The Schrödinger equation in rotating frames by using the stochastic variational method

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Abstract We give a pedagogical introduction of the stochastic variational method by considering the quantization of a non-inertial particle system. We show that the effects of fictitious forces are represented in the forms of vector fields which behave analogously to gauge fields in the electromagnetic interaction. We further discuss that the operator expressions for observables can be defined by applying the stochastic Noether theorem.

1 Introduction

The variational approach conceptually plays a fundamental role in elucidating the structure of classical mechanics, clarifying the origin of dynamics and the relation between symmetries and conservation laws. On the other hand, its operations in classical and quantum systems lack coherence. In fact, in classical mechanics the Lagrangian is usually given by $T - V$, where T and V are kinetic and potential terms, respectively, but in quantum mechanics the Lagrangian which is needed to derive Schrödinger's equation does not have such structure. That is, any clear and direct correspondence between classical and quantum mechanics does not seem to exist in the variational point of view.

However, if we extend the idea of the variational principle to the stochastic variable, it can describe classical and quantum behaviors in a unified way. This

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method is called the stochastic variational method (SVM), and was first proposed by Yasue [1–5] in order to reformulate Nelson's stochastic quantization [6]. This framework is, however, based on special techniques attributed to stochastic calculus, which is not familiar to physicists.

In this paper, we introduce this method by applying it to the quantization of a non-inertial particle system, which is still controversial. The appearance of the nontrivial interference effect of wave functions on a rotating non-inertial frame was experimentally observed in 1979 [7]. Later Sakurai pointed out that such an effect can be understood in thinking of the similarity between the Coriolis force and the Lorentz force [8]. So far, there are various approaches to derive the Schrödinger equation in a non-inertial frame [9–13].

2 Classical equations in non-inertial frames

Let us introduce a non-inertial frame in which the position is denoted by $\mathbf{q}$. Expressing the position in an inertial frame by $\mathbf{r}$. The transformation of these vectors is defined by

$$\mathbf{q} = \mathbf{R}(t)\,\mathbf{r} + \mathbf{c}(t), \tag{1}$$

where $\mathbf{c}(t)$ is a time-dependent translation, and $\mathbf{R}(t)$ is a general 3×3 rotation matrix satisfying $\mathbf{R}^T(t)\,\mathbf{R}(t) = 1$. Both of $\mathbf{r}$ and $\mathbf{q}$ are given by the Cartesian coordinate.

We usually consider a one particle system in the inertial frame. Applying the coordinate transformation (1), the same system observed in the non-inertial frame is characterized by the following Lagrangian:

$$L = \frac{M}{2}(\dot{\mathbf{q}} + \mathbf{A}(\mathbf{q},t) + \mathbf{B}(t))^2 - V(\mathbf{q}), \tag{2}$$

where V is the potential and

$$\mathbf{A}(\mathbf{q},t) = \mathbf{R}\dot{\mathbf{R}}^T(\mathbf{q}-\mathbf{c}), \quad \mathbf{B}(t) = -\dot{\mathbf{c}} \tag{3}$$

are vector fields we have introduced. The equations of motion obtained from this Lagrangian are given by

$$\mathbf{p} = M(\dot{\mathbf{q}} + \mathbf{A}(\mathbf{q},t) + \mathbf{B}(t)), \tag{4}$$

$$\partial_t \mathbf{p}^i = (\mathbf{R}\dot{\mathbf{R}}^T)^{ji}\mathbf{p}^j - \partial_i V(\mathbf{q}). \tag{5}$$

3 The stochastic variational method

The discussion in this section follows the pedagogical introduction of SVM given by Ref. [14]. For a review on SVM with an alternative quantization scheme, see Ref. [15].

In the variational principle for stochastic variables, a particle trajectory is no longer smooth and is seen as given by a zig-zag path in general. Consequently, the evolution of a particle trajectory is defined by the following forward stochastic differential equation (SDE),

$$d\mathbf{q}(t) = \left(\frac{\mathbf{p}(\mathbf{q}(t),t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t) \right) dt + \sqrt{2\nu} d\mathbf{W}_t \quad (dt > 0). \tag{6}$$

Here $\mathbf{p}(\mathbf{x},t)$ is an unknown field determined by the stochastic variation. Note that in what follows $\mathbf{x}$ is used to denote the spatial position in a non-inertial frame. The last term in Eq. (6) is the origin of the zig-zag motion and is called the noise term. The parameter ν characterizes the strength of this noise term. The property of $\mathbf{W}_t$ is given by the standard Wiener process, which is characterized by the following correlation properties:

$$E[d\mathbf{W}_t] = 0, \quad E[(dW_t^i)(dW_t^j)] = |dt|\delta^{ij}, \quad (i,j = x,y,z), \tag{7}$$

$$E[W_t^i dW_{t'}^j] = 0 \;\text{ for }\; (t \le t'), \tag{8}$$

where $E[\;\;]$ indicates the average of stochastic events.

The probabilistic nature of the particle distribution described by Eq. (6) is easily characterized by introducing the probability distribution defined by $\rho(\mathbf{q},t) = \int d^3\mathbf{q}_i\, \rho_I(\mathbf{q}_i) E[\delta^{(3)}(\mathbf{q} - \mathbf{q}(t))]$, where $\mathbf{q}(t)$ (more exactly $\mathbf{q}(t;\mathbf{q}_i)$) is the solution of Eq. (6) and $\rho_I(\mathbf{q}_i)$ is the initial particle distribution at an initial time t_i. As is well-known, the evolution equation of $\rho(\mathbf{q},t)$ is derived from the SDE (6) and is called the Fokker-Planck equation,

$$\partial_t \rho(\mathbf{x},t) = \nabla \cdot \left\{ -\left(\frac{\mathbf{p}(\mathbf{x},t)}{M} - \mathbf{A}(\mathbf{x},t) - \mathbf{B}(t) \right) + \nu \nabla \right\} \rho(\mathbf{x},t). \tag{9}$$

If the probability distribution evolves from $\rho_I(\mathbf{q})$ to $\rho_F(\mathbf{q}) \equiv \rho(\mathbf{q}(t_f),t_f)$ at a final time t_f following Eq. (9), the corresponding time-reversed process should describe the evolution from ρ_F to ρ_I. Suppose that this process is described by the backward SDE,

$$d\mathbf{q}(t) = \left(\frac{\tilde{\mathbf{p}}(\mathbf{q}(t),t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t) \right) dt + \sqrt{2\nu} d\mathbf{W}_t, \quad (dt < 0). \tag{10}$$

To reproduce Eq. (9) from the backward SDE, we find that the following consistency condition should be satisfied, $\mathbf{p}(\mathbf{x},t) = \tilde{\mathbf{p}}(\mathbf{x},t) + 2\nu \nabla \ln \rho(\mathbf{x},t)$.

We should stress that the usual definition of the particle velocity is not applicable, because $d\hat{\mathbf{r}}/dt$ is not well defined in the vanishing limit of dt due to the singular behavior of $\mathbf{W}_t$. The possible time differential in such a case was studied by Nelson [6] and it is known that there are two possibilities: One is the mean forward derivative

$$D\mathbf{q}(t) = \lim_{dt \to 0+} E\left[\frac{\mathbf{q}(t+dt) - \mathbf{q}(t)}{dt} \Big| \mathscr{P}_t \right], \tag{11}$$

and the other is the mean backward derivative,

$$\tilde{D}\mathbf{q}(t) = \lim_{dt\to 0-} E\left[\frac{\mathbf{q}(t+dt)-\mathbf{q}(t)}{dt}\Big|\mathscr{F}_t\right]. \tag{12}$$

These expectations are conditional averages, where $\mathscr{P}_t$ (resp. $\mathscr{F}_t$) indicates fixing the values of $\mathbf{r}(t')$ for $t' \leq t$ $(resp.\ t' \geq t)$. For the σ-algebra of all measurable events of $\mathbf{r}(t)$, $\{\mathscr{P}_t\}$ and $\{\mathscr{F}_t\}$ represent, respectively, increasing and decreasing families of sub-σ-algebras. Using these derivatives in Eqs. (6) and (10), we obtain, respectively,

$$D\mathbf{q}(t) = \frac{\mathbf{p}(\mathbf{q},t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t), \quad \tilde{D}\mathbf{q}(t) = \frac{\tilde{\mathbf{p}}(\mathbf{q},t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t). \tag{13}$$

4 Quantization in non-inertial frames

Let us apply the stochastic variation to the system given by the Lagrangian (2). Then the particle trajectory in Eq. (2) should be replaced by the stochastic one, as was discussed in the previous section. Due to the existence of two different time-derivatives D and $\tilde{D}$, there is an ambiguity when replacing the kinetic term. In this work, we adopt the following replacement,

$$L(\mathbf{q}, D\mathbf{q}, \tilde{D}\mathbf{q}) = \frac{m}{2}\left[\frac{(D\mathbf{q}(t)+\mathbf{A}+\mathbf{B})^2 + (\tilde{D}\mathbf{q}(t)+\mathbf{A}+\mathbf{B})^2}{2}\right] - V(\mathbf{q}(t)). \tag{14}$$

See Ref. [16] for a more precise discussion of this replacement.

The stochastic variation of the particle Lagrangian leads to the stochastic Euler-Lagrange equation

$$\tilde{D}\frac{\partial L}{\partial(D\mathbf{q}(t))} + D\frac{\partial L}{\partial(\tilde{D}\mathbf{q}(t))} - \frac{\partial L}{\partial \mathbf{q}(t)}\Bigg|_{\mathbf{q}(t)=\mathbf{x}} = 0. \tag{15}$$

Here $\mathbf{q}(t)$ is replaced by the position parameter $\mathbf{x}$ at the last step of the calculation. Substituting Eq. (14), we obtain

$$\left(\partial_t + \left(\frac{\mathbf{p}_m}{M} - \mathbf{A} - \mathbf{B}\right)\cdot\nabla\right)\mathbf{p}_m - 2M\nu^2\nabla\rho^{-1/2}\Delta\sqrt{\rho} = \mathbf{p}_m\cdot\nabla_i\mathbf{A} - \nabla_i V, \tag{16}$$

where $\mathbf{p}_m = (\mathbf{p}+\tilde{\mathbf{p}})/2$.

The result of this variation can be re-expressed in the form of the Schrödinger equation by introducing the wave function defined by $\Psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\theta(\mathbf{x},t)}$. Here $\rho(\mathbf{x},t)$ is the probability distribution introduced above Eq. (9), and the phase $\theta(\mathbf{x},t)$ is defined by $\mathbf{p}_m = 2M\nu\nabla\theta(\mathbf{x},t)$. Then we find that the evolution equation of the wave function is given by the following Schrödinger equation

$$i\hbar\partial_t \Psi(\mathbf{x},t) = \left[\frac{1}{2M}\Big(-i\hbar\nabla - M\big(\mathbf{A}(\mathbf{x},t)+\mathbf{B}(t)\big)\Big)^2 - \frac{M}{2}\Big(\mathbf{A}(\mathbf{x},t)+\mathbf{B}(t)\Big)^2 + V(\mathbf{x})\right]\Psi(\mathbf{x},t). \quad (17)$$

Here we choose $\nu = \hbar/(2M)$. One can see that the effect of the non-inertial forces appears in the vector fields $\mathbf{A}(\mathbf{x},t)$ and $\mathbf{B}(t)$ which behave like the gauge field in the electromagnetic interaction.

5 Observables

The dynamics described by the above Schrödinger equation satisfies Eherenfest's theorem. In fact, the time evolution of the expectation value of the operator $-\hbar\nabla$ is given by

$$\partial_t \langle -i\hbar\partial_i \rangle = \langle (\mathbf{R}\dot{\mathbf{R}}^T)^{ji}(-i\hbar\partial_j)\rangle - \langle \partial_i V\rangle. \quad (18)$$

One can see that if we can interpret $\hat{p} = -i\hbar\nabla$, the above equation corresponds to Eq. (5).

However, to be precise, it is non-trivial as to whether we can interpret $-\hbar\nabla$ as the momentum operator even in the non-inertial frame. In SVM, the operator representations of observables are defined through the conservation laws obtained from the stochastic Lagrangian (14).

For the sake of simplicity, let us consider the rotation around the z-axis, where

$$\mathbf{R}(t) = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) & 0 \\ -\sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{c}(t) = 0. \quad (19)$$

This non-inertial system still holds the invariance for the rotation if $V(\mathbf{x}) = V(|\mathbf{x}|)$. Then from the invariance of the stochastic action, we can obtain the angular momentum conservation of the present non-inertial system. For the infinitesimal rotation, $\mathbf{q}(t)$ is transformed as $\mathbf{q}(t) \longrightarrow \mathbf{q}(t) + \mathbf{A}(\phi(t))$, where $\mathbf{A}(\phi(t)) = \delta\dot{\phi}(-y,x,0)$.

On the other hand, if the action is invariant for the above rotation, we can show that the following quantity is conserved by applying the stochastic Noether theorem [17, 18],

$$Q = E\left[\mathbf{q}(t) \times \left(\frac{\partial L}{\partial(D\mathbf{q}(t))} + \frac{\partial L}{\partial(\tilde{D}\mathbf{q}(t))}\right)\right]. \quad (20)$$

Here $\times$ denotes the vector product. Substituting the result of the stochastic variation, the above equation is now expressed as

$$Q = \int d^3\mathbf{x}\ \Psi(\mathbf{x},t) L_z \Psi(\mathbf{x},t), \quad (21)$$

where the angular momentum operator is introduced, $L_z = -i\hbar(x\partial_y - y\partial_x)$. This result means that $-i\hbar\nabla$ can be interpreted as the momentum operator even in the non-inertial system.

6 Concluding remarks

We gave a brief summary of the stochastic variational method and showed that this is applicable to the quantization of the non-inertial particle system. Then we found that the Eherenfest's theorem is still satisfied even for the Schrödinger equation in the non-inertial frame, and thus the result is consistent with those in Refs. [9–11], but different from Refs. [12, 13].

The advantage of the present approach compared to Refs. [9–11] is that the operator representations for observables are systematically obtained by applying the stochastic Noether theorem.

Although the framework of SVM was originally proposed to reformulate Nelson's stochastic quantization, its applicability is not restricted to quantization. The derivation of the classical dissipative dynamics can be cast into the form of SVM: the Navier-Stokes-Fourier equation is obtained by employing the stochastic variation to the classical action of the Euler (ideal fluid) equation. See Refs. [18, 19] for details.

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