Generalized supersymmetry and the Lévy-Leblond equation

N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan

Abstract Symmetries of the Lévy-Leblond equation are investigated beyond the standard Lie framework. It is shown that the equation has two remarkable symmetries. One is given by the super Schrödinger algebra and the other by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra is achieved by transforming bosonic into fermionic operators in the super Schrödinger algebra and introducing second order differential operators as generators of symmetry.

1 Introduction

The purpose of the present work is to show that a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra is a symmetry of a simple equation of physics, the Lévy-Leblond equation (LLE), which is a non-relativistic wave equation of a spin $1/2$ particle [9]. In the process to prove the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry we also show that LLE has a supersymmetry given by the $\mathscr{N} = 1$ super Schrödinger algebra (see [3] and references therein).

$\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras (introduced in [12, 13], see also [14]) are natural generalizations of Lie superalgebras. We present their definition: Let $\mathfrak{g}$ be a vector space over $\mathbb{C}$ or $\mathbb{R}$ with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading structure, namely $\mathfrak{g}$ is the direct sum of four distinct subspaces labelled by an element of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group:

N. Aizawa, H. Tanaka
Osaka Prefecture University,Sakai, Osaka 599-8531 Japan,
e-mail: aizawa@p.s.osakafu-u.ac.jp,s_h.tanaka@p.s.osakafu-u.ac.jp

Z. Kuznetsova
UFABC, Av. dos Estados 5001, Bangu, cep 09210-580, Santo André (SP), Brazil,
e-mail: zhanna.kuznetsova@ufabc.edu.br

F. Toppan
CBPF, Rua Dr. Xavier Sigaud 150, Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil,
e-mail: toppan@cbpf.br

S. Duarte et al. (eds.), *Physical and Mathematical Aspects of Symmetries*,
https://doi.org/10.1007/978-3-319-69164-0_11

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} + \mathfrak{g}_{(0,1)} + \mathfrak{g}_{(1,0)} + \mathfrak{g}_{(1,1)}. \tag{1}$$

For two elements $\mathbf{a} = (a_1, a_2),\ \mathbf{b} = (b_1, b_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we define

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)\ (\mathrm{mod}(2,2)), \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \tag{2}$$

Definition 1. If $\mathfrak{g}$ admits a bilinear form $[\![\ ,\ ]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following three relations, then $\mathfrak{g}$ is called a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra:

1. $[\![\mathfrak{g}_{\mathbf{a}}, \mathfrak{g}_{\mathbf{b}}]\!] \subseteq \mathfrak{g}_{\mathbf{a}+\mathbf{b}}$,
2. $[\![X_{\mathbf{a}}, X_{\mathbf{b}}]\!] = -(-1)^{\mathbf{a}\cdot\mathbf{b}} [\![X_{\mathbf{b}}, X_{\mathbf{a}}]\!]$,
3. $[\![X_{\mathbf{a}}, [\![X_{\mathbf{b}}, X]\!]]\!] = [\![[\![X_{\mathbf{a}}, X_{\mathbf{b}},]\!], X]\!] + (-1)^{\mathbf{a}\cdot\mathbf{b}} [\![X_{\mathbf{b}}, [\![X_{\mathbf{a}}, X]\!]]\!]$,

where $X_{\mathbf{a}} \in \mathfrak{g}_{\mathbf{a}}$.

Two sub superalgebras exist (they are $\mathfrak{g}_{(0,0)} + \mathfrak{g}_{(0,1)}$ and $\mathfrak{g}_{(0,0)} + \mathfrak{g}_{(1,0)}$). This fact plays a crucial role when the symmetry of the LLE is identified with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra.

In contrast to ordinary Lie algebras and superalgebras, the number of papers in the literature discussing physical applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras is limited [8, 10, 15, 17, 18]. The equation discussed in this work is both simple and fundamental. Even so, we naturally encountered this unusual algebraic structure. This would suggest that $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras are natural objects in the investigation of symmetries.

The plan of this paper is as follows. In the next section we introduce the LLE and present its symmetries. We show that the LLE has a super Schrödinger symmetry. In §3 the supersymmetry is enhanced to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie symmetry.

2 LLE and its (super)symmetries

The LLE here considered is a non-relativistic wave equation for a spin $1/2$ free particle in 3D space. The wavefunction is a four-component spinor,

$$\psi(x) = {}^T(\varphi_1(x), \varphi_2(x)),$$

where φ_a is a $SU(2)$ spinor and $x = (t, x_1, x_2, x_3)$. We use the following form of LLE [4]:

$$\Omega\psi(x) = 0, \quad \Omega = -2i\alpha\partial_t + i\gamma_j\partial_{x_j} + 2m\beta, \tag{3}$$

where the sum over the repeated index $j = 1,2,3$ is understood; $\gamma_\mu, \alpha, \beta$ are 4×4 Dirac γ-matrices defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad (g_{\mu\nu}) = \mathrm{diag}(+,-,-,-), \quad \mu, \nu = 0,1,2,3 \tag{4}$$

and

$$\alpha = \frac{1}{2}(\gamma_0 + \gamma_4), \quad \beta = \frac{1}{2}(\gamma_0 - \gamma_4), \quad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{5}$$

One may take any four-dimensional representation of the γ-matrices. We do not distinguish upper and lower indices since we are working in a non-relativistic setting. LLE is the square root of the free Schrödinger equation, namely Ω^2 gives the free particle Schrödinger operator:

$$\Omega^2 = -4im\partial_t + \partial x_j^2. \tag{6}$$

We introduce now the symmetries of LLE. According to [4] we define them in terms of symmetry operators [4]:

Definition 2. Let $\mathscr{A}$ be an operator acting on the solution space of LLE. Namely, $\mathscr{A}$ maps a solution of LLE into another one:

$$\Omega\psi = 0 \quad \Longrightarrow \quad \Omega(\mathscr{A}\psi)\Big|_{\Omega\psi=0} = 0. \tag{7}$$

In this case $\mathscr{A}$ is called a symmetry operator.

In this definition $\mathscr{A}$ can be any kind of operator such as multiplication, differential, integral, etc. The traditional Lie point symmetry group of differential equations is generated by a subset of symmetry operators which is closed under commutations. Similarly, if a subset of symmetry operators forms a superalgebra or a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra, then the set generates a graded group of transformations in the solution space of LLE.

We restrict now $\mathscr{A}$ to a differential operator of finite order. In this case a sufficient condition of symmetry is given as follows. If $\mathscr{A}$ satisfies either the condition

$$[\Omega, \mathscr{A}] = \Lambda_{\mathscr{A}}(x)\Omega \tag{8}$$

or

$$\{\Omega, \mathscr{A}\} = \Gamma_{\mathscr{A}}(x)\Omega, \tag{9}$$

where $\Lambda_{\mathscr{A}}(x)$ or $\Gamma_{\mathscr{A}}(x)$ is a 4×4 matrix depending on the spacetime coordinates, then $\mathscr{A}$ is a symmetry operator.

We are looking for symmetry operators given by a first order differential operator. The results are summarized in the following two propositions:

Proposition 1. *The operators below are LLE symmetry operators satisfying the condition (8):*

$$\begin{aligned}
P_j &= \partial_{x_j}, \qquad G_j = t\partial_{x_j} + 2imx_j + \alpha\gamma_j, \qquad M = 2im, \\
H &= \partial_t, \qquad D = 2t\partial_t + x_j\partial_{x_j} + 2 - \frac{1}{2}\gamma_0\gamma_4, \\
K &= tD - t^2\partial_t + imx_jx_j + \alpha x_j\gamma_j, \\
J_{jk} &= x_j\partial_{x_k} - x_k\partial_{x_j} - \frac{1}{2}\gamma_j\gamma_k, \\
\tilde{X}_j &= -\varepsilon_{jkn}\Big([\alpha,\gamma_k]\partial_{x_n} + \frac{im}{2}[\gamma_k,\gamma_n]\Big).
\end{aligned} \tag{10}$$

The only two non-vanishing $\Lambda_{\mathscr{A}}(x)$ matrices are $\Lambda_D = 1$, $\Lambda_K = t$. For convenience the 4×4 unit matrix $\mathbf{1}_4$ is not explicitly indicated (e.g., $P_j = \mathbf{1}_4\,\partial_{x_j} \equiv \partial_{x_j}$).

Apart from the $\tilde{X}_j$'s, the remaining symmetry operators close a Lie algebra. $\mathfrak{h}(3) = \langle\, P_j, G_j, M \,\rangle$ is the three-dimensional Heisenberg Lie algebra with M as a central element. We have the non-relativistic conformal algebra $\mathfrak{sl}(2,\mathbb{R}) = \langle\, H, D, K \,\rangle$ and the spatial rotation $\mathfrak{so}(3) = \langle\, J_{jk} \,\rangle$. Combining together these three Lie algebras we get the Schrödinger algebra, whose structure is given by

$$(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \ni \mathfrak{h}(3),$$

with $\ni$ a semidirect sum of Lie algebras. We thus see that the Schrödinger group is a symmetry of LLE. This fact is already known in the literature. In [4] the Schrödinger algebra is presented as the maximal Lie symmetry of LLE. If the symmetry operators $\tilde{X}_j$ are included we are no longer able to close a Lie algebra. Their addition leads to a $\mathbb{Z}_2\times\mathbb{Z}_2$ graded Lie algebra. Before addressing the $\mathbb{Z}_2\times\mathbb{Z}_2$ structure we look at the LLE's supersymmetry.

Proposition 2. *The operators below are LLE symmetry operators satisfying the condition (9):*

$$\begin{aligned}
Q &= \frac{1}{\sqrt{-im}}\alpha\partial_t + \sqrt{-im}\beta, \\
S &= \frac{1}{\sqrt{-im}}\alpha\Big(t\partial_t + x_j\partial_{x_j} + \frac{3}{2}\Big) + \sqrt{-im}(t\beta + x_j\gamma_j), \\
X_j &= \frac{1}{\sqrt{-im}}\alpha\partial_{x_j} + \sqrt{-im}\gamma_j,
\end{aligned} \tag{11}$$

with only one non-vanishing $\Gamma_{\mathscr{A}}(x)$ matrix given by $\Gamma_S = -\alpha/\sqrt{-im}$.

The physical meaning of these symmetry operators becomes clear when computing their anticommutators:

$$\begin{aligned}
&\{Q,Q\} = 2H, \qquad \{S,S\} = 2K, \qquad \{X_j,X_k\} = \delta_{jk}M, \\
&\{Q,S\} = D, \qquad \{Q,X_j\} = P_j, \qquad \{S,X_j\} = G_j.
\end{aligned} \tag{12}$$

It follows that Q, S are, respectively, a supercharge and a conformal supercharge, with X_j a fermionic counterpart of $\mathfrak{h}(3)$. Indeed, the Schrödinger algebra of Proposition 1 and $\langle\, Q, S, X_j \,\rangle$ close the $\mathscr{N}=1$ super Schrödinger algebra. This is verified by direct computation of the (anti)commutation relations. The operator Q is already found in [4] without recognizing it as a supercharge. One may also show (we omit the proof for space reasons), that there exists no other supercharge $\overline{Q}$ satisfying

$$\{\overline{Q},\overline{Q}\} = 2H, \qquad \{Q,\overline{Q}\} = 0, \qquad \{\overline{Q},\Omega\} = \Gamma_{\overline{Q}}(x)\Omega,$$
$$[D,\overline{Q}] = -\overline{Q}, \qquad [J_{jk},\overline{Q}] = 0. \tag{13}$$

We thus have the theorem:

Theorem 1. *The $\mathscr{N}=1$ super Schrödinger algebra generates a symmetry supergroup of LLE and $\mathscr{N}=1$ is the maximal supersymmetry.*

The supersymmetry of LLE was conjectured many years ago in the study of the worldline supersymmetry of the spinning particle [5]. If the symmetry is defined according to Definition 2, then the conjecture is true. We mention here two other previous works on supersymmetry of LLE. In [6] it was shown that LLE coupled with an arbitrary static magnetic field has a super Schrödinger symmetry. In [7] the Dirac equation and the Deser-Jackiw-Templeton equation in a $(2+1)$ dimensional spacetime are unified in a single multiplet of $\mathfrak{osp}(1|2)$. It is shown that the non-relativistic limit of this system carries an $\mathscr{N}=2$ super Schrödinger symmetry.

3 The $\mathbb{Z}_2\times\mathbb{Z}_2$ graded symmetry of LLE

In this section we consider the symmetry of LLE with the $\tilde{X}_j$ operators. There are two key observations: (i) the $\tilde{X}_j$'s are obtained from the commutators of the fermionic generators X_j, $\tilde{X}_j = \frac{1}{2}\varepsilon_{jkn}[X_k,X_n]$; (ii) each pair $(Q,S),(P_j,G_j)$ is a $sl(2,\mathbb{R})$-doublet under the adjoint action. The observation (i) implies that we need to give up the super Schrödinger structure, while (ii) implies that we may regard (P_j,G_j) as fermionic since this treats all $sl(2,\mathbb{R})$ doublets on equal footing [16]. Therefore we introduce, from the anticommutators, the new operators

$$\tilde{P}_{jk} = \{P_j,P_k\}, \quad \tilde{G}_{jk} = \{G_j,G_k\}, \quad W_{jk} = \{P_j,G_k\},$$
$$X^P_{jk} = \{P_j,X_k\}, \quad X^G_{jk} = \{G_j,X_k\}. \tag{14}$$

They are second order differential operators; it is easy to verify that they are symmetry operators of LLE. Surprisingly, these second-order operators, together with the first-order operators in the super Schrödinger algebra, close a $\mathbb{Z}_2\times\mathbb{Z}_2$ graded Lie algebra $\mathscr{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. This means their (anti)commutators never produce higher order differential operators. The assignment of the grading is given by

$$\begin{aligned}
\mathfrak{g}_{00} &= \langle\, H, D, K, J_{jk}, \tilde{X}_j, W_{jk}, \tilde{P}_{jk}, \tilde{G}_{jk} \,\rangle, \\
\mathfrak{g}_{01} &= \langle\, P_j, G_j \,\rangle, \\
\mathfrak{g}_{10} &= \langle\, Q, S, X^P_{jk}, X^G_{jk} \,\rangle, \\
\mathfrak{g}_{11} &= \langle\, X_j \,\rangle.
\end{aligned} \tag{15}$$

One may verify, by direct but cumbersome computation of the (anti)commutators, that the algebra (15) satisfies Definition 1. We remark that the multiplication operator M has dropped out from this $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra.

Theorem 2. *The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra defined by the operators in (15) generates a symmetry group of LLE.*

We have shown, in summary, that LLE has a $\mathscr{N} = 1$ super Schrödinger symmetry and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded symmetry given by (15). The super Schrödinger algebra is not a subalgebra of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra, although they share the same symmetry operators. As a continuation of the present work, one may investigate symmetries of LLE with a potential, since it is known that Schrödinger equation with linear or quadratic potential has the same symmetry as the free equation [2, 11]. It is also an interesting problem to study symmetries of a LLE for an arbitrary space dimension. This would be done systematically by making use of the representation theory of Clifford algebra. These works are in progress. Part of these results are reported in [1].

Acknowledgements N. A. is supported by the grants-in-aid from JSPS (Contract No. 26400209). F. T. received support from CNPq (PQ grant No. 306333/2013-9).

References

1. N. Aizawa, Z. Kuznetsova, H. Tanaka, F. Toppan, Prog. Theor. Exp. Phys. **2016** (2016) 123A01, arXiv:1609.08224 [math-ph].
2. C. P. Boyer, Helv. Phys. Acta, **47** (1974) 589.
3. C. Duval and P. A. Horváthy, J. Math. Phys. **35** (1994) 2516.
4. W. Fushchych, R. Zhdanov, *Symmetries and Exact Solutions of Nonlinear Dirac Equations*, (Mathematical Ukraina Publisher, 1997).
5. J. P. Gauntlett, J. Gomis, P. K. Townsend, Phys. Lett. **B248** (1990) 288.
6. P. A. Horváthy, Int. J. Mod. Phys. **A3** (1993) 339-342, arXiv:0807.0513 [hep-th].
7. P. A. Horváthy, M. S. Plyushchay and M. Valenzuela, J. Math. Phys. **51** (2010) 092108.
8. P. D. Jarvis, M. Yang and B. G. Wybourne, J. Math. Phys. **28** (1987) 1192.
9. J.-M. Lévy-Leblond, Comm. Math. Phys. **6** (1967) 286.
10. J. Lukierski, V. Rittenberg, Phys. Rev. **D18** (1978) 385.
11. U. Niederer, Helv. Phys. Acta, **47** (1974) 167.
12. V. Rittenberg, D. Wyler, Nucl. Phys. **B139** (1978) 189.
13. V. Rittenberg, D. Wyler, J. Math. Phys. **19** (1978) 2193.
14. M. Scheunert, J. Math. Phys. **20** (1979) 712.
15. V. N. Tolstoy, Phys. Part. Nucl. Lett. **11** (2014) 933.
16. F. Toppan, J. Phys. Conf. Ser. **597** (2015) 012071.
17. M. A. Vasiliev, Class. Quantum Grav. **2** (1985) 645.
18. A. A. Zheltukhin, Teor. Mat. Fiz. **71** (1987) 218.