as follows

$$\frac{\partial \varphi_\tau^+}{\partial x} + 2\pi i\tau x\varphi_\tau^+ = (2\delta(x) - \delta(x-y) - \delta(x+y))\,\mathbf{e}\left(\frac{y^2 - x^2}{2}\tau\right). \tag{3.10}$$

Applying Fourier transform to both sides and making the change $\tau \mapsto -1/\tau$ yields

$$(-1/\tau)\frac{\partial \mathcal{F}(\varphi^+_{-1/\tau})(x,y)}{\partial x} - 2\pi i x \mathcal{F}(\varphi^+_{-1/\tau})(x,y)$$
$$= -\left(2\sqrt{-i\tau}\mathbf{e}\left(\frac{y^2}{2}\tau\right) - \delta(y-x) - \delta(x+y)\right)$$

Subtracting the previous two equations shows that the function defined by

$$\mathcal{D}^+_\tau(x,y) := \varphi^+_\tau(x,y) - \frac{\mathcal{F}(\varphi^+_{-1/\tau})(x,y)}{\tau} \tag{3.11}$$

also satisfies the differential equation (3.6). Note that $\delta(y \pm x) = \delta(y \pm x)\mathbf{e}(\frac{y^2-x^2}{2})$ and $\delta(x)\mathbf{e}(\frac{x^2}{2}\tau) = \delta(x)$. For each fixed $\tau_0 \in \mathcal{H}$, the function $\varphi^+_{\tau_0}$ is bounded with only jump singularities when either $x^2 = y^2$ or $x = 0$. Proposition 3.3 implies that $\mathcal{F}(\varphi^+_\tau)$ has the same property as well. So we can define $\mathcal{F}(\varphi^+_\tau)(x,y)$ on $x^2 = y^2$ such that $\mathcal{D}^+_\tau(x,y)$ is continuous when $y^2 = x^2 > 0$. By Proposition 3.1, we know that $\mathcal{D}^+_\tau = \mathcal{D}_\tau$ and have proved the following result.

Proposition 3.4 *For all $(x,y) \in \mathbb{R}^{1,1}$, the $L^\infty(\mathbb{R}^{1,1})$ function*

$$\tilde{\varphi}_\tau(x,y) := \varphi^+_\tau(x,y) - \varphi^*_\tau(x,y) = \mathrm{sgn}(x)\mathbf{e}\left(\frac{y^2-x^2}{2}\right)\left(\mathbb{1}_{y^2>x^2} - \mathrm{erfc}(\sqrt{2\pi v}|x|)\right) \tag{3.12}$$

satisfies

(1) $\tilde{\varphi}_{\tau+1}(x,y) = \mathbf{e}(\frac{y^2-x^2}{2})\tilde{\varphi}_\tau(x,y)$,
(2) $\mathcal{F}(\tilde{\varphi}_{-1/\tau})(x,y) = \tau\tilde{\varphi}_\tau(x,y)$,
(3) $\xi(\tilde{\varphi}_\tau(x,y)) = \varphi_\tau(x,y)$.

4 Real-Analytic Theta Series

In this section, we will construct weight 1 real-analytic theta series $\tilde{\vartheta}(\tau)$ that transforms with respect to ρ_{-L} and maps to $\vartheta(\tau)$ under ξ. Proposition 3.4 and the construction of $\vartheta_h(\tau)$ imply that we need to consider summing $\tilde{\varphi}_\tau(\iota_t(X))$ over $X \in L+h$ and integrating over $\mathbb{R}^\times_+$ with respect to $\frac{dt}{t}$. However, the sum and integral are both divergent. The problem with the sum is caused by isotropic elements in L. We will regularize the sum by considering a slight shift of the lattice L, and regularize the integral by adding the converging factor t^s as usual. The ideas are simple, but the procedure to carry it out is a bit complicated.

In the notations of Sects. 2 and 3, define the following series

$$\tilde{\Theta}_h(\tau, t; \varepsilon, \varepsilon') := \sum_{X \in L + h + \varepsilon\binom{1}{1}} \tilde{\varphi}_\tau\left(\iota_t(X)\right) \mathbf{e}\left(B\left(X, \varepsilon'\begin{pmatrix}1\\1\end{pmatrix}\right)\right) \tag{4.1}$$

for $\varepsilon, \varepsilon' \in (-\frac{1}{2}, \frac{1}{2})$. This series converges for $\varepsilon, \varepsilon' \in (-\frac{1}{2}, \frac{1}{2})$. It is modular for $\varepsilon, \varepsilon' \in (-\frac{1}{2}, \frac{1}{2})\backslash\{0\}$, but not continuous at $\varepsilon = 0$. Define Θ_h^* and Θ_h^+ as $\tilde{\Theta}_h$ in (4.1) with $\tilde{\varphi}_\tau$ replaced by φ_τ^* and φ_τ^+ respectively. To preserve the modularity, we define the theta series $\tilde{\Theta}(\tau, t) = \sum_{h \in L^*/L} \tilde{\Theta}_h(\tau, t)\mathfrak{e}_h$ by

$$\tilde{\Theta}_h(\tau, t) := c_h(0) + \tilde{\Theta}_h(\tau, t; 0, 0) = c_h(0) + \Theta_h^+(\tau, t; 0, 0) + \Theta_h^*(\tau, t; 0, 0), \tag{4.2}$$

where $c_h(0)$ is defined in Proposition 2.1. They have the following relationship.

Proposition 4.1 *For fixed* $\tau \in \mathcal{H}$, $t \in \mathbb{R}_+^\times$ *and* $h = \binom{h_1}{h_2} \in \mathbb{Z}^2$, *the series* $\tilde{\Theta}_h(\tau, t; \varepsilon, \varepsilon')$ *converges uniformly for* $(\varepsilon, \varepsilon')$ *in compact subsets of* $(-\frac{1}{2}, \frac{1}{2})\backslash\{0\} \times (-\frac{1}{2}, \frac{1}{2})$. *It is continuous for* $(\varepsilon, \varepsilon') \in (0, \min\{\frac{1}{1+t^2}, \frac{t^2}{1+t^2}\}) \times (-\frac{1}{2}, \frac{1}{2})$ *and satisfies*

$$\lim_{\varepsilon \to 0^+} \tilde{\Theta}_h(\tau, t; \varepsilon, \pm\varepsilon) - \frac{c_{-1}(h)}{2\pi i(\tau \mp 1)\varepsilon} = \tilde{\Theta}_h(\tau, t), \tag{4.3}$$

where $c_{-1}(h) \in \{0, 1, 2\}$ *is the number of* h_1, h_2 *that are divisible by* N.

Proof Since φ_τ^* decays like a Schwartz function, the series Θ_h^* converges absolutely and uniformly for $\varepsilon, \varepsilon' \in \mathbb{R}$, except for $h = \binom{0}{0}$, in which case, we have $\lim_{\varepsilon \to 0^+} \Theta_h^*(\tau, t; \varepsilon, \pm\varepsilon) = \Theta_h^*(\tau, t; 0, 0) + 1$. For Θ_h^+, notice that $B(X, Z_t)^2 - B(X, W_t)^2 = -\frac{2x_1x_2}{N}$ if $X = \binom{x_1}{x_2} \in V_\mathbb{R}$. So we can write

$$\Theta_h^+(\tau, t; \varepsilon, \varepsilon') = \sum_{\substack{n_1 \in N\mathbb{Z} + h_1 \\ n_2 \in N\mathbb{Z} + h_2 \\ n_1 n_2 \le -1}} + \sum_{\substack{n_1 \in N\mathbb{Z} + h_1 \\ n_2 \in N\mathbb{Z} + h_2 \\ (n_1+\varepsilon)(n_2+\varepsilon) < 0 \\ n_1 n_2 = 0}}$$
$$\operatorname{sgn}(t^{-1}(n_1 + \varepsilon) + t(n_2 + \varepsilon))\mathbf{e}\left(-\frac{(n_1 + \varepsilon)(n_2 + \varepsilon)}{N}\tau\right)\mathbf{e}\left(\frac{(n_1 + n_2 + 2\varepsilon)}{N}\varepsilon'\right). \tag{4.4}$$

Using the inequality $-(n_1 + \varepsilon)(n_2 + \varepsilon) > -\frac{n_1 n_2}{2}$ for $\varepsilon \in (-\frac{1}{2}, \frac{1}{2})$, we see that the first sum in Eq. (4.4) converges absolutely and uniformly for $(\varepsilon, \varepsilon')$ in compact subset of $(-\frac{1}{2}, \frac{1}{2})^2$. Note that the second sum is empty if and only if $N \nmid h_j$ for $j = 1, 2$.

Suppose $N \mid h_1$ and $\varepsilon > 0$. Then $n_2 \leq -1$ and the summand becomes

$$\operatorname{sgn}(t^{-1}\varepsilon + t(n_2+\varepsilon))\mathbf{e}\left(\frac{\varepsilon\tau - \varepsilon'}{N}(-n_2)\right)\mathbf{e}\left(\frac{2\varepsilon\varepsilon' - \varepsilon^2\tau}{N}\right)$$

Since $t > 0$, $\varepsilon < \min\{\frac{1}{1+t^2}, \frac{t^2}{1+t^2}\}$ and $n_2 \leq -1$, we have $t^{-1}\varepsilon + t(n_2+\varepsilon) < t^{-1}\varepsilon + t(-1+\varepsilon) < 0$. Then the second sum is just a geometric series and equals to

$$-\mathbf{e}\left(\frac{2\varepsilon\varepsilon' - \varepsilon^2\tau}{N}\right)\frac{\mathbf{e}\left((\varepsilon\tau - \varepsilon')\langle -\frac{h_2}{N}\rangle\right)}{1 - \mathbf{e}\left(\varepsilon\tau - \varepsilon'\right)}.$$

Using the power series expansion $-\frac{e^{ax}}{1-e^x} = x^{-1} - (\frac{1}{2} - a) + O(x)$, we see that we get a constant term $-\frac{1}{2} + \langle -\frac{h_2}{N}\rangle$ when $\varepsilon' = \pm\varepsilon$. When $h \neq \binom{0}{0}$, this is just $c_h(0)$. When $h = \binom{0}{0}$, there are twice this contribution, which sums to $c_h(0) + 1$. Now we are done since $\tilde{\Theta}_h(\tau, t; \varepsilon, \varepsilon') = \Theta_h^+(\tau, t; \varepsilon, \varepsilon') - \Theta_h^*(\tau, t; \varepsilon, \varepsilon')$. □

Proposition 4.2 *The theta function $\tilde{\Theta}(\tau, t)$ is a real-analytic modular form in τ of weight 1 with respect to ρ_{-L} and satisfies $\xi(\tilde{\Theta}(\tau, t)) = \Theta(\tau, t)$ and $\tilde{\Theta}_h(\tau, t) = O_\tau(1)$ for all $t \in \mathbb{R}_+^\times$.*

Proof The property $\xi(\tilde{\Theta}_h(\tau, t)) = \Theta_h(\tau, t)$ and the modularity in T are clear from the definition. For the modularity in S, we can apply Poisson summation to obtain

$$\frac{\tilde{\Theta}_h(-1/\tau, t; \varepsilon, \varepsilon')}{\tau} = \frac{\mathbf{e}(2\varepsilon\varepsilon')}{N}\sum_{\delta\in L^*/L}\mathbf{e}((\delta, h))\tilde{\Theta}_\delta(\tau, t; -\varepsilon', \varepsilon) \tag{4.5}$$

with $\varepsilon, \varepsilon' \in (-\frac{1}{2}, \frac{1}{2})\backslash\{0\}$. Using the identity $c_{-1}(h) = \frac{\sum_{\delta\in L^*/L}\mathbf{e}((\delta,h))c_{-1}(\delta)}{N}$, we obtain the desired modularity with respect to S after setting $\varepsilon' = -\varepsilon$, subtracting $\frac{c_{-1}(h)}{2\pi i(\tau-1)\varepsilon}$ from both sides and taking the limit $\varepsilon \to 0^+$. The asymptotic of $\tilde{\Theta}(\tau, t)$ in t can be seen from its definition, the decay of φ_τ^*, and the expression (4.4). □

Now to construct the preimage of $\vartheta_h(\tau)$ under ξ, we consider the integral

$$\tilde{\vartheta}_h(\tau; s) := \int_1^\infty \tilde{\Theta}_h(\tau, t)t^{-s}\frac{dt}{t} + \int_0^1 \tilde{\Theta}_h(\tau, t)t^s\frac{dt}{t}, \tag{4.6}$$

which converges for $\Re(s) > 0$ and can be analytically continued to $s \in \mathbb{C}$ via its Fourier expansion in τ. We are interested in the function

$$\tilde{\vartheta}_h(\tau) := \mathrm{Const}_{s=0}\tilde{\vartheta}_h(\tau; s). \tag{4.7}$$

It has the following desirable properties.

Theorem 4.3 *The function $\tilde{\vartheta}(\tau) := \sum_{h\in L^*/L}\tilde{\vartheta}_h(\tau)\mathfrak{e}_h$ is a harmonic Maass form of weight 1 with respect to ρ_{-L}, and maps to the Eisenstein series*

$\vartheta(\tau)$. It has the Fourier expansion $\tilde{\vartheta}_h(\tau) = \sum_{n\in\mathbb{Q}_{\geq 0}} \tilde{c}_h(n)q^n + c_h(0)\log v - \sum_{n\in\mathbb{Q}_{>0}} c_h(n)\Gamma(0, 4\pi vn)q^{-n}$, where $c_h(n) \in \mathbb{Q}$ are defined in Eq. (2.5) and $\tilde{c}_h(n)$ are defined by

$$\tilde{c}_h(0) := \begin{cases} c_h(0)\left(\log(\pi N) - \frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) - \log\frac{\Gamma\left(\langle\frac{h_1}{N}\rangle\right)\Gamma\left(\langle\frac{h_2}{N}\rangle\right)}{\Gamma\left(\langle-\frac{h_1}{N}\rangle\right)\Gamma\left(\langle-\frac{h_2}{N}\rangle\right)}, & N \mid h_1 \text{ or } N \mid h_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{c}_h(n) := \sum_{X=\binom{x_1}{x_2}\in L+h,\ -Q(X)=n} \operatorname{sgn}(x_1)\log\left|\frac{x_1}{x_2}\right|, \quad n > 0. \tag{4.8}$$

Proof The modularity statement follows from Proposition 4.2. For the Fourier expansion, we will first calculate the contribution of Θ_h^* in the integral defining $\tilde{\vartheta}_h(\tau, s)$, i.e.

$$\vartheta_h^*(\tau, s) := \int_1^\infty \Theta_h^*(\tau, t; 0, 0)t^{-s}\frac{dt}{t} + \int_0^1 \Theta_h^*(\tau, t; 0, 0)t^s\frac{dt}{t}.$$

Since the sum defining $\Theta_h^*(\tau, t; 0, 0)$ converges absolutely and uniformly in t, we can switch the sum and integral. It is then suffices to consider the integral

$$I_h^*(X, \tau, s) := \int_1^\infty \varphi_\tau^*(\iota_t(X))t^{-s}\frac{dt}{t} + \int_0^1 \varphi_\tau^*(\iota_t(X))t^s\frac{dt}{t} \tag{4.9}$$

for each $X = \binom{x_1}{x_2} \in L + h$. If $n := Q(X) = \frac{x_1x_2}{N} \neq 0$, then $\mathrm{Const}_{s=0}I_h^*(X, \tau, s) = I_h^*(X, \tau, 0)$ and

$$\begin{aligned} I_h^*(X, \tau, 0) &= \int_0^\infty \varphi_\tau^*(\iota_t(X))\frac{dt}{t} \\ &= \mathbf{e}(-n\tau)\int_0^\infty \operatorname{sgn}(x_1t^{-1} + x_2t)\operatorname{erfc}\left(\sqrt{\frac{\pi v}{N}}|x_1t^{-1} + x_2t|\right)\frac{dt}{t} \\ &= \operatorname{sgn}(x_1)\mathbf{e}(-n\tau) \\ &\qquad \int_0^\infty \operatorname{sgn}(w^{-1} + \operatorname{sgn}(n)w)\operatorname{erfc}\left(\sqrt{n\pi v}|w^{-1} + \operatorname{sgn}(n)w|\right)\frac{dw}{w}, \end{aligned}$$

where $w = \left|\frac{x_2}{x_1}\right|^{1/2} t$. For any $\alpha > 0$, we have $\int_0^\infty \operatorname{sgn}(w^{-1}-w)\operatorname{erfc}(\alpha|w^{-1}-w|)\frac{dw}{w} = 0$ and

$$\int_0^\infty \operatorname{erfc}(\alpha(w^{-1} + w))\frac{dw}{w} = \Gamma(0, 4\alpha^2),$$

with $\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt$ the incomplete gamma function. Therefore, we have

$$\mathrm{Const}_{s=0} I_h^*(X,\tau,s) = \mathrm{sgn}(x_1)\mathbf{e}(-Q(X)\tau)\Gamma(0,4\pi Q(X)v)$$

when X is not isotropic.

When $Q(X) = 0$, we know that $I_h^*(X,\tau,s) = 0$ if $x_1 = x_2 = 0$. So suppose $x_1 \neq 0$ and $x_2 = 0$. Simple estimate shows that

$$\left| I_h^*\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau, s\right) - \mathrm{sgn}(x_1)\int_0^\infty \mathrm{erfc}\left(\sqrt{\frac{\pi v}{N}}|x_1 t^{-1}|\right) t^{-s}\frac{dt}{t}\right| \ll e^{-cx_1^2}\cdot \int_1^\infty e^{-c(t^2-1)}|t^s - t^{-s}|\frac{dt}{t},$$

where $c = \frac{\pi v}{N}$. Using the formula $\int_0^\infty \mathrm{erfc}(\alpha t)t^s\frac{dt}{t} = \frac{\alpha^{-s}}{\sqrt{\pi}s}\Gamma\left(\frac{s+1}{2}\right)$, we have

$$\sum_{\substack{x_1\equiv h_1(N)\\ x_1\neq 0}} I_h^*\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau, s\right) = \frac{(N\pi v)^{-s/2}}{\sqrt{\pi}s}\Gamma\left(\frac{s+1}{2}\right)\left(H\left(s,\langle\frac{h_1}{N}\rangle\right) - H\left(s,\langle -\frac{h_1}{N}\rangle\right)\right).$$

The constant term at $s = 0$ of the right hand side is then given by $-c_h(0)\log v - \tilde{c}_h(0)$.

Now, we will consider the following integral

$$\vartheta_h^+(\tau,s) := \int_1^\infty \Theta_h^+(\tau,t;0,0)t^{-s}\frac{dt}{t} + \int_0^1 \Theta_h^+(\tau,t;0,0)t^s\frac{dt}{t}.$$

By the definition of φ_τ^+, it suffices to calculate as before

$$I_h^+(X,\tau,s) := \int_1^\infty \varphi_\tau^+(\iota_t(X))t^{-s}\frac{dt}{t} + \int_0^1 \varphi_\tau^+(\iota_t(X))t^s\frac{dt}{t} \tag{4.10}$$

for $X \in L+h$ with $-Q(X) > 0$. For $X = \binom{x_1}{x_2}$, this simplifies to

$$\begin{aligned} I_h^+(X,\tau,s) &= \mathbf{e}(-Q(X)\tau)\left(\int_1^\infty \mathrm{sgn}(t^{-1}x_1 + tx_2)t^{-s}\frac{dt}{t} + \int_0^1 \mathrm{sgn}(t^{-1}x_1+tx_2)t^s\frac{dt}{t}\right)\\ &= \mathrm{sgn}(x_1)\mathbf{e}(-Q(X)\tau)\left(r^s\int_r^\infty \mathrm{sgn}(w^{-1}-w)w^{-s}\frac{dw}{w}\right.\\ &\qquad \left. + r^{-s}\int_0^r \mathrm{sgn}(w^{-1}-w)w^s\frac{dw}{w}\right) \end{aligned}$$

after a change of variable $w = r \cdot t, r = |x_2/x_1|^{1/2}$. For $\Re(s) > 0$, we then have $I_h^+(X, \tau, s) = \mathrm{sgn}(x_1)\mathbf{e}(-Q(X)\tau)\frac{2(r^{-s}-1)}{s}$. Taking the limit as s goes to 0 then finishes the calculation. □

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A Note on the Growth of Nearly Holomorphic Vector-Valued Siegel Modular Forms

Ameya Pitale, Abhishek Saha, and Ralf Schmidt

Abstract Let F be a nearly holomorphic vector-valued Siegel modular form of weight ρ with respect to some congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$. In this note, we prove that the function on $\mathrm{Sp}_{2n}(\mathbb{R})$ obtained by lifting F has the moderate growth (or "slowly increasing") property. This is a consequence of the following bound that we prove: $\|\rho(Y^{1/2})F(Z)\| \ll \prod_{i=1}^{n}(\mu_i(Y)^{\lambda_1/2} + \mu_i(Y)^{-\lambda_1/2})$ where $\lambda_1 \geq \ldots \geq \lambda_n$ is the highest weight of ρ and $\mu_i(Y)$ are the eigenvalues of the matrix Y.

1 Introduction and Statement of Result

Let G be a connected reductive group over $\mathbb{Q}$ and K a maximal compact subgroup of $G(\mathbb{R})$. One of the properties that an automorphic form on $G(\mathbb{R})$ is required to satisfy is that it should be a slowly increasing function (also referred to as the *moderate growth property*). We now recall the definition of this property following [4].

A norm $\| \ \|$ on $G(\mathbb{R})$ is a function of the form $\|g\| = (\mathrm{Tr}(\sigma(g)^*\sigma(g)))^{1/2}$ where $\sigma : G(\mathbb{R}) \to \mathrm{GL}_r(\mathbb{C})$ is a finite-dimensional representation with finite kernel and image closed in $M_r(\mathbb{C})$ and such that $\sigma(K) \subseteq \mathrm{SO}_r$. For example, if $G = \mathrm{Sp}_{2n}$, we may take σ to be the usual embedding into $\mathrm{GL}_{2n}(\mathbb{R})$ while for $G = \mathrm{GL}_n$ we may take $\sigma(g) = (g, \det(g)^{-1})$ into $\mathrm{GL}_{n+1}(\mathbb{R})$. A complex-valued function ϕ on $G(\mathbb{R})$ is said to have the moderate growth property if there is a norm $\| \ \|$ on $G(\mathbb{R})$, a constant C, and a positive integer λ such that $|\phi(g)| \leq C\|g\|^{\lambda}$ for all $g \in G(\mathbb{R})$. This definition does not depend on the choice of norm.

In practice, automorphic forms on $G(\mathbb{R})$ are often constructed from classical objects (such as various kinds of "modular forms") and it is not always immediately clear that the resulting constructions satisfy the moderate growth property. For a

A. Pitale • R. Schmidt
Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
e-mail: apitale@math.ou.edu; rschmidt@math.ou.edu

A. Saha (✉)
School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK
e-mail: abhishek.saha@qmul.ac.uk

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classical modular form f of weight k on the upper half plane, one can prove the bound $|f(x+iy)| \leq C(1+y^{-k})$ for some constant C depending on f. Using this bound it is easy to show that the function ϕ_f on $\mathrm{SL}_2(\mathbb{R})$ attached to f has the moderate growth property. More generally, if F is a holomorphic Siegel modular form of weight k on the Siegel upper half space $\mathbb{H}_n$, Sturm proved the bound $|F(X+iY)| \leq C\prod_{i=1}^{n}(1+\mu_i(Y)^{-k})$ where $\mu_i(Y)$ are the eigenvalues of Y, which can be shown to imply the moderate growth property for the corresponding function Φ_F on $\mathrm{Sp}_{2n}(\mathbb{R})$.

Bounds of the above sort are harder to find in the literature for more general modular forms. In particular, when considering Siegel modular forms on $\mathbb{H}_n$, it is more natural to consider vector-valued modular forms. Such a vector-valued form comes with a representation ρ of $\mathrm{GL}_n(\mathbb{C})$ corresponding to a highest weight $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ of integers. Furthermore, for arithmetic purposes, it is sometimes important to consider more general modular forms where the holomorphy condition is relaxed to near-holomorphy. Recall that a nearly holomorphic modular form on $\mathbb{H}_n$ is a function that transforms like a modular form, but instead of being holomorphic, it is a polynomial in the entries of Y^{-1} with holomorphic functions as coefficients. The theory of nearly holomorphic modular forms was developed by Shimura in substantial detail and was exploited by him and other authors to prove algebraicity and Galois-equivariance of critical values of various L-functions. We refer the reader to the papers [1–3, 7–9] for some examples.

We remark that the moderate growth property for a certain type of modular form is absolutely crucial if one wants to use general results from the theory of automorphic forms to study these objects (as we did in our recent paper [6] in a certain case). It appears that a proof of the moderate growth property, while probably known to experts, has not been formally written down in the setting of nearly holomorphic vector-valued forms. In this short note, we fill this gap in the literature.

Consider a nearly holomorphic vector-valued modular form F of highest weight $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ with respect to a congruence subgroup. The function F takes values in a finite dimensional vector space V. For $v \in V$, denote $\|v\| = \langle v, v\rangle^{1/2}$ where we fix a $U(n)$-invariant inner product on V. We can lift F to a V-valued function $\vec{\Phi}_F$ on $\mathrm{Sp}_{2n}(\mathbb{R})$. For any linear functional $\mathcal{L}$ on V consider the complex valued function $\Phi_F = \mathcal{L} \circ \vec{\Phi}_F$ on $\mathrm{Sp}_{2n}(\mathbb{R})$. We prove the following result.

Theorem 1.1 *The function Φ_F defined above has the moderate growth property.*

The above theorem is a direct consequence of the following bound.

Theorem 1.2 *For any nearly holomorphic vector-valued modular form F as above, there is a constant C (depending only on F) such that for all $Z = X + iY \in \mathbb{H}_n$ we have*

$$\|\rho(Y^{1/2})F(Z)\| \leq C\prod_{i=1}^{n}(\mu_i(Y)^{\lambda_1/2} + \mu_i(Y)^{-\lambda_1/2}).$$

The proof of Theorem 1.2, as we will see, is elementary. It uses nothing other than the existence of a Fourier expansion, and is essentially a straightforward extension of arguments that have appeared in the classical case, e.g., in [5] or [10]. This argument is very flexible and can be modified to provide a bound for Siegel-Maass forms. With some additional work (which we do not do here), Theorem 1.2 can be used to derive a bound on the Fourier coefficients of F. We also remark that the bound in Theorem 1.2 can be substantially improved if F is a cusp form.

Notations

For a positive integer n and a commutative ring R, let $M_n^{\mathrm{sym}}(R)$ be the set of symmetric $n \times n$ matrices with entries in R. For $X, Y \in M_n^{\mathrm{sym}}(\mathbb{R})$, we write $X > Y$ if $X - Y$ is positive definite. Let $\mathbb{H}_n$ be the Siegel upper half space of degree n, i.e., the set of $Z = X + iY \in M_n^{\mathrm{sym}}(\mathbb{C})$ whose imaginary part Y is positive definite. For such Z and $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})$, let $J(g, Z) = CZ + D$.

For any complex matrix X we denote by X^* its transpose conjugate. For positive definite $Y = (y_{ij}) \in M_n^{\mathrm{sym}}(\mathbb{R})$, let $\|Y\| = \max_{i,j} |y_{ij}|$. We denote by $\mu_i(Y)$ the i-th eigenvalue of Y, in decreasing order.

2 Nearly Holomorphic Functions and Fourier Expansions

For a non-negative integer p, we let $N^p(\mathbb{H}_n)$ denote the space of all polynomials in the entries of Y^{-1} with total degree $\leq p$ and with holomorphic functions on $\mathbb{H}_n$ as coefficients. The space

$$N(\mathbb{H}_n) = \bigcup_{p \geq 0} N^p(\mathbb{H}_n)$$

is the space of *nearly holomorphic functions* on $\mathbb{H}_n$.

It will be useful to have some notation for polynomials in matrix entries. Let

$$R_n = \{(i,j) : 1 \leq i \leq j \leq n\}.$$

Let

$$T_n^p = \{\mathbf{b} = (b_{i,j}) \in \mathbb{Z}^{R_n} : b_{i,j} \geq 0, \sum_{(i,j) \in R_n} b_{i,j} \leq p\}.$$

For any $V = (v_{i,j}) \in M_n^{\text{sym}}(\mathbb{R})$, and any $\mathbf{b} \in T_n^p$, we define $[V]^{\mathbf{b}} = \prod_{(i,j)\in R_n} v_{i,j}^{b_{i,j}}$. In particular, a function F on $\mathbb{H}_n$ lies in $N^p(\mathbb{H}_n)$ if and only if there are holomorphic functions $G_{\mathbf{b}}$ on $\mathbb{H}_n$ such that

$$F(Z) = \sum_{\mathbf{b}\in T_n^p} G_{\mathbf{b}}(Z)[Y^{-1}]^{\mathbf{b}}.$$

Definition 2.1 For any $\delta > 0$, we define

$$V_\delta = \{Y \in M_n^{\text{sym}}(\mathbb{R}) : Y \geq \delta I_n\}.$$

Lemma 2.2 *Given any $Y \in V_\delta$, we have $\|Y^{-1}\| \leq \delta^{-1}$.*

Proof Note that for any positive definite matrix $Y' = (y'_{ij})$ we have $\|Y'\| = \max_{i,j} |y'_{ij}| = \max_i y'_{ii}$. This is an immediate consequence of the fact that each 2×2 principal minor has positive determinant and each diagonal entry is positive.

So it suffices to show that each diagonal entry of Y^{-1} is less than or equal to δ^{-1}. But the assumption $Y \geq \delta I_n$ implies that $Y^{-1} \leq \delta^{-1} I_n$, which implies the desired fact above. □

An immediate consequence of this lemma is that for any $\delta \leq 1$, $Y \in V_\delta$ and $\mathbf{b} \in T_n^p$, we have $|[Y^{-1}]^{\mathbf{b}}| \leq \delta^{-p}$.

Definition 2.3 We say that $F \in N^p(\mathbb{H}^n)$ has a nice Fourier expansion if there exists an integer N and complex numbers $a_{\mathbf{b}}(F, S)$ for all $0 \leq S \in \frac{1}{N} M_n^{\text{sym}}(\mathbb{Z})$, such that we have an expression

$$F(Z) = \sum_{\mathbf{b}\in T_n^p} \sum_{\substack{S\in \frac{1}{N}M_n^{\text{sym}}(Z) \\ S\geq 0}} a_{\mathbf{b}}(F, S) e^{2\pi i \text{Tr}(SZ)} [Y^{-1}]^{\mathbf{b}}$$

that converges absolutely and uniformly on compact subsets of $\mathbb{H}_n$.

Note that a key point in the above definition is that the sum is taken only over positive semidefinite matrices. The next proposition, which is well-known in the holomorphic case, shows that this implies a certain boundedness property for the function F.

Proposition 2.4 *Let $F \in N^p(\mathbb{H}^n)$ have a nice Fourier expansion. Then for any $\delta > 0$, the function $F(Z)$ is bounded in the region $\{Z = X + iY : Y \in V_\delta\}$.*

Proof We may assume that $\delta \leq 1$. By the notion of a nice Fourier expansion, for each $\mathbf{b} \in T_n^p$, the series

$$R_{\mathbf{b}}(Y) := \sum_{\substack{S\in \frac{1}{N}M_n^{\text{sym}}(Z) \\ S\geq 0}} |a_{\mathbf{b}}(F, S)| e^{-2\pi \text{Tr}(SY)}$$

converges for any $0 < Y \in M_n^{\mathrm{sym}}(\mathbb{R})$. For any Z in the given region, using Lemma 2.2, we get

$$|F(Z)| \leq \sum_{\mathbf{b} \in T_n^p} R_{\mathbf{b}}(Y) \delta^{-p}, \tag{1}$$

and so to prove the proposition it suffices to show that each $R_{\mathbf{b}}(Y)$ is bounded in the region $Y \geq \delta$. By positivity, we have

$$|a_{\mathbf{b}}(F, S)\, e^{-2\pi\, \mathrm{Tr}(SY)}| \leq R_{\mathbf{b}}(Y)$$

for each $Y > 0$ and each $S \in \frac{1}{N} M_n^{\mathrm{sym}}(\mathbb{Z})$. Therefore

$$|a_{\mathbf{b}}(F, S)| \leq R_{\mathbf{b}}(\delta I_n/2) e^{\delta\pi\, \mathrm{Tr}(S)}. \tag{2}$$

Next, note that if $Y \geq \delta I_n$, then $\mathrm{Tr}(SY) \geq \delta \mathrm{Tr}(S)$ for all $S \geq 0$. To see this, we write $Y = \delta I_n + Y_1^2$ where $Y_1 \geq 0$ is the square-root of $Y - \delta I_n$. As $S \geq 0$ we have $\mathrm{Tr}(Y_1 S Y_1) \geq 0$ and consequently $\mathrm{Tr}(SY) = \mathrm{Tr}(S\delta I_n) + \mathrm{Tr}(Y_1 S Y_1) \geq \mathrm{Tr}(S\delta I_n)$.

Using the above and (2), we have for all $Y \geq \delta I_n$

$$R_{\mathbf{b}}(Y) \leq R_{\mathbf{b}}(\delta I_n/2) \sum_{0 \leq S \in \frac{1}{N} M_n^{\mathrm{sym}}(\mathbb{Z})} e^{-\delta\pi\, \mathrm{Tr}(S)}.$$

As the sum $\sum_{0 \leq S \in \frac{1}{N} M_n^{\mathrm{sym}}(\mathbb{Z})} e^{-\delta\pi\, \mathrm{Tr}(S)}$ converges to a finite value (for a proof of this fact, see [5, p. 185]) this completes the proof that $R_{\mathbf{b}}(Y)$ is bounded in the region $Y \geq \delta I_n$. □

3 Bounding Nearly Holomorphic Vector-Valued Modular Forms

Let (ρ, V) be an irreducible, finite-dimensional, rational representation of $\mathrm{GL}_n(\mathbb{C})$ and $\langle, \rangle$ be a (unique up to multiples) $U(n)$-invariant inner product on V. In fact, the inner product $\langle, \rangle$ has the property that

$$\langle \rho(M) v_1, v_2 \rangle = \langle v_1, \rho(M^*) v_2 \rangle$$

for all $M \in \mathrm{GL}_n(\mathbb{C})$. (To see this, note that it's enough to check it on the Lie algebra level. It's true for the real subalgebra $u(n)$ and so by linearity it follows for all of $\mathrm{gl}(n, \mathbb{C})$.) For any $v \in V$, we define

$$\|v\| = \langle v, v \rangle^{1/2}.$$

As is well known, the representation ρ has associated to it an n-tuple $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ of integers known as the highest weight of ρ. We let d_ρ denote the dimension of ρ.

We define a right action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on the space of smooth V-valued functions on $\mathbb{H}_n$ by

$$(F\big|_\rho g)(Z) = \rho(J(g,Z))^{-1}F(gZ) \qquad \text{for } g \in \mathrm{Sp}_{2n}(\mathbb{R}),\ Z \in \mathbb{H}_n. \tag{3}$$

A *congruence subgroup* of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is a subgroup that is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$ and contains a principal congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$. For a congruence subgroup Γ and a non-negative integer p, let $N_\rho^p(\Gamma)$ be the space of all functions $F : \mathbb{H}_n \to V$ with the following properties.

1. For any $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$ and any linear map $\mathcal{L} : V \to \mathbb{C}$, the function $\mathcal{L} \circ (F\big|_\rho g)$ lies in $N^p(\mathbb{H}_n)$ and has a nice Fourier expansion.
2. F satisfies the transformation property

$$F\big|_\rho \gamma = F \qquad \text{for all } \gamma \in \Gamma. \tag{4}$$

Let $N_\rho(\Gamma) = \bigcup_{p\geq 0} N_\rho^p(\Gamma)$. We refer to $N_\rho(\Gamma)$ as the space of *nearly holomorphic Siegel modular forms* of weight ρ with respect to Γ. We put $N_\rho^{(n)} = \bigcup_\Gamma N_\rho(\Gamma)$, the space of all nearly holomorphic Siegel modular forms of weight ρ.

Recall that for any $Y > 0$ in $M_n^{\mathrm{sym}}(\mathbb{R})$, we let $\mu_1(Y) \geq \mu_2(Y) \geq \ldots \geq \mu_n(Y) > 0$ denote the eigenvalues of Y. We can now state our main result.

Theorem 3.1 *For any $F \in N_\rho^{(n)}$, there is a constant C_F (depending only on F) such that for all $Z = X + iY \in \mathbb{H}_n$ we have*

$$\|\rho(Y^{1/2})F(Z)\| \leq C_F \prod_{i=1}^{n} (\mu_i(Y)^{\lambda_1/2} + \mu_i(Y)^{-\lambda_1/2}).$$

In order to prove this theorem, we will need a couple of lemmas.

Lemma 3.2 *For any $v \in V$, and any $Y > 0$ in $M_n^{\mathrm{sym}}(\mathbb{R})$, we have*

$$\left(\prod_{i=1}^{n} \mu_i(Y)^{\lambda_{n+1-i}}\right) \|v\| \leq \|\rho(Y)v\| \leq \left(\prod_{i=1}^{n} \mu_i(Y)^{\lambda_i}\right) \|v\|.$$

Proof This follows from considering a basis of weight vectors. Note that it is sufficient to prove the inequalities for Y diagonal as any Y can be diagonalized by a matrix in $U(n)$ and our norm is invariant by the action of $U(n)$. □

Next, we record a result due to Sturm.

Lemma 3.3 (Proposition 2 of [10]) *Suppose that* $\mathbb{F}$ *is a fundamental domain for* $\mathrm{Sp}_{2n}(\mathbb{Z})$ *such that there is some* $\delta > 0$ *such that* $Z = X + iY \in \mathbb{F}$ *implies that* $Y \in V_\delta$*. Let* $\phi : \mathbb{H}_n \to \mathbb{C}$ *be any function such that there exist constants* $c_1 > 0$*,* $\lambda \geq 0$ *with the property that* $|\phi(\gamma Z)| \leq c_1 \det(Y)^\lambda$ *for all* $Z \in \mathbb{F}$ *and* $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$*. Then for all* $Z \in \mathbb{H}_n$ *we have the inequality*

$$|\phi(Z)| \leq c_\phi \prod_{i=1}^{n} (\mu_i(Y)^\lambda + \mu_i(Y)^{-\lambda}).$$

Proof of Theorem 3.1 Let F be as in the statement of the theorem, so that $F \in N_\rho(\Gamma)$ for some $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$. We let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be a set of representatives for $\Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{Z})$. Fix an orthonormal basis $v_1, v_2, \ldots, v_d$ of V and for any $G \in N_\rho^{(n)}$ define $G_i(Z) := \langle G(Z), v_i \rangle$. Note that $\|G(Z)\| = \left(\sum_{i=1}^d |G_i(Z)|^2\right)^{1/2}$.

Let $\mathbb{F}$ be as in Lemma 3.3. By Proposition 2.4, it follows that there is a constant C depending on F such that $|(F|_\rho \gamma_r)_i(Z)| \leq C$ for all $1 \leq r \leq t$, $1 \leq i \leq n$, and $Z \in \mathbb{F}$. Moreover, for any $Z = X + iY \in \mathbb{F}$ we have each $\mu_j(Y^{1/2}) \geq \delta^{1/2}$ and therefore $\left(\prod_{j=1}^n \mu_j(Y^{1/2})^{\lambda_j}\right) \leq \det(Y)^{\lambda_1/2} \delta^{\frac{1}{2}\sum_{j=2}^n (\lambda_j - \lambda_1)}$. Now consider the function $\phi(Z) = \|\rho(Y^{1/2})F(Z)\|$. For any $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$, there exists $\gamma_0 \in \Gamma$ and some $1 \leq r \leq t$ such that $\gamma = \gamma_0 \gamma_r$. An easy calculation shows that

$$\|\phi(\gamma Z)\| = \|\rho(Y^{1/2})(F|_\rho \gamma_r)(Z)\|.$$

So for all $Z \in \mathbb{F}$, $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ we have, using Lemma 3.2 and the above arguments,

$$\|\phi(\gamma Z)\| \leq \det(Y)^{\lambda_1/2} \delta^{\frac{1}{2}\sum_{i=2}^n (\lambda_i - \lambda_1)} d^{1/2} C.$$

So the conditions of Lemma 3.3 hold with $\lambda = \lambda_1/2$. This concludes the proof of Theorem 3.1. □

Corollary 3.4 *For any* $F \in N_\rho^{(n)}$*, there is a constant* C_F *(depending only on* F*) such that for all* $Z = X + iY \in \mathbb{H}_n$ *we have*

$$\|\rho(Y^{1/2})F(Z)\| \leq C_F \, (1 + \mathrm{Tr}(Y))^{n\lambda_1} \, (\det Y)^{-\lambda_1/2}.$$

Proof This follows immediately from Theorem 3.1 and the following elementary inequality, which holds for all positive integers λ, n and all positive reals $y_1, \ldots, y_n$:

$$\prod_{i=1}^{n} (1 + y_i^\lambda) \leq (1 + y_1 + \ldots + y_n)^{n\lambda}.$$

To prove the above inequality, note that $1 + y_i^\lambda \leq (1 + y_1 + \ldots + y_n)^\lambda$ for each i. Now take the product over $1 \leq i \leq n$. □

4 The Moderate Growth Property

Given any $F \in N_\rho^{(n)}$, we define a smooth function $\vec{\Phi}_F$ on $\mathrm{Sp}_{2n}(\mathbb{R})$ by the formula

$$\vec{\Phi}_F(g) = \rho(J(g, I))^{-1} F(gI),$$

where $I := iI_{2n}$.

Proposition 4.1 *Let $F \in N_\rho^{(n)}$ and $\vec{\Phi}_F$ be defined as above. Then there is a constant C such that for all $Z = X + iY \in \mathbb{H}_n$ we have*

$$\left\| \vec{\Phi}_F \left(\begin{bmatrix} Y^{1/2} & XY^{1/2} \\ & Y^{-1/2} \end{bmatrix} \right) \right\| \le C \prod_{i=1}^{n} (\mu_i(Y)^{\lambda_1/2} + \mu_i(Y)^{-\lambda_1/2}).$$

Proof This follows immediately from Theorem 3.1. □

A complex-valued function Φ on $\mathrm{Sp}_{2n}(\mathbb{R})$ is said to be slowly increasing if there is a constant C and a positive integer r such that

$$|\Phi(g)| \le C(\mathrm{Tr}(g^* g))^r$$

for all $g \in \mathrm{Sp}_{2n}(\mathbb{R})$.

Theorem 4.1 *Let $F \in N_\rho^{(n)}$ and $\vec{\Phi}_F$ be as defined above. For some linear functional $\mathcal{L}$ on V, let $\Phi_F = \mathcal{L} \circ \vec{\Phi}_F$. Then the function Φ_F has the moderate growth property.*

Proof Note that $|\Phi_F(g)| \le \|\mathcal{L}\| \, \|\vec{\Phi}_F(g)\|$. So it suffices to show that there is a constant C and a positive integer r such that

$$\|\vec{\Phi}_F(g)\| \le C(\mathrm{Tr}(g^* g))^r \tag{5}$$

for all $g \in \mathrm{Sp}_{2n}(\mathbb{R})$. Since both sides of this inequality do not change when g is replaced by gk, where k is in the standard maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$, we may assume that g is of the form $\begin{bmatrix} Y^{1/2} & XY^{1/2} \\ & Y^{-1/2} \end{bmatrix}$. Then the existence of appropriate C and r follows easily from Proposition 4.1. Indeed, we can take any $r \ge n\lambda_1/2$ and C the same constant as in Proposition 4.1. □

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Critical Values of L-Functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a Totally Real Field

A. Raghuram and Gunja Sachdeva

Abstract We prove an algebraicity result for all the critical values of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a totally real field F, which we derive from the theory of Rankin–Selberg L-functions attached to pairs of automorphic representations on $\mathrm{GL}_3 \times \mathrm{GL}_2$. This is a generalization and refinement of the results of Mahnkopf (J. Reine Angew. Math. 497:91–112, 1998) and Geroldinger (Ramanujan J. 38(3):641–682, 2015).

1 Introduction and Statement of the Main Theorem

To describe the main theorem proved in this paper in greater detail, we need some notations. Suppose $\mathbb{A}_F$ is the ring of adèles of F. Given a regular algebraic cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$, one knows (from Clozel [8]) that there is a pure dominant integral weight μ such that Π has a nontrivial contribution to the cohomology of some locally symmetric space of GL_3 with coefficients coming from the finite-dimensional representation $\mathcal{M}_\mu$ with highest weight μ. We denote this as $\Pi \in \mathrm{Coh}(G_3, \mu)$, for $\mu \in X_0^+(T_3)$, where T_3 is the diagonal torus of $G_3 = \mathrm{GL}_3/F$. Let $\Pi = \Pi_\infty \otimes \Pi_f$ be the decomposition of Π into its archimedean part Π_∞ and its finite part Π_f. The representation $\mathcal{M}_\mu$ is defined over a number field $\mathbb{Q}(\mu)$, and by Clozel [8], it is known that cuspidal cohomology has a $\mathbb{Q}(\mu)$-structure; hence the realization of Π_f as a Hecke-summand in cuspidal cohomology (in lowest possible degree) has a $\mathbb{Q}(\Pi)$-structure, for a number field $\mathbb{Q}(\Pi)$ known as the rationality field of Π. On the other hand, the Whittaker model $\mathcal{W}(\Pi_f)$ of the finite part of the representation admits a $\mathbb{Q}(\Pi)$-structure. By comparing these two $\mathbb{Q}(\Pi)$-structures, we get certain periods $p^{\epsilon_\Pi}(\Pi) \in \mathbb{C}^\times$; see, for example, [23]. Here $\epsilon_\Pi = (\epsilon_v)_{v \in S_\infty}$ is a collection of signs indexed by the set S_∞ of real places of F, and for GL_3/F, we know that Π_∞ uniquely determines ϵ_Π. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, one knows that ${}^\sigma\Pi \in \mathrm{Coh}(G_3, {}^\sigma\mu)$ and one can define periods simultaneously for all ${}^\sigma\Pi$. The reader is

A. Raghuram (✉) • G. Sachdeva
Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
e-mail: raghuram@iiserpune.ac.in; gunja.sachdeva@students.iiserpune.ac.in

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referred to [10, Sect. 3] for a review of such results. Henceforth, let $\mu \in X_0^+(T_3)$ stand for a dominant integral pure weight and consider $\Pi \in \mathrm{Coh}(G_3, \mu)$. The main theorem of this article is the following:

Theorem 1.1 *Let $\Pi \in \mathrm{Coh}(G_3, \mu)$ with $\varepsilon_{\Pi_v} = 1\!\!1$ for all $v \in S_\infty$ (see Proposition 2.1 for ε_{Π_v}), and let $\mu \in X_0^+(T_3)$ such that for each $\mu = (\mu_v)_{v \in S_\infty}$, $\mu_v = (n_v, 0, -n_v)$ with n_v a non-negative integer. Put $n = \min\{n_v\}$. Let $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a character of finite order, and define $\mathbb{Q}(\chi) := \mathbb{Q}\,(\{\text{values of } \chi\})$. Suppose that $m \in \mathbb{Z}$ is critical for $L_f(s, \Pi \otimes \chi)$, the finite part of the standard degree-3 L-function attached to Π and χ. Then*

$$m \in \begin{cases} \{1 - n_{ev}, \ldots, -3, -1;\ 2, 4, \ldots, n_{ev}\}, & \text{if } \chi \text{ is totally even,} \\ \{1 - n_{od}, \ldots, -4, -2, 0;\ 1, 3, \ldots, n_{od}\}, & \text{if } \chi \text{ is totally odd,} \end{cases}$$

where $n_{ev} = 2\left[\frac{n+1}{2}\right]$ = the largest even positive integer less than or equal to $n + 1$, and $n_{od} = 2\left[\frac{n}{2}\right] + 1$ = the largest odd positive integer less than or equal to $n + 1$. (If χ is even at one place and odd at another place then there are no critical points.) Fix a quadratic totally odd character ξ once and for all (which will be relevant only when χ is totally odd). Consider the four cases:

Case 1a. ***χ is totally even and*** *$m \in \{2, 4, \cdots, n_{ev}\}$. Define $\Omega_r^+(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(-1, \Pi)^{-1}$. There exists a nonzero complex number $P_\infty^1(\mu, m)$ depending only the weight μ and the critical point m such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^1(\mu, m)\, \Omega_r^+(\Pi)\, \mathcal{G}(\chi)^2,$$

where, by $\approx_{\mathbb{Q}(\Pi,\chi)}$, we mean up to an element of the number field which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\chi)$; and $\mathcal{G}(\chi)$ is the Gauß sum of χ.

Case 1b. ***χ is totally even and*** *$m \in \{1 - n_{ev}, \cdots, -3, -1\}$. Define $\Omega_l^+(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(2, \Pi)^{-1}$. There exists a nonzero complex number $P_\infty^2(\mu, m)$ such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^2(\mu, m)\, \Omega_l^+(\Pi)\, \mathcal{G}(\chi).$$

Case 2a. ***χ is totally odd and*** *$m \in \{1, 3, \cdots, n_{od}\}$. Define $\Omega_r^-(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(0, \Pi \otimes \xi)^{-1}$. There exists a nonzero complex number $P_\infty^3(\mu, m)$ such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^3(\mu, m)\, \Omega_r^-(\Pi)\, \mathcal{G}(\chi)^2 \mathcal{G}(\xi).$$

Case 2b. *χ **is totally odd and** $m \in \{1 - n_{od}, \cdots, -4, -2, 0\}$. Define $\Omega_l^-(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(1, \Pi \otimes \xi)^{-1}$. There exists a nonzero complex number $P_\infty^4(\mu, m)$ such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^4(\mu, m)\, \Omega_l^-(\Pi)\, \mathcal{G}(\chi).$$

Moreover, in each of the cases, the ratio of the L-value on the left hand side divided by all the quantities in the right hand side is equivariant for the action of $\mathrm{Aut}(\mathbb{C})$.

For $F = \mathbb{Q}$, $\mu = 0$ and $m = 1$, the case 2a above is the main rationality result in Mahnkopf [18]; and for $F = \mathbb{Q}$ and general μ, a weak form of the above theorem is implicit in the construction of the p-adic L-functions in Geroldinger [12]. Let's mention *en passant* that if $n = 0$ and χ is totally even, then there are no critical points. The proof of this theorem, following [18], is based on an integral representation for the value $L_f(m, \Pi \times \chi)$ which we derive from the Rankin–Selberg theory of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$, by taking Π on GL_3 and an induced representation on GL_2. To this end, let χ_1, χ_2 be two distinct idèle class characters defined as $\chi_i = |\ |^{d_i} \chi_i^\circ$, where χ_i° is any finite order character and define $\Sigma(\chi_1, \chi_2) := \mathrm{Ind}_{B_2(\mathbb{A}_F)}^{\mathrm{GL}_2(\mathbb{A}_F)}(\chi_1 |\ |^{1/2}, \chi_2 |\ |^{-1/2})$. Also assume that the representations are such that $s = 1/2$ is critical for the Rankin-Selberg L-function attached to $\Pi \times \Sigma(\chi_1, \chi_2)$. We note that

$$L(s, \Pi \times \Sigma(\chi_1, \chi_2)) = L(s + 1/2, \Pi \otimes \chi_1) L(s - 1/2, \Pi \otimes \chi_2).$$

Using results from [18] and [20], we can arrange for the data d_1, d_2, χ_1° and χ_2° (Proposition 3.6) so as to afford an interpretation of the critical L-value $L(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))$ as a Poincaré pairing between (the pull-back to GL_2) of a cuspidal cohomology class $\vartheta^\circ_{\Pi,\epsilon_\Pi}$ for Π and an Eisenstein cohomology class ϑ°_Σ for $\Sigma(\chi_1, \chi_2)$. See Theorem 3.29. Now we freeze one of the characters χ_1, χ_2, and let the other vary, to capture all the critical values $L(m, \Pi \otimes \chi)$. In section "The Main Identity for the Critical Values $L_f(m, \Pi \otimes \chi)$", for the each of the four cases above, we express $L(m, \Pi \otimes \chi)$ in terms of certain periods and the Poincaré pairing of $\vartheta^\circ_{\Pi,\epsilon_\Pi}$ and ϑ°_Σ, from which we deduce the required algebraicity result in Sect. 3.4.

Let's now briefly address the compatibility with motivic periods and motivic L-functions. Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(M)$. Suppose M is critical, then a celebrated conjecture of Deligne [9, Conj. 2.8] relates the critical values of its L-function $L(s, M)$ to certain periods that arise out of a comparison of the Betti and de Rham realizations of the motive. One expects a cohomological cuspidal automorphic representation Π to correspond to a motive $M(\Pi)$; one of the properties of this correspondence is that the standard L-function $L(s, \Pi)$ is the motivic L-function $L(s, M(\Pi))$ up to a shift in the s-variable; see Clozel [8, Sect. 4]. With the current state of technology, it seems impossible to compare our periods $p^\epsilon(\Pi)$ with Deligne's periods $c^\pm(M(\Pi))$. Be that as it may, one can still claim that Theorem 1.1 is compatible with Deligne's conjecture by considering the behavior of L-values under twisting by characters. Blasius

[2] and Panchishkin [19] have independently studied the behavior of $c^{\pm}(M(\Pi))$ upon twisting the motive $M(\Pi)$ by a Dirichlet character (more generally by Artin motives). Using Deligne's conjecture, they predict the behavior of critical values of motivic L-functions upon twisting by Dirichlet characters. This takes the following form in our situation which we state only when the twisting character is totally even:

Corollary 1.2 *Let* $\Pi \in \mathrm{Coh}(G_3, \mu)$ *and* $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ *be of finite order and which is totally even. If the critical point m is to the right of the center of symmetry then*

$$L_f(m, \Pi \otimes \chi) \;\approx\; L_f(m, \Pi)\mathcal{G}(\chi)^2,$$

but if the critical point m is to the left of the center of symmetry then we have

$$L_f(m, \Pi \otimes \chi) \;\approx\; L_f(m, \Pi)\mathcal{G}(\chi).$$

In both the cases the ratio is $\mathrm{Aut}(\mathbb{C})$*-equivariant.*

From the above relation between critical values for twisted L-functions with the corresponding values of the untwisted L-functions we may claim that our result is compatible with Deligne's conjecture. See also [22, Sect. 7] where such relations for twisted critical values are conjectured for symmetric power L-functions of a modular form.

Finally, we briefly discuss the case of symmetric square L-function for a Hilbert modular form. Let φ be a holomorphic cuspidal Hilbert modular form over F of weight $(k_1, \ldots, k_d)$. Suppose that all the k_j have the same parity, and that φ is not of CM-type. Then Theorem 1.1 applies to the symmetric square L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ attached to φ, twisted by a finite order Dirichilet character χ, by thinking of this L-function as the standard L-function of the symmetric-square transfer of φ—which is a cohomological cuspidal representation of G_3—twisted by χ. See Sect. 3.5.

2 Some Preliminaries

2.1 Notations

The Base Field

Let F denote a totally real number field of degree d_F, i.e., $[F : \mathbb{Q}] = d_F$, with ring of integers $\mathcal{O}$. For any place v we write F_v for the topological completion of F at v. Let S_∞ denote the set of all real places; hence $d_F = |S_\infty|$. Also the set of all real embeddings of F as a field into $\mathbb{C}$ is denoted by $\mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \mathbb{R})$. There is a canonical bijective map $\mathrm{Hom}(F, \mathbb{R}) \to S_\infty$, and for each $v \in S_\infty$, we fix an isomorphism $F_v \cong \mathbb{R}$. Further, if $v \notin S_\infty$, and $\mathfrak{p}$ denotes the prime ideal of

$\mathcal{O}$ corresponding to v, then we let $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$, and $\mathcal{O}_\mathfrak{p}$ the ring of integers of $F_\mathfrak{p}$. Sometimes, F_v is used for $F_\mathfrak{p}$ and similarly $\mathcal{O}_v$ for $\mathcal{O}_\mathfrak{p}$. The unique maximal ideal of $\mathcal{O}_\mathfrak{p}$ is $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ and is generated by a uniformizer $\varpi_\mathfrak{p}$. Let $\mathfrak{D}_F$ denote the absolute different of F, i.e., $\mathfrak{D}_F^{-1} = \{x \in F : T_{F/\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z}\}$. For any prime ideal $\mathfrak{p}$ of F define $r_\mathfrak{p} \geq 0$ by: $\mathfrak{D}_F = \prod_\mathfrak{p} \mathfrak{p}^{r_\mathfrak{p}}$. Let $\mathbb{A}_F$ stand for its adèle ring, with $\mathbb{A}_{Ff}$ and $\mathbb{A}_F^\times$ the ring of finite adèles and group of idèles, respectively. For brevity, let $\mathbb{A} := \mathbb{A}_\mathbb{Q}$ and $\mathbb{A}^\times := \mathbb{A}_\mathbb{Q}^\times$.

The Groups and Their Lie Algebras

The algebraic group GL_n/F will be denoted as $\underline{G}_n$, and we put $G_n = R_{F/\mathbb{Q}}(\underline{G}_n)$. An F-group will be denoted by an underline and the corresponding $\mathbb{Q}$-group via Weil restriction of scalars will be denoted without the underline; hence for any $\mathbb{Q}$-algebra A the group of A-points of G_n is $G_n(A) = \underline{G}_n(A \otimes_\mathbb{Q} F)$. Let $\underline{B}_n = \underline{T}_n\underline{U}_n$ stand for the standard Borel subgroup of $\underline{G}_n$ of all upper triangular matrices, where $\underline{U}_n$ is the unipotent radical of $\underline{B}_n$, and $\underline{T}_n$ the diagonal torus. The center of $\underline{G}_n$ will be denoted by $\underline{Z}_n$. These groups define the corresponding $\mathbb{Q}$-groups $G_n \supset B_n = T_nU_n \supset Z_n$. Observe that Z_n is not $\mathbb{Q}$-split (if $d_F > 1$), and we let S_n be the maximal $\mathbb{Q}$-split torus in Z_n; we have $S_n \cong \mathbb{G}_m$ over $\mathbb{Q}$.

Note that the group at infinity is

$$G_{n,\infty} := G_n(\mathbb{R}) = \prod_{v\in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v\in S_\infty} \mathrm{GL}_n(\mathbb{R}).$$

Suppose $C_{n,\infty} := \prod_{v\in S_\infty} \mathrm{O}(n)$ is the maximal compact subgroup of $G_n(\mathbb{R})$. Let $K_{n,\infty} = C_{n,\infty}Z_n(\mathbb{R})$. Let $K_{n,\infty}^0$ be the topological connected component of $K_{n,\infty}$. For any topological group $\mathfrak{G}$, we will let $\pi_0(\mathfrak{G}) := \mathfrak{G}/\mathfrak{G}^0$ stand for the group of connected components. We will identify $\pi_0(G_{n,\infty}) = \pi_0(K_{n,\infty}) \cong \prod_{v\in S_\infty}\{\pm 1\} = \prod_{v\in S_\infty}\{\pm\}$. Furthermore, we identify $\pi_0(G_n(\mathbb{R}))$ inside $G_n(\mathbb{R})$ via the $\delta_n' s$ where the matrix $\delta_n = \mathrm{diag}(-1, 1, \ldots, 1)$ represents the nontrivial element in $\mathrm{O}(n)/\mathrm{SO}(n)$. The character group of $\pi_0(K_{n,\infty})$ is denoted by $\widehat{\pi_0(K_{n,\infty})}$. For a real Lie group G, we denote its Lie algebra by $\mathfrak{g}^0$ and the complexified Lie algebra by $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{g}^0 \otimes_\mathbb{R} \mathbb{C}$. Thus, for example if G is the Lie group $\mathrm{GL}_n(\mathbb{R})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. We have $\mathfrak{g}_{n,\infty}$ and $\mathfrak{k}_{n,\infty}$ denote the complexified Lie algebras of $G_{n,\infty}$ and $K_{n,\infty}$, respectively.

Finite-Dimensional Representations

Consider $T_{n,\infty} = \prod_{v\in S_\infty} T_n(F_v) \cong \prod_{v\in S_\infty} T_n(\mathbb{R})$. Let $X^*(T_n) = X^*(T_{n,\infty})$ be the group of all algebraic characters of $T_{n,\infty}$, and let $X^+(T_n) = X^+(T_{n,\infty})$ be the subset of $X^*(T_n)$ which are dominant with respect to Borel subgroup B_n. A weight $\mu \in X^+(T_{n,\infty})$ is of the form: $\mu = (\mu_v)_{v\in S_\infty}$ such that for $v \in S_\infty$ we have $\mu_v = (\mu_1^v, \ldots, \mu_n^v)$, $\mu_i^v \in \mathbb{Z}$, $\mu_1^v \geq \ldots \geq \mu_n^v$; the character μ_v sends $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n(F_v)$ to $\prod_i t_i^{\mu_i^v}$. Further, if there is an integer $\mathrm{w}(\mu)$ such that for all $v \in S_\infty$ and any $1 \leq i \leq n$ we have $\mu_i^v + \mu_{n-i+1}^v = \mathrm{w}(\mu)$, then we say μ is a *pure weight* and call $\mathrm{w}(\mu)$ the *purity weight* of μ; denote the set of dominant integral pure weights as $X_0^+(T_{n,\infty})$. For $\mu \in X^+(T_{n,\infty})$, we let $\mathcal{M}_\mu$ stand for the irreducible finite-dimensional complex representation of $G_{n,\infty}$ for highest weight μ. Since $G_{n,\infty} = \prod_{v \in S_\infty} \mathrm{GL}_n(\mathbb{R})$, it is clear that $\mathcal{M}_\mu = \otimes_v \mathcal{M}_{\mu_v}$ with $\mathcal{M}_{\mu_v}$ being the irreducible finite-dimensional representation of $\mathrm{GL}_n(\mathbb{R})$ of highest weight μ_v.

Automorphic Representations

An irreducible representation of $G_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_F)$ is said to be *automorphic*, following Borel–Jacquet [4], if it is isomorphic to an irreducible subquotient of the representation of $G_n(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is *cuspidal* if it is a subrepresentation of the representation of $G_n(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) = \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F))$. Let V_π be the subspace of cusp forms realizing a cuspidal automorphic representation π. For an automorphic representation π of $G_n(\mathbb{A})$, we have $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is a representation of $G_{n,\infty}$ and $\pi_f = \otimes_{v \notin S_\infty} \pi_v$ is a representation of $G_n(\mathbb{A}_f)$. The central character of π will be denoted ω_π.

The Choice of Measures

Fix a global measure dg on $G_n(\mathbb{A})$, which is a product of local measures dg_v such that for a finite place v, we normalize dg_v by asking $\mathrm{vol}(\mathrm{GL}_n(\mathcal{O}_v)) = 1$, and at every infinite place we ask $\mathrm{vol}(C_{n,v}^0) = \mathrm{vol}(\mathrm{SO}(n)) = 1$.

Adèlic Characters and Gauss Sums

Fix an additive character $\psi_\mathbb{Q}$ of $\mathbb{Q}\backslash\mathbb{A}$, as in Tate's thesis, we define a character ψ of $F\backslash\mathbb{A}_F$ as: $\psi = \psi_\mathbb{Q} \circ Tr_{F/\mathbb{Q}}$. For a Hecke character χ of F, by which we mean a continuous homomorphism $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$, it's Gauß sum is denoted $\mathcal{G}(\chi_f)$ or even $\mathcal{G}(\chi)$ and is defined as in [21, Sect. 2.1].

2.2 Cuspidal Cohomology

For any open compact subgroup $K_f \subset G_n(\mathbb{A}_f)$ define a locally symmetric space

$$S_n(K_f) := G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K_{n,\infty}^0 K_f = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/K_{n,\infty}^0 K_f.$$

Let $\mu \in X^+(T_{n,\infty})$. The representation $\mathcal{M}_\mu$ defines a sheaf $\widetilde{\mathcal{M}}_\mu$ on $S_n(K_f)$. We are interested in the sheaf cohomology groups $H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_\mu)$. It is convenient to pass to the limit over all open compact subgroups K_f and define $H^\bullet(S_n, \widetilde{\mathcal{M}}_\mu) := \varinjlim_{K_f} H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_\mu)$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^\bullet(S_n, \mathcal{M}_\mu)$, called a Hecke action, and we can recover the cohomology of $S_n(K_f)$ by taking invariants: $H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_\mu) = H^\bullet(S_n, \widetilde{\mathcal{M}}_\mu)^{K_f}$. We can compute the sheaf cohomology groups via the de Rham complex, which upon reinterpreting in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S_n, \widetilde{\mathcal{M}}_\mu) \simeq H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \otimes \mathcal{M}_\mu).$$

With level structure K_f it takes the form:

$$H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_\mu) \simeq H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_\mu).$$

The inclusion $C^\infty_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [3], an injection in cohomology; this defines cuspidal cohomology:

$$H^\bullet_{\mathrm{cusp}}(S_n(K_f), \widetilde{\mathcal{M}}_\mu) \simeq H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_\mu).$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_{F,f})$-modules:

$$H^\bullet_{\mathrm{cusp}}(S_n, \widetilde{\mathcal{M}}_\mu) = \bigoplus_{\Pi} H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu) \otimes \Pi_f.$$

We say that Π contributes to the cuspidal cohomology of G_n with coefficients in $\mathcal{M}_\mu$ if Π has a nonzero contribution to the above decomposition. Equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by $\mathcal{M}_\mu$ has nontrivial relative Lie algebra cohomology. In this situation, we write $\Pi \in \mathrm{Coh}(G_n, \mu)$. It is well-known [8] that only pure weights support cuspidal cohomology.

Similarly, define

$$\tilde{S}_n(K_f) := G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/C^0_{n,\infty}K_f = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/C^0_{n,\infty}K_f,$$

where, $C^0_{n,\infty}$ is the connected component of the identity of the maximal compact subgroup $C_{n,\infty}$ of $G_n(\mathbb{R})$. We get a canonical fibration ϕ given by:

$$\begin{array}{ccc} \tilde{S}_n(K_f) & = & G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/C^0_{n,\infty}K_f \\ & & \downarrow \phi \\ S_n(K_f) & = & G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K^0_{n,\infty}K_f \end{array}$$

We will also consider the cohomology groups $H^\bullet(\tilde{S}_n(K_f), \mathcal{M}_\mu)$.

Let $\iota : \mathrm{GL}_{n-1} \to \mathrm{GL}_n$ be the map $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$. Then ι induces a map at the level of local and global groups and appropriate locally symmetric spaces of G_{n-1} and G_n, all of which will also be denoted by ι again. The pullback (of a function or a cohomology class) via ι will be denoted by ι^*.

Cuspidal Cohomology GL_3

Now we specialize to $n = 3$ and briefly review some well-known details that will be relevant later on. (See [21] for more details and further references.) For any integer $\ell \geq 1$, let D_ℓ stand for the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with lowest non-negative SO(2)-type given by the character $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \exp^{-i(\ell+1)\theta}$, and central character $a \mapsto \mathrm{sgn}(a)^{\ell+1}$.

Suppose $\mu \in X_0^+(T_3)$ is a pure dominant integral weight written as $\mu = (\mu_v)_{v\in S_\infty}$ with $\mu_v = (\mu_1^v, \mu_2^v, \mu_3^v)$ and let $\mathsf{w} = \mu_1^v + \mu_3^v = 2\mu_2^v$ be the purity weight of μ. Note that w is an even integer. Let's write it as $\mathsf{w} = 2\mathsf{w}^\circ$. Suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$, then it is clear that $\Pi \otimes |\ |^{\mathsf{w}^\circ} \in \mathrm{Coh}(G_3, \mu - \mathsf{w}^\circ)$ because

$$H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes |\ |^{\mathsf{w}^\circ} \otimes \mathcal{M}_\mu \otimes (\det)^{-\mathsf{w}^\circ}) = H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu).$$

The purity weight of $\mu - \mathsf{w}^\circ$ is 0, and furthermore $\Pi \otimes |\ |^{\mathsf{w}^\circ}$ is a unitary cuspidal representation. As far as L-functions (and their special values) are concerned, we have not lost any information since $L(s, \Pi \otimes |\ |^{\mathsf{w}^\circ}) = L(s + \mathsf{w}^\circ, \Pi)$. We will henceforth assume:

1. μ is a pure dominant integral weight with purity weight 0; so $\mu = (\mu_v)_{v\in S_\infty}$ with $\mu_v = (n_v, 0, -n_v)$ for a non-negative integer n_v.
2. $\Pi \in \mathrm{Coh}(G_3, \mu)$, i.e., Π is a unitary cuspidal automorphic representation of GL_3/F that has nontrivial cohomology with respect to $\mathcal{M}_\mu$.

The following well-known proposition records some basic information about the relative Lie algebra cohomology groups in this context.

Proposition 2.1 *Let $\mu \in X_0^+(T_3)$ be a pure dominant integral weight with purity weight 0; we write $\mu = (\mu_v)_{v\in S_\infty}$ with $\mu_v = (n_v, 0, -n_v)$ for an integer $n_v \geq 0$.*

Put $\ell_v = 2n_v + 2$. *Suppose* $\Pi \in \mathrm{Coh}(G_3, \mu)$. *Then for every* $v \in S_\infty$ *we have*

$$\Pi_v \;=\; \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} \left(D_{\ell_v} \otimes \varepsilon_{\Pi_v}\right),$$

where, $P_{(2,1)}$ *is the standard parabolic subgroup of* $\mathrm{GL}_3(\mathbb{R})$ *with Levi quotient* $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$, *and* ε_{Π_v} *is a quadratic character of* $\mathbb{R}^\times$. *In terms of the central character, we have* $\varepsilon_{\Pi_v}(-1) = -\omega_{\Pi_v}(-1)$. *(We also write* $\varepsilon_{\Pi_v} = \mathrm{sgn}^{e_{\Pi_v}}$ *with* $e_{\Pi_v} \in \{0, 1\}$.*)*

Define $b_3^F = 2d_F = 2[F : \mathbb{Q}]$. *The smallest degree* $\bullet$ *for which* $H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu) \neq 0$ *is* $\bullet = b_3^F$, *and in this degree, the cohomology group is one-dimensional, and as a* $K_{3,\infty}/K^0_{3,\infty}$*-module we denote it by* ϵ_Π, *which is a d-tuple of signs:* $(\mathrm{sgn}^{1+e_{\Pi_v}})_{v \in S_\infty}$.

2.3 Eisenstein Cohomology of GL_2

This is well-known by Harder [14], however, we need some of the details to be cast in a form that is useful for us and so we briefly present the details.

For $i = 1, 2$, let $\chi_i : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be algebraic Hecke characters; then $\chi_i = |\ |^{d_i} \chi_i^\circ$, where χ_1° and χ_2° are of finite order and $d_1, d_2 \in \mathbb{Z}$. Clearly, for every archimedean v, we have $\chi_{iv}^\circ = \epsilon_{iv} = (\mathrm{sgn})^{e_{iv}}$ for some $e_{iv} \in \{0, 1\}$. Define the globally induced representation

$$\Sigma(\chi_1, \chi_2) := \mathrm{Ind}_{B_2(\mathbb{A})}^{G_2(\mathbb{A})}(\chi_1|\ |^{1/2}, \chi_2|\ |^{-1/2}),$$

which decomposes into a restricted tensor product $\Sigma(\chi_1, \chi_2) = \otimes'_v \Sigma(\chi_{1,v}, \chi_{2,v})$, where $\Sigma(\chi_{1,v}, \chi_{2,v})$ denotes the normalized parabolically induced representation $\mathrm{Ind}_{B_2(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1,v}|\ |_v^{1/2}, \chi_{2,v}|\ |_v^{-1/2})$ of $\mathrm{GL}_2(F_v)$. Let $\Sigma_f(\chi_1, \chi_2) := \otimes_{v\nmid\infty} \Sigma(\chi_{1,v}, \chi_{2,v})$ and $\Sigma_\infty(\chi_1, \chi_2) := \otimes_{v|\infty} \Sigma(\chi_{1,v}, \chi_{2,v})$ denote the finite and infinite part of $\Sigma(\chi_1, \chi_2)$, respectively. For simplicity, let $V_{\chi_v} := \Sigma(\chi_{1,v}, \chi_{2,v})$.

Let $\mathcal{M}_\lambda$ be a finite dimensional representation of $G_{2,\infty}$, with highest weight $\lambda = (\lambda_v)_{v \in S_\infty} \in X^+(T_2)$; then $\mathcal{M}_\lambda$ decomposes as $\mathcal{M}_\lambda = \otimes_{v|\infty} \mathcal{M}_{\lambda_v}$, where $\mathcal{M}_{\lambda_v}$ is the finite-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ with highest weight λ_v. If we write $\lambda_v = (\lambda_{v,1}, \lambda_{v,2})$, with integers $\lambda_{v,j}$, and $\lambda_{v,1} \geq \lambda_{v,2}$, then $\mathcal{M}_{\lambda_v} = \mathrm{Sym}^{\lambda_{v,1}-\lambda_{v,2}}(\mathbb{C}^2) \otimes \det^{\lambda_{v,2}}$. Hence, the dimension of $\mathcal{M}_{\lambda_v}$ is $\lambda_{v,1} - \lambda_{v,2} + 1$, and it's central character is $t \mapsto t^{\lambda_{v,1}+\lambda_{v,2}}$. For each $v \in S_\infty$, we want to find $(\lambda_{v,1}, \lambda_{v,2})$ in terms of $d_1, d_2, \epsilon_{1v}, \epsilon_{2v}$ such that

$$H^\bullet(\mathfrak{g}_{2,v}, K^\circ_{2,v}; V_{\chi_v} \otimes \mathcal{M}_{\lambda_v}) \neq 0,$$

and, furthermore, we need to determine the action of $\pi_0(K_{2,\infty})$ on these cohomology groups.

Cohomology of Some Representations of $\mathrm{GL}_2(\mathbb{R})$

Let's fix some notational convention: for a representation π of GL_n (in any suitable local or global context), and for a real number t, by $\pi(t)$ we mean $\pi \otimes |\,|^t$, where $|\,|$ is the normalized local (resp., global) absolute value (resp., adèlic norm). Also, we will abbreviate the normalized parabolically induced representation $\mathrm{Ind}_{B_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi_1 \otimes \chi_2)$ simply as $\chi_1 \times \chi_2$. For $a \in \mathbb{Z}$, let $\xi_a : \mathbb{R}^\times \to \mathbb{R}^\times \subset \mathbb{C}^\times$ be the character defined as $\xi_a(t) = t^a$.

Take two integers $a \geq b$. We start with the following exact sequence of $(\mathfrak{gl}_2, \mathrm{O}(2)\mathbb{R}_+^\times)$-modules:

$$0 \longrightarrow D_{a-b+1}(\tfrac{a+b}{2}) \longrightarrow \xi_a(1/2) \times \xi_b(-1/2) \longrightarrow \mathcal{M}_{(a,b)} \longrightarrow 0.$$

Twisting by the sign character, while noting that twisting commutes with induction, we get

$$0 \longrightarrow D_{a-b+1}(\tfrac{a+b}{2}) \longrightarrow \xi_a(\mathrm{sgn})(1/2) \times \xi_b(\mathrm{sgn})(-1/2) \longrightarrow \mathcal{M}_{(a,b)} \otimes \mathrm{sgn} \longrightarrow 0.$$

Note that the discrete series representation D_ℓ is invariant under twisting by sgn-character. For brevity, let $\nu := (a,b)$; $\nu^\vee = (-b,-a)$; $V_\nu := \xi_a(\mathrm{sgn})(1/2) \times \xi_b(\mathrm{sgn})(-1/2)$; $D_{\nu^\vee} = D_{a-b+1}(\frac{a+b}{2})$; and $\mathcal{M}_\nu^- := \mathcal{M}_\nu \otimes \mathrm{sgn}$. The above exact sequence may then be written as

$$0 \longrightarrow D_{\nu^\vee} \xrightarrow{\ i\ } V_\nu \longrightarrow \mathcal{M}_\nu^- \longrightarrow 0. \tag{2.1}$$

Tensor this sequence by $\mathcal{M}_{\nu^\vee} = \mathcal{M}_\nu^\vee$, and apply $H^\bullet(-) := H^\bullet(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times; -)$ to get the following long exact sequence:

$$\begin{aligned}
0 &\to H^0(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^0(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to \\
&\to H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^1(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to \\
&\to H^2(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^2(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to \\
&\to H^3(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to \cdots
\end{aligned}$$

Now, we make precise all the above cohomology groups as $\mathrm{O}(2)/\mathrm{SO}(2)$-modules. Let $1\!\!1$ stand for the trivial character, and sgn the sign-character of $\mathrm{O}(2)/\mathrm{SO}(2)$. For the finite-dimensional modules $\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}$, first of all, since $H^0 = \mathrm{Hom}$, we easily see that

$$H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = \mathrm{sgn}\,.$$

Next, it follows from [29, Prop. I.4], that

$$H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = 1\!\!1,$$

and furthermore, one may see that $H^q(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = 0$ for $q \notin \{0, 2\}$. For the discrete series representation, it is well-known that

$$H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) = 1\!\!1 \oplus \mathrm{sgn}, \quad \text{and} \quad H^q(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) = 0, \text{ if } q \neq 1.$$

Also, since V_ν doesn't contain a finite-dimensional sub-representation we deduce $H^0(V_\nu \otimes \mathcal{M}_{\nu^\vee}) = 0$, whence, $H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee})$ sits in the short exact sequence:

$$0 \to H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_\nu^\vee) \to H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to 0.$$

Hence, as an $\mathrm{O}(2)/\mathrm{SO}(2)$-module we get

$$H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \;=\; 1\!\!1. \tag{2.2}$$

Furthermore, if $[D_{\nu^\vee}]^+$ denotes an eigenvector in $H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee})$ for the trivial action of $\mathrm{O}(2)$, then we may take its image under $i^\bullet$ (the map induced by the inclusion i in cohomology) as a generator $[V_\nu]$ for $H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee})$, i.e.,

$$i^\bullet[D_{\nu^\vee}]^+ \;=\; [V_\nu]. \tag{2.3}$$

To complete the picture, since the dimension of the symmetric space is 2, we have $H^q = 0$ for all $q \geq 3$, and that

$$H^2(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \;\cong\; H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \;=\; 1\!\!1.$$

What we especially will want later is summarized in the following

Proposition 2.5 *For integers $a \geq b$, we have as an* $\mathrm{O}(2)/\mathrm{SO}(2)$*-module:*

$$H^1(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times;\; (\xi_{-b}(\mathrm{sgn})(1/2) \times \xi_{-a}(\mathrm{sgn})(-1/2)) \otimes \mathcal{M}_{(a,b)}) \;=\; \mathbb{C}1\!\!1.$$

Cohomology of $\Sigma(\chi_1, \chi_2)$

Now we return to the global situation and use the above local details to get the following

Proposition 2.6 *Let $\chi_i = |\,|^{d_i}\chi_i^\circ$ be algebraic Hecke characters of F with $d_i \in \mathbb{Z}$ and χ_i° finite-order character. Suppose that $d_1 \geq d_2$, and for $v \in S_\infty$ suppose also that $\chi_{iv}^\circ = (\mathrm{sgn})^{e_{iv}}$ for $e_{iv} \in \{0,1\}$ such that $e_{iv} \not\equiv d_i \pmod 2$. Let $\lambda \in X_0^+(T_2)$ be the dominant integral 'parallel' weight determined by d_1, d_2 as: $\lambda = (\lambda_v)_{v \in S_\infty}$, where each $\lambda_v = (-d_2, -d_1)$. Then:*

(1) $H^\bullet(\mathfrak{g}_{2,\infty}, K_{2,\infty}^0; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda) \neq 0 \iff d_F \leq \bullet \leq 2d_F$.
Furthermore, in the extremal degrees of d_F and $2d_F$, the cohomology group is one-dimensional.

(2) The group $\pi_0(K_{2,\infty})$ *acts trivially on* $H^{d_F}(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1,\chi_2) \otimes \mathcal{M}_\lambda)$.

Proof For $v \in S_\infty$ we have

$$\begin{aligned}\Sigma(\chi_1,\chi_2)_v &= \chi_{1v}(1/2) \times \chi_{2v}(-1/2) \\ &= |\,|^{d_1}\,\mathrm{sgn}^{e_{1v}}(1/2) \times |\,|^{d_2}\,\mathrm{sgn}^{e_{2v}}(-1/2) \\ &= \xi_{d_1}(\mathrm{sgn})(1/2) \times \xi_{d_2}(\mathrm{sgn})(-1/2) \;=\; V_{(d_1,d_2)}.\end{aligned}$$

Now, (1) follows from Künneth formula [5] for relative Lie algebra cohomology and the fact that $H^q(V_\nu \otimes \mathcal{M}_{\nu^\vee}) = \mathbb{C}$ for $q = 1, 2$, and is 0 if $q \notin \{1, 2\}$, and (2) follows from Proposition 2.5. □

Eisenstein Cohomology Classes Corresponding to $\Sigma(\chi_1, \chi_2)$

This works exactly as in Mahnkopf [18, Sect. 1.1] with the additional book-keeping of having to work over a totally real field and a general coefficient system offering no additional complications; so, we merely record the details for later use. To begin, fix a generator $[\Sigma(\chi_1,\chi_2)_\infty] = [\Sigma_\infty]$ of the one-dimensional

$$H^{d_F}(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1,\chi_2) \otimes \mathcal{M}_\lambda) \;=\; \mathbb{C}[\Sigma_\infty].$$

Tensoring by $[\Sigma_\infty]$ and following it up by Eisenstein summation gives us a map:

$$\mathcal{F}_\Sigma \;:\; \Sigma(\chi_1,\chi_2)_f^{R_f} \;\longrightarrow\; H^{d_F}(S_2(R_f), \mathcal{M}_\lambda). \tag{2.4}$$

where R_f is any open-compact subgroup for which the R_f-invariants in $\Sigma(\chi_1,\chi_2)_f$, denoted as $\Sigma(\chi_1,\chi_2)_f^{R_f}$, is nonzero. (This is the map denote 'Eis' on [18, p. 96].) Furthermore, the map $\mathcal{F}_\Sigma$ is Aut($\mathbb{C}$)-equivariant.

3 Critical Values of L-Functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$

3.1 The Critical Set for L-Functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$

Consider $L(s, \Pi \times \chi)$ the standard degree-3 L-function attached to a cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ and a finite order character χ of $\mathbb{A}_F^\times/F^\times$. The critical set of $L(s, \Pi \times \chi)$ is the set of all integers m such that both $L_\infty(s, \Pi_\infty \times \chi_\infty)$ and $L_\infty(1 - s, \Pi_\infty^\vee \times \chi_\infty^\vee)$ are regular at $s = m$, i.e., have no poles at $s = m$, where

$$L_\infty(s, \Pi_\infty \times \chi_\infty) = \prod_{v \in S_\infty} L_v(s, \Pi_v \times \chi_v).$$

As in Proposition 2.1, let $\mu \in X_0^+(T_3)$ be a weight written as $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (n_v, 0, -n_v)$ for an integer $n_v \geq 0$, and suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$. For each $v \in S_\infty$, we have

$$\Pi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} (D_{\ell_v} \otimes \varepsilon_{\Pi_v}), \quad \text{recall that } \ell_v = 2n_v + 2,$$

and since χ has finite order, we have χ_v is a quadratic character of $\mathbb{F}_v^\times = \mathbb{R}^\times$ which we also denote as a signature ε_{χ_v}, and we write

$$\chi_v = \varepsilon_{\chi_v} = \mathrm{sgn}^{e_{\chi_v}}, \quad e_{\chi_v} \in \{0, 1\}.$$

Thus,

$$\Pi_v \otimes \chi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} (D_{\ell_v} \otimes \varepsilon_{\Pi_v}) \otimes \varepsilon_{\chi_v} = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} \left(D_{\ell_v} \otimes \varepsilon_{\Pi_v}\varepsilon_{\chi_v}\right).$$

Using a well-known recipe to attach local factors (see, for example, Knapp [17]), we can explicitly write down $L_\infty(s, \Pi_\infty \times \chi_\infty)$ and $L_\infty(1-s, \Pi_\infty^\vee \times \chi_\infty^\vee) = L_\infty(1-s, \Pi_\infty \times \chi_\infty)$; this latter equality being due to the fact that $\Pi_\infty \otimes \chi_\infty$ is self-dual. We will separate this into cases depending on whether $\varepsilon_{\Pi_v}\varepsilon_{\chi_v}$ is trivial or not.

Case 1. $\varepsilon_{\Pi_v} = \varepsilon_{\chi_v}, \forall v \in S_\infty$.

This case may also be described as $e_{\Pi_v} = e_{\chi_v}$ for all $v \in S_\infty$. We have $\Pi_v \otimes \chi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} (D_{\ell_v} \otimes 1\!\!1)$ and the associated L-factors are:

$$L_\infty(s, \Pi_\infty \times \chi_\infty) \approx \Gamma(s + n_v + 1)\Gamma\left(\frac{s}{2}\right),$$

$$L_\infty(1-s, \Pi_\infty^\vee \times \chi_\infty^\vee) \approx \Gamma(1 - s + n_v + 1)\Gamma\left(\frac{1-s}{2}\right),$$

where, by $\approx$, we mean up to nonzero constants and exponential functions which are holomorphic and nonvanishing everywhere and hence irrelevant for computing the critical points. It is an easy exercise now to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) = \{1 - n_{\mathrm{ev}}, \ldots, -3, -1, 2, 4, \ldots, n_{\mathrm{ev}}\}, \tag{3.1}$$

where,

$$n_{\mathrm{ev}} = 2\left[\frac{n+1}{2}\right] = \text{the largest even positive integer less than or equal to } n+1. \tag{3.2}$$

Note that if $n = 0$ (this is the case, for example if $\mu = 0$, i.e., the case of constant coefficients for the cohomology of GL_3) then the critical set is empty.

Case 2. $\varepsilon_{\Pi_v} \neq \varepsilon_{\chi_v}, \forall v \in S_\infty$.

This case may also be described as $e_{\Pi_v} \equiv e_{\chi_v} + 1 \pmod 2$ for all $v \in S_\infty$. We have $\Pi_v \otimes \chi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{\ell_v} \otimes \mathrm{sgn})$ and the associated L-factors are:

$$L_\infty(s, \Pi_\infty \times \chi_\infty) \approx \Gamma(s + n_v + 1)\Gamma\left(\frac{s+1}{2}\right),$$

$$L_\infty(1-s, \Pi_\infty^\vee \times \chi_\infty^\vee) \approx \Gamma(1 - s + n_v + 1)\Gamma\left(\frac{2-s}{2}\right).$$

It is an easy exercise now to see that

$$\text{Critical set for } L(s, \Pi \times \chi) \;=\; \{1 - n_{\mathrm{od}}, \ldots, -2, 0, 1, 3, \ldots, n_{\mathrm{od}}\} \tag{3.3}$$

where,

$$n_{\mathrm{od}} \;=\; 2\left[\frac{n}{2}\right] + 1 \;=\; \text{the largest odd integer less than or equal to } n+1. \tag{3.4}$$

Note that in this case, the critical set is always nonempty.

Case 3. There exist two places $v_1, v_2 \in S_\infty$ such that $\varepsilon_{\Pi_{v_1}} = \varepsilon_{\chi_{v_1}}$ and $\varepsilon_{\Pi_{v_2}} \neq \varepsilon_{\chi_{v_2}}$.

Then in the expression for $L_\infty(s, \Pi_\infty \times \chi_\infty)$ we would have as a factor: $\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$ and it is easy to see that in this situation there are no critical points; whence, we will not consider this case.

Remark 3.5 (Reduction to the Case of $\varepsilon_{\Pi_v} = 1\!\!1$*)* Take $\Pi \in \mathrm{Coh}(G_3, \mu)$ as before, and fix a nontrivial quadratic character η of F such that $\eta_v = \varepsilon_{\Pi_v}$ for all $v \in S_\infty$. (Such an η exists; consider the character attached to a quadratic extension obtained by adjoining the square root of an element that is negative for a prescribed set of embeddings—this element may be produced using weak-approximation in F.) Then $\Pi \otimes \eta$ also has cohomology with respect to μ, and it is easy to see that $\varepsilon_{\Pi_v \otimes \eta_v} = 1\!\!1$. Furthermore, to study the critical values of $L(s, \Pi \otimes \chi)$, it suffices to consider $L(s, (\Pi \otimes \eta) \otimes (\chi \otimes \eta))$. We are in **Case 1** or **Case 2** for the pair (Π, χ) exactly when we are in **Case 1** or **Case 2** for the pair $(\Pi \otimes \eta, \chi \otimes \eta)$. Henceforth, we will assume:

(1) $\mu = (\mu_v)_{v \in S_\infty}$, $\mu_v = (n_v, 0, -n_v)$ with $n_v \geq 0$, and
(2) $\Pi \in \mathrm{Coh}(G_3, \mu)$, and $\varepsilon_{\Pi_v} = 1\!\!1$ for all $v \in S_\infty$.

In this situation, **Case 1** is defined by $\varepsilon_{\chi_v} = 1\!\!1$, and **Case 2** by $\varepsilon_{\chi_v} = \mathrm{sgn}$ for all $v \in S_\infty$.

Before stating our next proposition, let's recall the following well-known branching-rule for finite-dimensional representations (see [13]):

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{C})}(\mathcal{M}_\lambda \otimes \mathcal{M}_\mu, \mathbb{C}) \neq 0 \iff \mu \succ \lambda^\vee, \quad \text{i.e.,} \quad \mu_1 \geq -\lambda_2 \geq \mu_2 \geq -\lambda_1 \geq \mu_3.$$

Proposition 3.6 *Let μ and Π be as in Remark 3.5, and let χ be a finite order character of $\mathbb{A}_F^\times/F^\times$. We fix once and for all, a totally odd quadratic Hecke character ξ of F, and make the following choices for Hecke characters $\chi_i = |\ |^{d_i}\chi_i^\circ$, with integers d_i and finite order characters χ_i°:*

Case 1. $\varepsilon_{\chi_v} = 1\!\!1$ *for all* $v \in S_\infty$.

Case 1a. $m \in \{2, 4, \ldots, n_{\mathrm{ev}}\}$, $d_1 = m-1$, $d_2 = -1$, $\chi_1^\circ = \chi$, *and* $\chi_2^\circ = 1\!\!1$; *put* $\lambda_v = (1, 1-m)$.

Case 1b. $m \in \{1-n_{\mathrm{ev}}, \ldots, -3, -1\}$, $d_1 = 1$, $d_2 = m$, $\chi_1^\circ = 1\!\!1$, *and* $\chi_2^\circ = \chi$; *put* $\lambda_v = (-m, -1)$.

Case 2. $\varepsilon_{\chi_v} = \mathrm{sgn}$ *for all* $v \in S_\infty$.

Case 2a. $m \in \{1, 3, \ldots, n_{\mathrm{od}}\}$, $d_1 = m-1$, $d_2 = 0$, $\chi_1^\circ = \chi$, *and* $\chi_2^\circ = \xi$; *put* $\lambda_v = (0, 1-m)$.

Case 2b. $m \in \{1-n_{\mathrm{od}}, \ldots, -2, 0\}$, $d_1 = 0$, $d_2 = m$, $\chi_1^\circ = \xi$, *and* $\chi_2^\circ = \chi$; *put* $\lambda_v = (-m, 0)$.

Then, in all the above four cases, we have

(1) $L(\frac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2))$ *is critical;*
(2) $H^1\left(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times; \Sigma(\chi_{1v}, \chi_{2v}) \otimes \mathcal{M}_{\lambda_v}\right) = \mathbb{C}1\!\!1$ *as an* $\mathrm{O}(2)/\mathrm{SO}(2)$*-module;*
(3) $\mu \succ \lambda^\vee$.

Proof The proof is a routine check in each case, and we will only briefly present the key details:

Case 1a. For the L-value we see that

$$L(\tfrac{1}{2}, \Pi\otimes\Sigma(\chi_1, \chi_2)) = L(1, \Pi\otimes\chi_1)L(0, \Pi\otimes\chi_2) = L(m, \Pi\otimes\chi)L(-1, \Pi),$$

and both the L-values on the right hand side are critical by (3.1). The induced representation may be written as

$$\Sigma(\chi_1, \chi_2)_v = \xi_{m-1}(\mathrm{sgn})(1/2) \times \xi_{-1}(\mathrm{sgn})(-1/2),$$

which has nontrivial cohomology with respect to $\lambda_v = (1, 1-m)$; see Proposition 2.5.

Case 1b. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) = L(m, \Pi \otimes \chi)L(2, \Pi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = \xi_1(\mathrm{sgn})(1/2) \times \xi_m(\mathrm{sgn})(-1/2).$$

Case 2a. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) \;=\; L(m, \Pi \otimes \chi)L(0, \Pi \otimes \xi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = \xi_{m-1}(\mathrm{sgn})(1/2) \times (\mathrm{sgn})(-1/2).$$

Case 2b. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) \;=\; L(m, \Pi \otimes \chi)L(1, \Pi \otimes \xi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = (\mathrm{sgn})(1/2) \times \xi_m(\mathrm{sgn})(-1/2).$$

3.2 The Analytic Theory of L-Functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$

The Global Integral

We will apply the Rankin–Selberg theory of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$ to the pair (Π, Σ), where Π is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ and $\Sigma = \Sigma(\chi_1, \chi_2)$ the induced representation defined above. Take a cusp form $\phi_\Pi \in V_\Pi$, and recall that a cusp form is a rapidly decreasing function. Let $\varphi_{\chi_1,\chi_2} \in \Sigma(\chi_1, \chi_2)$, and note that φ_{χ_1,χ_2} is a function on $B_2(\mathbb{Q})\backslash G_2(\mathbb{A})$. To ensure $G_2(\mathbb{Q})$-invariance we do an Eisenstein summation:

$$E(\varphi_{\chi_1,\chi_2}, g, s) \;:=\; \sum_{\gamma \in B_2(F)\backslash \mathrm{GL}_2(F)} |\alpha|^s \varphi_{\chi_1,\chi_2}(\gamma g).$$

It's well-known [14, p. 80] that $E(\varphi_{\chi_1,\chi_2}, g, s)$ converges for $\Re(s) \gg 0$, and has an analytic continuation to an entire function of s if $\chi_1 \neq \chi_2$. (In all the cases that will be relevant to us later on, based on the choices in Proposition 3.6, we will indeed have $\chi_1 \neq \chi_2$.) Put

$$E(\varphi_{\chi_1,\chi_2})(g) \;:=\; E(\varphi_{\chi_1,\chi_2}, g, 0).$$

Consider the global period integral:

$$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) \;:=\; \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \phi_\Pi(\iota(g)) E(\varphi_{\chi_1,\chi_2})(g) |\mathrm{det} g|^{s-1/2} dg. \qquad (3.5)$$

This integral converges for all $s \in \mathbb{C}$ since a cusp form has rapid decay whereas an Eisenstein series slowly increases.

To see the Eulerian nature of the above period integral, we pass to the Whitakker models of the representations. Fix a nontrivial additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$, and suppose that $w_\Pi \in \mathcal{W}(\Pi, \psi)$, and $w_E \in \mathcal{W}(\Sigma(\chi_1, \chi_2), \bar{\psi})$ are global Whittaker vectors corresponding to ϕ_Π and $E(\varphi_{\chi_1,\chi_2})$, respectively. Then

$$\begin{aligned} I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) &\overset{(\forall s\in\mathbb{C})}{=} \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \phi_\Pi(\iota(g))E(\varphi_{\chi_1,\chi_2})(g)|\det g|^{s-\frac{1}{2}}dg, \\ &\overset{(\Re(s)\gg 0)}{=} \int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g)) \\ &\quad \times \left(\int_{N_2(F)\backslash N_2(\mathbb{A})} E(\varphi_{\chi_1,\chi_2})(ng)\psi(n)dn \right) |\det g|^{s-\frac{1}{2}}dg, \\ &\overset{(\Re(s)\gg 0)}{=} \int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g))w_E(g)|\det g|^{s-\frac{1}{2}}dg. \end{aligned}$$

Now suppose ϕ_Π and φ_{χ_1,χ_2} are chosen so that w_Π and w_E are pure tensors, written as restricted tensors $\mathrm{w}_\Pi = \otimes' w_{\Pi_v}$ and $w_E = \otimes' w_{E_v}$, then we have

$$\begin{aligned} &\int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g))w_E(g)|\det g|^{s-\frac{1}{2}}dg \\ &= \prod_v \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v))w_{E_v}(g_v)|\det g_v|_v^{s-\frac{1}{2}}dg_v \\ &=: \prod_v \Psi(s, w_{\Pi_v}, w_{E_v}). \end{aligned}$$

We need to compute the local integrals $\Psi(s, w_{\Pi_v}, w_{E_v})$, especially at ramified places.

Choice of Local Whittaker Vectors for Induced Representations of GL_2

For $i = 1, 2$, let χ_i be algebraic Hecke characters of $F^\times \setminus \mathbb{A}_F^\times$. Fix a place v of F. Let F_v be the completion of F at v, with ring of integers $\mathcal{O}_v$ and maximal ideal $\mathfrak{p}_v$. Let $q_v = \#\mathcal{O}_v/\mathfrak{p}_v$ be the cardinality of the residue field, and ϖ_v denote a fixed generator of $\mathfrak{p}_v$. The normalized valuation **val** on F_v has the property that $\mathbf{val}(\varpi_v) = 1$. For the normalized absolute value we have $|\varpi_v| = q_v^{-1}$. Recall $\Sigma_v := \Sigma(\chi_{1v}, \chi_{2v}) := \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1v}|\,|_v^{1/2}, \chi_{2v}|\,|_v^{-1/2})$ is the induced representation of $\mathrm{GL}_2(F_v)$ on the space

$$V(\chi_{1v}, \chi_{2v}) := \{f : \mathrm{GL}_2(F_v) \to \mathbb{C} \mid f\left(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right)g\right) = |ab^{-1}|_v \chi_{1v}(a)\chi_{2v}(b)f(g)\}.$$

The action of $\mathrm{GL}_2(F_v)$ on $V(\chi_{1v},\chi_{2v})$ is by right translations. Since $B(F_v)N_2^-(F_v)$ is dense in $\mathrm{GL}_2(F_v)$, any $f\in V(\chi_{1v},\chi_{2v})$ is completely determined by its values on elements of the form $\left(\begin{smallmatrix}1&0\\ *&1\end{smallmatrix}\right)$. So we get a model for $\Sigma(\chi_{1v},\chi_{2v})$ obtained by restricting functions in $V(\chi_{1v},\chi_{2v})$ to $N_2^-(F_v)$. We denote the space of functions on $N_2^-(F_v)\simeq F_v$ by $V(\chi_{1v},\chi_{2v})^-$. We recall some well-known facts about 'new vectors' in induced representations (see [6] and [25, Prop. 2.1.2]):

Proposition 3.8 *Suppose the conductor of χ_{iv} is $Cond(\chi_{iv})=\mathfrak{f}_{\chi_{iv}}=\mathfrak{p}_v^{n_i}$, say. Then the conductor of $\Sigma(\chi_{1v},\chi_{2v})=\mathfrak{f}_{\Sigma_v}=\mathfrak{f}_{\chi_1}\mathfrak{f}_{\chi_2}=\mathfrak{p}_v^n$, with $n=n_1+n_2$. For $m\ge 0$ we define*

$$K_{01}(\mathfrak{p}^m)\ :=\ \{\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathcal{O}_v): c\equiv 0,\ d\equiv 1\ \ (\mathrm{mod}\ \mathfrak{p}^m)\},$$

with the understanding that $K_{01}(\mathfrak{p}^0)=\mathrm{GL}_2(\mathcal{O}_v)$. Then, the space of $K_{01}(\mathfrak{p}^n)$-invariant vectors in $\Sigma(\chi_{1v},\chi_{2v})$ is one-dimensional, say $\mathbb{C}f_v^{\mathrm{new}}$. Moreover, this 'new-vector' as a function on $N^-(F_v)$ may be taken to be of the following shape:

- *If χ_{1v} and χ_{2v} are ramified, then*
$$f_v^{\mathrm{new}}\left(\begin{smallmatrix}1&0\\ x&1\end{smallmatrix}\right)=\begin{cases}\chi_{1v}(x)^{-1}|x|^{-1/2}, & \text{if } val(x)=n_2,\\ 0, & \text{if } val(x)\neq n_2.\end{cases}$$
- *If χ_{1v} is unramified and χ_{2v} is ramified, then*
$$f_v^{\mathrm{new}}\left(\begin{smallmatrix}1&0\\ x&1\end{smallmatrix}\right)=\begin{cases}\chi_{1v}(\varpi_v)^{-n_2}|\varpi_v|^{-n_2/2}, & \text{if } val(x)\ge n_2,\\ 0, & \text{if } val(x)< n_2.\end{cases}$$
- *If χ_{1v} is ramified and χ_{2v} is unramified, then*
$$f_v^{\mathrm{new}}\left(\begin{smallmatrix}1&0\\ x&1\end{smallmatrix}\right)=\begin{cases}\chi_{1v}(x)^{-1}\chi_{2v}(x)|x|^{-2}, & \text{if } val(x)\le 0,\\ 0, & \text{if } val(x)> 0.\end{cases}$$
- *If χ_{1v} and χ_{2v} are unramified, then we take $f_v^{\mathrm{new}}=f_v^{\mathrm{sp}}$, the spherical vector, i.e., the vector fixed by $\mathrm{GL}_2(\mathcal{O}_v)$, normalized such that $f_v^{\mathrm{sp}}(k_v)=1$ for $k_v\in \mathrm{GL}_2(\mathcal{O}_v)$.*

Now we consider the new-vector f_v^{new} in the local Whittaker model. For the global additive character ψ, we will furthermore assume that the local ψ_v is unramified, i.e., the largest fractional ideal on which ψ_v is trivial is $\mathcal{O}_v$. For $f_v\in V(\chi_{1v},\chi_{2v})$, the corresponding ψ_v^{-1}-Whittaker function is given by the integral:

$$w_{f_v}(g)\equiv w_{f_v,\psi_v^{-1}}(g)\ =\ \int_{N(F_v)}\psi_v(n)f_v(\mathrm{w}_\circ^{-1}ng)dn,$$

where $\mathrm{w}_\circ=\left(\begin{smallmatrix}0&1\\ -1&0\end{smallmatrix}\right)$. The map $f_v\mapsto w_{f_v}$ identifies the local induced representation $\Sigma(\chi_{1v},\chi_{2v})$ with its Whittaker model $\mathcal{W}(\Sigma(\chi_{1v},\chi_{2v}),\psi_v^{-1})$. We have the following lemma for new vectors stated in terms of the Whittaker models:

Lemma 3.9 *The space of $K_{01}(\mathfrak{p}^n)$–invariant vectors in $\mathcal{W}(\Sigma(\chi_{1v},\chi_{2v}),\psi_v^{-1})$ is one-dimensional, and we may take as generator $w_{\Sigma_v}^{\mathrm{new}}:=w_{f_v^{\mathrm{new}}}$. Furthermore, there exists $t^*=diag(t,1)$ such that $w_{\Sigma_v}^{\mathrm{new}}(t^*)\neq 0$, for some $t\in F_v^\times$.*

Proof First part of the lemma follows from [25] and second part follows from Kirillov theory. □

We would like to take a convenient t^* and compute the value $w^{\mathrm{new}}_{\Sigma_v}(t^*)$. Towards this, to begin, suppose v is a finite unramified place, i.e., Σ_v admits a K_v-fixed vector which is unique up to scalars; take $w^{\circ}_{\Sigma_v}$ as the unique K_v-fixed vector such that $w^{\circ}_{\Sigma_v}(1) = 1$. On the other hand, $w_{f_v^{\mathrm{sp}}}$ is also a K_v-fixed vector, and so there exists a $C_v \in \mathbb{C}^\times$ such that $w_{f_v^{\mathrm{sp}}} = C_v w^{\circ}_{\Sigma_v}$; then $C_v = w_{f_v^{\mathrm{sp}}}(1)$. We have the well-known proposition [7, Thm. 5.4], [26, Chap. 5, p. 352]:

Proposition 3.10 *Suppose χ_{1v} and χ_{2v} are unramified characters, and f_v^{sp} is the spherical vector in the induced representation Σ_v, then $C_v = w_{f_v^{\circ}}(1) = L(2, \chi_{1v}\chi_{2v}^{-1})^{-1}$.*

Let's note that in the usual Casselman-Shalika formula, one sees the value at $s = 1$ of a local L-function, but recall in our case that for $\Sigma(\chi_1, \chi_2)$ the inducing representation is $\chi_1(1/2) \times \chi_2(-1/2)$ which accounts for the L-value at $s = 2$, since $L(1, \chi_{1v}(1/2)(\chi_{2v}(-1/2))^{-1}) = L(2, \chi_{1v}\chi_{2v}^{-1})$.

Let S_{χ_i} be the set of finite places where χ_i is ramified; then put $S_\Sigma = S_{\chi_1} \cup S_{\chi_2}$. Let $v \in S_\Sigma$. Applying Lemma 3.9, we take for $w^{\mathrm{new}}_{\Sigma_v}$ the unique $K_{01}(\mathfrak{p}^n)$-fixed vector normalized such that $w^{\mathrm{new}}_{\Sigma_v}(t^*) = 1$. Since the space of new-vectors is one-dimensional, there exists $A_v \in \mathbb{C}^\times$ such that $w_{f_v^{\mathrm{new}}} = A_v w^{\mathrm{new}}_{\Sigma_v}$. Hence, $A_v = w_{f_v}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The precise value of A_v is given by the following

Proposition 3.11 *Let f_v^{new} be the new vector in the induced representation Σ_v as in Proposition 3.8. Then*

$$A_v = w_{f_v^{\mathrm{new}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) = \begin{cases} q_v^{-n_2/2}\, \chi_{2v}(\varpi^{-n_2})\, \mathcal{G}(\chi_{2v}), & \textit{if } \chi_{2v} \textit{ is ramified,} \\ \mathrm{Vol}(\mathcal{O}_v), & \textit{if } \chi_{2v} \textit{is unramified.} \end{cases}$$

Proof Before we begin, let's recall the following well-known fact about local Gauß sums: if the conductor of ψ_v if $\mathcal{O}_v$, then

$$\int_{\mathcal{O}_v^\times} \psi_v(a\varepsilon)\chi_v(\varepsilon) d^\times\varepsilon = \begin{cases} \chi_v^{-1}(a\varpi^e)\mathcal{G}(\chi_v), & \text{if } \mathbf{val}_v(a) = -e, \\ 0, & \text{otherwise,} \end{cases}$$

where $e = \mathrm{cond}(\chi_v)$. Thus, by the definition of the Whittaker function we have

$$\begin{aligned} w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \int_{N(F_v)} f\left(\mathrm{w}_\circ^{-1} u \left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) \psi_v(u) du \\ &= \int_{F_v^\times} f\left(\mathrm{w}_\circ^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \psi_v(x) dx = \int_{F_v^\times} f\left(\left(\begin{smallmatrix} 0 & -1 \\ t & x \end{smallmatrix}\right)\right) \psi_v(x) dx. \end{aligned}$$

Make the substitution $x \longmapsto tx$, and use $\left(\begin{smallmatrix} 0 & -1 \\ t & tx \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^{-1} & -1 \\ 0 & tx \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$ to rewrite the last integral as

$$\begin{aligned}
&|t|_v \int_{F_v^\times} f\left(\left(\begin{smallmatrix} x^{-1} & -1 \\ 0 & tx \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \psi_v(tx)dx \\
&= |t|_v \int_{F_v^\times} \chi_{1v}(x)^{-1}\chi_{2v}(tx)|x^{-2}t^{-1}|_v f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\psi_v(tx)dx \\
&= \sum_{n\in\mathbb{Z}} \int_{\varpi_v^n \mathcal{O}_v^\times} \chi_{1v}(x)^{-1}\chi_{2v}(tx)|x^{-2}|_v f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\psi_v(tx)dx.
\end{aligned}$$

Now to compute $w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$ using the very last expression, we consider three cases:

(1) χ_{1v} **and** χ_{2v} **are both ramified.** In this case, by Proposition 3.8, we have $f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right) = 0$ for all x such that $\mathbf{val}(x^{-1}) \neq n_2$. Hence only the summand for $n = -n_2$ survives to get:

$$\begin{aligned}
w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \int_{\varpi_v^{-n_2}\mathcal{O}_v^\times} |x|_v^{-2}\chi_{1v}(x)^{-1}\chi_{2v}(tx)\chi_{1v}(x)|x|_v^{\frac{1}{2}}\psi_v(tx)dx \\
&= \int_{\varpi_v^{-n_2}\mathcal{O}_v^\times} |x|_v^{\frac{-3}{2}}\chi_{2v}(tx)\psi_v(tx)dx \quad (\text{put } x = \varpi_v^{-n_2}y) \\
&= |\varpi_v^{-n_2}|_v^{\frac{-3}{2}} \int_{\mathcal{O}_v^\times} \chi_{2v}(t\varpi_v^{-n_2}y)\psi_v(t\varpi_v^{-n_2}y)|\varpi_v^{-n_2}|_v dy \\
&= |\varpi_v|_v^{\frac{n_2}{2}}\chi_{2v}(\varpi_v^{-n_2})\int_{\mathcal{O}_v^\times} \chi_{2v}(ty)\psi_v(\varpi_v^{-n_2}ty)d^\times y.
\end{aligned}$$

Recall that on $\mathcal{O}_v^\times, dy = d^\times y$. Note that $\int_{\mathcal{O}_v^\times} \chi_{2v}(y)\psi_v(\varpi_v^{-n_2}ty)d^\times y \neq 0 \Leftrightarrow \text{val}(\varpi_v^{-n_2}t) = -n_2 \Leftrightarrow t \in \mathcal{O}_v^\times$. Put $ty = z$ to get

$$\begin{aligned}
w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= |\varpi_v|_v^{\frac{n_2}{2}}\chi_{2v}(\varpi_v^{-n_2})\int_{\mathcal{O}_v^\times} \chi_{2v}(z)\psi_v(\varpi_v^{-n_2}z)d^\times z \\
&= q_v^{-n_2/2}\chi_{2v}(\varpi^{-n_2})\mathcal{G}(\chi_{2v}).
\end{aligned}$$

(2) χ_{1v} **is unramified and** χ_{2v} **is ramified.** In this case, by Proposition 3.8, we have $f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right) = 0$, for $n > -n_2$. Hence we get

$$\begin{aligned}
w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \sum_{n\le -n_2}\int_{\varpi_v^n\mathcal{O}_v^\times} |x|_v^{-2}\chi_{1v}(x)^{-1}\chi_{2v}(tx)\chi_{1v}(\varpi_v^{-n_2})|\varpi_v|_v^{-n_2/2}\psi_v(tx)dx \\
&\quad \text{put } x = \varpi_v^n y; \\
&= \sum_{n\le -n_2} |\varpi_v|_v^{-n-n_2/2}\chi_{1v}(\varpi_v^{-n-n_2})\chi_{2v}(\varpi_v^n)\int_{\mathcal{O}_v^\times}\chi_{2v}(ty)\psi_v(t\varpi_v^n y)d^\times y.
\end{aligned}$$

For the inner integral we have: $\int_{\mathcal{O}_v^\times} \chi_{2v}(y)\psi_v(\varpi_v^n ty)d^\times y \neq 0 \Leftrightarrow \mathrm{val}(\varpi_v^n t) = -n_2$. Let's take $t \in \mathcal{O}_v^\times$, then only the summand for $n = -n_2$ will be non-zero, and we get:

$$\begin{aligned} w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= q_v^{-n_2/2}\chi_{2v}(\varpi_v^{-n_2}) \int_{\mathcal{O}_v^\times} \chi_{2v}(ty)\psi_v(t\varpi_v^{-n_2}y)d^\times y \\ &= q_v^{-n_2/2}\chi_{2v}(\varpi_v^{-n_2})\mathcal{G}(\chi_{2v}). \end{aligned}$$

(3) χ_{1v} **is ramified and** χ_{2v} **is unramified.** In this case, by Proposition 3.8, we have nonzero summands corresponding to $n \geq 0$:

$$\begin{aligned} w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \sum_{n\geq 0} \int_{\varpi_v^n \mathcal{O}_v^\times} |x|_v^{-2}\chi_{1v}(x)^{-1}\chi_{2v}(tx)\chi_{1v}(x)\chi_{2v}(x)^{-1}|x|_v^2\psi_v(tx)dx \\ &= \sum_{n\geq 0} \int_{\varpi_v^n \mathcal{O}_v^\times} \chi_{2v}(t)\psi_v(tx)dx. \end{aligned}$$

Now we take $t \in \mathcal{O}_v^\times$ so that $\chi_{2v}(t) = 1$ and $\psi_v(tx) = 1$ for $x \in \varpi_v^n\mathcal{O}_v^\times$ and $n \geq 0$; this gives:

$$\sum_{n\geq 0} \int_{\varpi_v^n \mathcal{O}_v^\times} dx = \mathrm{Vol}(\mathcal{O}_v).$$

□

For future reference let's define

$$A_\Sigma := \prod_{v \in S_\Sigma} A_v. \tag{3.6}$$

Integral Representation of $L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))$

Let's go back to the period integral in (3.5) and it's expression as a product of the local zeta integrals involving Whittaker vectors:

$$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) = \prod_v \Psi(s, w_{\Pi_v}, w_{E_v}). \tag{3.7}$$

We now make a judicious choice of Whittaker vectors and compute the zeta integrals as follows:

1. If $v \notin S_\Sigma \cup S_\infty$, take $w_{\Pi_v} = w^{\rm ess}_{\Pi_v}$ which is the essential vector as in [15], and $w_{E_v} = w_{f_v^{\rm sp}}$, then we have:

$$\begin{aligned}\Psi(s, w^{\rm ess}_{\Pi_v}, w_{f_v^{\rm sp}}) &= \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w^{\rm ess}_{\Pi_v}(\iota(g_v)) w_{f_v^{\rm sp}}(g_v)|\det(g_v)|_v^{s-\frac{1}{2}}\, dg_v \\ &= L(2, \chi_{1v}\chi_{2v}^{-1})^{-1} \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w^{\rm ess}_{\Pi_v}(\iota(g_v)) w_v^{\circ}(g_v)|\det(g_v)|_v^{s-\frac{1}{2}}\, dg_v \\ &= L(2, \chi_{1v}\chi_{2v}^{-1})^{-1} L(s, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v})).\end{aligned}$$

Let's define

$$L_\Sigma \;:=\; L_{S_\Sigma}(2, \chi_1\chi_2^{-1})^{-1} \;=\; \prod_{v\in S_\Sigma} L_v(2, \chi_{1v}\chi_{2v}^{-1})^{-1}. \tag{3.8}$$

2. If $v \in S_\Sigma$, take $w_{E_v} = w_{f_v^{\rm new}}$, and let w_{Π_v} be the unique Whittaker function whose restriction to $\iota(\mathrm{GL}_2(F_v))$ is supported on $N_2(F_v)t^*K_{01}(\mathrm{cond}(\Sigma_v))$, and on this double coset it's given by $w_{\Pi_v}(\iota(nt^*k)) = \psi(n)$, for all $n \in N_2(F_v)$ and all $k \in K_{01}(\mathrm{cond}(\Sigma_v))$.The existence and uniqueness of w_{Π_v} follows from Kirillov theory [1, Section 5]. So,

$$\begin{aligned}\Psi(s, w_{\Pi_v}, w_{f_v^{\rm ess}}) &= \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v)) w_{f_v^{\rm new}}(g_v)|\det(g_v)|_v^{s-\frac{1}{2}}\, dg_v \\ &= A_v \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v)) w^{\rm new}_{\Sigma_v}(g_v)|\det(g_v)|_v^{s-\frac{1}{2}}\, dg_v \\ &= A_v \mathrm{Vol}(K_{01}(\mathrm{cond}(\Sigma_v))).\end{aligned}$$

Let's define

$$V_\Sigma \;:=\; \prod_{v\in S_\Sigma} \mathrm{Vol}(K_{01}(\mathrm{cond}(\Sigma_v))). \tag{3.9}$$

3. If $v \in S_\infty$, let w_{Π_v} and w_{E_v} be arbitrary nonzero vectors. (Later these will be certain 'cohomological vectors'.)

Let's note that the function φ_{χ_1,χ_2} in the induced space $\Sigma(\chi_1, \chi_2)$ is taken accordingly:

$$\varphi_{\chi_1,\chi_2} = \varphi_\infty \otimes \varphi_f, \quad \varphi_f = \otimes_{v\notin S_\Sigma} f_v^{\rm sp} \otimes_{v\in S_\Sigma} f_v^{\rm new}, \tag{3.10}$$

with φ_∞ some cohomogical vector. Similarly, the cusp form ϕ_Π is chosen as:

$$\phi_\Pi = \phi_\infty \otimes \phi_f, \quad \phi_f = \otimes_{v \notin S_\infty} \phi_v, \quad \phi_v \text{ corresponds to } w_{\Pi_v}, \tag{3.11}$$

with ϕ_∞ some cohomogical vector.

With the above choice of Whittaker vectors (3.7) becomes (after multiplying and dividing by suitable local factors and after using the definitions in (3.6), (3.8) and (3.9)):

$$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) = \prod_{v \in S_\infty} \Psi_v(s, w_{\Pi_v}, w_{E_v}) \cdot \frac{A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{\prod_{v \in S_\Sigma} L(s, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v}))}$$
$$\cdot \frac{L_f(s, \Pi \times \Sigma(\chi_1, \chi_2))}{L_f(2, \chi_1 \chi_2^{-1})}.$$

For the factors for $v \in S_\infty$, suppose $s = 1/2$ is critical (as we will take a little later on), then by definition of criticality, $L(\frac{1}{2}, \Pi_v \times \Sigma_v)$ is finite. Also $\frac{\Psi(s, w_{\Pi_v}, w_{f_v})}{L_v(s, \Pi_v \times \Sigma_v)}$ is holomorphic for all $s \in \mathbb{C}$, hence

$$\Psi(\tfrac{1}{2}, w_{\Pi_v}, w_{f_v}) := \left(\frac{\Psi(s, w_{\Pi_v}, w_{f_v})}{L_v(s, \Pi_v \times \Sigma_v)}\right)|_{s=1/2} \cdot L(\tfrac{1}{2}, \Pi_v \times \Sigma_v)$$

is finite. Furthermore, the local L-factors are nonzero and the finite part of a global L-function $L_f(s, \Pi \times \Sigma(\chi_1, \chi_2))$ has an analytic continuation for all s. Hence we get, at $s = \frac{1}{2}$:

$$I(\tfrac{1}{2}, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) = \prod_{v \in S_\infty} \Psi_v(\tfrac{1}{2}, w_{\Pi_v}, w_{E_v}) \cdot \frac{A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma)} \cdot \frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))}{L_f(2, \chi_1 \chi_2^{-1})}, \tag{3.12}$$

where, $L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma) = \prod_{v \in S_\Sigma} L_v(\frac{1}{2}, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v}))$.

3.3 Cohomological Interpretation: L-Value as a Poincaré Pairing

In this section, we interpret the period integral $I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2}))$ in terms of Poincaré duality. More precisely, the vector w_{Π_f} will correspond to a cohomology class $\vartheta_{\Pi,\epsilon_\Pi}$ in degree $b_3^F = 2d_F$ (the bottom degree of the cuspidal range for G_3) on a locally symmetric space denoted by $S_3(K_f)$ for GL_3, and similarly $\varphi_f \in \Sigma(\chi_1, \chi_2)_f$ will correspond to a class ϑ_Σ in degree $b_2^F = d_F$. The class $\vartheta_{\Pi,\epsilon_\Pi}$, after dividing by a certain period, has good rationality properties. Pull back ϑ_Π along the proper map $\iota : \tilde{S}_2 \longrightarrow S_3$, and wedge (or cup) with ϑ_Σ, to give a top degree class on $\tilde{S}_2$ with coefficients in a tensor product sheaf. Now if $s = 1/2$ is critical which is the same as saying the constituent sheaves are compatible (which is the case when the weights interlace: $\mu \succ \lambda^\vee$), then we get a top-degree class on $\tilde{S}_2$ with constant coefficients. Apply Poincaré duality, i.e., fix an orientation on $\tilde{S}_2$ and integrate. One realizes then that this is essentially the above period integral. Interpreting the integral, and hence the L-value it represents, as a cohomological pairing permits us to study arithmetic properties of such special values, since this pairing is Galois equivariant. We now make all this precise.

The Cohomology Classes

Recall from Proposition 2.1, given any $\Pi \in \mathrm{Coh}(G_3, \mu)$ and for the signature ϵ_Π for Π, the cohomology group

$$H^{b_3^F}(\mathfrak{g}_{3,\infty}, K_{3,\infty}^0; \Pi_\infty \otimes \mathcal{M}_\mu)(\epsilon_\Pi) \neq 0,$$

and is one-dimensional. Fix a basis $[\Pi_\infty]$ of this one-dimensional space, and this gives us the following comparison isomorphism (see [23]):

$$\mathcal{F}_{\Pi_f} \equiv \mathcal{F}_{\Pi_f, \epsilon_\Pi, [\Pi_\infty]} : \mathcal{W}(\Pi_f) \longrightarrow H^{b_3^F}(\mathfrak{g}_{3,\infty}, K_{3,\infty}^0; V_\Pi \otimes \mathcal{M}_\mu)(\epsilon_\Pi). \tag{3.13}$$

The isomorphism $\mathcal{F}_{\Pi_f}$ is a $G_3(\mathbb{A}_f)$-equivariant map between irreducible modules, both of which have $\mathbb{Q}(\Pi)$ structures that are unique up to homotheties; we can adjust the map by a scalar—which is the period—so as to preserve rational structures; for more details see [23]. There is a nonzero complex number $p^{\epsilon_\Pi}(\Pi)$ attached to the datum $(\Pi_f, \epsilon_\Pi, [\Pi_\infty])$ such that the normalized map,

$$\mathcal{F}^\circ_{\Pi_f} := p^{\epsilon_\Pi}(\Pi)^{-1} \mathcal{F}_{\Pi_f}$$

is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e., the following diagram commutes

$$\begin{array}{ccc}
\mathcal{F}^\circ_{\Pi_f} : \mathcal{W}(\Pi_f) & \longrightarrow & H^{b_3^F}(\mathfrak{g}_3,\ K_{3,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon_\Pi) \\
\downarrow \sigma & & \downarrow \sigma \\
\mathcal{F}^\circ_{{}^\sigma\Pi_f} : \mathcal{W}({}^\sigma\Pi_f) & \longrightarrow & H^{b_3^F}(\mathfrak{g}_3,\ K_{3,\infty}^0; V_{{}^\sigma\Pi} \otimes \mathcal{M}_{{}^\sigma\mu,\mathbb{C}})(\epsilon_\Pi).
\end{array}$$

The complex number $p^{\epsilon_\Pi}(\Pi)$ is well-defined up to multiplication by elements of $\mathbb{Q}(\Pi)^\times$. The collection $\{p^{\epsilon_\Pi}({}^\sigma\Pi) : \sigma \in \mathrm{Aut}(\mathbb{C})\}$ is well-defined in $(\mathbb{Q}(\Pi) \otimes \mathbb{C})^\times / \mathbb{Q}(\Pi)^\times$. In terms of the un-normalized maps, we can write the above commutative diagram as

$$\sigma \circ \mathcal{F}_{\Pi_f} = \left(\frac{\sigma(p^{\epsilon_\Pi}(\Pi))}{p^{\epsilon_\Pi}({}^\sigma\Pi)} \right) \mathcal{F}_{{}^\sigma\Pi_f} \circ \sigma.$$

Now define the cohomology class attached to the global Whittaker vector w_{Π_f} as,

$$\vartheta_{\Pi,\epsilon_\Pi} := \mathcal{F}_{\Pi_f}(w_{\Pi_f}), \quad \text{and} \quad \vartheta^\circ_{\Pi,\epsilon_\Pi} = p^{\epsilon_\Pi}(\Pi)^{-1}\vartheta_{\Pi,\epsilon_\Pi}. \tag{3.14}$$

Let K_f be an open compact subgroup of $G_3(\mathbb{A}_f)$ which fixes w_{Π_f} and such that, $\Sigma(\chi_1, \chi_2)_f$ has vectors fixed under $R_f := \iota^* K_f$. Then $\vartheta_{\Pi,\epsilon_\Pi}$ is in $H^{b_3^F}(\mathfrak{g}_{3,\infty}, K^\circ_{3,\infty}; V_\Pi^{K_f} \otimes \mathcal{M}_\mu)(\epsilon_\Pi)$ and via certain standard isomorphisms [23, Section 3.3], we may identify it as a class in $H^{b_3^F}_{\mathrm{cusp}}(S_3(K_f), \widetilde{\mathcal{M}}_\mu)(\widetilde{\Pi}_f)$, where $\widetilde{\Pi}_f := \Pi_f \otimes \epsilon_\Pi$ is a representation of $G_3(\mathbb{A}_f) \otimes \pi_0(K_{3,\infty})$. Furthermore, since cuspidal cohomology injects into cohomology with compact support, we get $\vartheta_{\Pi,\epsilon_\Pi} \in H_c^{b_3^F}(S_3(K_f), \widetilde{\mathcal{M}}_\mu)$. Let's also recall from Proposition 2.1 that $b_3^F = 2d_F$.

On the other hand, recall the map in (2.4):

$$\mathcal{F}_{\Sigma_f} \;:\; \Sigma(\chi_1, \chi_2)_f^{R_f} \longrightarrow H^{d_F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda),$$

which is $\mathrm{Aut}(\mathbb{C})$-equivariant, that is, $\sigma \circ \mathcal{F}_\Sigma = \mathcal{F}_{{}^\sigma\Sigma} \circ \sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Define the class

$$\vartheta^\circ_\Sigma \;:=\; \mathcal{F}_\Sigma(\varphi_f), \tag{3.15}$$

where φ_f is defined in (3.10). Using the canonical map ϕ^* (the map induced by ϕ in cohomology)

$$H^{d_F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda) \xrightarrow{\phi^*} H^{d_F}(\tilde{S}_2(R_f), \widetilde{\mathcal{M}}_\lambda) \;,$$

we get $\phi^*\vartheta_\Sigma$ in $H^{d_F}(\tilde{S}_2(R_f), \widetilde{\mathcal{M}}_\lambda)$.

For the open compact subgroups K_f of $\mathrm{GL}_3(\mathbb{A}_f)$ and $R_f = \iota^*(K_f)$ of $\mathrm{GL}_2(\mathbb{A}_f)$, the map ι, being a proper map, induces a map between the cohomology with compact supports:

$$\iota^* : H_c^\bullet(S_3(K_f), \widetilde{\mathcal{M}}_\mu) \longrightarrow H_c^\bullet(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu).$$

Now consider the following diagram:

$$\begin{array}{ccc} \mathcal{W}(\Pi_f) \times \Sigma(\chi_1, \chi_2)_f & \longrightarrow & H_{\rm c}^{2d_F}(S_3(K_f), \widetilde{\mathcal{M}}_\mu) \times H^{d_F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda) \\ & & \Big\downarrow \iota^* \times \phi^* \\ & & H_{\rm c}^{2d_F}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu) \times H^{d_F}(\tilde{S}_2(R_f), \phi^*\widetilde{\mathcal{M}}_\lambda) \\ & & \Big\downarrow \wedge \\ & & H_{\rm c}^{3d_F}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda). \end{array}$$

Observe that $\dim(\tilde{S}_2(R_f)) = d_F \cdot \dim(\mathrm{GL}_2(\mathbb{R})^0/\mathrm{SO}(2)) = 3d_F$. Hence

$$\vartheta_{\Pi,\epsilon_\Pi} \wedge \vartheta_\Sigma \in H_{\rm c}^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda).$$

Compatibility of Sheaves

We now assume the hypotheses of Proposition 3.6. The interlacing condition of weights $\mu \succ \lambda^\vee$, gives the branching rule for finite-dimensional representations as

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{C})}(\iota^*\mathcal{M}_\mu, \mathcal{M}_{\lambda^\vee}) \neq 0$$

which gives a non-trivial pairing, $\langle \cdot, \cdot \rangle : \iota^*\mathcal{M}_\mu \times \mathcal{M}_\lambda \longrightarrow \mathbb{C}$ which in turn induces a pairing at the level of sheaves:

$$\langle \cdot, \cdot \rangle : \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda \longrightarrow \underline{\mathbb{C}}.$$

Now by composing this map with the $\wedge$-map gives

$$\langle \cdot, \cdot \rangle \circ \wedge : H_{\rm c}^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda) \longrightarrow H_c^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \underline{\mathbb{C}}).$$

The Global Pairing

We now have a top-degree class on an orientable manifold. We fix an orientation, compatibly on all the connected components; this was called the Harder-Mahnkopf cycle in [20, Sect. 3.2.3] (see also [21, Sect. 2.5.3.3]), and defined therein as

$$C(R_f) = \frac{1}{\mathrm{Vol}(R_f)} \sum_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} \det(R_f)} [\vartheta_{x,R_f}].$$

The action of $\delta_2 = (-1, 1)$ on this cycle $C(R_f)$ is given by $r^*_{\delta_2} C(R_f) = (-1)C(R_f)$. We can define the global pairing as:

$$\langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta_\Sigma \rangle_{C(R_f)} = \int_{C(R_f)} \iota^* \vartheta_{\Pi,\epsilon_\Pi} \wedge \phi^* \vartheta_\Sigma. \tag{3.16}$$

To evaluate this global pairing, we will write the cohomology classes as differential forms, and as in [20, Sect. 3.2.5], before we evaluate the global pairing we will need to discuss an analogous pairing involving the $(\mathfrak{g}_\infty, K^0_\infty)$-classes at infinity.

The Pairing at Infinity

(See [21, Sect. 2.5.3.6].) Recall, that we have fixed $[\Pi_\infty]$ a basis of the one-dimensional space $H^{b_3^F}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon_\Pi)$, and similarly, we have $[\Sigma_\infty]$ generating the one-dimensional space $H^{d_F}(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_{\lambda,\mathbb{C}})$. Define $d^F_n = \dim(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})$. To compute the pairing at infinity, we choose a basis $\{\mathrm{y}_j : 1 \le j \le d^F_2\}$ of $(\mathfrak{g}_{2,\infty}/\mathfrak{k}_{2,\infty})^*$ such that $\{\mathrm{y}_j : 1 \le j \le d^F_2 - 1\}$ is a basis of $(\mathfrak{g}_{2,\infty}/\mathfrak{c}_{2,\infty})^*$. Next, fix a basis $\{\mathrm{x}_i : 1 \le i \le d^F_3\}$ of $(\mathfrak{g}_{3,\infty}/\mathfrak{k}_{3,\infty})^*$, such that $\iota^*\mathrm{x}_j = \mathrm{y}_j$ for all $1 \le j \le d^F_2 - 1$, and $\iota^*\mathrm{x}_j = 0$ if $j \ge d^F_2$. We further note that $\mathrm{y}_1 \wedge \mathrm{y}_2 \wedge \cdots \wedge \mathrm{y}_{d^F_2}$ corresponds to a $G_2(\mathbb{R})^0$-invariant measure on $\tilde{S}_2(R_f)$. Let $\{m_\alpha\}$ (resp., $\{m_\beta\}$) be a $\mathbb{Q}$-basis for $\mathcal{M}_\mu$ (resp., $\mathcal{M}_\lambda$).

The class $[\Pi_\infty]$ is represented by a $K^0_{3,\infty}$-invariant element in $\wedge^{b_3^F}(\mathfrak{g}_{3,\infty}/\mathfrak{k}_{3,\infty})^* \otimes \mathcal{W}(\Pi_\infty) \otimes \mathcal{M}_{\mu,\mathbb{C}}$ which can be written as

$$[\Pi_\infty] = \sum_{\mathrm{i}=i_1<\cdots<i_{b_3^F}} \sum_\alpha \mathrm{x}_\mathrm{i} \otimes w_{\infty,\mathrm{i},\alpha} \otimes m_\alpha, \tag{3.17}$$

where $w_{\infty,\mathrm{i},\alpha} \in \mathcal{W}(\Pi_\infty, \psi_\infty)$. Similarly, $[\Sigma_\infty]$ is represented by a $K^0_{2,\infty}$-invariant element in

$$\wedge^{d_F}(\mathfrak{g}_{2,\infty}/\mathfrak{k}_{2,\infty})^* \otimes \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda$$

which we write as

$$[\Sigma_\infty] = \sum_{\mathrm{j}=j_1<\cdots<j_{d_F}} \sum_\beta \mathrm{y}_\mathrm{j} \otimes \varphi_{\infty,\mathrm{j},\beta} \otimes m_\beta. \tag{3.18}$$

Let $w_{\infty,\mathrm{j},\beta} \in \mathcal{W}(\Sigma_\infty(\chi_1, \chi_2), \psi^{-1}_\infty)$ be the Whittaker vector corresponding to $\varphi_{\infty,\mathrm{j},\beta}$. We now define a pairing at infinity by

$$\langle [\Pi_\infty], [\Sigma_\infty] \rangle = \sum_{\mathrm{i},\mathrm{j}} s(\mathrm{i},\mathrm{j}) \sum_{\alpha,\beta} \langle m_\alpha, m_\beta \rangle \Psi_\infty(1/2, w_{\infty,\mathrm{i},\alpha}, w_{\infty,\mathrm{j},\beta}), \tag{3.19}$$

where $s(\mathrm{i},\mathrm{j}) \in \{0,-1,1\}$ is defined by $\iota^*\mathrm{x_i} \wedge \mathrm{y_j} = s(\mathrm{i},\mathrm{j})\mathrm{y_1} \wedge \mathrm{y_2} \wedge \cdots \wedge \mathrm{y}_{d_2^F}$. Recall that the zeta integral at infinity $\Psi_\infty(1/2, w_{\infty,\mathrm{i},\alpha}, w_{\infty,\mathrm{j},\beta})$ is defined only after meromorphic continuation. However, the assumption that $s = 1/2$ is critical ensures that they are all finite, hence $\langle[\Pi_\infty], [\Sigma_\infty]\rangle$ is finite.

Lemma 3.26 $\langle[\Pi_\infty], [\Sigma_\infty]\rangle \neq 0.$

Proof It's easy to see that $\langle[\Pi_\infty], [\Sigma_\infty]\rangle = \prod_{v\in S_\infty}\langle[\Pi_v], [\Sigma_v]\rangle$; hence it is enough to prove nonvanishing locally for every $v \in S_\infty$. As in section "Cohomology of Some Representations of $\mathrm{GL}_2(\mathbb{R})$", for the discrete series representation, it is well known that

$$H^1(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times; D_{\lambda^\vee} \otimes \mathcal{M}_\lambda) \cong \mathbb{C}[D_\lambda]^+ \oplus \mathbb{C}[D_\lambda]^-.$$

Recall from (2.3) that $[D_\lambda]^+$ maps to $[\Sigma_v]$ under the map denoted $i^\bullet$ therein, and this map also kills $[D_\lambda]^-$. One can conclude that

$$\langle[\Pi_v], [\Sigma_v]\rangle = \langle[\Pi_v], [D_\lambda]^+\rangle.$$

Now Kasten and Schmidt [16] have proved $\langle[\Pi_v], [D_\lambda]^+\rangle \neq 0$, which proves the lemma. □

We are now justified in making the definition:

$$P_\infty(\mu, \lambda) := \frac{1}{\langle[\Pi_\infty], [\Sigma_\infty]\rangle}. \tag{3.20}$$

L-Value as a Global Pairing of Cohomology Classes

Using (3.13) and (3.17) we can write:

$$\vartheta_{\Pi,\epsilon_\Pi} = \sum_{\mathrm{i}}\sum_{\alpha} \mathrm{x_i} \otimes \phi_{\mathrm{i},\alpha} \otimes m_\alpha,$$

where the cusp form $\phi_{\mathrm{i},\alpha}$, in the ψ -Whittaker model of Π, looks like $w_{\Pi_f} \otimes w_{\infty,\mathrm{i},\alpha}$; recall from (3.11) that w_{Π_f} corresponds to ϕ_f. Similarly, using (2.4) and (3.18), we may write

$$\vartheta_\Sigma^\circ = \sum_{\mathrm{j}}\sum_{\beta} \mathrm{y_j} \otimes E_{\mathrm{j},\beta} \otimes m_\beta,$$

where the Eisenstein series $E_{\mathrm{j},\beta}$ is constructed by taking Eisenstein summation for the function $\varphi_f \otimes \varphi_{\infty,\mathrm{j},\beta}$ in the full induced representation $\Sigma(\chi_1, \chi_2)$, with φ_f as is in (3.10). We get the global pairing

$$\begin{aligned}\langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta_\Sigma^\circ \rangle_{C(R_f)} &= \textstyle\sum_{\mathrm{i,j}} \sum_{\alpha,\beta} s(\mathrm{i,j}) \langle m_\alpha, m_\beta \rangle \int_{\tilde{S}_2(R_f)} \phi_{\mathrm{i},\alpha}(\iota(g)) E_{\mathrm{j},\beta}(g)\, dg \\ &= \mathrm{vol}(R_f) \textstyle\sum_{\mathrm{i,j}} \sum_{\alpha,\beta} s(\mathrm{i,j}) \langle m_\alpha, m_\beta \rangle I(\frac{1}{2}, \phi_{\mathrm{i},\alpha}, E_{\mathrm{j},\beta}).\end{aligned}$$

The second equality is because of ϕ_f and φ_f are both R_f invariant, and also by our normalization of measure in section "The Choice of Measures" that $\mathrm{vol}(\mathrm{SO}(n)) = 1$. Using (3.12) we get:

$$\langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta_\Sigma^\circ \rangle_{C(R_f)} = \frac{1}{P_\infty(\mu,\lambda)} \cdot \frac{\mathrm{vol}(R_f) A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma)} \cdot \frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1,\chi_2))}{L_f(2, \chi_1\chi_2^{-1})}. \tag{3.21}$$

Dividing by the period $p^{\epsilon_\Pi}(\Pi)$ to get the rational class $\vartheta^\circ_{\Pi,\epsilon_\Pi}$ now proves the following

Theorem 3.29 *Let $\Pi \in Coh(G_3,\mu)$ with $\mu \in X_0^+(T_3)$, and $\Sigma(\chi_1,\chi_2)$ be the induced representation of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ *as in Sect. 2.3. Let $\lambda \in X_0^+(T_2)$ be the dominant integral 'parallel' weight determined by d_1, d_2 as: $\lambda = (\lambda_v)_{v \in S_\infty}$, where each $\lambda_v = (-d_2, -d_1)$. Assume that $s = 1/2$ is critical for $L(s, \Pi \times \Sigma(\chi_1,\chi_2))$ and that $\mu \succ \lambda^\vee$. Then there exist nonzero complex numbers $P_\infty(\mu,\lambda)$ and $p^{\epsilon_\Pi}(\Pi)$ such that*

$$\frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1,\chi_2))}{P_\infty(\mu,\lambda)\, p^{\epsilon_\Pi}(\Pi)\, L_f(2, \chi_1\chi_2^{-1})} = (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}.$$

This already shows that the left hand side is algebraic. Moreover we can study the action of the Galois group of $\mathbb{Q}$ on the various quantities.

The Main Identity for the Critical Values $L_f(m, \Pi \otimes \chi)$

Now recall the fact that the L-value at $s = 1/2$ attached to the pair of representations $(\Pi, \Sigma(\chi_1,\chi_2))$ decomposes as

$$L_f(\tfrac{1}{2}, \Pi \times \Sigma(\chi_1,\chi_2)) = L_f(1, \Pi \otimes \chi_1) L_f(0, \Pi \otimes \chi_2) = L_f(1+d_1, \Pi \otimes \chi_1^\circ) L_f(d_2, \Pi \otimes \chi_2^\circ)$$

Now we consider the four cases as delineated in Proposition 3.6. Before getting into details, let's comment that in each case the L-value $L(2, \chi_1\chi_2^{-1})$ in the denominator of the left hand side of the above theorem, which is the same as $L(2 + d_1 - d_2, \chi_1^\circ \chi_2^{\circ -1})$, is in fact a critical value of a classical Dirichlet L-function of a finite order Hecke character of F for the choices of d_j and $\chi_j^\circ, j = 1, 2$.

Case 1. χ is even, that is, $\varepsilon_{\chi_v} = 1\!\!1$ for all $v \in S_\infty$.

Case 1a. $m \in \{2, 4, \ldots, n_{\rm ev}\}$. Take $d_1 = m - 1$, $d_2 = -1$, $\chi_1^\circ = \chi$, and $\chi_2^\circ = 1\!\!1$. Put $\lambda_v = (1, 1 - m)$. Then Theorem 3.29 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_r^+(\Pi)\, L_f(2 + m, \chi)} = (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \tag{3.22}$$

where the modified period is defined as

$$\Omega_r^+(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(-1, \Pi)^{-1}. \tag{3.23}$$

Case 1b. $m \in \{1 - n_{\rm ev}, \ldots, -3, -1\}$. Take $d_1 = 1$, $d_2 = m$, $\chi_1^\circ = 1\!\!1$, and $\chi_2^\circ = \chi$. Put $\lambda_v = (-m, -1)$. Then Theorem 3.29 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_l^+(\Pi)\, L_f(3 - m, \chi^{-1})} = (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \tag{3.24}$$

where the modified period is defined as

$$\Omega_l^+(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(2, \Pi)^{-1}. \tag{3.25}$$

Case 2. χ is odd, that is, $\varepsilon_{\chi_v} = \mathrm{sgn}$ for all $v \in S_\infty$. In this case, we fix once and for all, a totally odd quadratic Hecke character ξ of F.

Case 2a. $m \in \{1, 3, \ldots, n_{\rm od}\}$. Take $d_1 = m - 1$, $d_2 = 0$, $\chi_1^\circ = \chi$, and $\chi_2^\circ = \xi$. Put $\lambda_v = (0, 1 - m)$. Then Theorem 3.29 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_r^-(\Pi)\, L_f(m + 1, \chi\xi^{-1})} = (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \tag{3.26}$$

where the modified period is defined as

$$\Omega_r^-(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(0, \Pi \otimes \xi)^{-1}. \tag{3.27}$$

Case 2b. $m \in \{1 - n_{\rm od}, \ldots, -2, 0\}$. Take $d_1 = 0$, $d_2 = m$, $\chi_1^\circ = \xi$, and $\chi_2^\circ = \chi$. Put $\lambda_v = (-m, 0)$. Then Theorem 3.29 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_l^-(\Pi)\, L_f(2 - m, \xi\chi^{-1})} = (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \tag{3.28}$$

where the modified period is defined as

$$\Omega_l^-(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(1, \Pi \otimes \xi)^{-1}. \tag{3.29}$$

3.4 Galois Equivariance and Proof of Theorem 1.1

The Action of Aut($\mathbb{C}$)

We study Galois equivariance, i.e., behaviour under the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ of all the quantities in the main identity for each of the four cases. Let's parse the Galois action on the various ingredients involved in the main identities:

- The Poincaré duality pairing $\langle\ ,\ \rangle$ is Galois-equivariant. (See, for example, [20, Prop. 3.14].)
- The Galois action on the class $\vartheta^\circ_{\Pi,\epsilon_\Pi}$. Due to our specific choice of finite Whittaker vectors w_{Π_f}, exactly as in [20, Prop. 3.15], we get

$$\sigma(\vartheta^\circ_{\Pi,\epsilon_\Pi}) \;=\; \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{{}^\sigma\Sigma_f})}\vartheta^\circ_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}} \;=\; \frac{\sigma(\mathcal{G}(\chi_1\chi_2))}{\mathcal{G}({}^\sigma\chi_1{}^\sigma\chi_2)}\vartheta^\circ_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}}.$$

  Furthermore, for Dirichlet characters χ_1 and χ_2, it's well-known (see [27, Lemma 8]) that

$$\sigma\left(\frac{\mathcal{G}(\chi_1\chi_2)}{\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)}\right) = \frac{\mathcal{G}({}^\sigma\chi_1{}^\sigma\chi_2)}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)}.$$

  Putting the above two together we get

$$\sigma(\vartheta^\circ_{\Pi,\epsilon_\Pi}) \;=\; \frac{\sigma(\mathcal{G}(\chi_1)\mathcal{G}(\chi_2))}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)}\,\vartheta^\circ_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}}. \tag{3.30}$$

- To understand the Galois action on the class ϑ°_Σ, we begin with the function φ_f which is the finite part of φ_{χ_1,χ_2} as defined in (3.10). Let's denote φ_f also as $\varphi_{\chi_{1f},\chi_{2f}}$. Now, $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts on φ_f by acting on all it's local components. The action of $\mathrm{Aut}(\mathbb{C})$ on Σ_v is given by acting on the values of a function in the induced space (see [18, Sect. 1.1]). For the local components of $\varphi_{\chi_{1f},\chi_{2f}}$ (with notations suitably modified) we get for the spherical vectors:

$${}^\sigma f_v^{\mathrm{sp}}(\chi_{1v},\chi_{2v}) \;=\; f_v^{\mathrm{sp}}({}^\sigma\chi_{1v},{}^\sigma\chi_{2v}),$$

  and from our choices of new vectors made in Proposition 3.8, we get for $v \in S_\Sigma \setminus S_{\chi_2}$:

$${}^\sigma f_v^{\mathrm{new}}(\chi_{1v},\chi_{2v}) \;=\; f_v^{\mathrm{new}}({}^\sigma\chi_{1v},{}^\sigma\chi_{2v}),$$

however, for $v \in S_{\chi_2}$—the set of ramified primes for χ_2, we get:

$$ {}^{\sigma}\!f_v^{\mathrm{new}}(\chi_{1v}, \chi_{2v}) \ = \ \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}} f_v^{\mathrm{new}}({}^{\sigma}\chi_{1v}, {}^{\sigma}\chi_{2v}). $$

(Note that the quantity $\sigma(q_v^{n_2/2})/q_v^{n_2/2}$ is ± 1.) Putting these together we get

$$ {}^{\sigma}\varphi_{\chi_{1f},\chi_{2f}} \ = \ \left(\prod_{v \in S_{\chi_2}} \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}} \right) \varphi_{{}^{\sigma}\chi_{1f},{}^{\sigma}\chi_{2f}}. $$

Since the Eisenstein map $\mathcal{F}_\Sigma$ in (2.4) is $\mathrm{Aut}(\mathbb{C})$-equivariant, we get

$$ \sigma(\vartheta_\Sigma^\circ) \ = \ \left(\prod_{v \in S_{\chi_2}} \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}} \right) \vartheta_{{}^{\sigma}\Sigma}^\circ. \tag{3.31} $$

- Now we look at Galois action on the quantity A_Σ. Recall from (3.6) that $A_\sigma = \prod_{v \in S_\Sigma} A_v$ and the values of A_v are computed in Proposition 3.11. Now the volume of $\mathcal{O}_v$, by our choice of measures, is rational. We easily deduce that

$$ \sigma(A_\Sigma) \ = \ \left(\prod_{v \in S_{\chi_2}} \frac{\sigma(q_v^{-n_2/2})}{q_v^{-n_2/2}} \right) \cdot \frac{\sigma(\mathcal{G}(\chi_2))}{\mathcal{G}({}^{\sigma}\chi_2)} A_{{}^{\sigma}\Sigma}. \tag{3.32} $$

From (3.31) and (3.32) we get

$$ \sigma(A_\Sigma\, \vartheta_\Sigma^\circ) \ = \ \frac{\sigma(\mathcal{G}(\chi_2))}{\mathcal{G}({}^{\sigma}\chi_2)} A_{{}^{\sigma}\Sigma}\, \vartheta_{{}^{\sigma}\Sigma}^\circ. \tag{3.33} $$

- The quantities L_Σ and $L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma)$ are Galois equivariant as they are finite products of local critical L-values; this follows from [20, Prop. 3.17].
- The volume terms $\mathrm{vol}(R_f)$ and V_Σ are rational numbers by our choice of measures.
- Finally, for a totally even Dirichlet character ϱ of F, and an even positive integer r, it's well-known that we have the following rationality result for the critical value $L_f(r, \varrho)$:

$$ \sigma\left(\frac{L_f(r, \varrho)}{(2\pi i)^r \mathcal{G}(\varrho)} \right) \ = \ \frac{L_f(r, {}^{\sigma}\varrho)}{(2\pi i)^r \mathcal{G}({}^{\sigma}\varrho)}. \tag{3.34} $$

Proof of Theorem 1.1

The line of proof is the same in each of the four cases, and so we only present the details in **Case 1a**, and leave the remaining three to the reader. To begin, rewrite (3.22) as

$$\frac{L_f(m, \Pi \otimes \chi)}{P^1_\infty(\mu, m)\,\Omega_r^+(\Pi)\mathcal{G}(\chi)} = \left(\mathrm{vol}(R_f)V_\Sigma\right) \cdot \left(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)\right) \cdot \left(\frac{L_f(2+m, \chi)}{(2\pi i)^{2+m}\mathcal{G}(\chi)}\right) \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^\circ_\Sigma \rangle,$$

where $P^1_\infty(\mu, m) = (2\pi i)^{2+m} P_\infty(\mu, m)$. Now apply $\sigma \in \mathrm{Aut}(\mathbb{C})$ to both sides. The first three parentheses on the right hand side are $\mathrm{Aut}(\mathbb{C})$-equivariant as explained above. For σ applied to the pairing of the cohomology classes, after using Galois-equivariance of the duality pairing, and after using (3.30) and (3.33) we get

$$\sigma\left(\langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^\circ_\Sigma \rangle\right) = \langle \vartheta^\circ_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma} \rangle \frac{\sigma(\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)^2)}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)^2}.$$

In the current situation of **Case 1a,** we have $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^\circ) = \mathcal{G}(\chi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^\circ) = \mathcal{G}(\mathbb{1}) = 1$. Hence,

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^1_\infty(\mu, m)\,\Omega_r^+(\Pi)\mathcal{G}(\chi)}\right) = \left(\mathrm{vol}(R_f)V_{{}^\sigma\Sigma}\right) \cdot \left(L_{{}^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2}, {}^\sigma\Pi \times {}^\sigma\Sigma)\right) \cdot \left(\frac{L_f(2+m, {}^\sigma\chi)}{(2\pi i)^{2+m}\mathcal{G}({}^\sigma\chi)}\right) \cdot \langle \vartheta^\circ_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma} \rangle \frac{\sigma(\mathcal{G}(\chi))}{\mathcal{G}({}^\sigma\chi)},$$

which is the same as

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^1_\infty(\mu, m)\,\Omega_r^+(\Pi)\mathcal{G}(\chi)}\right) = \frac{L_f(m, {}^\sigma\Pi \otimes {}^\sigma\chi)}{P^1_\infty({}^\sigma\mu, m)\,\Omega_r^+({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)} \cdot \frac{\sigma(\mathcal{G}(\chi))}{\mathcal{G}({}^\sigma\chi)},$$

and which in turn may be rewritten as

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^1_\infty(\mu, m)\,\Omega_r^+(\Pi)\mathcal{G}(\chi)^2}\right) = \frac{L_f(m, {}^\sigma\Pi \otimes {}^\sigma\chi)}{P^1_\infty({}^\sigma\mu, m)\,\Omega_r^+({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)^2},$$

proving the Galois-equivariant version of Theorem 1.1 in **Case 1a**. This implies in particular that

$$L_f(m, \Pi \otimes \chi) \approx_{\mathbb{Q}(\Pi,\chi)} P^1_\infty(\mu, m)\,\Omega_r^+(\Pi)\,\mathcal{G}(\chi)^2.$$

In the remaining cases of **1b, 2a** and **2b**, the proof is similar; let's suffice it to mention two details:

1. The appearance of Gauß sums come from the L-value $L(2, \chi_1\chi_2^{-1})$ and from the Galois action on cohomology classes as in (3.30) and (3.33), and total contribution is of the form

$$\mathcal{G}(\chi_1\chi_2^{-1})\mathcal{G}(\chi_1\chi_2)\mathcal{G}(\chi_2) \approx \mathcal{G}(\chi_1^2\chi_2) \approx \mathcal{G}(\chi_1)^2\mathcal{G}(\chi_2).$$

2. The power of $(2\pi i)$ to modify $P_\infty(\mu, m)$ comes from (3.34) as applied to

$$L_f(2, \chi_1\chi_2^{-1}) = L(2 + d_1 - d_2, \chi_1^\circ\chi_2^{\circ-1}).$$

   This gives us $P_\infty^2(\mu, m) = P_\infty(\mu, m)(2\pi i)^{3-m}$, $P_\infty^3(\mu, m) = P_\infty(\mu, m)(2\pi i)^{m+1}$ and $P_\infty^4(\mu, m) = P_\infty(\mu, m)(2\pi i)^{2-m}$, in cases **1b**, **2a**, and **2b**, respectively.

This concludes the proof of Theorem 1.1.

3.5 Example: Symmetric Square L-Function of a Hilbert Modular Form

Let φ be a primitive holomorphic cuspidal Hilbert modular form over F of weight $(k_1, \ldots, k_d)$, where $d = d_F$. Suppose that all the k_j have the same parity, and that φ is not of CM-type. The purpose of this section is to put on record that Theorem 1.1 applies to give a rationality result for all the critical values of the symmetric square L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ attached to φ, twisted by a finite order Dirichilet character χ.

We will work with the L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ in the automorphic context, toward which let $\Pi(\varphi)$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ attached to φ. By Gelbart–Jacquet [11] we know the existence of an isobaric automorphic representation $\mathrm{Sym}^2(\Pi(\varphi))$ of $\mathrm{GL}_3(\mathbb{A}_F)$, defined as $\mathrm{Sym}^2(\Pi(\varphi)) = \otimes_v' \mathrm{Sym}^2(\Pi_v(\varphi))$, where $\mathrm{Sym}^2(\Pi_v(\varphi))$ is an irreducible admissible representation of $\mathrm{GL}_3(F_v)$ obtained via the local Langlands symmetric-square transfer of $\Pi_v(\varphi)$. If φ is not of CM-type, i.e., $\Pi(\varphi)$ is not a dihedral representation then it's well-known that $\mathrm{Sym}^2(\Pi(\varphi))$ is cuspidal. If $L(s, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi)$ denotes the standard degree-3 L-function of $\mathrm{Sym}^2(\Pi(\varphi))$ twisted by χ then we have

$$L(s, \mathrm{Sym}^2\varphi, \chi) = L(s - k_0 + 1, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi), \tag{3.35}$$

where $k_0 = \min(k_j)$. This may be seen as in the verification of [24, Thm. 1.4, (1)].

For convenience, let's suppose that all the $k_j \geq 2$ are even. Then $\Pi(\varphi)$ is cohomological to the weight $\mu = (\mu_j)$ where $\mu_j = ((k_j - 2)/2, -(k_j - 2)/2)$.

Following [21], we may verify that $\mathrm{Sym}^2(\Pi(\varphi)) \in \mathrm{Coh}(G_3; \mathrm{Sym}^2(\mu))$ for the weight $\mathrm{Sym}^2(\mu) = (\mathrm{Sym}^2(\mu_j))$, where $\mathrm{Sym}^2(\mu_j) = (k_j - 2, 0, 2 - k_j)$.

In the notation of Theorem 1.1, we get $n = \min(n_j) = k_0 - 2$. For convenience again, let's consider the case when χ is totally even. Then the critical set for $L(s, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi)$ is given by

$$\{3 - k_0, \dots, -1;\ 2, 4, \dots, k_0 - 2\}.$$

From (3.35) we get that the critical set for $L(s, \mathrm{Sym}^2\varphi, \chi)$ is the set

$$\{2, 4, \dots, k_0 - 2;\ k_0 + 1, k_0 + 3, \dots, 2k_0 - 3\}.$$

This is to be interpreted as an empty set if $k_0 = 2$. If m is critical for $L(s, \mathrm{Sym}^2\varphi, \chi)$, and is on the right of the center of symmetry, i.e., $k_0 + 1 \leq m \leq 2k_0 - 3$ and m is odd, then **Case 1a.** of Theorem 1.1 takes the form:

$$L(m, \mathrm{Sym}^2\varphi, \chi) \approx_{\mathbb{Q}(\varphi,\chi)} P^1_\infty(\mathrm{Sym}^2(\mu), m - k_0 + 1)\, \Omega_r^+(\mathrm{Sym}^2(\Pi(\varphi))\, \mathcal{G}(\chi)^2. \tag{3.36}$$

It's an easy exercise to see that $\mathbb{Q}(\mathrm{Sym}^2(\Pi(\varphi)), \chi) \subset \mathbb{Q}(\Pi(\varphi), \chi) = \mathbb{Q}(\varphi, \chi)$. (See also statements (4) and (5) of [24, Thm. 1.4].) Moreover, we may state this result in a Galois-equivariant manner.

A comparison of (3.36) with the main result of Sturm [28] for the critical values of the symmetric-square L-functions for an elliptic modular form would lead us to speculate that our global period $\Omega_r^+(\mathrm{Sym}^2(\Pi(\varphi))$ is related in some explicit way to the Petersson inner product $\langle \varphi, \varphi \rangle$.

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Indecomposable Harish-Chandra Modules for Jacobi Groups

Martin Raum

Abstract We describe some indecomposable $(\mathfrak{g}, \mathrm{K})$-modules for Jacobi groups that admit an automorphic realization with possible singularities. A particular tensor product decomposition of universal enveloping algebras of Jacobi Lie algebras, which does not lift to the groups, allows us to study distinguished highest weight modules for the Heisenberg group. We encounter modified theta series as components of vector-valued Jacobi forms, whose arithmetic type is not completely reducible.

1 Introduction

Analytic properties of modular forms can be described in an advantageous way when passing to automorphic forms and inspecting the arising component at infinity, which is a representation of a real Lie group. The resulting notion of $(\mathfrak{g}, \mathrm{K})$-modules may be viewed as a systematic way of interpreting differential equations satisfied by the initially studied modular form.

Among the important contributions of representation theory to modular forms is the classification of irreducible local components in context of the local Langlands Correspondence. For the infinite place, it was achieved by Knapp and Zuckerman [12] and Langlands [13]. There are cases of automorphic Harish-Chandra modules, principal series that correspond to Eisenstein series in the classical language, for which local components at infinity are not irreducible. Their socle series and extension classes have not been determined in full generality, but many special cases are settled. In fact, any endeavor to handle extensions of Lie group representations has to be started with care: Gel'fand and Ponomarev [10] illustrated in the late '60s that the class of all indecomposables is prohibitively large.

The theory of harmonic weak Maaß forms [5, 26] has generated innumerable applications during the past decade. An overview of the status in 2009 can be found

M. Raum (✉)
Institutionen för Matematiska vetenskaper, Chalmers tekniska högskola och Göteborgs Universitet, SE-412 96 Göteborg, Sweden
e-mail: martin@raum-brothers.eu; raum@chalmers.se

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in [14]. Other variants of harmonic weak Maaß forms have entered the stage in the mean time. Generalizing the growth condition in [15], a notion of harmonic weak Maaß Jacobi forms was found in [3], which after correcting issues with singularities led to a classification in [4]. That theory was formulated for so-called one-variable Jacobi forms. Further generalizations to several elliptic variables were discussed in [7, 23]. The author has also initiated a theory of harmonic weak Siegel Maaß forms [22].

Connecting ideas from representation theory with the multitude of notions of harmonic weak Maaß forms that begin to emerge, we suggest the study of the $(\mathfrak{g}, \mathrm{K})$-modules connected to these Maaß forms. The first published instance of such a study can be found in [17]. It was extended and employed for a proof of existence of harmonic weak Siegel Maaß forms in [22]. In [2], Bringmann and Kudla attempted a classification of certain $(\mathfrak{g}, \mathrm{K})$-modules for $\mathrm{SL}_2(\mathbb{R})$ by employing a condition on their infinitesimal character. Their work suffices to cover most phenomena that have so far been recorded in the literature on real-analytic modular forms, but [8] features a "sesquiharmonic" Maaß form for the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ that does not fall into the scope of [2] as it produces a $(\mathfrak{g}, \mathrm{K})$-module with non-diagonalizable action of the Casimir operator. In fact, at the end of [2], there is an infinite family of $(\mathfrak{g}, \mathrm{K})$-modules for which it seems believable that corresponding weak Maaß forms can be constructed.

In this paper, we focus on $(\mathfrak{g}, \mathrm{K})$-modules for the Jacobi group that are in the sense of Corollary 2.2 supported on the Heisenberg Lie algebra. The decomposition there reads $\mathrm{ULie}(\mathrm{G}^{\mathrm{J}}(V)) \cong \mathrm{ULie}(\mathrm{SL}_2(\mathbb{R})) \otimes \mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$, where $\mathrm{G}^{\mathrm{J}}(V)$ and $\mathrm{H}(V)$ are a Jacobi group and a Heisenberg group, respectively, that are associated to a real quadratic space V, the universal enveloping Lie algebra is denoted by ULie, and the subscript $\mathfrak{z}$ denotes a localization at the center (see Sect. 2.6 for details). The most prominent case of representations supported on the Heisenberg Lie algebra arises from theta series. They generate $(\mathfrak{g}, \mathrm{K})$-modules that under the decomposition in Corollary 2.2 yield highest weight modules for the Heisenberg group. The parabolic subgroups associated with these highest weight modules are distinguished by the action of $\mathfrak{sl}_2$ on $\mathrm{H}(V)$. To include $(\mathfrak{g}, \mathrm{K})$-modules generated by real-analytic Andrianov-Siegel theta series, we suggest to consider highest weight modules of the Heisenberg group with respect to further parabolic subgroups. This mimics some of the essential features of cohomological representations of reductive groups [20]. A classification of such highest weight modules in terms of Verma modules is executed in Sect. 4. For each of them we provide Jacobi forms that generate it. Due to the relatively simple structure of the Heisenberg group, our highest weight modules amount to forcing meromorphic or antimeromorphic behavior in the elliptic variables of corresponding Jacobi forms. The Jacobi forms that we provide deserve further discussion, which we postpone for the moment. From a classical perspective the most interesting features are novel Jacobi slash actions, whose discussion we also postpone.

Beyond highest weight modules, the notion of Harish-Chandra modules is crucial for $(\mathfrak{g}, \mathrm{K})$-modules in general. Their definition in the case of reductive groups requires that K-isotypical components are finite dimensional. We explain in Sect. 3

why that definition for reductive groups cannot be extended in a straightforward way to the case of the Jacobi groups. In fact, it would be too restrictive, and include only meromorphic and antimeromorphic Jacobi forms, while excluding, for example, real-analytic indefinite theta series [24, 26]. We suggest an ad hoc definition that replaces the common notion of Harish-Chandra modules.

Our notion of Harish-Chandra modules for the Jacobi group leads us to the second classification result in Sect. 5. We impose an additional condition inspired by automorphic realizations: Weak modular realizations. Effectively, this condition means that we restrict to $(\mathfrak{g}, \mathrm{K})$-modules that correspond to real-analytic Jacobi forms with possible singularities. Such a condition is very much analogous to the condition of being automorphic in the sense that it leads to a weak automorphic form (i.e. an automorphic form with possible singularities). Weak modular realizations are the raison d'être of this paper: to start identifying a class of real-analytic weak modular forms whose theory can be founded not only on analysis but equally on real representation theory and geometric arguments as in the proof of Theorem 4.2 of [5].

Our investigation of indecomposable Harish-Chandra modules leads us naturally to consider indecomposable slash actions. Beyond the classical, scalar-valued Jacobi slash action $\big|_{k,m}$ there are vector-valued ones. This extension from scalar-valued to vector-valued Jacobi forms is well-known in the context of automorphic vector bundles on Shimura varieties and mixed Shimura varieties. Though, it has not yet spread very much in the literature on classical modular forms, it does emerge in applications. The most established case of vector-valued modular forms are associated to Weil representations, which occur in the context of theta series and theta lifts. Another example is provided by correlation functions of logarithmic conformal field theories. Skipping the details, we content ourselves with expressing our expectation that higher correlation functions of specific logarithmic conformal field theories have Fourier-Jacobi coefficients for the slash actions that we explain now.

We simplify this exposition compared to what we treat in Sect. 4, and employ the classical notion of Jacobi groups as in [9]. Inside the Jacobi group, there is a copy of $K \times Z := \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}$, which stabilizes the point $(i, 0)$ in the Jacobi upper half space. Each of its representations yields a slash action. So far, in absence of any need for a more general concept, only irreducible representations of Z have been considered. They are one-dimensional and therefore yield scalar-valued, classical slash actions. A typical reducible, indecomposable representation of Z can be realized by the action on polynomials $p(X) \mapsto p(X + \kappa)$, $\kappa \in Z$. We denote the associated slash action of weight k by $\big|_{k,0[1]}$. In general, we obtain slash actions $\big|_{k,m[d]}$ for any $d \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}$, and $m \in \mathbb{C}$.

We close the introduction with some open questions.

Degenerate Cases We assume throughout the paper that the quadratic space V is non-degenerate. This excludes, in particular, the case of Jacobi forms of index 0, which have applications to the counting of Feynman-like graphs [11]. Representations of the Heisenberg group, in this case descend to representation of the Abelian group $V \otimes W$ (see Sect. 2.2 for the notation), and the notion of highest weight modules continues to make sense. Extending Sect. 5 to this setting would be interesting.

Higher Dimensional Quadratic Spaces In Sect. 5 we restrict to the case of one-dimensional V. Including general V amounts to understanding singularities of $\widehat{\mu}$-functions that arise from [24] and extending holomorphicity statements from [7, 23] to differential operators of order greater than 1.

Reductive Groups This paper focuses on the representation theory of the Heisenberg group, which is particularly accessible when amended with our notion of Harish-Chandra modules. Recently, Bringmann and Kudla [2] studied $(\mathfrak{g}, \mathrm{K})$-modules for $\mathrm{SL}_2(\mathbb{R})$, but required that the Casimir operator act by scalars. All composition factors of the $(\mathfrak{g}, \mathrm{K})$-modules that they found are highest weight modules. Bringmann and Kudla at the end of their treatment give an infinite family of $(\mathfrak{g}, \mathrm{K})$-modules whose members have the very same set of composition factors but for which the Casimir operator does not act by scalars. It would be interesting to investigate how we can extend reasonably the notion of weak modular realizations to $\mathrm{SL}_2(\mathbb{R})$. Monodromy around the cusps should play a key role.

2 Preliminaries

2.1 The Metaplectic Group

Let $\mathrm{G} := \mathrm{G}(\mathbb{R}) := \mathrm{Mp}_2(\mathbb{R})$ be the connected double cover of $\mathrm{SL}_2(\mathbb{R})$. We will view G as the non-split central extension $0 \to \{\pm 1\} \to \mathrm{G} \to \mathrm{SL}_2(\mathbb{R}) \to 1$, and denote elements as pairs (g, ω) with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\omega : \mathbb{H} \to \mathbb{C}$ a choice of holomorphic square root of $c\tau + d$. Multiplication in this realization is given by $(g_1, \omega_1)(g_2, \omega_2) = (g_1 g_2,\, \omega_1 \circ g_2 \cdot \omega_2)$.

2.2 The Heisenberg Group

Let $V := V(\mathbb{R})$ be a non-degenerate, real quadratic space with quadratic form $q : V \to \mathbb{R}$ and bilinear form $\langle v_1, v_2 \rangle = q(v_1 + v_2) - q(v_1) - q(v_2)$. Denote its complexification $V \otimes \mathbb{C}$ by $V(\mathbb{C})$. Fix the standard sympletic lattice $W \cong \mathbb{Z}e_W \oplus \mathbb{Z}f_W$ with symplectic form $\omega(e_W, f_W) := 1$. For convenience, we extend ω to $V \otimes W$ by $\omega(v_1 \otimes w_1, v_2 \otimes w_2) := \frac{1}{2}\langle v_1, v_2 \rangle \omega(w_1, w_2)$. The dual of V is denoted by $V^\vee$, and given a lattice $L \subset V$, we let $L^\vee$ be the dual lattice.

We let the Heisenberg group $\mathrm{H}(V)$ attached to V be the one that is associated with the symplectic space $V \otimes W$. We write (λ, μ) for elements of $V \otimes W$ arising from the isomorphism of vector spaces $V \otimes W \cong V \oplus V$ that originates in the polarization of W. Specifically, we have a short exact sequence $0 \to \mathbb{R} \to \mathrm{H}(V) \to V \otimes W \to 0$, and multiplication in $\mathrm{H}(V)$ is given concretely by

$$(\lambda_1, \mu_1, \kappa_1)(\lambda_2, \mu_2, \kappa_2) = \big(\lambda_1 + \lambda_2, \mu_1 + \mu_2,\ \kappa_1 + \kappa_2 + \tfrac{1}{2}\langle \lambda_1, \mu_2 \rangle - \tfrac{1}{2}\langle \lambda_2, \mu_1 \rangle\big).$$

Throughout the paper we write Z for the center of $\mathrm{H}(V)$.

The action of $\mathrm{G} \cong \mathrm{Sp}(W(\mathbb{R})) \cong \mathrm{SL}_2(\mathbb{R})$ on W from the right extends to an action of G on $\mathrm{H}(V)$. In W-coordinates we have

$$(\lambda, \mu, \kappa)\,(g, \omega) \;=\; (a\lambda + c\mu, b\lambda + d\mu, \kappa).$$

2.3 The Real Jacobi Group

For a real quadratic space V as above, the real Jacobi group $\mathrm{G}^{\mathrm{J}}(V)$ is the extension $0 \to \mathrm{H}(V) \to \mathrm{G}^{\mathrm{J}}(V) \to \mathrm{G} \to 1$ of G by $\mathrm{H}(V)$. Multiplication in $\mathrm{G}^{\mathrm{J}}(V)$ is defined by

$$\big((g_1, \omega_1), h_1\big) \cdot \big((g_2, \omega_2), h_2\big) \;=\; \big((g_1 g_2, \omega_1 \circ g_2 \cdot \omega_2),\ h_1 g_2 \cdot h_1\big).$$

Recall from the introduction that we write $Z \cong \mathbb{R}$ for the center of $\mathrm{G}^{\mathrm{J}}(V)$.

We often consider elements of $\mathrm{H}(V)$ as elements of $\mathrm{G}^{\mathrm{J}}(V)$ via the inclusion $\mathrm{H}(V) \subset \mathrm{G}^{\mathrm{J}}(V)$. Shorthand notation for elements of G viewed as elements of $\mathrm{G}^{\mathrm{J}}(V)$ is provided by the following section to the canonical projection $\mathrm{G}^{\mathrm{J}}(V) \twoheadrightarrow G$:

$$\mathrm{G} \ni g \longmapsto \big(g, (0,0,0)\big) \in \mathrm{G}^{\mathrm{J}}(V).$$

2.4 The Maximal Compact Subgroup

In the literature known to the author the group $\mathrm{Spin}_2(\mathbb{R}) \times \mathbb{R} \subset \mathrm{G}^{\mathrm{J}}(V)$ (or $\mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}$ when working without double covers) was chosen as an analogue of the maximal compact subgroup in the reductive case. This is most natural when considering Jacobi forms as functions on $\mathbb{H}^{\mathrm{J}}(V) = \mathbb{H} \times V(\mathbb{C})$. It also forces a diagonalizable action of the center $Z \subset \mathrm{G}^{\mathrm{J}}(V)$. Since in this paper, we also consider functions on $\widetilde{\mathbb{H}}^{\mathrm{J}}(V) = \mathbb{H} \times V(\mathbb{C}) \times \mathbb{R}$, we set $K = \mathrm{Spin}_2(\mathbb{R}) \subset \mathrm{G}^{\mathrm{J}}(V)$. This provides us with a richer supply of Jacobi forms, allowing for central representations that are not completely reducible.

2.5 The Jacobi Modular Group

Write $\Gamma := \mathrm{Mp}_2(\mathbb{Z})$ for the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under the natural projection $\mathrm{G} \to \mathrm{SL}_2(\mathbb{R})$. If $L \subset V$ is a lattice, the discrete Jacobi group $\Gamma^{\mathrm{J}}(L)$ is defined as

$$\Gamma^{\mathrm{J}}(L) = \big\{\gamma^{\mathrm{J}} = \big((\gamma,\omega),(\lambda,\mu,\kappa)\big) \,:\, \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ \omega(\tau)^2 \\ = c\tau + d,\ \lambda,\mu \in L,\ \kappa \in \tfrac{1}{2}\langle L, L\rangle\big\} \;\subset\; \mathrm{G}^{\mathrm{J}}(V).$$

As opposed to most other authors, we have to include elements γ^{J} with $\kappa \neq 0$, while the Jacobi group is usually defined as the quotient of our $\Gamma^{\mathrm{J}}(L)$ by its center. Note also that if L is not even, then we enlarge the center correspondingly: we have denoted by $\langle L, L\rangle \subseteq \mathbb{Q}$ the image of L under the bilinear form associated to V.

2.6 The Jacobi Lie Algebra

Recall that we assume V non-degenerate. We write $\mathfrak{g}^{\mathrm{J}}(V)$ for the Lie algebra attached to $\mathrm{G}^{\mathrm{J}}(V)$. One reference for it is [7], which extends some of the theory in [1].

The Lie algebra of $G = \mathrm{Sp}(W)$ admits an $\mathfrak{sl}_2$-triple (e, h, f) with matrix realizations

$$e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \tag{2.1}$$

which arise from W-coordinates. The Lie brackets are $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$.

We can identify the underlying vector spaces of $\mathrm{H}(V)$ and $\mathrm{Lie}(\mathrm{H}(V))$ via the Lie exponential. Under this identification we have the Lie bracket $[(v_1 \otimes w_1, \kappa_1), (v_2 \otimes w_2, \kappa_2)] = 2\omega(v_1 \otimes w_1, v_2 \otimes w_2)z$, where z is a generator of the center $\mathfrak{z}$ of $\mathrm{Lie}(\mathrm{H}(V))$.

The following ideas are explained in more detail in [7] and originate in [6, 16]. We let $\mathrm{U}(\mathfrak{g}^{\mathrm{J}}(V))$ and $\mathrm{ULie}(\mathrm{H}(V))$ be the universal enveloping algebras of $\mathfrak{g}^{\mathrm{J}}(V)$ and $\mathrm{Lie}(\mathrm{H}(V))$. Let

$$\mathrm{U}(\mathfrak{g}^{\mathrm{J}}(V))_{\mathfrak{z}} \;:=\; \mathrm{U}(\mathfrak{g}^{\mathrm{J}}(V)) \otimes_{\mathfrak{z}} \mathrm{Frac}(\mathfrak{z}),$$

where $\mathrm{Frac}(\mathfrak{z})$ is the fraction field of $\mathrm{U}(\mathfrak{z})$. We denote by $1_V \in V \otimes V^\vee$ the canonical diagonal. Consider the map η from $\mathfrak{sl}_2$ to $\mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$:

$$\eta(e) \;:=\; \frac{1}{2z}\langle 1_V \otimes e_W,\, 1_V \otimes e_W\rangle, \quad \eta(f) \;:=\; -\frac{1}{2z}\langle 1_V \otimes f_W,\, 1_V \otimes f_W\rangle, \\ \eta(h) \;:=\; \frac{1}{2z}\big(\langle 1_V \otimes e_W,\, 1_V \otimes f_W\rangle + \langle 1_V \otimes f_W,\, 1_V \otimes e_W\rangle\big), \tag{2.2}$$

where the scalar product is taken with respect to the middle components $V^\vee$ of $V \otimes V^\vee \otimes W$.

Lemma 2.1 (cf. Section 5.1 of [7]) *The map η is an isomorphism of the Lie algebras $\mathfrak{sl}_2$ and $\eta(\mathfrak{sl}_2)$, where the image is equipped with the commutator bracket of* $\mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$.

Proof The map is an isomorphism of vector spaces, so that it remains to check the Lie brackets:

$$\big[\eta(e), \eta(f)\big] \;=\; \eta(h), \quad \big[\eta(h), \eta(e)\big] \;=\; 2\eta(e), \quad \big[\eta(h), \eta(e)\big] \;=\; 2\eta(e).$$

We derive the first equality, and leave the others to the reader. Fix a basis $(v_i)_i$ of V, and let $(v_i^\vee)_i$ be the dual basis. We have $1_V = \sum_i v_i v_i^\vee$. Let $a_{ij} = \langle v_i, v_j \rangle$, so that we have $a_{ij}^{-1} = \langle v_i^\vee, v_j^\vee \rangle$, where a_{ij}^{-1} is the (i,j)- th entry of the matrix $(a_{ij})_{i,j}$. To simplify notation, we write $e_i = v_i \otimes e_W \in V \otimes W$ and $f_i = v_i \otimes f_W$. We then have

$$\eta(e)\eta(f) \;=\; -\frac{1}{4z^2} \sum_{i,j,k,l} a_{ij}^{-1} a_{ij}^{-1} e_i e_j f_k f_l.$$

We apply commutation relations step-by-step and obtain

$$-\frac{1}{4z^2} \sum_{i,j,k,l} a_{ij}^{-1} a_{ij}^{-1} \big(e_i f_k e_j f_l - a_{kj} e_i f_l z\big) = -\frac{1}{4z^2} \sum_{i,j,k,l} a_{ij}^{-1} a_{ij}^{-1} \big(f_k e_i e_j f_l - a_{ik} e_j f_l z - a_{kj} e_i f_l z\big)$$

$$= -\frac{1}{4z^2} \sum_{i,j,k,l} a_{ij}^{-1} a_{ij}^{-1} \big(f_k e_i f_l e_j - a_{lj} f_k e_i z - a_{ik} e_j f_l z - a_{kj} e_i f_l z\big)$$

$$= -\frac{1}{4z^2} \sum_{i,j,k,l} a_{ij}^{-1} a_{ij}^{-1} \big(f_k f_l e_i e_j - a_{li} f_k e_j z - a_{lj} f_k e_i z - a_{ik} e_j f_l z - a_{kj} e_i f_l z\big)$$

The first term is $\eta(f)\eta(e)$, and the remaining terms equal $\eta(h)$ after employing $\sum_{i,l} a_{ij}^{-1} a_{il} a_{kl}^{-1} = a_{jk}$. ■

Corollary 2.2 (cf. Section 5.1 of [7]) *The map*

$$\mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}} \longrightarrow \mathrm{U}(\mathfrak{g}^{\mathrm{J}}(V))_{\mathfrak{z}}, \quad a \otimes b \longmapsto \big(a - \eta(a)\big)\, b$$

is an isomorphism.

The above decomposition allows us to study the infinitesimal representation theory of $\mathrm{G}^{\mathrm{J}}(V)$ in terms of that of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{H}(V)$. Both theories are well-known, and their connections to the real Jacobi group have been observed previously [1, 7, 15].

2.7 Parabolic Subalgebras

As a replacement for parabolic subalgebras in the reductive case, we consider a decomposition of Lie(H(V)) in terms of eigenspaces with respect to

$$\mathfrak{k} \;:=\; \mathrm{span}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \subset \mathrm{Lie}(G).$$

The following is in complete analogy with the definition of theta stable parabolic subalgebras on page 57 of [20].

Write $\mathrm{Lie}(\mathrm{H}(V)) \;:=\; \mathfrak{m}^{\mathrm{H}} \oplus \mathfrak{z}$, where $\mathfrak{m}^{\mathrm{H}}$ is the space $V \otimes W$ in Lie(H(V)). The action of G on W yields a Lie algebra action of $\mathfrak{k}$ on $\mathfrak{m}^{\mathrm{H}}$. The eigenspaces with respect to this action, in coordinates, are $V \otimes (1, \pm i)$. Given any orthogonal decomposition $V = V^+ \oplus V^-$, we set

$$\begin{aligned} \mathfrak{m}^{\mathrm{H}}(V^+, V^-) \;&:=\; \big(V^+ \otimes (1, +i)\big) \oplus \big(V^- \otimes (1, -i)\big) \;\subset\; \mathrm{Lie}(\mathrm{H}(V)), \\ \mathfrak{q}^{\mathrm{H}}(V^+, V^-) \;&:=\; \mathfrak{m}^{\mathrm{H}}(V^+, V^-) \oplus \mathfrak{z} \;\subset\; \mathrm{Lie}(\mathrm{H}(V)). \end{aligned} \tag{2.3}$$

2.8 $(\mathfrak{g}, \mathrm{K})$-*Modules*

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. Given a compact subgroup K of G, we adopt Lepowsky's definition of $(\mathfrak{g}, \mathrm{K})$-modules from page 80 of [21]: A vector space V that is a $\mathfrak{g}$ and a K-module is called a $(\mathfrak{g}, \mathrm{K})$-module if the following conditions are satisfied:

1. For all $v \in V$, $k \in \mathrm{K}$, and $x \in \mathfrak{g}$, we have $k(xv) = (\mathrm{Ad}(k)x)v$.
2. For all $v \in V$, the span of Kv is finite dimensional and K acts continuously on span Kv.
3. For all $x \in \mathrm{Lie}(\mathrm{K})$ and $v \in V$, we have $xv = \big(\frac{d}{dt} \exp(tx)v\big)_{t=0}$.

2.9 Upper Half Spaces

We write $\mathbb{H} := \{\tau \in \mathbb{C} \,:\, \Im(\tau) > 0\}$ and $\mathbb{H}^{\mathrm{J}}(V) := \mathbb{H} \times V(\mathbb{C}) = \{(\tau, z) \,:\, \tau \in \mathbb{H}, z \in V(\mathbb{C})\}$ for the Poincaré upper half plane and the Jacobi upper half space. We write $x = \Re(\tau)$, $y = \Im(\tau)$, $u = \Re(z)$, and $v = \Im(z)$, throughout. In accordance with our choice of maximal compact subgroup K, we also consider $\widetilde{\mathbb{H}}^{\mathrm{J}}(V) := \mathbb{H} \times V(\mathbb{C}) \times \mathbb{R}$ with elements typically denoted by (τ, z, x'), $x' \in \mathbb{R}$.

Remark 2.3 The notation x' is connected to Siegel modular forms of genus 2. The Siegel upper half space of genus 2 consists of certain matrices $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$, and the real part of τ' is denoted by x'. This x' is directly related to the x' in this paper via a variant of the Fourier-Jacobi expansion that holds for arbitrary vector-valued Siegel modular forms, including all indecomposable arithmetic types.

2.10 Slash Actions

There is an action of G and $\mathrm{G}^{\mathrm{J}}(V)$ on $\mathbb{H}$ and $\mathbb{H}^{\mathrm{J}}(V)$, respectively.

$$(g, \omega)\,\tau \;:=\; g\,\tau \;:=\; \frac{a\tau + b}{c\tau + d},$$
$$g^{\mathrm{J}}\,(\tau, z) \;:=\; (g, h)\,(\tau, z) \;:=\; \big((g, \omega), (\lambda, \mu, \kappa)\big)\,(\tau, z) \;:=\; \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\Big) \tag{2.4}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $(g, \omega) \in \mathrm{Mp}_2(\mathbb{R})$, and $h = (\lambda, \mu, \kappa) \in \mathrm{H}(V)$.

Cocycles for the corresponding actions on $C^\infty(\mathbb{H})$ and $C^\infty(\mathbb{H}^{\mathrm{J}}(V))$ are parametrized by $k \in \frac{1}{2}\mathbb{Z}$ and $m \in \mathbb{C}$ (see [7]). For $f \in C^\infty(\mathbb{H})$ and $\phi \in C^\infty(\mathbb{H}^{\mathrm{J}}(V))$, we define slash actions:

$$\big(f\big|_k\,(g, \omega)\big)(\tau) \;:=\; \omega(\tau)^{-2k} f(g\tau), \tag{2.5}$$
$$\big(\phi\big|_{k,m}\,g^{\mathrm{J}}\big)(\tau, z) \;:=\; \big(\phi\big|_{k,m}\,((g, \omega), h)\big)(\tau, z) \;:=\; \omega(\tau)^{-2k} e\big(mj(g^{\mathrm{J}}, (\tau, z))\big)\,\phi\big((g, h)(\tau, z)\big) \tag{2.6}$$

with

$$j\Big(\big((g, \omega)(\lambda, \mu, \kappa)\big),\, (\tau, z)\Big) \;:=\; \frac{-cq(z + \lambda\tau + \mu)}{c\tau + d} + \langle z, \lambda\rangle + q(\lambda)\tau + \tfrac{1}{2}\langle \lambda, \mu\rangle + \kappa. \tag{2.7}$$

Here and throughout $e(x)$ stands for $\exp(2\pi i\, x)$, where $x \in \mathbb{C}$.

2.11 Functions on the Jacobi Group

We use $(\mathfrak{g}, \mathrm{K})$-modules to describe analytic properties of Jacobi forms. Let $\phi : \mathbb{H}^{\mathrm{J}}(V) \to \mathbb{C}$ be a smooth function with possible singularities. Let $k \in \frac{1}{2}\mathbb{Z}$ and $m \in \mathbb{C}$. Then we let

$$\big(\mathrm{A}_{k,m}(\phi)\big)(g^{\mathrm{J}}) \;:=\; \big(\mathrm{A}_{\mathbb{R},k,m}(\phi)\big)(g^{\mathrm{J}}) \;:=\; \big(\phi\big|_{k,m}\,g^{\mathrm{J}}\big)(i, 0). \tag{2.8}$$

We write $\varpi_{k,m}(\phi)$ for the $(\mathfrak{g}, \mathrm{K})$-module that is generated by $\mathrm{A}_{k,m}(\phi)$. If k and m are clear from the context, we suppress them. Given a $(\mathfrak{g}, \mathrm{K})$-module ϖ, we say that a function ϕ on the Jacobi upper half space as above has analytic type ϖ if $\varpi(\phi) \cong \varpi$.

2.12 Singularities at Torsion Points

Consider the rational structure $V(\mathbb{Q}) := L \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq V$ of V that arises from a lattice $L \subset V$. A smooth function $\phi : \mathbb{H}^{\mathrm{J}}(V) \to \mathbb{C}$ with possible singularities is said to have singularities at torsion points (with respect to L), if the singularities are locally of codimension at least 1 and the singular locus is locally given by $s^\vee(z) = \alpha\tau + \beta$ for some $s^\vee \in V(\mathbb{Q})^\vee$ and $\alpha, \beta \in \mathbb{Q}$.

2.13 Weak Jacobi Forms

Fix an integral lattice $L \subset V$. For the purpose of this paper a weak Jacobi form of analytic type ϖ, Jacobi index mL ($m \in \mathbb{C}$), and weight $k \in \frac{1}{2}\mathbb{Z}$, is smooth function $\phi : \mathbb{H}^{\mathrm{J}}(V) \to \mathbb{C}$ with possible singularities at torsion points of analytic type ϖ such that

$$\phi\big|_{k,m} g^{\mathrm{J}} \;=\; \phi \quad \text{for all } g^{\mathrm{J}} \in \Gamma^{\mathrm{J}}(L)$$

and $\phi\big|_{k,m}(\lambda, \mu, 0)\big|_{z=0} = \mathscr{O}(1)$ for every $\lambda, \mu \in V(\mathbb{Q})$ if ϕ is not singular along $z = \lambda\tau + \mu$.

Remarks 2.4

(1) It is common to call certain holomorphic Jacobi forms with relaxed growth conditions weak Jacobi forms—cf. Chapter III of [9]. We overlap this terminology in order to tentatively unify notions of "*weakly* holomorphic modular forms" and "harmonic *weak* Maaß forms".

(2) Most authors fix a realization of the Jacobi group, by fixing a standard quadratic space V and allowing cocycles to depend on an additional quadratic form. This approach would force us to extend the center of $\mathrm{G}^{\mathrm{J}}(V)$ from $\mathbb{R}$ to $V \otimes V$. Details can be found in [7]. Since the isomorphism class of $\mathrm{G}^{\mathrm{J}}(V)$ as a topological group does not depend on the quadratic form q, reconciling those two approaches is merely a matter of normalization. Our approach appears as the better one, if we aim at studying the Jacobi group on its own, since we avoid using two unrelated quadratic forms on V. When examining Jacobi forms in relation to, say, the standard symplectic groups, it appears better to use the realization of $\mathrm{G}^{\mathrm{J}}(V)$ determined by a fixed embedding into $\mathrm{Sp}_n(\mathbb{R})$ with $n = 1 + \dim(V)$.

3 Harish-Chandra ULie(H(V))-Modules

Given a real reductive group G with Lie algebra $\mathfrak{g}$ and maximal compact subgroup K, a $(\mathfrak{g}, \mathrm{K})$-module ϖ is called a Harish-Chandra module if its K-isotypical components are finite dimensional. Harish-Chandra modules are important in the theory of reductive groups, but the notion has not yet been transferred to the Jacobi group in a meaningful way.

First of all notice that holomorphic Jacobi forms (in the sense of [9] and [25]) generate $(\mathfrak{g}, \mathrm{K})$-modules that have finite dimensional K-isotypical components. On first sight, this suggests that we adapt the notion of Harish-Chandra modules in a straightforward way, asking for finite dimensional K-isotypical components.

3.1 *Infinite Dimensional K-Types*

We illustrate two kinds of issues. The first one arises from unitarizable representations of the Heisenberg group $\mathrm{H}(V)$, if the dimension of V exceeds 1. The second one arises from the outer tensor product of unitarizable representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{H}(V)$. We encounter it for any nontrivial V.

For indefinite V and an orthogonal decomposition $V = V^+ \oplus V^-$ into a positive definite and a negative definite space, consider the theta series

$$\theta_{L,0}\big(V^+, V^-;\, \tau, z\big) \;:=\; y^{\dim\, V^-} \sum_{l \in L} e\big(q(l_+)\tau + \langle l_+, z\rangle \;+\; q(l_-)\overline{\tau} + \langle l_-, \overline{z}\rangle\big), \tag{3.1}$$

where subscripts $\pm$ refer to the orthogonal projections onto $V^\pm$. It is a well-behaved Jacobi form, but the $(\mathfrak{g}, \mathrm{K})$-module associated to it has infinite dimensional K-types. Under the isomorphism in Corollary 2.2, we have

$$\varpi\big(\theta_L(V^+, V^-)\big) \;\cong\; \mathbb{1} \otimes \varpi_\theta(V^+, V^-) \quad \text{as a module for} \quad \mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$$

where the second tensor factor is unitarizable, but K-types are not finite dimensional.

Skewholomorphic Jacobi forms [18] are analogues of holomorphic Jacobi forms whose original purpose was to accommodate a parity condition in [19]. Let ϕ be a skewholomorphic Jacobi form of weight k. Then we have

$$\varpi\big(y^{k-\frac{1}{2}}\phi\big) \;\cong\; \overline{\varpi}(f_\phi) \otimes \varpi_\theta \quad \text{as a module for} \quad \mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}.$$

Here f_ϕ corresponds to ϕ under the theta decomposition, $\overline{\varpi}(f_\phi)$ is an antiholomorphic representation and ϖ_θ is an irreducible representation of $\mathrm{ULie}(\mathrm{H}(V))$, where V is one-dimensional. Both $\overline{\varpi}(f_\phi)$ and ϖ_θ have finite dimensional K-types. Their outer tensor product, however, has infinite dimensional K-types.

Infinite dimensional K-types have not yet been much of a problem. When insisting on the square integrable representations of nonzero Jacobi index as in [15] one automatically obtains unitary representations of $\mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$. The Stone-von Neumann Theorem then yields sufficient control of the representations of $\mathrm{H}(V)$. To the author's knowledge the only representation theoretic account of real representations that are not square integrable was carried out in [4]. In this study, the assumption that the Casimir operator acts by scalars narrowed down possibilities sufficiently.

Our goal is to formalize and extend the theory in [4] of the $\widehat{\mu}$-function from [26]. In such a context finiteness conditions on K-isotypical components are excessively restrictive. Any theory of Jacobi forms building on a notion of $(\mathfrak{g}^{\mathrm{J}}(V), K)$-modules with finite dimensional K-types remains limited to holomorphic (or meromorphic) and antiholomorphic (or antimeromorphic) Jacobi forms.

3.2 Highest Weight Modules

Fix, for the time being, a real reductive group G with a maximal compact subgroup $\mathrm{K} \subseteq G$, Lie algebra $\mathfrak{g}$, and choice of positive roots. A $(\mathfrak{g}, \mathrm{K})$-module is called a highest weight module if it is generated by a vector that is annihilated by the action of all positive root spaces in $\mathfrak{g}$. The prototypical examples in our context are the modules $\mathrm{A}_{\mathfrak{q}}(\lambda)$ of [20]. Notice that their highest weight vector is annihilated by the complement of $\mathfrak{q}$ (cf. Theorems 2.5 (c) and 5.3 (c) of op. cit.). Highest weight modules are Harish-Chandra modules. From [20], we record the fact that cohomological modules are highest weight modules.

Definition 3.1 A $\mathrm{ULie}(\mathrm{H}(V))$-module ϖ is a highest weight module if there is an orthogonal decomposition $V = V^+ \oplus V^-$ and a cyclic vector $w \in \varpi$ such that $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)w = 0$.

Remarks 3.2

(1) The theta series $\theta_L(V^+, V^-)$ in (3.1) yield highest weight modules $\varpi(\theta_L(V^+, V^-))$.
(2) Pitale has shown in [15] that any square integrable Jacobi form ϕ for one-dimensional V gives rise to a $(\mathfrak{g}^{\mathrm{J}}(V), K)$-module

$$\varpi(\phi) \;\cong\; \varpi(f_\phi) \otimes \varpi_\theta,$$

where ϖ_θ is a highest weight module in the sense of Definition 3.1. This result most likely extends to any nondegenerate V.
(3) In classical language, a highest weight module for $V = V^+ \oplus V^-$ corresponds to a Jacobi form that is meromorphic in z_- and antimeromorphic in z_+. In [7], Jacobi forms that are holomorphic in z were called semi-holomorphic.

In light of the discussions in Sect. 3.1, in [15], and in [2], we propose the following definition.

Definition 3.3 A ULie(H(V))-module ϖ is a Harish-Chandra module if it has finite Jordan-Hölder length and its composition factors are highest weight modules in the sense of Definition 3.1.

Remarks 3.4

(1) It is an interesting question whether or not Definition 3.3 can be rephrased more intrinsically.
(2) Our motivation for Definition 3.3 is to provide an analogy both to cohomological $(\mathfrak{g}, \mathrm{K})$-modules and to $(\mathfrak{g}, \mathrm{K})$-modules for $\mathrm{SL}_2(\mathbb{R})$ arising from harmonic weak Maaß forms. The later correspond to nonvanishing classes in de Rham cohomology [5]. A similar phenomenon was observed for Siegel Maaß forms [22]. Do these classes correspond to any natural one in a suitable $(\mathfrak{g}, \mathrm{K})$-cohomology?

Indecomposable highest weight modules for ULie(H(V)) arise from any orthogonal decomposition $V = V^+ \oplus V^-$ and any indecomposable representation π of $\mathfrak{z}$. Assume that π is not nilpotent, and write $\kappa \to e(m\kappa)$, $m \neq 0$ for the corresponding character of Z. The dimension d of π determines it uniquely up to twists by characters. We can extend π to a representation of $\mathfrak{q}^{\mathrm{H}}(V^+, V^-)$, by letting $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ act by zero. The Verma modules

$$\varpi_{m[d]}(V^+, V^-) \;:=\; \mathrm{ULie}(\mathrm{H}(V)) \otimes_{\mathfrak{q}^{\mathrm{H}}(V^+, V^-)} \pi$$

are indecomposable: For characters π this is clear from the Stone-von Neumann Theorem, and the general case follows from indecomposability of π with respect to $\mathrm{U}(\mathfrak{z})$, which is central in ULie(H(V)).

Proposition 3.5 *Let $V = V^+ \oplus V^-$ be an orthogonal decomposition and let ϖ be an indecomposable highest weight module of* ULie(H(V)) *with respect to* $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ *with highest weight vector w. Let π be the* $\mathfrak{q}^{\mathrm{H}}(V^+, V^-)$*-representation generated by w. Then we have*

$$\varpi \;\cong\; \mathrm{ULie}(\mathrm{H}(V)) \otimes_{\mathfrak{q}^{\mathrm{H}}(V^+, V^-)} \pi. \tag{3.2}$$

Proof Since w is a highest weight vector, $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ acts on π as zero. It is therefore indecomposable as a $\mathfrak{z}$-module. Consequently, the Verma module in (3.2) is indecomposable, as argued above. ∎

Remark 3.6 From the perspective of harmonic weak Maaß Jacobi forms, it is natural to ask for Verma modules attached to finite dimensional representation π of $\mathfrak{q}^{\mathrm{H}}(V^+, V^-)$ for which $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ does not act as zero. We leave the proof of the following statement to the reader: Let π be a finite dimensional representation of $\mathfrak{q}^{\mathrm{H}}(V^+, V^-)$ such that $\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ acts as zero on its semisimplification. Let π' be the $\mathfrak{q}^{\mathrm{H}}(V^+, V^-)$-representation that agrees with π as a $\mathfrak{z}$-module and on which

$\mathfrak{m}^{\mathrm{H}}(V^+, V^-)$ acts as zero. Then we have

$$\mathrm{ULie}(\mathrm{H}(V)) \otimes_{\mathfrak{q}^{\mathrm{H}}(V^+,V^-)} \pi \;\cong\; \mathrm{ULie}(\mathrm{H}(V)) \otimes_{\mathfrak{q}^{\mathrm{H}}(V^+,V^-)} \pi' .$$

4 Jacobi Forms that Generate Highest Weight Modules

Section 3 can be viewed as the abstract treatment of the analytic behavior of harmonic weak Maaß Jacobi forms in the elliptic variable z. To connect $(\mathfrak{g}^{\mathrm{J}}(V), K)$-modules and Jacobi forms, we introduce suitable slash actions for the Jacobi group $\mathrm{G}^{\mathrm{J}}(V)$.

As a special case, Proposition 4.4 says that Jacobi forms on $\mathbb{H}^{\mathrm{J}}(V)$ necessarily lead to $(\mathfrak{g}^{\mathrm{J}}(V), K)$-modules with central character. For this reason, we consider functions on $\widetilde{\mathbb{H}}^{\mathrm{J}}(V) = \mathbb{H} \times V(\mathbb{C}) \times \mathbb{R}$ instead. We start by lifting functions on $\mathbb{H}^{\mathrm{J}}(V)$ to functions on $\widetilde{\mathbb{H}}^{\mathrm{J}}(V)$. These lifts can be viewed as an intermediate object between $\phi : \mathbb{H}^{\mathrm{J}}(V) \to \mathbb{C}$ and $\mathrm{A}_{k,m}(\phi) : \mathrm{G}^{\mathrm{J}}(V) \to \mathbb{C}$. Given a function $\phi : \mathbb{H}^{\mathrm{J}}(V) \to \mathbb{C}$, set

$$\widetilde{\phi}(\tau, z, x') \;:=\; e(mx')\, \phi(\tau, z). \tag{4.1}$$

The action of $\mathrm{G}^{\mathrm{J}}(V)$ act on $\widetilde{\mathbb{H}}^{\mathrm{J}}(V)$ is given by

$$\begin{aligned} g^{\mathrm{J}}\,(\tau, z, x') &:= \big((g, \omega), (\lambda, \mu, \kappa)\big)\,(\tau, z, x') \\ &:= \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}, x' + j(g^{\mathrm{J}}, (\tau, z))\Big). \end{aligned}$$

This extends the action of $\mathrm{G}^{\mathrm{J}}(V)$ on $\mathbb{H}^{\mathrm{J}}(V)$, which is given in (2.4). In complete analogy with (2.6), the slash actions $|_{k,m}$ on $\mathrm{C}^{\infty}(\widetilde{\mathbb{H}}^{\mathrm{J}}(V))$ are defined by

$$\big(\phi\big|_{k,m}\,((g, \omega), h)\big)(\tau, z, x') \;:=\; \omega(\tau)^{-2k} e\big(mj(g^{\mathrm{J}}, (\tau, z))\big)\, \phi\big((g, h)(\tau, z, x')\big).$$

A straightforward computation shows that

$$\widetilde{\phi|_{k,m} g^{\mathrm{J}}} \;=\; \widetilde{\phi}\big|_{k,0} g^{\mathrm{J}} \quad \text{for all } g^{\mathrm{J}} \in \mathrm{G}^{\mathrm{J}}(V). \tag{4.2}$$

In particular, the lift of functions from $\mathbb{H}^{\mathrm{J}}(V)$ to $\widetilde{\mathbb{H}}^{\mathrm{J}}(V)$ preserves any suitable notion of Jacobi forms.

Remark 4.1 The slash action on the right hand side of (4.2) does not depend on m. Contributions of the Jacobi index m to the transformation law already arise from the factor $e(mx')$ on the right hand side of (4.1), where it is otherwise suppressed from the notation.

4.1 Slash Actions

Consider the space $\mathrm{Poly}(X, d)$ of polynomials in X of degree at most $d \in \mathbb{Z}_{\geq 0}$ with complex coefficients. The Z-representations

$$\sigma_{m[d]}(\kappa)P(X) \;:=\; e(m\kappa) \cdot P(X - \kappa)$$

exhaust the finite dimensional, indecomposable representations of Z when m runs through $\mathbb{C}$ and d runs through $\mathbb{Z}_{\geq 0}$. For each $k \in \mathbb{Z}$, $m \in \mathbb{C}$, and $d \in \mathbb{Z}_{\geq 0}$, we define a slash action $|_{k,m[d]}$ on the space $\mathscr{C}^\infty(\widetilde{\mathbb{H}}^{\mathrm{J}}(V) \to \mathrm{Poly}(X, d))$ of smooth functions from $\widetilde{\mathbb{H}}^{\mathrm{J}}(V)$ to $\mathrm{Poly}(X, d)$:

$$\big(f\big|_{k,m[d]}((g, \omega), h)\big)(\tau, z, x') \;:=\; \omega(\tau)^{-2k}\sigma_{m[d]}\big(j(g^{\mathrm{J}}, (\tau, z))\big)\phi\big((g, h)(\tau, z, x')\big). \tag{4.3}$$

Recall the lift $\mathrm{A}_{k,m}$ of functions on $\mathbb{H}^{\mathrm{J}}(V)$ to functions on $\mathrm{G}^{\mathrm{J}}(V)$ in Eq. (2.8). We denote its generalization to functions $\widetilde{\mathbb{H}}^{\mathrm{J}}(V) \to \mathrm{Poly}(X, d)$ by $\mathrm{A}_{k,m[d]}$:

$$\big(\mathrm{A}_{k,m[d]}(\widetilde{\phi})\big)(g^{\mathrm{J}}) \;:=\; \big(\mathrm{A}_{\mathbb{R},k,m[d]}(\widetilde{\phi})\big)(g^{\mathrm{J}}) \;:=\; \big(\widetilde{\phi}\big|_{k,m[d]}\, g^{\mathrm{J}}\big)(i, 0) \in \mathrm{Poly}(X, d). \tag{4.4}$$

We write $\varpi_{k,m[d]}(\widetilde{\phi})$ for the $(\mathfrak{g}^{\mathrm{J}}(V), K)$-module that is generated by the X-coefficients of $\mathrm{A}(\widetilde{\phi})$.

4.2 Jacobi Forms

Jacobi forms for the slash actions $\big|_{k,m[d]}$ can be expressed in terms of the classical ones. Observe that the invariance condition in the next definition includes invariance with respect to Z. In applications to Fourier-Jacobi coefficients of Siegel modular forms this invariance condition is satisfied naturally.

Definition 4.2 Let ϖ be a $(\mathfrak{g}^{\mathrm{J}}(V), K)$-module. A smooth function $\widetilde{\phi} : \widetilde{\mathbb{H}}^{\mathrm{J}}(V) \to \mathrm{Poly}(X, d)$ is called a Jacobi form of analytic type ϖ, weight k, Jacobi index mL, and central depth d if the following conditions are satisfied:

(i) We have $\widetilde{\phi}\big|_{k,m[d]}\, \gamma^{\mathrm{J}} = \widetilde{\phi}$ for all $\gamma^{\mathrm{J}} \in \Gamma^{\mathrm{J}}(L)$ and all $\gamma^{\mathrm{J}} \in Z$.
(ii) We have $\varpi(\widetilde{\phi}) \cong \varpi$.
(iii) For every $\lambda, \mu \in V(\mathbb{Q})$ we have $\widetilde{\phi}\big|_{k,m[d]}\, (\lambda, \mu, 0)\big|_{z=0} = \mathscr{O}(1)$.

We denote the corresponding space by $\mathrm{J}(\varpi, k, \sigma_{m[d]}(L))$.
Let ϖ_k denote the $(\mathfrak{g}, \mathrm{K})$-module of $\mathfrak{sl}_2$ with highest weight vector corresponding to modular weight k (i.e. the discrete series D_{k-1} in the notation of Section 5.6.4 of [21]). Fix an orthogonal decomposition $V = V^+ \oplus V^-$. Suppressing this

decomposition from our notation, we set

$$\mathrm{J}\big(k, \sigma_{m[d]}(L)\big) \;:=\; \bigoplus_{j=0}^{d} \mathrm{J}\big(\varpi_k \otimes \varpi_{m[j]}(V^+, V^-),\, k, \sigma_{m[d]}(L)\big).$$

In the case of $d = 0$ and $V^+ = \{0\}$, $V^- = V$, this coincides with the classical definition of holomorphic Jacobi forms of weight k and index mL.

Remark 4.3 One can extend the above definition to Jacobi forms for subgroups of $\Gamma^{\mathrm{J}}(L)$ or insert arithmetic types ρ (i.e. a representation of $\Gamma^{\mathrm{J}}(L)$) into the definition.

Proposition 4.4 *Fix $k \in \mathbb{Z}$, $m \in \mathbb{C}$, and $d \in \mathbb{Z}_{\geq 0}$. The map*

$$\Big(\mathrm{J}\big(k, \sigma_m(L)\big)\Big)^{d+1} \longrightarrow \mathrm{J}\big(k, \sigma_{m[d]}(L)\big), \quad \big(\phi_0, \ldots, \phi_d\big) \longmapsto \sum_{j=0}^{d} (X - x')^j \phi_j$$

is an isomorphism.

Proof This is a straightforward consequence of the Z-invariance of $X - x'$:

$$\begin{aligned} X - x'\big|_{k,m[d]}(0,0,\kappa) &= \sigma_{m[d]}(-\kappa)\big(X - (x' + \kappa)\big) \;=\; \big((X + \kappa) - (x' + \kappa)\big) \\ &= X - x'. \end{aligned}$$

■

Remark 4.5 Proposition 4.4 allows us to transfer the notion of theta decompositions from usual Jacobi forms to Jacobi forms for indecomposable slash actions.

The previous remark shows that theta series remain the primary example of Jacobi forms also in the setting of indecomposable slash actions. Assume that L is integral. We adopt notation from [24]. In particular, let $\operatorname{disc} L$ be the discriminant module of L and ρ_L the corresponding Weil representation. Its representation space has basis $\mathfrak{e}_l$, $l \in \operatorname{disc} L$. Let $V = V^+ \oplus V^-$ be an orthogonal decomposition into a positive definite and a negative definite subspace. Set

$$\begin{aligned} \theta_L(V^+, V^-;\, \tau, z) &:= y^{\dim V^-} \sum_{l \in L^\vee} \mathfrak{e}_l\, e\big(q(l_+)\tau + \langle l_+, z\rangle \;+\; q(l_-)\overline{\tau} + \langle l_+, \overline{z}\rangle\big) \\ &\in \mathrm{J}\big(\tfrac{\dim V}{2}, \sigma_1(L) \otimes \rho_L\big). \end{aligned}$$

We have

$$\varpi\big(\theta_L(V^+, V^-)\big) \;\cong\; \mathbb{1} \otimes \varpi_{1[0]}(V^-, V^+) \quad \text{as a module for} \quad \mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{U}\mathrm{Lie}(\mathrm{H}(V))_{\mathfrak{z}}.$$

Note that V^+ and V^- are exchanged on the right hand side. This unfortunate notation originates in the eigenvalues of $\mathfrak{k}$ on $\mathfrak{m}^{\mathrm{H}}$.

For any $d \in \mathbb{Z}_{\geq 0}$,we have

$$(X - x')^d \, \theta_L\big(V^+, V^-;\ \tau, z\big) \ \in \ \mathrm{J}\big(\tfrac{\dim V}{2}, \sigma_{1[d]} \otimes \rho_L\big)$$

and

$$\varpi\big((X - x')^d \, \theta_L(V^+, V^-)\big) \ \cong \ \mathbb{1} \otimes \varpi_{1[d]}(V^-, V^+) \quad \text{as a module for}$$

$\mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{ULie}(\mathrm{H}(V))_{\mathfrak{z}}$.

5 Weakly Modular Harish-Chandra ULie(H(V))-Modules

In this section we restrict to the case of one-dimensional V. Without loss of generality, we may further assume that V is negative definite. Proposition 4.4 says that that all ULie(H(V))-modules $\varpi_{m[d]}$ have an “automorphic and modular realization”. In this section we study which Harish-Chandra ULie(H(V))-modules have weak modular realizations.

From the results in [4], we conclude that the one-variable $\widehat{\mu}$-function generates a $(\mathfrak{g}^{\mathrm{J}}(V), K)$-module whose attached ULie(H(V))-module fits into the exact sequence

$$\varpi_{1[0]}(V, \{0\}) \hookrightarrow \varpi(\widehat{\mu}) \twoheadrightarrow \varpi_{1[0]}(\{0\}, V).$$

In the multi-variable case, analogs of the $\widehat{\mu}$-function were found in [23], which yield ULie(H(V))-modules that are two-step extensions of highest weight modules. By following the strategy in loc. cit. further Harish-Chandra ULie(H(V))-modules can be obtained from [24]. If V has signature (d^+, d^-), their Jordan-Hölder length is bounded by $1 + d^-$.

Combining the notion of weak Jacobi forms from Sect. 2.13 and Definition 4.2 we obtain

Definition 5.1 Let ϖ be a $(\mathfrak{g}^{\mathrm{J}}(V), K)$-module. A smooth function $\widetilde{\phi} : \widetilde{\mathbb{H}}^{\mathrm{J}}(V) \to \mathrm{Poly}(X, d)$ with possible singularities at torsion points is called a Jacobi form of analytic type ϖ, index mL, central depth d, and weight k if the following conditions are satisfied:

(i) We have $\widetilde{\phi}\big|_{k,m[d]}\, \gamma^{\mathrm{J}} = \widetilde{\phi}$ for all $\gamma^{\mathrm{J}} \in \Gamma^{\mathrm{J}}(L)$ and all $\gamma^{\mathrm{J}} \in Z$.
(ii) We have $\varpi(\widetilde{\phi}) \cong \varpi$.
(iii) For every $\lambda, \mu \in V(\mathbb{Q})$ we have $\widetilde{\phi}\big|_{k,m[d]}\, (\lambda, \mu, 0)\big|_{z=0} = \mathcal{O}(1)$ if $\widetilde{\phi}$ is not singular along $z = \lambda\tau + \mu$.

We denote the corresponding space by $\mathrm{J}^{!}(\varpi,\ k, \sigma_{m[d]}(L))$.
We propose a provisional definition of weakly modular (i.e. automorphic) ULie(H(V))-modules: A ULie(H(V))-module ϖ is called weakly modular if it is generated by a weak Jacobi form (of analytic type ϖ) in the sense of Definition 5.1.

Remark 5.2 Note that because of the singularities it is not clear whether every ULie(H(V))-module that admits a weak automorphic realization also admits a weak modular realization.

As evidence that weakly modular Harish-Chandra ULie(H(V))-modules are an interesting class of modules, we offer the next theorem. It largely promotes statements of [4] to the setting of the present paper.

Theorem 5.3 *Let ϖ be an indecomposable, weakly modular Harish-Chandra* ULie(H(V))*-module that admits an infinitesimal character. Then ϖ fits into an exact sequence*

$$\varpi_\iota \hookrightarrow \varpi \twoheadrightarrow \varpi_\pi,$$

where ϖ_ι and ϖ_π are highest weight modules.

Remark 5.4 The bound on the socle length that results from Theorem 5.3 results from our restriction that V is one-dimensional. Recall that examples that arise from [24] have socle length at most $1 + d^-$, where (d^+, d^-) is the signature of V. In [8], we have seen "sesquiharmonic" Maaß forms for $\mathrm{Mp}_2(\mathbb{R})$ that generate $(\mathfrak{g}, \mathrm{K})$-modules of socle length 3.

Proof of Theorem 5.3 If ϖ is irreducible we are done. Otherwise, let $\varpi_0 \subset \varpi_1 \cdots \subset \varpi$ be the socle filtration of ϖ. Since ϖ is weakly modular, ϖ_1 is too. Consider a modular realization ϕ_1 of it. By Theorem 1.1 (3) and Theorem 1.3 of [4], it suffices to treat the case that ϕ_1 is a $\widehat{\mu}$-function. Notice that it has meromorphic singularities of order 1 with nonvanishing residues. Suppose that ϖ has socle length at least 2. Then there is a modular realization of ϖ_2, and we may assume that it is a preimage of ϕ_1 under $\mathfrak{m}^{\mathrm{H}}(V, \{0\})$. Equation (5) in [4] yields a renormalized (and complex conjugate) action of $\mathfrak{m}^{\mathrm{H}}(V, \{0\})$ on ϕ_2. From that very equation, we see that ϕ_2 cannot exist due to the singularities of ϕ_1. This implies that $\varpi = \varpi_1$, and finishes the proof. ∎

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Multiplicity One for Certain Paramodular Forms of Genus Two

Mirko Rösner and Rainer Weissauer

Abstract We show that certain paramodular cuspidal automorphic irreducible representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, which are not CAP, are globally generic. This implies a multiplicity one theorem for paramodular cuspidal automorphic representations. Our proof relies on a reasonable hypothesis concerning the non-vanishing of central values of automorphic L-series.

1 Introduction

Atkin-Lehner theory defines a one-to-one correspondence between cuspidal automorphic irreducible representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ with archimedean factor in the discrete series and normalized holomorphic elliptic cuspidal newforms on the upper half plane, that are eigenforms for the Hecke algebra. As an analogue for the symplectic group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, a local theory of newforms has been developed by Roberts and Schmidt [19] with respect to paramodular groups.

However, still lacking for this theory is the information whether paramodular cuspidal automorphic irreducible representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ occur in the cuspidal spectrum with multiplicity one. Furthermore, holomorphic paramodular cusp forms, i.e. those invariant under some paramodular subgroup of $\mathrm{Sp}(4, \mathbb{Q})$, do not describe all holomorphic Siegel modular cusp forms. Indeed, at least if the weight of the modular forms is high enough, one is lead to conjecture that the paramodular holomorphic cusp forms exactly correspond to those holomorphic modular cusp forms for which their local non-archimedean representations, considered from an automorphic point of view, are generic representations. Under certain technical restrictions, we show that this is indeed the case.

M. Rösner (✉) • R. Weissauer
Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany
e-mail: mroesner@mathi.uni-heidelberg.de; weissauer@mathi.uni-heidelberg.de

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To be more precise, suppose $\Pi = \otimes_v \Pi_v$ is a paramodular cuspidal automorphic irreducible representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, which is not CAP and whose archimedean factor Π_∞ is in the discrete series. Under the assumption of the hypothesis below we prove that the local representations Π_v are generic at all non-archimedean places. Furthermore, we show that the hypothesis implies that Π occurs in the cuspidal spectrum with multiplicity one and is uniquely determined by almost all of its local factors Π_v. The hypothesis imposed concerns the non-vanishing of central L-values and is crucial for our approach.

Hypothesis 1.1 *Suppose* Π *is a globally generic unitary cuspidal automorphic irreducible representation of* $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ *and* α *and* $\beta > 0$ *are real numbers. Then there is a unitary idele class character* $\mu : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$*, locally trivial at a prescribed non-archimedean place of* $\mathbb{Q}$*, such that the twisted Novodvorsky* L*-function*

$$L_{\mathrm{Nvd}}(\Pi, \mu, s) \tag{1}$$

does not vanish at $s = 1/2 + i(\alpha + k\beta)$ *for some integer* k*.*

The analogous hypothesis for the group GL(4) would imply our hypothesis, see Proposition 4.4. The corresponding statement for GL(2) is well-known [28, Thm. 4]. For $\mathrm{GL}(r)$, $r = 1, 2, 3$, compare [7, 10]. An approximative result for GL(4) has been shown by Barthel and Ramakrishnan [2], later improved by Luo [13]: Given a unitary globally generic cuspidal automorphic irreducible representation Π of $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$, a finite set S of $\mathbb{Q}$-places and a complex number s_0 with $\mathrm{Re}(s_0) \neq 1/2$ there are infinitely many Dirichlet characters μ such that μ_v is unramified for $v \in S$ and the completed L-function $\Lambda((\mu \circ \det) \otimes \Pi, s)$ does not vanish at $s = s_0$.

We remark, there is good evidence for our result (Theorem 4.5) on genericity of paramodular representations. In fact, the generalized strong Ramanujan conjecture for cuspidal automorphic irreducible representations $\Pi = \otimes'_v \Pi_v$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ (not CAP) predicts that every local representation Π_v should be tempered. But paramodular tempered local representations Π_v at non-archimedean places are always generic by Lemma 3.2.

2 Preliminaries

The group $\mathbf{G} = \mathrm{GSp}(4)$ (symplectic similitudes of genus two) is defined over $\mathbb{Z}$ by the equation

$$g^t J g = \lambda J$$

for $(g, \lambda) \in \mathrm{GL}(4) \times \mathrm{GL}(1)$ and $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since λ is uniquely determined by g, we write g for (g, λ) and obtain the similitude character

$$\mathrm{sim} : \mathbf{G} \to \mathrm{GL}(1)\,, \quad g \mapsto \lambda\,.$$

Fix a totally real number field $F/\mathbb{Q}$ with integers $\mathfrak{o}$ and adele ring $\mathbb{A}_F = \mathbb{A}_\infty \times \mathbb{A}_{\text{fin}}$. For the profinite completion of $\mathfrak{o}$ we write $\mathfrak{o}_{\text{fin}} \subseteq \mathbb{A}_{\text{fin}}$. The paramodular group $K^{\text{para}}(\mathfrak{a}) \subseteq \mathbf{G}(\mathbb{A}_{\text{fin}})$ attached to a non-zero ideal $\mathfrak{a} \subseteq \mathfrak{o}$ is the group of all

$$g \in \begin{pmatrix} \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{a}^{-1}\mathfrak{o}_{\text{fin}} \\ \mathfrak{a}\mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} \\ \mathfrak{a}\mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} \\ \mathfrak{a}\mathfrak{o}_{\text{fin}} & \mathfrak{a}\mathfrak{o}_{\text{fin}} & \mathfrak{a}\mathfrak{o}_{\text{fin}} & \mathfrak{o}_{\text{fin}} \end{pmatrix} \cap \mathbf{G}(\mathbb{A}_{\text{fin}}), \qquad \text{sim}(g) \in \mathfrak{o}_{\text{fin}}^\times .$$

An irreducible smooth representation $\Pi = \Pi_\infty \otimes \Pi_{\text{fin}}$ of $\mathbf{G}(\mathbb{A}_F)$ is called paramodular if Π_{fin} admits non-zero invariants under $K^{\text{para}}(\mathfrak{a})$ for some non-zero ideal $\mathfrak{a}$.

Two irreducible automorphic representations are said to be weakly equivalent if they are locally isomorphic at almost every place. A cuspidal automorphic irreducible representation of GSp(4) is CAP if it is weakly equivalent to a constituent of a globally parabolically induced representation from a cuspidal automorphic irreducible representation of the Levi quotient of a proper parabolic subgroup. In that case we say that Π is strongly associated to this parabolic. The three standard proper parabolic subgroups of $\mathbf{G}$ are the Borel $\mathbf{B}$, Siegel parabolic $\mathbf{P}$ and Klingen parabolic $\mathbf{Q}$:

$$\mathbf{B} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathbf{G}\,, \qquad \mathbf{P} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \cap \mathbf{G}\,, \qquad \mathbf{Q} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathbf{G}\,.$$

3 Poles of Local Spinor Factors

Fix a local nonarchimedean place v of F with completion F_v, valuation character $\nu(x) = |x|_v$ for $x \in F_v$, residue field $\mathfrak{o}_v/\mathfrak{p}_v$ of order q and uniformizer $\varpi \in \mathfrak{p}_v$. In this section we consider preunitary irreducible admissible representations Π_v of $G = \mathbf{G}(F_v)$. The non-supercuspidal Π_v have been classified by Sally and Tadic [22] and we use their notation. Roberts and Schmidt [19] have designated them with roman numerals. Π_v is called paramodular if it admits non-zero invariants under the local factor at v of some paramodular group.

Remark 3.1 Every paramodular Π_v has unramified central character.
Indeed, the intersection of $K_v^{\text{para}}(\mathfrak{a})$ with the center of G is isomorphic to $\mathfrak{o}_v^\times$.

Lemma 3.2 *For tempered preunitary irreducible admissible representations Π_v the following assertions are equivalent:*

i) Π_v *is generic and has unramified central character,*
ii) Π_v *is paramodular.*

Proof By Remark 3.1, we can assume that Π_v has trivial central character. Then this is a result of Roberts and Schmidt [19, 7.5.8]. □

Recall that for every smooth character $\chi : F_v^\times \to \mathbb{C}^\times$ the local Tate L-factor is

$$L(\chi, s) = \begin{cases} (1 - \chi(\varpi)q^{-s})^{-1} & \chi \text{ unramified,} \\ 1 & \chi \text{ ramified.} \end{cases}$$

For a generic irreducible admissible representation Π_v of G and a smooth complex character μ of $F_v^\times$, Novodvorsky [15] has defined a local degree four spinor L-factor $L_{\mathrm{Nvd}}(\Pi_v, \mu, s)$. Piatetskii-Shapiro and Soudry [17, 18] have given a construction of a local degree four spinor L-factor $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$ for infinite-dimensional irreducible admissible representations of G.[1] This L-factor depends on the choice of a Bessel model (Λ, ψ). The Bessel models have been classified by Roberts and Schmidt [20]. Poles of $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$ are called regular if they occur as poles of certain zeta integrals [18, §2]; the other poles are exceptional. For generic Π_v every pole is regular [17, Thm. 4.3].

Lemma 3.3 *For every generic irreducible admissible representation Π_v of G, the quotient $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)/L_{\mathrm{Nvd}}(\Pi_v, \mu, s)$ is holomorphic. If Π_v is also preunitary and irreducible Borel induced and if Λ is unitary, then $L_{(\Lambda,\psi)}(\Pi_v, \mu, s) = L_{\mathrm{Nvd}}(\Pi_v, \mu, s)$.*

Proof $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$ has only regular poles [17, Thm. 4.4]. For the case of non-split Bessel models, the regular poles have been determined explicitly by Danisman[2] [3–5]. For split Bessel models, see [21]. The poles of $L_{\mathrm{Nvd}}(\Pi_v, \mu, s)$ have been determined by Takloo-Bighash [27] and cancel each pole of $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$. Irreducible fully Borel induced Π_v are theta lifts from $\mathrm{GSO}(2, 2)$, see [8, Thm. 8.2vi)], so the second statement holds by Piatetskii-Shapiro and Soudry [18, Thm. 2.4]. This also follows from [21] and [5]. □

Lemma 3.4 *Let Π_v be a preunitary non-generic irreducible admissible representations of G, that is not one-dimensional, and μ a unitary character. Then $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$ has a regular pole on the line* $\mathrm{Re}(s) = 1/2$ *exactly in the following cases:*

- *IIb* $\Pi_v \cong (\chi \circ \det) \rtimes \sigma$ *for a pair of characters χ, σ that are either both unitary or satisfy $\chi^2 = \nu^{2\beta}$ for $0 < \beta < \frac{1}{2}$ with unitary $\chi\sigma$. The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factor $L(\nu^{-1/2}\chi\sigma, s)$, so they occur if and only if $\chi\sigma$ is unramified.*
- *IIIb* $\Pi_v \cong \chi \rtimes (\sigma \circ \det)$ *for unitary characters σ and χ with $\chi \neq 1$. The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factors $L(\nu^{-1/2}\sigma, s)$ and $L(\nu^{-1/2}\sigma\chi, s)$, so they occur for unramified σ or $\sigma\chi$, respectively.*

[1] Unfortunately, detailed proofs of their results are not available.

[2] Danisman assumes odd characteristic. This is used in the proof of [3, Prop. 4.3], but not necessary.

Vb,c $\Pi_v \cong L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ *for unitary characters* σ *and* ξ *with* $\xi^2 = 1 \neq \xi$. *The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factor* $L(\nu^{-1/2}\sigma, s)$ *and appear for unramified* σ.

Vd $\Pi_v \cong L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ *for unitary characters* σ *and* ξ *with* $\xi^2 = 1 \neq \xi$. *The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factors* $L(\nu^{-1/2}\sigma, s)$ *and* $L(\nu^{-1/2}\xi\sigma, s)$*, and occur for unramified* σ *or* $\xi\sigma$*, respectively.*

VIc $\Pi_v \cong L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ *for unitary* σ. *The Tate factor* $L(\nu^{-1/2}\sigma, s)$ *gives rise to regular poles with* $\mathrm{Re}(s) = 1/2$ *when* σ *is unramified.*

VId $\Pi_v \cong L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$ *for unitary* σ. *The Tate factor* $L(\nu^{-1/2}\sigma, s)^2$ *gives rise to double regular poles with* $\mathrm{Re}(s) = 1/2$ *when* σ *is unramified.*

XIb $\Pi_v \cong L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$*, where* π *is a preunitary supercuspidal irreducible admissible representation of* $\mathrm{GL}(2, F_v)$ *with trivial central character and* σ *is a unitary character. The regular poles with* $\mathrm{Re}(s) = 1/2$ *occur with the Tate factor* $L(\nu^{-1/2}\sigma, s)$ *when* σ *is unramified.*

Proof For non-split Bessel models, see Danisman [3–5]. For split Bessel models, see [21]. □

Lemma 3.5 *Up to isomorphism, the paramodular non-generic preunitary irreducible admissible representations* Π_v *of G are exactly the following:*

IIb $(\chi \circ \det) \rtimes \sigma$*, for characters* χ, σ *such that* $\chi\sigma$ *is unramified and either both are unitary or* $\chi^2 = \nu^{2\beta}$ *for* $0 < \beta < \frac{1}{2}$ *with unitary characters* $\chi\sigma$*,*

IIIb $\chi \rtimes (\sigma \circ \det)$*, for unramified unitary characters* χ, σ *with* $\chi \neq 1$*,*

IVd $\sigma \circ \mathrm{sim}$*, for unramified unitary characters* σ*,*

Vb,c $L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$*, for* ξ *with* $\xi^2 = 1 \neq \xi$ *and unramified unitary* σ*,*

Vd $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$*, for unramified unitary characters* σ, ξ *with* $\xi^2 = 1 \neq \xi$*,*

VIc $L(\nu^{1/2}\,\mathrm{St}, \nu^{-1/2}\sigma)$*, for unramified unitary characters* σ*,*

VId $L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$*, for unramified unitary characters* σ*,*

XIb $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$*, for a supercuspidal preunitary irreducible admissible representation* π *of* $\mathrm{GL}(2, F_v)$ *with trivial central character and an unramified unitary character* σ.

Proof By Remark 3.1, we can assume that the central character is trivial. For non-supercuspidal Π_v, see Tables A.2 and A.12 of Roberts and Schmidt [19]. Supercuspidal non-generic Π_v are not paramodular by Lemma 3.2. □

Proposition 3.6 *Let* Π_v *be a paramodular preunitary irreducible admissible representation of G, that is not one-dimensional and* μ *a unitary character. The following assertions are equivalent:*

i) Π_v *is non-generic,*

ii) the spinor L-factor $L_{(\Lambda,\psi)}(\Pi_v, \mu, s)$ *has a pole on the line* $\mathrm{Re}(s) = 1/2$.

Proof By the previous two lemmas, for non-generic Π_v there is a regular pole on the line $\mathrm{Re}(s) = 1/2$. If Π_v is generic, poles do not occur on the line $\mathrm{Re}(s) = 1/2$ by Lemma 3.3 and [27]. □

The proposition fails without the paramodularity assumption. For example, type Vd is always non-generic, but has a pole in $\mathrm{Re}(s) = 1/2$ if and only if σ or $\xi\sigma$ are unramified.

4 Global Genericity

Let $F = \mathbb{Q}$ with adele ring $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$. In the following, $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ is a cuspidal automorphic irreducible representation of $\mathbf{G}(\mathbb{A})$, not CAP, with central character ω_Π, such that Π_∞ belongs to the discrete series. Global Bessel models (Λ, ψ) of Π are always unitary. We want to show that if Π is paramodular, then Π_v is locally generic at every nonarchimedean place v.

The Euler product $L_{(\Lambda,\psi)}(\Pi, \mu, s) = \prod_v L_{(\Lambda_v,\psi_v)}(\Pi_v, \mu_v, s)$ converges for s in a right half plane and admits a meromorphic continuation to $\mathbb{C}$ [17, Thm. 5.3]. This is the global degree four spinor L-series of Piatetskii-Shapiro and Soudry.

Proposition 4.1 (Generalized Ramanujan) *The spherical local factors Π_v of Π_{fin} are isomorphic to irreducible tempered principal series representations $\chi_1 \times \chi_2 \rtimes \sigma$ for unramified unitary complex characters χ_1, χ_2, σ of $\mathbb{Q}_v^\times$.*

Proof See [31, Thm. 3.3]. □

Proposition 4.2 *Π is weakly equivalent to a unique globally generic cuspidal automorphic irreducible representation Π_{gen} of $\mathbf{G}(\mathbb{A})$ whose archimedean local component $\Pi_{\mathrm{gen},\infty}$ is the generic constituent in the local archimedean L-packet of Π_∞. The lift $\Pi \mapsto \Pi_{\mathrm{gen}}$ commutes with character twists by unitary idele class characters. The central characters of Π_{gen} and Π coincide.*

Proof See [29, Thm. 1]; the proof relies on certain Hypotheses A and B shown in [31]. The lift commutes with twists because Π_{gen} is unique. The central characters are weakly equivalent, so they coincide globally by strong multiplicity one for $\mathrm{GL}(1, \mathbb{A})$. □

Proposition 4.3 *If Π is not CAP and not a weak endoscopic lift, then the discrete series representation Π_∞ is contained in an archimedean local L-packet $\{\Pi_\infty^W, \Pi_\infty^H\}$ such that the multiplicities of $\Pi_\infty^W \otimes \Pi_{\mathrm{fin}}$ and $\Pi_\infty^H \otimes \Pi_{\mathrm{fin}}$ in the cuspidal spectrum coincide. Here Π_∞^H denotes the holomorphic constituent and Π_∞^W the generic constituent.*

Proof By Proposition 4.2, Π is weakly equivalent to a globally generic representation Π' of $\mathbf{G}(\mathbb{A})$, which satisfies multiplicity one [12]. Now [30, Prop. 1.5] implies the statement. □

Proposition 4.4 *Suppose Π is globally generic. Then there is a unique globally generic automorphic irreducible representation $\tilde{\Pi}$ of $\mathrm{GL}(4, \mathbb{A})$ with partial Rankin-Selberg L-function*

$$L^S(\tilde{\Pi}, s) = L^S_{(\Lambda,\psi)}(\Pi, 1, s)$$

for a sufficiently large set S of places. This lift is local in the sense that $\tilde{\Pi}_v$ *only depends on* Π_v*. It commutes with character twists by unitary idele class characters.*

Proof For the existence and locality of the lift, see Asgari and Shahidi [1]; uniqueness follows from strong multiplicity one for GL(4). It remains to be shown that $\Pi \mapsto \tilde{\Pi}$ commutes with character twists. Indeed, by Proposition 4.1, almost every local factor is of the form $\Pi_v \cong \chi_1 \times \chi_2 \rtimes \sigma$ with unitary unramified characters χ_1, χ_2, σ. Its local lift $\tilde{\Pi}_v$ is the parabolically induced GL$(4, \mathbb{A})$-representation

$$\tilde{\Pi}_v \ \cong\ \chi_1\chi_2\sigma \times \chi_1\sigma \times \chi_2\sigma \times \sigma\ ,$$

[1, Prop. 2.5]. Therefore, the lift $\Pi_v \mapsto \tilde{\Pi}_v$ commutes with local character twists at the unramified Π_v. Strong multiplicity one for GL(4) implies the statement. □

Theorem 4.5 *Suppose* $\Pi = \otimes_v \Pi_v$ *is a paramodular unitary cuspidal irreducible automorphic representation of* $\mathbf{G}(\mathbb{A}_\mathbb{Q})$ *that is not CAP nor weak endoscopic. We assume that* Π_∞ *is in the discrete series and that Hypothesis 1.1 holds. Then* Π_v *is locally generic at all nonarchimedean places v.*

Proof By Proposition 4.3 we may assume without loss of generality that Π_∞ is a holomorphic discrete series representation. By Propositions 4.2 and 4.3 there exists a cuspidal automorphic irreducible representation Ξ, weakly equivalent to Π with archimedean factor $\Xi_\infty \cong \Pi_\infty$, such that Ξ_v is locally generic at every non-archimedean place $v < \infty$. There exist global Bessel models (Λ, ψ) and (Λ_Ξ, ψ_Ξ) for Π and Ξ, respectively. By assumption Π is paramodular, so by a twist we can assume that the central character is trivial. The central characters of Π and Ξ coincide, so Ξ is also paramodular [19, Thm. 7.5.4].

Let S be the finite set of places, including ∞, such that $\Xi_v \cong \Pi_v$ is spherical for every place $v \notin S$. For these $v \notin S$, the local L-factors coincide by Proposition 4.1 and Lemma 3.3. For every idele class character μ this implies

$$\frac{L_{(\Lambda,\psi)}(\Pi, \mu, s)}{L_{(\Lambda_\Xi,\psi_\Xi)}(\Xi, \mu, s)} = \prod_{v \in S} \frac{L_{(\Lambda_v,\psi_v)}(\Pi_v, \mu_v, s)}{L_{(\Lambda_{\Xi,v},\psi_{\Xi,v})}(\Xi_v, \mu_v, s)}\ . \tag{2}$$

Now assume there is at least one non-archimedean place $w \in S$ where Π_w is not generic. By Proposition 3.6, the right hand side of (2) must have an arithmetic progression $(s_k)_{k\in\mathbb{Z}}$ of poles $s_k = 1/2 + i(\alpha + k\beta)$ with $\beta = 2\pi/\ln(p_w)$ and some real α depending only on Π_w and μ_w. Indeed, we show below that the L-factors of $\Pi_\infty \cong \Xi_\infty$ do not have poles on the line $\Re(s) = 1/2$ for any Bessel model.

Π is not CAP, so $L_{(\Lambda,\psi)}(\Pi, \mu, s)$ is holomorphic. Hence $L_{(\Lambda_\Xi,\psi_\Xi)}(\Xi, \mu, s_k) = 0$ for every k by (2). Especially, the partial L-function $L^S_{(\Lambda_\Xi,\psi_\Xi)}(\Xi, \mu, s)$ vanishes at $s = s_k$ for every k and every finite set S.

If Hypothesis 1.1 is true, there is $k \in \mathbb{Z}$ and a unitary idele class character μ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ with $\mu_w = 1$ such that for a sufficiently large finite set S, including all the archimedean and ramified places, the partial L-function $L^S_{\mathrm{Nov}}(\Xi, \mu, s_k) \neq 0$ does not vanish. For $v \notin S$ the local L-factors of Novodvorsky and Piatetskii-Shapiro coincide by Proposition 4.1 and Lemma 3.3, so the same non-vanishing assertion holds for $L^S_{(\Lambda_\Xi,\psi_\Xi)}(\Xi, \mu, s_k)$. This is a contradiction, so the place w does not exist.

It remains to be shown that the archimedean L-factor of Π_∞ in the holomorphic discrete series of lowest weight $l \geq l' \geq 3$ does not admit poles on the line $Re(s) = 1/2$. Regular and exceptional poles can be defined analogous to the non-archimedean case. By [17, Thm. 4.2], exceptional poles only occur on the line $\Re(s) = -1/2$ because the central character is unitary. By the archimedean analogue of [3, Prop. 2.5], the regular poles are the poles of the archimedean regular zeta integrals

$$\zeta_{reg}(v, \mu, s) = \int_{\mathbb{R}_{>0}} \varphi_v(\lambda)\mu(\lambda)\lambda^{s-3/2} \mathrm{d}^\times\lambda$$

attached to the Bessel functions $\varphi_v(\lambda) = \ell(\Pi_\infty(diag(\lambda, \lambda, 1, 1)v)$ for $v \in \Pi_\infty$ and the Bessel functional ℓ. In fact these zeta integrals are holomorphic for $Re(s) > -3/2$ and to show this it suffices to estimate the growth of $\varphi_v(\lambda)$ as $\lambda \to 0$.

The holomorphic lowest weight vector $v_{hol} \in \Pi_\infty$ satisfies $|\varphi_{v_{hol}}(\lambda)| \leq C\lambda^{(l+l')/2} \exp(-c\lambda)$ with positive real constants C and c [26, (1-26)]. Since $(l + l')/2 \geq 3$, the zeta integral $\zeta_{reg}(v_{hol}, \mu, s)$ converges for $\Re(s) > -3/2$. The other K_∞-types τ of Π_∞ are obtained from v_{hol} by repeatedly applying the differential operator (Maaß operator)

$$E_+ : C^\infty(\mathbf{G}(\mathbb{R}))_\tau \to C^\infty(\mathbf{G}(\mathbb{R}))_{\tau\otimes Sym^2},$$

see [14, §5.1]. It is easy to see that this can only improve convergence. □

Corollary 4.6 *In the situation of the theorem, if* Π_∞ *is generic, then* Π *is globally generic.*

Proof Π is locally generic at every place. By Proposition 4.2, there is a globally generic automorphic representation Π_{gen}, weakly equivalent to Π. By a result of Jiang and Soudry [12], $\Pi = \Pi_{\mathrm{gen}}$ in the cuspidal spectrum. □

A cuspidal automorphic irreducible representation Π of $\mathbf{G}(\mathbb{A})$, not CAP, is a weak endoscopic lift if there is a pair of cuspidal automorphic irreducible representations σ_1, σ_2 of $\mathrm{GL}(2, \mathbb{A})$ with the same central character, and local spinor L-factor

$$L_{(\Lambda_v,\psi_v)}(\Pi_v, \mu, s) = L(\sigma_{1,v}, s)L(\sigma_{2,v}, s)\,.$$

at almost every place [31, §5.2]. This condition does not depend on the global Bessel model (Λ, ψ) by Proposition 4.1 and Lemma 3.3.

Proposition 4.7 *Suppose a paramodular cuspidal irreducible automorphic representation $\Pi \cong \otimes_v \Pi_v$ of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$ is a weak endoscopic lift with local archimedean factor Π_∞ in the discrete series. Then Π is globally generic.*

Proof σ_1 and σ_2 are attached to holomorphic elliptic modular forms [31, Cor. 4.2]. They are locally tempered at every place v by the Ramanujan conjecture (Deligne). The local endoscopic lifts Π_v are also tempered [31, §4.11]. At the non-archimedean places Π_v is then generic by Lemma 3.2. Then the archimedean factor Π_∞ is generic [31, Thm. 5.2]. Hence Π is globally generic [31, Thm. 4.1], [12]. □

5 Multiplicity One and Strong Multiplicity One

We show the multiplicity one theorem and the strong multiplicity one theorem for paramodular cuspidal automorphic representations of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$ under certain restrictions. It is well-known that strong multiplicity one fails without the paramodularity assumption [6, 11].

Lemma 5.1 *A cuspidal automorphic irreducible representation Π of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$, that is strongly associated to the Borel or Klingen parabolic subgroup, is never paramodular.*

Proof Every such representation is a theta lift $\Pi = \theta(\sigma)$ of an automorphic representation σ of $\mathrm{GO}_T(\mathbb{A}_\mathbb{Q})$ for an anisotropic binary quadratic space T over $\mathbb{Q}$, see Soudry [24]. Let d_T be the discriminant of T, then T is rationally equivalent to $(K, t \cdot \mathrm{N}_K)$ for the norm N_K of the quadratic field $K = \mathbb{Q}(\sqrt{-\mathrm{d}_T})$ and a squarefree integer t. Fix a non-archimedean place v of $\mathbb{Q}$ that ramifies in K. The norm form on $K_w = K \otimes \mathbb{Q}_v$ remains anisotropic. By Lemma A.1, the local Weil representation of $\mathbf{G}(\mathbb{Q}_v) \times \mathrm{GO}_T(\mathbb{Q}_v)$ is not paramodular. Thus the global Weil representation is not paramodular either. Since paramodular groups are compact, the functor of passing to invariants is exact and therefore the paramodular invariant subspace of $\Pi = \theta(\sigma)$ is zero. □

Theorem 5.2 (Multiplicity One) *Suppose Π is a paramodular cuspidal automorphic irreducible representation of $\mathbf{G}(\mathbb{A})$ with archimedean factor Π_∞ in the discrete series. If Hypothesis 1.1 holds, Π occurs in the cuspidal spectrum with multiplicity one.*

Proof A weak endoscopic lift occurs in the cuspidal spectrum with multiplicity at most one [31, Thm. 5.2]. If Π is CAP, it is strongly associated to the Siegel parabolic by Lemma 5.1. Then it is a Saito-Kurokawa lift in the sense of Piatetskii-Shapiro [16] and occurs with multiplicity one [9, (5.10)]. If Π is neither CAP nor weak endocopic, we can assume that Π_∞ is generic by Proposition 4.3. By Corollary 4.6, Π is globally generic and the assertion holds by a result of Jiang and Soudry [12]. □

Theorem 5.3 (Strong Multiplicity One) *Suppose two paramodular automorphic cuspidal irreducible representations* Π_1, Π_2 *of* $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ *are locally isomorphic at almost every place. Assume that the archimedean local factors are either both in the generic discrete series or both in the holomorphic discrete series of* $\mathbf{G}(\mathbb{R})$. *If Hypothesis 1.1 holds,* $\Pi_1 = \Pi_2$ *coincide in the cuspidal spectrum.*

Proof Suppose Π_1 and Π_2 are not CAP. After possibly replacing the archimedean factor by the generic constituent in its local L-packet, we can assume that both Π_1 and Π_2 are globally generic by Corollary 4.6 and Proposition 4.7. Strong multiplicity one holds for globally generic representations [12, 25].

If Π_1 and Π_2 are CAP, they are strongly associated to the Siegel parabolic by Lemma 5.1 and occur as Saito-Kurokawa lifts in the sense of Piatetskii-Shapiro [16]. For each non-archimedean place v, the local factors $\Pi_{1,v}$ and $\Pi_{2,v}$ are non-generic and belong to the same Arthur packet. Exactly one constituent of this Arthur packet is non-tempered [23, §7]. Both local factors $\Pi_{1,v}$ and $\Pi_{2,v}$ are non-tempered by Lemma 3.2 and therefore isomorphic. The local factors $\Pi_{1,\infty}$, $\Pi_{2,\infty}$ at the archimedean place are in the discrete series, so they are isomorphic to the unique discrete series constituent of the archimedean Arthur packet [23, §4]. Thus Π_1 and Π_2 are locally isomorphic at every place. They coincide in the cuspidal spectrum by Theorem 5.2. □

Appendix 1: The Weil Representation

Let K/F be a ramified quadratic field extension of a local nonarchimedean number field F with principal ideals $\mathfrak{p}_K$ and $\mathfrak{p}$. The anisotropic binary quadratic form $T = (K, N_{K/F})$ defines the F-bilinear form $(x,y)_T = (x\bar{y} + \bar{x}y)/2$ for $x, y \in K$ where $\bar{\ }$ is the Galois conjugation on K/F. Fix a non-trivial additive character ψ of F with conductor $\mathfrak{p}^c$, the largest broken ideal in the kernel of ψ. The Schrödinger model of the smooth Weil representation ω of $\mathbf{G}(F) \times \mathrm{GO}_T(F)$ is given by the space of Schwarz-Bruhat functions $\phi \in \mathcal{S}(K^2 \times F^\times)$ with the action of $\mathbf{G}(F)$ given on generators by

$$\omega(\begin{smallmatrix} I_2 & sw \\ 0 & I_2 \end{smallmatrix})\phi(x,t) = \psi(t\sum_{i,j} s_{ij}(x_i, x_j)_T)\phi(x,t) ,$$

$$\omega(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix})\phi(x,t) = \gamma^2(\psi \circ T)\int_{K^2} \phi(y,t)\psi(2t\sum_i (x_i, y_i)_T)\mathrm{d}_t y .$$

Here $\gamma(\psi \circ T)$ is the Weil index. The Haar measure is normalized, depending on t, such that ω preserves the L^2-scalar product in $\mathcal{S}(K^2 \times F^\times)$. The action of $h \in \mathrm{GO}(T)$ is by

$$\omega(h)\phi(x,t) = |\det h|^{-1}\phi(h^{-1}x, N_{K/F}(h)t) .$$

Lemma A.1 *The Weil representation ω of $\mathbf{G}(F) \times \mathrm{GO}_T(F)$ is not paramodular.*

Proof Suppose $\phi \in \mathcal{S}(K^2 \times F^\times)$ is invariant under $K^{\mathrm{para}}(\mathfrak{p}^n)$ for some n. For (x, t) in the support of ϕ we must have $\psi(t \sum_{i,j} s_{ij}(x_i, x_j)_T) = 1$ for every $\left(\begin{smallmatrix} I_2 & sw \\ 0 & I_2 \end{smallmatrix}\right) \in K^{\mathrm{para}}(\mathfrak{p}^n)$. This implies $t\varpi_F^{-n}(x_1, x_1)_T \in \mathfrak{p}^c$ and $t(x_2, x_2)_T \in \mathfrak{p}^c$, so $\phi(\,\cdot\,, t)$ has support in

$$\mathfrak{p}_K^{c-v_F(t)+n} \times \mathfrak{p}_K^{c-v_F(t)} .$$

By the same argument, $\omega\left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)\phi(\,\cdot\,, t)$ has support in $\mathfrak{p}_K^{c-v_F(t)-n} \times \mathfrak{p}_K^{c-v_F(t)}$. By Fourier theory, $\phi(\,\cdot\,, t)$ is constant on cosets modulo

$$\mathfrak{p}_K^{c-v_F(t)+n}\mathfrak{d}_{K/F}^{-1} \times \mathfrak{p}_K^{c-v_F(t)}\mathfrak{d}_{K/F}^{-1}$$

with the relative different ideal $\mathfrak{d}_{K/F} = \mathfrak{p}_K$. This implies $\phi = 0$. □

Appendix 2: On Regular Poles Attached to Split Bessel Models for GSp(4)

For infinite-dimensional representations Π of $G = \mathrm{GSp}(4, F)$ with central character ω_Π, where F is a local non-Archimedean field, Piatetskii-Shapiro [17] has constructed a local L-factor

$$L_{(\Lambda,\psi)}(\Pi, s)$$

attached to a choice of a Bessel model (Λ, ψ). To be precise, fix the standard Siegel parabolic subgroup $P = MN$ in G with Levi M and unipotent radical N as above. For a non-degenerate linear form ψ of N, the connected component $\tilde{T}$ of the stabilizer of ψ in M is isomorphic to the unit group $L^\times$ for a quadratic extension L/F. A Bessel character is a pair (Λ, ψ) where Λ is a character of $\widetilde{T}$. The coinvariant space $\Pi_{(\Lambda,\psi)}$ with respect to the action of $\widetilde{T}N$ by (Λ, ψ) is at most one-dimensional [17, Thm. 3.1], [20, Thm. 6.3.2]. If it is non-zero, we say Π has a Bessel model. Such a Bessel model is called anisotropic or split, depending on whether L is a field or not. The Bessel models have been classified by Roberts and Schmidt [20].

The local factor $L_{(\Lambda,\psi)}(\Pi, s)$ has a regular part $L^{\mathrm{reg}}_{(\Lambda,\psi)}(\Pi, s)$ and an exceptional part $L^{\mathrm{ex}}_{(\Lambda,\psi)}(\Pi, s)$. For generic Π it coincides with its regular part [17, Thm. 4.3]. Danisman [3–5] has shown that the regular part does not depend on the choice of an anisotropic Bessel model. Especially, for generic Π and anisotropic Bessel models the L-factor $L_{(\Lambda,\psi)}$ coincides with the L-factor that was constructed by Novodvorsky [15] in a completely different way. One may therefore expect that the L-factor does not depend on the choice of any Bessel model. This expectation was formulated by Piatetski-Shapiro and Soudry [18, p.1] and proven for the case of Borel induced Π [18, Thm. 2.4]. Further motivation originated from the results of Danisman.

In [21] we determine the regular part of $L_{(\Lambda,\psi)}(\Pi, s)$ for split Bessel models. For non-generic Π there are finitely many split Bessel models. The regular part of the L-factors attached to arbitrary Bessel models is the product of Tate L-factors given in Table 1, as shown in [21]. Hence the expectation holds true for non-generic Π, i.e. $L^{\text{reg}}_{(\Lambda,\psi)}(\Pi, s)$ is independent of any Bessel model.

In general, however, this is false. For generic Π there are infinitely many split Bessel models and for certain cases the attached L-factor is a divisor of Novodvorsky's L-factor. For the precise results we refer to [21].

Table 1 Regular part of L-factors for non-generic infinite-dimensional Π

Type	Π	$L^{\text{reg}}_{(\Lambda,\psi)}(\Pi, s)$
IIb	$(\chi_1 \circ \det) \rtimes \sigma$	$L(s,\sigma)L(s,\nu^{-1/2}\chi_1\sigma)L(s,\chi_1^2\sigma)$
IIIb	$\chi_1 \rtimes (\sigma \circ \det)$	$L(s,\nu^{-1/2}\chi_1\sigma)L(s,\nu^{-1/2}\sigma)$
IVb	$L(\nu^2, \nu^{-1}\sigma St)$	$L(s,\nu^{3/2}\sigma)$
IVc	$L(\nu^{3/2}St, \nu^{-3/2}\sigma)$	$L(s,\nu^{-3/2}\sigma)L(s,\nu^{1/2}\sigma)$
Vb	$L(\nu^{1/2}\xi St, \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$
Vc	$L(\nu^{1/2}\xi St, \xi\nu^{-1/2}\sigma)$	$L(\nu^{-1/2}\xi\sigma)$
Vd	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$L(\nu^{-1/2}\xi\sigma)$
VIb	$\tau(T, \nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$
VIc	$L(\nu^{1/2}St, \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$
VId	$L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$
VIIIb	$\tau(T,\pi)$	1
IXb	$L(\nu\xi, \nu^{-1/2}\pi)$	1
XIb	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$

Let us briefly recall the notation. Up to equivalence, we can assume the split Bessel model is given by $\psi\begin{pmatrix} I_2 & s \\ 0 & I_2 \end{pmatrix} = \psi(tr(s)/2)$ for a non-trivial additive character ψ of F and some character Λ of $\tilde{T} = \{diag(t_1, t_2, t_1, t_2) \,|\, t_1, t_2 \in F^\times\}$. For fixed central character ω_Π, every Λ is uniquely determined by its restriction to $\rho(t_1) = \Lambda(t_1, 1, t_1, 1)$. We have a decomposition $N = \tilde{N} \times S$ where $\tilde{N}$ is in the kernel of ψ and S commutes with $\widetilde{T}$. Let $T = \{\text{diag}(1, 1, t, t) \,|\, t \in F^\times\} \subseteq G$. For every TS-module E of finite length it can be shown that the canonical morphism $E^S \to E_S$ from S-invariants to S-coinvariants is injective and we consider the finite-dimensional quotient as a T-module

$$\mathcal{L}(E) = E_S/E^S \,.$$

For an irreducible representation Π of G, the space of coinvariants with respect to the action of $\widetilde{TN}$ by ρ defines a TS-module $\widetilde{\Pi} = \Pi_\rho$ of finite length. It turns out that the quotient $\mathcal{L}(\widetilde{\Pi})$ completely determines the regular part of the L-function: For T-characters χ let $a(\chi)$ be the multiplicity of χ in the semisimplification of the T-module $\mathcal{L}(\widetilde{\Pi})$. The regular part of the L-factor is the product of Tate L-factors

$$L^{\mathrm{reg}}_{(\Lambda,\psi)}(\Pi,s)=\prod_{\chi}L(\nu^{-3/2}\omega_\Pi\chi^{-1},s)^{a(\chi)}\ .$$

Notice $a(\chi)\neq 0$ implies that χ occurs as a T-character in the unnormalized Siegel Jacquet module Π_N because $\widetilde{\Pi}_S=(\Pi_N)_{\tilde{T},\Lambda}$. For unitary generic Π this connection with the Siegel-Jacquet module easily implies that $L_{(\Lambda,\psi)}(s,\Pi)$ does not have a pole on the critical line $\Re(s)=1/2$. Indeed, these poles come from characters χ with $|\chi|=\nu^{-1}$. For generic unitary Π it follows from the list of constituents in the Siegel-Jacquet module [19, A.3], that they do not occur. For our application in Lemma 3.3, this is crucial.

To calculate the factor $a(\chi)$, we study the T-module $\mathcal{L}(\widetilde{\Pi})$. For semisimple $\widetilde{\Pi}$ we would have $\mathcal{L}(\widetilde{\Pi})=0$, so the non-trivial L-factors come from indecomposable extensions of TS-modules. We observe that the L-factor is the expected one if and only if $\widetilde{\Pi}^S$ vanishes. The analogous assertion in the anisotropic case holds true by Proposition 4.7 of Danisman [3]. In the split case the necessary information is provided by a combination of various techniques, as for instance P_3-theory in the sense of Roberts and Schmidt [19], a detailed study of the Siegel-Jacquet module and the analysis of induced representations for which Π is an irreducible quotient.

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Restriction of Hecke Eigenforms to Horocycles

Ho Chung Siu and Kannan Soundararajan

Abstract We prove a sharp upper bound on the L^2-norm of Hecke eigenforms restricted to a horocycle, as the weight tends to infinity.

1 Introduction

A central problem in "quantum chaos" is to understand the limiting behavior of eigenfunctions. An important example that has attracted a lot of attention is that of Maass cusp forms with large Laplace eigenvalue on the modular surface $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$. Let ϕ denote such a Maass form, with eigenvalue λ, and normalized to have L^2-norm 1: that is, $\int_X |\phi(z)|^2 \frac{dxdy}{y^2} = 1$. Then the Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak [15] states that the measure $\mu_\phi = |\phi(z)|^2 \frac{dxdy}{y^2}$ tends to the uniform measure on X as $\lambda \to \infty$. If ϕ is also assumed to be an eigenfunction of all the Hecke operators, then QUE holds by the work of Lindenstrauss [13], with a final step on escape of mass provided by Soundararajan [19]. Thus, the measure μ_ϕ does not concentrate on subsets of X with small measure, but is uniformly spread out. A finer problem is to understand how much the measure can concentrate on sub-manifolds; for example, on a geodesic, or a closed horocycle, or even at just a point (that is, bounding the L^∞ norm). The letter of Sarnak to Reznikov [16] draws attention to such restriction problems, and these problems (and generalizations) have been studied extensively in recent years, see for example [1, 2, 4, 11, 12, 20–22].

This note is concerned with a related question for holomorphic modular forms for $SL_2(\mathbb{Z})$ that are also eigenfunctions of all Hecke operators, when the weight k becomes large. Let f be a Hecke eigenform of weight k on the modular surface X, with L^2-norm 1: that is,

$$\int_X y^k |f(z)|^2 \frac{dxdy}{y^2} = 1.$$

H.C. Siu • K. Soundararajan (✉)
Department of Mathematics, Stanford University, Stanford, CA 94305, USA
e-mail: soarersiuhc@gmail.com; ksound@stanford.edu

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To f, we associate the measure $\mu_f = y^k|f(z)|^2\frac{dxdy}{y^2}$. The analog here of QUE states that μ_f tends to the uniform measure $\frac{3}{\pi}\frac{dxdy}{y^2}$ as $k \to \infty$, and this is known to hold by the work of Holowinsky and Soundararajan [6]. As with Maass forms, one may now ask for finer restriction theorems for holomorphic Hecke eigenforms. We study the problem of bounding the L^2-norm of Hecke eigenforms on a fixed horocycle, and establish the following uniform bound.

Theorem 1 *Let f be a Hecke eigenform of weight k on $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ with L^2-norm normalized to be 1. Let $\delta > 0$ be fixed. Uniformly in the range $1/k \leq y \leq k^{1/2-\delta}$ we have*

$$\int_0^1 y^k|f(z)|^2 dx \leq C(\delta),$$

for some constant $C(\delta)$.

Our result gives a uniform bound for the L^2-norm restricted to horocycles, answering a question from Sarnak [16]. In the Maass form situation, Ghosh et al. [4] establish weaker restriction bounds (of size λ^{ϵ}) for the corresponding problem, and Sarnak [16] notes that uniform boundedness there follows from the Ramanujan conjecture and a sub-convexity bound (in eigenvalue aspect) for the Rankin-Selberg L-function $L(s, \phi \times \phi)$. One might hope to strengthen and extend Theorem 1 in the following two ways. First, Young [22, Conjecture 1.4] has conjectured that for any fixed $y > 0$, the restriction of μ_f to the horocycle $[0, 1] + iy$ still tends to the uniform measure, as $k \to \infty$: in particular, as $k \to \infty$

$$\int_0^1 y^k|f(z)|^2 dx \to \frac{3}{\pi}.$$

Second, one might expect that two different eigenforms f and g of weight k are approximately orthogonal on the horocycle $[0, 1] + iy$, so that (as $k \to \infty$)

$$\int_0^1 y^k f(x+iy)\overline{g(x+iy)}dx \to 0.$$

Our proof, which relies crucially on bounds for mean-values of non-negative multiplicative functions in short intervals, does not allow us to address these refined conjectures.

2 Preliminaries

Let f be a Hecke eigenform of weight k on $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$. Write

$$L(s,f) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} = \prod_p\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1},$$

where $\lambda_f(n)$ are the Hecke eigenvalues for f, and α_p, $\beta_p = \alpha_p^{-1}$ are the Satake parameters. Our L-function has been normalized such that the Deligne bound reads $|\lambda_f(n)| \le d(n)$ (the divisor function), or equivalently that $|\alpha_p| = |\beta_p| = 1$.

The symmetric square L-function $L(s, \mathrm{sym}^2 f)$ is defined by

$$L(s, \mathrm{sym}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} = \prod_p \Big(1 - \frac{\alpha_p^2}{p^s}\Big)^{-1} \Big(1 - \frac{1}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_p^2}{p^s}\Big)^{-1}.$$

From the work of Shimura [17] we know that $L(s, \mathrm{sym}^2 f)$ has an analytic continuation to the entire complex plane, and satisfies a functional equation connecting s and $1-s$: namely, with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$,

$$\Lambda(s, \mathrm{sym}^2 f) = \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+k-1)\Gamma_{\mathbb{R}}(s+k)L(s, \mathrm{sym}^2 f) = \Lambda(1-s, \mathrm{sym}^2 f).$$

Moreover, Gelbart and Jacquet [3] have shown that $L(s, \mathrm{sym}^2 f)$ arises as the L-function of a cuspidal automorphic representation of $GL(3)$. Invoking the Rankin-Selberg L-function attached to $\mathrm{sym}^2 f$, a standard argument establishes the classical zero-free region for $L(s, \mathrm{sym}^2 f)$, with the possible exception of a real Landau-Siegel zero (see Theorem 5.42 of [8]). The work of Hoffstein and Lockhart [5] (especially the appendix by Goldfeld, Hoffstein and Lieman) has ruled out the existence of Landau-Siegel zeroes for this family. Thus, for a suitable constant $c > 0$, the region

$$\mathcal{R} = \Big\{ s = \sigma + it : \sigma \ge 1 - \frac{c}{\log k(1+|t|)} \Big\}$$

does not contain any zeroes of $L(s, \mathrm{sym}^2 f)$ for any Hecke eigenform f of weight k.

Lastly, we shall need a "log-free" zero-density estimate for this family, which follows from the work of Kowalski and Michel (see [9], and also the recent works of Lemke Oliver and Thorner [10], and Motohashi [14]).

Lemma 2 *There exist absolute constants B, C, and c such that for all* $1/2 \le \alpha \le 1$, *and any T we have*

$$|\{\rho = \beta + i\gamma : \ L(\rho, \mathrm{sym}^2 f) = 0, \ \beta \ge \alpha, \ |\gamma| \le T\}| \le C(T+1)^B k^{c(1-\alpha)}.$$

The special value $L(1, \mathrm{sym}^2 f)$ shows up naturally when comparing the L^2 normalization and Hecke normalization of a modular form. Suppose f has been normalized in such a way that

$$\int_X y^k |f(z)|^2 \frac{dx\,dy}{y^2} = 1.$$

Then the Fourier expansion of $f(z)$ is given by (see, for example, Chapter 13 of [7])

$$f(z) = C_f \sum_{n=1}^{\infty} \lambda_f(n)(4\pi n)^{\frac{k-1}{2}} e(nz), \tag{1}$$

where

$$C_f = \Big(\frac{2\pi^2}{\Gamma(k)L(1,\mathrm{sym}^2 f)}\Big)^{1/2}.$$

We can now state our main lemma, which refines Lemma 2 of [6], and allows us to estimate $L(1, \mathrm{sym}^2 f)$ by a suitable Euler product. Below we use the notation $g \asymp h$ to denote $g \ll h$ and $h \ll g$.

Lemma 3 *For any Hecke eigenform f of weight k for the full modular group, we have*

$$L(1, \mathrm{sym}^2 f) \asymp \exp\Big(\sum_{p\le k} \frac{\lambda_f(p^2)}{p}\Big).$$

Proof Let $1 \le \sigma \le \frac{5}{4}$, and consider for some $c > 0$ and $x \ge 1$, the integral

$$\frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} -\frac{L'}{L}(s+\sigma, \mathrm{sym}^2 f)(s+1)\Gamma(s)x^s ds, \tag{2}$$

which we shall evaluate in two ways. Here we shall take $x = k^A$ for a suitably large constant A. On one hand, we write

$$-\frac{L'}{L}(s, \mathrm{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\Lambda_{\mathrm{sym}^2 f}(n)}{n^s}$$

where $\Lambda_{\mathrm{sym}^2 f}(n) = 0$ unless $n = p^k$ is a prime power, in which case

$$\Lambda_{\mathrm{sym}^2 f}(p^k) = (\alpha_p^{2k} + 1 + \beta_p^{2k}) \log p,$$

so that $|\Lambda_{\mathrm{sym}^2 f}(n)| \le 3\Lambda(n)$ for all n. Using this in (2), and integrating term by term, using

$$\frac{1}{2\pi i}\int_{(c)} (s+1)\Gamma(s)y^s ds = e^{-1/y}\Big(1+\frac{1}{y}\Big),$$

we obtain

$$\frac{1}{2\pi i}\int_{(c)} -\frac{L'}{L}(s+\sigma, \mathrm{sym}^2 f)(s+1)\Gamma(s)x^s ds = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^2 f}(n)}{n^\sigma} e^{-n/x}\Big(1+\frac{n}{x}\Big). \tag{3}$$

On the other hand, shift the line of integration in (2) to $\Re(s) = -3/2$. We encounter poles at $s = 0$, and at $s = \rho - \sigma$ for non-trivial zeroes $\rho = \beta + i\gamma$ of $L(s, \mathrm{sym}^2 f)$. Computing these residues, we see that (2) equals

$$\begin{aligned} -\frac{L'}{L}(\sigma, \mathrm{sym}^2 f) &- \sum_\rho x^{\rho-\sigma}(\rho-\sigma+1)\Gamma(\rho-\sigma) \\ &+ \frac{1}{2\pi i}\int_{(-3/2)} -\frac{L'}{L}(s+\sigma, \mathrm{sym}^2 f)x^s(s+1)\Gamma(s)ds. \end{aligned} \tag{4}$$

Differentiate the functional equation of $L(s, \mathrm{sym}^2 f)$ logarithmically, and use Stirling's formula. Thus with $s = -\frac{3}{2} + it$ we obtain that

$$-\frac{L'}{L}(s+\sigma, \mathrm{sym}^2 f) \ll \log(k(1+|t|) + \left|\frac{L'}{L}(1-s-\sigma, \mathrm{sym}^2 f)\right| \ll \log(k(1+|t|)).$$

Therefore the integral in (4) may be bounded by $O((\log k)x^{-3/2})$, and we conclude that

$$\begin{aligned} \sum_n \frac{\Lambda_{\mathrm{sym}^2 f}(n)}{n^\sigma} e^{-n/x}\Big(1+\frac{n}{x}\Big) &= -\frac{L'}{L}(\sigma, \mathrm{sym}^2 f) \\ &- \sum_\rho x^{\rho-\sigma}(\rho+1-\sigma)\Gamma(\rho-\sigma) + O(x^{-3/2}\log k). \end{aligned} \tag{5}$$

We now bound the sum over zeros in (5). Write $\rho = \beta + i\gamma$, and split into terms with $n \le |\gamma| < n+1$, where $n = 0, 1, 2, \ldots$. If $n \le |\gamma| < n+1$, we may check using the exponential decay of the Γ-function that

$$|\rho - \sigma + 1||\Gamma(\rho-\sigma)| \ll (\sigma-\beta)^{-1}e^{-n}.$$

Therefore the contribution of zeros from this interval is

$$\ll \sum_{n\le|\gamma|<n+1} \frac{x^{\beta-\sigma}}{\sigma-\beta}e^{-n}.$$

Splitting the zeros further based on $1-(j+1)/\log k \le \beta < 1-j/\log k$ (and using the zero free region, so that $\sigma - \beta \gg (j+1)/\log k$) the above is

$$\ll e^{-n}\sum_{j=0}^{\log k} \frac{x^{1-\sigma-j/\log k}}{(j+1)/\log k}|\{\beta+i\gamma : 1-(j+1)/\log k \le \beta < 1-j/\log k,\ n \le |\gamma| < n+1\}.$$

Now using the log-free zero density estimate from Lemma 2, and recalling that $x = k^A$, the quantity above is

$$\ll e^{-n} x^{1-\sigma} \log k \sum_{j=0}^{\log k} \frac{e^{-jA}}{j+1} (n+1)^B k^{c(j+1)/\log k} \ll (n+1)^B e^{-n} x^{1-\sigma} \log k,$$

provided $A \geq c + 1$ is large enough. Now summing over n, we conclude that the sum over zeros in (5) is $\ll x^{1-\sigma} \log k$.

Use this bound in (5), and integrate that expression over $1 \leq \sigma \leq 5/4$. It follows that

$$\log L(1, \mathrm{sym}^2 f) = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^2 f}(n)}{n \log n} e^{-n/x}\Big(1 + \frac{n}{x}\Big) + O(1) = \sum_{p \leq x} \frac{\lambda_f(p^2)}{p} + O(1),$$

since the contribution of prime powers above is easily seen to be $O(1)$, and since

$$\sum_{p \leq x} \frac{1}{p} \Big| 1 - e^{-p/x}\Big(1 + \frac{p}{x}\Big)\Big| + \sum_{p > x} \frac{1}{p} e^{-p/x}\Big(1 + \frac{p}{x}\Big) = O(1).$$

Exponentiating, we obtain

$$L(1, \mathrm{sym}^2 f) \asymp \exp\Big(\sum_{p \leq x} \frac{\lambda_f(p^2)}{p}\Big) \asymp \exp\Big(\sum_{p \leq k} \frac{\lambda_f(p^2)}{p}\Big),$$

since $x = k^A$, and $\sum_{k < p \leq k^A} 1/p \ll 1$. This concludes our proof. □

3 Proof of Theorem 1

The Fourier expansion (1) and the Parseval formula give

$$\int_0^1 y^k |f(z)|^2 dx = \frac{C_f^2}{4\pi} \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n} (4\pi n y)^k e^{-4\pi n y}$$
$$\ll \frac{1}{\Gamma(k) L(1, \mathrm{sym}^2 f)} \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n} (4\pi n y)^k e^{-4\pi n y}. \tag{6}$$

For $\xi \geq 0$, note that

$$\frac{\xi^k e^{-\xi}}{\Gamma(k)} \asymp \sqrt{k}\Big(\frac{\xi}{k}\Big)^k e^{k-\xi} \ll \begin{cases} \sqrt{k} \exp(-(k-\xi)^2/(4k)) & \text{if } \xi \leq 2k \\ \sqrt{k}(e/2)^{k-\xi} & \text{if } \xi > 2k, \end{cases} \tag{7}$$

where the first bound follows because $\log(1+t) \le t - t^2/4$ for $|t| \le 1$ (with $t = (\xi - k)/k$), and the second bound from $\log(1+t) \le t \log 2$ for $t \ge 1$.

The estimate (7) with $\xi = 4\pi ny$ shows that the sum in (6) is concentrated around values of n with $|4\pi ny - k|$ about size $\sqrt{k}$. To flesh this out, let us first show that the contribution to (6) from n with $4\pi ny \ge 2k$ is negligible. Using the second bound in (7), such terms n contribute (using that $L(1, \mathrm{sym}^2 f) \gg (\log k)^{-1}$, which follows from Lemma 3 or [5])

$$\ll \frac{1}{L(1, \mathrm{sym}^2 f)} \sum_{n \ge k/(2\pi y)} \frac{\lambda_f(n)^2}{n} \sqrt{k}(e/2)^{k - 4\pi ny}$$

$$\ll \sqrt{k} \log k \sum_{n \ge k/(2\pi y)} \frac{\lambda_f(n)^2}{n} \frac{1}{n} e^{-k/10} \ll e^{-k/20}.$$

This contribution to (6) is clearly negligible.

It remains to handle the contribution from those n with $4\pi ny \le 2k$. Divide such n into intervals of the form $j\sqrt{k} \le |4\pi ny - k| < (j+1)\sqrt{k}$, where $0 \le j \ll \sqrt{k}$. We use the first bound in (7) with $\xi = 4\pi ny$, and in the range $j\sqrt{k} \le |4\pi ny - k| < (j+1)\sqrt{k}$ this gives

$$\frac{1}{\Gamma(k)} \frac{(4\pi ny)^k}{n} e^{-4\pi ny} \ll \frac{\sqrt{k} e^{-j^2/4}}{n} \ll \frac{y}{\sqrt{k}} e^{-j^2/8},$$

provided $y \ge 1/k$ say. Thus the contribution from the terms $j\sqrt{k} \le |4\pi ny - k| < (j+1)\sqrt{k}$ is

$$\ll \frac{y e^{-j^2/8}}{\sqrt{k} L(1, \mathrm{sym}^2 f)} \sum_{j\sqrt{k} \le |4\pi ny - k| < (j+1)\sqrt{k}} \lambda_f(n)^2. \tag{8}$$

At this stage, we appeal to a result of Shiu (see Theorem 1 of [18]) bounding averages of non-negative multiplicative functions in short intervals.

Lemma 4 *Let g be a non-negative multiplicative function with (i) $g(p^l) \le A^l$ for some constant A, and (ii) $g(n) \ll_\epsilon n^\epsilon$ for any $\epsilon > 0$. Then for any $\delta > 0$, if $x^\delta \le z \le x$, we have*

$$\sum_{x < n \le x+z} g(n) \ll_{A,\delta} \frac{z}{\log x} \exp\Big(\sum_{p \le x} \frac{g(p)}{p} \Big).$$

Applying this lemma in (8), in the range $y \le k^{1/2-\delta}$, we may bound that quantity by

$$\ll \frac{y e^{-j^2/8}}{\sqrt{k} L(1, \mathrm{sym}^2 f)} \frac{\sqrt{k}}{y \log k} \exp\Big(\sum_{p \le k} \frac{\lambda_f(p)^2}{p} \Big).$$

Since $\lambda_f(p)^2 = \lambda_f(p^2) + 1$, the above bound when combined with Lemma 3 yields $\ll e^{-j^2/8}$, and summing this over all j gives $\ll 1$. Thus we conclude that the quantity in (6) is bounded, completing the proof of our theorem.

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On the Triple Product Formula: Real Local Calculations

Michael Woodbury

Abstract We consider a triple of admissible representations π_j for $j = 1, 2, 3$ of $\mathrm{GL}_2(\mathbb{R})$ of weights k_j with $k_1 \geq k_2 + k_3$. Test vectors are given, and using a formula of Michel-Venkatesh explicit values for local trilinear forms are computed for these vectors. Using this we determine the real archimedean local factors in Ichino's formula for the triple product L-function. Applications both new and old to subconvexity, quantum chaos and p-adic modular forms are discussed.

1 Introduction

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. We consider a triple of GL_2 automorphic representations π_1, π_2, π_3 over F such that the product of the central characters is trivial. Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and denote by $\Lambda(s, \Pi)$ the corresponding (completed) L-function corresponding to the natural 8-dimensional tensor product representation of the L-group $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. This L-function has a distinguished history. Indeed, if π_3, for example, corresponds to an Eisenstein series and π_1 and π_2 to modular forms with q-expansions $f = \sum_{n\geq 1} a_f(n)q^n$ and $g = \sum_{n\geq 1} a_g(n)q^n$, then, up to some additional Gamma factors, $\Lambda(s, \Pi)$ is the Rankin-Selberg convolution L-function

$$L(s, f \times g) = \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

whose importance in number theory can hardly be overstated. Thinking of this as a triple product L-function was an important point of view taken in the work of Michel and Venkatesh in [17] in which they established subconvexity bounds for GL_2 type L-functions simultaneously in all aspects.

M. Woodbury (✉)
Columbia University, 2990 Broadway, New York, NY 10027, USA
e-mail: woodbury@math.columbia.edu

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The case in which all three representations are cuspidal was first taken up by Garrett in [5] and by Piateski-Shapiro and Rallis in [21]. Garrett, by essentially integrating a triple of cusp forms f, g, h with Fourier coefficients as above against a certain Eisenstein series for Sp_6, was able to give an integral representation for the triple product L-function

$$L(s, f \times g \times h) = \sum_{n \geq 1} \frac{a_f(n) a_g(n) a_h(n)}{n^s},$$

and he used this to prove a functional equation and meromorphic continuation. This work has since been extended by many authors. (See for example [6, 7, 25] and [11].) The main result of [11] (to be described below) is the culmination of these formulas. It has the advantage of being valid for any choice of test vectors; however, from the standpoint of number theoretic applications, Watson's more explicit result has been particularly applicable in number theory and quantum chaos precisely due to its more explicit form. Most notably among these applications are subconvexity results (See for example [1]) and to the so-called Quantum Unique Ergodicity conjecture which is now a theorem of Holowinsky and Soundarajan (see [8] and [23] and [25]).

To describe Ichino's formula, let us write $\pi_j = \otimes_v \pi_{j,v}$ as a (restricted) tensor product over the places v of F, with each $\pi_{j,v}$ an admissible representation of $\mathrm{GL}_2(F_v)$. Let $\langle \cdot, \cdot \rangle_v$ be a (Hermitian) form on π_j. Then, assuming that $\varphi_j = \otimes \varphi_{j,v} \in \pi_{j,v}$ is factorizable,[1] for each v we can consider the form obtained by integrating the matrix coefficient associated to $\varphi_v = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$:

$$I'_v(\varphi_v) = \int_{\mathrm{PGL}_2(F_v)} \langle \pi_v(g_v)\varphi_{1,v}, \varphi_{1,v}\rangle_v \langle \pi_v(g_v)\varphi_{2,v}, \varphi_{2,v}\rangle_v \langle \pi_v(g_v)\varphi_{3,v}, \varphi_{3,v}\rangle_v dg_v, \tag{1.1}$$

and the normalization

$$I_v(\varphi_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(1/2, \Pi_v)} I'_v(\varphi_v). \tag{1.2}$$

We call I'_v and I_v trilinear forms although this is somewhat of an abuse of language since it actually defines a quadratic form on the triple product.

Ichino proved (in the case that each π_i is cuspidal) that there is a constant C (depending only on the choice of measures) such that

$$\frac{\left| \int_{[\mathrm{GL}_2]} \varphi_1(g)\varphi_2(g)\varphi_3(g)\, dg \right|^2}{\prod_{j=1}^{3} \int_{[\mathrm{GL}_2]} |\varphi_j(g)|^2\, dg} = \frac{C}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{\Lambda(1/2, \Pi)}{\Lambda(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle_v} \tag{1.3}$$

[1] As a restricted tensor product, we have chosen vectors $\varphi^0_{i,v} \in \pi_v$ for all but finitely many places v. We require that the local inner forms must satisfy $\langle \varphi^0_{i,v}, \varphi^0_{i,v} \rangle_v = 1$ for all such v.

whenever the denominators are nonzero. Note that the notation $[\mathrm{GL}_2]$ represents the quotient $\mathbb{A}^\times\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A})$. By the choice of normalizations, the product on the right hand side of (1.3) is in fact a finite product as each factor is identically 1 when all of the input data is unramified.

In order to derive number theoretic applications from Ichino's formula, it is necessary to compute (or at least control) the local factors at the infinite and ramified finite places. This is the topic of the author's PhD thesis [26] wherein Watson's formulas and explicit generalizations are derived from (1.3) by computing local trilinear forms. (In the nonarchimedean case, for example, the trilinear forms were computed for triples of representations with—potentially distinct—squarefree level.) Using this, a certain hypothesis of Venkatesh from [24] was proved thereby leading to subconvexity results analogous to [1], but in the level instead of eigenvalue aspect. This topic is further taken up by Hu in [9] and [10] wherein higher ramification is considered with applications similar to those of [18].

In addition to the results outlined above, the triple product L-function plays an important role in the work of Darmon, Lauder and Rotger (see [3]) as well as others in relation to the so-called elliptic Stark conjecture, a generalization of Stark's conjecture that is closely related to the Birch-Swinnerton-Dyer conjecture. In this work it is critical to know that up to a computable power of π, the central critical value of the completed triple product L-function is rational. This work takes as a necessary starting step the evaluation of the right hand side of (1.3) in the case that the triple of representations comes from two weight one modular forms and a weight two modular form—a case which was not covered by Watson. The relevant calculation at the infinite case is one of the results of the current paper.

To be more explicit, in this paper we treat the question of determining test vectors at the real infinite places and compute the corresponding trilinear forms. This work builds in particular on the results of [26] and the appendix to [22]. We also remark that our choice of test vectors is inspired greatly by Popa [19]. Moreover, although we give results in most cases only for one choice of test vectors, using [16] one can deduce the values of the trilinear form at other test vectors as well.

Since we will be considering only the local case from this point onward, unless otherwise specified, we drop the subscript v from all local objects. Hence, for example, $I(f_1 \otimes f_2 \otimes f_3)$, $L(s, \Pi)$ etc. refer to the local normalized trilinear and L-factors of (1.2) at a real place. With this in place, the following is the main result of this paper.

Theorem 1 *Suppose that π_j for $j = 1, 2, 3$ are irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$ of weights k_j for which the product of central characters is trivial.*[2] *If we assume*[3] *that $k_1 \geq k_2 + k_3$ then there exists a choice of test vectors $f^{(j)} \in \pi_j$ such that $f = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}$ satisfies $I(f) \neq 0$. (See*

[2] This implies directly that $k_1 + k_2 + k_3$ is even.

[3] By Prasad [20], this assumption is necessary as otherwise I_v is identically zero.

Propositions 3.2, 3.3, 3.4 and 3.5 for the choice of vectors and explicit values of $I(f)$ in each case.)

In particular,[4] *if $k_1 = k_2 + k_3$ there exist $f_j \in \pi_j$ such that*

$$\frac{I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)})}{\langle f^{(1)}, f^{(1)} \rangle \langle f^{(2)}, f^{(2)} \rangle \langle f^{(3)}, f^{(3)} \rangle} = c(\Pi) = \begin{cases} 2 & \text{if } k_1 \geq 2, \\ 1 & \text{otherwise.} \end{cases} \tag{1.4}$$

Equivalently, setting $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, there exist $f^{(j)} \in \pi_j$ such that $\langle f^{(j)}, f^{(j)} \rangle = 1$ for each $j = 1, 2, 3$, and for which the matrix coefficients $\Phi_j(g) = \langle \pi_1(g) f^{(j)}, f^{(j)} \rangle$ satisfy

$$I'(f) = \int_{\mathrm{PGL}_2(\mathbb{R})} \Phi_1(g)\Phi_2(g)\Phi_3(g)\, dg = c(\Pi) \frac{\Gamma_{\mathbb{R}}(2)^2 L(\frac{1}{2}, \Pi)}{L(1, \Pi, \mathrm{Ad})}. \tag{1.5}$$

The right hand side of (1.5) is given explicitly in terms of the parameters of π_j and the Gamma function in Table 2.

Remark 1 In the applications in [22] and [3] it is essential that one has an exact formula (in the latter case at least up to rational factor) for the triple product L-function.

Remark 2 With the hindsight of Ichino's formula which linked the local trilinear forms to certain zeta integrals on the group Sp_6, the evaluation of the archimedean local trilinear forms in the case that two or more of the representations are (weight zero) principal series was essentially worked out by Ikeda in [13] and by Watson [25] as evaluations of these zeta integrals. Moreover, [12] gives Theorem 1 for $k_3 > 1$. As such, the principal new contribution of our work here is to give a generalized and uniform treatment. Moreover, the calculations in the case of $k_2 = 1$ and/or $k_3 = 1$ as well as the more general results in Propositions 3.2, 3.3, 3.4 and 3.5 are new. Besides giving these new results, we believe that the present proofs illustrate how the method is widely and easily applicable.

The proof of Theorem 1 is obtained on a case by case basis considering all possible combinations of representations π_1, π_2, π_3. We give an overview of the relevant representation theoretic background in Sect. 2 and then compute the trilinear forms in Sect. 3. The normalizing factor in (1.2) relating I' and I can be calculated following the prescription for the local Langlands correspondence given in [15]. We include an overview of this theory and record the relevant factors for each of the possible cases in an Appendix.

[4]In this special case we also assume that if $k_j = 0$ for all j then a certain invariant $\epsilon = 0$ defined in Sect. 3.1 in terms of the representations π_j.

2 Background and Notation

We now set notation and give definitions for the representation theory of $\mathrm{GL}_2(\mathbb{R})$ that will be used in the sequel. This theory is well known. See [2] or [14] for complete details.

2.1 *Admissible Representations of* $\mathrm{GL}_2(\mathbb{R})$

Given an automorphic representation $\otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$, as discussed in the introduction, for all real places v of F the local factor π_v is an admissible $(\mathfrak{gl}_2, K)$-module where $\mathfrak{gl}_2$ is the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$ and

$$K = \mathrm{SO}(2) = \left\{\kappa_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)\right\}.$$

Using a slight abuse of language we refer to such a module as an admissible representation of $\mathrm{GL}_2(\mathbb{R})$.

Let $\psi_n : K \to \mathbb{C}$ be given by $\psi_n(\kappa_\theta) = e^{in\theta}$. Recall that restricting any irreducible admissible representation π to K there exists a nonnegative integer $\mathrm{wt}(\pi)$ such that

$$\pi_j \mid_K \simeq \bigoplus_{\substack{|n| \geq \mathrm{wt}(\pi) \\ n \equiv \mathrm{wt}(\pi) \pmod 2}} \mathbb{C}\psi_n.$$

An element $\phi \in \pi$ is said to have *weight* n if ϕ corresponds, via this isomorphism, to an element in $\mathbb{C}\psi_n$. The integers n appearing in the decomposition above are called the *weights* of π. Accordingly, we say that π has *even* or *odd weight* depending on whether $\mathrm{wt}(\pi)$ is even or odd respectively.

We define the following subgroups of $\mathrm{GL}_2(\mathbb{R})$:

$$A = \left\{a(y) = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid y \in \mathbb{R}^\times\right\},$$
$$Z = \left\{z(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) \mid u \in \mathbb{R}^\times\right\},$$
$$N = \left\{n(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\right\}.$$

We can construct all such representations via the *induced representations* which are defined in terms of (quasi-)characters $\chi_j : \mathbb{R}^\times \to \mathbb{C}^\times$ of the form $\chi_j(x) = \mathrm{sgn}(x)^{\epsilon_j}|x|^{s_j}$ where $\mathrm{sgn} : \mathbb{R}^\times \to \{\pm 1\}$ is the sign character $x \mapsto x/|x|$, $\epsilon_j \in \{0, 1\}$ and $s_j \in \mathbb{C}$. Then

$$\mathcal{B}(\chi_1, \chi_2) := \left\{ f : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{c} f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g) = \chi_1(a)\chi_2(d)|\tfrac{a}{d}|^{1/2} f(g) \\ \text{for all } g \in \mathrm{GL}_2(\mathbb{R}), \\ f \text{ is smooth and } K\text{-finite.} \end{array} \right\}.$$

It is easy to see that for any $f \in \mathcal{B}(\chi_1, \chi_2)$,

$$f(z(u)a(y)g) = \operatorname{sgn}(u)^{\delta}|u|^{\mu}\operatorname{sgn}(y)^{\epsilon_1}|y|^{s}f(g)$$

where $\delta \in \{0, 1\}$ is such that $\delta \equiv \epsilon_1 + \epsilon_2 \pmod 2$, $s = \frac{1}{2}(1 + s_1 - s_2)$ and $\mu = s_1 + s_2$. Given this, we define $\pi_{\delta,\epsilon}(s, \mu) := \mathcal{B}(\chi_1, \chi_2)$ where $\chi_1 = \operatorname{sgn}^{\epsilon}|\cdot|^{s+\frac{\mu-1}{2}}$ and $\chi_2 = \operatorname{sgn}^{\delta-\epsilon}|\cdot|^{-s+\frac{\mu+1}{2}}$. We also use the notation $\pi_{\delta,\epsilon}(s) := \pi_{\delta,\epsilon}(s, 0)$. The above makes clear that the central character of $\pi_{\delta,\epsilon}(s, \mu)$ is given by $\operatorname{sgn}^{\delta}|\cdot|^{\mu}$.

Twisting by the determinant we have

$$|\det(\cdot)|^{\frac{\mu}{2}} \otimes \pi_{\delta,\epsilon}(s, 0) \simeq \pi_{\delta,\epsilon}(s, \mu),$$

and hence it follows that $\pi_{\delta,\epsilon}(s)$ is the unique such twist of $\pi_{\delta,\epsilon}(s, \mu)$ such that the central character is $\operatorname{sgn}^{\delta}$.

We denote by $f_{m,s}$ the weight m vector in $\pi_{\delta,\epsilon}(s)$ satisfying $f_{m,s}(\kappa_\theta) = e^{im\theta}$. Note that this is nonzero if and only if $m \equiv \delta \pmod 2$. The set of all such vectors forms a basis.

There exists an intertwining operator from $\pi = \mathcal{B}(\chi_1, \chi_2) = \pi_{\delta,\epsilon}(s, \mu)$ to $\widetilde{\pi} := \mathcal{B}(\chi_2, \chi_1) = \pi_{\delta,\delta-\epsilon}(1 - s)$. If $\operatorname{Re}(s) > \frac{1}{2}$, this is given by $M(s) : \pi \to \widetilde{\pi}$ defined via

$$\big(M(s)f\big)(g) := \int_{-\infty}^{\infty} f(wn(x)g)\, dx, \tag{2.1}$$

where $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We drop μ from the notation as the map is independent of the choice of μ within the class of twists of $\pi_{\delta,\epsilon}(s)$. By analytic continuation, $M(s)$ extends to other values of s, μ. It sends the weight m vector $f_m \in \pi$ to a multiple of $\widetilde{f}_m \in \widetilde{\pi}$. As long as $\pi_{\delta,\epsilon}(s, \mu)$ is irreducible (which is the case unless $s = \frac{k}{2}$ or $s = 1 - \frac{k}{2}$ with $k > 1$ an integer satisfying $k \equiv \delta \pmod 2$) the map $M(s)$ is an isomorphism.

Given $\pi = \mathcal{B}(\chi_1, \chi_2) = \pi_{\delta,\epsilon}(s, \mu)$ the contragradient is $\widehat{\pi} = \mathcal{B}(\chi_1^{-1}, \chi_2^{-1}) = \pi_{\delta,\epsilon}(1 - s, -\mu)$ with pairing

$$(\cdot, \cdot) : \pi \times \widehat{\pi} \to \mathbb{C}, \qquad (f, h) := \int_K f(\kappa)h(\kappa)\, d\kappa \tag{2.2}$$

(We normalize the measure $d\kappa$ on K such that $\operatorname{vol} K = 1$.) In the case of the unitary principal series, the characters $\overline{\chi_j} = \chi_j^{-1}$, and so we can identify the contragradient with the complex conjugate $\overline{\pi}$. Then the we have a Hermitian form $\langle f, g\rangle := (f, \overline{g})$ on π. In general, for unitary representations π one has $\widehat{\pi} = \overline{\widetilde{\pi}}$. Using the intertwining operator $M(s)$ one can define a Hermitian form $\langle f, g\rangle := (f, \overline{cM(s)g})$ for a suitable constant c.

For global applications (i.e., to be applied towards (1.3)), one only needs the results of this paper for choices of (s, μ) such that these representations are unitarizable. Note that $\pi_{\delta,\epsilon}(s, \mu)$ is unitarizable if and only if $\pi_{\delta,\epsilon}(s)$ is unitarizable and the central character is unitary, i.e., $\mu \in i\mathbb{R}$. Therefore up to twists by unitary characters, the unitary representations are differentiated as follows.

- If $s = \frac{1}{2} + \nu$ with $\nu \in i\mathbb{R}^\times$, $\pi_{\delta,\epsilon}(s)$ is called an even or odd weight *(unitary) principal series* according as $\delta = 0$ or 1 respectively. Since $\pi_{\delta,\epsilon}(\frac{1}{2} + \nu) \simeq \pi_{\delta,\delta-\epsilon}(\frac{1}{2} - \nu)$, in the case of $\delta = 1$, it suffices to consider $\pi_{1,\epsilon}(\frac{1}{2} + it)$ only in the case of $\epsilon = 0$.
- If $\delta = 0$ and $s = \frac{1}{2} + \nu$ and $s' = \frac{1}{2} + \nu'$ with $\nu, \nu' \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, we have that $\pi_{0,\epsilon}(s) \simeq \pi_{0,\epsilon'}(s')$ if and only if $\epsilon = \epsilon'$ and $s' = 1 - s$ (meaning that $\nu' = -\nu$). These are called *complementary series*.
- If $s = \frac{k}{2}$ or $s = 1 - \frac{k}{2}$ for some $k \geq 1$ then $\pi_{\delta,\epsilon}(s) = 0$ unless $k \equiv \delta \pmod 2$. (The choice of ϵ is irrelevant.) Then, for such s with $k > 1$, $\pi_{\delta,\epsilon}(s)$ is not irreducible; however, there is a representation π^k_{dis}, called the *(holomorphic) weight k discrete series*, which is an irreducible $(\mathfrak{gl}_2, O(2))$-module. The weight k discrete series is isomorphic to a subrepresentation if $s = \frac{k}{2}$ and a quotient if $s = 1 - \frac{k}{2}$ if $k > 1$, and $\pi^1_{\text{dis}} \simeq \pi_{1,0}(\frac{1}{2})$. We refer to π^1_{dis} as a *limit of discrete series*.

To conclude this section we record the action of the Laplace-Beltrami operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta}, \tag{2.3}$$

and the raising and lowering operators

$$R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right), \quad L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}\right) \tag{2.4}$$

on $f \in \pi = \pi_{\delta,\epsilon}(s)$ in terms of the coordinates $n(x)a(y)\kappa_\theta$ on $\mathrm{GL}_2(\mathbb{R})$. These act via

$$\Delta f = s(1-s)f \qquad \text{(for all } f \in \pi),$$

and

$$Rf_{m,s} = \left(s + \frac{m}{2}\right)f_{m+2,s}, \qquad Lf_{m,s} = \left(s - \frac{m}{2}\right)f_{m-2,s}. \tag{2.5}$$

2.2 Whittaker Models and Functions

Given an irreducible admissible representation $\pi = \pi_{\delta,\epsilon}(s)$ or $\pi = \pi^k_{\text{dis}}$ and a character $\psi : \mathbb{R} \to \mathbb{C}^\times$, there is a unique space $\mathcal{W}(\pi, \psi)$ of Schwartz functions $W : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ such that

$$W(n(x)g) = \psi(x)W(g) \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{R}), \tag{2.6}$$

and, under the action $\rho(g)W(h) = W(hg)$, $\pi \simeq W(\pi, \psi)$.

We denote by $W_m \in \mathcal{W}(\pi, \psi)$ the unique up to constant vector of weight m, i.e., the vector which satisfies $\rho(\kappa_\theta)W_m = e^{im\theta}W_m$. There exists an explicit intertwiner $\pi \to \mathcal{W}(\pi, \psi)$ given by

$$f \mapsto W_f(g) = \int_{\mathbb{R}} f(wn(x)g)\overline{\psi(x)}\, dx.$$

Hence, if $f_{m,s} \in \pi_{\delta,\epsilon}(s)$ as defined in the previous section, the functions $W_{f_{m,s}}$ satisfy the same relations as given in (2.5) for the raising and lowering operators. Rather than work with this intertwiner directly, we simply require that $W_m \in \mathcal{W}(\pi, \psi)$ be a weight m vector such that

$$RW_m = \left(s + \frac{m}{2}\right) W_m, \qquad LW_m = \left(s - \frac{m}{2}\right) W_{m-2},$$

and

$$\rho(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))W_m = (-1)^\delta W_{-m} \tag{2.7}$$

hold for all m. This defines the collection $\{W_m\}$, therefore, up to a common constant multiple. Moreover, if $\pi = \pi_{\delta,\epsilon}(s)$ one sees via (2.6) that

$$W_m(a(-y)) = (-1)^{\epsilon+\delta} W_m(a(y)), \tag{2.8}$$

so in this case $W_m(y)$ is determined by its values on $y > 0$. In the case $\pi = \pi^k_{\mathrm{dis}}$, we will see that $W_m(a(y))$ is nonzero either for $y > 0$ or $y < 0$ (depending on ψ and m).

Following the strategy of [19] (which itself is based on [14]), in Proposition 2.1 we describe certain functions $W \in \mathcal{W}(\pi, \psi)$. We do so in terms of the *modified Bessel function*, $K_\nu(y)$, which[5] up to a constant is the unique solution with moderate growth (as $y \to \infty$) to the differential equation

$$0 = f''(y) + \frac{1}{y} f'(y) - \left(\frac{\nu^2}{y^2} + 1\right) f(y). \tag{2.9}$$

Fixing the constant, we take for $y > 0$

$$K_\nu(y) = \tfrac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(t+\frac{1}{t}\right)} t^\nu\, d^\times t,$$

which is easily seen to satisfy (2.9) and have exponential decay as $y \to \infty$.

[5]Contrary to commonly used notation, in [19] this function is referred to as J_ν.

In the sequel, we will make use of the identities

$$\int_0^\infty K_\nu(y/2)K_\mu(y/2)y^s d^\times y = 2^{2s-3}\frac{\Gamma(\frac{s+\mu+\nu}{2})\Gamma(\frac{s-\mu+\nu}{2})\Gamma(\frac{s+\mu-\nu}{2})\Gamma(\frac{s-\mu-\nu}{2})}{\Gamma(s)}, \tag{2.10}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\mu)| + |\mathrm{Re}(\nu)|$, and

$$\int_0^\infty e^{-y/2}K_\nu(y/2)y^s d^\times y = \pi^{1/2}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+\frac{1}{2})}, \tag{2.11}$$

which holds whenever $\mathrm{Re}(s) > |\mathrm{Re}\,\nu|$. These are equations (6.8.48) and (6.8.28) of [4] respectively. We will also need the additional fact that

$$\frac{d}{dz}K_\nu(z) = -\frac{1}{2}\big(K_{\nu-1}(z) + K_{\nu+1}(z)\big) \tag{2.12}$$

$$= \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z). \tag{2.13}$$

Proposition 2.1 *Suppose that $\psi(x) = e^{\gamma ix/2}$ with $\gamma \in \{\pm\}$. Let $W^\gamma_{\pm k} \in \mathcal{W}(\pi^k_{\mathrm{dis}}, \psi)$ be vectors of weight $\pm k$ respectively. Up to scalar, for $k \geq 0$ these are given by*

$$W^-_{-k}(a(y)) = W^+_k(a(y)) = \begin{cases} y^{k/2}e^{-y/2} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

and $W^-_k(a(y)) = W^+_{-k}(a(y)) = (-1)^k W^+_k(a(-y))$.

Writing $s = \frac{1}{2} + \nu$, we may choose $W^\gamma_0, W^\gamma_{-2}, W^\gamma_2 \in \mathcal{W}(\pi_{0,\epsilon}(s), \psi)$ such that

$$W^\gamma_0(a(y)) = \mathrm{sgn}(y)^\epsilon |y|^{1/2}K_\nu(|y|/2), \tag{2.15}$$

$$(W^\gamma_{-2} - W^\gamma_2)(a(y)) = \mathrm{sgn}(y)^{\epsilon+1}|y|^{3/2}K_\nu(|y|/2), \tag{2.16}$$

and

$$(W^\gamma_{-2} + W^\gamma_2)(a(y)) = \frac{\mathrm{sgn}(y)^{\epsilon+1}}{4}\Big(2|y|^{-1/2}K_\nu(|y|/2) - |y|^{1/2}\big(K_{\nu-1}(|y|/2) + K_{\nu+1}(|y|/2)\big)\Big). \tag{2.17}$$

Finally, we may choose $W^\gamma_{\pm 1} \in \mathcal{W}(\pi_{1,0}(s), \psi)$ such that

$$(W^\gamma_{-1} + W^\gamma_1)(a(y)) = yK_{-\frac{1}{2}+\nu}(|y|/2), \tag{2.18}$$

and

$$(W^\gamma_{-1} - W^\gamma_1)(a(y)) = |y|K_{\frac{1}{2}+\nu}(|y|/2). \tag{2.19}$$

Remark 3 Note that our choice of character ψ is not the same as that given in [19] resulting in slightly different formulas. One advantage of our choice (as will be shown) is the corresponding functions W_m will be solutions of the classical differential equation of Whittaker.

Proof Given that π has central character sgn^δ, any function $W_m \in \mathcal{W}(\pi, \psi)$ of weight m satisfies

$$W_m(z(u)n(x)a(y)\kappa_\theta) = \mathrm{sgn}(u)^\delta e^{i(\gamma \frac{x}{2} + m\theta)} W_m(a(y)). \tag{2.20}$$

Suppose that $\lambda = s(1-s)$ is the eigenvalue of the action of the Laplace operator Δ on π. Then combining this with the definition of Δ from (2.3) applied to (2.20), it is easy to see that $w_m(y) = W_m(a(y))$ satisfies the differential equation

$$w'' + \left[-\frac{1}{4} + \frac{\gamma m}{2y} + \frac{\lambda}{y^2} \right] w = 0. \tag{2.21}$$

which, writing $s = \frac{1}{2} + \nu$, has solutions $W_{m,\nu}(y)$ and $W_{-m,\nu}(-y)$, the so-called Whittaker functions. Since only $W_{m,\nu}$ has moderate growth as $y \to \infty$, together with (2.8), we find—provided that[6] $W_m^\gamma \in \mathcal{W}(\pi_{\delta,\epsilon}(s), \psi)$ and $s \notin \frac{1}{2}\mathbb{Z}$—that for $m \geq 0$,

$$W_m^\gamma(a(y)) = \begin{cases} W_{m,\nu}(y) & \text{if } y > 0, \\ (-1)^{\epsilon+\delta} W_{m,\nu}(-y) & \text{if } y < 0. \end{cases}$$

Combined with (2.7), this defines $W_m(a(y))$ for all $m \equiv \delta \pmod 2$ and for all $y \neq 0$.

Applying the operators R and L given in (2.4) to W_m we find that

$$\left(\nu + \frac{1 \pm m}{2} \right) w_{m\pm 2} = \pm \left(\frac{m-y}{2} \right) w_m + y w_m'. \tag{2.22}$$

If $\pi = \pi_{\mathrm{dis}}^k$, one has that W_k must be annihilated by L. (This fact is true for $k = 1$ as well.) Using this leads to the differential equation

$$2y w_k'(y) + \left(y - \frac{k}{2} \right) w_k = 0,$$

which can be solved using elementary methods. The restriction on the growth leads immediately to (2.14). The formula for W_{-k} follows from (2.7).

For the remainder of the proof we note that the choice of γ effects only the sign of m appearing in (2.21). This means that W_m^- is the weight m vector such

[6] It is necessary, of course, that $m = \mathrm{wt}(\pi) + 2n$ for some $n \in \mathbb{Z}_{\geq 0}$.

that $W_m^-(a(y)) = W_{-m}^+(a(y))$. For the purpose of the rest of the proof, therefore, it suffices to take $\gamma = +$ and we drop it from the notation.

Now we consider the case of $\pi = \pi_{0,\epsilon}(s)$. If we let $w_0(y) = y^{1/2}f(y/2)$ and plug this into (2.21), after simplifying, we arrive at Eq. (2.9). Since $f = K_\nu$ has moderate growth as $y \to \infty$, clearly $y^{1/2}K_\nu(y/2)$ does as well. Equation (2.15) follows by applying (2.8).

Next, we apply (2.22) in the case of $m = 0$ which yields

$$2sw_2 = -yw_0 + 2yw_0' \quad \text{and} \quad 2sw_{-2} = yw_0 + 2yw_0'.$$

Hence

$$w_{-2}(y) - w_2(y) = \frac{yw_0(y)}{s}, \qquad w_{-2}(y) + w_2(y) = \frac{2yw_0'(y)}{s}.$$

Note that since π is a principal series, $s \neq 0$. In the first case, using the formula for $w_0(y)$ from above gives (2.16). Then (2.17) is obtained similarly using (2.12).

Finally, we now assume $\pi = \pi_{1,0}(s)$. Applying (2.22) in the case of $m = \pm 1$ leads to the system of equations

$$\nu w_1 = \left(\frac{-1-y}{2}\right) w_{-1} + yw_{-1}'(y),$$
$$\nu w_{-1} = \left(\frac{y-1}{2}\right) w_1 + yw_1'(y).$$

We now set $f = w_1 + w_{-1}$ and $g = w_1 - w_{-1}$, so that adding and subtracting these two formulas we find that

$$\begin{aligned}(2\nu+1)f &= yg + 2yf' \\ (2\nu-1)g &= -yf - 2yg'.\end{aligned} \tag{2.23}$$

This simplifies further to

$$f'' - \frac{1}{y}f' - \left(\frac{(2\nu-1)^2-4}{4y^2} + \frac{1}{4}\right)f = 0.$$

Plugging $f(y) = yK(y/2)$ into the above, we find that K satisfies the differential equation

$$0 = K''(y) + \frac{1}{y}K'(y) - \left(\frac{(\nu-\frac{1}{2})^2}{y^2} - 1\right)K(y).$$

Comparing this with (2.9), the formula (2.18) for $f = W_1 + W_{-1}$ follows readily, using the fact that $f(y)$ is odd. (That f is odd is a direct consequence of (2.7).) On the other hand, using (2.13) and (2.23), we see that $g = W_1 - W_{-1}$ satisfies

$$\begin{aligned} yg(y) &= (2\nu + 1)f(y) - 2yf'(y) \\ &= (2\nu + 1)K_{\nu-\frac{1}{2}}(y/2) - 2y(-(y/2)K_{\nu+\frac{1}{2}}(y/2) + (\nu + 1/2)K_{\nu-1/2}(y/2)) \\ &= y^2K_{\nu+\frac{1}{2}}(y/2). \end{aligned}$$

This is valid for $y > 0$ and leads directly to (2.19) since g is an even function. □

Proposition 2.2 *The norms of the vectors from Proposition 2.1 are as follows. The vector $W_k \in \mathcal{W}(\pi^k_{\text{dis}}, \psi)$ satisfies $\langle W_k, W_k\rangle = (k-1)!$. The vectors $W_{\pm\ell} \in \mathcal{W}\pi_{\delta,\epsilon}(\frac{1}{2} + \nu)$ with $\ell = 0, 1, 2$ and $\ell \equiv \delta \pmod 2$ satisfy $\langle W_\ell, W_\ell\rangle = \pi\Gamma(\frac{1+\ell}{2} + \nu)\Gamma(\frac{1+\ell}{2} - \nu)$.*

Proof If π is a discrete series or a unitary principal series then the inner product on $\mathcal{W}(\pi, \psi)$ is given by

$$\langle W, W'\rangle = \int_K \int_{\mathbb{R}^\times} W(a(y)\kappa)\overline{W(a(y)\kappa)}d^\times y\, d\kappa.$$

Thus, using the integral representation $\Gamma(s) = \int_0^\infty y^s e^{-y} d^\times y$ for the Gamma function, we see in the case of the discrete series π^k_{dis} that

$$\langle W_k, W_k\rangle = \int_0^\infty e^{-y}y^{k-1}\, dy = \Gamma(k) = (k-1)!.$$

We write the norms of each of the functions (2.15), (2.16) and (2.19) in terms of (2.10). For $W = W_0$ this is completely straightforward. The case of $W = W_{\pm 2}$ is somewhat more complicated, and so we go through the proof in detail. First, writing $w_m(y) = W_m(a(y))$, note that if we set $f_- = w_{-2} - w_2$ and $f_+ = w_{-2} + w_2$, then f_- is an odd function of y and f_+ is even. Thus

$$\begin{aligned} \langle W_{\pm 2}, W_{\pm 2}\rangle &= \int_{\mathbb{R}^\times} w_{\pm 2}(y)\overline{w_{\pm 2}(y)}d^\times y \\ &= \int_{\mathbb{R}^\times} \left(\frac{f_+(y) \pm f_-(y)}{2}\right)\left(\overline{\frac{f_+(y) \pm f_-(y)}{2}}\right) d^\times y \\ &= \frac{1}{4}\int_{\mathbb{R}^\times} \left(f_+(y)\overline{f_+(y)} + f_-(y)\overline{f_-(y)}\right) d^\times y \\ &= \frac{1}{2}\int_0^\infty \left(f_+(y)\overline{f_+(y)} + f_-(y)\overline{f_-(y)}\right) d^\times y. \end{aligned}$$

One calculates easily using (2.10) that

$$\int_0^\infty f_+(y)\overline{f_+(y)}\,d^\times y = \pi\Gamma(\tfrac{3}{2}+it)\Gamma(\tfrac{3}{2}-it).$$

The calculation of $\int_0^\infty f_-(y)\overline{f_-(y)}\,d^\times y$ is similar (but messier) and gives the same result. Putting this together leads to the claimed result for $W_{\pm 2}$.

The case of $W_{\pm 1}$ is similar. We leave the details to the reader. □

Remark 4 Note that

$$\int_K W_m(a(y)\kappa)\overline{W_n(a(y)\kappa)}\,d\kappa = 0$$

unless $m = n$. In particular this implies that $\langle W_\ell \pm W_{-\ell}, W_\ell \pm W_{-\ell}\rangle = 2\pi\Gamma(\frac{1+\ell}{2} + \nu)\Gamma(\frac{1+\ell}{2} - \nu)$ if $\ell \neq 0$.

3 Computing Trilinear Forms

For $j = 1, 2, 3$, let π_j be irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$. We assume that the product of their central characters is trivial. Thus, without loss of generality, if ω_j is the central character of π_j we may assume that $\omega_j(z(u)) = \mathrm{sgn}(u)^{\delta_j}$ for $\delta_j \in \{0, 1\}$ satisfying

$$\delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod 2. \tag{3.1}$$

As a matter of notation, we will denote an element of π_j by $f^{(j)}$, and similarly elements of $\mathcal{W}(\pi_j, \psi)$ will be denoted by $W^{(j)}$. The calculation of the trilinear form is simplified by using the Whittaker models of π_j for $j = 1, 2$ due to the following result of [17].

Proposition 3.1 (Michel-Venkatesh) *Let π_1, π_2, π_3 be tempered representations of $\mathrm{GL}_2(\mathbb{R})$ with π_3 a principal series. Fix isometries $\pi_1 \to \mathcal{W}(\pi_1, \psi)$ and $\pi_2 \to \mathcal{W}(\pi_2, \overline{\psi})$ for $\psi(x) = e^{ix/2}$. Via these isometries, associating to $f^{(j)} \in \pi_j$ for $j = 1, 2$ vectors $W^{(j)}$ in the Whittaker models, the form $\ell_{\mathrm{RS}} : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ given by*

$$\ell_{\mathrm{RS}}(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \sqrt{4\pi}\int_K \int_{\mathbb{R}^\times} W^{(1)}(a(y)\kappa)W^{(2)}(a(y)\kappa)f^{(3)}(a(y)\kappa)|y|^{-1}\,d^\times y\,d\kappa \tag{3.2}$$

satisfies $|\ell_{\mathrm{RS}}|^2 = I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)})$ where I' is as in (1.1).

Remark 5 The constant $\sqrt{4\pi}$ is an artifact of the fact that the formula given in [17] (in which this constant does not appear) is valid in the particular case that

$\psi(x) = e^{2\pi ix}$. Adjusting to our case of $\psi(x) = e^{ix/2}$ has the effect of multiplying by this constant.

To ease notation we will assume henceforth that $\langle f^{(j)}, f^{(j)} \rangle = 1$. This implies that the map $\pi_j \to \mathcal{W}(\pi_j, \psi)$ given by $f^{(j)} \mapsto W^{(j)}/\langle W^{(j)}, W^{(j)} \rangle^{1/2}$ is an isometry to which we may apply Proposition 3.1.

As remarked in [17], the non-tempered case (including the complementary series) can also be treated with Proposition 3.1 via a polarization which we describe now. In this generality, we associate to $f \in \pi$ a vector $\widetilde{f} \in \widehat{\pi}$ such that up to constant $\widetilde{f} = \overline{M(s)f}$ and $(f, \widetilde{f}) = 1$. In the case of the weight m vector $f = f_{m,s} \in \pi_{\delta,\epsilon}(s)$, this implies that $\widetilde{f} = f_{-m,1-s}$. We denote by $\widetilde{W}^{(j)}$ the image of $\widetilde{f}^{(j)}$ in $\mathcal{W}(\widetilde{\pi}_j, \psi)$ as above.

So, under the assumption that $f^{(j)}$ and $\widetilde{f}^{(j)}$ satisfy $(f^{(j)}, \widetilde{f}^{(j)}) = 1$, we see that the polarized form of Proposition 3.1 gives

$$I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)})\ell_{\mathrm{RS}}(\widetilde{W}^{(1)} \otimes \widetilde{W}^{(2)} \otimes \widetilde{f}^{(3)})}{(W^{(1)}, \widetilde{W}^{(1)})(W^{(2)}, \widetilde{W}^{(2)})}. \tag{3.3}$$

Following our convention for choosing $\widetilde{f}$ from f, the calculation of norms given in Proposition 2.2 gives the correct values for $(W, \widetilde{W})$ even in the case that π is not unitarizable. In the sequel, we will use this polarized form throughout.

Remark 6 For $f = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}$, the trilinear from $\frac{I'(f)}{\langle f, f \rangle}$ is clearly invariant under scaling $f^{(j)}$ by a nonzero constant, hence in defining the particular choice of test vectors in the sequel (or equivalently in Proposition 2.1) the exact choice of scalar is not so important. We refer to a choice such that $\langle f, f \rangle = (f, \widetilde{f}) = 1$ as *normalized*.

For the remainder of this section we adopt the notation $\mathrm{wt}(\pi_j) = k_j$, and we assume that if $\{\ell, m, n\} = \{1, 2, 3\}$ with $k_\ell = \max\{k_1, k_2, k_3\}$ then $k_\ell \geq k_m + k_n$. The condition on the central characters implies that $k_1 + k_2 + k_3$ is even.

3.1 The Case of Three Principal Series

We consider first the situation in which $\pi_j = \pi_{\delta_j,\epsilon_j}(\frac{1}{2} + \nu_j)$ for all $j = 1, 2, 3$ with $\delta_1 + \delta_2 + \delta_3$ even. Attached to such a triple we define $\epsilon \in \{0, 1\}$ be such that $\epsilon \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 \pmod 2$.

Proposition 3.2 *Let π_j be principal series representations with ϵ as above. Then we may arrange that $(\delta_1, \epsilon_1) = (0, \epsilon)$ and $(\delta_j, \epsilon_j) = (\delta, \epsilon')$ for $j = 2, 3$. If $\delta = 0$ there exists a choice of normalized test vectors $f^{(j)} \in \pi_j$ such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \left(\frac{4\pi^4}{\lambda_j}\right)^{\epsilon} \tag{3.4}$$

where λ_j is the eigenvalue of the Laplace-Beltrami operator on π_j for either $j = 2$ or $j = 3$, i.e. $\lambda_j = \frac{1}{4} - \nu_j^2$.

When $\delta = 1$ *there is a choice of normalized test vectors such that* $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$.

Remark 7 Strictly speaking our proof is only valid for parameters ν_j such that certain integrals of the type (2.9) and (2.10) are convergent. To get the more general result one must employ analytic continuation. When the parameters correspond to unitary representations, the proof below is complete.

Proof Suppose first that $\delta = 0$. Let $W^{(1)} = W_0$,

$$W^{(2)} = \begin{cases} W_0 & \text{if } \epsilon = 0 \\ \frac{W_{-2}-W_2}{\sqrt{2}} & \text{if } \epsilon = 1, \end{cases} \qquad \text{and} \qquad f^{(3)} = \begin{cases} f_0 & \text{if } \epsilon = 0 \\ \frac{f_{-2}+f_2}{\sqrt{2}} & \text{if } \epsilon = 1. \end{cases}$$

Note that with these choices the restriction of $W^{(1)}W^{(2)}f^{(3)}$ to A is an even function. This is because for *any* $f \in \pi_3$ the restriction to A satisfies $f(a(y)) = c\,\mathrm{sgn}(y)^\epsilon |y|^{\frac{1}{2}+\nu_3}$ for some constant c.

By Proposition 2.2 and Remark 4, $\langle W^{(j)}, W^{(j)}\rangle = \pi\Gamma(\frac{1}{2}+\nu_j)\Gamma(\frac{1}{2}-\nu_j)$ for $j = 1, 2$ and any choice of ϵ, ϵ'. Also, note that $\langle f^{(3)}, f^{(3)}\rangle = 1$ in any case.

We claim that

$$\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) = \frac{2^{\epsilon+2\nu_3}\sqrt{\pi}}{(\frac{1}{2}+\nu_3)^\epsilon} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{1+2\epsilon}{2} + \frac{\gamma_1\nu_1+\gamma_2\nu_2+\nu_3}{2})}{\Gamma(\frac{1+2\epsilon}{2}+\nu_3)}. \tag{3.5}$$

We verify this in the case of $\epsilon = 1$ by computing

$$\begin{aligned} \ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) &= \sqrt{\pi}\int_{\mathbb{R}^\times} (W_0W_{-2}f_2 - W_0W_2f_{-2})(a(y))|y|^{-1}d^\times y \\ &= \sqrt{\pi}\int_{\mathbb{R}^\times} W_0(W_{-2}-W_2)(a(y))\,\mathrm{sgn}(y)|y|^{-\frac{1}{2}+\nu_3}d^\times y \\ &= 2\sqrt{\pi}\int_0^\infty y^{\frac{3}{2}+\nu_3}K_{\nu_1}(y/2)K_{\nu_2}(y/2)\,d^\times y \\ &= 2^{1+2\nu_3}\sqrt{\pi}\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{3}{4}+\frac{\gamma_1\nu_1+\gamma_2\nu_2+\nu_3}{2})}{\Gamma(\frac{3}{2}+\nu_3)} \\ &= \frac{2^{1+2\nu_3}\sqrt{\pi}}{\frac{1}{2}+\nu_3}\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{3}{4}+\frac{\gamma_1\nu_1+\gamma_2\nu_2+\nu_3}{2})}{\Gamma(\frac{1}{2}+\nu_3)}. \end{aligned}$$

The other case is similar.

Computing $\ell_{\mathrm{RS}}(\widetilde{W}^{(1)} \otimes \widetilde{W}^{(2)} \otimes \widetilde{f}^{(3)})$ as above has the net effect of giving exactly the same result except with ν_3 replaced by $-\nu_3$. Thus, combining (3.5) with

Proposition 2.2, we see that (3.3) now gives

$$I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \left(\frac{4}{\lambda_3}\right)^{\epsilon} \frac{\prod_{\gamma_j=\pm} \Gamma(\frac{1+2\epsilon}{4} + \frac{\gamma_1\nu_1+\gamma_2\nu_2+\gamma_3\nu_3}{2})}{\pi \prod_{j=1}^{3} \Gamma(\frac{1}{2}+\nu_j)\Gamma(\frac{1}{2}-\nu_j)}.$$

Finally, we divide by the normalizing factor in Table 2 for Π^1 and thus obtain the desired result.

Now suppose that $\delta = 1$. We choose $f^{(1)} = f_{0,\frac{1}{2}+\nu_1}$,

$$f^{(2)} = \frac{f_{1,\frac{1}{2}+\nu_2} - (-1)^{\epsilon} f_{-1,\frac{1}{2}+\nu_2}}{\sqrt{2}}, \quad \text{and} \quad f^{(3)} = \frac{f_{1,\frac{1}{2}+\nu_3} - f_{-1,\frac{1}{2}+\nu_3}}{\sqrt{2}}.$$

Thus $W^{(1)} = W_0$ and $W^{(2)} = \frac{W_1-(-1)^{\epsilon}W_{-1}}{\sqrt{2}}$. Note again that having made these choices the product $W^{(1)}W^{(2)}f^{(3)}$ has the property that its restriction to A is an even function.

By a computation very similar to that above, we find

$$\begin{aligned}
\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) &= \sqrt{\pi} \int_{\mathbb{R}^{\times}} \big(W_0 W_1 f_{-1} + (-1)^{\epsilon} W_{-1} f_1\big)(a(y))|y|^{-1}\, d^{\times}y \\
&= \sqrt{\pi} \int_{\mathbb{R}^{\times}} \big(W_0(W_1 + (-1)^{\epsilon} W_{-1})\big)(a(y))\, \mathrm{sgn}(y)|y|^{\nu_3-\frac{1}{2}}\, d^{\times}y \\
&= 2\sqrt{\pi} \int_0^{\infty} y^{1+\nu_3} K_{\nu_1}(y/2) K_{\nu_2-\frac{1}{2}+\epsilon}(y/2)\, d^{\times}y \\
&= \sqrt{\pi}\, 2^{2\nu_3} \frac{\prod_{\gamma\in\pm} \Gamma(\frac{1+2\epsilon}{2} + \frac{\gamma\nu_1+\nu_2+\nu_3}{2})\Gamma(\frac{3-2\epsilon}{2} + \frac{\gamma\nu_1-\nu_2+\nu_3}{2})}{\Gamma(1+\nu_3)}.
\end{aligned}$$

Multiplying by the appropriate polarizing factor, and dividing by the appropriate norms as before, we find that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{\prod_{\gamma_j\in\pm} \Gamma(\frac{1+2\epsilon}{2} + \frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})\Gamma(\frac{3-2\epsilon}{2} + \frac{\gamma_1\nu_1+\gamma_2(\nu_2-\nu_3)}{2})}{\pi \prod_{\gamma\in\pm} \Gamma(\frac{1}{2}+\gamma\nu_1)\Gamma(1+\gamma\nu_2)\Gamma(1+\gamma\nu_3)}.$$

Since this agrees with the corresponding factor in Table 2 it follows that the normalized trilinear form satisfies $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$ as claimed. □

Remark 8 As discussed in the introduction, the case of three weight zero principal series representations was treated by Watson in [25] but only in the case $\delta = 0$. We remark that the way in which he uses ϵ agrees with our notation. He did not give a test vector in the case that $\epsilon = 1$, but showed that if one takes $f^{(j)}$ to be the weight zero vector for each of $j = 1, 2, 3$, the resulting trilinear from will be zero. This is immediately evident from our method above, as the resulting function $\big(W^{(1)}W^{(2)}f^{(3)}\big)(a(y))$ will be an odd function of y.

3.2 The Case of Two Principal Series and a Discrete Series

Note that this case was worked out in [22] when k is even and both principal series are weight zero. We extend the result here to arbitrary k and allow that the principal series be odd.

Proposition 3.3 *Suppose that* $\pi_1 = \pi_{\mathrm{dis}}^k$ *and* $\pi_j = \pi_{\delta_j,\epsilon_j}(\frac{1}{2}+\nu_j)$ *such that* $k+\delta_2+\delta_3$ *is even. Then if* $\delta = \delta_2 + \delta_3 \le 1$ *there exists a choice of normalized test vectors* $f^{(j)} \in \pi_j$ *such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{(2\pi)^{k-1}}{\pi^{\delta-1}} \frac{\Gamma(\frac{1+\delta_m}{2} + \nu_m)\Gamma(\frac{1+\delta_m}{2} - \nu_m)}{\Gamma(\frac{k+1}{2} + \nu_m)\Gamma(\frac{k+1}{2} - \nu_m)}, \tag{3.6}$$

where $\{\ell, m\} = \{2, 3\}$ *satisfies* $\delta_\ell = 0$.

Otherwise, (if $\delta_2 = \delta_3 = 1$*),*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 2\lambda_{k,j}(2\pi)^{k-2} \frac{\Gamma(1 + \nu_j)\Gamma(1 - \nu_j)}{\Gamma(\frac{k}{2} + 1 + \nu_j)\Gamma(\frac{k}{2} + 1 - \nu_j)}, \tag{3.7}$$

for $j = 2$ *or* $j = 3$ *and* $\lambda_{k,j} = (\frac{k}{2})^2 - \nu_j^2$.

Proof We arrange the representations and take test vectors such that

$$\begin{aligned} W^{(1)} &= W_k \in \mathcal{W}(\pi_{\mathrm{dis}}^k, \psi), \\ W^{(2)} &= W_{-\delta_2}^- \in \mathcal{W}(\pi_{\delta_2,\epsilon_2}(\tfrac{1}{2} + \nu_2), \overline{\psi}), \\ f^{(3)} &= f_{-k+\delta_2,\frac{1}{2}+\nu_3} \in \pi_{\delta_3,\epsilon_3}(\tfrac{1}{2} + \nu_3). \end{aligned}$$

Since $W_k(a(y))$ is supported on $y > 0$, we see that

$$\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(1)}) = 2\sqrt{\pi} \int_0^\infty y^{\frac{k-1}{2}+\nu_3} e^{-y/2} W_{-\delta_2}^-(a(y))\, d^\times y$$

In the case that $\delta_2 = 0$, one follows the same procedure as in the proof of Proposition 3.2 to arrive at

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi}{(k-1)!} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{k}{2} + \gamma_2\nu_2 + \gamma_3\nu_3)}{\Gamma(\frac{k}{2} + \nu_3)\Gamma(\frac{k}{2} - \nu_3)\Gamma(\frac{1}{2} + \nu_2)\Gamma(\frac{1}{2} - \nu_2)},$$

from which (3.7) follows as before.

We now consider the case $\delta_2 = 1$, for which

$$W_{-1}^-(a(y)) = \frac{|y|}{2}\Big(K_{\nu_2-\frac{1}{2}}(|y|/2) + \mathrm{sgn}(y)K_{\nu_2+\frac{1}{2}}(|y|/2)\Big).$$

Therefore,

$$\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(1)}) = \sqrt{\pi} \int_0^\infty y^{\frac{k+1}{2}+\nu_3} e^{-y/2} \Big(K_{\nu_2 - \frac{1}{2}}(|y|/2) + K_{\nu_2+\frac{1}{2}}(|y|/2) \Big)\, d^\times y$$
$$= 2\pi (\frac{k}{2} + \nu_3) \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{k}{2} + \gamma_2\nu_2 + \gamma_3\nu_3)}{\Gamma(\frac{k}{2} + 1 + \nu_3)},$$

where in the final step we have used (2.11) and the functional equation $s\Gamma(s) = \Gamma(s+1)$ in order to simplify. Again polarizing this and dividing by the appropriate norms, this implies that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi\lambda}{(k-1)!} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{k}{2} + \gamma_2\nu_2 + \gamma_3\nu_3)}{\prod_{\gamma \in \pm} \Gamma(\frac{k}{2} + 1 + \gamma\nu_3)\Gamma(\frac{1}{2} + \gamma\nu_2)},$$

where $\lambda = (\frac{k}{2})^2 - \nu_3^2$. Dividing this by the appropriate normalizing factor from Table 2 and simplifying gives (3.7) in the case of $j = 3$. Switching the roles of π_2 and π_3 gives the other case. □

3.3 The Case of One Principal Series and Two Discrete Series

We now assume that $\pi_j = \pi_{\mathrm{dis}}^{k_j}$ for $j = 1, 2$ and $\pi_3 = \pi_{\delta,\epsilon}(\frac{1}{2} + \nu)$.

Proposition 3.4 *Let $\pi_j = \pi_{\mathrm{dis}}^{k_j}$ for $j = 1, 2$ with $k_1 \geq k_2$. Let π_3 be a principal series representation of weight zero if $k_1 + k_2$ is even and of weight one otherwise. Then there exists a choice of normalized test vectors $f^{(j)} \in \pi_j$ such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{(2\pi)^{k_1-k_2}}{\pi^\delta} \frac{\Gamma(\frac{1+\delta}{2} + \nu)\Gamma(\frac{1+\delta}{2} - \nu)}{\Gamma(\frac{k_1-k_2+1}{2} + \nu)\Gamma(\frac{k_1-k_2+1}{2} - \nu)}. \tag{3.8}$$

where $\lambda = \frac{1}{2} - \nu^2$ is the eigenvalue of Δ on π_3.

Proof Let $\delta = \mathrm{wt}(\pi_3)$. We take $f^{(1)}$ to be the weight k_1 vector, $f^{(2)}$ the weight $-k_2$ vector and $f^{(3)}$ the weight $k_2 - k_1$ vector. Then $W^{(1)} = W_{k_j}^{+}$ and $W^{(2)} = W_{-k_2}^{-}$. Since

$$\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) = \sqrt{4\pi} \int_0^\infty y^{\frac{k_1+k_2}{2} - \frac{1}{2} + \nu} e^{-y}\, d^\times y = \sqrt{4\pi}\,\Gamma\left(\tfrac{k_1+k_2-1}{2} + \nu\right),$$

using (3.3) we find that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi\Gamma(\frac{k_1+k_2-1}{2} + \nu)\Gamma(\frac{k_1+k_2-1}{2} - \nu)}{(k_1-1)!(k_2-1)!}.$$

We again divide by the normalizing factor for Π^3 from Table 2 to obtain the desired result. □

3.4 The Case of Three Discrete Series

Let us assume that $\pi_j = \pi_{\text{dis}}^{k_j}$ for $j = 1, 2$ and $\pi_3 = \pi_{\delta,0}(\frac{k_3}{2})$ where $\delta \in \{0, 1\}$ has the same parity as k_3, so that $\pi_{\text{dis}}^{k_3} \subset \pi_3$. Note then that $\widetilde{\pi_3} = \pi_{\delta,0}(1 - \frac{k_3}{2})$ which has $\pi_{\text{dis}}^{k_3}$ as a quotient.

In this situation, the form ℓ_{RS} descends to a trilinear form on $\pi_{\text{dis}}^{k_1} \otimes \pi_{\text{dis}}^{k_2} \otimes \pi_{\text{dis}}^{k_3}$, and we will take as a hypothesis that the polarization of the form $|\ell_{\text{RS}}|^2$ in fact gives the correct trilinear form on $\pi_{\text{dis}}^{k_1} \otimes \pi_{\text{dis}}^{k_2} \otimes \pi_{\text{dis}}^{k_3}$. Note that this is unconditionally true if $k_3 = 1$, and in the special case that $k_1 = k_2 + k_3$ the answer that we obtain by this method is correct.

Proposition 3.5 *Let $\pi_j = \pi_{\text{dis}}^{k_j}$ for $j = 1, 2, 3$ and $k_1 - (k_2 + k_3) = 2m \geq 0$. There exists a choice of normalized test vectors $f^{(j)} \in \pi_j$ such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{2(2\pi)^{2m}}{\binom{k_3+m-1}{k_3-1}}. \tag{3.9}$$

Proof As test vectors, we choose $f^{(1)}$ to be the weight k_j vector, $f^{(2)}$ to be the weight $-k_2$ vector, and $f^{(3)} = f_{k_2-k_1}$. Then the computation of ℓ_{RS} proceeds exactly as in the previous section but with $\nu = \frac{k_3-1}{2}$. This immediately implies that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi\Gamma(\frac{k_1+k_2+k_3}{2} - 1)\Gamma(\frac{k_1+k_2-k_3}{2})}{(k_1 - 1)!(k_2 - 1)!}$$

in the case at hand. Note, however, that the normalizing factor of the previous section does not agree with that here unless $k_3 = 1$. That is to say that the triple product local L-factor $L(\frac{1}{2}, \Pi)$ for $\Pi = \pi_{\text{dis}}^{k_1} \otimes \pi_{\text{dis}}^{k_2} \otimes \pi_{\delta,0}(\frac{s}{2})$ does not specialize to that for $\Pi = \pi_{\text{dis}}^{k_1} \otimes \pi_{\text{dis}}^{k_2} \otimes \pi_{\text{dis}}^{k_3}$ as $s \to k_3$. Moreover, the adjoint L-factor $L(s, \pi_{\delta,0}(\frac{k_3}{2}), \text{Ad}) = L(s, \pi_{\delta,0}(1 - \frac{k_3}{2}), \text{Ad})$ has a pole at $s = \frac{1}{2}$ if $k_3 \geq 2$. In any event, dividing by the correct normalizing factor gives the result. □

Remark 9 One may ask why the above proof doesn't also apply in the case that $k_1 < k_2 + k_3$. By Prasad [20], the form must be zero in this case, although at first glance it may not appear to be so. However, it is easy to see from Proposition 2.1 that for weight m_j vectors $f^{(j)}$ if the form $\ell_{\text{RS}}(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) \neq 0$ then the weights m_1 and m_2 must have opposite parity. Thus, in the special case that $m_1 = k_1$ and $m_2 = -k_2$, the vector $f^{(3)}$ has weight $-k_3 + m$ where $m = k_3 + k_2 - k_1$ which corresponds to a vector in $\pi_{\text{dis}}^{k_3}$ if and only if $k_1 \geq k_2 + k_3$.

3.5 Proof of Theorem 1

The calculations of the previous sections cover all possible cases π_1, π_2, π_3 satisfying the hypotheses of Theorem 1, and in each case the corresponding test vectors $f^{(j)}$ are shown to satisfy $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) \neq 0$.

In particular, if all three representations are principal series, then $k_1 = k_2 = \delta$ and $k_3 = 0$. In the case that $\delta = 0$ if we assume moreover that $\epsilon = 0$, from Proposition 3.2 one sees by (3.4) that $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$. In the case $\delta = 1$ Proposition 3.2 says immediately that $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$.

If exactly one of the representations is a discrete series then either $k_1 = 1$ and the other representations are a weight 1 and a weight 0 principal series, or $k_1 = 2$ and both of the other representations are weight 1 principal series. In the first case, the result follows from Proposition 3.3 by applying (3.6) in the case that $k = 1, \delta_2 = 0$ and $\delta_3 = 1$. In the latter case one applies (3.7) with $k = 2$, from which we obtain

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 2\lambda_{2,j} \frac{\Gamma(1+\nu_j)\Gamma(1-\nu_j)}{\Gamma(2+\nu_j)\Gamma(2-\nu_j)} = 2,$$

since $\Gamma(2+\nu_j)\Gamma(2-\nu_j) = (1-\nu_j^2)\Gamma(1+\nu_j)\Gamma(1-\nu_j) = \lambda_{2,j}\Gamma(1+\nu_j)\Gamma(1-\nu_j)$.

If two of the representations are discrete series, then we may assume that π_3 is a principal series of weight $k_3 = \delta \in \{0, 1\}$ and $k_1 = k_2 + \delta > 1$. This case corresponds to Proposition 3.4. Specializing (3.8) gives $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$ if $\delta = 0$, and $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 2$ if $\delta = 1$, as claimed.

Finally, the case that all three representations are discrete series is treated in Proposition 3.5. The assumption that $k_1 = k_2 + k_3$ means that $m = 0$ in which case the right hand side of (3.9) is obviously 2. □

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Appendix: Normalizing *L*-Factors

The goal of this appendix is to record the normalizing L-factors for the triple product L-function appearing in (1.2). These factors are determined by applying the local Langlands correspondence relating finite dimensional semisimple representations of the Weil group $W_{\mathbb{R}}$ to admissible representations of $\mathrm{GL}_2(\mathbb{R})$ as detailed in [15]. The local factors will be described in terms of

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

We recall the following elementary facts:

$$\overline{\Gamma(s)} = \Gamma(\bar{s}), \quad \Gamma_{\mathbb{R}}(1) = 1, \quad \Gamma_{\mathbb{R}}(2) = \frac{1}{\pi}, \quad \Gamma_{\mathbb{C}}(m) = \frac{(m-1)!}{2^{m-1}\pi^m}.$$

Local Langlands Parameters for $\mathbf{GL_2(\mathbb{R})}$

We recall briefly the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$. (See [15] for complete details.) Let $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$ be the Weil group. For $\delta \in \{0, 1\}$ and $t \in \mathbb{C}$, we have the 1-dimensional representation of $W_{\mathbb{R}}$ given by

$$\rho_1(\delta, t) : \begin{array}{c} z \mapsto |z|^t \\ j \mapsto (-1)^\delta. \end{array}$$

Moreover, if $m \in \mathbb{Z}$ and $t \in \mathbb{C}$ we have the 2-dimensional representation

$$\rho_2(m, t) : \begin{array}{c} re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{im\theta} & 0 \\ 0 & r^{2t}e^{-im\theta} \end{pmatrix} \\ j \mapsto \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix} \end{array},$$

which is easily checked to be irreducible except when $m = 0$. The following is a simple exercise.

Lemma A.1 *Every semisimple finite-dimensional representation of $W_{\mathbb{R}}$ is a direct sum of irreducibles of the type ρ_1 and ρ_2 as defined above. Under the operations of direct sum and tensor product, the following is a complete set of relations.*

$$\begin{aligned} \rho_2(m, t) &\simeq \rho_2(-m, t) \\ \rho_2(0, t) &\simeq \rho_1(0, t) \oplus \rho_1(1, t) \\ \rho_1(\delta_1, t_1) \otimes \rho_1(\delta_2, t_2) &\simeq \rho_1(\delta, t_1 + t_2) \qquad \delta \equiv \delta_1 + \delta_2 \pmod 2) \\ \rho_1(\delta, t_1) \otimes \rho_2(m, t_2) &\simeq \rho_2(m, t_1 + t_2) \\ \rho_2(m_1, t_1) \otimes \rho_2(m_2, t_2) &\simeq \rho_2(m_1 + m_2, t_1 + t_2) \oplus \rho_2(m_1 - m_2, t_1 + t_2). \end{aligned}$$

Moreover, if $\widetilde{\rho}$ denotes the contragradient of ρ then

$$\widetilde{\rho_1(\delta, t)} \simeq \rho_1(\delta, -t), \quad \textit{and} \quad \widetilde{\rho_2(m, t)} \simeq \rho_2(m, -t).$$

Given an irreducible admissible representation π of $\mathrm{GL}_2(\mathbb{R})$ we associate to it a representation $\rho(\pi)$ of $W_{\mathbb{R}}$. For example, if $\mathcal{B}(\chi_1, \chi_2) = \mathcal{B}(\mathrm{sgn}^{\epsilon_1}|\cdot|^{s_1}, \mathrm{sgn}^{\epsilon_2}|\cdot|^{s_2})$

Table 1 Representations of $W_{\mathbb{R}}$ attached to admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$

π	$\rho(\pi)$	$\mathrm{Ad}(\rho(\pi))$
$\pi_{\delta,\epsilon}(\frac{1}{2}+\nu)$	$\rho_1(\epsilon,\nu)\oplus\rho_1(\overline{\delta+\epsilon},-\nu)$	$\rho_1(0,0)\oplus\rho_1(\delta,2\nu)\oplus\rho_1(\delta,-2\nu)$
π_{dis}^k	$\rho_2(k-1,0)$	$\rho_1(1,0)\oplus\rho_2(2k-2,0)$

is irreducible the corresponding representation of $W_{\mathbb{R}}$ is $\rho_1(\epsilon_1,s_1)\oplus\rho_1(\epsilon_2,s_2)$. We record how this correspondence works in Table 1 for representations with central character sgn^{δ}. (We let $\overline{\delta+\epsilon}\in\{0,1\}$ be the reduction of $\epsilon+\delta$ modulo 2.) Note that the third column of the table is calculated using Lemma A.1 and the identity

$$\mathrm{Ad}(\rho)\simeq\rho\otimes\widetilde{\rho}\ominus\rho_1(0,0).$$

Triple Product and Adjoint L-Factors

We associate to each of $\rho_1(\delta,t)$ and $\rho_2(m,t)$ the L-functions

$$L(s,\rho_1(\delta,t))=\Gamma_{\mathbb{R}}(s+\delta+t),\qquad L(s,\rho_2(m,t))=\Gamma_{\mathbb{C}}(s+\tfrac{m}{2}+t).\tag{A.1}$$

More generally, given $\rho\simeq\rho_1\oplus\rho_2\oplus\cdots\oplus\rho_r$ a (semisimple) representation of $W_{\mathbb{R}}$ with ρ_j irreducible we define

$$L(s,\rho)=\prod_{j=1}^{r}L(s,\rho_j).$$

Using this definition it follows, setting $L(s,\pi,\mathrm{Ad})=L(s,\mathrm{Ad}(\rho(\pi)))$ and combining (A.1) with Table 1, that

$$L(1,\pi,\mathrm{Ad}))=\begin{cases}\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(k) & \text{if } \pi=\pi_{\mathrm{dis}}^k,\\ \dfrac{\Gamma(\frac{1+\delta}{2}+\nu)\Gamma(\frac{1+\delta}{2}-\nu)}{\pi^{1+\delta}} & \text{if } \pi=\pi_{\delta,\epsilon}(\frac{1}{2}+\nu).\end{cases}\tag{A.2}$$

Recall that we are considering admissible representations π_1,π_2,π_3 of $\mathrm{GL}_2(\mathbb{R})$ such that $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$ has trivial central character. This means that we may assume without loss of generality that the central character of each π_j is of the form sgn^{δ_j} with $\delta_1+\delta_2+\delta_3\equiv 0\pmod 2$.

Proposition A.2 *Consider* $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$ *a triple product of admissible* $\mathrm{GL}_2(\mathbb{R})$ *representations. Let*

$$L(s,\Pi)=L(s,\rho(\pi_1)\otimes\rho(\pi_2)\otimes\rho(\pi_3)),$$
$$L(s,\Pi,\mathrm{Ad})=L(s,\mathrm{Ad}(\rho(\pi_1))\otimes\mathrm{Ad}(\rho(\pi_2)\otimes\mathrm{Ad}(\rho(\pi_3)).$$

Table 2 Normalizing factors for triple product L-function at a real place

Π	$\frac{\Gamma_{\mathbb{R}}(2)^2 L(\frac{1}{2},\Pi)}{L(1,\Pi,\mathrm{Ad})}$
Π^1	$\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{1+2\epsilon}{4}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})\Gamma(\frac{1+2\epsilon+2\delta(1-2\epsilon)}{4}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})}{\pi^{1+4\epsilon(1-\delta)}\prod_{j=1}^{3}\Gamma(\frac{1+\delta_j}{2}+\nu_j)\Gamma(\frac{1+\delta_j}{2}-\nu_j)}$
Π^2	$\frac{(2\pi)^{3-k}\pi^{\delta_2+\delta_3-2}}{(k-1)!}\frac{\prod_{\gamma_j=\pm1}\Gamma(\frac{k}{2}+\gamma_2\nu_2+\gamma_3\nu_3)}{\prod_{j=1}^{2}\Gamma(\frac{1+\delta_j}{2}+\nu_j)\Gamma(\frac{1+\delta_j}{2}-\nu_j)}$
Π^3	$\frac{2\pi^{\delta}}{(2\pi)^{k_1-k_2-1}}\frac{\prod_{\gamma\in\pm}\Gamma(\frac{k_1+k_2-1}{2}+\gamma\nu)\Gamma(\frac{k_1-k_2+1}{2}+\gamma\nu)}{(k_1-1)!(k_2-1)!\Gamma(\frac{1+\delta}{2}+\nu)\Gamma(\frac{1+\delta}{2}-\nu)}$
Π^4	$\frac{\Gamma(\frac{k_1+k_2+k_3}{2}-1)\Gamma(\frac{k_1+k_2-k_3}{2})\Gamma(\frac{k_1-k_2+k_3}{2})\Gamma(\frac{k_1-k_2-k_3}{2}+1)}{(2\pi)^{k_1-k_2-k_3-1}(k_1-1)!(k_2-1)!(k_3-1)!}$

The normalizing factors relating I to I' in (1.2) for

$$\Pi^1 = \pi_{0,\epsilon}(\tfrac{1}{2}+\nu_1)\otimes\pi_{\delta,\epsilon'}(\tfrac{1}{2}+\nu_2)\otimes\pi_{\delta,\epsilon'}(\tfrac{1}{2}+\nu_3),$$

$$\Pi^2 = \pi_{\mathrm{dis}}^{k}\otimes\pi_{\delta_2,\epsilon_2}(\tfrac{1}{2}+\nu_2)\otimes\pi_{\delta_3,\epsilon_3}(\tfrac{1}{2}+\nu_3),$$

$$\Pi^3 = \pi_{\mathrm{dis}}^{k_1}\otimes\pi_{\mathrm{dis}}^{k_2}\otimes\pi_{\delta,\epsilon}(\tfrac{1}{2}+\nu) \qquad \textit{(with } k_1\geq k_2+\delta\textit{)},$$

$$\Pi^4 = \pi_{\mathrm{dis}}^{k_1}\otimes\pi_{\mathrm{dis}}^{k_2}\otimes\pi_{\mathrm{dis}}^{k_3} \qquad \textit{(with } k_1\geq k_2+k_3\textit{)}$$

are given by Table 2.

Proof A simple exercise in applying Lemma A.1 gives the following.

$$\rho(\Pi^1) = \Big(\bigoplus_{\gamma_j\in\pm}\rho_1(\epsilon,\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3))\Big)\oplus\Big(\bigoplus_{\gamma_j\in\pm}\rho_1(\overline{\epsilon+\delta},\gamma_1\nu_1+\gamma_2(\nu_2-\nu_3))\Big)$$

$$\rho(\Pi^2) = \bigoplus_{\gamma_j\in\pm}\rho_2(k-1,\gamma_2\nu_2+\gamma_3\nu_3)$$

$$\rho(\Pi^3) = \rho_2(k_1+k_2-2,\nu)\oplus\rho_2(k_1+k_2-2,\nu)\oplus\rho_2(k_1-k_2,-\nu)\oplus\rho_2(k_1-k_2,-\nu)$$

$$\rho(\Pi^4) = \rho_2(k_1+k_2+k_3-3,0)\oplus\rho_2(k_1+k_2-k_3-1,0)$$
$$\oplus\,\rho_2(k_1-k_2+k_3-1,0)\oplus\rho_2(k_1-k_2-k_3+1,0)$$

Combining each of these with the appropriate factors for $L(1,\pi,\mathrm{Ad})$ from (A.2) together with $\Gamma_{\mathbb{R}}(2)^2$ gives the result. □

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An Introduction to the Theory of Harmonic Maass Forms

Claudia Alfes-Neumann

Abstract In this note we give a short introduction to the theory of harmonic Maass forms. We start by introducing modular forms and Maass forms and then present the notion of (vector valued) harmonic Maass forms as developed by Bruinier and Funke in [4]. We end by giving two recent applications of this theory.

1 Modular Forms

In this section we introduce modular forms. For a thorough introduction to the topic see for example [9, 10, 12, 13].

By $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ we denote the complex upper half-plane. The special linear group $\mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : ad - bc = 1\}$ acts on $\mathbb{H}$ by fractional linear transformations

$$z \mapsto Mz = \frac{az+b}{cz+d}, \quad \text{where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We note that the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices T and S, where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Throughout, we let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$ and we write $q := e^{2\pi i z}$.

C. Alfes-Neumann (✉)
Fakultät für Eleklrotechnik, Informatik und Mathematik, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn
e-mail: alfes@math.uni-paderborn.de

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Definition 1.1 Let $k \in \mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$, if:

(1) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
(2) f is holomorphic at the cusp ∞.

Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we find $f(z+1) = f(z)$ and therefore $f(z)$ has a Fourier expansion. By condition (2) it is of the form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

Remark 1.2 The most common generalizations of this definition are the following:

(1) We can also take $k \in \frac{1}{2}\mathbb{Z}$. Note that we have to change the transformation law in condition (1) of Definition 1.1 (see for example [12]).
(2) We can replace $\mathrm{SL}_2(\mathbb{Z})$ by other (congruence) groups. Most frequently, we will replace $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\},$$

where $N \geq 1$ is an integer. Note that we have to consider more than one cusp in condition (2) of Definition 1.1 in this case.
(3) We can require a different transformation behaviour in condition (1). For example, we can require $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \chi(d) f(z)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where χ is a Dirichlet character.
(4) We can change condition (2) in Definition 1.1.

- If we require that f vanishes at the cusp ∞, we obtain a *cusp form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier expansion at ∞ is of the form

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n.$$

- If we allow f to have poles of finite order at the cusp ∞, we obtain a *weakly holomorphic modular form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier expansion at ∞ is of the form

$$f(z) = \sum_{n=m}^{\infty} a(n) q^n,$$

for some $m \in \mathbb{Z}$.

We denote the space of modular forms of weight k for $\Gamma_0(N)$ by $M_k(N)$, the space of cusp forms of weight k for $\Gamma_0(N)$ by $S_k(N)$ and the space of weakly holomorphic modular forms of weight k for $\Gamma_0(N)$ by $M_k^!(N)$.

1.1 Modular Forms and Number Theoretic Functions

Number theoretic functions often occur as the Fourier coefficients of modular forms. In many cases this knowledge leads to a better understanding of the corresponding number theoretic function. In particular, this is one reason why the study of modular forms is crucial in many areas of mathematics.

For example, one can consider as number theoretic functions $a(n)$ for $n \in \mathbb{N}$:

1. the function $r_2(n) = \#\{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$ which counts the number of ways n can be written as a sum of two squares;
2. the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ for $k \in \mathbb{N}$;
3. the partition function $p(n)$ which counts the number of ways n can be written as a sum of integers less or equal to n.

We can then consider the formal generating series

$$f(q) = \sum_{n=1}^{\infty} a(n)q^n.$$

Often one is "lucky" and this generating series is a modular form when viewed as a function in z, where $q = e^{2\pi iz}$.

Since $M_k(N)$ and $S_k(N)$ are finite dimensional vector spaces, it is a finite computation to verify a given identity between the Fourier coefficients of different modular forms. Often this leads to a "nice" formula for $a(n)$.

Coming back to our examples we obtain:

1. The function $\Theta^2(z) = \sum_{n=0}^{\infty} r_2(n)q^n$ is a modular form of weight 1. Comparing the coefficients of $\Theta^2(z)$ with the Fourier coefficients of another modular form in the space, we obtain the formula
$$r_2(n) = 4 \sum_{\substack{d|n \\ d>0 \text{ odd}}} (-1)^{\frac{d-1}{2}}.$$
This reproves an old theorem of Fermat, namely that every prime $p \equiv 1 \pmod 4$ is the sum of two squares.
2. The Eisenstein series
$$E_{2k}(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz + d)^{2k}} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n \qquad (1.1)$$

is a modular form of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, when $k \geq 2$. Here, B_{2k} denotes the $2k$-th Bernoulli number. Using that $E_4(z)^2 = E_8(z)$ we obtain the following identity between divisor sums

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120}.$$

3. Let $\eta(z)$ be the Dedekind η-function. We have that

$$q^{-1/24} \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\eta(z)}$$

is a weakly holomorphic modular form of weight $-1/2$. Although, this space is not finite dimensional Bruinier and Ono [7] were able to show that one can exploit this fact to obtain an algebraic formula for $p(n)$ (see Sect. 4.1).

A second reason for the popularity of modular forms is that they appear in a variety of subdisciplines of mathematics and physics such as Lie theory, combinatorics, string theory, algebraic geometry and others. See for example Zagier's part of [9] or [15].

2 Maass Forms

In 1949 Hans Maass defined so-called Maass forms, generalizations of modular forms that are not required to be holomorphic on $\mathbb{H}$ any more.

Definition 2.1 Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a Fuchsian group of the first kind (i.e. discrete with finite covolume in $\mathrm{SL}_2(\mathbb{Z})$). A function $f : \mathbb{H} \to \mathbb{C}$ is called a *Maass form on* Γ *with eigenvalue* $\lambda = r(1-r) \in \mathbb{C}$, if:

(1) $f(Mz) = f(z)$ for all $M \in \Gamma$.
(2) $\Delta_0 f = r(1-r)f$, where $\Delta_0 = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.
(3) $\int_{\mathcal{F}(\Gamma \backslash \mathbb{H})} |f(z)|^2 \frac{dxdy}{y^2} < \infty$, where $\mathcal{F}(\Gamma \setminus \mathbb{H})$ denotes a fundamental domain for the action of Γ on $\mathbb{H}$.

Remark 2.2 If we additionally require that the 0-th Fourier coefficient $\int_0^1 f(x+iy)dx$ of a Maass form f vanishes (at all cusps), we obtain a *Maass cusp form on* Γ *with eigenvalue* $\lambda = r(1-r) \in \mathbb{C}$.

Note that for arbitrary Γ it is not even clear that Maass cusp forms exist. Selberg was able to show that they do exist when Γ is a congruence subgroup of level $N \geq 1$.

We summarize some more famous results on Maass forms (see [11]):

1. It is known that the eigenvalue $\lambda = r(1-r)$ either satisfies $\Re(r) = \frac{1}{2}$ or $\frac{1}{2} \leq r \leq 1$, which implies $\lambda \geq \frac{1}{4}$, or $\lambda < \frac{1}{4}$ respectively. The latter eigenvalues are called exceptional and there can only be finitely many of them. A famous conjecture of Selberg predicts that actually all eigenvalues are $\geq 1/4$ when Γ is a congruence subgroup.
2. The spectral decomposition with respect to Δ of the Hilbert space of square-integrable functions can be described via Maass forms.

3 Harmonic Maass Forms

In their work *On Two Geometric Theta Lifts* [4] Bruinier and Funke came up with the following space of automorphic forms in a sense combining the notion of modular forms and Maass forms.

Definition 3.1 Let $k \in \mathbb{Z}$. A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a *harmonic (weak) Maass form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$, if:

(1) $f(Mz) = (cz+d)^k f(z)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
(2) $\Delta_k f = 0$, where $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is the hyperbolic Laplace operator of weight k.
(3) $f(z) = O(e^{\epsilon y})$ as $y \to \infty$ for some $\epsilon > 0$.

Again, we can generalize this definition to include half-integral weights or other groups than $\mathrm{SL}_2(\mathbb{Z})$.

We denote the space of harmonic Maass forms of weight k for $\Gamma_0(N)$ by $H_k(N)$. If we replace condition (3) by

(3') there is a Fourier polynomial $P_f(z) \in \mathbb{C}[q^{-1}]$ such that

$$f(z) - P_f(z) = O(e^{-\epsilon y})$$

as $y \to \infty$ for some $\epsilon > 0$,

we obtain the space $H_k^+(N)$ of harmonic Maass forms of weight k for $\Gamma_0(N)$.

Caution This is the original notation as used by Bruinier and Funke in [4]. Note that both, functions in $H_k^+(N)$ and $H_k(N)$, are called harmonic Maass forms. Other authors often denote these spaces by different names.

Obviously, we have

$$S_k(N) \subset M_k(N) \subset M_k^!(N) \subset H_k^+(N) \subset H_k(N).$$

Lemma 3.2 *Let $k \neq 1$. Every $f \in H_k(N)$ has a Fourier expansion of the shape*

$$F(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) y^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) H(2\pi n y) e^{2\pi i n x},$$

where $H(w) = e^{-w} \int_{-2w}^{\infty} e^{-t} t^{-k} dt$. For $k = 1$ we have to replace $c_f^-(0) y^{1-k}$ by $c_f^-(0) \log(y)$.

If $f \in H_k^+(N)$, $k < 1$, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi |n| y) q^n,$$

where $\Gamma(\alpha, y) = \int_y^{\infty} e^{-t} t^{\alpha-1} dt$ is the incomplete Γ-function.

Proof Let $k \neq 1$ (the proof for $k = 1$ is similar). We briefly sketch the proof (see [4] for more details). Since $f(z)$ is periodic, we have $f(z) = \sum_{n \in \mathbb{Z}} c_f(n, y) e^{2\pi i n x}$. Writing $C(2\pi n y) := c_f(n, y)$ and using that $\Delta_k f = 0$ we obtain

$$\frac{\partial^2}{\partial w^2} C(w) - C(w) + \frac{k}{w} \left(\frac{\partial}{\partial w} C(w) + C(w) \right) = 0. \tag{3.1}$$

If $n = 0$, then $c_f(0, y)$ is a linear combination of 1 and y^{1-k} and if $n \neq 0$, then (3.1) has two linearly independent solutions, namely e^{-w} and $H(w)$. □

Remark 3.3 In the above situation, we say that f splits into a holomorphic part f^+ and a non-holomorphic part f^-, where

$$f^+(z) = \sum_{n \gg -\infty} c_f^+(n) q^n, \text{ and}$$

$$f^-(z) = c_f^-(0) y^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) H(2\pi n y) e^{2\pi i n x}.$$

3.1 Examples of Harmonic Maass Forms

Example 3.4 (The Eisenstein Series $E_2^*(z)$) We define

$$E_2^*(z) := -\frac{3}{\pi y} + E_2(z),$$

where $E_2(z)$ is the Eisenstein series of weight 2 as defined in (1.1).

Lemma 3.5 *The function $E_2^*(z)$ is a harmonic Maass form of weight 2 for $SL_2(\mathbb{Z})$ in $H_2(1)$.*

Proof Here we briefly sketch the proof of this lemma. We have to show that $E_2^*(z+1) = E_2^*(z)$ (which is clear) and $E_2^*\left(-\frac{1}{z}\right) = z^2 E_2^*(z)$. For the second claim we employ Hecke's trick (which is necessary since E_2 does not converge absolutely). Therefore, we let

$$G_2(s,z) := \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^2|mz+n|^{2s}}, \quad \text{for } s \in \mathbb{C} \text{ with } \Re(s) > 0.$$

Then $G_2(s,z)$ converges absolutely and therefore $G_2(s,Mz) = (cz+d)^2|cz+d|^{2s}G_2(s,z)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. It remains to show that $\lim_{s\to 0} G_2(s,z) = \frac{\pi^2}{6}E_2^*(z)$. In order to do so, one defines

$$I_s(z) := \int_{-\infty}^{\infty} \frac{1}{(z+t)^2|z+t|^{2s}} dt$$

and then shows the following:

- $G_2(s,z) - \sum_{m=1}^{\infty} I_s(mz)$ converges absolutely and locally uniformly,
- $\lim_{s\to 0} G_2(s,z) - \sum_{m=1}^{\infty} I_s(mz) = \frac{\pi^2}{6}E_2(z)$,
- $\lim_{s\to 0} \sum_{m=1}^{\infty} I_s(mz) = -\frac{\pi}{2y}$. □

Example 3.6 (Zagier's Weight 3/2 Eisenstein Series) We let $-n$ be a discriminant and $H(n)$ be the usual Hurwitz class number. We define

$$G(z) = -\frac{1}{12} + \sum_{n=1}^{\infty} H(n)q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta\left(4\pi n^2 y\right) q^{-n^2},$$

where $\beta(s) = \int_1^{\infty} t^{-3/2}e^{-st}dt$.

Using Hecke's trick as in Example 3.4 one can prove the following theorem.

Theorem 3.7 *Zagier's Eisenstein series $G(z)$ is a harmonic Maass form of weight $3/2$ for $\Gamma_0(4)$ in $H_{3/2}(4)$.*

Example 3.8 (Ramanujan's Mock Theta Functions) These functions were introduced by Ramanujan in his famous last letter to Hardy. We present two of the functions Ramanujan wrote down:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \qquad \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{n+1}^2},$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$.

Although Ramanujan claimed that these functions transform like modular forms, it remained mysterious for a long time what exactly he meant by that. In 2002, Sander Zwegers was able to "complete" Ramanujan's mock modular forms by adding a nonholomorphic function and to show that they then satisfy the desired transformation behaviour.

Theorem 3.9 (Theorem 3.6 in [18]) *Define*

$$F(z) = (q^{-1/24}f(q), 2q^{1/2}\omega(q^{1/2}), 2q^{1/2}\omega(-q^{1/2}))^t$$

and

$$G(z) = 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{(g_1(w), g_0(w), -g_2(w))^t}{\sqrt{-i(w+z)}}dw,$$

where g_0, g_1, g_2 *are weight* $3/2$ *unary theta functions defined in [18]. Let* $H(z) = F(z) - G(z)$*. Then*

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z),$$

$$H\left(-\frac{1}{z}\right) = = \sqrt{-iz}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z).$$

In other words, H is a vector valued harmonic Maass form of weight $1/2$ in H^+. Therefore, Ramanujan's mock theta functions are the holomorphic parts of weight $1/2$ harmonic Maass forms in H^+.

3.2 *Important Differential Operators*

We define the Maass raising operator R_k and the Maass lowering operator L_k by

$$R_k := 2i\frac{\partial}{\partial z} + \frac{k}{y},$$

$$L_k := -2iy^2\frac{\partial}{\partial \bar{z}}.$$

The following lemma summarizes some essential properties of these operators.

Lemma 3.10 *For a continuously differentiable function f and $M \in SL_2(\mathbb{R})$ we have*

$$R_k(f|_kM) = (R_k(f))|_{k+2}M,$$
$$L_k(f|_kM) = (L_k(f))|_{k-2}M,$$

where $(f|_kM)(z) = (cz+d)^{-k}f(Mz)$. Moreover,

$$-\Delta_k = L_{k+2} \circ R_k + k = R_{k-2} \circ L_k.$$

The ξ-Operator and a Pairing Defined by Bruinier and Funke

Define

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}} = y^{k-2}\overline{L_k}.$$

Bruinier and Funke showed that this differential operator relates harmonic Maass forms to usual modular forms.

Theorem 3.11 ([4]) *We have*

$$\xi_k : H_k^+(N) \to S_{2-k}(N)$$

and

$$\xi_k : H_k(N) \to M_{2-k}^!(N).$$

Moreover, $\xi_k(f) = \xi_k(f^-)$ and ξ_k is surjective.

Recall the Petersson inner product. Let $f, g \in M_k(N)$ and

$$(f,g)_k := \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2},$$

whenever the integral converges.

Proposition 3.12 ([4]) *For $g \in M_k(N)$ and $f \in H_{2-k}^+(N)$ define*

$$\{g,f\} := (g, \xi_{2-k}(f))_k.$$

Then the following hold:

- *This induces a non-degenerate pairing between $H_{2-k}^+(N)/M_{2-k}^!(N)$ and $S_k(N)$.*
- *If $g(z) = \sum_{n\geq 0} a_\ell(n)q^n$ and $f^+(z) = \sum_{n\gg -\infty} b_\ell^+(n)q^n$ denote the Fourier expansion of g and f^+ respectively at the cusp ℓ, then*

$$\{g,f\} = \sum_\ell \sum_{n\leq 0} a_\ell(-n)b_\ell^+(n).$$

Remark 3.13 Some authors prefer to use the notion of a mock modular form and the shadow of a mock modular form. Here a *mock modular form* is the holomorphic part of a harmonic Maass form. For $f \in H_k^+(N)$ we call $\xi_k(f) \in S_{2-k}(N)$ its *shadow*. A *mock theta function* is a mock modular form of weight $1/2$ or $3/2$ whose shadow is a unary theta function.

The Differential Operator *D*

We let

$$D := \frac{1}{2\pi i}\frac{\partial}{\partial z}.$$

Note that D does not preserve modularity (but by adding a "correction term" it does, see the definition of R_k). Moreover we have $D(f) = D(f^+)$ for $f \in H_k(N)$, so the role played by D for the holomorphic part is similar to the role played by ξ for the nonholomorphic part.

Theorem 3.14 ([8]) *For $k \in \mathbb{N}_{\geq 2}$ we have*

$$D^{k-1} : H_{2-k}(N) \to M_k^!(N),$$

and the image consists of those $h \in M_k^!(N)$ which are orthogonal to cusp forms with respect to the (regularized) Petersson inner product and which also have constant term 0 *at all cusps of* $\Gamma_0(N)$.

3.3 Vector Valued Harmonic Maass Forms

Let V be a rational quadratic space with a non-degenerate bilinear form $(,)$ of signature (b^+, b^-) and let $Q(X) = \frac{1}{2}(X, X)$ be the associated quadratic form. We let $L \subset V$ be an even lattice of full rank and denote by L' its dual lattice. Then L'/L is a finite abelian group, called the discriminant group of L.

We let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic group. It is a double cover of $\mathrm{SL}_2(\mathbb{R})$ and consists of pairs (M, ϕ) with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\phi : \mathbb{H} \to \mathbb{C}$ holomorphic such that $\phi(z)^2 = cz + d$. Here, we denote by $\sqrt{w} = w^{1/2}$ the principal branch of the square root. Multiplication is defined via

$$(M, \phi(z))(M', \phi'(z)) = (MM', \phi(M'z)\phi'(z)) \text{ for } (M, \phi), (M', \phi') \in \mathrm{Mp}_2(\mathbb{Z}).$$

The map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{cz + d}\right)$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. Note that

$$\mathrm{Mp}_2(\mathbb{Z}) = \left\langle T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{z} \right) \right\rangle.$$

We let $\mathbb{C}[L'/L] = \left\{ \sum_{h \in L'/L} a_h \mathfrak{e}_h \, : \, a_h \in \mathbb{C} \right\}$ be the group algebra of L. Here, $\mathfrak{e}_h$ denotes the standard basis elements of $\mathbb{C}[L'/L]$.

There is a unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ which is defined through the action of the generators $T, S \in \mathrm{Mp}_2(\mathbb{Z})$ given by

$$\rho_L(T)\mathfrak{e}_h = e^{2\pi i Q(h)} \mathfrak{e}_h,$$

$$\rho_L(S)\mathfrak{e}_h = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e^{-2\pi i (h,h')} \mathfrak{e}_{h'}.$$

The representation ρ_L is called the Weil representation attached to L.

Definition 3.15 A smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic Maass form of weight k with respect to the representation* ρ_L *and the group* $\mathrm{Mp}_2(\mathbb{Z})$ if:

(1) $f(Mz) = \phi(z)^{2k} \rho_L(M, \phi) f(z)$ for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.
(2) $\Delta_k f = 0$.
(3) $f(z) = O(e^{\epsilon y})$ as $y \to \infty$ for some $\epsilon > 0$.

We denote this space by H_{k,ρ_L}. The spaces $H^+_{k,\rho_L}, M^!_{k,\rho_L}, M_{k,\rho_L}, S_{k,\rho_L}$ are defined correspondingly.

The Fourier expansion of a function $f \in H^+_{k,\rho_L}$ is then given by

$$\sum_{h \in L'/L} \sum_{n \gg -\infty} c_f^+(n,h) q^n \mathfrak{e}_h + \sum_{h \in L'/L} \sum_{n<0} c_f^-(n,h) H(2\pi n y) e^{2\pi i n x} \mathfrak{e}_h.$$

Remark 3.16

(1) For a unimodular, even lattice, we recover the definition of scalar valued harmonic Maass forms.
(2) The components of harmonic Maass forms in H_{k,ρ_L} are scalar valued harmonic Maass forms (since ρ_L factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ - b^-$ is even (where N is the level of L) and a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ - b^-$ is odd).

Example 3.17 We let

$$V = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in \mathbb{Q}^{2\times 2} \right\}$$

with $Q(X) = N\det(X)$. Then, (V, Q) is a rational quadratic space of signature $(1, 2)$. Moreover we let

$$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \; : \; a, b, c \in \mathbb{Z} \right\}$$

with dual lattice

$$L' = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \; : \; a, b, c \in \mathbb{Z} \right\}.$$

Then $L'/L \simeq \mathbb{Z}/2N\mathbb{Z}$. The function

$$G(z) := \sum_{r \in \mathbb{Z}/12\mathbb{Z}} \chi_{12}(r)\eta(z)^{-1}\mathfrak{e}_r$$

is a weakly holomorphic modular form of weight $-1/2$ transforming with respect to ρ_L with principal part $q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 - \mathfrak{e}_7 + \mathfrak{e}_{11})$.

4 Applications

In this section we present two applications of the theory of harmonic Maass forms. First, we review how Bruinier and Ono [7] derived an algebraic formula for the partition function $p(n)$. Second, we present a result of Bruinier and Ono [6] and Alfes et al. [2] relating the Fourier coefficients of weight $1/2$ harmonic Maass forms to the vanishing of the central value and the central derivative of an L-function of an elliptic curve.

For more applications of the theory of harmonic Maass forms see [16]. Moreover, a book by Bringmann, Folsom, Ono and Rolen will appear soon [3].

4.1 An Algebraic Formula for $p(n)$

We let V, L, L' be as in Example 3.17 and define the *Heegner divisor of discriminant* (m, h) on $\Gamma_0(N) \setminus \mathbb{H}$ by

$$Z(m, h) = \sum_{X \in \Gamma_0(N) \setminus L_{m,h}} Z(X),$$

where $L_{m,h} = \{X \in L + h \; : \; Q(X) = m\}$ and $Z(X)$ is the image of $D_X = \mathrm{Span}(X) \in \mathbb{H}$ in $\Gamma_0(N) \setminus \mathbb{H}$ counted with multiplicity $1/|\bar{\Gamma}_X|$. Here, Γ_X denotes the stabilizer of X in $\Gamma_0(N)$ and $\bar{\Gamma}_X$ is its image in $\mathrm{PSL}_2(\mathbb{Z})$. For more on such divisors see Eric Hofmann's notes in this volume.

Remark 4.1 An element $X = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \in L_{m,h}$ gives rise to a quadratic form $[a, b, Nc]$ of discriminant $D = b^2 - 4Nac$ satisfying $b \equiv h \pmod{2N}$ via

$$M(X) = \begin{pmatrix} a & b/2 \\ b/2 & Nc \end{pmatrix} = X \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix},$$

(and vice versa). Then, we arrive at the often more familiar notion of CM points (i.e. solutions $z \in \mathbb{H}$ of $Q(z, 1) = 0$, where $Q = [a, b, Nc]$).

We now consider a theta lift which employs the so-called Kudla-Millson theta function $\Theta_L(\tau, z, \varphi_{KM})$ as an integration kernel. For more on theta lifts see Eric Hofmann's notes on his course.

The function $\Theta_L(\tau, z, \varphi_{KM})$ is a non-holomorphic modular form of weight $3/2$ for $\mathrm{Mp}_2(\mathbb{Z})$ and representation ρ_L in the variable τ. In z it is a $\Gamma_0(N)$-invariant $(1, 1)$-form on $\mathbb{H}$. Moreover, it decays square exponentially in z towards the cusps.

Thus, it makes sense to define the Kudla-Millson theta lift of a function $F \in H_0^+(N)$ by

$$I^{KM}(\tau, F) = \int_{\Gamma_0(N)\backslash\mathbb{H}} F(z)\Theta_L(\tau, z, \varphi_{KM}). \tag{4.1}$$

Theorem 4.2 ([5]) *For $F \in H_0^+(N)$ the Kudla-Millson lift $I^{KM}(\tau, F)$ is a harmonic Maass form of weight $3/2$ for ρ_L in $H_{3/2,\rho_L}$. The coefficient of index (m, h), $m > 0$, is given by the so-called trace of the CM values of F, namely by*

$$\mathbf{t}(F; m, h) := \sum_{z \in Z(m,h)} F(z).$$

Remark 4.3 The goal of Bruinier and Funke was to generalize a result of Zagier [17] stating that the generating series of the traces of singular moduli is a weakly holomorphic modular form of weight $3/2$ and to combine this with the fact that the Eisenstein series of weight $3/2$ in Example 3.6 is essentially the generating series of the traces of CM values of the constant function 1.

And as it turns out, we have $I^{KM}(\tau, j - 744) = g_1$ (in Zagier's notation, see [17]) and $I^{KM}(\tau, 1) = E_{3/2}(\tau)$.

In 2010 Bruinier and Ono [7] defined the following extension of the Kudla-Millson lift, now taking weight -2 forms as an input. Let $F \in H_{-2}^+(N)$ and define

$$I^{KM}(\tau, F) = L_{3/2,\tau} \int_{\Gamma_0(N)\backslash\mathbb{H}} R_{-2,z}F(z)\Theta_L(\tau, z, \varphi_{KM})d\mu(z).$$

Theorem 4.4 (**[7]**) *In this situation, $I^{KM}(\tau, F)$ is a harmonic Maass form of weight $-1/2$ for ρ_L in $H^{+}_{-1/2,\rho_L}$ and the coefficient of index (m, h), $m > 0$, of the holomorphic part is given by*

$$-\frac{1}{2m}\frac{1}{4\pi}\mathbf{t}(R_{-2}F; m, h).$$

In order to obtain an algebraic formula for the partition function $p(n)$ we want to realize $G(\tau) := \sum_{r\in\mathbb{Z}/12\mathbb{Z}} \chi_{12}(r)\eta(\tau)^{-1}\mathfrak{e}_r$ in the image of $I^{KM}(\tau, F)$ for a "nice enough" F. (It turns out that "nice enough" means that F has to be weakly holomorphic and its Fourier coefficients need to be in a certain number field.)

Roughly, the proof then goes as follows:

- We compute the lift of Poincaré series (whose principal part is non-trivial only at the cusp ∞). These form a basis of the space of harmonic Maass forms of weight -2 with non-trivial principal part only at the cusp ∞.
- We construct a linear combination of Poincaré series F whose lift has the same principal part as $G(\tau)$ (using the theory of Atkin-Lehner involutions).
- It turns out that

$$F(z) = \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2}.$$

- Since the principal part of $C \cdot I^{KM}(\tau, F) - G(\tau)$ is zero for a suitable constant C, we obtain that $C \cdot I^{KM}(\tau, F) = G(\tau)$ via the bilinear pairing of Bruinier and Funke.

Therefore,

$$p(n) = \frac{1}{24n-1}\sum_{z\in Z(24n-1,1)} P(z),$$

where $P := \frac{1}{4\pi}R_{-2}F(z)$.

Now using the language of quadratic forms Bruinier and Ono showed that for $D > 0$ with $(D, 6) = 1$, $r^2 \equiv -D \pmod{24}$ and a primitive quadratic form Q of discriminant $-D$ with $6|a$ and $b \equiv 1 \pmod{12}$, the number $6 \cdot D \cdot P(\alpha_Q)$ is an algebraic integer contained in the ring class field corresponding to the order O_D, where α_Q denotes the corresponding CM point.

Remark 4.5

(1) Even though we did not need harmonic Maass forms explicitly when deriving the formula for $p(n)$ above, they and the underlying theory are crucial when proving certain properties of the Kudla-Millson lift.
(2) In [14] Larson and Rolen showed that the factor 6 in the statement above can be omitted, that is $D \cdot P(\alpha_Q)$ is always an algebraic integer.

4.2 Fourier Coefficients of Weight 1/2 Harmonic Maass Forms and the Vanishing of Central L-Derivatives of Elliptic Curves

We let E be an elliptic curve over $\mathbb{Q}$ of conductor p, i.e.

$$E:\ y^2 = x^3 + ax + b,\ a, b \in \mathbb{Q}.$$

Moreover, we let $L(E, s)$ be the Hasse-Weil zeta function of E. In the following we consider twists of E for a fundamental discriminant Δ, that is

$$E(\Delta):\ \Delta y^2 = x^3 + ax + b.$$

By the famous Modularity Theorem it is well-known that for every elliptic curve E, there is a cusp form $G_E(z) = \sum_{n\geq 1} a_E(n)q^n \in S_2(p)$ such that

$$L(E(\Delta), s) = L(G_E, \chi_\Delta, s) = \sum_{n=1}^{\infty} \chi_\Delta(n) a_E(n) n^{-s},$$

where χ_Δ is the usual Kronecker character.

In 2006 Bruinier and Ono proved a theorem relating the vanishing of the central value and central derivative of such an L-function to the coefficients respectively the algebraicity of the coefficients of weight $1/2$ harmonic Maass forms.

Even though the statement is phrased in terms of vector valued forms we chose to present the result in the language of scalar valued forms for ease of notation.

Theorem 4.6 ([6]) *Let $f \in H^+_{1/2}(4p)$ be a harmonic Maass form of weight $1/2$ for ρ_L and let $g = \xi_{1/2}(f) \in S_{3/2}(4p)$. Moreover, we let $G \in S_2(p)$ be the cusp form which is the image of g under the Shimura correspondence. That is, we are in the following situation:*

$$\begin{array}{ccc} & & G_E \in S_2(p) \\ & & \uparrow \text{\scriptsize Shimura lift} \\ f \in H^+_{1/2}(4p) & \xrightarrow{\xi_{1/2}} & g \in S_{3/2}(4p). \end{array}$$

We write $f(\tau) = \sum_{n\gg -\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1/2, 4\pi|n|v)q^n$ for the Fourier expansion of f. Moreover, assume that $L(G, 1) = 0$. Then the following hold:

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then $L(G, \chi_\Delta, 1) = C \cdot c_f^-(\Delta)^2$ for an explicit constant C.*

(2) *If* $\Delta > 0$ *is a fundamental discriminant for which* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L'(G, \chi_\Delta, 1) = 0 \;\Leftrightarrow\; c_f^+(\Delta) \in \mathbb{Q}.$$

The proof of the first part follows from the fact that Waldspurger and Kohnen and Zagier described the coefficients of the cusp form $g = \xi(f)$ in this way. The proof of the second part is much harder and makes use of generalized Borcherds products, algebraicity results for differentials of the third kind and the Gross-Zagier formula.

The author, Griffin, Ono and Rolen proved a more intrinsic version of this theorem. Let $G_E \in S_2(p)$ be a cusp form of weight 2 corresponding to an elliptic curve E. They defined a canonical preimage F_E of G_E under ξ_0 and made use of a theta lift similar to the Kudla-Millson theta lift (see [2] for the definition of the lift I). Then the following diagram is Hecke-equivariant:

$$\begin{array}{ccc}
F_E \in H_0^+(p) & \xrightarrow{\;\xi_0\;} & G_E = \xi_0(F_E) \in S_2(p) \\
\Big\downarrow I & & \Big\downarrow \text{Shintani lift} \\
I(F_E) \in H_{1/2}^+(4p) & \xrightarrow{\;\xi_{1/2}\;} & \xi_{1/2}(I(F_E)) \in S_{3/2}(4p).
\end{array}$$

We briefly describe the canonical preimage of G_E. We let Λ_E be the lattice associated to the elliptic curve E via the analytic parametrization. We recall the Weierstrass ζ-function

$$\zeta(\Lambda_E; t) := \frac{1}{t} + \sum_{w \in \Lambda_E \setminus \{0\}} \left(\frac{1}{t-w} + \frac{1}{w} + \frac{t}{w^2} \right).$$

Furthermore, we make use of the modular parametrization. We let $\mathcal{E}_E(t)$ be the Eichler integral of a cusp form G_E defined as

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} G_E(\tau) d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} \cdot q^n.$$

Moreover, we let $S(\Lambda_E) := \lim_{s \to 0^+} \sum_{w \in \Lambda_E \setminus \{0\}} \frac{1}{w^2 |w|^{2s}}$. Eisenstein observed that the function

$$\zeta^*(\Lambda_E; t) = \zeta(\Lambda_E; t) - S(\Lambda)t - \frac{\pi}{a(\Lambda_E)} \bar{t}$$

is lattice invariant, where $a(\Lambda_E)$ is the area of a fundamental parallelogram for Λ_E. This implies that

$$\mathcal{W}_E^*(z) := \zeta^*(\Lambda_E, \mathcal{E}_E(z))$$

is modular of weight 0. We have the following theorem.

Theorem 4.7 *There is a modular function $M_E(z)$ on $\Gamma_0(p)$ with algebraic Fourier coefficients for which $\mathcal{W}_E^*(z) - M_E(z)$ is a harmonic Maass form of weight 0 on $\Gamma_0(p)$. We call the function $\mathcal{W}_E(z) = \mathcal{W}_E^*(z) - M_E(z)$ a Weierstrass harmonic Maass form.*

We write $I(\tau, \mathcal{W}_E(z)) = \sum_{n \gg -\infty} c_E^+(n) q^n + \sum_{n<0} c_E^-(n) \Gamma(1/2, 4\pi |n| v) q^n$ for the Fourier expansion of $I(\tau, \mathcal{W}_E(z))$.

Theorem 4.8 ([2]) *With the same notation as above the following are true:*

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(E(\Delta), 1) = 0 \ \textit{ if and only if } \ c_E^-(\Delta) = 0.$$

(2) *If $\Delta > 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L'(E(\Delta), 1) = 0 \ \textit{ if and only if } \ c_E^+(\Delta) \ \textit{ is in } \ \mathbb{Q}.$$

Remark 4.9 Using the properties of the lift I and the explicit description of the Fourier coefficients of $I(\tau, \mathcal{W}_E(z))$ we can give a different proof for this theorem than Bruinier and Ono (see [1]).

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Elementary Introduction to *p*-Adic Siegel Modular Forms

Siegfried Böcherer

Abstract We give an introduction to the theory of Siegel modular forms mod p and their p-adic refinement from an elementary point of view, following the lines of Serre's presentation (J.-P. Serre, Formes modulaires et fonctions zeta p-adiques. In: Modular Functions of One Variable III. Lecture Notes in Mathematics, vol. 350. Springer, New York, 1973) of the case $SL(2)$.

1 Introduction

In the late sixties of the last century Serre [18] and Swinnerton-Dyer [22] created a theory of p-adic modular forms, which was soon reformulated and refined by Katz [12] in a geometric language. Later on S. Nagaoka and others started to generalize that theory (in the classical language) to Siegel modular forms. In these notes we give a naive introduction, emphasizing level changes and generalizations of Ramanujan's theta operator (i.e. derivatives). Compared with the theory for elliptic modular forms at some points new techniques are necessary. Also some aspects do not appear at all in the degree one case, in particular mod p singular modular forms and also vector-valued modular forms. We will focus on the scalar-valued modular forms, but the vector-valued case will arise naturally in the context of derivatives. We will not enter into the intrinsic theory for the vector-valued case (see e.g. [11] and other papers by the same author); all vector-valued modular forms which appear in our notes arise from scalar-valued ones.

Our naive point of view is that p-adic modular forms encode number theoretic properties (congruences) of Fourier coefficents of Siegel modular forms. We understand that there is a much more sophisticated geometric point of view; in these notes we completely ignore the geometric theory (see e.g. [11, 24, 25]).

S. Böcherer (✉)
Kunzenhof 4B, 79117 Freiburg, Germany
e-mail: boecherer@math.uni-mannheim.de

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2 Basics on Siegel Modular Forms

Mainly to fix notation, we summarize basic facts on Siegel modular forms here. The reader should consult [1, 9, 14] for details.

The symplectic group

$$Sp(n,\mathbb{R}) := \{M \in GL(2n,\mathbb{R}) \mid J_n[M] = J_n\}$$

acts on the Siegel upper half space

$$\mathbb{H}_n := \{Z = Z^t = X + iY \in \mathbb{C}^{(n,n)} \mid Y > 0\}$$

by

$$(M,Z) \longmapsto M < Z >:= (AZ+B)(CZ+D)^{-1}.$$

Here J_n denotes the alternating form given by the $2n \times 2n$ matrix $J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and for matrices U, V we put $U[V] := V^t UV$ whenever it makes sense; we decompose the matrix M into block matrices of size n by $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

There are good reasons to look at vector-valued automorphy factors:

For a finite-dimensional polynomial representation $\rho : GL(n,\mathbb{C}) \longrightarrow GL(V_\rho)$ we consider $V = V_\rho$-valued functions $F : \mathbb{H}_n \longrightarrow V$; the group $Sp(n,\mathbb{R})$ acts on such functions from the right via

$$\left(F \mid_\rho M\right)(Z) := \rho(CZ+D)^{-1} F(M < Z >).$$

As usual, we write $|_k M$ instead of $F |_\rho M$ if $\rho = \det^k$.

We write $\Gamma^n = Sp(n,\mathbb{Z})$ for the full modular group and for $N \geq 1$ we define the principle congruence subgroup of level N by

$$\Gamma(N) := \{M \in \Gamma^n \mid M \equiv 1_{2n} \bmod N\}.$$

We will denote by Γ any group which contains some $\Gamma(N)$ as a subgroup of finite index; typically we will consider the groups

$$\Gamma_0(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv 0 \bmod N\}$$

and

$$\Gamma_1(N) := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv 0 \bmod N, \det(A) \equiv \det(D) \equiv 1 \bmod N\}.$$

The space $M^n_\rho(\Gamma)$ of Siegel modular forms of degree n for ρ consists of all holomorphic functions $F : \mathbb{H}_n \longrightarrow V$, which satisfy the transformation properties $F \mid_\rho \gamma = F$ for all $\gamma \in \Gamma$; only for $n = 1$ we need additional conditions in the cusps, for $n > 1$ such conditions are automatically satisfied ("Koecher principle").

The functions $F \in M^n_\rho(\Gamma)$ are periodic, i.e. $F(Z+S) = F(Z)$ for all $S \in N \cdot \mathbb{Z}^{(n,n)}_{sym}$, their Fourier expansion is then conveniently written in the form

$$F(Z) = \sum_T a_F(T) e^{2\pi i \frac{1}{N} trace(TZ)}. \tag{1}$$

Here T runs over the set $\Lambda^n_{\geq}$ of all symmetric half-integral matrices of size n, which are positive semidefinite.

If we want to emphasize the formal aspects of such a Fourier expansion, then we can view (1) as a formal series as follows:

With $Z = (z_{ij}) \in \mathbb{H}_n$ we put $q_{i,j} = e^{2\pi\sqrt{-1}z_{ij}}$ and we write for $T \in \Lambda^n_{\geq}$

$$q^T := \prod_{i<j} q_{ij}^{2t_{ij}} \prod_j q_{jj}^{t_{jj}}.$$

We consider the q_{ij} as formal variables and we may then view

$$\sum_T a_F(T) q^T$$

as an element of

$$\mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]] \qquad \text{with} \quad q_j := q_{jj}.$$

We mention two typical examples of number-theoretic interest:

Example 1 (Siegel Eisenstein Series) We consider $\rho = \det^k$ with an even integer $k > n+1$ and

$$E^n_k(Z) := \sum_M 1 \mid_k M = \sum_M \det(CZ + D)^{-k};$$

here M runs over $Sp(n, \mathbb{Z})_\infty \backslash Sp(n, \mathbb{Z})$, where $Sp(n, \mathbb{Z})_\infty$ is defined by the condition $C = 0$.

This defines an element of $M^n_k(\Gamma^n)$ with rational Fourier coefficients with bounded denominators (this is not obvious!).

Example 2 (Theta Series) Let $S \in 2 \cdot \Lambda_{>}^{m}$ be a positive definite even integral matrix of size $m = 2k$ and of level N (i.e. N is the smallest positive integer such that $N \cdot S^{-1} \in \Lambda_{>}^{m}$). Then

$$\vartheta_S^n(Z) := \sum_{R \in \mathbb{Z}^{(m,n)}} e^{\pi i trace(X^t SXZ)}$$

defines an element of

$$M_k^n(\Gamma_0(N), \epsilon_S) := \{F \in M_k^n(\Gamma_1(N)) \mid F \mid_k \gamma = \epsilon_S(\det(D)) \cdot F \quad \forall \gamma \in \Gamma_0(N)\}$$

with the quadratic character

$$\epsilon_S(.) = \left(\frac{(-1)^k \det(S)}{.} \right).$$

It is obvious that such theta series have integral Fourier coefficients.

For a subring $\mathcal{R}$ of $\mathbb{C}$ we denote by $\overline{M_k^n(\Gamma)(\mathcal{R})}$ the $\mathcal{R}$ submodule of all modular forms with all their Fourier coefficients in $\mathcal{R}$. This notion can be extended in an obvious way to the vector-valued case after fixing a basis of the representation space of ρ.

Let ξ_N denote a primitive root of unity and denote by $\mathcal{O}_{\xi_N}$ the ring of integral elements in the N-th cyclotomic field. Then we have the following

Fundamental Property

$$M_k(\Gamma(N)) = M_k^n(\Gamma(N))(\mathcal{O}_{\xi_N}) \otimes \mathbb{C},$$

in particular, the field of Fourier coefficients of a modular form is finitely generated and all modular forms and the Fourier coefficients of a modular form in $M_k^n(\Gamma(N))(\overline{\mathbb{Q}})$ *have bounded denominators.*

The property above will be crucial at several points below (sometimes implicitly). We take this for granted and refer to the literature [20]. In some cases (squarefree levels and large weights) elementary proofs are available, using the solution of the basis problem ("all modular forms are linear combinations of the theta series introduced above", see [3]).

Remark We note here two *important differences* between elliptic modular forms and Siegel modular forms of higher degree:

No Obvious Normalization For $n > 1$ there is no good notion of "first Fourier coefficient" and (even for Hecke eigenforms) we cannot normalize modular forms in a reasonable arithmetic way (note that a normalization by requesting the Petersson product to be one is not an arithmetic normalization!).

Hecke Eigenvalues and Fourier Coefficients Fourier coefficients and Hecke eigenvalues are different worlds for $n > 1$. We briefly explain the reason in the simplest case (scalar-valued modular forms of level one): For $g \in GSp^+(n, \mathbb{Q})$ with $g^t J_n g = \lambda \cdot J_n$ we consider for $\Gamma^n = Sp(n, \mathbb{Z})$ the double coset $\Gamma^n \cdot g \cdot \Gamma^n = \bigcup \Gamma^n \cdot g_i$

with representatives $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}$ with $A_i^t \cdot D_i = \lambda$. Then we define a Hecke operator acting on $F \in M_k^n(\Gamma^n)$ by

$$F \longmapsto G := F \mid \Gamma^n \cdot g \cdot \Gamma^n := \sum_i \det(D_i)^{-k} F((A_i \cdot Z + B_i) \cdot D_i^{-1}.)$$

We may plug in the Fourier expansion $F = \sum a_F(T) q^T$ and we get for the Fourier coefficients of $a_G(S)$ a formula of type

$$a_G(S) = \text{a linear combination of } a_F(T) \text{ with } D_i^{-1} T A_i = S,$$

in particular, S and T are rationally equivalent up to a similitude factor.

The conclusion is that Hecke operators give relations between Fourier coefficients only within a rational similitude class of positive definite matrices $T \in \Lambda_>^n$. For $n \geq 2$, the set $\Lambda_>^n$ however decomposes into *infinitely* many such rational similitude classes. In some sense this is a situation similar to the perhaps more familiar case of degree one modular forms of half-integral weight.

Our aim here will be to study congruences among Fourier coefficients of Siegel modular forms (not congruences among eigenvalues!).

The reader interested in congruences for eigenvalues should consult the work of Katsurada [13], who studies congruences between eigenvalues of different types of automorphic forms (lifts and non-lifts); in a different direction (connection to Galois representations) one may look at the work of Weissauer [23].

3 Congruences

3.1 The Notion of Congruences of Modular Forms

For a prime p we denote by ν_p the (additive) p-adic evaluation $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$, normalized by $\nu_p(p^t) = t$. For a modular form $F = \sum_T a_F(T) q^T \in M_k^n(\Gamma_1(N))(\mathbb{Q})$ we put

$$\nu_p(F) := \inf\{\nu_p(a_F(T)) \mid T \in \Lambda^n\}.$$

By the boundedness of denominators, this number is $> -\infty$.

We defined this notion only for scalar-valued modular forms with Fourier coefficients in $\mathbb{Q}$, but we can easily generalize it to modular forms with Fourier coefficients in $\mathbb{C}$ by extending ν_p to the field generated by the Fourier coefficients. Furthermore, we can define it also for vector-valued modular forms after fixing coordinates and taking the minimum of ν_p on the coordinates (this depends on the choice of coordinates!).

Definition For $F, G \in M^n_k(\Gamma_1(N))(\mathbb{Q})$ we define

$$F \equiv G \bmod p^m \iff \nu_p(F - G) > \nu(F) + m.$$

Note that this definition avoids trivial congruences.

Remark In case of Hecke eigenforms, such congruences for modular forms imply congruences for eigenvalues (but not the other way around!).

3.2 Congruences and Weights

A first observation is that such congruences cannot occur among modular forms of arbitrary weights:

Theorem I *For a prime p and a positive integer N coprime to p we consider $\Gamma = \Gamma_1(N) \cap \Gamma_0(p^l)$. Then for $F_i \in M^n_{k_i}(\Gamma)$(with $i = 1, 2$ a congruence $F_1 \equiv F_2 \bmod p^m$ implies a congruence among the weights:*

$$k_1 \equiv k_2 \bmod \begin{cases} (p-1)p^{m-1} & \text{if } p \neq 2 \\ 2^{m-2} & \text{if } p = 2, \quad m \geq 2. \end{cases}$$

For $n = 1$ this is a result of Katz [12, Corollary 4.4.2]. The case of general degree can be deduced from that by associating to F and G suitable elliptic modular forms f and g with the same weights (possibly with larger level) and satisfying the same congruence (see [6] for details).

As a special case, we mention

Corollary *For an odd prime p a modular form $F \in M^n_k(\Gamma)(\mathbb{Q})$ with Γ as above, can be congruent mod p^m to a constant only if $(p-1) \cdot p^{m-1} \mid k$ holds.*

3.3 Mod p Singular Modular Forms

Singular modular forms are a topic which is specific for higher degree, see [9]; there is an analogue mod p:

Definition We call a modular form $F = \sum a_F(T)q^T \in M^n_k(\Gamma)(\mathbb{Q})$ with $\nu_p(F) = 0$ a mod p-singular modular form of rank r, $0 \leq r \leq n-1$ iff $a_F(T) \equiv 0 \bmod p$ for all $T \in \Lambda^n$ with $rank(T) > r$ and if there exists $T_0 \in \Lambda^n$ with $rank(T_0) = r$ such that $a_F(T_0) \not\equiv 0 \bmod p$.

Theorem II *If $F \in M^n_k(\Gamma_0(N))$ is mod p singular of rank r, then*

$$2k - r \equiv 0 \bmod (p-1)p^{m-1}$$

if p is odd.

The proof is inspired by the method used to prove a similar statement for true singular modular forms [9]: One considers a Fourier-Jacobi-expansion $F(Z) = \sum_{S\in\Lambda^r_{\geq}} \phi_S(z_1, z_2)e^{2\pi i trace(Sz_4)}$ with

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix}, \quad z_1 \in \mathbb{H}_{n-r}, \quad z_4 \in \mathbb{H}_r.$$

We choose $T_0 \in \Lambda^n$ with rank r such that $a_F(T_0) \neq 0 \bmod p$; without loss of generality we may assume that T_0 equals $\begin{pmatrix} 0 & 0 \\ 0 & S_o \end{pmatrix}$ with $S_o \in \Lambda^r_{>}$. The "theta expansion" of the special Fourier-Jacobi coefficient ϕ_{S_0} allows us to arrive at a modular form h of degree r and weight $k - \frac{r}{2}$ which is constant mod p. We may then apply the corollary to h^2.

Example Let S be a positive definite even integral quadratic form in m variables. We assume that S has an integral automorphism σ of order p (the existence of such quadratic forms will be considered below). Let l be the maximal number of linearly independent fixed points of σ. Then ϑ^n_S is mod p singular of rank l.

Other types of examples can be constructed using Siegel Eisenstein series; here divisibility properties of certain Bernoulli numbers play an important role, see [4].

3.4 Existence Theorem

In degree 1 the Clausen-von Staudt property of Bernoulli numbers B_{p-1} implies that the Eisenstein series of weight $p - 1$

$$E_{p-1}(z) = 1 - \frac{2p-2}{B_{p-1}} \sum_{n=1} \left(\sum_{d|n} d^{p-2} \right) e^{2\pi i n z}$$

is congruent 1 mod p for $p \geq 5$. In higher degree the situation is more complicated, the Siegel Eisenstein series of weight $p - 1$ is not necessarily congruent 1 modulo p for irregular primes, see [16].

Before stating a general existence theorem we introduce the "zero dimensional cusps" for a group $\Gamma_0(p)$. It is a consequence of the Bruhat decomposition for the symplectic group over a finite field that a complete set of representatives for the double cosets

$$\Gamma_0(p)\backslash Sp(n, \mathbb{Z})/Sp(n, \mathbb{Z})_\infty$$

is given by the $n+1$ elements

$$\omega_i := \begin{pmatrix} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{pmatrix} \qquad (0 \leq i \leq n). \tag{2}$$

The theorem below assures the existence of level p modular forms congruent to 1 mod p and with nice behaviour mod p in the other cusps. This is a very usefull technical tool. The proof will be based on the existence of certain quadratic forms with automorphisms of order p. The advantage of theta series (when compared with Eisenstein series) is that the Fourier expansions in *all* cusps are accessible. This point of view is new even for degree one.

We briefly recall the theta transformation formula relevant for us: Let S be an even integral symmetric matrix, positive definite, $\det(S) = p^{2r}$ of size $m = 2k$ and $0 \leq j \leq n$. Then

$$\vartheta_S^n \mid_k \omega_j = w(S)^j \cdot p^{-jr} \sum_X e^{\pi i S[X] Z}.$$

Here $w(S) = \pm 1$ is the Hasse-Witt invariant of S and X runs over

$$\underbrace{S^{-1}\cdot\mathbb{Z}^m \times \cdots \times S^{-1}\cdot\mathbb{Z}^m}_{j} \times \underbrace{\mathbb{Z}^m \ldots \mathbb{Z}^m}_{n-j}.$$

Theorem III

a) *p odd:* $\exists F \in M^n_{p-1}(\Gamma_0(p)) : F \equiv 1 \bmod p$

b) $p \geq n+3 : \exists F_{p-1} \in M^n_{p-1}(Sp(n,\mathbb{Z})) : F_{p-1} \equiv 1 \bmod p$

c) $p \geq n+3 : \exists k_p : \exists \mathcal{F} \in M^n_{k_p}(\Gamma_0(p)) :$

$$\mathcal{F} \equiv 1 \bmod p \quad \textit{and} \quad \mathcal{F} \mid_{k_p} \omega_i \equiv 0 \bmod p \quad (1 \leq i \leq n).$$

Proof (sketch)

a) We consider the root lattice

$$A_{p-1} := \{(x_1, \ldots, x_p) \in \mathbb{Z}^p \mid \sum_i x_i = 0\}$$

inside the standard euclidean space $\mathbb{R}^p$. We can act on this lattice by the symmetric group S_p; the only lattice point fixed by a $\sigma \in S_p$ of order p is $\mathbf{0}$.

In particular, the orthogonal sum $A_{p-1} \perp A_{p-1}$ corresponds to an even integral positive definite symmetric matrix S of determinant p^2 with an (integral) automorphism of order p without nontrivial fixed point. The theta series ϑ_S^n has the requested properties [5].

b) We put $T := p \cdot S^{-1}$ with S from above, then

$$F_{p-1} := \pm p^{(p-2)n - \frac{n(n+1)}{2}} \sum_{\gamma \in \Gamma_0(p) \in \backslash Sp(n,\mathbb{Z})} \vartheta_T^n \mid_{p-1} \gamma.$$

The sign depends on the Hasse invariant of the underlying quadratic space.

c) This is more complicated: One has to use not only the lattice $A_{p-1} \perp A_{-1}$ but several lattices $\mathcal{L}_1 \ldots \mathcal{L}_{n+1}$ with determinants $p^2, \ldots, p^{2n+2}$ (all with automorphisms of order p without nonzero fixed points). One can construct such lattices from certain ideals in the cyclotomic field generated by p-th roots of unity. In a first step one may then use linear combinations of theta series for such lattices to construct modular forms $G_i \in M_{p-1}^n(\Gamma_0(p))$ such that

$$G_i \mid_{p-1} \omega_j \equiv 1 \bmod p \quad (0 \le j \le i)$$
$$G_i \mid_{p-1} \omega_{i+1} \equiv 0 \bmod p.$$

Typically, the G_i have high powers of p in the denominators of their Fourier coefficients in the cusps ω_j with $j > i + 1$. We may then construct $\mathcal{F}_{k_p}$ by taking suitable products of powers of the G_i.

3.5 *The Ring of Modular Forms Mod p d'après Raum-Richter*

The existence theorem above is an ingredient in the following beautifull recent result (the proof goes beyond our elementary approach).

We define the ring $\tilde{M}^{n,p}$ of modular forms mod p as the image of the ring $\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)})$ under the reduction map˜mod p

$$F = \sum a_F(T)q^T \longmapsto \sum_T \widetilde{a_F(T)}q^T.$$

After Faltings/Chai the ring $\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)})$ of modular forms with coefficients in $\mathbb{Z}_{(p)}$ is finitely generated:

$$\oplus_k M_k^n(\Gamma^n)(\mathbb{Z}_{(p)}) \simeq \mathbb{Z}_{(p)}[X_1, \ldots, X_r]/C$$

with some ideal C describing the relations. One may in particular write the modular form F_{p-1} as a polynomial B in the generators $X_1, \ldots X_r$ (or rather their images mod C).

Theorem of Raum-Richter [17]
For $p \geq n+3$ we have

$$\tilde{M}^{n,p} \simeq \mathbb{F}_p[X_1, \ldots, X_r]/\tilde{C} + < \tilde{B} - 1 > .$$

We can rephrase this by saying that by reduction mod p, the only new relation among the generators is the one coming from $F_{p-1} \equiv 1 \bmod p$.

4 *p*-Adic Modular Forms and Level Changes

Definition A formal series

$$F = \sum_{T \in \Lambda^n_{\geq}} a(T) q^T \qquad (a(T) \in \mathbb{Z}_p)$$

is called p-adic modular form if there is a sequence F_j of level one modular forms $F_j \in M^n_{k_j}(Sp(\Gamma^n))(\mathbb{Z}_{(p)})$ such that the sequence (F_j) converges p-adically to F, i.e. $\nu_p(F - F_j) \longrightarrow \infty$, which means that all the sequences $a_{F_j}(T)$ converge p-adically to $a(T)$ uniformly in T.

Some Comments

- It follows from our Theorem I that such a p-adic modular form has a weight in $\mathbb{Z}/(p-1) \cdot \mathbb{Z} \times \mathbb{Z}_p$.
- One can generalize the notion of p-adic modular form to the vector-valued case in an obvious way.
- Clearly, all level one Siegel modular forms with Fourier coefficients in $\mathbb{Z}_p$ are p-adic modular forms.
- It can happen, that such a p-adic limit is itself a modular form, possibly with nontrivial level: A nice example is exhibited by Nagaoka [15] following an observation by Serre in the degree one case [18]: the sequence of Eisenstein series $\left(E^n_{k_m}\right)_{m \in \mathbb{N}}$ with $k_m = 1 + \frac{p-1}{2} p^{m-1}$ converges p-adically to a weight one modular form for $\Gamma_0(p)$, if $p \equiv 3 \bmod 4$, more precisely, it is proportional to the genus Eisenstein series for the genus of positive binary quadratic forms of discriminant $-p$.

Proposition *All modular forms $F \in M^n_k(\Gamma_0(p))(\mathbb{Z}_{(p)})$ are p-adic (p any odd prime).*
We give here a proof for $p \geq n+3$ and refer to [7] for a different proof covering the general case.
We use the existence of a modular form $\mathcal{F}_{k_p}$ as in Theorem IIIc)
and we consider for $N \in \mathbb{N}$ a "trace function"

$$G_N := \sum_{\gamma \in \Gamma_0(p) \backslash Sp(n,\mathbb{Z})} \left(F \cdot \mathcal{F}^N_{k_p}\right) \mid_{k+Nk_p} \gamma.$$

According to (2), G_N decomposes naturally into $n+1$ summands

$$G_N = \sum_i G_{N,i} \quad \text{with} \quad G_{N,i} := \sum_{\gamma_i} \left(F \cdot \mathcal{F}_{k_p}^N \right) \mid_{k+Nk_p} (\omega_i \cdot \gamma_i),$$

where the γ_i run over certain elements of $Sp(n, \mathbb{Z})_\infty$.

For $i \geq 1$ we have $\nu_p(\mathcal{F}_{k_p}^N \mid_{Nk_p}) \geq N$ and therefore $G_{N,i}$ will be divisible by a high power of p if N is large (the denominators which possibly appear in the Fourier expansion of $F \mid_k \omega_i$ will be compensated. As for $G_{N,0} = F \cdot \mathcal{F}_{k_p}^N$ we observe that $\mathcal{F}_{k_p}^N$ is congruent one modulo p^m provided that N is chosen as $N = p^{m-1}$.

We therefore get that G_N is a level one form congruent to F modulo a high power of p provided that $N = p^m$ with m sufficiently large.

The proposition can be generalized to prime power levels:

Proposition *A modular form $F \in M_k^n(\Gamma_0(p^m))$ is p-adic (p odd, m arbitrary).*

We can use the $U(p)$-operator, defined on Fourier series by

$$\sum a(T)q \longmapsto \sum a(p \cdot T)q^T.$$

Such an operator maps modular forms for $\Gamma_0(p^m)$ to modular forms for $\Gamma_0(p^{m-1})$, provided that $m \geq 2$. It is sufficient to show that F is congruent to a modular form for $\Gamma_0(p^{m-1})$ modulo high powers of p, $m \geq 2$. One can start from the elementary observation

$$F^p \mid U(p) \equiv F \bmod p$$

and then apply the same procedure (with $\mathcal{F}$ as in Theorem IIIa)) to

$$\frac{1}{p}\left(F \cdot \mathcal{F} - F^p \mid U(p)\right)$$

to get a congruence mod p^2; iteration gives the desired result; this proof is a straightforward generalization of the one by Serre [19] for degree one.

Remark There is a delicate difference between the two propositions: the first one generalizes in an obvious way to vector-valued situations, whereas for the second proposition a substitute for taking a p-th power is necessary. A natural choice is taking the p-th symmetric tensor; one can get results along this line, but the notion of p-adic modular form has to be generalized, because one varies the representation space V_ρ.

5 Derivatives

In general, derivatives of modular forms are not modular (by derivatives we mean here holomorphic derivatives!)

But there are bilinear holomorphic differential operators, usually called "Rankin-Cohen" operators, e.g. for $n = 1$ and integral weights k, l with $l \neq 0$

$$[\ ,\ ]_{k,l} : \begin{cases} M_k^1(\Gamma) \times M_l^1(\Gamma) \longrightarrow M_{k+l+2}^1(\Gamma) \\ (f, g) \qquad\qquad \longmapsto \frac{1}{2\pi i}\left(f' \cdot g - \frac{k}{l} f \cdot g'\right) \end{cases}$$

We explain how one can use such Rankin-Cohen-operators to prove that derivatives of modular forms are p-adic modular forms; our proof is different from the usual one which uses the Eisenstein series of weight 2, see [18]; note that we cannot expect in higher degree to find a function analogous to the weight 2 Eisenstein series. We advertise here that the Rankin-Cohen operators, together with modular forms congruent one mod p are an appropriate substitute, which also works in higher degree.

To get a congruence mod p in degree one, we may use

$$[f, \mathcal{F}]_{k,p-1} \equiv \frac{1}{2\pi i} f' \bmod p$$

with $\mathcal{F}$ as in Theorem IIIa). For congruences mod p^m, this does not work with $\mathcal{F}^{p^{m-1}}$, because of $l = (p-1)p^{m-1}$ in the denominator of the Rankin-Cohen-operator. We can avoid this problem, if we use the operator V, defined by $g \mid V(t)(z) := g(t \cdot z)$ and consider

$$[f, \mathcal{F}^{p^{m-1}} \mid V(p^m)]_{k,(p-1)p^{m-1}} \equiv \frac{1}{2\pi i} f' \bmod p^m.$$

Here we increase the level by the operator $V(p^m)$; this can be avoided by using a modular form $\mathcal{H}$ of level one and some weight h satisfying

$$\mathcal{H} \equiv \mathcal{F}^{p^{m-1}} \mid V(p^m) \bmod p^m.$$

Then $[f, \mathcal{H}]_{k,h} \equiv \frac{1}{2\pi i} f' \bmod p^m$ holds. Note that the existence of $\mathcal{H}$ is assured by our proposition and by Theorem I, the weight of $\mathcal{H}$ is under control. Clearly this line of reasoning also works for higher derivatives. Furthermore, this proof contains all the ingredients for generalization to higher degree:

First we introduce a symmetric $n \times n$ matrix ∂ of partial derivatives on $\mathbb{H}_n$:

$$(\partial)_{i,j} := \begin{cases} \frac{\partial}{\partial z_{ii}} & \text{if } i = j \\ \frac{1}{2}\frac{\partial}{\partial z_{ij}} & \text{if } i \neq j \end{cases}$$

We fix a weight k and a (possibly vector-valued) automorphy factor ρ and $l = (p-1)p^{m-1}$ with suitable m. Let $Hol(\mathbb{H}_n, V_\rho; |_\rho)$ denote the vector space of all holomorphic V_ρ-valued functions on $\mathbb{H}_n$, equipped with the action of $Sp(n, \mathbb{R})$ defined by the automorphy factor ρ; if $\rho = \det^k$, we just write $Hol(\mathbb{H}_n; |_k)$ We consider a bilinear holomorphic differential operator

$$[\ ,\ ]_{k,l} : Hol(\mathbb{H}_n; |_k) \times Hol(\mathbb{H}_n; |_l) \longrightarrow Hol(\mathbb{H}_n, V_\rho, |_{\rho\otimes\det^{k+l}}),$$

which is equivariant for the action of $Sp(n, \mathbb{R})$, i.e.

$$[F\ |_k\ g, G\ |_l\ g]_{k,l} = [F, G]_{k,l}\ |_{\rho\otimes\det^{k+l}}\ g$$

for all holomorphic functions F, G and all $g \in Sp(n, \mathbb{R})$, in particular, it maps $(F, G) \in M_k^n(\Gamma) \times M_l^n(\Gamma)$ to an element of $M^n_{\rho\otimes\det^{k+l}}(\Gamma)$.

We impose the following three conditions

(RC1) $[F, G]_{k,l}$ is a polynomial in the derivatives of F and G, more precisely, there exists a V_ρ-valued polynomial with rational coefficients in two matrix variables $R_1, R_2 \in \mathbb{C}^{n,n}_{sym}$, homogeneous of degree λ, such that

$$[F, G]_{k,l} = (2\pi i)^{-\lambda}\mathcal{P}(\partial_{Z_1}, \partial_{z_2})(F(Z_1) \cdot G(Z_2))_{|Z=Z_1=Z_2}$$

(RC2) We write $\mathcal{P} = \sum_j \mathcal{P}_j$ where the $\mathcal{P}_j$ are homogenous of degree j when viewed as polynomials in the second variable R_2 alone. Then $\mathcal{P}_0$ should be independent of l.

(RC3) The coefficients of $\mathcal{P}$ depend continuously on l (p-adically)

Comment The existence of such bilinear differential operators is not a problem if we stay away from finitely many values of k and l; this is a matter of invariant theory, see [8, 10]. The condition (RC2) however is delicate and has to be checked case by case as far as I can see.

Using such a Rankin-Cohen operator, we can now define analogues of Ramanujan's theta-operator

$$f = \sum a_t q^t \longmapsto \theta(f) = \frac{1}{2\pi i} f' = \sum_t t \cdot a(t) q^t.$$

For a Rankin-Cohen operator $[\ ,\ ]_{k,l}$ and $F \in M_k^n(\Gamma)$ we define a V_ρ-valued operator by

$$\Theta_{k,\rho}(F) := (2\pi i)^{-\lambda}\mathcal{P}_0(F).$$

Exactly by the same reasoning as for degree one we may show now

Theorem IV *For a modular form $F \in M_k^n(Sp(n,\mathbb{Z}))(\mathbb{Z}_{(p)})$ and a Rankin-Cohen operator $[\ ,\ ]_{k,l}$ with properties (RC1). (RC2), (RC3) the theta operator defines a V_ρ-valued p-adic modular form $\Theta_{k,\rho}(F)$.*

To explain our principle examples, we introduce some convenient notation following [9, III.§6]: For $0 \le i \le n$ and a $n \times n$ matrix A let $A^{[i]}$ be the matrix of size $\binom{n}{i} \times \binom{n}{i}$ consisting of the determinants of all submatrices of size i.

Examples For $0 \le i \le n$ and $F = \sum a_F(T)q^T \in M_k^n(\Gamma)$ we put

$$\Theta^{[i]}F := \sum_T a_F(T) \cdot T^{[i]} q^T$$

For $F \in M_k^n(\Gamma_0(p^r))(\mathbb{Z}_{(p)})$ this expression $\Theta^{[i]}(F)$ is congruent mod p^m to a level one modular form with automorphy factor

$$\det{}^{k+(p-1)p^{m'}} \otimes \underbrace{(\overbrace{2,\ldots,2}^{r},0,\ldots,0)}_{\text{highest weight of } \rho}$$

for a sufficiently large m', in particular, $\Theta^{[i]}F$ is a p-adic (vector-valued) modular form. This is in particular true for

$$\Theta^{[n]}(F) = \sum_T a_F(T)\det(T)q^T$$

and

$$\Theta^{[1]}(F) = \sum_T a_F(T) \cdot Tq^T.$$

In fact, the corresponding Rankin-Cohen bracket for $\Theta[i](F)$ can be constructed completely explicitly: We define polynomials $Q_{i,j}(R,S)$ in variables $R, S \in \mathbb{C}_{sym}^{(n,n)}$ by

$$(R + xS)^{[i]} = \sum_{j=0}^{i} Q_{i,j}(R,S)x^j.$$

Then there is an explicit linear combination of the

$$Q_{i,j}(\partial_{Z_1}, \partial_{z_2})(F(Z_1)) \cdot G(Z_2)_{Z_1=Z_2}$$

with leading term $(\Theta^{[i]}F) \cdot G$.

Remark If $F \in M_k^n(\Gamma)(\mathbb{Z}_{(p)})$ is mod p singular of rank r, then $\Theta^{[r+1]}(F) \equiv 0 \bmod p$ holds, but not only mod p singular modular forms have this property: let S be a positive definite quadratic forms in $m = 2k$ variables with $rank_{\mathbb{F}_p}(S) = n - j < n$;

we assume that S has no nontrivial integral automorphism. The theta series $\vartheta_S^n = \sum_T a(T)q^T$ is not mod p singular, because $a(S) = 2$. On the other hand, one has

$$\Theta^{[n-j+1]}\vartheta_S^n \equiv \cdots \equiv \Theta^{[n]}\vartheta_S^n \equiv 0 \bmod p.$$

6 Outlook: Quasimodular Forms

There is a sophisticated theory of nearly holomorphic modular forms due to Shimura [21]; they behave like modular forms, but they are no longer holomorphic: they are polynomials in the entries of Y^{-1} with holomorphic coefficients. A very famous example is the nonholomorphic Eisenstein series of weight 2:

$$1 - \frac{3}{\pi iy} - 24\sum \sigma_1(n)q^n.$$

A quasimodular form is then defined as the constant term of such a nearly holomorphic function. Using the calculus of Rankin-Cohen operators and the full theory of nearly holomorphic modular forms, one can then show that such quasimodular forms are also p-adic modular forms [2].

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Liftings and Borcherds Products

Eric Hofmann

Abstract This chapter serves as a brief introduction to the theory of theta-liftings with the main focus on Borcherds' singular theta-lift and the construction of Borcherds products. Thus, after a few initial examples for liftings, we proceed to develop the tools needed to understand how the Borcherds lift works. Namely, we go through the construction of symmetric domains for orthogonal groups, introduce vector-valued modular forms and explain the definition of the Siegel theta-function. Then, we give a detailed treatment of the regularization recipe for the theta-integral and of the proof for the key properties of the additive lift: the location and type of its singularities. Finally, in the closing section, we sketch how to obtain a multiplicative lifting and the Borcherds' products.

1 Introduction

The present course notes are based on three lectures held by the author during a preparatory course for the conference 'L-functions and automorphic forms'. Their purpose is to give a brief introduction to theta-liftings, in which input functions (usually modular forms) are 'lifted' by integrating them against a suitable theta-function. The main focus lies on the singular theta-lift of Borcherds [4], which leads up to the construction of Borcherds products through a multiplicative lifting. This lifting yields meromorphic modular forms for an indefinite orthogonal group of signature $(2, n)$, $n \geq 2$, which take their zeros and poles along certain arithmetically defined divisors called Heegner divisors and which posses absolutely convergent infinite product expansions (called 'Borcherds product expansions').

Special cases of such infinite products were already obtained by Borcherds in an earlier paper [3], however using completely different methods. This construction was originally motivated by the theory of generalized Lie (super-)algebras (see e.g. [36] or [15]).

E. Hofmann (✉)
Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, D-69120 Heidelberg, Germany
e-mail: hofmann@mathi.uni-heidelberg.de

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The singular theta-lift we will concentrate on takes weakly holomorphic modular forms (see Definition 1.1 below) for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ and lifts them to modular forms for an indefinite orthogonal group.

It should be mentioned that the theoretical reason, why such a lifting using a theta-function is possible, is that $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SO}(2, n)$ form what is called a dual reductive pair in the sense of Howe [see 25].

Overview

In the first section, we give a few examples of liftings that can be realized as theta lifts. This includes a special case of Borcherds' original construction from [3].
In Sect. 3, we go through the construction of symmetric domains for indefinite orthogonal groups of signature $(2, n)$, $n \geq 1$. Further, we define orthogonal modular groups related to lattices (Sect. 3.2) and introduce Heegner divisors (Sect. 3.3). The section closes with a definition of orthogonal modular forms (see p. 350).
The main section, Sect. 4 (p. 351) covers the singular theta-lift:
First, we study the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$, a representation of which is used to define vector valued modular forms (see p. 352), generalizing the usual definition of scalar valued modular forms, see Definition 1.1.
Next, in Sect. 4.2, we introduce the Siegel theta-function which is employed in the lifting, and formulate the theta-integral. We will indicate, why in this particular case it is necessary to consider, on the one hand, weakly holomorphic forms as input functions, and, on the other hand, to use a regularized integral.
The regularization procedure is described in detail in Sect. 4.3. We derive one of the main properties of Borcherds' singular lifting, namely the location and type of its singularities (Theorem 4.2). Also we briefly outline some of the main steps used in the actual evaluation of the theta-lift, without going into further detail (see p. 362).
Finally, in Sect. 4.4 the singular theta-lift is used to define the multiplicative lifting:

Borcherds products are explained as solutions of a multiplicative Cousin problem, namely of finding a meromorphic functions with divisor supported on the singularities of the singular theta-lift. We formulate a version of Borcherds' theorem [4, Theorem 13.3], with a simplified form of the infinite product expansion.

1.1 Basic Definitions and Notation

Throughout these notes, as usual, the integers are denoted by $\mathbb{Z}$, and the positive integers by $\mathbb{N}$. Also, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ denotes the reals, and $\mathbb{C}$ the complex numbers.

We recall some basic definitions from the theory of modular forms, details of which can be found in many places, for example in [2, 16, 26], [12, part I] or [31].

As usual, the complex upper half-plane is denoted by $\mathbb{H} = \{z \in \mathbb{C}; \Im z > 0\}$. Throughout, τ will be used to denote a point in $\mathbb{H}$, with $\tau = u + iv$, with u and v the real and the imaginary part of τ, respectively. Also, we denote by $\mathbb{H}^*$ the union of $\mathbb{H}$ with its rational boundary points, i.e. $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$.

The special linear group $\mathrm{SL}_2(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right); a, b, c, d \in \mathbb{Z}, ad - bc = 1\right\}$ operates on $\mathbb{H}$ by fractional linear transformations,

$$M\tau = \frac{a\tau + b}{c\tau + c} \quad \text{if} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A standard fundamental domain for this operation is given by

$$\mathcal{F} := \left\{\tau = u + iv;\ |z| > 1, -\tfrac{1}{2} < u < \tfrac{1}{2}\right\}.$$

Also, recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Beside $\mathrm{SL}_2(\mathbb{Z})$, known as the full (elliptic) modular group, subgroups of finite index are also called modular groups. These include the families of *congruence subgroups*, most importantly $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ for N a positive integer, their *level*:

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; c \equiv 0 \bmod N\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N\right\},$$

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; b \equiv c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N\right\}.$$

Note that $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1)$.

Let Γ be a modular group. The Γ-equivalence classes of $\mathbb{Q} \cup \{i\infty\}$ are called the *cusps* of Γ. The equivalence class of $\{i\infty\}$ is usually referred to as *the cusp at* ∞. Note that for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, this is the only cusp.

Now, we recall the definition of modular forms.

Definition 1.1 Let Γ be a modular group, k an integer and χ a group character of Γ. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a holomorphic modular form of weight k for Γ, with character χ, denoted $f \in \mathrm{M}_k(\Gamma, \chi)$ if

1. $f(M\tau) = \chi(M)(c\tau + d)^k f(\tau)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
2. f is holomorphic at all cusps.

If further f vanishes at all cusps, f is called a cusp form. The space of cusp forms (for Γ, with weight k and character χ) is denoted $\mathrm{S}_k(\Gamma, \chi)$.

Contrastingly, if instead of satisfying condition 2. f is only meromorphic at the cusps, f is called a weakly holomorphic modular form. The space of weakly holomorphic modular forms is denoted $\mathrm{M}^!_k(\Gamma, \chi)$.

Clearly, we have $\mathrm{S}_k(\Gamma, \chi) \subset \mathrm{M}_k(\Gamma, \chi) \subset \mathrm{M}^!_k(\Gamma, \chi)$. Similarly, the notations $\mathrm{S}_k(\Gamma)$, $\mathrm{M}_k(\Gamma)$ and $\mathrm{M}^!_k(\Gamma)$ are used, if the character is trivial.

More generally, we will also consider modular forms of half-integer weight. For this, if $k \in \frac{1}{2}\mathbb{Z}$, one has to replace condition 1. in the definition and require, in its place

$$f(M\tau) = \chi(M)j(M,\tau)^{2k}f(\tau) \qquad \text{for all} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

with a suitable automorphy factor $j(M,\tau)$. In particular, if $\Gamma = \Gamma_0(4N)$ the automorphy factor is given by $j(M,\tau) = \theta_0(M\tau)/\theta_0(\tau)$, where $\theta_0(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the usual Jacobi theta function [see 26, Chapter IV].

Finally, modular forms have Fourier expansions since the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, one of the two generators of $\mathrm{SL}_2(\mathbb{Z})$, acts on $\mathbb{H}$ as $\tau \mapsto \tau + 1$. As every modular group Γ, being of finite index, contains some power of T, modular forms are periodic with positive integer periods, and hence can be expanded as Fourier series around the cusp at infinity and around all other cusps. Thus, for example if Γ is modular group with $T \in \Gamma$ (e.g. one of the $\Gamma_0(N)$'s), and k an integer, the Fourier expansion of $f \in \mathrm{M}^!_k(\Gamma)$ around ∞ takes the form

$$f(\tau) = \sum_{m \gg -\infty} a(m)q^m, \qquad q = e(\tau) = e^{2\pi i\tau}.$$

There are only finite many non-zero terms with $m < 0$. Further, if $f \in \mathrm{M}_k(\Gamma)$, then $a(m) \neq 0$ only for $m \geq 0$. Finally, if f is a cusp form, $a(m) \neq 0$ implies $m > 0$.

For an overview of further notation, the reader is advised to consult (Table 1).

2 Examples of Liftings

In this section we will give some examples for liftings, all of which can, in fact, be realized as theta-liftings, using the theory of Howe duality (which is beyond the scope of the present course notes [see 25]). However, this is not the only way such liftings can be formulated, and indeed, the examples in this section were originally constructed using other methods.

2.1 Convolution of L-Series

Our first two examples were discovered in the 1970s using convolution of L-series:

1. The *Shimura lift*, discovered by Goro Shimura [37] which takes certain half-integer weight cusp form of level $4N$ ($N \geq 1$) to integral weight modular forms of level $2N$.

Table 1 Some frequently used notation

$\mathbb{H}$	The complex upper half-plane (p. 334)
$\mathcal{F}$	A standard fundamental domain $\subseteq \mathbb{H}$ (p. 334)
$\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$	Principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (p. 335)
$\mathrm{S}_k(\Gamma)$, $\mathrm{M}_k(\Gamma)$, $\mathrm{M}^!_k(\Gamma)$	Spaces of modular forms for a modular group Γ
$\mathrm{S}_k(\Gamma,\chi)$, $\mathrm{M}_k(\Gamma,\chi)$, $\mathrm{M}^!_k(\Gamma,\chi)$	(see Definition 1.1)
$\theta_0 = \sum_{n\in\mathbb{Z}} q^{n^2}$	The Jacobi theta-function
$\mathrm{M}^+_k(\Gamma)$	A Kohnen plus-space (see p. 338)
$X_0(N)$ $X_0(N)^*$	A modular curve $\simeq \Gamma_0(N)\backslash\mathbb{H}$, its compactification
$V = V(\mathbb{Q})$, $V(\mathbb{R})$	A quadratic space over $\mathbb{Q}$, with $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ and signature $(2, n)$ (see p. 343)
$q(\cdot)$, $(\cdot,\cdot)$ $x^2 = (x,x) = 2q(x)$	The quadratic and the bilinear form of V, (see p. 343)
$\mathrm{SO}(V)$, $\mathrm{O}(V)$ $\mathrm{O}^+(V)$	The orthogonal and the special orthogonal group of V, The spinor kernel in $\mathrm{SO}(V)$ (see p. 343)
$V(\mathbb{C}) = V(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$	The complexification of $V(\mathbb{R})$
$\mathbb{D}$, $\mathcal{K}$, $\mathcal{H}$	Models for the symmetric domain of $\mathrm{SO}(V)(\mathbb{R})$, see Sect. 3.1
ℓ, ℓ'	Isotropic lattice vectors in V with $(\ell,\ell') = 1$
$V_0(\mathbb{R}) = V(\mathbb{R}) \cap \ell \cap \ell'$	A Lorentzian subspace
Z, z, Z_ℓ, $w(z)$	See Sect. 3.1, p. 345
L, L', L/L'	A lattice in V, its dual and the discriminant group, (p. 348)
Γ_L	The discriminant kernel and
X_Γ	the modular variety $\Gamma_L\backslash\mathbb{D}$. (p. 348)
$\mathbb{D}_\lambda$ $\mathcal{Z}(\mu, m)$	A primitive Heegner divisor (Definition 3.2, p. 349), a Heegner divisor of index (μ, m)
$\mathrm{Mp}_2(\mathbb{R})$	The metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$ (p. 351)
$\mathrm{Mp}_2(\mathbb{Z}) = \widetilde{\mathrm{SL}_2}(\mathbb{Z})$, $\widetilde{\Gamma_0}(N)$	The pre-images of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(N)$ in $\mathrm{Mp}_2(\mathbb{R})$
ρ_L, ρ_L^*	The Weil representation and its dual
$\mathbb{C}[L'/L]$	The group algebra of L'/L
S_{k,ρ_L}, M_{k,ρ_L}, $\mathrm{M}^!_{k,\rho_L}$	Space of vector valued modular forms (p. 352)
H_{k,ρ_L}, H^+_{k,ρ_L}	Spaces of harmonic Mass forms, see (p. 353)
$\langle\cdot,\cdot\rangle$	A hermitian pairing on $\mathbb{C}[L'/L]$
$\Theta_L(z,\tau)$	The Siegel theta function for L (p. 355)
$d\mu = \frac{du dv}{v^2}$	The left-invariant measure on $\mathbb{H}$
$\int^{reg}$	Regularized integral (Sect. 4.3)
$\Phi(z,f)$	The singular theta lift of f
$\Psi(z,f)$	The multiplicative lift of f (Sect. 4.4)

2. The *Doi-Naganuma correspondence*, between modular forms for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ and modular forms for the Hilbert modular group, constructed by Koji Doi and Hidehisa Naganuma [see 17].

Shimura's Lifting

Let us turn to the Shimura lift first, an overview of which can be found e.g. in [35, Chapt. 3].

Suppose that N and κ are positive integers, with N square-free, and that χ is a character modulo N. Further, assume that g is a cusp form of half-integer weight contained in $\mathrm{S}_{\kappa+\frac{1}{2}}(\Gamma_0(4N), \chi)$, with Fourier expansion given by

$$g(\tau) = \sum_{n=1}^{\infty} b(n)q^n.$$

Let t be a positive square-free integer and define a Dirichlet character Ψ_t by setting

$$\Psi_t(n) := \chi(n) \cdot \left(\frac{-1}{n}\right)^{\kappa} \left(\frac{t}{n}\right) \qquad (n \in \mathbb{N}).$$

Denote by $L(s, \Psi_t) = \sum_{n>0} \Psi_t(n)n^{-s}$ the Dirichlet L-series attached to Ψ_t.

Further, let $\{a_t(n)\}_{n=1,2,\dots}$ be a sequence of complex numbers given by

$$\sum_{n=1}^{\infty} \frac{a_t(n)}{n^s} = L(s - \kappa + 1, \Psi_t) \cdot \sum_{n=1}^{\infty} \frac{b\left(tn^2\right)}{n^s}.$$

Then, the q-expansion with coefficients $a_t(n)$ defines a modular form, called the *Shimura lift* of g:

$$\sum_{n=1}^{\infty} a_t(n)q^n =: S_{t,\kappa}(g)(\tau),$$

contained in $\mathrm{M}_{2\kappa}(\Gamma_0(2N), \chi^2)$. Further, if $\kappa \geq 2$, the lift $S_{t,\kappa}(g)$ is a cusp form, whereas for $\kappa = 1$, $S_{t,\kappa}(g)$ is cuspidal only for certain g. (More precisely, for g contained in the orthogonal complement of the subspace spanned by unary theta series [see 35, p. 53].)

In 1975, Shinji Niwa [see 33] refined Shimura's lifting and realized it as a theta-lift.

Kohnen's Theory We introduce the Kohnen plus space $\mathrm{M}^{+}_{\kappa+\frac{1}{2}}(\Gamma_0(4N))$. It consists of modular forms with Fourier expansions of the form

$$g(z) = \sum_{(-1)^{\kappa} n \equiv 0,1 \bmod 4} b(n)\, q^n, \tag{1}$$

with coefficients $b(n) \neq 0$ only for n which satisfy $(-1)^\kappa n \equiv 0, 1 \pmod 4$. The plus space was introduced by Winfred Kohnen as he studied the properties of the Shimura lift with respect to Hecke operations [see 27].

Furthermore, extending Shimura's results in [28, 29], he showed that the two spaces of newforms $\mathrm{S}^{+,new}_{\kappa+\frac{1}{2}}(\Gamma_0(4N))$ and $\mathrm{S}^{new}_{2\lambda}(\Gamma_0(N))$ are isomorphic. The isomorphism is given by a linear combination of Shimura lifts. Some authors refer to this Hecke-invariant isomorphism as the 'Shimura correspondence'.

The Doi-Naganuma Correspondence

Our next example is due to Doi and Naganuma [17] and was discovered at around the same time as Shimura's lifting. See [13, Sections 1.7, 1.10] and [12, II. Section 3.1] for details.

In order to formulate the correspondence, we briefly recall some facts about Hilbert modular forms [see 12, II. Sections 1.3, 1.6]: Let $d > 1$ be a square-free integer and denote by K the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. We shall assume that the narrow class number of K is one.

Denote by $\mathcal{O}_K$ the ringer of integers in K and by $\mathfrak{d}^{-1}$ the inverse different ideal. Further, for $a \in K$ denote by a' the Galois conjugate of a.

The special linear group $\mathrm{SL}_2(K)$ is embedded into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ through the two real embeddings of K. It acts on $\mathbb{H}\times\mathbb{H}$ through fractional linear transformations. For $z = (z_1, z_2) \in \mathbb{H}^2$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right) \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K).$$

The Hilbert modular group $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ acts properly discontinuously.

Let k be an integer. A holomorphic Hilbert modular form F for Γ_K of (parallel) weight k is a holomorphic function $F : \mathbb{H}^2 \to \mathbb{C}$ which transforms according to

$$F(\gamma z) = \left(cz_1 + d\right)^k \left(c'z_2 + d'\right)^k F(z) \qquad \text{for all} \quad \gamma \in \Gamma_K, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

We denote by $\mathrm{M}_{H,k}(\Gamma_K)$ the space of holomorphic Hilbert modular forms of weight k for Γ_K. Note that by the Koecher principle [see 12, II. Theorem 1.20] if a Hilbert modular form F is holomorphic on $\mathbb{H}^2$, it is automatically holomorphic at the cusp ∞, and indeed at all cusps. Here, as usual, by the cusps of $\mathbb{H}^2$, we mean the Γ_K-equivalence classes of elements in $\mathbb{P}^1(K)$.

We will describe what it means for a Hilbert modular form F to holomorphic at the cusp ∞ using the Fourier expansion. From this one can obtain the description for the other cusps through conjugation, noting that for any $\kappa \in \mathbb{P}^1(K)$, one can take $\rho \in \mathrm{SL}_2(K)$ with $\rho\infty = \kappa$.

Since the stabilizer of ∞ in Γ_K contains a finite index subgroup acting by translations [see 12, p. 113], for $F \in \mathrm{M}_{H,k}(\Gamma_L)$, with the transformation behavior (2), this implies the existence of a Fourier expansion of the following form:

$$F(z) = a(0) + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} a(\nu) e\left(\mathrm{tr}(\nu z)\right).$$

Here, the sum ranges over totally positive ν (denoted $\nu \gg 0$), if, as implied by the Koecher principle, F is holomorphic at ∞. Then, one further sets $F(\infty) = a(0)$. Finally F is called a cusp form, if in addition to F being holomorphic, one has $F(\infty) = 0$.

Now, given $F \in \mathrm{M}_{H,k}(\Gamma_K)$ with Fourier coefficients $a(\nu)$, we introduce a Dirichlet series denoted $L(s, F)$ as follows:

$$L(s, F) := \sum_{\substack{\nu \in \mathfrak{d}^{-1}/U \\ \nu \gg 0}} a(\nu)\, \mathrm{N}(\nu \mathfrak{d})^{-s}.$$

Here, U denotes the set of squares of totally positive units in $\mathcal{O}_K$, while for an ideal $\mathfrak{a}$ the norm is denoted $\mathrm{N}(\mathfrak{a})$.

Now, we are ready to describe the Doi-Naganuma lifting: Suppose $f(\tau) = \sum_{n \geq 0} a(n) q^n$ is a Hecke eigenform in $\mathrm{M}_k(\Gamma_0(1))$, with even weight k. Let $L(s,f)$ be the attached Dirichlet series and denote by $L(s,f,\chi_d)$ a twist by the quadratic character $\chi_d = \left(\frac{d}{\cdot}\right)$:

$$L(s,f) = \sum_{n>0} a(n) n^{-s}, \quad L(s,f,\chi_D) = \sum_{n>0} \chi_d(n) a(n) n^{-s}.$$

Denote by $L_{DN}(s)$ the product of these two Dirichlet series,

$$L_{DN}(s,f) := L(s,f) \cdot L(s,f,\chi_d).$$

Then, there is a Hilbert modular form $DN(f) \in \mathrm{M}_{H,k}(\Gamma_K)$, the *Doi-Naganuma lift* of f, with precisely this Dirichlet series, so that $L(s, DN(f)) = L_{DN}(s,f)$.

Remark 2.1 Of course this is not exactly the way Doi and Naganuma originally stated their result in [17]. In 1973, Naganuma obtained the following version [see 32]: Assume that $d = p$ is a prime and let $K = \mathbb{Q}(\sqrt{p})$. Let $f(\tau) = \sum_n a(n) q^n$ be a normalized Hecke eigenform in $\mathrm{M}_k(\Gamma_0(p), \chi_p)$, with χ_p a character of order two, and let $f^\rho(\tau) = \sum_n \overline{a(n)} q^n$. Then, we have $L(s, DN(f)) = L(s,f) \cdot L(s,f^\rho)$ and $DN(f) \in \mathrm{M}_{H,k}(\Gamma_K)$.

2.2 Borcherds Products

In [3] Richard E. Borcherds introduced his famous multiplicative lifting. The methods he used are totally unrelated to either convolutions of L-series or, indeed, theta-correspondences. In contrast to this, Borcherds' later, much more general construction in [4], which we will study in Sect. 4, is formulated as a theta-lift.

For now, though, we describe only a special case from [3]: Here, the input functions for the multiplicative lifting are contained in $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$, i.e they are weakly holomorphic modular forms of level 4, weight $\frac{1}{2}$ and satisfy a plus-space condition like in (1). They are lifted to meromorphic modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, which have infinite product expansions, and, in their Fourier expansion (around the cusp at infinity), integral Fourier coefficients and leading coefficient one. Further, they take their zeros and poles along linear combinations of rational divisors, called Heegner divisors:

Heegner Divisors (Classical) Heegner divisors are subsets of $\mathbb{H}$ arising as the pre-images under $\mathbb{H} \to X_0(N)$ ($N \in \mathbb{N}$) of certain rational divisors on the modular curve $X_0(N) \simeq \Gamma_0(N)\backslash\mathbb{H}$, for a precise definition [see 20, Section IV.1].

In the present setting, the level N is 1, and Heegner divisors are given as follows: Let D be a negative integer, with D a square modulo 4. Let a, b, c with $a > 0$ be integers satisfying $b^2 - 4ac = D$. Thus, a, b, c are the coefficients of an integral binary quadratic form, with D as its discriminant.

A point $\tau \in \mathbb{H}$ satisfying $a\tau^2 + b\tau + c = 0$ is then called a CM-point of discriminant D. Finally, the Heegner divisor of discriminant D consists of all CM-points of that discriminant. Often, it is useful to consider divisors supported at cusps as Heegner divisors, too.

We will encounter a generalization of this concept of Heegner divisors in Sects. 3 and 4 below.

The Multiplicative Lifting Let $\tilde{H}(\tau)$ denote the following generating series

$$\tilde{H}(\tau) := \sum_{\substack{n \equiv 0,3 \bmod 4 \\ n \geq 0}} H(n) q^n,$$

where $H(n)$ are the usual Hurwitz class numbers. They are modified class numbers given as follows [see 14, Section 5.3.2]: For $n = 0$ one sets $H(0) = -\frac{1}{12}$. Otherwise, for $n > 0$, if $h(-n)$ is the usual class number of primitive positive definite quadratic forms with discriminant $-n$, then

$$H(n) = \sum_{d^2 | n} w\left(\frac{n}{d^2}\right) \cdot h\left(-\frac{n}{d^2}\right) \qquad \text{where} \quad w(n) = \begin{cases} \frac{1}{3} & n = 3, \\ \frac{1}{2} & n = 4, \\ 1 & n > 4. \end{cases}$$

In particular, if $-n < -4$ is a fundamental discriminant, $H(n) = h(-n)$.

Now, let $f(\tau)$ be a weakly holomorphic modular form contained in $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$, and assume that the Fourier expansion of f around the cusp at ∞ is given by $\sum_{n>n_0} a(n)q^n$ with integer coefficients $a(n)$, with $a(n) = 0$ unless $n \equiv 0, 1 \pmod 4$. Then, the Borcherds lift $\Psi(\tau,f)$ of f is a meromorphic modular form of weight $a(0)$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ which has an absolutely converging infinite product expansion (a 'Borcherds product') as follows [see 3, Theorem 14.1]:

$$\Psi(\tau,f) = q^{-h} \prod_{n=1}^{\infty} (1-q^n)^{a(n^2)}. \tag{3}$$

Here, h denotes the constant coefficient of the product $f(\tau)\tilde{H}(\tau)$.

Further $\Psi(\tau,f)$ has integer coefficients in its Fourier expansion around infinity, and leading coefficient one. Also, its divisor is supported on a linear combination of Heegner divisors or possibly the cusp. More precisely, if $\tau \in \mathbb{H}$ is a CM-point of discriminant $D < 0$, its multiplicity in $\mathrm{div}(\Psi(\tau,f))$ is given by $\sum_{n>0} a(Dn^2)$.

We note two further important properties:

1. The map $\Psi : f \mapsto \Psi(\tau,f)$ is *multiplicative*, with $\Psi(f+g) = \Psi(f)\Psi(g)$.
2. Any meromorphic modular form for the modular group $\mathrm{SL}_2(\mathbb{Z})$, the divisor of which is a linear combination of Heegner divisors (possibly including the cusp), can be realized as a Borcherds product $\Psi(f)$ for some $f \in \mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$.

By these two properties, the map Ψ becomes an *isomorphism* between the additive group $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$ and the multiplicative group of meromorphic modular forms satisfying the conditions given above for $\Psi(\tau,f)$.

Examples We present some examples following [3, Section 14] and [35, Section 4.2]. A basis for the space $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$ consists of functions $\{f_d\}_{d\equiv 0,3(4)}$ given by

$$f_0(\tau) = 1 + \sum_{n>0} 2q^{n^2}, \qquad f_d(z) = q^{-d} + \sum_{D>0} a(D,d)\, q^D, \quad d = 3, 4, 7, \ldots . \tag{4}$$

Note that $f_0(\tau)$ is simply the Jacobi theta-function $\theta_0(\tau)$. Given f_0 and f_3, further f_d's can be obtained inductively by observing that $f_{d-4}(\tau)j(4\tau)$ has the leading term q^{-d}. (For an explicit formula defining $f_3(\tau) = q^{-3} - 248q + \ldots$, see [35, (4.4), p. 70] or [3, Example 2, p. 202]).

From (3) and (4), one has:

$$\Psi(\tau,f_d) = q^{-H(d)} \prod_{n=1}^{\infty} (1-q^n)^{a(n^2,d)}.$$

with $H(d)$ a Hurwitz class number as defined above. For applications of this formula see [35, Chapter 4].

Now, for two examples:

1. Let $f(z) = 12f_0(\tau) = 12\theta_0(\tau)$. Then, $f(z) = 12 + 24q + 24q^4 + \dots$ and for $\Psi(\tau, f)$, we have

$$\Psi(\tau, f) = q \prod_{n>0} (1 - q^n)^{24} = \Delta(\tau),$$

   which is just the usual modular discriminant function, with divisor supported at the cusp.
2. Consider $g(\tau) = 4f_0(\tau) + f_3(\tau)$. Then, one finds that $\Psi(\tau, g) = E_4(\tau)$, the Eisenstein series of weight 4, since this is the only holomorphic modular form of weight 4 with leading coefficient one. Modulo the action of $\mathrm{SL}_2(\mathbb{Z})$, the divisor $\mathrm{div}(\Psi(g))$ is, of course, given by $\zeta = \frac{1}{2}(1 + \sqrt{-3})$.

3 Orthogonal Groups

We give a brief introduction to the theory of symmetric domains for indefinite orthogonal groups and of orthogonal modular forms. Further details on these topics can be found in a number of places, for instance [6, 23] or [18].

In this section, let $V = V(\mathbb{Q})$ be a quadratic space over $\mathbb{Q}$ of signature $(2, n)$, $n \geq 1$, endowed with a non-degenerate indefinite bilinear form, denoted $(\cdot, \cdot)$. Let $q(x) = \frac{1}{2}(x, x)$ be the attached quadratic form. Further, we will often the notation $x^2 = (x, x)$. Denote by $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$, the real quadratic space obtained from $V(\mathbb{Q})$ by extension of scalars, with $(\cdot, \cdot)$ likewise extended to a real-valued form. For later use, we also introduce the notation $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$ for the complexified space with $(\cdot, \cdot)$ extended to a complex bilinear form.

The orthogonal group of V is denoted $\mathrm{O}(V)$. Considered as an algebraic group defined over $\mathbb{Q}$, its set of real points is given by $\mathrm{O}(V)(\mathbb{R})$, the orthogonal group of $V(\mathbb{R})$. Similarly, the special orthogonal groups of $V(\mathbb{Q})$ and $V(\mathbb{R})$ are denoted by $\mathrm{SO}(V)$ and $\mathrm{SO}(V)(\mathbb{R})$, respectively.

Now, there is an exact sequence with the spin group Spin_V, wherein θ denotes the spinor norm:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V(\mathbb{Q}) \longrightarrow \mathrm{SO}(V) \xrightarrow{\ \theta\ } \mathbb{Q}^\times/\mathbb{Q}^{\times 2}. \tag{5}$$

Looking at the sets of real points, the image of $\mathrm{Spin}_V(\mathbb{R})$ (which, of course is the kernel of θ) is the connected component of the identity in $\mathrm{SO}(V)(\mathbb{R})$. It is referred to as the *spinor kernel* and denoted $\mathrm{O}^+(V)(\mathbb{R})$.

3.1 Models for the Symmetric Domain of SO(V)

Let K_{SO} be a maximal compact (path-connected) subgroup of $\mathrm{SO}(V)(\mathbb{R})$. A symmetric domain for the operation of $\mathrm{SO}(V)(\mathbb{R})$ on $V(\mathbb{R})$ is given by the quotient

$$\mathrm{SO}(V)(\mathbb{R})/K_{SO}.$$

It is isomorphic to the *Grassmannian* of two-dimensional positive definite oriented subspaces, called the Grassmannian model:

$$\mathbb{D} := \{v \subset V(\mathbb{R});\ \dim v = 2, q\ |_v \geq 0, v \text{ oriented}\}.$$

Note that $\mathbb{D}$ has two connected components, they correspond to the two choices of orientation and are stabilized by the spinor-kernel $\mathrm{O}^+(V)(\mathbb{R})$.

Also, each $v \in \mathbb{D}$, through the decomposition $V(\mathbb{R}) = v \oplus v^{\perp}$, fixes an isometry between $V(\mathbb{R})$ and the standard pseudo-Euclidean space $\mathbb{R}^{2,n}$, with quadratic form $q(x) = \frac{1}{2}\left(x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2\right)$. Denoting the special orthogonal groups of $\mathbb{R}^{2,n}$, $\mathbb{R}^{2,0}$ and $\mathbb{R}^{0,n}$ by $\mathrm{SO}(2,n)$, $\mathrm{SO}(2)$ and $\mathrm{SO}(n)$, respectively, we obtain an isomorphism

$$\mathrm{SO}(V)(\mathbb{R})/K_{SO} \simeq \mathrm{SO}(2,n)/\left(\mathrm{SO}(2) \times \mathrm{SO}(n)\right).$$

Remark For the orthogonal group $\mathrm{O}(V)(\mathbb{R})$ a symmetric domain is given by

$$\mathrm{O}(V)(\mathbb{R})/K_O \simeq \mathrm{O}(2,n)/\left(\mathrm{O}(2) \times \mathrm{O}(n)\right),$$

with K_O a maximal compact subgroup. In this case, the Grassmannian model consists simply of the two-dimensional positive-definite subspaces of $V(\mathbb{R})$ (without orientation), and there is only one connected component.

The Projective Cone Model

Let $V(\mathbb{C})$ be the complexified space $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$, as above. Further, denote by $\mathbb{P}V(\mathbb{C})$ the projective space

$$\mathbb{P}V(\mathbb{C}) = \left(V(\mathbb{C}) \setminus \{0\}\right)/\mathbb{C}^{\times},$$

and by $\pi : V(\mathbb{C}) \setminus \{0\} \longrightarrow \mathbb{P}V(\mathbb{C})$ the canonical projection.

The *positive cone model* $\mathcal{K}$ is defined as the following subset of $\mathbb{P}V(\mathbb{C})$:

$$\mathcal{K} := \left\{[Z] \in \mathbb{P}V(\mathbb{C});\ (Z, Z) = 0, \left(Z, \bar{Z}\right) > 0\right\},$$

a complex projective manifold of dimension n with two connected components.

Given $Z \in V(\mathbb{C})$ with $\pi(Z) \in \mathcal{K}$, write Z in the form $Z = X + iY$ with $X, Y \in V(\mathbb{R})$. From the definition of $\mathcal{K}$, we have

$$(X, Y) = 0 \qquad \text{and} \quad X^2 = Y^2 > 0.$$

In other words, if $\pi(Z) \in \mathcal{K}$, the real and the imaginary part of Z constitute an orthogonal, normalized and *oriented* basis for a two-dimensional positive subspace of $V(\mathbb{R})$.

Thus, immediately, we have an isomorphism between the models $\mathbb{D}$ and $\mathcal{K}$ given by a real-analytic map:

$$\begin{aligned} \mathcal{K} &\longrightarrow \quad \mathbb{D} \\ [Z] &\longmapsto \mathbb{R}X + \mathbb{R}Y. \end{aligned}$$

We take note of the following properties of $\mathcal{K}$:

1. The special orthogonal group acts on $\mathcal{K}$, with $g[Z] = [gZ]$ for $g \in \mathrm{SO}(V)(\mathbb{R})$.
2. There is an element of order two which interchanges the two connected components of $\mathcal{K}$, thus acting by complex conjugation. In contrast to this, the action of the spinor kernel $\mathrm{O}^+(V)(\mathbb{R})$ stabilizes the connected components.

The Tube Domain Model

Suppose there are two isotropic vectors $\ell, \ell' \in V(\mathbb{Q})$, with $(\ell, \ell') = 1$. Later on, we will further require there to be an integral lattice $L \subset V$ with $\ell \in L$ and that ℓ' is in contained in the dual lattice L' (see Sect. 3.2).

Consider the subspace $V_0(\mathbb{R}) = V(\mathbb{R}) \cap \ell^\perp \cap \ell'^\perp$. This is a Lorentzian space, as the restriction $(\cdot,\cdot)\ |_{V_0}$ is a quadratic form with signature $(1, n-1)$. The complexification $V_0(\mathbb{C})$ is a complex quadratic space with the extension of $(\cdot,\cdot)\ |_{V_0}$, as usual. Now, the *tube domain model* is defined as the set

$$\mathcal{H} := \{z = x + iy \in V_0(\mathbb{C});\ q(y) > 0\}\,. \tag{6}$$

There is an isomorphism between $\mathcal{H}$ and $\mathcal{K}$ given by

$$\mathcal{H} \xrightarrow{\sim} \mathcal{K} : z \longmapsto \left[Z_\ell(z) := z + \ell' - q(z)\ell\,\right].$$

Whence further,

$$\mathcal{H} \xrightarrow{\sim} \mathbb{D} :\ z \longmapsto w(z) := \mathbb{R}\Re Z_\ell(z) + \mathbb{R}\Im Z_\ell(z).$$

A first non-trivial example for this construction is the following:

Example 3.1 Let $n = 1$. Then, $V_0(\mathbb{C}) = \mathbb{C}$ and we have

$$\mathcal{H} = \left\{z = x + iy \in \mathbb{C}; (\Im z)^2 > 0\right\} \simeq \mathbb{H} \cup \overline{\mathbb{H}}.$$

We remark at this point, that it may sometimes be useful to restrict to one connected component, as the example shows.

The action of $G = \mathrm{SO}(V)(\mathbb{R})$ on $\mathcal{H}$ is described by the following diagram (with $g \in G$):

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{[Z] \mapsto g[Z]} & \mathcal{K} \\ {\scriptstyle z \mapsto [Z_\ell(z)]} \uparrow & & \uparrow {\scriptstyle gz \mapsto [Z_\ell(gz)]} \\ \mathcal{H} & \xrightarrow[z \mapsto gz]{} & \mathcal{H} \end{array}$$

In order for this diagram to commute, we must have

$$\left[gZ_\ell(z)\right] = \left[Z_\ell(gz)\right] \quad (\forall g \in G, \forall z \in \mathcal{H}) .$$

Thus, an automorphy factor $j(g, z) : G \times \mathcal{H} \to \mathbb{C}$ is defined by setting

$$gZ_\ell(z) = j(g, z)Z_\ell(gz) \quad (g \in G, z \in \mathcal{H}).$$

Note that if g is actually contained in $g \in \mathrm{SO}(V_0)(\mathbb{R})$, this automorphy factor is trivial.

Example 3.2 Again, let $n = 1$. Further, let the *level* N be an integer, $N \geq 1$. We consider the space

$$V = \{x \in \mathrm{Mat}(2 \times 2, \mathbb{Q})\ ;\ \mathrm{tr}(x) = 0\} ,$$

with the quadratic form $q(x) = -N \det(x)$ and the bilinear form $(x, y) = +N \,\mathrm{tr}(xy)$. Setting

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \quad \ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{we get} \quad V_0 = \mathbb{Q} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also, clearly, $\ell^2 = \ell'^2 = 0$ and $(\ell, \ell') = 1$. The isomorphisms between the tube domain, the projective cone and the Grassmannian model are given by

$$\begin{array}{ccccc} \mathcal{H} = \mathbb{H} \cup \bar{\mathbb{H}} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathbb{D} \\ z = x + iy & \longmapsto & \left[\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\right] & \longmapsto & \mathbb{R}\Re\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} + \mathbb{R}\Im\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}. \end{array}$$

Now, consider the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices A with $\det(A) = \pm 1$. One can define an isometric action on $V(\mathbb{R})$ by setting

$$(A, X) \mapsto AXA^{adj},$$

where A^{adj} denotes the usual adjoint matrix of A, i.e. with $AA^{adj} = \det(A)E_2$. Thus, there is a homomorphism $\{A \in \mathrm{GL}_2(\mathbb{R});\ \det(A) = \pm 1\} \longrightarrow \mathrm{O}(V)(\mathbb{R})$. Its kernel is a subgroup of order two, as clearly A and $-A$ have the same image. We note that $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}^+(V)(\mathbb{R})$.

On $\mathcal{K}$, the action is given as follows: Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ with $\det A = \pm 1$. Then,

$$A\left[\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\right] = \left[A\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}A^{adj}\right] = \left[\begin{pmatrix} (az+b)(cz+d) & -(az+b)^2 \\ (cz+d)^2 & -(az+b)(cz+d) \end{pmatrix}\right].$$

The automorphy factor thus is given by $j(g, z) = (cz + d)^2$. Also, we see that the action on $\mathcal{H}$ is compatible with the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} \cup \bar{\mathbb{H}}$ through Möbius transformations $z \mapsto \frac{az+b}{cz+d}$.

Example 3.3 Let $n = 2$. A commonly used model for this case is the following

$$V = \mathrm{Mat}(2 \times 2, \mathbb{Q}), \qquad q(X) = -\det(X).$$

After setting

$$\ell = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \ell' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

the subspace V_0 is given by

$$V_0 = \left\{ \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix};\ x_1, x_2 \in \mathbb{Q} \right\}.$$

Now, a subset of $\widetilde{\mathcal{K}}$ of $V(\mathbb{C})$ with $\pi(\widetilde{\mathcal{K}}) = \mathcal{K}$ is given by

$$\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}; z_1, z_2 \in \mathbb{C} \right\}.$$

Hence, for the tube domain, we have

$$\mathcal{H} = \left\{ \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \in \widetilde{\mathcal{K}};\ \Im z_1 \cdot \Im z_2 > 0 \right\} \simeq \left(\mathbb{H} \times \mathbb{H}\right) \cup \left(\overline{\mathbb{H}} \times \overline{\mathbb{H}}\right).$$

We can define an isometric action of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ on $V(\mathbb{R})$ by setting

$$(A, B)X = AXB^{adj} \qquad (A, B \in \mathrm{SL}_2(\mathbb{R}), X \in V(\mathbb{R}))\,,$$

with B^{adj} the adjoint matrix of B. From this we get a homomorphism $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}(V)(\mathbb{R})$, which can be shown to be an isogeny. Its image is the connected component $\mathrm{O}(V)(\mathbb{R})^+$ and the kernel is a subgroup of order 4 [see 18, p. 15]. The action on $\mathcal{H}$ is compatible with the usual action of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C} \times \mathbb{C}$:

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) : \quad (z_1, z_2) \longmapsto \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right).$$

We remark that through $\mathrm{SL}_2(K) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ (see p. 339), one has a homomorphism from $\mathrm{SL}_2(K)$ to $\mathrm{O}(V)(\mathbb{R})^+$. Hence, the symmetric domain of the Hilbert modular group can be considered as a connected component of $\mathcal{H}$.

3.2 Lattices and Modular Groups

In the following, let L be an even integral lattice in V, meaning that $\lambda^2 \in 2\mathbb{Z}$ for all $\lambda \in L$ (i.e. $q(\lambda) \in \mathbb{Z}$ for all λ). Let L' be the dual lattice of L, defined as

$$L' = \{v \in V(\mathbb{R});\ (\lambda, v) \in \mathbb{Z} \quad \text{for all } \lambda \in L\} \supseteq L.$$

The quotient L'/L is called the *discriminant group* of L. Let $\mathrm{SO}(L)$ be the group of isometries of L in $\mathrm{SO}(V)$. By $\Gamma_L \subset \mathrm{SO}(L)$, we denote the *discriminant kernel* of L, the subgroup acting trivially on the discriminant group. By a *modular group* we shall understand a subgroup $\Gamma \subset \mathrm{SO}(L)$ which is commensurable with the discriminant kernel. In particular, a modular group has finite index in $\mathrm{SO}(L)$.

Let us introduce one further notation. As in Sect. 3.1, let ℓ, ℓ' be isotropic vectors with $(\ell, \ell') = 1$ and, further, assume that $\ell \in L$ and $\ell' \in L'$. Then, we denote by L_0 the Lorentzian lattice given by $L \cap \ell^\perp \cap \ell'^\perp$. Note that $V_0(\mathbb{Q}) = L_0 \otimes \mathbb{Q}$, where V_0 is the Lorentzian space used in the construction of the tube domain.

Definition 3.1 Let $\Gamma \subseteq \Gamma_L$ be a modular group. The quotient $X_\Gamma = \Gamma \backslash \mathbb{D}$ is called the (non-compact) *modular variety* associated to Γ. By the theory of Baily-Borel, there is a compactification, which we denote by X_Γ^*. See [18, Chapter II]. For a more general background [see 5, Sections I.4, I.5].

Remark 3.1 The compactified modular variety X_Γ^* gives rise to a Shimura variety [see 7, Section 1.5] (of course, one has to take the non-archimedian places into account for this, too).

Example 3.4 In the setup of Examples 3.1 and 3.2, and using the same notation, the following set L is an even integral lattice and L' its dual:

$$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix};\ a,b,c \in \mathbb{Z} \right\}, \qquad L' = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix};\ a,b,c \in \mathbb{Z} \right\}.$$

The discriminant group L'/L is isomorphic to $\mathbb{Z}/2N\mathbb{Z}$. It is easily verified that $\Gamma_0(N)$ acts trivially on the discriminant group and, in fact, $\Gamma_0(N) = \Gamma_L \cap \mathrm{O}^+(L)$.

In classical language, the modular variety corresponding to the quotient $\Gamma\backslash\mathcal{H}$ is given by the modular curve $X_0(N)$. In particular, for $N = 1$, we have $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^* \simeq X_0(1)^*$, [see 16, Section 2.4, Section 7]. Its points correspond to isogeny classes of elliptic curves (more generally, the points of $X_0(N)$ describe cyclic N-isogenies of elliptic curves).

3.3 Special Cycles

For the following, [cf. 23, Section 2.1.2] or [cf. 6, p. 119]. Let $W \subset V(\mathbb{R})$ be a negative definite one-dimensional subspace. Then, a codimension-one sub-Grassmannian is given by

$$\mathbb{D}_W := \{v \in \mathbb{D};\ v \perp W\} \subset \mathbb{D}.$$

It defines a codimension-one submanifold of the projective cone $\mathcal{K}$, also denoted by $\mathbb{D}_W$ which, in turn, corresponds to a subset of the tube domain. In the following, if w is a negative definite vector, we further simplify notation by setting $\mathbb{D}_w := \mathbb{D}_{\mathbb{R}w}$.

Example 3.5 Taking up the $n = 1$ Examples 3.1, 3.2 and 3.4 set $N = 2$ and consider

$$w = \begin{pmatrix} b/4 & c/2 \\ -a/2 & -b/4 \end{pmatrix}, \quad \text{with} \quad a,b,c \in \mathbb{Z} \quad \text{and} \quad b^2 - 4ac < 0.$$

Then,

$$\begin{aligned} \mathbb{D}_w &= \left\{ z \in \mathbb{H} \cup \bar{\mathbb{H}};\ 2\,\mathrm{tr}\left(w \cdot \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \right) = 0 \right\} \\ &= \left\{ z \in \mathbb{H} \cup \bar{\mathbb{H}};\ az^2 + bz + c = 0 \right\}. \end{aligned}$$

Then, $\mathbb{D}_w$ consists of CM-points in $\mathbb{H} \cup \bar{\mathbb{H}}$. (By a common abuse of notation, $\mathbb{D}_w$ is also used to denote the subset of the tube domain.)

The case where W is defined by a lattice vector is particularly important. As before, let L be an even integral lattice, and L' it dual. We define:

Definition 3.2

1. Assume that λ is a lattice vector with $\lambda \in L'$ and with $q(\lambda) = m, m \in \mathbb{Z}_{<0}$. Then, $\mathbb{D}_\lambda$ is called the *primitive Heegner divisor* attached to λ.
2. Let $\gamma \in L'/L$ be an element of the discriminant group and m a negative integer. The *Heegner divisor of index* (γ, m) is defined as

$$\mathcal{Z}(\gamma, m) := \sum_{\substack{\lambda \in \gamma + L \\ q(\lambda) = m}} \mathbb{D}_\lambda. \tag{7}$$

The sum runs over a system of representatives for $\gamma \in L'/L$.

Note that the sum in (7) is Γ_L-invariant. Thus, $\mathcal{Z}(\gamma, m)$ is, in fact, the pre-image under the canonical projection of a divisor on the modular variety $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$. Usually, the term Heegner divisor is used both for the divisor on X_{Γ_L} and for its pre-image. Also by abuse of notation, both are denoted $\mathcal{Z}(\gamma, m)$.

3.4 Modular Forms

We use the notation established before. Hence, let L be an even integral lattice, and $\Gamma_L \subset \mathrm{SO}(L)$ the discriminant kernel of L. Also assume that the isotropic vectors from Sect. 3.1 are lattice vectors, with $\ell \in L$, $\ell' \in L'$. Then, the tube domain is contained in $V_0(\mathbb{C}) = L_0 \otimes \mathbb{C}$ with $L_0 = L \cap \ell^\perp \cap \ell'^\perp$.

Definition 3.3 Let k be an integer and Γ an orthogonal modular group. A function $f : \mathcal{H} \to \mathbb{C}$ is called a holomorphic modular form of weight k on Γ, if the following conditions are satisfied:

1. $f(\gamma z) = j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma$.
2. f is holomorphic on $\mathcal{H}$.
3. f is holomorphic on the boundary of $\mathcal{H}$.

Note that by the Koecher principle [see 18, Theorem IV.3.6], for holomorphic modular forms, the third condition can be omitted if $n > 2$. More generally, the Koecher principle is valid, if the Witt-rank of $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. the dimension of a maximal totally isotropic subspace, is less than n. (For example, this is the case for Hilbert modular forms, cf. p. 339.)
Meromorphic (etc.) modular forms are defined similarly, with 2. and 3. replaced by suitable conditions on $\mathcal{H}$ and on the boundary components. Also, the definition can easily be extended to accommodate for half-integral weights and multiplier systems.

We will not say much about the properties of modular forms for orthogonal groups, but let us at least mention that they admit Fourier expansions:

If f is a modular form for a modular group Γ, as in Definition 3.3, there is a lattice M in V_0 such that $f(z+\mu) = f(z)$ for all $\mu \in M$. For example, if $\Gamma = \Gamma_L$, then $M = L_0$. Thus, f has a Fourier expansion of the form

$$f(z) = \sum_{\mu \in M'} a(\mu) e\left((\mu, z)\right).$$

Due to the Koecher principle or, if necessary, by condition 3., μ's with $a(\mu) \neq 0$ satisfy a positivity condition [see 18, Section IV.3].

4 The Singular Theta Lift

For this section, recall our convention that $\tau = u + iv$ denote a point in the complex upper half-plane $\mathbb{H}$. In the following, $\sqrt{\tau} = \tau^{1/2}$ is the principal branch of the complex square root, with $\arg(\sqrt{\tau}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Further, z shall denote a point in $\mathcal{H}$ and $w(z)$ the attached positive definite subspace in $\mathbb{D}$.

We would like to mention some general references, which, among them, cover most of this section: Beside the original works of Borcherds [4] and of Bruinier [6], these are [36] and the lecture notes [7].

4.1 The Weil Representation

Consider the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$, the double cover of $\mathrm{SL}_2(\mathbb{R})$. It can be written as the set of pairs $(M, \phi(\tau))$, with $M \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ a holomorphic square root of $c\tau + d$. In particular, $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the elements

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \text{and} \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

The center of $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$Z = S^2 = (TS)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

If Γ is an elliptic modular group, we denote the pre-image under $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ by $\widetilde{\Gamma}$, i.e. $\widetilde{\Gamma_1} = \mathrm{Mp}_2(\mathbb{Z})$, $\widetilde{\Gamma_0}(N)$ etc.

Now, there is a representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$, defined through the action of the above generators on the basis elements $\mathfrak{e}_\mu$:

$$\rho_L(T)\mathfrak{e}_\mu = e\left(q(\mu)\right)\mathfrak{e}_\mu,$$
$$\rho_L(S)\mathfrak{e}_\mu = \frac{\sqrt{i}^{n-2}}{\sqrt{|L'/L|}}\sum_{\nu\in L'/L} e\left(-(\mu,\nu)\right)\mathfrak{e}_\nu.$$

Also, the action of Z is given by $\rho_L(Z)\mathfrak{e}_\mu = i^{n-2}\mathfrak{e}_\mu$.

Essentially, ρ_L is the Weil representation, for more details we refer to Shintani [see 38] and, for a description using the language of adeles, to [7, Sections 3.1, A].

Remark 4.1 If n is even, the representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ factors through a representation of $\mathrm{SL}_2(\mathbb{Z})$. Also, the representation factors over the finite group $\mathrm{Mp}_2(\mathbb{Z}/N_L\mathbb{Z})$, where N_L is the level L, defined as the the smallest positive integer N satisfying $Nq(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$; if n is even, ρ_L factors over $\mathrm{SL}_2(\mathbb{Z}/N_L\mathbb{Z})$.

We denote the standard hermitian scalar product on $\mathbb{C}[L'/L]$ by $\langle\cdot,\cdot\rangle$, i.e.

$$\Big\langle \sum_{\mu\in L'/L} a_\mu\mathfrak{e}_\mu, \sum b_\mu\mathfrak{e}_\mu\Big\rangle = \sum_{\mu\in L'/L} a_\mu\overline{b_\mu}. \tag{8}$$

With this, for $\mu,\nu \in L'/L$ and $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$, the matrix coefficient $\rho_{\mu\nu}(M,\phi)$ of the representation ρ_L is given by

$$\rho_{\mu\nu}(M,\phi) = \big\langle \rho_L(M,\phi)\mathfrak{e}_\mu, \mathfrak{e}_\nu\big\rangle.$$

Finally, the dual representation ρ_L^* for $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ given in terms of its matrix coefficients is the complex conjugate of the matrix $\big(\rho_{\mu\nu}(M,\phi)\big)_{\mu,\nu\in L'/L}$.

We briefly recall the definitions of vector-valued modular forms for the representation ρ_L, more details can be found in the course notes of Claudia Alfes-Neumann [1].

Definition 4.1 Let $k \in \frac{1}{2}\mathbb{Z}$ be a half-integer. A smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ which transforms under ρ_L according to

$$(M,\phi)f(\tau) = \phi(\tau)^{2k}\rho_L(M,\phi)f(M\tau), \quad ((M,\phi) \in \mathrm{Mp}_2(\mathbb{Z}))$$

is called

1. a *weakly holomorphic modular form*, if f is holomorphic on $\mathbb{H}$ and meromorphic at the cusp ∞,
2. a *holomorphic modular form*, if f is holomorphic on $\mathbb{H}$ and at the cusp, Further, f is called a *cusp form* if f is holomorphic and vanishing at the cusp.

We denote the by $\mathrm{S}_{k,\rho_L} \subset \mathrm{M}_{k,\rho_L} \subset \mathrm{M}^!_{k,\rho_L}$ the spaces of cusp forms, holomorphic modular forms and weakly holomorphic modular forms transforming under the Weil representation, respectively.
We remark that, similarly, vector valued modular forms can be defined for the dual representation ρ_L^*, i.e S_{k,ρ_L^*}, M_{k,ρ_L^*} and $\mathrm{M}^!_{k,\rho_L^*}$.

Next, following [9, Section 3] we introduce harmonic Maass forms.

Definition 4.2
Let $k \in \frac{1}{2}\mathbb{Z}$ A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic Maass form (or harmonic weak Maass form) with representation ρ_L for $\mathrm{Mp}_2(\mathbb{Z})$ if

1. $(M,\phi)f(\tau) = \phi(\tau)^{2k}\rho_L(M,\phi)f(M\tau)$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$,
2. There is a $C > 0$ such that $f(\tau) = \mathrm{O}\big(e^{Cv}\big)$ as $v \to \infty$ (uniformly in u),
3. f is annihilated by the weight-k Laplace operator, $\Delta_k f(\tau) = 0$, with

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + iku\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

We denote the space of harmonic Maass forms by H_{k,ρ_L}.

Further we denote by H^+_{k,ρ_L} the subspace of harmonic Maass forms f, which additionally to 1.–3. satisfy the following condition: The image of f under the ξ-operator

$$\xi_k := 2iv^k\frac{\partial}{\partial \bar{z}}$$

is a cusp form for the dual representation ρ_L^* with $\xi_k(f)(\tau) \in \mathrm{S}_{2-k,\rho_L^*}$.
Note that the component functions of an elliptic modular form $f = \sum_\mu f_\mu \mathfrak{e}_\mu$ are scalar valued elliptic modular forms of the appropriate type (i.e. weakly holomorphic, holomorphic or cuspidal) and the same weight for (at least) the principal congruence subgroup $\Gamma(N_L)$, where the level N_L is determined as in Remark 4.1. The same applies for harmonic Maass forms.

Due to invariance under $T \in \mathrm{Mp}_2(\mathbb{Z})$, a weakly holomorphic modular form f with representation ρ_L, admits a Fourier expansion around the cusp ∞ of the following form:

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\mu) \\ m \gg -\infty}} c(\mu,m)q^m \mathfrak{e}_\mu, \tag{9}$$

with only finitely many $m < 0$ for which $c(\mu,m) \neq 0$. If f is a holomorphic modular form, then $c(\mu,m) \neq 0$ only for $m \geq 0$, and for a cusp form, $c(\mu,m) \neq 0$ only for $m > 0$.

The Fourier expansion of a harmonic Maass form $f \in \mathrm{H}^+_{k,\rho_L}$ ($k \neq 1$) consists of a holomorphic part f_+ similar to (9) and a non-holomorphic part f_- involving certain special functions, see for example [1, Section 3]. We will need the Fourier expansion only in the case where $k < 1$, for which it takes the following form:

$$\begin{aligned} f(\tau) &= f^{+}(\tau) + f^{-}(\tau) \\ &= \sum_{\mu \in L'/L} \sum_{m \gg -\infty} c^{+}(m, \mu) q^{m} \mathfrak{e}_{\mu} + \sum_{\mu \in L'/L} \sum_{m<0} c^{-}(m, \mu) \Gamma\left(1-k, 4\pi |m| v\right) q^{m} \mathfrak{e}_{\mu}, \end{aligned} \tag{10}$$

with the incomplete Gamma function $\Gamma(a, x) = \int_x^\infty e^{-r} r^{a-1} dr$ [cf. 34, 8.2.2].

Remark 4.2 As Bruinier and Funke have shown [see 9] the condition $\xi(f)(\tau) \in \mathrm{S}_{2-k,\rho_L^*}$ for $f \in \mathrm{H}^{+}_{k,\rho_L}$ has immediate consequences for the growth behavior of f: Denote by $P(f)$ the principal part of f, i.e. the Fourier polynomial given by

$$P(f)(\tau) := \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Z} + q(\mu) \\ 0 > m \gg -\infty}} c(\mu, m) q^{m} \mathfrak{e}_{\mu}.$$

Then, for $f \in \mathrm{H}^{+}_{k,\rho_L}$, $f - P(f)$ decays exponentially as $v \to \infty$. For the Fourier expansion given in (10) (for $k \neq 1$), this is can also be seen from the asymptotic behavior of the incomplete Gamma function.

4.2 Siegel Theta Functions

In this section we want to introduce the Siegel theta-function attached to the lattice L, integrating against which will yield the theta-lift. For a concise yet very readable treatment in the language of representation theory see [30].

Definite Theta Functions To begin, we start with a simple example for a theta-function attached to a lattice. For this, let M be a positive definite even lattice, of rank $l \geq 1$ and endowed with a quadratic form $q(\cdot)$. Then, generalizing the well known Jacobi theta-function $\theta_0(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$, one sets, if M is unimodular

$$\Theta_M(\tau) = \sum_{\lambda \in M} q^{\frac{1}{2}\lambda^2} = \sum_{\lambda \in M} e\big(q(\lambda)\,\tau\big).$$

Otherwise, if M'/M is non-trivial, one sets

$$\Theta_M(\tau) = \sum_{\mu \in M'/M} \sum_{\lambda \in \mu + M} e\big(q(\lambda)\,\tau\big) \mathfrak{e}_{\mu}.$$

Clearly, in both cases the series converges absolutely and uniformly and hence defines a holomorphic function on $\mathbb{H}$. Using Poisson summation, it is fairly straightforward to show that $\Theta_M(\tau)$ transforms as a modular form of weight $l/2$.

If, contrastingly, the lattice is indefinite, to assure absolute convergence of the theta-series, we have to replace $q(\lambda)$ by a majorant.

The Siegel Theta Function Thus, let L be an indefinite even lattice, as in Sect. 3, with $L \subset V$ and with $V = L \otimes \mathbb{Q}$ an indefinite quadratic space of signature $(2, n)$. The quadratic form is again denoted $q(\cdot)$. We will now attach an absolutely convergent theta-series to L and at the same time obtain a function on $\mathbb{H} \times \mathbb{D}$.

Recall that $\mathbb{D}$ consists of maximal positive definite (oriented) subspaces. Given a maximal positive definite subspace $w \subset V(\mathbb{R})$, we decompose $V = w \oplus w^{\perp}$. Naturally, $w^{\perp}$ is negative definite. Writing $a \in V(\mathbb{R})$ as $a_w + a_{w^{\perp}}$, the majorant $q_w(a)$ is given by $q(a_w) - q(a_{w^{\perp}})$.

Further, recall that to every $z \in \mathcal{H}$, we can associate a positive definite subspace $w(z) \in \mathbb{D}$. To simplify notation, we write a_z and $a_{z^{\perp}}$ for the projections $a_{w(z)}$ and $a_{w(z)^{\perp}}$, respectively. Now, for $\tau = u + iv \in \mathbb{H}$, we define

$$\frac{1}{2}(x,x)_{z,\tau} := q(x)\,u + q_{w(z)}(x)v = q(x_z)\,\tau + q\big(x_{z^{\perp}}\big)\,\bar{\tau}. \qquad (x \in V(\mathbb{R}))\,.$$

Then, for every $z \in \mathcal{H}$, the following function, called the *Gaussian*, is rapidly decreasing,

$$\phi(x,z,\tau) := e\left(\tfrac{1}{2}(x,x)_{z,\tau}\right), \tag{11}$$

in other words, ϕ is a Schwartz function on $V(\mathbb{R})$.

This leads to the following definition of a theta-function attached to L:

Definition 4.3 The Siegel theta-function $\Theta_L(\tau, z) : \mathbb{H} \times \mathbb{D} \to \mathbb{C}[L'/L]$ is given by

$$\Theta_L(\tau,z) = \sum_{\mu \in L'/L} \theta_\mu(\tau,z)\mathfrak{e}_\mu, \tag{12}$$

with component functions

$$\theta_\mu(\tau,z) = \sum_{\lambda \in \mu + L} \phi(\lambda,z,\tau) = \sum_{\lambda \in \mu + L} e\left(\tau q(\lambda_z) + \bar{\tau} q\big(\lambda_{z^{\perp}}\big)\right). \tag{13}$$

Due to the rapid decay of the Gaussian, the series defining $\Theta_L(\tau, z)$ is absolutely convergent. Its transformation behavior is given by the following theorem, which can be proved using Poisson summation [see 4, Theorem 4.1].

Theorem 4.1 *For* $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi(\tau)\right) \in \mathrm{Mp}_2(\mathbb{Z})$*, we have*

$$\Theta_L(\gamma\tau, z) = \phi(\tau)^2\overline{\phi(\tau)}^n \rho_L(\gamma)\Theta_L(\tau, z).$$

Also, $\Theta_L(\tau, z)$ *is invariant under* $\mathrm{SO}(V)(\mathbb{R})$.

It is worth mentioning that the theorem in [4] is much more general. For example, Borcherds allows for arbitrary signature (p, n) and also covers Siegel theta-functions with a fairly general harmonic polynomial as an additional factor together with $\phi(x, z, \tau)$.

The Theta Integral Now, let $f \in \mathrm{M}_{k,\rho_L}$ be a modular form transforming under the Weil representation ρ_L, and consider the theta-integral given by

$$\int_{\mathcal{F}} \langle f(\tau), \Theta_L(\tau, z)\rangle v\, d\mu \qquad (\text{with} \quad d\mu = \frac{du\, dv}{v^2}). \tag{14}$$

Here, $\mathcal{F}$ denotes a fundamental domain for the operation of $\mathrm{Mp}_2(\mathbb{Z})$, while $\langle\cdot,\cdot\rangle$ is the hermitian scalar product on $\mathbb{C}[L'/L]$ from (8). Note that $d\mu$ is the left-invariant Haar measure for the operation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{H}$.

By Theorem 4.1, if f has weight $k = 1 - n/2$, the expression under the integral is invariant under $\mathrm{Mp}_2(\mathbb{Z})$. Thus, we may expect to evaluate the integral by using unfolding.

However, there are two problems:

1. The space $\mathrm{M}_{1-\frac{n}{2},\rho_L}$ is often trivial. Indeed, if $n > 2$, then $\mathrm{M}_{1-\frac{n}{2},\rho_L} = \{0\}$.
2. A possible solution is to extend to $\mathrm{M}^!_{1-\frac{n}{2},\rho_L}$, allowing f to be weakly holomorphic. However this entails a new difficulty: The integral in (14) no longer converges. (Hence the name 'singular' theta-lift.)

Thus, if we admit weakly holomorphic modular form contained in $\mathrm{M}^!_{1-\frac{n}{2},\rho_L}$ as input functions, which is desirable, we have to replace the theta-integral in (14) by a suitably regularized integral. This is what we will do in the next section.

Remark 4.3 To avoid these difficulties, one can also use a more refined kernel function instead of the Gaussian. Most commonly, one introduces a homogeneous polynomial as a further factor, the degree of which then enters into the transformation behavior of the theta-function.

An example for this is the following kernel function, φ_r ($r \in \mathbb{N}$) defined as

$$\varphi_r(\lambda, z, \tau) := \frac{(\lambda, \overline{w(z)})}{(y, y)^r}\phi(\lambda, z, \tau) \quad (z \in \mathcal{H}).$$

With this kernel function, the theta-integral is $\mathrm{Mp}_2(\mathbb{Z})$-invariant for input functions of weight $k = 1 - \frac{n}{2} + r$. Indeed, for suitable $r > 1$, the space M_{k,ρ_L} is non-trivial. Also, in this case the theta-integral converges without need for any regularization. The kernel function φ_r leads to the Shintani-Oda-Gritsenko lifting, see [38]

4.3 The Regularized Theta Lift

We set $k = 1 - \frac{n}{2}$. Somewhat more generally, following Bruinier and Funke [9], we extend $\mathrm{M}^!_{k,\rho_L}$ to H^+_{k,ρ_L}, the space of harmonic Maass forms introduced in Sect. 4.1. Recall from (10) the Fourier expansion for a harmonic Maass form $f \in \mathrm{H}^+_{k,\rho_L}$ (note that $k < 1$):

$$\begin{aligned} f(\tau) &= \sum_{\mu \in L'/L} \left(f^+_\mu(\tau) + f^-_\mu(\tau)\right) \mathfrak{e}_\mu \\ &= \Big[\sum_{m \gg -\infty} c^+(m,\mu) q^m + \sum_{m<0} c^-(m,\mu) \Gamma\left(1-k, 4\pi |m| v\right) q^m \Big] \mathfrak{e}_\mu, \end{aligned} \tag{15}$$

where we denote by f^+_μ and f^-_μ the components of the holomorphic part f^+ and the non-holomorphic part f^- of f, respectively. Recall that each component function $f_\mu = f^+_\mu + f^-_\mu$ is a scalar valued harmonic Maass form.

Note the asymptotic behavior of the non-holomorphic part for $v \to \infty$: Since the incomplete Γ-functions (or, more generally the M-Whittaker functions they are related to) are of rapid decay [see 34, Sections 8.11, 8.12 and 13.21], f^- decays rapidly, too. (Also, see Remark 4.2.)

Thus, for the question of convergence or non-convergence of the theta-integral in (14), only the f^+ part plays a role. So, to formulate the necessary regularization recipe, we look at the integral $\int_{\mathcal{F}} \langle f^+, \Theta_L \rangle v \, d\mu$.

The regularization we describe is due to Harvey, Moore [22] and Borcherds [4], [see also 6, Section 2.2]. For $t \in \mathbb{R}_{>0}$, define the truncated fundamental domain $\mathcal{F}_t$ as follows

$$\mathcal{F}_t := \mathcal{F} \cap \{\tau \in \mathbb{H}; \Im\tau \le t\} = \left\{\tau = u + iv; |\tau| > 1, -\tfrac{1}{2} < u < \tfrac{1}{2}, 0 < v \le t\right\}.$$

Clearly $\mathcal{F}_t$ is compact. Hence, since Θ_L and f^+ are holomorphic as functions of τ on $\mathbb{H}$, the definite integral

$$\int_{\mathcal{F}_t} \langle f^+, \Theta_L \rangle v \, d\mu$$

is well-defined. One can take the limit $t \to \infty$ and, providing it exists, define the regularized integral accordingly.

Actually, the constant coefficient $c^+(0,0)$ still poses a problem, as we will see presently. But, excluding this coefficient, the following regularization can be used.

Definition 4.4 (Regularization 1) If the constant term $c^+(0,0)$ in the Fourier expansion of f vanishes, the regularized integral is defined as

$$\int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle v \, d\mu := \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v \, d\mu.$$

We note that since the integral is definite, we are allowed to interchange the order of integration.

To see why it is necessary to require $c^+(0,0) = 0$, consider the Fourier expansions of $f^+(\tau)$ and of $\overline{\Theta}_L(\tau)$. (Note that the expression under the integral is periodic with period length 1):

$$\begin{aligned} f^+(\tau) &= \sum_{\mu \in L'/L} \sum_m c^+(\mu, m) e(m\tau) \mathfrak{e}_\mu, \\ \overline{\Theta}_L(\tau) &= \sum_{\mu \in L'/L} \sum_{\lambda \in \mu + L} e^{-4\pi v q(\lambda_z)} e(-q(\lambda)\, \tau) \mathfrak{e}_\mu. \end{aligned} \tag{16}$$

Due to absolute convergence, we many integrate term by term. Thus,

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f^+, \Theta_L \rangle(\tau) du &= \sum_{\mu \in L'/L} \sum_m c^+(\mu, m) \sum_{\lambda \in \mu + L} e^{-4\pi q(\lambda_z) v} \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(u(m - q(\lambda))\right) du \\ &= \sum_{\mu \in L'/L} \sum_{\lambda \in \mu + L} e^{-4\pi q(\lambda_z) v}\, c^+(\mu, q(\lambda)). \end{aligned} \tag{17}$$

Hence, the contribution of the constant term to the integral over $\mathcal{F}_t$ is given by

$$c^+(0,0) \lim_{t \to \infty} \int_{\mathcal{F}_t} v \frac{dv du}{v^2} = c^+(0,0) \lim_{t \to \infty} \int_{v=0}^{t} \frac{dv}{v} = c^+(0,0) \left[\int_{v=0}^{1} \frac{dv}{v} + \lim_{t \to \infty} \int_1^t \frac{dv}{v} \right].$$

Clearly, on the right hand side, the first integral is divergent, as is the limit of the second integral.

Thus, a slightly more elaborate regularization recipe is needed here, which of course also works if $c^+(0,0) = 0$:

Definition 4.5 (Regularization 2) If for $s \in \mathbb{C}$ with $\Re(s) \gg 0$ the limit

$$g(s) = \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v^{1-s} d\mu$$

exists and has a meromorphic continuation on $\mathbb{C}$, then the regularized integral is defined as the constant term of the Laurent expansion of $g(s)$ at[1] $s = 0$, denoted $\mathcal{C}_{s=0}[g(s)]$:

$$\int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle v\, d\mu := \mathcal{C}_{s=0} \left[\lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v^{1-s}\, d\mu \right].$$

[1] If 0 happens to be a pole, yet another, slight variation of this recipe is needed, see [6].

The regularized lift will give us a smooth function $\Phi(z,f)$ which still has some singularities. Beside determining their location, we also want to describe the behavior of $\Phi(z,f)$ around these singularities. For the following it is somewhat more natural to consider the regularized integral as a function on $\mathbb{D}$, rather than on $\mathcal{H}$.

We define the *type of a singularity* as follows:

Definition 4.6 Let $U \subset \mathbb{D}$ be an open subset and f, g functions on a dense open subset of U. We say that f has a singularity of type g, if $f - g$ can be continued to a real analytic function on U. In this case, we write $f \simeq_U g$.

Let $f \in \mathrm{H}^+_{k,\rho_L}$ be a harmonic Maass form with Fourier expansion as in (15). Further, assume that $c^+(\mu, m) \in \mathbb{Z}$ for all $m < 0$. We define a Heegner divisor associated to f by setting

$$\mathcal{Z}(f) := \sum_{\mu \in L'/L} \sum_{m<0} c^+(\mu, m)\mathcal{Z}(\mu, m), \tag{18}$$

where the $\mathcal{Z}(\mu, m)$ are the Heegner divisors of index (μ, m) from Definition 3.2.

Theorem 4.2 (Borcherds-Bruinier, cf. [4, Theorem 6.2], [6, Theorem 2.12]) *The function* $\Phi(z,f)$ *given by the regularized integral*

$$\Phi(z,f) = \int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle \, v \, d\mu, \tag{19}$$

considered as a function on $\mathbb{D}$, *is real-analytic on* $\mathbb{D} \setminus \operatorname{sup}(-2\mathcal{Z}(f))$ *and takes singularities of logarithmic type along the divisor* $-2\mathcal{Z}(f)$ *(i.e. for every* $w \in \mathbb{D}$*, there is a neighborhood* $w \in U \subset \mathbb{D}$ *and a local equation* $\mathrm{Div}(g) = -2\mathcal{Z}(f)\mid_U$ *with a meromorphic function* g*, such that* $\Phi \simeq_U \log|g|$*).*

We give a brief sketch of the calculations involved in the proof, following [7]:

Proof To determine the divisor of $\Phi(z,f)$, we need to work out the integral up to smooth functions. First, split up the integral into two parts, one over $z \in \mathcal{F}$ with $\Im z \leq 1$ and one over z with $\Im z > 1$.

$$\Phi(z,f) = \int_{\mathcal{F}_1}^{reg} \langle f, \Theta_L \rangle v \, d\mu + \int_{\mathcal{F}_{>1}}^{reg} \langle f, \Theta_L \rangle v \, d\mu.$$

Clearly, the first integral is smooth, and it suffices to consider the second integral. Further, due to the rapid decay of the non holomorphic part, only the contribution of f^+ matters here. Thus, consider

$$\lim_{t\to\infty} \int_{v=1}^{t} \int_{u=-\frac{1}{2}}^{\frac{1}{2}} \langle f^+, \Theta_L \rangle v^{1-s} d\mu. \tag{20}$$

Since the expression under the integral is periodic in the indeterminate τ, we can insert the Fourier expansion of f^+ and $\overline{\Theta_L}$ and carry out integration over u, as above.

With (17) we get:

$$\lim_{t\to\infty}\int_{v=1}^{t}\sum_{\lambda\in L'} e^{-4\pi q(\lambda_z)v}c^+(\lambda,q(\lambda))\frac{dv}{v^{s+1}}.$$

We now split the sum into three parts: First, the sum over $\lambda\neq 0$ with $q(\lambda)\geq 0$, second the term for $\lambda=0$ and third the sum over λ with $q(\lambda)<0$. Also, since the integral is definite, we can interchange the order of integration. Absolute convergence allows the limit to be taken term-wise.

So, first consider

$$\int_{v=1}^{t}\sum_{\substack{0\neq\lambda\in L'\\ q(\lambda)\geq 0}} e^{-4\pi q(\lambda_z)v}c^+(\lambda,q(\lambda))\frac{dv}{v^{s+1}}. \tag{21}$$

We will estimate the growth of the sum under the integral: Applying the Hecke estimate [see e.g. 12, I. Proposition 8] to the Fourier coefficients $c^+(\lambda,q(\lambda))$, we see that their asymptotic behavior as $q(\lambda)$ increases is $\mathrm{O}(e^{c\sqrt{q(\lambda)}})$ with some constant $c>0$. We rewrite the argument of the exponential as follows:

$$-4\pi q(\lambda_z)\,v = 2\pi\left[q\big(\lambda_{z^\perp}\big)-q(\lambda_z)\right]v - 2\pi q(\lambda)\,v.$$

Note that the first term is a negative define quadratic form. It follows that the asymptotic behavior of $c^+(\lambda,q(\lambda))e^{-4\pi q(\lambda_z)v}$ is given by $\mathrm{O}(e^{-q(\lambda)})$. Hence, the integral (21) contributes only a smooth function.

Now, for the term with $\lambda=0$: We get the integral expression

$$c^+(0,0)\int_{v=1}^{t}\frac{dv}{v^s},$$

of which, after regularization, only a constant remains.

Finally, from the third sum, with $q(\lambda)<0$, we get the following contribution to the regularized integral

$$\sum_{\substack{\lambda\in L'\\ q(\lambda)\leq 0}} c^+(\lambda,q(\lambda))\,\mathcal{C}_{s=0}\left[\int_{v=1}^{\infty}e^{-4\pi q(\lambda_z)v}\frac{dv}{v^{1+s}}\right].$$

We can express this in terms of the incomplete Γ-function, $\Gamma(a,x)=\int_x^\infty e^{-r}r^{a-1}dr$ [cf. 34, 8.2.2], with $a=0$ and $x=4\pi|q(\lambda_z)|$. Thus, after regularization and *up to smooth functions*, $\Phi(z,f)$ is given by

$$\Phi(z,f)\simeq\sum_{\substack{\lambda\in L'\\ q(\lambda)\leq 0}} c^+(\lambda,q(\lambda))\Gamma\left(0,4\pi|q(\lambda_z)|\right).$$

Now, we study the behavior of $\Phi(z,f)$ locally around a given point $w(z_0) \in \mathbb{D}$. From the definition of $\Gamma(a,x)$, by partial integration,

$$\Gamma(0,x) = -[e^{-r}\log(r)]_x^\infty + \int_x^\infty e^{-r}\log(r)\,dr.$$

one can see that near $x = 0$, the function $\Gamma(0,x)$ behaves like $-\log(x)$ and is otherwise smooth. Thus, we write the above sum as follows:

$$\sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu,m)\Bigg[\sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\not\perp z_0}}\Gamma\left(0,4\pi|\lambda_z^2|\right) + \sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\perp z_0}}\Gamma\left(0,4\pi|\lambda_z^2|\right)\Bigg].$$

The first, sum over all λ with $\lambda \not\perp w(z_0)$ contributes a function which is smooth on a small neighborhood of $w(z_0)$. This can be shown using reduction theory. The remaining λ with $\lambda \perp w(z_0)$ generate a positive definite sublattice, and thus the second sum is finite. Hence, locally near $w(z_0)$ and up to smooth functions $\Phi(z,f)$ is given by the finite sum

$$-\sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu,m)\sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\perp z_0}}\log|\lambda_z^2|.$$

We conclude that the divisor of $\Phi(z,f)$ is given by a *locally finite* sum of the primitive Heegner divisors $\mathbb{D}_\lambda$, and get $\mathrm{div}(\Phi) = -2\mathcal{Z}(f)$. Also, clearly, the singularities are of logarithmic type, as claimed. This completes the proof. □

Remark 4.4 Beside its singularities, the function $\Phi(z,f)$ has a number of further remarkable properties. Just to mention a few:

1. Bruinier showed that $\Phi(z,f)$ is an eigenfunction of the $\mathrm{SO}(V)(\mathbb{R})$-invariant Laplacian [6, Theorem 4.6, 4.7]. He further used this result to construct a lifting into the cohomology [see 6, Chapter 5].
2. Also, $\Phi(z,f)$ can be used to define a smooth $(1,1)$-form on the modular variety X_Γ, which satisfies a current equation. Naturally, this leads to various geometric applications for example in Arakelov theory [see 11].

   In particular, in the special case where f is a weakly holomorphic modular form, this current equation implies that

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left[\Phi(z,f) + c^+(0,0)\log|y|^2\right] = 0 \quad (0 \le i,j \le n).$$

   This means that $\Phi(z,f)$ is pluriharmonic on $\mathcal{H} \setminus \mathcal{Z}(f)$.

The Evaluation of the Integral The calculations involved in the evaluation of the regularized theta integral are quite involved and too extensive to reproduce here. But, at least, we want to outline some of the main points. Let f be a weakly holomorphic modular form with $f \in \mathrm{M}^!_{k,\rho_L}$, $k = 1 - n/2$.

Borcherds observed that the Siegel theta-function Θ_L can be expressed through the Siegel theta-function of the smaller, Lorentzian lattice $L_0 = L \cap \ell^\perp \cap \ell'^\perp$. Likewise, using the Fourier expansion of the input function f one can define a vector valued modular form $f_{L_0} : \mathbb{C} \to \mathbb{C}[L_0'/L_0]$ transforming under the Weil representation of the lattice L_0 [see 4, Theorems 5.2 and 5.3]. Using partial Poisson summation, Borcherds then decomposed the regularized theta-integral [4, Theorem 7.1], with one part given by, essentially, the regularized theta-lift for signature $(1, n-1)$ of f_{L_0}, with Θ_{L_0} as the theta-function, and further terms which are evaluated by unfolding.

The evaluation of the Lorentzian part (actually, in there, there is again a contribution of a positive definite lattice contained in L_0, which however evaluates to a constant) gives piecewise polynomial functions. The singularities can be evaluated quite similarly to the proof of Theorem 4.2, except that in the end their type is not logarithmic. They also lie along Heegner divisors, which dissect the symmetric domain of the Lorentzian orthogonal group $\mathrm{SO}(1, n)$ into connected components. On each connected component, many terms cancel, leaving only piecewise linear functions, which Borcherds gathers into a term involving a Weyl vector [cf. 4, Section 10]. This is where the Weyl chambers—connected components of $\mathcal{H}$ with wall-crossing occurring between them—and the Weyl vector terms in Theorem 4.4 below originate from: in the contribution of the Lorentzian part.

4.4 Borcherds Products

Our main references for the following are [4, Section 13] and [6, Section 3.2]. In this section, we assume the signature of V to be $(2, n)$ with $n \geq 2$. Further, let f denote a weakly holomorphic modular form with $f \in \mathrm{M}^!_{k,\rho_L}$, with $k = 1 - \frac{n}{2}$.

We define $\Psi(z, f)$ as a meromorphic function on $\mathcal{H}$ with $\mathrm{div}(\Psi) = \mathcal{Z}(f)$ by setting

$$\Phi(z, f) + c^+(0, 0) \log|y|^2 = -2 \log|\Psi(z, f)|.$$

To see why this works, we note that the multiplicative Cousin problem is universally solvable on $\mathcal{H}$ [see 19, Section V.2], since the components of $\mathcal{H}$ are convex. Hence there exists a meromorphic function g with divisor $\mathcal{Z}(f)$; for this, one has to show that $\Phi(z, f)$ is pluriharmonic i.e. all mixed second derivatives $\partial_i \bar{\partial}_j \Phi$ $(1 \leq i, j \leq n-1)$ vanish (see Remark 4.4).

Then, $\Phi - \log|g|$ extends to a pluriharmonic real analytic function on $\mathcal{H}$. Further, this implies that there is a holomorphic function h with $\Re(h) = \Psi - \log|g|$ [see 21, Section IX.C], and one can set $\Psi = e^h g$. (For a detailed version of this argument [see 6, p. 82ff] or [cf. also 8, Lemma 6.6]).

Since $\Phi(z,f)$ is invariant, $\Psi(z,f)$ transforms under Γ_L according to

$$\Psi(\gamma z,f) = \sigma(\gamma) \cdot j(\gamma,z)^{c^+(0,0)/2}\Psi(z,f),$$

with some multiplier system σ. It can be shown that σ has at most finite order, using a result of Margulis (for $n > 2$). (For $n = 2$ an embedding trick has to be employed first.) See [4, Lemma 13.1], [6, Section 3.4]. Thus, Ψ is a meromorphic modular form of weight $c^+(0,0)/2$.

Now we are ready to formulate Borcherds' celebrated result [4, Theorem 13.3]:

Theorem 4.3 (Borcherds) *Let $f \in \mathrm{M}^!_{k,\rho_L}$ be a weakly holomorphic modular form with Fourier expansion $f = \sum_{\mu.m} c^+(\mu,m)q^n$, satisfying*[2] *$c^+(\mu,m) \in \mathbb{Z}$ for all $m \le 0$. Then, there is a meromorphic function $\Psi(z,f)$ on $\mathcal{H}$ with the following properties:*

i) $\Psi(z,f)$ is a modular form of weight $c^+(0,0)/2$ with respect to Γ_L with a multiplier system of (at most) finite order.
ii) The divisor of $\Psi(z,f)$ is given by

$$\mathrm{Div}(\Psi(z,f)) = \mathcal{Z}(f),$$

$$\textit{where} \quad \mathcal{Z}(f) = \sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu.m)\mathcal{Z}(\mu.m)$$

is the Heegner divisor associated to f, see (18).
iii) For $z \in \mathcal{H}$ with $|y|^2 \gg 0$ and z in the complement of the set of poles, $\Psi(z,f)$ has an absolutely convergent infinite product expansion.

To simplify notation, instead of the general product expansion for $\Psi(z,f)$ from [4, Theorem 13.3.5], we will give a simplified version. Consider the following setup:

Assume that $L = L_0 \oplus H$, the direct sum of a lattice L_0 of signature $(1,n-1)$ and a hyperbolic plane H, i.e. a unimodular lattice of signature (1,1). We set $V_0(\mathbb{R}) = L_0 \otimes_{\mathbb{Z}} \mathbb{R}$ (so $\mathcal{H}$ is adapted to L_0). Then, part iii) of Theorem 4.3 can be formulated as follows:

Theorem 4.4 *For $z \in \mathcal{H}$ with $|y|^2 \gg 0$ and z in the complement of the set of poles, the absolutely convergent infinite product expansion of $\Psi(z,f)$ takes the following form:*

$$\Psi(z,f) = e\big((\rho_W(f),z)\big) \prod_{\substack{\lambda\in L_0' \\ (\lambda,W)>0}} \Big[1 - e\left((\lambda,z)\right)\Big]^{c^+(\lambda,q(\lambda))}.$$

[2] If we want to avoid a rational weight for $\Psi(z,f)$, we must further assume that $c^+(0,0) \in 2\mathbb{Z}$. In this case, the multiplier system in i) is a character [see 6, Theorem 3.22 i)].

Here, $W \subset V_0(\mathbb{R})$ denotes a Weyl chamber for f and $\rho_W(f) \in V_0(\mathbb{R})$ is the Weyl vector attached to W and f.

The Weyl chambers occurring in the theorem are connected components of $\mathcal{H}$; together with the associated Weyl vectors, they can often be determined explicitly, using results of Bruinier [see 6, p. 88]. It is worth noting, that while the Weyl vector parts and the infinite product parts differ depending on the Weyl chamber, the product as a whole is actually the same for all Weyl chambers.

Remark 4.5 Assume that the signature of $V = L \otimes \mathbb{Q}$ is $(2, n)$ with $n \geq 3$. Then, by the Koecher principle, if in the sum in ii) all coefficients $c^+(\mu, m)$ are positive, it follows that $\Psi(z, f)$ is a holomorphic orthogonal modular form.

Contrastingly, if $n = 2$, as the Koecher principle fails in general, this line of reasoning only works for those lattices L where the Witt rank is smaller than n, see Definition 3.3.

Finally, for the case $n = 1$, excluded above, Theorem 4.3 is mostly still correct, except for one caveat: The multiplier system is not guaranteed to have finite order. Bruiner and Ono give a precise criterion for this [see 10, Section 6], which in the present setting can be stated as follows: The order is finite if for all $m < 1$ the Fourier coefficients $c^+(\mu, m)$ of the input function f are rational. As can further be shown, this is equivalent to f being perpendicular to the subspace spanned by unary theta series.

Example 4.1 Let L be the even unimodular lattice of signature $(2, 2)$. Then, L is given by the direct sum of two hyperbolic planes, and the Witt rank here is 2. The space of input functions is given by $\mathrm{M}_0^!(\Gamma(1)) = \mathbb{C}[j]$, where $j = j(\tau)$ is the modular invariant.

For example, let $J(\tau) = j(\tau) - 744$. Then,

$$\Psi(z, J) = j(z_1) - j(z_2) = q_1^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} \left(1 - q_1^m q_2^n\right)^{c(mn)},$$

with $q_1 = e(z_1)$ and $q_2 = e(z_2)$. A complete treatment of this case is carried out in [24].

Example 4.2 We now turn to the case $n = 1$, see Remark 4.5. Consider the following lattice of the form introduced in Example 3.4 (here, $N = 1$):

$$L = \left\{ \begin{pmatrix} b & a \\ c & -b \end{pmatrix};\ a, b, c \in \mathbb{Z} \right\}.$$

Then,

$$L'/L = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}.$$

As the space of input functions $M^!_{\frac{1}{2},\rho_L}$ is isomorphic to $M^{!,+}_{\frac{1}{2}}(\Gamma_0(4))$, we recover the examples of Borcherds products from Sect. 2.2. (It can also be shown that the criterion of Bruinier and Ono mentioned in Remark 4.5 is satisfied.)

We remark that as elliptic modular forms these products have double the weight that they have as orthogonal modular forms. The reason for this is that $SL_2(\mathbb{R})$ is isomorphic to $\mathrm{Spin}_V(\mathbb{R})$ and the map from $\mathrm{Spin}_V(\mathbb{R})$ to $O^+(V)(\mathbb{R})$ is two-to-one [cf. 10, Section 5] or [cf. 4, Example 14.4].

Remark 4.6 In [10], Bruinier and Ono study a generalization of Borcherds' construction for signature $(2, 1)$, using a *twisted* Siegel theta function and with harmonic Maass forms as input functions. One of their results [10, Theorem 6.1] is the existence of generalized Borcherds products, which, however can have multiplier systems of infinite order.

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