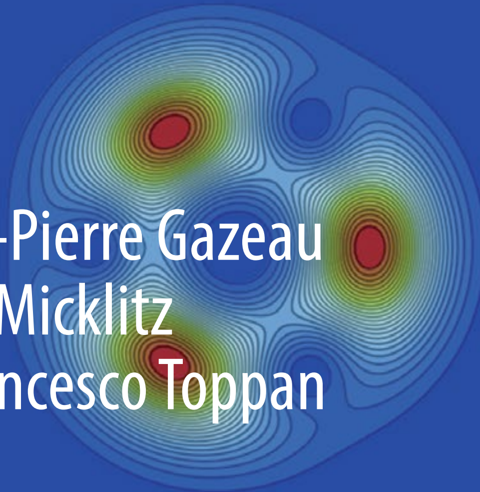

Sergio Duarte · Jean-Pierre Gazeau
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Editors

Physical and Mathematical Aspects of Symmetries

Proceedings of the 31st International Colloquium in Group Theoretical Methods in Physics

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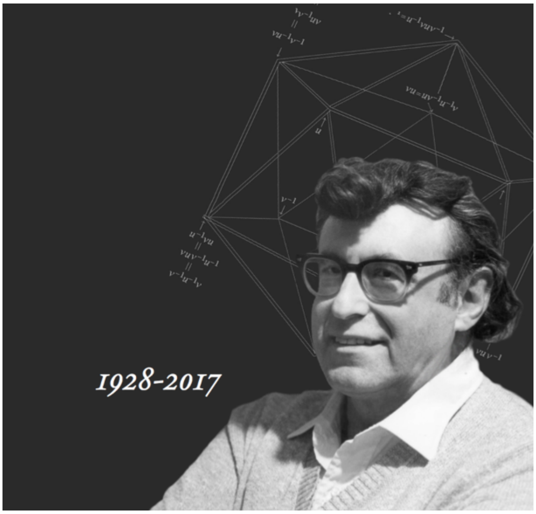
Dedication to Bertram Kostant



Bertram Kostant, Professor Emeritus of Mathematics at MIT, died at the Hebrew Senior Rehabilitation Center in Roslindale on Thursday February 2 at the age of 88.

He was a Professor of Mathematics at MIT from 1962 until 1993 when he officially retired yet continued his active life in research, traveling and lecturing at various universities and conferences around the world. Kostant's legacy spans well over five decades, highlighting his originality and creativity in 107 published papers. His remarkable ability to connect seemingly diverse ideas led to brilliant results that formed the cornerstone of rich and fruitful theories both in mathematics and theoretical physics. It has been said, "Bert's gift to the world showed a deep passion for truth, for understanding, and for beauty, and an unshakeable faith that these things are woven together."

Bertram Kostant was born on May 24, 1928 in Brooklyn, New York. He graduated from Peter Stuyvesant High School in 1945. After studying chemistry for two years at Purdue University, he switched to mathematics having fallen in love with the subject in the classes of Arthur Rosenthal and Michael Golomb, who were recent immigrants from Germany. In 1950 he earned a bachelor's degree with distinction in mathematics.

Kostant was awarded an Atomic Energy Commission Fellowship for graduate studies at the University of Chicago. There he found a stimulating environment. The various influences on him included Marshall Stone, Adrian Albert, Shing Shen Chern, Paul Halmos, Irving Kaplansky, Irving Segal, but above all, via André Weil he was exposed to the French Revolution led by the Bourbaki group with their stunning innovations in thinking about and writing down mathematics. Ed Spanier's course on Group Theory used Chevalley's text—a treasure in Kostant's library. It marked a turning point in his life, and as he often said, "the sheer beauty of it all resonated with me." And it was indeed Kostant's entrée into Lie groups for the next sixty-five years. His fundamental and varied mathematical work entailed many aspects of Lie theory, which pervades almost all of mathematics, and is marked by simplicity and elegance. Notable among the Lie areas touched upon are the following: algebraic groups and invariant theory, the geometry of homogeneous spaces, representation theory, geometric quantization and symplectic geometry, Lie algebra cohomology, Hamiltonian mechanics, modular forms, and much more.

Kostant received an M.S. degree in Mathematics in 1951, and in 1954 his Ph.D. under Irving Segal. His thesis was on "Representations of a Lie algebra and its enveloping algebra on a Hilbert space."

Between 1953 and 1956 Kostant was a Member at the Institute for Advanced Study in Princeton. In 1955-56 he was a Higgins Lecturer at Princeton University. In Princeton, his lifelong passion for Lie groups—the continuous families of symmetries at the core of great parts of geometry, mathematical physics, and even algebra—began to blossom. He investigated the "holonomy groups" arising in differential geometry, and at the same time worked to deepen our understanding of the structure of the (deceptively named!) simple Lie algebras. From 1956 to 1962 Kostant was a faculty member at the University of California at Berkeley, where he became a full professor in 1962. He was a Member of the Miller Institute for Basic Research, 1958-59.

In 1962 in he joined the faculty at MIT, where he remained for the rest of his life. Early on, Norman Levinson urged him to build the MIT Department in Representation Theory. Kostant eagerly welcomed the task, attracting new graduate students and excellent mathematicians to come to MIT. He was devoted to his weekly Lie Seminars, with both colleagues and graduate students in attendance; over the years he had encouraged more than twenty Ph.D. students. He also served as a mentor to many postdocs and young faculty members.

In the early 1960s, Kostant began to develop the "method of coadjoint orbits" and "geometric quantization" (GQ 1965) relating symplectic geometry to infinite-dimensional representation theory. Geometric quantization "provided a way to pass between the geometric pictures of Hamiltonian mechanics and the Hilbert spaces of quantum mechanics. These deep and complicated subjects with their profound connections have been at the heart of several very different mathematical disciplines ever since." Kostant's great contribution was also to relate such complex ideas to much simpler mathematics. Again and again he was able to make powerful use of these relationships. For example, in the early 1960s he proved a purely algebraic result about "tridiagonal" matrices. In the 1970s, he used that result and the ideas of geometric quantization to study Whittaker models (which are at the heart of the theory of automorphic forms) and the Toda lattice (a widely studied model for one-dimensional crystals).

Kostant received many awards and honors. He was a Guggenheim Fellow in 1959-60 (in Paris), and a Sloan Fellow in 1961-63. In 1962 he was elected to the American Academy of Arts and Sciences, and in 1978 to the National Academy of Sciences. In 1982 he was a Fellow of the Sackler Institute for Advanced Studies at Tel Aviv University. In 1990 he was awarded the Steele Prize of the American Mathematical Society, in recognition of his 1975 paper, "On the existence and irreducibility of certain series of representations."

In 2001, he was the Chern Lecturer in Berkeley. In 1989, the University of Cordoba, Argentina named him Honorary Professore. In 1992, the University of Salamanca in Spain named him Doctor Honoris Causa; in 1997, Purdue University gave him an honorary Doctor of Science degree. Purdue cited Kostant for his fundamental contributions to mathematics and the inspiration he and his work have provided to generations of researchers.

In May 2008, the Pacific Institute for Mathematical Sciences hosted a conference: "Lie Theory and Geometry: the Mathematical Legacy of Bertram Kostant," at the University of British Columbia, celebrating the life and work of Kostant in his 80th year. In 2012 he was elected to the inaugural class of Fellows of the American Mathematical Society.

In June 2016 Kostant traveled to Rio for the Colloquium on Group Theoretical Methods in Physics, where he received the prestigious Wigner Medal, "for his fundamental contributions to representation theory that led to new branches of mathematics and physics." Michio Jimbo of Rikkyo University, Tokyo, Chair of the Selection Committee said: "the lifelong achievements of Bertram Kostant have had a profound impact in pure mathematics". At the same time, his work miraculously has been finding its way to physics. Kostant's winning the award perfectly suits the spirit of Wigner who coined the famous phrase, "the unreasonable effectiveness of mathematics in the physical sciences."

Professor Kostant is survived by his wife Ann of 49 years; children Abbe Kostant Smerling of Lexington, Massachusetts; Steven Kostant of Chevy Chase, Maryland; Elizabeth Loew of Stoughton, Massachusetts; David Amiel of Glendale, California; Shoshanna Kostant of Boston, Massachusetts; and nine grandchildren and two great-grandchildren.

The MIT Mathematics Department held a memorial event on May 11 at 3:30 in the MIT Chapel. Further information will be posted on the MIT Mathematics Department website: math.mit.edu.

Preface

The 31^{st} International Colloquium on Group Theoretical Methods in Physics (also shortened as “Group 31”) was held in Rio de Janeiro, Brazil, from June 19 to June 25, 2016. This was the first time that a colloquium of the prestigious and nowadays traditional ICGTMP series, which started in 1972 in Marseille, France, took place in South America.

The aim of the ICGTMP Colloquia is to provide a forum for physicists, mathematicians, and scientists of related disciplines who either develop or apply methods in group theory (further information on the history of the Colloquia and its recent development is found at the ICGTMP homepage http://icgtmp.blogs.uva.es/).

The Group 31 Colloquium was hosted by the *Centro Brasileiro de Pesquisas Físicas* (CBPF), a Federal Research Institute which, since its creation in 1949, has been essential for Brazilian science in promoting research and scientific interchange. The Group 31 Colloquium, consisted of three venues, was located in different areas of Rio de Janeiro. The main activity (registration, parallel and poster sessions) took place at CBPF in the Urca neighborhood, while plenary sessions were held at the Auditorium of the *Fundação Casa de Rui Barbosa* in Botafogo. The Award Ceremony for the Wigner Medal and the Weyl Prize was held on June 22 in the new landmark of Rio de Janeiro, the *Museu do Amanhã* science museum, next to the waterfront of Pier Mauá. The last day of the colloquium a general public event was also held at *Museu do Amanhã*.

In recent years Brazil experienced a scientific boost (measured, e.g., by the number of scientific publications and their impact) which has been unparalleled in its history. To be sure, the group theoretical community was both a beneficiary and a promoter of this scientific rise. One of the motivations to organize the Group Theoretical Colloquium in Brazil was indeed to offer a unique opportunity to the growing, although scattered on a vast subcontinental nation and not yet fully organized, community of researchers working in the country (and profiting, as well, researchers from other South American nations). In this respect the colloquium was a great success, with more than 140 participants, equally split into Brazilians and foreigners. It is particularly remarkable that all continents were represented, this is a sign of the relevance of this scientific topic and of the world-wide esteem that the colloquium is held by our colleagues. This success was made possible, in particular, by grants received by TWAS, supporting participation of scientists from developing countries, and ICTP, supporting participants from Latin American countries outside of Brazil. The main sponsor of the event has been the CAPES Federal Agency which offered a substantial contribution to the Local Organizing Committee. Important logistic support with free use of the facilities was provided by CBPF, Fundação Casa de Rui Barbosa, and Museu do Amanhã.

The scientific program of Group 31 was particularly rich, with eleven plenary talks, thirteen parallel sessions with both oral and poster presentations, two laudatio speeches in honor of, respectively, the Wigner Medalist and the Weyl Prize winner, and two memorial talks. The memorial talks were held to honor renowned colleagues Laurence Boyle and Syed Twareque Ali, members of the ICGTMP Standing Committee, who sadly passed away.

Before the Colloquium, as a parallel program, a two-day Satellite Workshop on Mathematical Physics was organized on June 16 and 17 in S. Paulo by ICTP-SAIFR (the International Center for Theoretical Physics-South American Institute for Fundamental Research).

At the Inauguration of the colloquium welcome speeches were given by Mariano del Olmo, Chairman of the ICGTMP Standing Committee, Ronald Shellard, Director of the CBPF and Luiz Davidovich, President of the Brazilian Academy of Science.

A distinctive innovation of Group 31, with respect to previous colloquia, was the creation of a special prize reserved for the most interesting posters presented by Master and Ph.D. students, with the aim of promoting active participation of the new generation.

The nowadays traditional Wigner Medal and Weyl Prize Award Ceremony, held in the splendid and prestigious frame of *Museu do Amanhã*, was the highlight of the Colloquium. The Wigner Medal, established in 1978 and administered by *The Group Theory and Fundamental Physics Foundation* located at the University of Texas at Austin and represented by Arno R. Bohm, recognizes and awards outstanding contributions through group theoretical and representation methods. The 2016 Wigner Medal was awarded to Bertram Kostant. Quoting Michio Jimbo of Rikkyo University, Tokyo, chair of the selection committee, *the lifelong achievements of Bertram Kostant have had profound impact in pure mathematics. At the same time his work miraculously has been finding his way to physics, suiting the spirit of Wigner who coined the famous phrase "the unreasonable effectiveness of mathematics in the physical sciences"*.

During his time as chairman (1994-2008), Heinz-Dietrich Dœbner convinced the Standing Committee of the International Colloquium on Group Theoretical Methods in Physics that it would be necessary for the future development of our field to acknowledge young researchers who presented outstanding work and to motivate them to continue and diversify their activity. Hence, the Weyl prize, established in 2002 by the Standing Committee, is awarded to young scientists who have performed original work in understanding physics through symmetries. A Selection Committee, chaired by Edward Frenkel of the University of California, Berkeley, awarded the 2016 Hermann Weyl Prize to Vasily Pestun of l'Institut des Hautes Études Scientifiques for his groundbreaking results in the study of supersymmetric gauge theories.

Francesco Toppan
Chairman of the Local Organizing Committee

Acknowledgments

A Colloquium of this type requires a large organization and would not have been possible without the active support of CBPF, its administration, the staff members, and the great help provided by its students. The Local Organizing Committee thanks them all for their contribution to a successful Group 31 Colloquium.

The Local Organizing Committee also thanks the colloquium other sponsors: Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES); Museu do Amanhã; Fundação Casa de Rui Barbosa; The World Academy of Sciences (TWAS); Abdus Salam International Centre for Theoretical Physics (ICTP); and Fundação Roberto Marinho.

The Local Organizing Committee

Organization of the volume

This volume contains contributions to the *31^{st} International Colloquium on Group Theoretical Methods in Physics* held on June 19–25, 2016 in Rio de Janeiro, Brazil.

Following the spirit of previous years, the colloquium covered a broad range of current topics from the fields of mathematical and theoretical physics. The variety of themes, joined by a common conceptual rather than a research theme, reflects well in the plenary talks given during the colloquium and range from cosmological problems to “the problem of life”.

Two prestigious prizes were awarded during the colloquium: Bertram Kostant was honored with the “Wigner Medal”, and Vasily Pestun received the “Hermann Weyl prize”. A short description of the awardees and *exposés of the Laudatios* open the volume.

A selection of the *plenary talks* is presented in the first section of the volume. The contributions are organized in alphabetical order and were not subjected to size restrictions or to a refereeing process.

Regular talks given during the colloquium are found in the following section of *longer papers*. During the event, these talks were grouped into mathematics- and physics-oriented contributions, each further organized into one of five parallel sessions. While such a division has obvious advantages for the organization of the colloquium, we opted for an alphabetical presentation in order to facilitate their localization. Longer contributions were restricted to a maximum of 10 pages. They have undergone an independent refereeing process and editorial decisions, as a result of which most, but not all of them have been included.

Poster clips presented during the event resulted in *shorter paper* contributions, which make up the third section of the volume. Shorter papers, restricted to a maximum of two pages, underwent the same refereeing process as longer papers, and also appear in alphabetical order.

During the event a *best poster prize* was awarded to three young researchers. The first prize went to Grace Akinwande Itunuoluwa (AIMS, Senegal) for her poster “Finding a dictionary between Tensor Models and GEM crystallization manifolds”. The second prize went to Diego Vidal (UNAM, Mexico) for the poster “Gravity from quantum space-time”. The third and final awardee was Florencia Benitez

Martinez (U. de la República, Uruguay) for her poster “Primordial tensor modes of the early Universe”. The Judging Committee was formed by Sylvie Paycha (Postdam, Germany), Sebastião Alves Dias (CBPF, Brazil) and José A. Helayël-Neto (CBPF, Brazil).

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2016 Wigner Medal is attributed to Bertram Kostant. Ceremony at the Museu do Amanã, Rio de Janeiro. From left to right: Olivier Fudym, Francesco Toppan, Shoshanna Kostant, Arno Bohm, Ann Kostant, Gerald Goldin, Bertram Kostant, Piotr Kielanowski, Abbe Kostant Smerling, Michelle Vergne, Jean-Pierre Gazeau, Mariano Del Olmo and Vasily Pestun. Image by Alvaro Farias.

2016 Weyl Prize is attributed to Vasily Pestun (left) by Mariano Del Olmo (right). Ceremony at the Museu do Amanã, Rio de Janeiro. Image by Alvaro Farias.

Laudatio of Bertram Kostant

Michèle Vergne
with Anthony Joseph and Shrawan Kumar

The 2016 Wigner Medal has been awarded to Bertram Kostant of the Massachusetts Institute of Technology (USA) for his fundamental contributions to the representation theory of Lie algebraic systems. Many of his results have led to new developments both in Mathematics and, as emphasized here, in Theoretical Physics.

For this occasion, let me highlight some of the themes in Kostant's work directly related to particle physics: Geometric quantization, convexity, and completely integrable systems. This brief account has been prepared with the help of Anthony Joseph and Shrawan Kumar.

The fundamental problem of quantum mechanics, as inaugurated by Dirac, is the passage from Hamiltonian mechanics to unitary representations of the symmetry group. Quantum mechanics should explain why some states of some physical systems take discrete values, and was directly motivated by the quantum theory of matter—at the time new—since it is the unitary transformations that preserve the all important probability density.

Valentine Bargmann and Eugene Wigner, the first recipients in 1978 of the Wigner medal, would have been delighted by the choice of the new laureate. Indeed, in his fundamental paper *Quantization and Unitary representations* (1970) B. Kostant showed that only those Hamiltonian manifolds admitting a prequantum line bundle, now called the Kostant line bundle, are candidates for giving rise to unitary representations of the symmetry group. Applied to the Poincaré group, this provided

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a clear theoretical understanding as to why a massive elementary particle must have a discrete spin.

Convexity theorems are important in determining the domain where experiments should be done.

For completely integrable systems, the Hamiltonian equation is solvable in an explicit fashion, the classical example being that of the Kepler laws of planetary motion. The Toda lattice, originally introduced as a simple model for a one-dimensional crystal, was generalized by Kostant into a multi-dimensional completely integrable system defined for any semisimple Lie algebra. A very simple and brilliant idea of Kostant produces a maximal algebra of Poisson commuting functions. Furthermore, the representation theory of semisimple groups allows us to compute the evolution law of the system.

Let me comment in more detail on Geometric Quantization, its history and its recent developments along the lines of Kostant's theory. There were many ways, apparently very different, to construct unitary representations of Lie groups. For example, the unitary representation of the Heisenberg group in the Bargmann-Fock space of holomorphic functions on the n-dimensional complex vector space, the Borel-Weil-Bott construction of the irreducible representations of a compact Lie group K on the $\overline{\partial}$ cohomology of flag manifolds with line bundles, Kirillov's construction of unitary representations of unipotent Lie groups by polarizing coadjoint orbits, Harish-Chandra's construction of unitary representations of real semisimple Lie groups based on differential equations and induction. Kostant saw that all these constructions are part of the unique scheme of quantum mechanics: passing from a classical phase space to a Hilbert space. Kostant realized the fundamental fact that any coadjoint orbit of a Lie group gives a Hamiltonian system. These systems are the most basic ones: any Hamiltonian manifold with a transitive action of a Lie group covers a coadjoint orbit, and those that are quantizable cover an orbit satisfying some discrete integrality conditions.

Furthermore, Kostant explained quantum conditions in terms of Chern classes of line bundles: a quantizable manifold is a symplectic manifold equipped with a prequantum line bundle, now called the Kostant line bundle. It could be "quantized" as a unitary representation of the underlying Lie group of symmetry if a suitable "polarization" could be found. This separates (removes) one half of the variables of phase space, a process that encapsulated Dirac's original insight.

Building on the Bargmann-Fock realization of representations of the Heisenberg group and of the quantum harmonic oscillator, Kostant considered complex polarizations, and the notion that the corresponding Hilbert space of sections is to be found among holomorphic sections, or going into cohomological constructions among solutions of a Dirac operator.

As a first successful use of geometric quantization, Kostant (with Auslander) classified the unitary representations of real class 1 simply-connected solvable Lie groups. Geometric quantization greatly generalizes provided one allows for cohomological methods and the study of the complex structure associated to a polarization.

Kostant's study of the homology of certain nilpotent Lie algebras encompasses the Borel-Weil-Bott theorem for compact Lie groups, and is used as a fundamental tool in constructing unitary representations for any real semisimple Lie group. Finally, as shown by Duflo, and also following the deep work of many authors, notably Schmid on the discrete series, geometric quantization of admissible coadjoint orbits of maximum dimension produces most (but not all) unitary representations of any real Lie group.

Geometric quantization applies to any Hamiltonian manifold. The main intrinsic object is the Kostant line bundle, together with its connection. This provides a moment map, and a notion of reduction. The most basic pieces of geometric quantization theory are quantization of coadjoint orbits. It was shown by Meinrenken-Sjamaar how to associate to any Kostant line bundle on a manifold with a compact group of symmetry a quantum model made up of these basic pieces and reflecting the semi-classical properties obtained at the asymptotic limit.

Severe difficulties may arise in quantizing a general Hamiltonian manifold with an arbitrary symmetry group, involving the absence of a suitable polarization and the verification of unitarity. The quantization of "small" coadjoint orbits or real semisimple Lie groups are of particular interest because they lead to many relations outside of those of the Lie algebra which are often just those of a physical system. The quantization of those orbits is difficult to construct. It may seem paradoxical that it is more difficult to quantize small coadjoint orbits than orbits of maximal dimensions. This is because they are small dimensional manifolds, but with a large group of symmetries and it may not be possible to integrate the full group of symmetries with a group of symmetries of the quantized space. If the Hamiltonian space is just one point with a trivial line bundle, then the quantization is just the trivial representation of the group G.

Models of quantization are usually produced by producing several models with different groups of symmetry, and then piecing these models together. This is the way that the metaplectic representation, a representation of the full group of symmetries of the simplest phase space T^*R^n, was constructed by Segal-Shale-Weil using the uniqueness of the canonical commutation relation. The following is one of the most fundamental representations, namely, the quantization of the minimal orbit of the symplectic group. With R. Brylinski, Kostant constructed uniform Fock space models for quantizing minimal orbits. Kostant showed that the smallest non-trivial orbit (for a semisimple Lie algebra) is defined by quadratic relations, thereby giving rise to a so-called quadratic algebra. This result is of great importance. In particular, this quadratic algebra was shown to be Koszul, which meant that it could be rather readily quantized — Gerstenhaber's ghastly infinite set of quantization conditions thereby reduces to just three. Imitating this, symplectic reflection algebras were defined and have proved to be central to the understanding of several physical systems, notably the Knizhnik-Zamolodchikov equations arising in the study of quantum many-body problems.

Pursuing the work of Valentine Bargmann on the complementary series, Kostant computed a remarkable determinant (for real Lie groups) whose description still provides one of the best tests for unitarity of the complementary series. The Kostant determinant had many other generalizations, notably by Parthasarathy-Ranga Rao-Varadarajan, Shapovalev, Jantzen and Kac. These have been used many times as a criteria for irreducibility and unitarity, and even for some infinite dimensional Lie groups.

Kostant's work on Lie algebra cohomology is an essential tool in representation theory. It was for example influential on Vogan's algebraic approach to the classification of irreducible representations of semisimple Lie groups, in particular in the success of the Atlas team in finding all unitary representations of the split form of E_8. Let us quote Bert: "Dealing with E_8 is like looking at a diamond...from one direction, one sees $2s$ all over the place, from another direction, one sees $3s$, from a third direction, one sees $5s$,... it is magnificent, it is a symphony in the numbers 2,3 and 5".

Let us recall at this point that the classification of all irreducible unitary representations of a real Lie semisimple Lie group is still an open problem.

Can "everything" be quantized? Yes, if one abandons the idea of unitarity. Deformation quantization is in some sense an infinitesimal version of geometric quantization, and it might not be possible to integrate the symmetries. Using the powerful techniques of Feynman graphs, Kontsevich showed that deformation quantization allows us to produce a quantization of the commutation relations of any Poisson manifold as a formal series. This striking result of Kontsevich relies in part on the fundamental Hochschild-Kostant-Rosenberg theorem identifying the Hochschild homology of an affine regular algebra.

One important object of quantization is the study of the spectra of matrices. The simplest case of representation theory is to study the decomposition of a Hermitian space under the action of a Hermitian matrix. Horn-Schur showed that the diagonal of a Hermitian matrix with prescribed spectrum always lies in some convex polytope, the vertices being obtained when the matrix itself is diagonal. Convexity results are important notably in studying measurements related to quantum computers. Kostant generalizes convexity results for linear projections, and also in the context of the Iwasawa decomposition $G = KAN$ of a real Lie group. It led to a further decomposition $G = KNK$ of the latter. Moreover it provided a generalization of the Golden-Thompson rule which was widely used in the C^* algebra approach to quantum field theory.

Let us now discuss completely integrable systems. One of Kostant's most influential articles is his paper on the *Toda Lattice* in 1978. The fact that the solution of the Toda lattice problem can be solved by the representation theory of the corresponding semisimple Lie algebra is a result of atonishing beauty and significance. It was the ingenuity of Kostant who could see at the time the interplay between coadjoint orbits of the Borel subgroup, a Hamiltonian manifold with a solvable Lie group of symmetry, and invariant polynomials of the corresponding semisimple Lie algebra, two of Kostant's favorite subjects. As is the case with several of Kostant's ideas, it is a brilliant, surprising, yet very simple idea.

The Toda lattice, originally introduced as a simple model for a one-dimensional crystal, was transformed by Kostant into a multi-dimensional completely integrable system defined for any semisimple Lie algebra. The quantization of a transversal slice, first undertaken by Kostant in the regular case and leading to the Whittaker model, was further developed by many in greater generality. This eventually allowed researchers (Premet, Losev, and others) to show that any nilpotent orbit could be quantized. When applied to Kac-Moody infinite dimensional Lie algebras, the Drinfeld-Sokolov generalization of the Toda system leads to W-algebras, the latter being important in the study of conformal field theory.

In this short talk, it is impossible to mention Kostant's various contributions to Pure Mathematics. Kostant is counted as one of the most remarkable mathematicians of the latter half of the last century in Lie Theory. Every paper by Kostant has a life of its own, being the precursor of many developments in representation theory of semisimple Lie groups and quantum groups, some developments being completely unexpected. His works and ideas have inspired innumerable mathematicians.

Each of the papers is a bright star in the dark sky of our knowledge. And, over the years, it has formed a beautiful constellation.

Thank you, Bert, for all this beautiful mathematics.

Reflecting on mathematics and mathematical physics

Bertram Kostant

Let me begin by expressing my appreciation to the Scientific Committee for awarding me the 2016 Wigner Medal, and to those on the Organizing Committee who have made it possible for me to be here tonight as well as all of you who are sharing with me in this great honor. The carefully chosen wording on this beautiful medal —"For fundamental work in representation theory that led to new branches of mathematics and physics" — resonates deeply with me, and captures the spirit of Wigner's phrase, "the unreasonable effectiveness of mathematics in the physical sciences."

I want to thank Michèle Vergne, distinguished member of the French Academy of Sciences, my colleague at MIT, and longtime friend for coming to Rio to give the Laudatio and speak about my work.

And I thank my wife Ann and two of my daughters, Abbe and Shoshanna, for coming with me to this memorable event. I would not have made it quite so easily without them. And they too insisted on being here.

I'd now like to go down memory lane with some unforgettable meetings related to mathematical physics.

I met Wigner many years ago in Princeton. Among the topics of conversation were the representation theory of the Lorenz group and the Poincaré group. I had discussions with Bargmann as well on a variety of subjects. Back in the 70s and 80s Bleuler and Doebner and so many colleagues from around the world regularly invited me to conferences in Group Theoretical Methods in Physics, which became an important part of my life, as I often recall those unforgettable meetings in Switzerland, Clausthal, and Salamanca, among others.

So, how did I get involved with group theory and Lie Groups in particular? It all began at the University of Chicago back in the late 40s and early 50s when I was a graduate student in mathematics. This was an historic time, exciting years for mathematics and physics. I met and spoke with Fermi and other physicists, and in mathematics, this period has been called the Stone Age, named for Marshall Stone

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who built an outstanding mathematics department. Those years became a turning point in my life.

André Weil also played a key role. He was among the many French mathematicians who led another French revolution, this time in mathematics, and brought critical ideas to the US. Weil spread the Bourbaki way of writing down and thinking about mathematics, and the sheer beauty of it all resonated with me immediately. I took a course with Ed Spanier who used Chevalley's book, which I marked up and still treasure—and that was my initiation into Lie theory.

At Princeton after Chicago, I had conversations with Oppenheimer, Hermann Weyl, Von Neumann, and many others, and I often think back to Von Neumann whose engaging personality and warmth towards me was especially gratifying, as was the time he gave me to deal with some of my questions.

Fast forward to 1955, on one unforgettable day, a week before he died, I met Einstein at the Institute. I'll tell you a short story about my encounter with him. It was on a Good Friday. Einstein realized that his driver had the day off. I was at the Institute and offered to drive him home. (In our conversation Einstein admitted his lack of mathematical knowledge.) We talked about a lot of different things and then he asked me what I was interested in. I told him I was interested in Lie Theory. Einstein looked at me, raised a shaking finger, nodded, and said "That will be very important some day."

Years later, I had stimulating conversations with Dirac in Florida. Dirac had invented a square root of the wave operator, and as a first-order operator it later gave rise by others to the theory of anti-particles. I was pleased to hear him tell me that the motivation for the operator was that it was mathematically beautiful.

This is but a very short glimpse into some of my enduring memories, which I've been privileged to share with you. Again, I want to thank the Scientific and Organizing Committees for this great honor.

Laudatio of Vasily Pestun

Luc Vinet

Dear colleagues and distinguished guests,

The Hermann Weyl Prize was established by the Standing Committee of the International Colloquium on Group Theoretical Methods in Physics in 2002 and is awarded every two years to recognize young scientists who have performed original work of significant scientific quality in the area of understanding physics through symmetries. To be eligible for the Weyl Prize, the candidate should be either under thirty-five years of age, or be within five years of having received the doctoral degree, at the time of the deadline of the application.

This year the members of the selection committee were:

- Edward Frenkel, UC Berkeley (Chair)
- Gitta Kutyniok, Berlin
- Neli Stoilova, Sofia
- Francesco Toppan, Rio de Janeiro
- Luc Vinet, Montreal

The Chair of our committe could not be here today and has asked me to introduce the 2016 winner of the prize which I am delighted to do.

It should first be said that the committee had a rather difficult task since there was a number of outstanding nominees that were all deserving to receive the prize. It is thus quite telling that in the end the members of the committee unanimously agreed to choose Vasily Pestun as the winner.

Vasily Pestun is currently a permanent professor at the IHS in Paris. Prior to this appointment he obtained his PhD in Physics from Princeton University under the supervision of Edward Witten; he has been a Junior Fellow at Harvard University and a member of the Institute for Advanced Study in Princeton. He has also received many awards including an ERC starting grant.

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Vasily Pestun is without doubt a leading mathematical physicist. His field of research is quantum field theory, its symmetries, and the use of symmetries in finding exact solutions of quantum field theory.

The groundbreaking result of Vasily Pestun is the computation of partition function of N=4 super-Yang-Mills theory on the four dimensional sphere. By an ingenious use of supersymmetric localization, he showed that this partition function, as well as the expectation values of the great circle supersymmetric Wilson loop, can be cast in the form of the correlation function of a two dimensional conformal field theory. This result led to tremendous activity in the field of BPS/CFT correspondence, with major discoveries in both two dimensional and four dimensional quantum field theories.

Let me now quote from the letter of recommendation of Nikita Nekrasov, himself a Weyl prize winner. Nikita writes:

> Perhaps the most important consequence of Dr. Pestun's work was its impact on the theoretical physics community. More than to 700 papers were written following up his work, extending it in various direction here I had to change Nekrasovs text because the number of citations has grown by more than a hundred in the last 6 months. Continuing with Nikitas comments : people found that the localization approach used by Pestun for the theories on spheres can be extended to the theories on ellipsoids, giving additional parameters to the partition functions one can play with. The structure of the partition function found by V. Pestun led to the discovery of the four-dimensional version of the tt-fusion found by Cecotti and Vafa in 1992, which was resisting generalizations for almost 15 years!

More recently Pestun has given a complete and definite treatment of the ordinary and quantized Seiberg-Witten geometry of 4- and 5-dimensional quiver gauge theory. The quantized Seiberg-Witten geometry and a connection with quantum integrable systems arise in the Nekrasov-Shatashvili limit. Pestun and collaborators have developed a very elegant and powerful way of untangling the complexities of this limit. It is based on the idea of q-characters that goes back to Frenkel and Reshetikhin almost 20 years ago, and is now rapidly gaining importance and appreciation in the mathematical physics community.

Furthermore with Kimura, Pestun has defined a general notion of deformed W-algebra, which makes sense for any quiver and specialized it to the algebra considered by Frenkel and Reshetikhin for ADE quiver. He then connected the conjectural formulas for the generating fields to the geometry of the corresponding Nakajima variety and also to the notion of qq-characters investigated by Nekrasov in recent years.

Commenting on these results of Pestun the Fields medallist Sacha Okounkov wrote:

> This beautiful construction completes a very important circle of ideas and represents very important progress in understanding the structure of the deformed W-algebras and in applying it to solve important problems in mathematical physics.

Okunkov concludes his recommendation letter by the following:

> Vasily Pestun is a highly original, exceptionally gifted, and very successful researcher working on the interface of supersymmetric gauge theories and what you may call geometric representation theory. Both of these topics obviously relate to symmetries, but from very different perspectives and in very different ways. The way in which they become intertwined in

> Pestun's work is really beautiful and innovative, I therefore consider him an exceptionally fitting candidate for the Hermann Weyl Prize.

In the official award statement, Edward Frenkel the Chair of the committe had these words:

> Vasily Pestun's original contributions opened new opportunities for fruitful interaction between mathematics and quantum physics. It is quite fitting that his work is honored by the prize named after Hermann Weyl, a pioneer in both of these fields who used to say that in his research, he always tried to unite the true and the beautiful.

Ladies and Gentlemen, please welcome the 2016 Weyl prize winner Vasily Pestun.

Part II
Plenary invited articles

Phenomenology of neutrinos and macroscopic bodies in non-commutative spacetime

Giovanni Amelino-Camelia

Abstract Over the last decade the efforts in quantum-gravity phenomenology have been intensified significantly, and spacetime noncommutativity has inspired quite a few of the relevant proposals. I here focus on two recent developments for quantum-gravity phenomenology inspired by spacetime noncommutativity, which concern neutrino observations and the description of the total momentum of a macroscopic body.

1 Introduction

The field of quantum-gravity phenomenology [3] has experienced strong growth over the last decade. Several proposals have been put forward for types of experiments and observations which might have the peculiar qualities needed to be sensitive to the minute quantum-gravity-scale effects. Among the formalisms which proved most fruitful in inspiring some of these phenomenological avenues a prominent role is played by theories with spacetime noncommutativity and the associated description of relativistic symmetries [1], which can be given in particular by Hopf algebras [5, 14, 16] (quantum groups). I here want to focus on two projects of this type, inspired by spacetime noncommutativity, which kept me busy recently and might have rather broad implications.

I have written elsewhere (see, e.*g*., Ref. [3] and references therein) about a phenomenology focused on propagation of photons in a quantum spacetime, for which indeed certain spacetime-noncommutativity models have provided a good part of the inspiration [5]. We are now starting to open a new window on the Universe. The first cosmological high-energy neutrinos have been observed. I here offer a short

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summary of (and perspective on) the first steps which have been taken for analyzing data from the perspective of neutrino propagation in a quantum spacetime.

Besides writing about neutrino phenomenology I shall also offer some observations on the description of the total momentum of a macroscopic body in a quantum spacetime. When there are nonlinearities in momentum space the description of macroscopic bodies can be pathologic: the same nonlinearities producing minute effects for microscopic particles (totally or for the most part not observable for us) could produce, if applicable, a picture of macroscopic bodies in sizable conflict with what is observed. Some spacetime-noncommutativity models do produce nonlinearities in momentum space, but, as I shall here show, these nonlinearities do not affect the total momentum of a macroscopic body.

2 Neutrino phenomenology

In-vacuo dispersion has been discussed extensively in the context of some much-studied models of spacetime quantization (see, e.*g*., [1,3,6,7,11,12] and references therein), and particularly spacetime noncommutativity [5]. These results can inspire the hypothesis that the time needed for a ultrarelativistic particle to travel from a given source to a given detector receives a quantum-spacetime correction, here denoted with Δt. I here follow Ref. [10], so I focus on the class of scenarios whose predictions for Δt can all be described, for corresponding choices of the parameters η and δ, in terms of the formula (working in units with the speed-of-light scale "c" set to 1)

$$\Delta t = \eta \frac{E}{M_P} D(z) \pm \delta \frac{E}{M_P} D(z)\,. \tag{1}$$

Here the redshift- (z-)dependent $D(z)$ carries the information on the distance between source and detector, for which it is customary to take exploratively the form [12]

$$D(z) = \int_0^z d\zeta \frac{(1+\zeta)}{H_0\sqrt{\Omega_\Lambda + (1+\zeta)^3\Omega_m}}\,, \tag{2}$$

where Ω_Λ, H_0 and Ω_0 denote, as usual, respectively the cosmological constant, the Hubble parameter and the matter fraction. M_P denotes the Planck scale ($\simeq 1.2 \cdot 10^{28} eV$) while the values of the parameters η and δ in (1) characterize the specific scenario one intends to study. The notation "$\pm\delta$" reflects the fact that δ parametrizes the size of quantum-uncertainty (fuzziness) effects.

The parameters η and δ are expected to take values somewhere in a neighborhood of 1, but values as large as 10^3 are plausible if the solution to the quantum-gravity problem is somehow connected with the unification of non-gravitational forces while values significantly smaller than 1 find support in some renormalization-group arguments. In general, η and δ can take different values for different particles [3] and in particular, one should allow for a dependence of η and δ on the helicity [3] of the neutrino.

The relevant phenomenology could be particularly powerful [10] for neutrinos produced by GRBs (gamma-ray bursts). For the analysis of candidate GRB neutrinos possibly affected by in-vacuo dispersion it is convenient to introduce a "distance-rescaled time delay" Δt^* defined as

$$\Delta t^* \equiv \Delta t \frac{D(1)}{D(z)} \tag{3}$$

so that (1) can be rewritten as

$$\Delta t^* = \eta \frac{E}{M_P} D(1) \pm \delta \frac{E}{M_P} D(1)\,. \tag{4}$$

If one measures a certain Δt for a candidate GRB neutrino and the redshift z of the relevant GRB is well known, then one gets a firm determination of Δt^* by simply rescaling the measured Δt by the factor $D(1)/D(z)$. When the redshift of the relevant GRB is not known accurately one will be able to convert a measured Δt into a determined Δt^* with accuracy governed by how much one is able to still assume about the redshift of the relevant GRB.

In order to select some GRB-neutrino candidates one needs [9, 10] a temporal window and criteria of directional selection. The analysis of Ref. [10] focuses on neutrinos with energies between 60 TeV and 500 TeV, allowing for a temporal window of 3 days, and the directional criteria for the selection of GRB-neutrino candidates on the signal direction PDF depending on the space angle difference between GRB and neutrino: $P(\nu, GRB) = (2\pi\sigma^2)^{-1}\exp(-\frac{|\mathbf{x}_\nu - \mathbf{x}_{GRB}|^2}{2\sigma^2})$, a two dimensional circular Gaussian whose standard deviation is $\sigma = \sqrt{\sigma_{GRB}^2 + \sigma_\nu^2}$, denoting of course with σ_{GRB} and σ_ν respectively the uncertainties in the direction of observation of the GRB and of the neutrino. One then requests [10] that a GRB-neutrino candidate should be such that the pair composed by the neutrino and the GRB is at angular distance compatible within a 2σ region.

Evidently whenever η and/or δ do not vanish one should expect on the basis of (4) a correlation between the $|\Delta t^*|$ and the energy of the candidate GRB neutrinos.

Ref. [10] considered four years of operation of IceCube, from June 2010 to May 2014. Since the determination of the energy of the neutrino plays such a crucial role in the analysis one focuses only on IceCube "shower events". There are 21 such events within our 60-500 TeV energy window, and 9 of them fit the requirements of Ref. [10] for candidate GRB neutrinos. For some of these 9 candidates the selection criteria produce multiple GRB-neutrino candidates, which one can handle by focusing on the case that provides the highest correlation.

For the majority of GRBs relevant for the analysis the redshift was not measured. For the rather rare cases of short GRBs this can be handled [10] by assuming the redshift of 0.6. For long GRBs one typically will have some in the relevant sample for which the redshift is known, and as argued in Ref. [10] one can use those known values of redshift for obtaining at least a rough estimate of the redshift of long GRBs for which the redshift is unknown.

Following these prescriptions one finds a correlation of 0.951 between $|\Delta t^*|$ and energy, for the 9 GRB-neutrino candidates found in Ref. [10]. This is a strikingly high value of correlation, which invites one to ask [10] how likely it would be to have accidentally data with such good agreement with the expectations of the quantum-spacetime models here contemplated. Ref. [10] proposed that one needs to estimate how often a sample composed exclusively of background neutrinos would produce accidentally 9 or more GRB-neutrino candidates with correlation comparable to (or greater than) those found in data. This estimate was performed in Ref. [10] finding that background neutrinos could produce accidentally 9 or more GRB-neutrino candidates with correlation ≥ 0.951 only in 0.03% of cases.

These numbers are somewhat impressive but of course we should assess them prudently. These numbers already take into account the fact that the analysis involves only a few neutrinos, but somehow I still feel that because this is all about just a few neutrinos we should be more cautious than the numbers appear to suggest. There is no reason to rush to any conclusions, since more data is being gathered by IceCube and will soon be reported. Still, it is interesting to take for a moment as a working assumption that indeed these results are a true manifestation of in-vacuo dispersion. In that hypothesis what would most surprise me is that our tentative formulas provide such a good match. In particular, most results on in-vacuo dispersion, including those based on spacetime noncommutativity, were obtained for flat/non-expanding spacetimes, while of course these data analyses require to factor in the Universe expansion. The formula given above for $D(z)$ is our best guess so far of how the effects of in-vacuo dispersion interface with spacetime expansion. For contexts where the universe is expanding at an accelerating rate we have some theory support for this $D(z)$ (at least as one among a few possibilities [8]) by rather compelling arguments applicable to de Sitter expansion. However, at high redshifts, according to the current picture of cosmology, the Universe should be described by a phase of decelerating expansion, and we have no solid result on which to anchor our description of in-vacuo dispersion. For the case of decelerating expansion one still assumes the validity of $D(z)$ without any support from theory. I feel that we urgently need studies of in-vacuo dispersion applicable to cases in which spacetime is in decelerated expansion.

3 On the description of macroscopic bodies in a non-commutative spacetime

An emerging characteristic of quantum-gravity research over the last decade has been a gradual shift of focus toward manifestations of the Planck scale on momentum space, particularly pronounced in some approaches to quantum gravity. In particular for some research lines based on spacetime noncommutativity several momentum-space structures have been in focus, including the possibility of deformed laws of composition of momenta. There has been growing interest in the conceptual implications and possible phenomenological implications [3] of nonlinear laws on momentum space and particularly nonlinear laws of composition of momenta. However, this interest is being tempered by concerns that a nonlinear law of addition of momenta might produce a pathological description of the total momentum of a macroscopic body (see, e.*g*., Refs. [2,13,15]). This issue has been often labelled as the "soccer-ball problem" [2]: the quantum-gravity pictures lead one to expect nonlinearities of the law of composition of momenta which are suppressed by the Planck scale ($\sim 10^{28} eV$) and would be unobservably small for particles at energies we presently can access, but in the analysis of a macroscopic body, such as a soccer ball, one might have to add up very many of such minute nonlinearities, potentially producing a conflict with experimentally-established facts.

I here show that previous discussions of this soccer-ball problem failed to appreciate the differences between two roles for laws of composition of momentum in physics. Previous results supporting a nonlinear law of addition of momenta relied exclusively on the role of a law of momentum composition in the description of spacetime locality. The notion of total momentum of a multi-particle system is not a manifestation of locality, but rather reflects translational invariance in interacting theories. After being myself confused about these issues for quite some time [2] I feel I am now in a position to address them. For definiteness I do this focusing here on a specific simple model affected by nonlinearities for a law of composition of momenta, a 2+1-dimensional model with pure-spatial κ-Minkowski noncommutativity [14, 16], so that the time coordinate is left unaffected by the deformation and the two spatial coordinates, x_1 and x_2, are governed by

$$[x_1, x_2] = i\ell x_1 \tag{5}$$

(with the deformation scale ℓ expected to be of the order of the inverse of the Planck scale).

3.1 Soccer-ball problem and sum of momenta from locality

The ingredients needed for seeing a nonlinear law of composition of momenta emerging from noncommutativity of type (5) are very simple. Essentially one needs only to rely on results establishing that functions of coordinates governed by (5) still admit a rather standard Fourier expansion

$$\Phi(x) = \int d^4k \; \tilde{\Phi}(k) \; e^{ik_\mu x^\mu}$$

and that the notion of integration on such a noncommutative space preserves many of the standard properties, including

$$\int d^4x \; e^{ik_\mu x^\mu} = (2\pi)^4 \delta^{(4)}(k) \,. \tag{6}$$

It is a rather standard exercise for practitioners of spacetime noncommutativity to use these tools in order to enforce locality within actions describing classical fields. For example, one might want to introduce in the action the product of three (possibly identical, but in general different) fields, Φ, Ψ, Υ, insisting on locality in the sense that the three fields be evaluated "at the same quantum point x", i.*e*., $\Phi(x)\Psi(x)\Upsilon(x)$. There is still no consensus on how one should formulate the more interesting quantum-field version of such theories, and it remains unclear to which extent and in which way our ordinary notion of locality is generalized by the requirement of evaluating "at the same quantum point x" fields intervening in a product such as $\Phi(x)\Psi(x)\Upsilon(x)$. Nonetheless for the classical-field case there is a sizable literature consistently adopting this prescription for locality. Important for my purposes here is the fact that, with such a prescription, locality inevitably leads to a nonlinear law of composition of momenta, as I show explicitly in the following example:

$$\begin{aligned}
&\int d^4\hat{x}\, \Phi(\hat{x})\, \Psi(\hat{x})\, \Upsilon(\hat{x}) = \qquad (7)\\
&= \int d^4\hat{x} \int d^4k \int d^4p \int d^4q\, \tilde{\Phi}(k)\, \tilde{\Psi}(p)\, \tilde{\Upsilon}(q)\, e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} e^{iq_\rho \hat{x}^\rho}\\
&= \int d^4\hat{x} \int d^4k \int d^4p \int d^4q\, \tilde{\Phi}(k)\, \tilde{\Psi}(p)\, \tilde{\Upsilon}(q) e^{i(k\oplus p\oplus q)_\mu \hat{x}^\mu}\\
&= (2\pi)^4 \int d^4k \int d^4p \int d^4q\, \tilde{\Phi}(k)\, \tilde{\Psi}(p)\, \tilde{\Upsilon}(q)\, \delta^{(4)}(k\oplus p\oplus q)
\end{aligned}$$

where $\oplus$ is such that

$$(k\oplus p)_0 = k_0 + p_0 \qquad (8)$$

$$(k\oplus p)_2 = k_2 + p_2 \qquad (9)$$

$$(k\oplus p)_1 = \frac{k_2+p_2}{1-e^{\ell(k_2+p_2)}}\left[\frac{1-e^{\ell k_2}}{k_2 e^{\ell p_2}} k_1 + \frac{1-e^{\ell p_2}}{p_2} p_1\right]. \qquad (10)$$

This result is rooted in one of the most studied aspects of such noncommutative spacetimes, which is their "generalized star product" [4]. This is essentially a characterization of the properties of products of exponentials induced by rules of noncommutativity of type (5). Specifically, one easily arrives at (7) (with $\oplus$ such that, in particular, (10) holds) by just observing that from the defining commutator (5) it follows that

$$\begin{aligned}
&\log\left[\exp\left(ik_2\hat{x}_2 + ik_1\hat{x}_1\right)\exp\left(ip_2\hat{x}_2 + ip_1\hat{x}_1\right)\right] = \qquad (11)\\
&= i\hat{x}_2(p_2+k_2) + i\hat{x}_1 \frac{k_2+p_2}{1-e^{\ell(k_2+p_2)}}\left(\frac{1-e^{\ell k_2}}{k_2 e^{\ell p_2}} k_1 + \frac{1-e^{\ell p_2}}{p_2} p_1\right).
\end{aligned}$$

The so-called soccer-ball problem concerns the acceptability of laws of composition of type (10). Since one assumes that the deformation scale ℓ is of the order of the inverse of the Planck scale, applying (10) to microscopic/fundamental particles has no sizable consequences: of course (10) gives us back to good approximation $(k \oplus p)_1 \simeq k_1 + p_1$ whenever $|\ell k_2| \ll 1$ and $|\ell p_2| \ll 1$. But if a law of composition such as (10) should be used also when we add very many microparticle momenta in obtaining the total momentum of a multiparticle system (such as a soccer ball) then the final result could be pathological even when each microparticle in the system has momentum much smaller than $1/\ell$.

3.2 Sum of momenta from translational invariance

As clarified in the brief review of known results given in the previous subsection, a nonlinear law of composition of momenta arises in characterizations of locality, as a direct consequence of the form of some star products. My main point here is that a different law of composition of momenta is produced by the analysis of translational invariance, and it is this other law of composition of momenta which is relevant for the characterization of the total momentum of a multi-particle system. Here too I shall just use known facts about the peculiarities of translation transformations in certain noncommutative spacetimes, but exploit them for obtaining results relevant for the description of the total momentum of a multi-particle system.

A first hint that translation transformations should be modified in certain non-commutative spacetimes comes from noticing that (5) is incompatible with the standard Heisenberg relations $[p_j, x_k] = i\delta_{jk}$. Indeed, if one adopts (5) and $[p_j, x_k] = i\delta_{jk}$ one then easily finds that some Jacobi identities are not satisfied. The relevant Jacobi identities are satisfied if one allows for a modification of the Heisenberg relations which balances the noncommutativity of the coordinates:

$$[p_1, x_1] = i\ , \ \ [p_1, x_2] = 0\ , \ \ [p_2, x_2] = i\ , \tag{12}$$

$$[p_1, x_2] = -i\ell p_1\ . \tag{13}$$

One easily finds that combining (5), (12) and (13) all Jacobi identities are satisfied.

Additional intuition for these nonstandard properties of the momenta p_j comes from actually looking at which formulation of translation transformations preserves the form of the noncommutativity of coordinates (5). Evidently the standard description

$$x_2 \to x_2' = x_2 + a_2\ , \ \ x_1 \to x_1' = x_1 + a_1$$

is not a symmetry of (5):

$$[x_1', x_2'] = [x_1 + a_1, x_2 + a_2] = i\ell x_1 = i\ell(x_1' - a_1). \tag{14}$$

Unsurprisingly what does work is the description of translation transformations using as generators the p_j of (12)-(13), which as stressed above satisfy the Jacobi-identity criterion. These deformed translation transformations take the form

$$x_1' = x_1 - ia_1[p_1, x_1] - ia_2[p_2, x_1] = x_1 + a_1 \; ,$$
$$x_2' = x_2 - ia_1[p_1, x_2] - ia_2[p_2, x_2] = x_2 + a_2 - \ell a_1 p_1 \; , \tag{15}$$

and indeed are symmetries of the commutation rules (5):

$$[x_1', x_2'] = [x_1 + a_1, x_2 + a_2 - \ell a_1 p_1] =$$
$$= i\ell x_1 - \ell a_1 [x_1, p_1] = i\ell(x_1 + a_1) = i\ell x_1' \; . \tag{16}$$

My main observation is that in order for us to be able to even contemplate the total momentum of a multiparticle system, we must be dealing with a case where translational invariance is ensured: total momentum is the conserved charge for a translationally invariant multi-particle system. Surely the introduction of translationally invariant multi-particle systems must involve some subtleties due to the noncommutativity of coordinates, and these subtleties are directly connected to the new properties of translation transformations (13), but they are not directly connected to the properties of the star product (11) and the associated law of composition of momenta (10). For my purposes, it is best to show the implications of this point very simply and explicitly, focusing on a system of two particles interacting via a harmonic potential.

I start by noticing that evidently one does not achieve translational invariance through a description of the form

$$\mathscr{H}_{non-transl} = \frac{(p_1^A)^2 + (p_2^A)^2 + (p_1^B)^2 + (p_2^B)^2}{2m} + \frac{\rho}{2}[(x_1^A - x_1^B)^2 + (x_2^A - x_2^B)^2]$$

where indices A and B label the two particles involved in the interaction via the harmonic potential. As stressed above, translation transformations consistent with the coordinate noncommutativity (5), must be such that (see (15)) $x_1 \to x_1 + a_1$ and $x_2 \to x_2 + a_2 - \ell a_1 p_1$, and as a result by writing the harmonic potential with $(x_1^A - x_1^B)^2 + (x_2^A - x_2^B)^2$ one does not achieve translational invariance.

One does get translational invariance by adopting instead

$$\mathscr{H} = \frac{(p_1^A)^2 + (p_2^A)^2 + (p_1^B)^2 + (p_2^B)^2}{2m} + \frac{\rho}{2}[(x_1^A - x_1^B)^2 + (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2].$$

This is trivially invariant under translations generated by p_2, which simply produce $x_1 \to x_1$ and $x_2 \to x_2 + a_2$. And it is also invariant under translations generated by p_1, since they produce $x_1 \to x_1 + a_1$ and $x_2 \to x_2 - \ell a_1 p_1$, so that $x_2 + \ell x_1 p_1$ is left unchanged:

$$x_2 + \ell x_1 p_1 \to x_2 - \ell a_1 p_1 + \ell(x_1 + a_1) p_1 = x_2 + \ell x_1 p_1 .$$

It is interesting for my purposes to see which conserved charge is associated with this invariance under translations of the hamiltonian $\mathscr{H}$. This conserved charge will describe the total momentum of the two-particle system governed by $\mathscr{H}$, i.*e*., the center-of-mass momentum. It is easy to see that this conserved charge is just the standard $\mathbf{p}^A + \mathbf{p}^B$. For the second component one trivially finds that indeed

$$[p_2^A + p_2^B, \mathscr{H}] = 0$$

and the same result also applies to the first component:

$$\begin{aligned}
[p_1^A + p_1^B, \mathscr{H}] &\propto [p_1^A + p_1^B, (x_1^A - x_1^B)^2] + \\
&+ [p_1^A + p_1^B, (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2] = \\
&= [p_1^A + p_1^B, (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2] \propto \\
&\propto [p_1^A + p_1^B, x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B] \\
&= -i\ell p_1^A + i\ell p_1^A + i\ell p_1^B - i\ell p_1^B = 0
\end{aligned} \tag{17}$$

where the only non-trivial observation I have used is that (5) leads to $[p_1, x_2 + \ell x_1 p_1] = -i\ell p_1 + i\ell p_1 = 0$.

The result (17) shows that indeed $\mathbf{p}^A + \mathbf{p}^B$ is the momentum of the center of mass of my translationally-invariant two-particle system, i.*e*., it is the total momentum of the system.

The concerns about total momentum that had been voiced in discussions of the Planck-scale soccer-ball problem were rooted in the different sum of momenta relevant for locality, the $\oplus$ sum discussed in the previous section. It was feared that one should obtain the total momentum by combining single-particle momenta with the nonlinear $\oplus$ sum. The result (17) shows that this expectation was incorrect. One can also directly verify that indeed $\mathbf{p}^A \oplus \mathbf{p}^B$ is not a conserved charge for my translationally-invariant two-particle system, and specifically, taking into account (10), one finds that

$$[(\mathbf{p}^A \oplus \mathbf{p}^B)_1, \mathscr{H}] \neq 0.$$

4 Outlook

The results I here summarized for candidate GRB neutrinos are evidently intriguing and set the stage for a very active research line, considering that IceCube will take much more data and other neutrino telescopes (such as KM3NeT) are at advanced stage of scheduling. By 2019 IceCube alone should put this sort of analyses in position to work with more than twice the amount of data so far available. In general I expect that the new opportunities provided by the birth of neutrino astrophysics will affect fundamental physics very strongly.

The description of macroscopic bodies in a quantum spacetime has been a very active area of study, to which I here contributed novel results for the description of total momentum. The "soccerball problem" fades away. From a conceptual perspective it is also interesting that the analysis I here reported makes us appreciate how our current theories are built on a non-trivial correspondence between the momentum-space manifestations of locality and translational invariance. I hope future studies will allow us to understand more in depth the subtleties of this correspondence, which were here only preliminarily exposed. This might be achieved by also taking as guidance the fact that in Galilean relativity all laws of composition of momenta and velocities are linear, and there is a linear relationship between velocity and momentum. Within Galilean-relativistic theories one could choose to never speak of momentum, and work exclusively in terms of velocities, with apparently a single linear law of composition of velocities. In our current post-Galilean theories, the relationship between momentum and velocity is non-linear (and velocities are composed non-linearly, while the laws of composition of momenta remain linear) and we then manage to better appreciate the differences between the logical roles of the composition law for momenta and those of the composition law for velocities.

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The dynamical evolution in quantum physics and its semi-group

Arno Bohm

Abstract Experiments on quantum systems are usually divided into preparation of states and the registration of observables. Using the traditional mathematical methods (the Hilbert space or the Schwartz space of distribution theory), it is *not* possible to distinguish mathematically between observables and states. The Hilbert space boundary conditions for the dynamical equation lead by mathematical theorems (Stone-von-Neumann) to unitary group evolution with $-\infty < t < +\infty$. In contrast, the set-up of a scattering experiments calls for time-asymmetric boundary conditions. Therefore, a new axiom of quantum theory is needed. This is the Hardy space axiom, which uses a pair of Hardy spaces, one of them for states (defined by the experimental preparation procedure), and the other for observables (defined by detectors). The Paley-Wiener theorem for Hardy spaces then leads to semi-groups and time asymmetry. It introduces a finite beginning of time t_0 for a time asymmetric quantum theory, which can be observed by an ensemble of onset times $t_0^{(i)}$ of dark periods in Dehmelt's quantum jump experiments with single ions [1].

1 Time symmetric quantum theory

Time in quantum theory is usually assumed to extend over $-\infty < t < +\infty$.

Time Asymmetric Quantum Theory [2] is a quantum theory in which:

- the time t has a preferred direction: $t_0 = 0 \le t < \infty$
- the energy E (eigenvalue of the "essentially" selfadjoint Hamiltonian H) can take (discrete and continuous) values in the complex energy planes: $E \to z \in \mathbb{C}_\pm$ (1)

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The **conventional** mathematical theory for quantum physics [3]: is time-symmetric

- the time t extends over $-\infty < t < +\infty$
- the energy E is real (spectrum of selfadjoint Hamiltonian H) and it is bounded from below ("stability of matter"): $0 = E_0 \leq E < \infty$.

This is a consequence of the choice of the *boundary conditions* for the dynamical equations

$$i\hbar\frac{\partial}{\partial t}\psi = -H\psi \qquad \text{(2a)} \quad \text{or} \quad i\hbar\frac{\partial}{\partial t}\phi = H\phi \qquad \text{(2b)}$$

the Heisenberg eq. for observables $|\psi\rangle\langle\psi|$ and Schrödinger eq. for states ϕ.

To find the solutions of differential equations one needs *Boundary Conditions* (B.C.). In Standard Quantum Mechanics these B.C. are usually chosen to be given by the
Hilbert Space Axiom:

$$\text{set of states } \{\phi\} = \text{set of observables } \{\psi\} = \mathscr{H} = \text{Hilbert space (norm-complete)} \tag{3}$$

This means energy wave functions $\langle E|\phi\rangle = \phi(E)$, $\langle E\,|\,\psi\rangle = \psi(E)$ are *Lebesgue* square integrable functions of energy, i.e., to one state ϕ does not correspond one wave function $\phi(E)$ but infinitely many that differ from each other on a set of measure zero, e.g., at all rational numbers E.

To avoid these complications one uses in quantum physics only smooth Schwartz space functions $\phi(E)$ and $\psi(E)$ and the convergence is defined not by one norm, but by a countable number of norms, e.g. for the harmonic oscillator by the definition of norms:

$$(\phi,\psi)_n = (\phi,(H+E_0)^n\psi), \quad n = 0,1,2,3\ldots, \tag{4}$$

where H is the energy operator or the Nelson operator for the quantum system.

In the Dirac formulation one uses the Schwartz space Φ with countable norms, as e.g., defined by (4):

1) The solutions of both the Heisenberg equation as well as the solutions of the Schrödinger equation (observable and state) have a Dirac basis vector expansion for processes with continuous E:

$$\phi = \sum_{j,j_3,\eta}\int dE\ |E,j,j_3,\eta\rangle\langle E,j,j_3,\eta\,|\phi\rangle = \int dE\,|E\rangle\langle E\,|\phi\rangle, \tag{5}$$

(an analogue of $\mathbf{x} = \sum_{i=1}^{3}\mathbf{e}_i x^i$).

The basis vectors $|E,j,j_3,\eta\rangle$ are "eigenkets" of the energy operator H (and a complete system of commuting operators H, J^2, J_3, η^{op}), using angular momentum j, j_3 and possibly other quantum numbers η:

$$\langle\phi\,|\,H\,|E,j,j_3,\eta\rangle = E\langle\phi\,|E,j,j_3,\eta\rangle \qquad \text{for all vectors } \phi,\ \psi\in\Phi,\ |E\rangle\in\Phi^\times.$$

To each vector ϕ corresponds then one function $\langle E\,|\phi\rangle = \phi(E)$ of E (and additional quantum numbers, like j, j_3).

2) The components of ϕ, i.e. the bra-kets $\langle E\,|\phi\rangle = \phi(E)$ are smooth, rapidly decreasing functions of E ("Schwartz function" $\in \mathscr{S}_{\mathbb{R}_+}$), and

$$\begin{aligned} &\text{one has a triplet of function spaces} && \{\phi(E)\} = \mathscr{S} \subset L^2 \subset \mathscr{S}^\times \\ &\text{and a triplet of abstract vector spaces} && \{\phi\} = \Phi \subset \mathscr{H} \subset \Phi^\times \end{aligned} \tag{6a}$$

called a Gelfand Triplet or Rigged Hilbert Space (RHS) [4]. In the Dirac formalism one uses the **same** RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$ for the state vectors ϕ as well as for the observables $|\psi\rangle\langle\psi|$:

$$\{\phi\} = \{\psi\} = \Phi = \text{abstract Schwartz space.} \tag{6b}$$

Is there a physical reason that the solutions of the dynamical (Schrödinger or Heisenberg) equation which fulfill the Hilbert space boundary condition (3), as well as those fulfilling the Schwartz space boundary condition, $\phi \in \Phi$, $\psi \in \Phi$, are given by the two time evolution groups?:

$$\phi(t) = e^{-iHt}\phi(0), \quad \psi(t) = e^{iHt}\psi(0), \qquad -\infty < t < +\infty. \tag{7}$$

The conclusion is: For standard quantum mechanics, even when amended with the Dirac formalism in a Schwartz-Rigged Hilbert Space, the time extends over $-\infty < t < +\infty$ and there is *no* finite beginning of time $t_0 > -\infty$, as required for time asymmetric quantum theory in condition (1) of Sect. 1. Therefore, the dynamical equations (2b) for the state vectors $\phi(t)$ and the dynamical equation (2a) for the observable (i.e., the solution of the Heisenberg equation (2a)) obey the unitary group evolution (1).

The question is: Could there be for quantum theory other boundary conditions of the dynamical equation, that do not lead to the unitary group evolution like (1), but to a quantum theory with a preferred direction of time, starting at a finite $t_0 < t < \infty$, which our Universe seems to posses as the big bang time t_0?

2 Dynamical equations of states and observables

The two fundamental entities of quantum theory are states (denoted by ϕ^+ or by operator ρ) and observables (denoted by operator A or vector ψ^-). The time evolution is expressed using two contrasting pictures:

In the Schrödinger picture / ***In the Heisenberg picture***

One solves

The Schrödinger equation

$$i\hbar \frac{\partial \phi^+(t)}{\partial t} = H\phi^+(t) \quad (8)$$

for the state vector $\phi^+(t)$,

or

the von Neumann equation for the state operator ("density operator") $\rho(t)$

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H,\rho(t)]. \quad (9)$$

The Heisenberg equation

$$i\hbar \frac{\partial A(t)}{\partial t} = -[H,A(t)] \quad (10)$$

for the observables $A(t)$,

or

the Heisenberg equation for special case $A_{\psi^-} = |\psi^-(t)\rangle\langle\psi^-(t)|$ for the vector observable ψ^-

$$i\hbar \frac{\partial}{\partial t}\psi^-(t) = -H\psi^-(t). \quad (11)$$

The Schrödinger equation (1) is the special case of the von Neumann equation (1) for the case: $\rho(t) = |\phi^+(t)\rangle\langle\phi^+(t)|$.

The Heisenberg equation (4) is the special case of the Heisenberg equation (2) for the "observable vector" $\psi^-(t)$ in the special case $A_{\psi^-} = |\psi^-(t)\rangle\langle\psi^-(t)|$.

State operator ρ or the state vector ϕ^+, as well as the observable (-operators) A or the observable vector ψ^-, represent physical apparatuses in laboratory experiments.

The theoretical predictions which need to be compared with the experimental data are the Born probabilities:

$$\mathscr{P}_\rho(A(t)) \equiv \mathrm{Tr}(A(t)\,\rho) = \mathrm{Tr}(A\,\rho(t)). \quad (12)$$

In the special case that A is the projection operator $A = |\psi^-\rangle\langle\psi^-|$ onto the 1-dimensional subspace spanned by $|\psi^-\rangle$ and $\rho(t) = |\phi(t)\rangle\langle\phi(t)|$, one gets

$$\mathscr{P}_{\phi^+}(\psi^-(t)) = \mathrm{Tr}(|\psi^-(t)\rangle\langle\psi^-(t)||\phi^+\rangle\langle\phi^+| = |\langle\psi^-|\phi^+(t)\rangle|^2, \quad (13)$$

which represents the probability of the observable $\psi^-(t)$ in the state ϕ^+.

3 Meaning of states ρ or ϕ^+, and of observable A or ψ^-

States and observables are associated with two different aspects of scattering experiments. Scattering experiments test the structure of micro physical systems.

States: are described in the theory by "density operators" ρ or by state vectors ϕ^+ for pure states $\rho = |\phi^+\rangle\langle\phi^+|$.

Observables: are described by operators $A = A^\dagger$, or also by "observable vectors" ψ^- (i.e., vectors that obey the Heisenberg equation (4)).

In the experiment [5]:

1. *States* ρ (or the pure *in*-states ϕ^+ of scattering experiment)	*are defined* experimentally	by the preparation apparatus (e.g., accelerator).

Preparation of a state $|\phi^+\rangle\langle\phi^+|$ or ρ

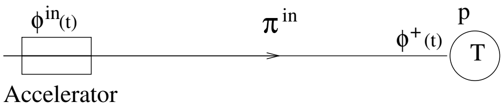

Fig. 1: Accelerator and target T define the in-state.

Due to interaction of beam and target T, the in-state ϕ^+ becomes an "uncontrolled out-state" $\phi^+(t) \to \phi^{\text{out}}$ outside the interaction region

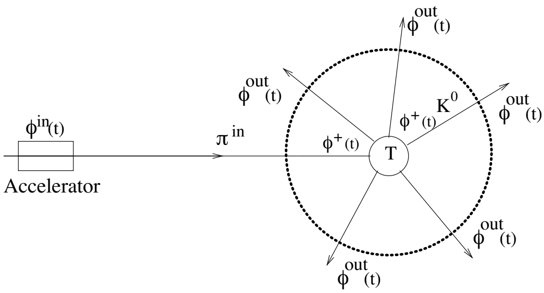

Fig. 2: The preparation of the uncontrolled out-state: $\phi^{\text{out}} = S\phi^{\text{in}}$

States fulfill the Schrödinger equation.

2. *Observables* A, $|\psi^-\rangle\langle\psi^-|$ (*out*-observable, often misleadingly called "out-state")	*are defined* experimentally	by registration apparatus (e.g. detector)

Example: Preparation and decay of K_s^0 in the reaction [6] ("formation" of a "resonance" or of an unstable state K_s^0)

$$\pi^- \, p \to \Lambda K_S^0, \quad K_S^0 \to \pi^+\pi^-, \, \pi^0\pi^0$$

Registration of an observable $|\psi^-\rangle\langle\psi^-|$ or A

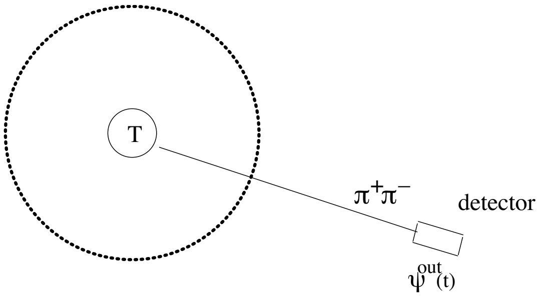

Fig. 3: The detector is built according to what it needs to register (e.g, a $\pi^+\pi^-$-detector registers or "counts" the observable $A = |\psi^-(t)\rangle\langle\psi^-(t)| \to |\psi^{out}\rangle\langle\psi^{out}| = |\pi^+\pi^-\rangle\langle\pi^+\pi^-|$ = two π's (usually the objects π^+ and π^- are detected at different places (scattering angles)).

3. The scattering experiment combines the preparation apparatus in Fig. 1 (accelerator) with a registration apparatus (detector) to count the clicks at the detector $|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}|$.

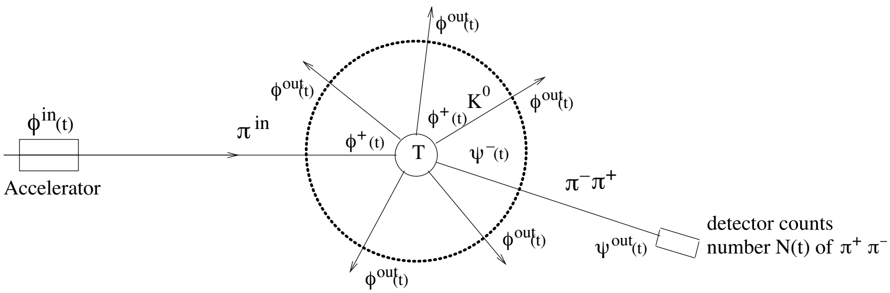

Fig. 4: Combining the preparation of the state ϕ and the registration of an observable ψ in a scattering experiment: One places the detector of Fig. 3 behind the target in Fig. 2 and counts the number $N(t) = N(t;\Delta\Omega, E^{out})$ of π^+, π^- at the time t into the solid angle $\Delta\Omega$.

Observables fulfill the Heisenberg equation

4 Theoretical description of states and observables

4.1 Born probabilities

As exhibited in the Figures 1, 2, 3 and 3, there are two different kinds of quantum theoretical entities associated to the two different kinds of physical apparatuses:

States ρ (in-states vectors ϕ^+)	are prepared	by the preparation apparatus, (Figs. 1, 2) and governed by the Schrödinger Eq. (1).
Observables A (or ψ^-)	are registered	by the detector and governed by Heisenberg Eq. (4)
Experimental (observed) quantities are the *probabilities* for observable A in state ρ.	They are calculated in the theory as Born Probabilities solving dynamic equations (1) or (4) and (1) or (2).	They are measured as ratios of large number of detector counts ("relative frequencies") $N(t)/N$.

$$\mathscr{P}_\rho(A(t)) \equiv \mathrm{Tr}(A(t)\,\rho_0) = \mathrm{Tr}(A_0\,\rho(t)) \approx N(t)/N. \tag{14a}$$

In the special case of an observable $|\psi^-\rangle\langle\psi^-|$ in the pure state $\phi^+(t)$ the probability of the observable ψ^- in the state ϕ^+ is:

$$\mathscr{P}_{\phi^+}(\psi^-(t)) \equiv \underbrace{|\langle\psi^-(t)|\phi^+\rangle|^2}_{\text{in Heisenberg's picture}} = \underbrace{|\langle\psi^-|\phi^+(t)\rangle\langle\phi^+(t)|\psi^-\rangle}_{\text{in Schrödinger's picture}} \tag{14b}$$

The agreement of theory $\mathscr{P}_\rho(A(t))$ with the experimental counting rate $N(t)$ of Fig. 3 is given by

$$\mathscr{P}_\rho(A(t)) \;\text{ or }\; |\langle\psi^-(t)|\phi^+\rangle|^2 \approx \frac{N(t)}{N}. \tag{15}$$

The theory-calculated probabilities $\mathscr{P}_\rho(A(t))$ must agree $\approx$ with the registered detector counts $N(t)/N$; this must hold for an ensemble of N detector counts, where N are "large" numbers.

To make a comparison of ***experimental*** counting rates $\frac{N(t)}{N}$ and the ***theoretical*** probabilities $|\langle\psi^-(t)|\phi^+\rangle|^2$, one needs ***to solve*** either the Heisenberg equation (4) for the out observable $\psi^-(t)$ or one needs to solve the Schrödinger equation (1) for the in-state $\phi^+(t)$ and then calculate the Born probabilities: $\mathrm{Tr}(A(t)\,|\phi^+\rangle\langle\phi^+|)$ $= |\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle\psi^-|\phi^+(t)\rangle|^2$ using the solution of Heisenberg eq. (4), or the solution of Schrödinger eq. (1), respectively.

4.2 Solutions of the dynamical equations and their boundary conditions

To solve a differential equation (Schrödinger Eq. (1) and (1) or Heisenberg Eq. (2) and (4)), requires the choice of **Boundary Conditions**. This means one has to choose the ***mathematical spaces***, to which the solutions of equations (1) and of equations (4) need to belong.

The right choice of a Boundary Condition is most important for quantum physics. The boundary condition will make the difference for the prediction of the theory and thus it is our choice of the Boundary Conditions, which will determine the theory of our choice. In the historical development of quantum theory, the following boundary conditions were used for the solutions of the dynamical equations (1) and (4) or (2) and (1):

1. Hilbert space boundary condition of von Neumann (called "Hilbert space axiom"): [1]

$$\text{Set of state vectors } \{\phi\} = \text{Set of observables } \{\psi\} \doteqdot \mathscr{H} = \text{Hilbert space} \quad (16a)$$

2. Schwartz space boundary condition of the Dirac formalism chooses the Schwartz space for the states as well as the observables:

$$\text{Set of state vectors } \{\phi\} = \text{Set of observables } \{\psi\} \doteqdot \Phi = \text{Schwartz space} \quad (16b)$$

From the standard boundary condition (16a) as well as (16b) follows by the Stone-von Neumann theorem, for Hilbert space and by a similar theorem for the Schwartz space, *the (unitary) group evolution* (1); (10) and (17b).

The solutions of the Schrödinger equation under the condition $\phi \in \mathscr{H}$ as well as under $\phi \in \Phi$ are given by

$$\phi(t) = U^{\dagger}(t)\phi = \mathrm{e}^{-iHt/\hbar}\phi\,, \text{ with } -\infty < t < +\infty \text{ for } \phi \in \mathscr{H} \text{ and } \phi \in \Phi. \quad (17a)$$

The same holds for the Schwartz space boundary condition $\phi \in \Phi$:

$$\phi(t) = U^{\dagger}_{\Phi}(t)\ \phi = \mathrm{e}^{-iHt/\hbar}\phi \text{ with } -\infty < t, +\infty \text{ for } \phi \in \Phi,$$
$$\text{where } U^{\dagger}_{\Phi}(t) = U^{\dagger}(t)\ |_{\Phi} \text{ is the restriction of } U^{\dagger} \text{ to the subspace } \Phi \subset \mathscr{H}.$$

Similar results hold for the solutions of the Heisenberg equation under these boundary conditions:

$$\psi(t) = U(t)\ \psi = \mathrm{e}^{iHt/\hbar}\psi, \text{ with } -\infty < t < +\infty \text{ for } \psi \in \mathscr{H}\,, \quad (17b)$$

and $\psi(t) = U_{\Phi}(t)\ \psi = \mathrm{e}^{iHt/\hbar}\psi$ for $\psi \in \Phi$.

[1] "Complete" Hilbert space of von Neumann means the integrals which define the scalar product are Lebesgue integrals and not just Riemann integrals, but physicists do not want to deal with Lebesgue integrals anyway.

This result (10) and (17b) has been well known (Stone-von Neumann theorem [3, 7]) for the Hilbert space boundary condition and it can also be proven for the Schwartz space boundary condition ψ, $\phi \in \Phi$.

From this, follows that a theory which uses the *Hilbert space boundary condition* (16a) *or* the theory based on the *Schwartz space boundary condition* (16b) (i.e., the mathematical version of the Dirac formalism), *"predict"* the Born probabilities

$\mathscr{P}_\rho(A(t)) = |\langle\psi^-(t)|\phi^+\rangle|^2$ to detect an observable $A(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$ in the state ϕ^+ for all t: $-\infty < t < +\infty$;

This would mean it predicts the probabilities, also for times $t < t_0$, *before* the state ϕ^+ will be prepared by the accelerator and target T at a time t_0. This would, however violates causality, ***because***:

The detector in Fig. 3 cannot detect anything relevant to the scattering process ***before*** the times t_0, at which the Accelerator will be turned on.

Thus, a theory that makes predictions for $-\infty < t < +\infty$ cannot be "quite" right. It would violate the causality principle of quantum physics, which asserts [8]:

A state ϕ^+ needs to be prepared first, by a time t_0, *before* an observable $|\psi^-(t)\rangle\langle\psi^-(t)|$ can be measured in that state ϕ^+ at times $t \geq t_0$ by the detector counts, $N(t)/N$. There cannot be any $K^0_S \to \pi^+\pi^-$ counted in the detector of Fig. 3 before the accelerator has been turned on and π^{in} has hit the target T.

Thus the experimental result, as well as our intuitive feeling of causality suggests that the Born probabilities of the observable $A(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$ in the prepared state ϕ^+:

$$\begin{aligned}\mathscr{P}_{\phi^+}(A(t)) &= \mathrm{Tr}(A(t)\,|\phi^+\rangle\langle\phi^+|) = \mathscr{P}_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|)\\ &= |\langle\psi^-(t)|\phi^+(t_0)\rangle|^2 = |\langle\psi^-(t)|\phi^+\rangle|^2\,, \qquad (18)\\ &\text{make physical sense only for times } t \text{ later than } t_0, \text{ i.e., for } t > t_0\,.\end{aligned}$$

Here the time $t_0(=0)$ is the *time at which the state ϕ^+ is prepared, and only after this time t_0 can the observable ψ^- be registered in the state for $t > t_0$.*

Since a quantum state represents an ***ensemble*** of (large number of) micro systems in the lab, this beginning of time t_0 represents usually also an ensemble of ***finite*** times, $t_0 \leftrightarrow \{t_0^{(i)}\}$, where the $t_0^{(i)}$ are in general different times on a clock in the lab. (Such $t_0^{(i)}$ for single particles have been observed as the onset times of the dark periods in Dehmelt's quantum jump experiments with single ions in a Paul traps).

Comparison with the quantum jump experiments means that t_0 represents the ensemble $\{t_0^{(i)}\}$ of beginnings of time for the i-th individual quantum particles.

All this suggest that one must ***not*** solve the dynamical differential equation under the standard Hilbert space or under the standard Schwartz space boundary conditions which lead to (10), (17b), but under *new boundary conditions* which lead to "beginnings of time" t_0, and thus to the *semi-group time evolution* like

$$\psi^-(t) = \mathscr{U}(t-t_0)\,\psi(t_0) \quad \text{with the finite beginning of time } t_0 \leq t < +\infty, \qquad (19)$$

where t_0 is an ensemble of finite times (generally different times $t_0^{(i)}$ on the clocks in the lab.). The solution of the dynamical equations under Hardy Space Boundary Condition leads to semigroup evolution of the dynamical euqations of quantum physics.

4.3 New Hardy space boundary conditions

New boundary condition means that one chooses in place of the historical Hilbert space $\mathscr{H}$ or in place of the Schwartz space $\boldsymbol{\Phi}$, a new space for state vectors $\{\phi^+\}$ representing the preparation apparatus (e.g., the accelerator of Fig. 3) and another new space for the observables $\{\psi^-\}$ representing the registration apparatus (e.g., the detector in Fig. 3).

These new spaces must NOT be given:

by the Hilbert space axiom:

$$\{\phi\} = \{\psi\} \doteqdot \mathscr{H} \text{ of von Neumann, } (L^2\text{-integrable}). \tag{20H}$$

And they must also not be given by the Schwartz space axiom:

$$\{\phi\} = \{\psi\} \doteqdot \boldsymbol{\Phi} \text{ of the Dirac formulation} \tag{20Φ}$$

which would lead to the Schwartz Rigged Hilbert Space:

$$\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times \text{ of Sec. 1.}$$

Thus a new mathematical Axiom is needed for quantum physics; this new axiom will be based on a ***pair*** of mathematical spaces, one for the prepared states in Figs. 1 and 2 and the other for the detected observables in Figs. 3 and 3.

There is ***no*** reason that the set of accelerator prepared states $\{\phi^+\}$, as well as the set of detector registerd observables $\{\psi^-\}$, should both be represented by the ***same*** mathematical space (e.g., by the Schwartz space $\boldsymbol{\Phi}$, or by the Hilbert space $\mathscr{H}$). Thus, rather than using one and the same representation space, $\mathscr{H}$ or $\boldsymbol{\Phi}$ for the in-state vectors ϕ^+, ***as well as*** for the out-observable vectors ψ^-, as done in (20H) and similar for the Dirac formulation in (20$\boldsymbol{\Phi}$), it would be much more natural to represent the accelerator prepared ***states*** (Fig. 1 and Fig. 2) and the detected ***observables*** (Fig. 3 and Fig. 3) by two ***different*** mathematical representation spaces.

The new boundary conditions for the two differential equations (Schrödinger or Heisenberg) of quantum mechanics need to be given by two different spaces, one for the states ϕ^+ and the other for the observables ψ^-. For these two spaces one can choose the *Hardy space boundary conditions* as the new Axiom for the fundamental dynamical equations (1), (4), or (1), (2) of quantum mechanics.

There are two different Hardy spaces [9] that are conjugate to each other. This suggests the following new axiom which allows us to distinguish mathematically the prepared states from the observables, using the pair of Hardy spaces:

The space of state vectors $\{\phi^+\}$ representing the ***accelerator-prepared states*** of Fig. 1 and 2, is the Hardy space Φ_- of the lower complex energy plane (2nd sheet of the analytic S-matrix):

$$\text{prepared in-states } \{\phi^+\} \doteqdot \Phi_- \text{ Hardy space solutions of the Schrödinger eq.} \tag{21.1-}$$

The space of observable vectors $\{\psi^-\}$ representing the ***detector-registered observables*** is the Hardy space Φ_+ of the upper complex energy plane (2nd sheet of the analytic S-matrix):

$$\text{detected out-observables } \{\psi^-\} \doteqdot \Phi_+ \text{ Hardy space solutions of the Heisenberg eq.} \tag{21.1+}$$

The amusing miss-match in the notation for the physical vectors and their mathematical representation spaces:

$$\phi^+ \in \Phi_- \quad \text{(lower complex plane),} \tag{21.2-}$$

$$\psi^- \in \Phi_+ \quad \text{(upper complex plane),} \tag{21.2+}$$

has its origin in the two different conventions used for the Hardy spaces in mathematics, and for the state vectors in physics:

Mathematicians notation of Hardy spaces	Physicists notation for the vectors of the scattering theory	
Hardy space $\Phi_- = \{\phi^+\}$ is realized by the smooth Hardy function $\phi^+(E) = \langle^+E\,|\phi^+\rangle \in (\mathscr{H}^2_- \cap S)_{\mathbb{R}_+}$ on $\mathbb{C}_-$, i.e., on the lower complex plane 2nd sheet of the S-matrix.	$\{\phi^+\}$ represents the accelerator prepared states and thus the Lippmann-Schwinger kets $\|E^+\rangle \in \Phi_-^\times$.	(21.3-)
Hardy space $\Phi_+ = \{\psi^-\}$ is realized by the smooth Hardy function $\psi^-(E) = \langle^-E\,\|\psi^-\rangle \in (\mathscr{H}^2_+ \cap S)_{\mathbb{R}+}$ on $\mathbb{C}_+$[2], i.e., on the upper complex plane 2nd sheet of the S-matrix.	$\{\psi^-\}$ represents the detector registered observables $\|\psi^-\rangle\langle\psi^-\|$ and thus the Lippmann-Schwinger kets are $\|E^-\rangle \in \Phi_+^\times$, the dual space of Φ_+	(21.3+)

Both, the state vectors ϕ^+ and the observable vectors ψ^-, represent two ***entirely different*** physical aspects of the experimental apparatus as displayed by the comparison of Figs. 1 and 2 with Figs. 3 and 3. Therefore we have *no* reason to suspect that the vectors $\{\phi^+\}$ representing states and the vectors $\{\psi^-\}$ representing observables, should be described by the ***same*** mathematical spaces, namely both by the Hilbert space or both by the Schwartz space, as is usually done.

[2] For the Hardy space we consider only the spaces, which can be realized by the "smooth Hardy functions" $\mathscr{H}_\pm \cap S$, i.e., Hardy class intersected with Schwartz function spaces.

The Lippmann-Schwinger kets had been postulated in analogy to the Dirac kets (5) as the in-plane wave "states" and the out-plane wave "states" $|E^+\rangle$ and $|E^-\rangle$, $H\ |E^\pm\rangle = E\ |E^\pm\rangle$, $K\ |E\rangle = E\ |E\rangle$ which fulfill the *Lippmann-Schwinger* equations [10]:

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - K \pm i\varepsilon} V |E^\pm\rangle \qquad H = K + V. \tag{22}$$

In analogy to the Dirac basis vector expansion (5) of Section 1 for the Schwartz space, the *nuclear spectral theorem* [4] of the topological vector space for in-state vectors ϕ^+ (*representing the preparation apparatus, e.g., an accelerator* in the scattering experiment of Fig. 3) would then be given by

$$\phi^+ = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^+\rangle\langle^+E, j, j_3, \eta|\phi^+\rangle$$
$$= \int_0^\infty dE |E^+\rangle\langle^+E|\phi^+\rangle = \int dE\ |E^+\rangle\phi^+(E), \quad \text{(23-)}$$

j, j_3 denote the angular momentum and η denotes some additional (species) quantum numbers.

The $|E^-\rangle = |E, j, j_3, \eta^-\rangle$ are taken as basis systems for out-vectors (*representing the observables* $|\psi^-\rangle\langle\psi^-|$ registered by the *detector*); thus for the observables $|\psi^-\rangle\langle\psi^-|$ of Figs. 3 and 3 one would then have the basis vector expansion:

$$\psi^- = \sum_{j,j_3,\eta} \int_0^\infty dE' |E', j, j_3, \eta^-\rangle\langle^-E', j, j_3, \eta|\psi^-\rangle$$
$$= \int_0^\infty dE' |E'^-\rangle\langle^-E'|\psi^-\rangle = \int dE'\ |E'^-\rangle\psi^-(E'). \quad \text{(23+)}$$

This is done in perfect analogy to the Dirac basis vector expansion justified by the "nuclear spectral theorem" for Schwartz space: for every ϕ or $\psi \in \Phi$:

$\phi = \int dE\ |E\rangle\langle E|\phi\rangle$, The Dirac $|E\rangle$ are mathematically defined as continuous
$\psi = \int dE\ |E\rangle\langle E|\psi\rangle$, antilinear functional of Φ: $|E\rangle \in \Phi^\times$.

The shortcoming of the Schwartz space axiom is that the set of states $\{\phi\}$ of Figs. 1 and 2, and the set of observables $\{\psi\}$ of Figs. 3 and 3 cannot be mathematically distinguished from each other if one uses just the ***one*** Schwartz space (or one $\mathscr{H}$).

The new idea suggested by the Lippmann-Schwinger kets (20), which is also dictated by the Gamow vector, is to associate the ***set of physical states*** ϕ^+ (defined by the preparation apparatus, Figs. 1 and 2) ***with the mathematical Hardy space***, which is called Φ_-. Thus the new physical axiom of scattering theory is

$$\text{set of state vectors } \{\phi^+\} \ \doteqdot\ \Phi_- = \text{ Hardy on } \mathbb{C}_- \text{ (2nd sheet of the } S\text{-matrix).} \quad \text{(24-)}$$

And similarly for the set of observables ψ^-,

$$\text{set of } \{\psi^-\} \doteqdot \Phi_+ = \text{Hardy on } \mathbb{C}_+ \text{ (2nd sheet of the } S\text{-matrix).} \tag{24+}$$

This new Hardy space axiom (24±) is an enormous step forward, because comparing this Hardy space axiom (24±) with Figs. 1 and 2 for the states and with Figs. 3 and 3 for the observables, the Schwartz space axiom (6b) and the Hilbert space axiom (3) do not feel right, because one should also distinguish mathematically between the prepared states $\{\phi^+\}$ and registered observables $\{\psi^-\}$.

Since ψ^- is the observable (defined by the detector, Figs. 3 and 3), and ϕ^+ represents the state (defined by the accelerator, Figs. 1 and 2), the matrix element $(\psi^-|\phi^+)$ is *the probability amplitude* to detect the observable $|\psi^-\rangle\langle\psi^-|$ in the state $|\phi^+\rangle\langle\phi^+|$ and therfore,

$|(\psi^-|\phi^+)|^2$ is the probability to detect the observable $|\psi^-(t)\rangle\langle\psi^-(t)|$ in the state ϕ^+, which according to standard quantum theory (6), (14a), (14b) and (15) is:

$$|(\psi^-(t)|\phi^+)|^2 \sim \frac{N(t)}{N} \text{ (\textbf{\textit{is measured by the detector counts}})} \geq 0 \text{ for } t > t_0 \tag{25}$$

as stated in (6) or for the general case in (5).

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Algebraic structures on the moduli spaces in gauge theories

Vasily Pestun

Abstract The partition function of a four-dimensional supersymmetric gauge theory on a four-sphere is factorizable in holomorphic and antiholomorphic blocks similar to the correlation functions of the two-dimensional conformal field theories. The holomorphic blocks are controlled by the geometry of the moduli spaces of vacua in 4d supersymmetric gauge theory, and this reveals a deep connection with algebraic structures of quantum integrable systems, two-dimensional conformal field theories and their q-deformations.

1 Introduction

Some of the pressing questions in the studies of quantum gauge theories are: what can we do beyond perturbation theory, are there hidden algebraic structures, what are the exactly computable quantities?

In the remarkable work of Belavin, Polyakov and Zamolochikov [1] the correlation functions of some operators $\mathscr{O}_i$ in two-dimensional conformal field theories have been shown to have the factorizable form, schematically

$$\langle \prod_i \mathscr{O}_i \rangle = \int \mu(a) Z(a) \overline{Z(a)} \tag{1}$$

where the variable a labels the primary fields of the theory, the $\mu(a)$ is a certain integration measure determined by the physical content of the theory, whereas functions $Z(a)$, called *conformal blocks* are determined by the Virasoro symmetry algebra of the 2d conformal field theory.

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Since then the algebraic approach to the two-dimensional conformal field theories and their relatives, such as massive deformations and 2d lattice integrable models, turned out to be very powerful with many mathematical and physical results that are not possible to survey in this lecture.

Some of the key mathematical structures that have been encountered on the way are:

- conserved quantities
- holomorphic factorization
- conformal block and operator product expansion
- affine Kac-Moody Lie algebras, $\mathscr{W}$-algebras (generalization of Virasoro)
- quantum groups, R-matrix

Recently, many of these facets of solvable models have been detected in two current programs to access gauge theories beyond the perturbation theory;

1. large N (planar) limit of Yang-Mills gauge theory, specifically $\mathscr{N}=4$ super Yang-Mills,
2. vacuum or BPS sector of $\mathscr{N}=2$ supersymmetric gauge theories.

The *large N approach* is based on the gauge-string duality [2] [3] [4] [5] under which computation in large N gauge theory is mapped to the computation in the two-dimensional sigma model on the world-sheet of confining string, and further the two-dimensional sigma-model is solved using the 2d integrability tools.

The vacuum or *BPS approach* [6] [7] [8] is based on the analysis of the supersymmetry implications on the geometry of the vacua in supersymmetric gauge theories.

The focus of this lecture is on the geometrical and algebraic structures arising from the *BPS approach* also known as *localization*.

2 The structures of 4d $\mathscr{N}=2$ theories

By now the following features of 4d $\mathscr{N}=2$ theories have been understood relatively well:

- *vacua sector = complex algebraic integrable system*
- *holomorphic factorization*
- *emergence of quantum algebras*:
  1. CFT type (Virasoro, $\mathscr{W}$-algebra, Kac-Moody)
  2. Lattice model type (R-matrix, spin-chains, quantum groups).

2.1 Vacua sector = complex algebraic integrable system

In [6] [9] [10] [11] it was shown that the Coulomb branch $\mathscr{M}_{\mathbb{R}3\times S1}$ of the moduli space of vacua of 4d $\mathscr{N}=2$ theory on $\mathbb{R}^3\times S^1$ is fibered over the Coulomb branch of vacua $\mathscr{M}_{\mathbb{R}^4}$ of the same $\mathscr{N}=2$ gauge theory on $\mathbb{R}^4$,

$$\mathscr{M}_{\mathbb{R}^3\times S^1}\to\mathscr{M}_{\mathbb{R}^4}. \tag{2}$$

Moreover, the space $\mathscr{M}_{\mathbb{R}^3\times S^1}$ has dimension twice that of the base space $\mathscr{M}_{\mathbb{R}^4}$. It is holomorphic symplectic and the fibers of the fibration (2) are holomorphic Lagrangian abelian varieties. The holomorphic symplectic structure on $\mathscr{M}_{\mathbb{R}^3\times S^1}$ is determined from the hyperKähler structure at a certain point on the twistor sphere of complex structures usually called complex structure I.

2.2 Holomorphic factorization

In [12] using supersymmetric path integral version of the localization formula of Atiyah-Bott [13] and Berline-Vergne [14] it was shown that the partition function of $\mathscr{N}=2$ gauge theory on four-sphere is computed by the finite-dimensional integral over the Cartan of the Lie algebra of the gauge group

$$Z_{S^4}=\int\mu(a)Z(a)\overline{Z(a)} \tag{3}$$

where $\mu(a)$ is a certain measure computed from the Atiyah-Singer index theorem for transversally elliptic operators [15]. This result established that the partition functions of the $\mathscr{N}=2$ supersymmetric gauge theories on a four-sphere have factorization property similar to the partition functions of the two-dimensional conformal field theories. The *holomorphic or chiral block* $Z(a)$ was identified with Nekrasov's partition function of the equivariant topological Donaldson-Witten gauge theory, also called gauge theory in the Ω-background [8]. This function $Z(a)$ can be found by the cohomological computation on the moduli space of the BPS configurations of the gauge theory called instantons [16], [17], [7], [8], [18] [19].

For the review of supersymmetric localization in gauge theories in different dimensions leading to the result similar in spirit to the factorization equation (3), see the review of collected papers [20]

The factorization (3) suggests that $Z(a)$ is like a conformal block of some algebra of symmetries of the 2d conformal field theory. What is this algebra exactly and where does it come from?

3 Quantum algebras

3.1 Quantum algebras: CFT type

For certain $\mathcal{N} = 2$ gauge theories, the answer to the last question of the previous paragraph has been discovered in a beautiful paper by Alday-Gaiotto-Tachikawa [21].

Namely, for the 4d $\mathcal{N} = 2$ gauge theories of the class named $S(C, \mathfrak{g})$, AGT identified the holomorphic block (Nekrasov's partition function) $Z(a)$ in the four-sphere partition function (3) with the conformal block in the correlation function (1) for a certain operator in two-dimensional conformal field theory of Toda for Lie algebra $\mathfrak{g}$ on the Riemann surface C. The 4d $\mathcal{N} = 2$ supersymmetric gauge theory $S(C, \mathfrak{g})$ is defined as a quantum field theory obtained by the compactification of the $(0,2)$ supersymmetric self-dual 6d tensor theory of ADE type $\mathfrak{g}$ on the Riemann surface C, possibly with punctures and certain data at the punctures [22].

The integrable system (2) which corresponds to the supersymmetric gauge theories of class $S(\mathfrak{g}, C)$ is Hitchin system [23] on C for the Lie algebra $\mathfrak{g}$. The phase space $\mathscr{M}_{\mathbb{R}^3 \times S^1}$ is identified with the moduli space $\mathscr{M}_{\mathrm{Hit}}(C, \mathfrak{g})$ of G-Higgs bundles on C, and the base $\mathscr{M}_{\mathbb{R}^4}$ is the space of action variables, or in other words, it is the space where Hitchin Hamiltonians are taking values.

The function $Z(a)$ is a conformal block of the algebra called $\mathscr{W}(\mathfrak{g})$-algebra which is a generalization of the Virasoro symmetry algebra to higher rank, so that $\mathscr{W}(\mathfrak{sl}_2) = \mathbf{Vir}$.

Consequently, the relation between the gauge theory of class $S(C, \mathfrak{g})$, the integrable system and the conformal theory can be summarized by the diagram

$$\begin{array}{ccc} & \text{4d QFT: } \mathscr{M}_{\mathbb{R}^3 \times S^1} \text{ for } S(C, g) & \\ \swarrow\!\!\nearrow & & \searrow\!\!\nwarrow \\ \mathscr{M}_{\mathrm{Hit}}(C, \mathfrak{g}) & \longleftrightarrow & \mathscr{W}(\mathfrak{g})\text{-algebra on C} \end{array} \tag{4}$$

The link between $\mathscr{M}_{\mathrm{Hit}}(C, \mathfrak{g})$ and the $\mathscr{W}(\mathfrak{g})$ algebra is understood after Drinfeld-Sokolov [24], Feigin-Frenkel [25], Nekrasov-Witten [26], Teschner [27]. Namely, the $\mathscr{W}$-algebra that emerges is the quantized algebra on the space of *opers*. The space of opers is obtained by Poisson reduction from a hyperplane in a coadjoint Kac-Moody Lie algebra by the loop nilpotent algebra, while that hyperplane is identified with the space of $G_{\mathbb{C}}$-flat connections $\partial_z + A$ on a punctured disc, see E. Frenkel 2002 lecture [28]. The space of $G_{\mathbb{C}}$-flat connection as a complex variety is isomorphic to $\mathscr{M}_{\mathrm{Hit}}(C, \mathfrak{g})$ for a different choice of the complex structure on the twistor sphere, usually called J in contrast to the complex structure I in which $\mathscr{M}_{\mathrm{Hit}}(C, \mathfrak{g})$ has the geometry of the algebraic integrable system

What replaces the $\mathscr{W}(\mathfrak{g})$-algebra for the moduli space of vacua $\mathscr{M}_{\mathbb{R}^3 \times S^1}$ for generic $\mathcal{N} = 2$ gauge theory, of not necessarily class S?

3.2 Quantum algebras: generic proposition

Proposition. To the generic hyperKähler moduli space of vacua $\mathscr{M} \equiv \mathscr{M}_{\mathbb{R}^3 \times S^1_R}$ there is associated the two-parametric $\mathscr{W}$-algebra $\mathscr{W}_{\varepsilon_1,\varepsilon_2}(M)$ which is defined as the ε_1-quantized algebra of holomorphic functions on space $\mathscr{M}$ in the complex structure $\zeta = R\varepsilon_2$ where $\zeta \in \mathbb{CP}^1$ is the twistor parameter of the twistor sphere of complex structures on the hyperKähler manifold $\mathscr{M}$ in the limit $R \to 0$.

Besides previously mentioned papers, this proposition links to the works of Kontsevich-Soibelman [29], Gukov-Witten [30], Kapustin-Witten [31], Gaiotto-Moore-Neitzke [32], Fock-Goncharov [33], Gaiotto [34], Cecotti-Neitzke-Vafa [35].

In particular, the point $\zeta = 0$ is the complex structure I in which $\mathscr{M}$ is the complex phase space of an integrable system.

The global holomorphic sections of the quantized algebra of holomorphic functions $\mathscr{M}$ are identified with the quantum commuting Hamiltonians of the quantum integrable system with quantum Planck constant $\hbar = \varepsilon_1$ [36]. The non-zero ε_2-parameter deforms the commutative algebra of quantum Hamiltonians of an integrable system into an associative algebra of quantum integrals of motion of auxiliary low-dimensional quantum field theory: $\mathscr{W}$-algebra. For the 4d gauge theory of class $S(C,\mathfrak{g})$, this low-dimensional theory is two-dimensional quantum Toda field theory.

3.3 Quantum algebras: lattice model type

The generic proposition of Section 3.2 can be tested more precisely in the different class of theories rather than class $S(C,\mathfrak{g})$, namely in the class $\mathscr{N}=2$ gauge theories called *quiver gauge theories* [37].

The quiver gauge theory is defined by a graph Γ with some data assigned to the nodes and edges. To each node i we assign a positive integer n_i, which is a rank of the factor of the gauge group $U(n_i)$ associated to this node, and a complex number $\mathfrak{q}_i$ with $|\mathfrak{q}_i| < 1$ which is the exponentiated coupling constant for the given gauge group factor $U(n_i)$. To each edge $e : i \to j$ we assign a complex number m_e which is a mass of the hypermultiplet in the bi-fundamental representation $(\bar{\mathbf{n}}_i, \mathbf{n}_j)$ between the nodes i and j.

Using arguments from string theory and brane dualities, the phase space of the integrable system underlying this class of theories was identified by Kapustin-Cherkis [38] as the moduli space of the G_Γ monopoles on $\mathbb{R}^2 \times S^1$. The derivation of this result from the BPS-style localization computations on the moduli space of quiver instantons was found in [19].

Here G_Γ is the Lie group with the simply-laced (ADE) Dynkin graph isomoprhic to Γ.

For a Riemann surface C, the moduli space of monopoles on $C \times S^1$ can be thought of as a moduli space of group version of the moduli space of Higgs bundles on C [39].

Namely, one defines this space similar to Hitchin as the space of pairs (holomorphic G-bundle on C, Higgs field $g(x)$), except that now the Higgs field $g(x)$ is taken to be a meromorphic *Lie group valued* field, a section of $\mathrm{Ad}\, G$, rather than a *Lie algebra valued* field $\phi(x)$, a section of $\mathrm{ad}\, \mathfrak{g} \otimes K_C$ in the usual Hitchin case. (Here x denotes a complex coordinate on C.)

In the case of the usual Lie algebra valued Hitchin system, the ring of the commuting Hamiltonians is generated by the global sections of polynomial adjoint invariant functions on the Lie algebra $\mathfrak{g}$ evaluated on the Lie algebra valued Higgs field $\phi(x)$. The ring of the adjoint invariant functions is generated by the fundamental invariants of degrees $m_i + 1$ where m_i are the Coxeter exponents of G.

In the case of the group valued Hitchin system the ring of the commuting Hamiltonians is generated by the global sections of polynomial adjoint invariant functions on the Lie group G evaluated on the group valued Higgs field $g(x)$. The ring of adjoint invariant functions on the group is generated by the characters $\chi_i = \mathrm{tr}_{R_I}$ of the fundamental representations R_i, that is highest weight irreducible representation with the highest weight given by a fundamental weight.

In the case of $C = \mathbb{C} \simeq \mathbb{R}^2$ with no punctures, the global holomorphic sections of $\mathrm{tr}_{R_i} g(x)$ are polynomials of degrees n_i defined by the ranks of the gauge group factors $U(n_i)$ of the $\mathscr{N} = 2$ 4d gauge theory

$$\mathrm{tr}_{R_i} g(x) = x^{n_i} + u_{i,1} x^{n_i - 1} + \dots u_{i,n_i}, \qquad i \in \text{nodes of } \Gamma. \tag{5}$$

The coefficients $(u_{i,1}, \dots, u_{i,n_i})_i$ are the Poisson commuting Hamiltonian functions on the complex phase space of an integrable system: the moduli space of monopoles on $C \times S^1$.

Moreover, the phase space of monopole integrable system can be identified with a symplectic leaf in the Poisson-Lie loop group $\{g(x)\}$ [40]. The Poisson structure on this Poisson-Lie group is of quasi-triangular type defined either by a classical rational type r-matrix if $C \simeq \mathbb{C}$ is the complex affine line, or by a trigonometric r-matrix if $C \simeq \mathbb{C}^\times \simeq \mathbb{C}/\mathbb{Z}$ is a cylinder, or by an elliptic r-matrix if $C \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is an elliptic curve.

The quantization of such a Poisson-Lie group produces the famous quantum groups of Drinfeld [41] and Jimbo [42] which are the quasi-triangular Hopf algebras (with quantum R-matrix) underlying the solvability of the quantum spin chains of various types and the Bethe ansatz [43]. The commuting Hamiltonians, as operators of the quantum group in a representation on a physical Hilbert space W, can be constructed by taking the trace of R-matrix $\mathrm{tr}_V R_{V,W}$ over an auxiliary space V. Their commutativity is implied by the Yang-Baxter equation which is satisfied by the R-matrix.

The explicit algebraic objects that replace the characters $\chi_i = \mathrm{tr}_{R_i}$ after the quantization have been constructed by Frenkel-Reshetikhin [44] and were called q-characters for affine quantum algebra $U_q(\hat{\mathfrak{g}})$ associated to the Poisson-Lie group of trigonometric type for $C = \mathbb{C}^{\times}$. Here the parameter $q = \exp(\varepsilon_1)$ is the exponentiated Planck constant. At the same time, the commutative algebra of the q-characters was identified by Frenkel-Reshetikhin with the q-deformation of the classical $\mathscr{W}$-algebra [44]. The *polynomiality* conjecture of q-characters of [44] has been proven in [45].

On the other hand, the same algebraic objects, the q-characters, were obtained from the study of the equivariant cohomology of instanton moduli spaces in quiver gauge theories on $\mathbb{C}_{q_1,q_2}$ in [46] in the limit $q_2 = 1$, which justifies the q-version of the triangle relation analogous to (4):

4d QFT: $\mathscr{M}_{\mathbb{R}^3\times S^1}$ for
Γ-quiver gauge theory

$\mathscr{M}_{\text{Monopoles}}(C \times S^1, \mathfrak{g}_\Gamma) \leftrightsquigarrow \mathscr{W}_{q_1,q_2}(\mathfrak{g}_\Gamma)$-algebra on C (6)

Furthermore, following the approach in [19, 46], the two-parametric q_1, q_2-deformation of the characters, called q_1, q_2-characters were obtained from the geometry of the quiver instanton moduli spaces in [47], and in [48] it was shown that gauge-theory construction of q_1, q_2-characters is isomorphic to the Frenkel-Reshetikhin definition [49] of q-deformed $\mathscr{W}$-algebras $\mathscr{W}_{q_1,q_2}(\mathfrak{g})$. This supports the two-parametric (q_1, q_2) relation (6).

While the geometric Langlands program can be embedded into the context of the diagram (4) [50], [31], [26] relating to the quantization of the Hitchin system, differential equations and conformal field theories, the quantum field theory context for the q-geometric Langlands program [49] relating to the quantization of the system of periodic monopoles, difference equations and lattice models is provided by the diagram (6).

The horizontal arrow in (6) denotes that $\mathscr{W}_{q_1,q_2}(\mathfrak{g}_\Gamma)$ is obtained by a quantization of the Poisson algebra of functions on the space of q-opers [51, 52], and that the Poisson structure on the space of q-opers naturally arises from the Poisson structure on the moduli space of monopoles on the twisted product $C \times_q S^1$ for $C \simeq \mathbb{C}^{\times}$. In turn, the symplectic structure on the monopole moduli space on the twisted product $C \times_q S^1$ comes from the hyperKäler rotation on the P^1-twistor sphere of complex structures on the monopole moduli space on the direct product space $C \times S^1$. This justifies the generic Proposition 3.2 in the context of quiver gauge theories and monopoles integrable systems on $C \times S^1$. Also, the construction of $\mathscr{W}_{q_1,q_2}(\mathfrak{g}_\Gamma)$ algebras from quiver gauge theories on $\mathbb{C}^2_{q_1,q_2}$ gives a natural q_2-deformation of the commutative K-theory ring of the representation theory of $U_{q_1}(L\mathfrak{g})$, obtained in a geometrical way by Nakajima from the quiver variety associated to the same quiver [53], into an associative non-commutative algebra. The representation theory meaning of this algebra remains to be clarified.

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Statistical mechanics for complex systems: On the structure of q-triplets

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Abstract A plethora of natural, artificial and social complex systems exists which violate the basic hypothesis (e.g., ergodicity) of Boltzmann-Gibbs (BG) statistical mechanics. Many of such cases can be satisfactorily handled by introducing non-additive entropic functionals, such as $S_q \equiv k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}\left(q \in \mathscr{R}; \sum_{i=1}^{W} p_i = 1\right)$, with $S_1 = S_{BG} \equiv -k\sum_{i=1}^{W} p_i \ln p_i$. Each class of such systems can be characterized by a set of values $\{q\}$, directly corresponding to its various physical/dynamical/geometrical properties. A most important subset is usually referred to as the q-triplet, namely $(q_{\text{sensitivity}}, q_{\text{relaxation}}, q_{\text{stationary state}})$, defined in the body of this paper. In the BG limit we have $q_{\text{sensitivity}} = q_{\text{relaxation}} = q_{\text{stationary state}} = 1$. For a given class of complex systems, the set $\{q\}$ contains only a few independent values of q, all the others being functions of those few. An illustration of this structure was given in 2005 [Tsallis, Gell-Mann and Sato, Proc. Natl. Acad. Sc. USA **102**, 15377; TGS]. This illustration enabled a satisfactory analysis of the Voyager 1 data on the solar wind. But the general form of these structures still is an open question. This is so, for instance, for the challenging q-triplet associated with the edge of chaos of the logistic map. We introduce here a transformation which sensibly generalizes the TGS one, and which might constitute an important step towards the general solution.

1 Introduction

The pillars of contemporary theoretical physics may be considered to be Newtonian, quantum and relativistic mechanics, Maxwell electromagnetism, and Boltzmann-

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Gibbs (BG) statistical mechanics (microscopic theory consistent with thermodynamics). Statistical mechanics is in turn grounded in *electromechanics* (meaning by this the set of all mechanics and electromagnetism) and in the theory of probabilities. The BG theory can be formally constructed by adopting the BG entropic functional $S_{BG} = -k\sum_{i=1}^{W} p_i \ln p_i$, with $\sum_{i=1}^{W} p_i = 1$, k being a conventional positive constant (usually taken to be the Boltzmann constant k_B). This hypothesis is known to be fully satisfactory for dynamical systems satisfying simple properties such as ergodicity.

For more complex systems, the BG entropy can be inadequate, even plainly misleading. When this happens, must we abandon the statistical mechanical approach? It was advanced in 1988 [1] that this is not necessary. Indeed, it suffices to consider entropic functionals different from S_{BG}, and reconstruct statistical mechanics on more general grounds. The so-called *nonextensive statistical mechanics* follows along this path, based on the entropy $S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$ $(q \in \mathscr{R}; S_1 = S_{BG})$. It can be easily verified that, if A and B are any two probabilistically independent systems (i.e., $p_{ij}^{A+B} = p_i^A p_j^B$), then $\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k}$. In other words, S_q is *nonadditive* for $q \neq 1$, in contrast with S_{BG} which is *additive*.

The optimization of S_q under appropriate constraints yields distributions such as the q-exponential one $p_q(x) \propto [1-(1-q)\beta x]^{1/(1-q)} \equiv e_q^{-\beta x}$ or the q-Gaussian one $p_q(x) \propto e_q^{-\beta x^2}$ (see [2] for an introductory text). This and similar generalizations of the BG statistical mechanics have been shown to provide uncountable predictions, verifications, and applications in natural, artificial and social complex systems. A regularly updated bibliography as well as selected theoretical, experimental, observational, and computational papers can be seen at http://tsallis.cat.cbpf.br/biblio.htm Among recent applications we mention the experimental validation [3] (accomplished in granular matter) of a 20-year-old prediction, the emergence of neat q-statistical behavior in high-energy collisions at LHC/CERN along 14 experimental decades (see [4] for instance), a notable numerical discovery in the celebrated standard map [5], and the connection with networks (see [6] for instance).

2 q-triplets

The solution of the differential equation

$$\frac{dy}{dx} = a_1 y \;\; (y(0)=1) \tag{1}$$

is given by $y = e^{a_1 x}$. The solution of the more general equation

$$\frac{dy}{dx} = a_q y^q \;\; (y(0)=1) \tag{2}$$

is given by $y = e_q^{a_q x}$. These facts in the realm of nonextensive statistical mechanics suggested a conjecture in 2004 [7], namely that there could exist in nature q-triplets as indicated in Table 1 and [8]. The first verification of the conjecture was done in 2005 by NASA researchers Burlaga and Vinas in the solar wind [9].

	x	a	$y(x)$
Stationary state distribution	E_i	$-\beta$	$Z_{q_{\text{stationary state}}}\, p(E_i) = e_{q_{\text{stationary state}}}^{-\beta_{q_{\text{stationary state}}} E_i}$
Sensitivity to the initial conditions	t	$\lambda_{q_{\text{sensitivity}}}$	$\xi(t) = e_{q_{\text{sensitivity}}}^{\lambda_{q_{\text{sensitivity}}} t}$
Typical relaxation of observable O	t	$-1/\tau_{q_{\text{relaxation}}}$	$\Omega(t) \equiv \frac{O(t)-O(\infty)}{O(0)-O(\infty)} = e_{q_{\text{relaxation}}}^{-t/\tau_{q_{\text{relaxation}}}}$

Table 1: Three possible physical interpretations of Eq. (2) within nonextensive statistical mechanics. In the BG limit we have $q_{\text{sensitivity}} = q_{\text{relaxation}} = q_{\text{stationary state}} = 1$. For one dimensional dynamical systems it is $q_{\text{entropy production}} = q_{\text{sensitivity}}$, where $q_{\text{entropy production}}$ denotes the index q for which S_q increases linearly with time t. From [8].

Since then a plethora of q-triplets and directly related quantities have been found in solar plasma [10–13], the ozone layer [14], El Niño/Southern Oscillations [15], geological faults [16], finance [17, 18], DNA sequence [19], logistic map (see [20–31]), and elsewhere [32, 33].

3 Connections between q-indices

Some very basic points can be addressed at this stage: How many indices q can be systematically defined? How many of them are independent? Through what relations can all the others be calculated? To what specific physical/mathematical/probabilistic/dynamical property is each of them associated?

As we shall see, there are many more than three relevant q-indices. Nevertheless, the q-triplet plays a kind of guiding role in questions such as what is the correct entropy to be used, at what rhythm does it relax to a stationary state, and how can this stationary state be characterized. Consistently, in the BG limit all the indices q are expected to be equal among them and equal to unity.

Inspired by the specific values for the q-triplet observed by NASA [9], a path was developed in [34]. Two self-dual transformations admitting $q = 1$ as a fixed point were introduced, namely the *additive duality* $q \to 2-q$ and the *multiplicative duality* $q \to 1/q$. These simple transformations had already appeared in various contexts in nonextensive statistical mechanics (see [2] and references therein). The novelty in [34] is that they were used to systematically construct a mathematical structure, which we describe in what follows. We first define the transformations μ and ν:

$$\mu \;\to\; q_2(q) = 2-q \;\to\; \frac{1}{1-q_2(q)} = \frac{1}{q-1}\,, \tag{3}$$

$$\nu \;\to\; q_0(q) = \frac{1}{q} \;\to\; \frac{1}{1-q_0(q)} = \frac{1}{q-1} + 1\,. \tag{4}$$

The subindices 2 and 0 will become clear soon. We straightforwardly verify $\mu^2 = \nu^2 = 1$, $\nu\mu = (\mu\nu)^{-1}$. Also, we can analogously define $(\mu\nu)^m$ and $(\nu\mu)^n$ with integer numbers(m,n). This set of transformations enables (see [2,34]) the definition of a simple structure (hereafter referred to as the TGS structure). The NASA q-triplet for the solar wind found an elegant description within this structure, as shown later on in this paper. Not so the logistic-map edge-of-chaos q-triplet, and others. As a possible way out of this limitation, a generalization of the TGS structure was proposed in [8], which we review now.

Let us consider the following transformation:

$$q_a(q) = \frac{(a+2)-aq}{a-(a-2)q} \;\;(a \in \mathscr{R})\,, \tag{5}$$

or, equivalently,

$$\frac{1}{1-q_a(q)} = \frac{1}{q-1} + 1 - \frac{a}{2}\,, \tag{6}$$

or, even

$$\frac{2}{2-a}\frac{1}{1-q_a(q)} = \frac{2}{2-a}\frac{1}{q-1} + 1\,. \tag{7}$$

We straightforwardly verify that $q_2 = 2-q$ (*additive duality*) and $q_0 = 1/q$ (*multiplicative duality*) [2, 34, 38, 39]. Also, we generically verify *selfduality*, i.e., $q_a(q_a(q)) = q\,,\forall(a,q)$, as well as the BG fixed point, i.e., $q_a(1) = 1\,,\forall a$: See the figure in [8]. The duality (5) is in fact a quite general ratio of linear functions of q which satisfies these two important properties (selfduality and BG fixed point). It transforms biunivocally the interval $[1,-\infty)$ into the interval $[1,\frac{a}{a-2}]$. Moreover, for $a = 3$ and $a = 5$ we recover respectively $q_3 = \frac{5-3q}{3-q}$ [35] and $q_5 = \frac{7-5q}{5-3q}$ [36].

Let us combine now two[1] transformations of the type (5) (or, equivalently, (6)):

$$\mu \;\to\; q_a(q) = \frac{(a+2)-aq}{a-(a-2)q} \;\to\; \frac{1}{1-q_a(q)} = \frac{1}{q-1} + 1 - \frac{a}{2}\,, \tag{8}$$

[1] Of course, it is also possible to combine, along similar lines, three or more such transformations.

and

$$\nu \;\rightarrow\; q_b(q) = \frac{(b+2)-bq}{b-(b-2)q} \;\rightarrow\; \frac{1}{1-q_b(q)} = \frac{1}{q-1} + 1 - \frac{b}{2}\,, \tag{9}$$

with $b \neq a$. It follows that

$$\mu\nu \;\rightarrow\; q_a(q_b(q)) = \frac{(b-a)-(b-a-2)q}{(b-a+2)-(b-a)q} \;\rightarrow\; \frac{1}{1-q_a(q_b(q))} = \frac{1}{1-q} + \frac{b-a}{2}\,, \tag{10}$$

and

$$\nu\mu \;\rightarrow\; q_b(q_a(q)) = \frac{(a-b)-(a-b-2)q}{(a-b+2)-(a-b)q} \;\rightarrow\; \frac{1}{1-q_b(q_a(q))} = \frac{1}{1-q} + \frac{a-b}{2}\,, \tag{11}$$

with $\mu^2 = \nu^2 = 1$, $\nu\mu = (\mu\nu)^{-1}$, and $q_a(q_a(q)) = q\,, \forall (a,q)$.

For integer values of m and n, we can straightforwardly establish

$$(\mu\nu)^m \;\rightarrow\; q_{a,b}^{(m)}(q) \equiv q_a(q_b(q_a(q_b(...)))) = \frac{m(b-a)-[m(b-a)-2]q}{[m(b-a)+2]-m(b-a)q} \tag{12}$$

$$\rightarrow\; \frac{1}{1-q_{a,b}^{(m)}(q)} = \frac{1}{1-q_a(q_b(q_a(q_b(...))))} = \frac{1}{1-q} + m\frac{b-a}{2}\,, \tag{13}$$

and

$$(\nu\mu)^n \;\rightarrow\; q_{b,a}^{(n)}(q) \equiv q_b(q_a(q_b(q_a(...)))) = \frac{n(a-b)-[n(a-b)-2]q}{[n(a-b)+2]-n(a-b)q} \tag{14}$$

$$\rightarrow\; \frac{1}{1-q_{b,a}^{(n)}(q)} = \frac{1}{1-q_b(q_a(q_b(q_a(...))))} = \frac{1}{1-q} + n\frac{a-b}{2}\,. \tag{15}$$

As we see, $q_{a,b}^{(1)} = q_a(q_b(q))$ and $q_{b,a}^{(1)} = q_b(q_a(q))$.

For $a \neq b$ and any integer values for (m,n), the above general relations can be conveniently rewritten as follows:

$$\frac{2}{b-a}\frac{1}{1-q_{a,b}^{(m)}(q)} = \frac{2}{b-a}\frac{1}{1-q} + m \quad (m = 0, \pm 1, \pm 2, ...)\,, \tag{16}$$

and

$$\frac{2}{a-b}\frac{1}{1-q_{b,a}^{(n)}(q)} = \frac{2}{a-b}\frac{1}{1-q} + n \quad (n = 0, \pm 1, \pm 2, ...)\,. \tag{17}$$

For $m = n = 1$ and $(a,b) = (2,0)$ we recover the simple transformations $q_{2,0}^{(1)} = 2 - \frac{1}{q}$ (see Eq. (7) in [37], and footnote on page 15378 of [34]) and $q_{0,2}^{(1)} = \frac{1}{2-q}$.

We can also check that with $m = 0, \pm 1, \pm 2, ...$, $(\mu\nu)^m\mu$ and $\nu(\mu\nu)^m$ correspond respectively to

$$\frac{2}{b-a}\frac{1}{1-q_{a,b}^{(m,\mu)}(q)}-\frac{2-a}{2(b-a)}=-\Big[\frac{2}{b-a}\frac{1}{1-q}-\frac{2-a}{2(b-a)}\Big]-m\,, \tag{18}$$

and

$$\frac{2}{b-a}\frac{1}{1-q_{a,b}^{(\nu,m)}(q)}-\frac{2-b}{2(b-a)}=-\Big[\frac{2}{b-a}\frac{1}{1-q}-\frac{2-b}{2(b-a)}\Big]+m\,. \tag{19}$$

Analogously we can check that with $n=0,\pm1,\pm2,...$, $(\nu\mu)^n\nu$ and $\mu(\nu\mu)^n$ correspond respectively to

$$\frac{2}{a-b}\frac{1}{1-q_{b,a}^{(n,\nu)}(q)}-\frac{2-b}{2(a-b)}=-\Big[\frac{2}{a-b}\frac{1}{1-q}-\frac{2-b}{2(a-b)}\Big]-n\,, \tag{20}$$

and

$$\frac{2}{a-b}\frac{1}{1-q_{b,a}^{(\mu,n)}(q)}-\frac{2-a}{2(a-b)}=-\Big[\frac{2}{b-a}\frac{1}{1-q}-\frac{2-a}{2(a-b)}\Big]+n\,. \tag{21}$$

As we see, the structures that are involved exhibit some degree of complexity. Let us therefore summarize the frame within which we are working. If we have an unique parameter (noted a) to play with, we can only transform q through Eq. (5). If we have two parameters (noted a and b) to play with, we can transform q in several ways, namely through Eqs. (13), (15), (18), (19), (20) and (21), with $m=0,\pm1,\pm2,...$ and $n=0,\pm1,\pm2,...$; the cases $m=0$ and $n=0$ recover respectively Eqs. (8) and (9). The particular choice $(a,b)=(2,0)$ recovers the TGS structure introduced in [2, 34, 38, 39]. Also, the particular choice $(a,b)=(-1,0)$ within the transformation (10) recovers the transformation $q\rightarrow\frac{1+q}{3-q}$, which plays a crucial role in the q-generalized Central Limit Theorem [40]; coincidentally (or not), the relation $b-a=1$ recovers the $\gamma=1/2$ case of Eq. (32) of [8] (see also [41–43]).

To make the approach introduced in [8] even more powerful, we may introduce now the *most general self-dual ratio of linear functions of q, which has the $q=1$ fixed point*. It is given by

$$q_{a_1,a_2}(q)=\frac{a_1-a_2q}{a_2-(2a_2-a_1)q}\quad(a_1\in\mathscr{R};a_2\in\mathscr{R})\,, \tag{22}$$

or, equivalently,

$$\frac{1}{1-q_{a_1,a_2}(q)}=\frac{1}{q-1}+1+\frac{a_2}{a_2-a_1}\,, \tag{23}$$

or, even,

$$\frac{a_2-a_1}{2a_2-a_1}\frac{1}{1-q_{a_1,a_2}(q)}=\frac{a_2-a_1}{2a_2-a_1}\frac{1}{q-1}+1\,. \tag{24}$$

The particular case

$$(a_1,a_2)=(a+2,a) \tag{25}$$

recovers the transformation introduced in Eq. (5) [8]. All the steps from Eq. (8) to Eq. (21) can easily be generalized, involving now four parameters, (a_1, a_2, b_1, b_2), instead of only two, (a, b). It becomes clear that the 4-parameter structure that can be constructed with the transformation (24) remains isomorphic to the set Z of integer numbers. Of course, to go from the 4-parameter structure to the 2-parameter structure we need to assume also, analogously to Eq. (25), that $(b_1, b_2) = (b+2, b)$.

4 Some final remarks

Essentially, we reproduce here the final remarks in [8]. The data observed in [9] for the solar wind are consistent with the q-triplet [34]:
$(q_{\text{sensitivity}}, q_{\text{stationary state}}, q_{\text{relaxation}}) = (-0.5, 7/4, 4)$.

If we identify, in Eq. (10), $(q, q^{(1)}_{a,b}) \equiv (q_{\text{sensitivity}}, q_{\text{relaxation}})$ we can verify that, for $a-b=2$, the data are consistently recovered. Moreover, if we use once again Eq. (10) and $a-b=2$, but identifying now $(q, q^{(1)}_{a,b}) \equiv (q_{\text{relaxation}}, q_{\text{stationary state}})$, once again the data are consistently recovered. The particular case $(a,b) = (2,0)$ was first proposed in [34]. In other words, it is possible to consider this q-triplet as having only one independent value, say $q_{\text{sensitivity}}$; from this value we can calculate $q_{\text{relaxation}}$ by using Eq. (10); and from $q_{\text{relaxation}}$ we can calculate $q_{\text{stationary state}}$ by using once again Eq. (10). This discussion can be summarized as follows:

$$\frac{1}{1-q_{\text{sensitivity}}} - \frac{1}{1-q_{\text{relaxation}}} = \frac{1}{1-q_{\text{relaxation}}} - \frac{1}{1-q_{\text{stationary state}}} = \frac{a-b}{2} = 1\,. \quad (26)$$

It is occasionally convenient to use the ε-triplet defined as $(\varepsilon_{\text{sensitivity}}, \varepsilon_{\text{stationary state}}, \varepsilon_{\text{relaxation}}) = (1-q_{\text{sensitivity}}, 1-q_{\text{stationary state}}, 1-q_{\text{relaxation}})$. Let us mention that an amazing set of relations was found among these by [44], namely

$$\varepsilon_{\text{stationary state}} = \frac{\varepsilon_{\text{sensitivity}} + \varepsilon_{\text{relaxation}}}{2}\,, \quad (27)$$

$$\varepsilon_{\text{sensitivity}} = \sqrt{\varepsilon_{\text{stationary state}}\,\varepsilon_{\text{relaxation}}}\,, \quad (28)$$

$$\varepsilon^{-1}_{\text{relaxation}} = \frac{\varepsilon^{-1}_{\text{sensitivity}} + \varepsilon^{-1}_{\text{stationary state}}}{2}\,. \quad (29)$$

The emergence of the three *Pythagorean means* in this specific q-triplet remains still today enigmatic. One could advance that these relations hide some unexpected symmetry, but its nature remains today completely unrevealed.

Let us now focus on a different system, namely the well-known logistic map at its edge of chaos (also referred to as the Feigenbaum point). The numerical data for this map yield the q-triplet $(q_{\text{sensitivity}}, q_{\text{stationary state}}, q_{\text{relaxation}}) = (0.244487701..., 1.65 \pm 0.05, 2.249784109...)$ [21,28,46–48].

An heuristic relation has been found [45] between these three values, namely (using $\varepsilon \equiv 1-q$):

$$\varepsilon_{\text{sensitivity}} + \varepsilon_{\text{relaxation}} = \varepsilon_{\text{sensitivity}}\, \varepsilon_{\text{stationary state}} \,. \tag{30}$$

Indeed, this relation straightforwardly implies

$$q_{\text{stationary state}} = \frac{q_{\text{relaxation}} - 1}{1 - q_{\text{sensitivity}}} \,. \tag{31}$$

Through this relation we obtain $q_{\text{stationary state}} = 1.65424...$ which is perfectly compatible with 1.65 ± 0.05. In the generalized structure that we have developed here above, we have five free parameters (q, a_1, a_2, b_1, b_2) (or only three free parameters (q, a, b) in the more restricted version presented in [8]) in addition to the integer numbers (m, n). It is therefore trivial to make analytical identifications with $(q_{\text{sensitivity}}, q_{\text{stationary state}}, q_{\text{relaxation}})$ such that Eq. (30) is satisfied.

The real challenge, however, is to find a general theoretical frame within which such identifications (and, through the freedom associated with (m, n), infinitely many more, related to physical quantities) become established on a clear basis, and not only through conjectural possibilities; as a simple illustration of such q indices being associated to specific properties, we may mention the relation [49–51] $q_{\text{stationary state}} = \frac{\tau+2}{\tau}$, hence $q_{\text{stationary state}} - 1 = 2(q_{avalanche\, size} - 1)$ with $\tau \equiv 1/(q_{avalanche\, size} - 1)$. Such a frame of systematic identifications remains up to now elusive and certainly constitutes a most interesting open question. Along this line, a connection that might reveal promising is that if we assume that q is a complex number (see, for instance, [52, 53]), then Eq. (5) corresponds to nonsingular [with $(a+2)(a-2) - a^2 = -4 \neq 0, \forall a$] Moebius transformations, which form the Moebius group, defining an automorphism of the Riemann sphere.

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Unconventional supersymmetry: Local SUSY without SUGRA

Jorge Zanelli

Eugene Wigner defined particle physics as a study of group representations. Assuming spacetime to be essentially flat and therefore invariant under global (rigid) Poincaré transformations, it was Wigner's genius to observe that elementary particle states must correspond to irreducible representations of the Poincaré group. Hence, the intrinsic particle properties mass and spin (M, J) should correspond to the eigenvalues of the Casimir operators that classify those representations.

In the Standard Model, fundamental interactions result from locally realized internal symmetries (gauge groups). It has been a long-sought idea that spacetime and internal symmetries could be combined in a natural way through a "super" symmetry. The simplest implementation of supersymmetry (SUSY) has two fundamental weaknesses:

a) It predicts for each fermionic matter field a bosonic one in the same gauge representation and with the same mass, and vice-versa;
b) In spite of decades of intensive search, no experimental evidence of SUSY has been found yet.

The fact that no trace of SUSY has been observed so far has been excused by saying that it is a broken symmetry at experimentally accessible energies, but it must be unbroken at sufficiently high energy. A statement of this sort can never be falsified because it can always be said that the energy range for SUSY restoration is such high energy that it remains unobserved, which puts SUSY on a doubtful scientific basis.

In this work, we consider a gauge theory based on a superalgebra that includes an internal gauge symmetry, the local Lorentz invariance and supersymmetry gen-

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erators. The important distinctive features between this theory and standard supersymmetry are:

- The number of fermionic and bosonic states are not necessarily equal.
- There are no fermionic superpartners of gauge bosons (bosoninos), or bosonic partners of matter (s-leptons).
- Although this supersymmetry originates in a local gauge theory and gravity is included, there are no gravitini.
- Fermions acquire mass from the coupling to the background while bosons remain massless.

The existence of bosonic SUSY-invariant vacua depends on the existence of globally defined Killing spinors. Hence the fact that supersymmetry is not manifest in a given situation might be understood as a consequence of the absence of Killing spinors, a contingent phenomenon rather than a mysterious breaking of a local symmetry.

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Part III
Articles

Analysis of the production of exotic bottomonium-like resonances via heavy-meson effective theory

L. M. Abreu

Abstract We analyze the production of exotic bottomonium-like resonances in the processes involving initial bottomed meson states $\bar{B}^{(*)}B^{(*)}$, by using effective Lagrangians taking as guiding principles heavy quark symmetries. In this scenario, we consider the $Z_b(10610)$ and $Z_b'(10650)$ as bound states of $(\bar{B}^0 B^{*+} + B^+ \bar{B}^{*0})$ and $(B^{*+}\bar{B}^{*0})$ channels, respectively, and obtain the amplitudes of relevant processes.

1 Introduction

About five years ago, Belle Collaboration discovered two charged exotic states $Z_b^{\pm}(10610)$ and $Z_b'^{\pm}(10650)$ (denoted henceforth as $Z_b^{\pm}$ and $Z_b'^{\pm}$), in $\Upsilon(5S) \to \Upsilon(nS)\pi^+\pi^-$ $(n = 1,2,3)$ and $\Upsilon(5S) \to h_b(mS)\pi^-\pi^-$ $(m = 1,2)$ decays [1,2]. Their favored quantum numbers are $I^G(J^P) = 1^+(1^+)$. The masses averaged over the five channels are $m_{Z_b^{\pm}} = 10607.2 \pm 2.0$ MeV and $m_{Z_b'^{\pm}} = 10652.21.5$ MeV [6], being close to the $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds, respectively. Also, the charge neutral partner of $Z_b(10610)$ Belle Collaboration has found in Dalitz plot analysis of $\Upsilon(5S) \to \Upsilon(2S)\pi^0\pi^0$, with mass being $m_{Z_b^0} = 10609 \pm 6$ MeV, suggesting that the three sets of Z_b resonances might form isospin triplets and need at least four quarks as minimal constituents. Besides, Belle reported the observation of these two $Z_b^{(\prime)}$ in $\Upsilon(5S) \to (B\bar{B}^* + c.c.)\pi$ and $\Upsilon(5S) \to B^*\bar{B}^*\pi$ decays [4].

Many interesting theoretical discussions concerning the structure and properties of Z_b states have been made. In this sense, in the present work we are interested in analyzing the hadronic effects on the production of $Z_b^{(\prime)}$ resonances. The inspiration relies on previous works, in which it is discussed the interaction between the exotic $X(3872)$ state and light hadrons, since it can be absorbed by the comoving

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light mesons or produced from the interaction between heavy mesons [5–12]. Thus, here we investigate the processes $\bar{B}B \to \pi Z_b$, $\bar{B}^* B \to \pi Z_b^{(\prime)}$ and $\bar{B}^* B^* \to \pi Z_b^{(\prime)}$ within the framework of Heavy-Meson Effective Theory (HMET), i.e.,, with effective Lagrangians constructed, taking as guiding principles heavy quark symmetries. The leading-order amplitudes are determined and discussed.

2 Formalism

Here we introduce the effective theory known as Heavy-Meson Effective Theory (HMET). It is characterized by effective Lagrangians respecting chiral, heavy-quark spin, Lorentz, parity and charge conjugation symmetries, being given by

$$\mathscr{L} = \mathscr{L}_M + \mathscr{L}_{Z^{(\prime)}}. \tag{1}$$

In Eq. (1), $\mathscr{L}_M$ is the lowest-order effective Lagrangian carrying the kinetic terms and couplings between light- and heavy-meson fields [13–20],

$$\begin{aligned}\mathscr{L}_M = & -i \,\mathrm{Tr}\left[\bar{H}^{(Q)b} v \cdot \mathscr{D}_b^a \, H_a^{(Q)}\right] - i \,\mathrm{Tr}\left[H^{(\bar{Q})b} v \cdot \mathscr{D}_b^a \, \bar{H}_a^{(\bar{Q})}\right] \\ & + ig \,\mathrm{Tr}\left[\bar{H}^{(Q)b} H_a^{(Q)} \gamma^\mu \gamma^5\right] (\mathscr{A}_\mu)_b^a + ig \,\mathrm{Tr}\left[H^{(\bar{Q})b} \bar{H}_a^{(\bar{Q})} \gamma^\mu \gamma^5\right] (\mathscr{A}_\mu)_b^a, \end{aligned} \tag{2}$$

where we have introduced the superfields:

$$\begin{aligned} H_a^{(Q)} &= \left(\frac{1 + v_\mu \gamma^\mu}{2}\right)\left(P_{a\mu}^{*(Q)} \gamma^\mu - P_a^{(Q)} \gamma^5\right), \\ H^{(\bar{Q})a} &= \left(P_\mu^{*(\bar{Q})a} \gamma^\mu - P^{(\bar{Q})a} \gamma^5\right)\left(\frac{1 - v_\mu \gamma^\mu}{2}\right), \\ \bar{H}^{(Q)a} &= \gamma^0 H_a^{(Q)\dagger} \gamma^0, \qquad \bar{H}_a^{(\bar{Q})} = \gamma^0 H^{(\bar{Q})\dagger a} \gamma^0, \end{aligned} \tag{3}$$

with $Q = c, b$ being the index with respect to the heavy-quark flavor group $SU(2)_{HF}$; v the velocity parameter; a the triplet index of the $SU(3)_V$ group; and $P_a^{(Q/\bar{Q})}$ and $P_{a\mu}^{*(Q/\bar{Q})}$ the pseudoscalar and vector heavy-meson fields forming a $\bar{\mathbf{2}}$ representation of isospin group, i.e.,

$$P_a^{(b)} = \left(B^-, \bar{B}^0\right), \qquad P_a^{(\bar{b})} = \left(B^+, B^0\right), \tag{4}$$

for the bottomed pseudoscalar meson field, and analogously for the vector case. The heavy vector meson fields obey the transversality conditions: $v \cdot P_a^{*(Q/\bar{Q})} = 0$.

Also, in Eq. (1) we have defined

$$(\mathscr{D}_\mu)^a_b = \left[\partial_\mu + \frac{1}{2}\left(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger\right)\right]^a_b, \quad (\mathscr{A}_\mu)^a_b = \frac{1}{2}\left(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger\right)^a_b,$$
$$\xi = e^{\frac{i}{f}M}, \quad M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

In Eq. (5), M represents the light meson fields, with the ξ-field transforming as $L\xi U^\dagger = U\xi R^\dagger$ under chiral transformations; g and f are coupling and pion decay constants, respectively.

The term $\mathscr{L}_M$ in Eq. (1) is the lowest-order effective Lagrangian coupling the $Z^{(\prime)\mu}$ fields to $B^{(*)}$-mesons:

$$\mathscr{L}_{Z^{(\prime)}} = -i\frac{z^{(\prime)}}{2}\mathrm{Tr}\left[\mathscr{Z}^{(\prime)a}_{b\mu}\bar{H}^{(\bar{Q})}_a\gamma^\mu\bar{H}^{(Q)a}\right] + c.c., \tag{6}$$

where $z^{(\prime)}$ is the coupling constant; $\mathscr{Z}^{(\prime)}_\mu$ is the superfield representing Z_b and Z_b' states [18,20,21],

$$\mathscr{Z}^{(\prime)}_\mu = \left(\frac{1+v_\rho\gamma^\rho}{2}\right)Z^{(\prime)}_\mu\gamma^5\left(\frac{1-v_\sigma\gamma^\sigma}{2}\right), \tag{7}$$

with $Z^{(\prime)\mu}$ being a 2×2 matrix representing quantized fields associated to the $Z^{(\prime)\mu}$ state:

$$Z^{(\prime)}_\mu = \begin{pmatrix} \frac{1}{\sqrt{2}}Z^{(\prime)0}_\mu & Z^{(\prime)+}_\mu \\ Z^{(\prime)-}_\mu & -\frac{1}{\sqrt{2}}Z^{(\prime)0}_\mu \end{pmatrix}. \tag{8}$$

In Table 1 we outline transformation properties of the superfields under the relevant symmetries.

Now we can determine scattering amplitudes of the processes $\bar{B}^{(*)}B^{(*)} \to \pi Z^{(\prime)}_b$. Following Ref. [21], we assume that Z_b couples to the components $(\bar{B}^0 B^{*+} + B^+\bar{B}^{*0})$, while Z_b' couples only to the channel $(B^{*+}\bar{B}^{*0})$. In this sense, there is no contribution to $\bar{B}B \to \pi Z_b'$, since we do not consider the $B\bar{B}^* Z_b'$ vertex.

Then, based on the effective Lagrangians introduced above, we determine the leading-order amplitudes, i.e., the amplitudes associated to processes represented by one-heavy meson exchange diagrams. We fix the velocity parameter to be $v^\mu = (1,\mathbf{0})$, which in the present formalism means the transition to a non-relativistic approach. Also, we approximate the sum over the polarizations to $\sum\varepsilon^i\varepsilon^{*j} \sim \delta^{ij}$. Therefore, with these assumptions, the squared transition amplitudes, averaged over the spins and isospins of the particles in the initial and final states, can be written as:

Table 1: Transformations of the superfields under chiral, heavy-quark spin, Lorentz, parity and charge conjugation symmetries. U is a matrix acting on unbroken $SU(2)_V$ group; $S^{(Q)}$ is a rotation matrix acting on heavy-quark spin (HQS); $S^{(\bar{Q})}$ is a rotation matrix acting on heavy-antiquark spin; $D = D(\Lambda)$ is the spinor representation of Lorentz transformation Λ; $C = i\gamma^2\gamma^0$ is the charge conjugation matrix. The negative charge conjugation for the $\mathscr{Z}$ field holds for neutral components.

Transformation / field	$H_a^{(Q)}$	$\bar{H}^{(Q)a}$	$H^{(\bar{Q})a}$	$\bar{H}_a^{(\bar{Q})}$	$Z_{\mu ab}^{(\prime)}$
Chiral	$H_b^{(Q)}U_{ba}^\dagger$	$U^{ab}\bar{H}^{(Q)b}$	$U^{ab}H^{(\bar{Q})b}$	$\bar{H}_b^{(\bar{Q})}U_{ba}^\dagger$	$U_{ac}Z_{\mu cd}^{(\prime)}U_{db}^\dagger$
HQS	$S^{(Q)}H_a^{(Q)}$	$\bar{H}^{(Q)a}S^{(Q)\dagger}$	$H^{(\bar{Q})a}S^{(\bar{Q})\dagger}$	$S^{(\bar{Q})}H^{(\bar{Q})a}$	$S^{(Q)}Z_{\mu ab}^{(\prime)}S^{(\bar{Q})\dagger}$
Lorentz	$DH_a^{(Q)}D^{-1}$	$D\bar{H}^{(Q)a}D^{-1}$	$DH^{(\bar{Q})a}D^{-1}$	$D\bar{H}_a^{(\bar{Q})}D^{-1}$	$\Lambda_\mu^\nu DZ_{\nu ab}^{(\prime)}D^{-1}$
Parity	$-H_a^{(Q)}$	$-\bar{H}^{(Q)a}$	$-H^{(\bar{Q})a}$	$-\bar{H}_a^{(\bar{Q})}$	$-Z_{ab}^{(\prime)\mu}$
Charge Conjugation	$CH^{(\bar{Q})aT}C$	$C\bar{H}_a^{(\bar{Q})T}C$	$CH_a^{(Q)T}C$	$C\bar{H}^{(Q)aT}C$	$-CZ_{ab}^{(\prime)\mu}C = -Z_{ab}^{(\prime)\mu}$

$$
\begin{aligned}
\overline{\left|\mathscr{M}_1^{(\bar{B}B\to\pi Z_b)}\right|^2} &= \frac{1}{4}\frac{g^2z^2}{f^2}\frac{|\mathbf{p}_\pi|^2}{\left(\tilde{E}_{B^*}-E_\pi-\Delta\right)^2},\\
\overline{\left|\mathscr{M}_2^{(\bar{B}^*B\to\pi Z_b)}\right|^2} &= \frac{1}{16}\frac{g^2z^2}{f^2}\frac{|\mathbf{p}_\pi|^2}{\left(\tilde{E}_{B^*}-E_\pi\right)^2},\\
\overline{\left|\mathscr{M}_3^{(\bar{B}^*B^*\to\pi Z_b)}\right|^2} &= \frac{5}{72}\frac{g^2z^2}{f^2}|\mathbf{p}_\pi|^2\frac{|\mathbf{p}_\pi|^2}{\left(\tilde{E}_{B^*}-E_\pi+\Delta\right)^2},\\
\overline{\left|\mathscr{M}_4^{(\bar{B}^*B\to\pi Z_b')}\right|^2} &= \frac{1}{16}\frac{g^2z'^2}{f^2}\frac{|\mathbf{p}_\pi|^2}{\left(\tilde{E}_{B}-E_\pi-\Delta\right)^2},\\
\overline{\left|\mathscr{M}_5^{(\bar{B}^*B^*\to\pi Z_b')}\right|^2} &= \frac{7}{72}\frac{g^2z'^2}{f^2}|\mathbf{p}_\pi|^2\frac{|\mathbf{p}_\pi|^2}{\left(\tilde{E}_{B^*}-E_\pi\right)^2}.
\end{aligned}
\tag{9}
$$

In Eq. (9), $\mathbf{p}_\pi$ and $E_\pi = \sqrt{m_\pi^2 + |\mathbf{p}_\pi|^2}$ are the tri-momentum and energy of the pion, $\tilde{E}_{B^{(*)}} = p_{B^{(*)}}^2/2m_{B^{(*)}}$ is the kinetic energy of incoming particle 1 for every respective reaction and $\Delta = m_{B^*} - m_B$.

Taking the isospin-spin averaged squared transition amplitudes of the processes discussed above in CM frame, the four-vectors associated to the incoming bottomed mesons are $p_1 = (E_1, \mathbf{p})$, $p_2 = (E_2, -\mathbf{p})$; and to outgoing particles are $p_3 = (E_\pi, \mathbf{p}_\pi)$ and $p_4 = (E_Z, -\mathbf{p}_\pi)$. The total energy of incoming particles can be approximated to $E_1 + E_2 \approx m_1 + m_2 + E_{CM}$, where $E_{CM} = |\mathbf{p}|^2/2\mu_{12}$ is the collision energy, with μ_{12} being the reduced mass of incoming bottomed mesons [5]. Making use of conservation of energy, the pion momentum can be written as function of collision energy:

$|\mathbf{p}_\pi| \approx \{[m_1 + m_2 - m_Z + E_{CM}]^2 - m_\pi^2\}^{\frac{1}{2}}$. Thus, using these definitions in CM frame, Eq. (9) can be given properly as function of E_{CM}.

3 Results

In order to analyze the $Z_b^{(\prime)}$-production in Eq. (9) as function of collision energy E_{CM}, we use the following values for physical quantities and coupling constants [6,21,22]: $m_\pi = 137.3$ MeV; $m_B = 5279.45$ MeV; $m_{B^*} = 5324.83$ MeV; $m_Z = 10607.2$ MeV; $m_{Z'} = 10652.2$ MeV; $g = 0.6$; $f = 92.2$ MeV.

Focusing on z and z' coupling constants, the values considered here are those obtained in Ref. [21] for original coupling constants with dimensions of $E^{-\frac{1}{2}}$ within the HMET approach: 0.79 GeV$^{-\frac{1}{2}}$ and 0.62 GeV$^{-\frac{1}{2}}$, respectively. Nonetheless, we notice that the squared amplitudes shown in Eq. (9) must be multiplied by the factor $\sqrt{8 m_{B^{(*)}} m_{B^{(*)}} m_{Z^{(\prime)}}}$ to account for the non-relativistic normalization of the heavy-meson and $\mathscr{Z}$ fields [17]. Then we incorporate this factor in the definition of the z and z' couplings, yielding values with dimensions of E^1.

In addition, it is important to delimit the region of validity of the present approach. Since the relevant scales for HMET are the heavy scale M (M being the mass of the heavy meson) and the physical scale $\Lambda_\chi = 4\pi f_\pi \sim 1$ GeV, p_π is requested to be much less than Λ_χ, which safely occurs considering $p_\pi \lesssim 200$ MeV. Thus, taking the threshold and the upper bound of the pion momentum, we can estimate the allowed ranges of validity for the Z_b production processes: 185.6 MeV $\leq E_{CM}^{(1)} \lesssim 300$ MeV, 140.2 MeV $\leq E_{CM}^{(2)} \lesssim 250$ MeV and 94.8 MeV $\leq E_{CM}^{(3)} \lesssim 200$ MeV for each respective reaction; while for the Z_b' production processes we have: 185.2 MeV $\leq E_{CM}^{(4)} \lesssim 300$ MeV, 139.8 MeV $\leq E_{CM}^{(5)} \lesssim 250$ MeV.

The squared transition amplitudes in Eq. (9) are plotted in Fig. 1 as function of collision energy E_{CM}. In the case of Z_b-production, it can be noticed that all processes have magnitudes of the same order with respect to the allowed range of E_{CM}, but the $\bar{B}B \to \pi Z_b$ acquires the greatest magnitude in greater values of collision energy. Taking the upper limits of collision energy E_{CM} for each respective reaction (i.e., considering $|\mathbf{p}_\pi| \simeq 200$ MeV), the $\bar{B}B$ channel yields the biggest magnitude by a factor about 3 and 1.5 with respect to other reactions $\bar{B}^*B$ and $\bar{B}^*B^*$, respectively.

In the case of Z_b' production (right panel in Fig. 1), the process $\bar{B}^*B^*$ has the greatest magnitude in respective allowed range of E_{CM} when compared to $\bar{B}^*B$ process (we remind that $\bar{B}B \to \pi Z_b'$ has a vanishing magnitude). Working with the upper limits of E_{CM} for each respective reaction (engendering $|\mathbf{p}_\pi| \simeq 200$ MeV), the $\bar{B}^*B^*$ process yields the biggest values by a factor about 2 with respect to $\bar{B}^*B$ channel.

Also, we see that the squared magnitudes associated to the Z_b production acquires larger values with respect to the allowed range of E_{CM} with respect to the Z_b' production. In particular, at $|\mathbf{p}_\pi| \simeq 200$ MeV (taking the upper limits of collision energy E_{CM} of each reaction), the ratio between the Z_b and Z_b' production squared

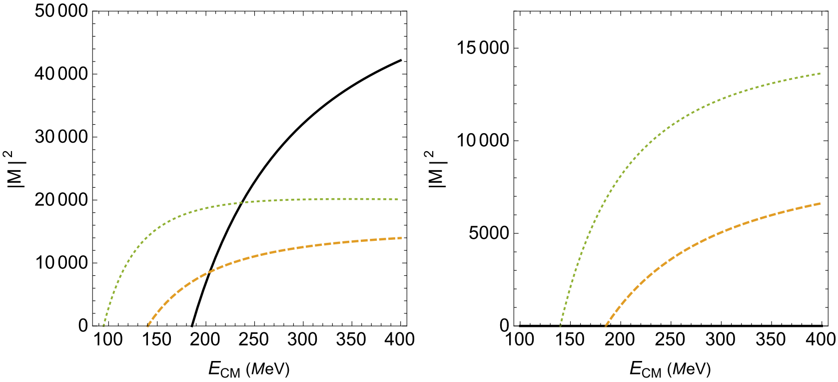

Fig. 1: Left panel: squared transition amplitudes $\overline{|\mathscr{M}_1|^2}$, $\overline{|\mathscr{M}_2|^2}$ and $\overline{|\mathscr{M}_3|^2}$ (solid, dashed and dotted lines, respectively), defined Eq. (9), as a function of collision energy. Right panel: squared transition amplitudes $\overline{|\mathscr{M}_4|^2}$ and $\overline{|\mathscr{M}_5|^2}$ (dashed and dotted lines, respectively), defined in Eq. (9), as a function of collision energy.

amplitudes is about 2, due to the different magnitudes of coupling constants and multiplicative factors in amplitudes given in Eq. (9).

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An alternative construction for the Type-II defect matrix for the sshG

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Abstract In this paper we construct a Type-II defect (super) matrix for the supersymmetric sinh-Gordon model as a product of two Type-I defect (super) matrices. We also show that the resulting defect matrix corresponds to a fused defect.

1 Introduction

Integrable classical field theories with defects and its connection with Type-I and Type-II Bäcklund transformations (BT) has been widely studied in recent years by using mainly the Lagrangian formalism and the defect matrix approach [1]– [7]. The classical integrability is ensured by the derivation of modified higher order conserved quantities, which requires explicit solutions for the corresponding defect matrices.

On the other hand, the supersymmetric extensions for Liouville and sinh-Gordon (sshG) models with Type-I and Type-II defects has been also discussed in [12]– [15], and their associated defect matrices constructed.

More recently, it has been proposed in [18] that Type-II defect matrices could be constructed as a product of two Type-I defect matrices. This proposal was checked for the bosonic case of the mKdV hierarchy.

The aim of this paper is to verify this proposal for the sshG model and show that the resulting defect matrix corresponds to a fused defect.

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2 Type-I and Type II defect formulation

The Lagrangian density describing the $N=1$ sshG model with Type-I defects located at $x=x_1$ can be written as follows:

$$\mathscr{L} = \theta(x_1-x)\mathscr{L}_1 + \delta(x-x_1)\mathscr{L}_{D_1} + \theta(x-x_1)\mathscr{L}_0, \tag{1}$$

with

$$\begin{aligned}\mathscr{L}_p &= \frac{1}{2}(\partial_x\phi_p)^2 - \frac{1}{2}(\partial_t\phi_p)^2 + i\psi_p(\partial_x+\partial_t)\psi_p - i\bar{\psi}_p(\partial_x-\partial_t)\bar{\psi}_p \\ &\quad +4\left[\cosh(2\phi_p)-1\right] - 8i\bar{\psi}_p\psi_p\cosh\phi_p, \qquad (2)\\ \mathscr{L}_{D_1} &= \frac{1}{2}(\phi_0\partial_t\phi_1 - \phi_1\partial_t\phi_0) - i\psi_1\psi_0 - i\bar{\psi}_1\bar{\psi}_0 + 2ig_1\partial_t g_1 + B_0^{(1)} + B_1^{(1)}, \qquad (3)\end{aligned}$$

where ϕ_p is a real scalar field, and ψ_p, $\bar{\psi}_p$ are the components of a Majorana spinor field in the regions $x>x_1$ $(p=0)$ and $x<x_1$ $(p=1)$ respectively, and g_1 an auxiliary fermionic field defined at the defect point. The defect potentials are given by,

$$B_0^{(1)} = 2\sigma_1\cosh(\phi_0+\phi_1) + \frac{2}{\sigma_1}\cosh(\phi_0-\phi_1), \tag{4}$$

$$B_1^{(1)} = 2i\sqrt{2}g_1\left[\sqrt{\sigma_1}\cosh\left(\frac{\phi_0+\phi_1}{2}\right)(\bar{\psi}_0+\bar{\psi}_1) + \frac{1}{\sqrt{\sigma_1}}\cosh\left(\frac{\phi_0-\phi_1}{2}\right)(\psi_0-\psi_1)\right].$$

where σ_1 represent the Bäcklund parameter. Besides the bulk field equations, we get the following defect equations at $x=x_1$:

$$\begin{aligned}\partial_t\phi_0 - \partial_x\phi_1 &= 2\sigma_1\sinh(\phi_0+\phi_1) - \frac{2}{\sigma_1}\sinh(\phi_0-\phi_1) \qquad (5)\\ &\quad + \sqrt{2\sigma_1}ig_1\left[\sinh\left(\frac{\phi_0+\phi_1}{2}\right)(\bar{\psi}_0+\bar{\psi}_1) - \frac{1}{\sigma_1}\sinh\left(\frac{\phi_0-\phi_1}{2}\right)(\psi_0-\psi_1)\right],\\ \partial_x\phi_0 - \partial_t\phi_1 &= 2\sigma_1\sinh(\phi_0+\phi_1) + \frac{2}{\sigma_1}\sinh(\phi_0-\phi_1) \qquad (6)\\ &\quad + \sqrt{2\sigma_1}ig_1\left[\sinh\left(\frac{\phi_0+\phi_1}{2}\right)(\bar{\psi}_0+\bar{\psi}_1) + \frac{1}{\sigma_1}\sinh\left(\frac{\phi_0-\phi_1}{2}\right)(\psi_0-\psi_1)\right],\\ \psi_0+\psi_1 &= 2\sqrt{\frac{2}{\sigma_1}}\cosh\left(\frac{\phi_0-\phi_1}{2}\right)g_1, \qquad (7)\\ \bar{\psi}_0-\bar{\psi}_1 &= -2\sqrt{2\sigma_1}\cosh\left(\frac{\phi_0+\phi_1}{2}\right)g_1, \qquad (8)\\ \partial_t g_1 &= \sqrt{\frac{\sigma_1}{2}}\left[\frac{1}{\sigma_1}\cosh\left(\frac{\phi_0-\phi_1}{2}\right)(\psi_1-\psi_0) - \cosh\left(\frac{\phi_0+\phi_1}{2}\right)(\bar{\psi}_0+\bar{\psi}_1)\right]. \qquad (9)\end{aligned}$$

These defect conditions preserve the integrability of the system after considering defect contributions to the conserved quantities [14]. The generating function for an infinite set of modified conserved quantities depends on the existence of the defect matrix K_1 connecting two field configurations, namely $\Psi^{(0)} = K_1 \Psi^{(1)}$, satisfying the following equations:

$$\partial_\pm K_1 = K_1 A_\pm^{(1)} - A_\pm^{(0)} K_1, \tag{10}$$

where $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$, l is a spectral parameter, and $\Psi^{(p)}$ are vector-valued fields satisfying the associated auxiliary linear problem, $\partial_\pm \Psi^{(p)} = -A_\pm^{(p)} \Psi^{(p)}$. The Lax pair $A_\pm^{(p)}$ are 3×3 graded matrices valued in the $sl(2,1)$ Lie superalgebra, which can be written in the following form:

$$A_+^{(p)} = \left(\begin{array}{cc|c} \lambda^{1/2} - \partial_+ \phi_p & -1 & \sqrt{i}\bar{\phi}_p \\ -\lambda & \lambda^{1/2} + \partial_+ \phi_p & l^{1/2}\sqrt{i}\bar{\phi}_p \\ \hline l^{1/2}\sqrt{i}\bar{\phi}_p & \sqrt{i}\bar{\phi}_p & 2\lambda^{1/2} \end{array}\right), \tag{11}$$

$$A_-^{(p)} = \left(\begin{array}{cc|c} \lambda^{-1/2} & -l^{-1} e^{2\phi_p} & l^{-1/2}\sqrt{i}\psi_p e^{\phi_p} \\ -e^{-2\phi_p} & l^{-1/2} & \sqrt{i}\psi_p e^{-\phi_p} \\ \hline -\sqrt{i}\psi_p e^{-\phi_p} & -\sqrt{i} l^{-1/2} \psi_p e^{\phi_p} & 2l^{-1/2} \end{array}\right). \tag{12}$$

Therefore, we find that a suitable solution for the type-I defect matrix K can be written in the following explicit form [14]:

$$K_1 = c_1 l^{1/2} \left(\begin{array}{cc|c} 1 & \frac{\sigma_1}{l} e^{\phi_1 + \phi_0} & -\sqrt{\frac{2i\sigma_1}{l}} e^{\frac{\phi_1+\phi_0}{2}} g_1 \\ \sigma_1 e^{-(\phi_1+\phi_0)} & 1 & -\sqrt{2i\sigma_1} e^{-\frac{(\phi_1+\phi_0)}{2}} g_1 \\ \hline \sqrt{2i\sigma_1} e^{-\frac{(\phi_1+\phi_0)}{2}} g_1 & \sqrt{\frac{2i\sigma_1}{l}} e^{\frac{(\phi_1+\phi_0)}{2}} g_1 & 1 - \frac{\sigma_1}{l^{1/2}} \end{array}\right), \tag{13}$$

where c_1 is a free constant parameter.

Now, the Type-II defect for the $N = 1$ sshG model can be constructed by considering initially a two-defects system of Type-I at different points, and then fusing them to the same point by taking a limit in the Lagrangian density [15]– [17]. Let us consider one of the defects placed at $x = x_1$ and the other at $x = x_2$. The Lagrangian density for this system can be written as,

$$\begin{aligned} \mathscr{L} = {} & \theta(x_1 - x)\mathscr{L}_1 + \delta(x - x_1)\mathscr{L}_{D_1} + \theta(x - x_1)\theta(x_2 - x)\mathscr{L}_0 \\ & + \delta(x - x_2)\mathscr{L}_{D_2} + \theta(x - x_2)\mathscr{L}_2, \end{aligned} \tag{14}$$

where $\mathscr{L}_p$, with $p = 0, 1, 2$, is given by eq. (2), and the two type-I defect Lagrangian densities at $x = x_k$, $k = 1, 2$, are given by eq. (3). Now, we have two auxiliary fermionic fields g_k, and two free parameters σ_k, with $k = 1, 2$, defined at the defect positions, respectively. At the Lagrangian level, the fusing of defects can be performed by taking the limit $x_2 \to x_1$. After some manipulations, it was shown that the fused defect is equivalent to a type-II defect [15], and takes the following form:

$$\mathscr{L}_D = \phi_- \partial_t l_0 - \frac{1}{2}\phi_- \partial_t \phi_+ + \frac{i}{2}(\bar{\psi}_+ \bar{\psi}_- - \psi_+ \psi_-) + i f_1 \partial_t f_1 + i \tilde{f}_1 \partial_t \tilde{f}_1 + B, \quad (15)$$

with $\phi_\pm = \phi_1 \pm \phi_2$, $\psi_\pm = \psi_1 \pm \psi_2$, and $B = B_0^{(+)} + B_0^{(-)} + B_1^{(+)} + B_1^{(-)}$ the defect potentials,

$$B_0^{(+)} = m\sigma \left[e^{(\phi_+ - l_0)} + e^{-(\phi_+ - l_0)} \left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau \right) \right], \quad (16)$$

$$B_0^{(-)} = \frac{m}{\sigma} \left[e^{-l_0} + e^{l_0} \left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau \right) \right], \quad (17)$$

$$B_1^{(+)} = -i\sqrt{m\sigma} \left[\left(e^{\frac{(\phi_+ - l_0)}{2}} + e^{-\frac{(\phi_+ - l_0)}{2}} \cosh\tau \right) \bar{\psi}_+ f_1 + e^{-\frac{(\phi_+ - l_0)}{2}} \sinh\left(\frac{\phi_-}{2}\right) \bar{\psi}_+ \tilde{f}_1 \right]$$
$$+ im\sigma \left(1 + e^{-(\phi_+ - l_0)} \cosh\tau \right) \cosh\left(\frac{\phi_-}{2}\right) f_1 \tilde{f}_1, \quad (18)$$

$$B_1^{(-)} = -i\sqrt{\frac{m}{\sigma}} \left[\left(e^{-\frac{l_0}{2}} + e^{\frac{l_0}{2}} \cosh\tau \right) \psi_+ \tilde{f}_1 - e^{\frac{l_0}{2}} \sinh\left(\frac{\phi_-}{2}\right) \psi_+ f_1 \right]$$
$$+ \frac{im}{\sigma} \left(1 + e^{l_0} \cosh\tau \right) \cosh\left(\frac{\phi_-}{2}\right) f_1 \tilde{f}_1, \quad (19)$$

where it has been used $\sigma_1 = \sigma e^{-\tau}, \sigma_2 = \sigma e^{\tau}$, and the reparametrizations

$$\phi_0 \to -l_0 + \frac{\phi_+}{2} - \ln\left[\cosh\left(\frac{\phi_-}{2} - \tau\right)\right] - \frac{i}{2}\operatorname{sech}\left(\frac{\phi_-}{2} - \tau\right) f_1 \tilde{f}_1, \quad (20)$$

$$f_1 = \mu_+ g_2 + \mu_- g_1, \qquad \tilde{f}_1 = \mu_- g_2 - \mu_+ g_1, \qquad \mu_\pm = \left[\frac{1 + e^{\pm(\phi_- - 2\tau)}}{2}\right]^{-\frac{1}{2}}. \quad (21)$$

From the above defect Lagrangian we can write the defect conditions at $x_1 = x_2$,

$$(\partial_x - \partial_t)\phi_+ = \partial_t l_0 - m\left[\sigma e^{-(\phi_+ - l_0)} + \frac{1}{\sigma} e^{l_0}\right] \sinh\phi_- - im\left(\sigma + \frac{1}{\sigma}\right) \sinh\left(\frac{\phi_-}{2}\right) f_1 \tilde{f}_1$$
$$+ i\sqrt{m\sigma}\, e^{-\frac{(\phi_+ - l_0)}{2}} \cosh\left(\frac{\phi_-}{2}\right) \bar{\psi}_+ \tilde{f}_1 - i\sqrt{\frac{m}{\sigma}} e^{\frac{l_0}{2}} \cosh\left(\frac{\phi_-}{2}\right) \psi_+ f_1$$
$$- im\left[\sigma e^{-(\phi_+ - l_0)} + \frac{1}{\sigma} e^{l_0}\right] \cosh\tau \sinh\left(\frac{\phi_-}{2}\right) f_1 \tilde{f}_1, \quad (22)$$

$$(\partial_x + \partial_t)\phi_- = 2m\sigma \left[e^{-(\phi_+ - l_0)} \left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau \right) - e^{(\phi_+ - l_0)} \right]$$
$$+ i\sqrt{m\sigma} \left(e^{\frac{(\phi_+ - l_0)}{2}} - e^{-\frac{(\phi_+ - l_0)}{2}} \cosh\tau \right) \bar{\psi}_+ f_1$$
$$- i\sqrt{m\sigma}\, e^{-\frac{(\phi_+ - l_0)}{2}} \sinh\left(\frac{\phi_-}{2}\right) \bar{\psi}_+ \tilde{f}_1$$
$$+ 2im\sigma\, e^{-(\phi_+ - l_0)} \cosh\tau \cosh\left(\frac{\phi_-}{2}\right) f_1 \tilde{f}_1, \quad (23)$$

$$
\begin{aligned}
(\partial_x - \partial_t)\phi_- = & \frac{2m}{\sigma}\left[e^{-l_0} - e^{l_0}\left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau\right)\right] \\
& - i\sqrt{\frac{m}{\sigma}}\left[\left(e^{-\frac{l_0}{2}} - e^{\frac{l_0}{2}}\cosh\tau\right)\psi_+\tilde{f}_1 + e^{\frac{l_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ f_1\right] \\
& - \frac{2im}{\sigma}e^{l_0}\cosh\tau\cosh\left(\frac{\phi_-}{2}\right)f_1\tilde{f}_1,
\end{aligned} \tag{24}
$$

$$
\psi_- = \sqrt{\frac{m}{\sigma}}\left[e^{\frac{l_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)f_1 - \left(e^{-\frac{l_0}{2}} + e^{\frac{l_0}{2}}\cosh\tau\right)\tilde{f}_1\right], \tag{25}
$$

$$
\bar{\psi}_- = \sqrt{m\sigma}\left[\left(e^{\frac{(\phi_+ - l_0)}{2}} + e^{-\frac{(\phi_+ - l_0)}{2}}\cosh\tau\right)f_1 + e^{-\frac{(\phi_+ - l_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\tilde{f}_1\right], \tag{26}
$$

$$
\begin{aligned}
\partial_t f_1 = & -\frac{\sqrt{m\sigma}}{2}\left(e^{\frac{(\phi_+ - l_0)}{2}} + e^{-\frac{(\phi_+ - l_0)}{2}}\cosh\tau\right)\bar{\psi}_+ + \frac{1}{2}\sqrt{\frac{m}{\sigma}}e^{\frac{l_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ \\
& - \frac{m}{2}\left[\left(\sigma + \frac{1}{\sigma}\right) + \left(\sigma e^{-(\phi_+ - l_0)} + \frac{1}{\sigma}e^{l_0}\right)\cosh\tau\right]\cosh\left(\frac{\phi_-}{2}\right)\tilde{f}_1,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\partial_t \tilde{f}_1 = & -\frac{\sqrt{m\sigma}}{2}e^{-\frac{(\phi_+ - l_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\bar{\psi}_+ - \frac{1}{2}\sqrt{\frac{m}{\sigma}}\left(e^{-\frac{l_0}{2}} + e^{\frac{l_0}{2}}\cosh\tau\right)\psi_+ \\
& + \frac{m}{2}\left[\left(\sigma + \frac{1}{\sigma}\right) + \left(\sigma e^{-(\phi_+ - l_0)} + \frac{1}{\sigma}e^{l_0}\right)\cosh\tau\right]\cosh\left(\frac{\phi_-}{2}\right)f_1.
\end{aligned} \tag{28}
$$

In order to derive the associated Type-II defect super-matrix for the model, we propose [18] to construct it as a product of two Type-I defect matrices, such that

$$
\Psi^{(2)} = K_1(\sigma_2)\Psi^{(0)} = K_1(\sigma_2)K_1(\sigma_1)\Psi^{(1)} = K_2(\sigma,\tau)\Psi^{(1)}, \tag{29}
$$

where $K_2(\sigma,\tau) = K_1(\sigma_2)K_1(\sigma_1)$. Therefore, by a direct computation we find that the components k_{ij} of the fused defect matrix K_2 are given by:

$$
k_{11} = c\left(l + \sigma^2 e^{-\phi_-} + 2i\sigma e^{-\frac{\phi_-}{2}}(g_1 g_2)l^{1/2}\right), \tag{30}
$$

$$
k_{12} = c\sigma e^{\phi_0}\left(e^{(\phi_1 - \tau)} + e^{(\phi_2 + \tau)} + 2ie^{\frac{\phi_+}{2}}(g_1 g_2)\right), \tag{31}
$$

$$
k_{13} = -c\sigma\sqrt{2i\sigma}\,e^{\frac{\phi_0}{2}}\left(e^{\phi_2 - \frac{(\phi_1 - \tau)}{2}}g_1 - e^{\frac{(\phi_2 - \tau)}{2}}g_2\right) \tag{32}
$$

$$
-c\sqrt{2i\sigma}l^{1/2}e^{\frac{\phi_0}{2}}\left(e^{\frac{(\phi_1 - \tau)}{2}}g_1 + e^{\frac{(\phi_2 + \tau)}{2}}g_2\right) \tag{33}
$$

$$
k_{21} = c\sigma e^{-\phi_0}\left(e^{-(\phi_1 + \tau)} + e^{-(\phi_2 - \tau)} + 2ie^{-\frac{\phi_+}{2}}g_1 g_2\right), \tag{34}
$$

$$
k_{22} = c\left(l + \sigma^2 e^{\phi_-} + 2i\sigma e^{-\frac{\phi_-}{2}}g_1 g_2\right), \tag{35}
$$

$$
\begin{aligned}
k_{23} = & -c\sqrt{2i\sigma}\,l\,e^{-\frac{\phi_0}{2}}\left(g_1 e^{-\frac{(\phi_1 + \tau)}{2}} + g_2 e^{-\frac{(\phi_2 - \tau)}{2}}\right) \\
& + c\sigma\sqrt{2i\sigma}\,l^{1/2}e^{-\frac{\phi_0}{2}}\left(g_2 e^{-\frac{(\phi_2 + \tau)}{2}} - g_1 e^{\frac{(\phi_1 + \tau)}{2} - \phi_2}\right),
\end{aligned} \tag{36}
$$

$$k_{31} = c\sqrt{2i\sigma}\, l\, e^{-\frac{\phi_0}{2}}\left(g_1 e^{-\frac{(\phi_1+\tau)}{2}} + g_2 e^{-\frac{(\phi_2-\tau)}{2}}\right)$$
$$+c\sigma\sqrt{2i\sigma}\, l^{1/2} e^{-\frac{\phi_0}{2}}\left(g_2\, e^{\frac{(\phi_2-\tau)}{2}-\phi_1} - g_1\, e^{\frac{(\phi_1+\tau)}{2}}\right), \tag{37}$$

$$k_{32} = c\sigma\sqrt{2i\sigma}\, e^{\frac{\phi_0}{2}}\left(g_2\, e^{-\frac{(\phi_2+\tau)}{2}+\phi_1} - g_1\, e^{\frac{(\phi_1+\tau)}{2}}\right)$$
$$+c\sqrt{2i\sigma}\, l^{1/2} e^{\frac{\phi_0}{2}}\left(e^{\frac{(\phi_1-\tau)}{2}} g_1 + e^{\frac{(\phi_2+\tau)}{2}} g_2\right), \tag{38}$$

$$k_{33} = c\left(l+\sigma^2 - 2\sigma l^{1/2}\left(\cosh(\tau) - 2ig_1g_2\cosh\left(\frac{\phi_-}{2}\right)\right)\right), \tag{39}$$

where $c = c_1c_2$. By straightforward comparison with eq. (A.80)–(A.89) in [15], it is not difficult to see that the fused defect matrix derived as product of two type-I defect matrices is equivalent (up to $l^{1/2}$) to the type-II defect matrix previously found in [15], after reparametrazing the auxiliary fields given as in eqs. (20) and (21).

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Generalized supersymmetry and the Lévy-Leblond equation

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Abstract Symmetries of the Lévy-Leblond equation are investigated beyond the standard Lie framework. It is shown that the equation has two remarkable symmetries. One is given by the super Schrödinger algebra and the other by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra is achieved by transforming bosonic into fermionic operators in the super Schrödinger algebra and introducing second order differential operators as generators of symmetry.

1 Introduction

The purpose of the present work is to show that a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra is a symmetry of a simple equation of physics, the Lévy-Leblond equation (LLE), which is a non-relativistic wave equation of a spin $1/2$ particle [9]. In the process to prove the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry we also show that LLE has a supersymmetry given by the $\mathscr{N} = 1$ super Schrödinger algebra (see [3] and references therein).

$\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras (introduced in [12, 13], see also [14]) are natural generalizations of Lie superalgebras. We present their definition: Let $\mathfrak{g}$ be a vector space over $\mathbb{C}$ or $\mathbb{R}$ with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading structure, namely $\mathfrak{g}$ is the direct sum of four distinct subspaces labelled by an element of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group:

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$$\mathfrak{g} = \mathfrak{g}_{(0,0)} + \mathfrak{g}_{(0,1)} + \mathfrak{g}_{(1,0)} + \mathfrak{g}_{(1,1)}. \tag{1}$$

For two elements $\mathbf{a} = (a_1, a_2),\ \mathbf{b} = (b_1, b_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we define

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)\ (\mathrm{mod}(2,2)), \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \tag{2}$$

Definition 1. If $\mathfrak{g}$ admits a bilinear form $[\![\ ,\ ]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following three relations, then $\mathfrak{g}$ is called a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra:

1. $[\![\mathfrak{g}_\mathbf{a}, \mathfrak{g}_\mathbf{b}]\!] \subseteq \mathfrak{g}_{\mathbf{a}+\mathbf{b}}$,
2. $[\![X_\mathbf{a}, X_\mathbf{b}]\!] = -(-1)^{\mathbf{a}\cdot\mathbf{b}} [\![X_\mathbf{b}, X_\mathbf{a}]\!]$,
3. $[\![X_\mathbf{a}, [\![X_\mathbf{b}, X]\!]]\!] = [\![[\![X_\mathbf{a}, X_\mathbf{b},]\!], X]\!] + (-1)^{\mathbf{a}\cdot\mathbf{b}} [\![X_\mathbf{b}, [\![X_\mathbf{a}, X]\!]]\!]$,

where $X_\mathbf{a} \in \mathfrak{g}_\mathbf{a}$.

Two sub superalgebras exist (they are $\mathfrak{g}_{(0,0)} + \mathfrak{g}_{(0,1)}$ and $\mathfrak{g}_{(0,0)} + \mathfrak{g}_{(1,0)}$). This fact plays a crucial role when the symmetry of the LLE is identified with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra.

In contrast to ordinary Lie algebras and superalgebras, the number of papers in the literature discussing physical applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras is limited [8, 10, 15, 17, 18]. The equation discussed in this work is both simple and fundamental. Even so, we naturally encountered this unusual algebraic structure. This would suggest that $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras are natural objects in the investigation of symmetries.

The plan of this paper is as follows. In the next section we introduce the LLE and present its symmetries. We show that the LLE has a super Schrödinger symmetry. In §3 the supersymmetry is enhanced to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie symmetry.

2 LLE and its (super)symmetries

The LLE here considered is a non-relativistic wave equation for a spin 1/2 free particle in 3D space. The wavefunction is a four-component spinor,

$$\psi(x) = {}^T(\varphi_1(x), \varphi_2(x)),$$

where φ_a is a $SU(2)$ spinor and $x = (t, x_1, x_2, x_3)$. We use the following form of LLE [4]:

$$\Omega\psi(x) = 0, \quad \Omega = -2i\alpha\partial_t + i\gamma_j\partial_{x_j} + 2m\beta, \tag{3}$$

where the sum over the repeated index $j = 1,2,3$ is understood; $\gamma_\mu, \alpha, \beta$ are 4×4 Dirac γ-matrices defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad (g_{\mu\nu}) = \mathrm{diag}(+,-,-,-), \quad \mu,\nu = 0,1,2,3 \tag{4}$$

and

$$\alpha = \frac{1}{2}(\gamma_0 + \gamma_4), \quad \beta = \frac{1}{2}(\gamma_0 - \gamma_4), \quad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{5}$$

One may take any four-dimensional representation of the γ-matrices. We do not distinguish upper and lower indices since we are working in a non-relativistic setting. LLE is the square root of the free Schrödinger equation, namely Ω^2 gives the free particle Schrödinger operator:

$$\Omega^2 = -4im\partial_t + \partial x_j^2. \tag{6}$$

We introduce now the symmetries of LLE. According to [4] we define them in terms of symmetry operators [4]:

Definition 2. Let $\mathscr{A}$ be an operator acting on the solution space of LLE. Namely, $\mathscr{A}$ maps a solution of LLE into another one:

$$\Omega\psi = 0 \quad \Longrightarrow \quad \Omega(\mathscr{A}\psi)\Big|_{\Omega\psi=0} = 0. \tag{7}$$

In this case $\mathscr{A}$ is called a symmetry operator.

In this definition $\mathscr{A}$ can be any kind of operator such as multiplication, differential, integral, etc. The traditional Lie point symmetry group of differential equations is generated by a subset of symmetry operators which is closed under commutations. Similarly, if a subset of symmetry operators forms a superalgebra or a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra, then the set generates a graded group of transformations in the solution space of LLE.

We restrict now $\mathscr{A}$ to a differential operator of finite order. In this case a sufficient condition of symmetry is given as follows. If $\mathscr{A}$ satisfies either the condition

$$[\Omega, \mathscr{A}] = \Lambda_{\mathscr{A}}(x)\Omega \tag{8}$$

or

$$\{\Omega, \mathscr{A}\} = \Gamma_{\mathscr{A}}(x)\Omega, \tag{9}$$

where $\Lambda_{\mathscr{A}}(x)$ or $\Gamma_{\mathscr{A}}(x)$ is a 4×4 matrix depending on the spacetime coordinates, then $\mathscr{A}$ is a symmetry operator.

We are looking for symmetry operators given by a first order differential operator. The results are summarized in the following two propositions:

Proposition 1. *The operators below are LLE symmetry operators satisfying the condition (8):*

$$\begin{aligned}
P_j &= \partial_{x_j}, \qquad G_j = t\partial_{x_j} + 2imx_j + \alpha\gamma_j, \qquad M = 2im,\\
H &= \partial_t, \qquad D = 2t\partial_t + x_j\partial_{x_j} + 2 - \frac{1}{2}\gamma_0\gamma_4,\\
K &= tD - t^2\partial_t + imx_jx_j + \alpha x_j\gamma_j,\\
J_{jk} &= x_j\partial_{x_k} - x_k\partial_{x_j} - \frac{1}{2}\gamma_j\gamma_k,\\
\tilde{X}_j &= -\varepsilon_{jkn}\Big([\alpha,\gamma_k]\partial_{x_n} + \frac{im}{2}[\gamma_k,\gamma_n]\Big).
\end{aligned} \tag{10}$$

The only two non-vanishing $\Lambda_{\mathscr{A}}(x)$ matrices are $\Lambda_D = 1$, $\Lambda_K = t$. For convenience the 4×4 unit matrix $\mathbf{1}_4$ is not explicitly indicated (e.g., $P_j = \mathbf{1}_4\,\partial_{x_j} \equiv \partial_{x_j}$).

Apart from the $\tilde{X}_j$'s, the remaining symmetry operators close a Lie algebra. $\mathfrak{h}(3) = \langle\, P_j, G_j, M \,\rangle$ is the three-dimensional Heisenberg Lie algebra with M as a central element. We have the non-relativistic conformal algebra $\mathfrak{sl}(2,\mathbb{R}) = \langle\, H, D, K \,\rangle$ and the spatial rotation $\mathfrak{so}(3) = \langle\, J_{jk} \,\rangle$. Combining together these three Lie algebras we get the Schrödinger algebra, whose structure is given by

$$(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \niplus \mathfrak{h}(3),$$

with $\niplus$ a semidirect sum of Lie algebras. We thus see that the Schrödinger group is a symmetry of LLE. This fact is already known in the literature. In [4] the Schrödinger algebra is presented as the maximal Lie symmetry of LLE. If the symmetry operators $\tilde{X}_j$ are included we are no longer able to close a Lie algebra. Their addition leads to a $\mathbb{Z}_2\times\mathbb{Z}_2$ graded Lie algebra. Before addressing the $\mathbb{Z}_2\times\mathbb{Z}_2$ structure we look at the LLE's supersymmetry.

Proposition 2. *The operators below are LLE symmetry operators satisfying the condition (9):*

$$\begin{aligned}
Q &= \frac{1}{\sqrt{-im}}\alpha\partial_t + \sqrt{-im}\beta,\\
S &= \frac{1}{\sqrt{-im}}\alpha\Big(t\partial_t + x_j\partial_{x_j} + \frac{3}{2}\Big) + \sqrt{-im}(t\beta + x_j\gamma_j),\\
X_j &= \frac{1}{\sqrt{-im}}\alpha\partial_{x_j} + \sqrt{-im}\gamma_j,
\end{aligned} \tag{11}$$

with only one non-vanishing $\Gamma_{\mathscr{A}}(x)$ matrix given by $\Gamma_S = -\alpha/\sqrt{-im}$.

The physical meaning of these symmetry operators becomes clear when computing their anticommutators:

$$\begin{aligned}
&\{Q,Q\} = 2H, \qquad \{S,S\} = 2K, \qquad \{X_j,X_k\} = \delta_{jk}M,\\
&\{Q,S\} = D, \qquad \{Q,X_j\} = P_j, \qquad \{S,X_j\} = G_j.
\end{aligned} \tag{12}$$

It follows that Q, S are, respectively, a supercharge and a conformal supercharge, with X_j a fermionic counterpart of $\mathfrak{h}(3)$. Indeed, the Schrödinger algebra of Proposition 1 and $\langle\, Q, S, X_j \,\rangle$ close the $\mathscr{N}=1$ super Schrödinger algebra. This is verified by direct computation of the (anti)commutation relations. The operator Q is already found in [4] without recognizing it as a supercharge. One may also show (we omit the proof for space reasons), that there exists no other supercharge $\overline{Q}$ satisfying

$$\{\overline{Q},\overline{Q}\} = 2H, \qquad \{Q,\overline{Q}\} = 0, \qquad \{\overline{Q},\Omega\} = \Gamma_{\overline{Q}}(x)\Omega,$$
$$[D,\overline{Q}] = -\overline{Q}, \qquad [J_{jk},\overline{Q}] = 0. \tag{13}$$

We thus have the theorem:

Theorem 1. *The $\mathscr{N}=1$ super Schrödinger algebra generates a symmetry supergroup of LLE and $\mathscr{N}=1$ is the maximal supersymmetry.*

The supersymmetry of LLE was conjectured many years ago in the study of the worldline supersymmetry of the spinning particle [5]. If the symmetry is defined according to Definition 2, then the conjecture is true. We mention here two other previous works on supersymmetry of LLE. In [6] it was shown that LLE coupled with an arbitrary static magnetic field has a super Schrödinger symmetry. In [7] the Dirac equation and the Deser-Jackiw-Templeton equation in a $(2+1)$ dimensional spacetime are unified in a single multiplet of $\mathfrak{osp}(1|2)$. It is shown that the non-relativistic limit of this system carries an $\mathscr{N}=2$ super Schrödinger symmetry.

3 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded symmetry of LLE

In this section we consider the symmetry of LLE with the $\tilde{X}_j$ operators. There are two key observations: (i) the $\tilde{X}_j$'s are obtained from the commutators of the fermionic generators X_j, $\tilde{X}_j = \frac{1}{2}\varepsilon_{jkn}[X_k, X_n]$; (ii) each pair $(Q,S),(P_j,G_j)$ is a $sl(2,\mathbb{R})$-doublet under the adjoint action. The observation (i) implies that we need to give up the super Schrödinger structure, while (ii) implies that we may regard (P_j,G_j) as fermionic since this treats all $sl(2,\mathbb{R})$ doublets on equal footing [16]. Therefore we introduce, from the anticommutators, the new operators

$$\tilde{P}_{jk} = \{P_j,P_k\}, \quad \tilde{G}_{jk} = \{G_j,G_k\}, \quad W_{jk} = \{P_j,G_k\},$$
$$X^P_{jk} = \{P_j,X_k\}, \quad X^G_{jk} = \{G_j,X_k\}. \tag{14}$$

They are second order differential operators; it is easy to verify that they are symmetry operators of LLE. Surprisingly, these second-order operators, together with the first-order operators in the super Schrödinger algebra, close a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra $\mathscr{G}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. This means their (anti)commutators never produce higher order differential operators. The assignment of the grading is given by

$$\begin{aligned}
\mathfrak{g}_{00} &= \langle\, H, D, K, J_{jk}, \tilde{X}_j, W_{jk}, \tilde{P}_{jk}, \tilde{G}_{jk} \,\rangle, \\
\mathfrak{g}_{01} &= \langle\, P_j, G_j \,\rangle, \\
\mathfrak{g}_{10} &= \langle\, Q, S, X^P_{jk}, X^G_{jk} \,\rangle, \\
\mathfrak{g}_{11} &= \langle\, X_j \,\rangle.
\end{aligned} \tag{15}$$

One may verify, by direct but cumbersome computation of the (anti)commutators, that the algebra (15) satisfies Definition 1. We remark that the multiplication operator M has dropped out from this $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra.

Theorem 2. *The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra defined by the operators in (15) generates a symmetry group of LLE.*

We have shown, in summary, that LLE has a $\mathscr{N} = 1$ super Schrödinger symmetry and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded symmetry given by (15). The super Schrödinger algebra is not a subalgebra of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra, although they share the same symmetry operators. As a continuation of the present work, one may investigate symmetries of LLE with a potential, since it is known that Schrödinger equation with linear or quadratic potential has the same symmetry as the free equation [2, 11]. It is also an interesting problem to study symmetries of a LLE for an arbitrary space dimension. This would be done systematically by making use of the representation theory of Clifford algebra. These works are in progress. Part of these results are reported in [1].

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Investigating the effect of cognitive stress on cardiorespiratory synchronization

Maia Angelova, Philip Holloway and Laurie Rauch

Abstract Synchrograms have been used to investigate the effects of cognitive stress, induced by the Stroop test, on the phase synchronization of the cardiac and respiratory systems. The cardiorespiratory interactions have been investigated during a rest and cognitive stressful task, namely the Stroop test, and found that cardiorespiratory synchronization decreased during cognitive stress. Synchrogram techniques and the Hilbert transform have been used to analyse phase syncronization. Our results support the hypothesis that respiration is key for improving the feedback between the cardiac and respiratory systems.

1 Introduction

The cardiac and respiratory systems are known to be coupled by several mechanisms [4]. The interaction between these two systems involves a large number of feedback and feedforward mechanisms. In healthy subjects, the heart rate increases during inspirations and decreases with expiration – a well known, and well studied phenomenon [1], known as respiratory sinus arrhythmia (RSA). Although this arrhythmia is termed respiratory it is important to note that the variations in heart rate are not directly caused by respiration itself. The modulation of the heart rate is thought to be a result of several influences, most notably the results of a reflection of the blood pressure waves via the baroreceptor feedback loop in the heart rate [7].

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Along with RSA, another phenomena rapidly gaining interest is that of cardiorespiratory synchronizations. Earlier studies support its existence [2, 16, 21, 28]. As shown in [3] cardiorespiratory synchronization and RSA represent different aspects of the interaction between the cardiac and respiratory systems.

Cognitive stress is known to affect the physiological functioning of the cardiovascular system suppressing heart rate variability (HRV) [6, 9, 27]. In physiology, HRV is the variation in the time interval between heartbeats, measured by the variation in the beat-to-beat interval [10]. Raschke et al. suggested that synchronization between the cardiac and respiratory systems would be at its strongest during states of relaxation and stated that this coordination was easily disturbed under conditions of stress or disease [19]. However, there is little knowledge on the effect of cognitive stress on cardiorespiratory synchronizations. In this study, the participants were asked to complete a Stroop test in order to impose stress and draw attention away from consciously controlling one's breathing and instead focus on completing the task. The expectation is to see an increase in synchronizations during periods of control – whether it be forced deep breathing or during unconscious control. In both scenarios the cardiorespiratory systems are trying to maintain homeostasis.

The paper is organised as follows. Section 2 introduces the experimental data and data collection methods. Section 3 considers the analysis techniques applied, followed by the results in Section 4 and final conclusions in Section 5.

2 Data

The study was undertaken with 15 healthy participants, age 24 to 58. It investigated the effect of cognitive stress with measurements before and during the Stroop test. ECG and respiration signals were recorded from all participants during a period of normal breathing where no restrictions or conditions were enforced and the subject was instructed to breathe at a rate comfortable to them. After 5 minutes of resting, the subjects were asked to complete a Stroop test. The scores from these Stroop tests were recorded. ECG was measured via 3 electrodes – placed in Einthoven's triangle configuration – and was recorded at 1000Hz. The respiratory signal was recorded via a force transducer fixed to a belt around the chest. Subjects were asked to expel air from their lungs as the transducer was first fit, and then were instructed to breathe normally. ECG and respiratory signals were recorded simultaneously for ten minutes – five minutes prior to a Stroop test and five minutes during the test, using AcqKnowledge software (version 2). The resultant time series were noisy and strongly non-stationary.

The Stroop test [25] was used to investigate the participants psychological capacities. Essentially, participants are given the name of a colour, for example red, which may or may not be written in the same colour ink. They are then asked to state the colour of a word rather than read the word, for example, if the colour red is written in blue ink the subject would be required to answer blue.

3 Methods

Synchronization is a basic phenomenon in nature [13, 20]. Through the detection of synchronous states one may be able to achieve a better understanding of physiological functioning. There are different types of synchronizations such as amplitude synchronizations or frequency synchronizations. Pikovsky et al. suggested that the properties associated with phase synchronization in chaotic oscillators are very similar to noise in noisy oscillators [17], therefore analysis of phase synchronizations would allow studying both chaotic and noisy signals, such as respiration or ECG, under one common framework.

In the case of physiological signals, detecting phase-locking is not a simple task; moreover, recording such signals via non-invasive means can result in synchronicities being hidden by considerable background noise. Therefore, an adapted definition is used here to investigate phase-locking synchronizations:

$$\phi_{n,m} = |n\Phi_h - m\Phi_r| \cong \text{const}, \tag{1}$$

where the heart beats n times in m respiratory cycles, and h and r denote heart and respiration phase respectively. In these cases, the $m:n$ phase-locking manifests itself as a variation of $\phi_{n,m}$ around a horizontal plateau [26]. The phase $\phi(t)$ can be easily estimated from any mono-component time series, however, a problem arises if the signal contains multiple component or time-varying spectra, thus making phase estimation difficult.

To study the phase synchronization of the cardiorespiratory system, we use the Hilbert transform (HT). It is far superior than Fourier-based methods, which are the simplest and most popular methods of decomposing a signal into energy-frequency distributions. However, these methods lose track of time-localised events and are proven ineffective when analysing physiological systems with non-stationary processes. The HT, y_i, can be written for any function x_i as follows:

$$y_i(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x_i(t')}{t-t'} dt', \tag{2}$$

where P indicates the Cauchy principal value. Gabor et al. determined that an analytical function can be formed with the HT pair [8],

$$z_i(t) = x_i(t) + iy_i(t) \equiv A_i(t)e^{i\phi_i(t)}, \tag{3}$$

with amplitude $A_i(t)$ and instantaneous phase $\phi_i(t)$,

$$A_i(t) = \sqrt{x_i^2(t) + y_i^2(t)}, \quad \phi_i(t) = tan^{-1}\left(\frac{y_i(t)}{x_i(t)}\right). \tag{4}$$

The instantaneous frequency can be written as the time derivative of the phase [11],

$$\omega = \frac{d\phi_i(t)}{dt}. \tag{5}$$

One of the main advantages of the HT is that it can find the phase of a single oscillation directly. In estimating the instantaneous phase, an assumption is made that the system studied can be modelled as weakly-coupled oscillators. This implies that the relative phase of the oscillators changes slowly with respect to their motion around the limit cycle, resulting in a slow convergence to a steady state phase-locking [18, 24]. We also assume that their interactions can be investigated by analysing such phases [14]. We should note that the HT is not the only method to estimate phase relationships; this can also be done by using wavelet transform or marked events methods [5, 15, 24].

In 1998, Schafer et al. developed the cardiorespiratory synchrogram in order to analyse $n:m$ synchronizations in the cardiorespiratory systems, in which the heart beats n times in m respiratory cycles [22, 23]. The synchrogram analysis is very effective to study phase synchronization between a point process (heartbeat) and a continuous signal (respiration). The HT was used to calculate the instantaneous phase of the respiration signal Φ_{nr} from (4). We then regarded the respiratory phase at times t_k – the r-peak of the k^{th} heartbeat. The cardiorespiratory synchrogram can be constructed by observing the phase of the respiration at each t_k, and wrapping the phase into a $[0, 2\pi m]$ interval. In the simplest case of $n:1$ synchronization, there are n heartbeats in each respiratory cycle. Plotting these relative phases $\Psi_{n,1}$ as a function of time against t_k, we observe n horizontal lines (representing the number of heartbeats) in one respiratory cycle. The relative phase is given by

$$\Psi_{n,m}(t_k) = \frac{1}{2\pi}[\Phi_{nr}(t_k) mod 2\pi m]. \qquad (6)$$

4 Results

ECG and respiratory signals were recorded simultaneously for ten minutes, five minutes prior to a Stroop test and five minutes during the test. Figure 1 illustrates the results gained from such recordings for one participant. Initially, ($t < 60$ sec), there is no synchronization as the participant is getting settled. From 60 sec to 300 sec pronounced regions of 6:1 synchronization can be seen with total length of 160 sec. During the Stroop test ($t >$300 sec), virtually no areas of coordination are present, which may explain the high number of Stroop mistakes (40) for this individual. All participants (except one) displayed longer regions of synchronization during the rest stage ($t < 300$ sec) with some regions lasting over 3 minutes. For the majority of participants, most prominent synchronizations were 4:1 (average 95, stdev 48) and 5:1 (average 99, stdev 72). 3:1 locking (average 75; stdev 15) was observed for three individuals and 6:1 (average 100, stdev 60) for two. During the Stroop test ($t > 300$ sec) synchronizations between the cardiorespiratory systems declined. Seven participants displayed shorter areas of synchronization: 3:1 (average 68, stdev 40), 4:1 (average 54, stdev 30), two showed prolonged 5:1 locking (average 105, stdev 15). Synchronizations were not observed for six individuals. The length of the regions of synchronization in this stage was found to correlate with the subjects performance in the Stroop test, with stronger synchronizations seen in those who performed better (R-sq =76%, p-value=0.00).

5 Conclusion

In this work we investigated the effect of cognitive stress induced via the Stroop test on cardiorespiratory synchronization using synchrograms and HT. Our analysis showed that synchronizations exist during the resting stage to some extent for each individual. Some individuals displayed considerably more synchronization than others, possibly a result of a multitude of factors; from general health to better command of their respiration and deep breathing. On the whole, synchronization between the cardiorespiratory systems declined during the Stroop test. Therefore, we conclude that cognitive stress causes a decrease or in some cases a loss of synchronization of the cardiorespiratory systems. This, however, varied from individual to individual with some participants still displaying prolonged periods of synchronization during the Stroop test. The length of synchronization present was found to correlate with the subjects performance in the Stroop test with stronger synchronizations seen in those who made a small number of mistakes.

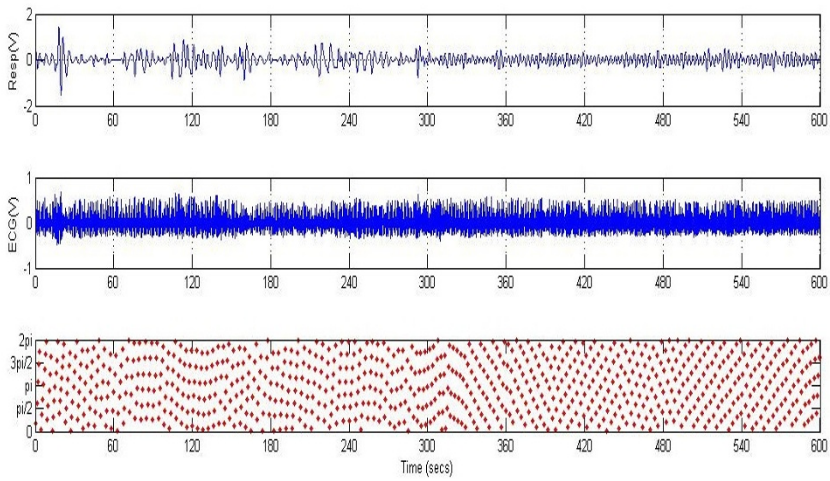

Fig. 1: Cardiorespiratory synchrogram illustrating results for one participant. In the rest stage pronounced regions of 6:1 locking can be seen for $60 < t < 300$ sec, however in the latter half of the signal (the Stroop) all synchronizations have been lost.

The study confirmed that finding phase-locking regions with synchrograms and HT is an effective way to investigate cardiorespiratory dynamics. Respiration has been shown to be the driving force behind cardiorespiratory coupling [12]. Our results support the hypothesis that control over one's respiration is essential for improving the feedback between the cardiac and respiratory systems and possibly for improving physiological function.

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Generalization of conserved charges for Toda models

Rita C. Anjos

Abstract The soliton solutions to Toda models receive a zero curvature representation of their equations of motion, i.e. there exist potentials, (A_μ), that are functionals of the fields of the theory and which belong to a Kac-Moody algebra G such that the zero curvature condition is equivalent to the equations of motion. For the construction of the soliton solutions and conserved charges it is required an integer gradation of the Kac-Moody algebra and a "vacuum solution", such that the potentials evaluated on it belong to an Abelian subalgebra. The conserved charges are then constructed using the dressing method.

1 Introduction

Several methods have been used to calculate solutions of Toda models and hence obtain the conserved charges [1–3]. The soliton solutions to the affine Toda equation of motion using Hirota's method can be derived by an ansatz. A large number of authors have obtained soliton solutions using τ-functions [4, 5].

The aim of this article is to give an explicit expression of conserved charges to the affine Toda model $sl(3)$ and $sl(N)$ following the construction given in [6, 7]. We calculate the charges for Toda model $sl(3)$ and generalized for Toda model $sl(N)$ for N-soliton [7].

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2 Dressing method

Dressing transformations are zero curvature symmetries, symmetries of non-linear 1+1 dimension differential equations. The zero curvature condition or Lax-Zakahov-Shabat equation is given by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\phi A_\mu + [A_\mu, A_\nu] = 0, \tag{1}$$

where A_μ is a Lie algebra valued vector field. This equation constitutes the conservation laws in $1+1$ dimensions. Gradation of the algebra ensures that potentials can de decomposed:

$$\hat{\mathscr{G}} = \bigoplus_n \hat{\mathscr{G}}_n \qquad\qquad [\mathscr{G}_n, \mathscr{G}_m] \subset \mathscr{G}_{n+m}. \tag{2}$$

The equation (1) is invariant under gauge transformations generated by element of group g:

$$A^h_\mu \to A_\mu = g A_\mu g^{-1} - \partial_\mu g g^{-1}, \tag{3}$$

where g is an element of the Lie group associated with A_μ. The general idea of the method is to consider the problem as a factorization problem. From equation

$$A^{vac}_\mu = -\partial_\mu \psi_{vac} \psi^{-1}_{vac}, \tag{4}$$

with

$$\psi_{vac} = e^{x_+ E_1} e^{-x_- E_{-1}}, \tag{5}$$

we consider a constant group element h, obtained by exponentiating the generators of the $sl(3)$ Kac-Moody algebra [8], which admit the Gauss decomposition

$$\psi_{vac} h \psi^{-1}_{vac} = G^{-1}_{-} G_0 G_+, \tag{6}$$

where G_-, G_0 and G_+ are group elements obtained by exponentiating the generators of the negative, zero and positive grades, respectively [6,7]. In addition, it introduces the variables:

$$\psi_h \equiv G_0 G_- \psi_{vac} h = G_+ \psi_{vac} \tag{7}$$

$$\bar{\psi}_h \equiv G_- \psi_{vac} h = G^{-1}_0 G_+ \psi_{vac} \tag{8}$$

and the corresponding potentials

$$A^h_\mu = -\partial_\mu \psi_h \psi^{-1}_h \tag{9}$$

$$\bar{A^h_\mu} = -\partial_\mu \bar{\psi}_h \bar{\psi}_h{}^{-1}. \tag{10}$$

The potentials A^h_μ and A_μ are connected by two gauge transformations, which contain only elements with non-negative and non-positive grades that preserve the gradation structure.

2.1 Conserved charges for Toda model $sl(3)$

The affine Toda theories are integrable relativistic models in two dimensions. The models can be obtained by reduction from the Wess-Zumino-Novikov-Witten model [4]. The Conformal affine Toda model is constructed by adding the two extras fields in the Affine Toda models. The potentials for Toda model $sl(3)$ are

$$\begin{aligned} A_+ = -\tfrac{e\eta}{3}[F_1^1(e^{2\varphi_1-\varphi_2}+\gamma^2 e^{2\varphi_2-\varphi_1}+\gamma e^{-\varphi_1-\varphi_2})+ \\ +F_1^2(e^{2\varphi_1-\varphi_2}+\gamma e^{2\varphi_2-\varphi_1}+\gamma^2 e^{-\varphi_1-\varphi_2})+ \\ +E_1(e^{2\varphi_1-\varphi_2}+e^{2\varphi_2-\varphi_1}+e^{-\varphi_1-\varphi_2})] \end{aligned} \tag{11}$$

and

$$A_- = -\partial_-[\frac{1}{3}F_0^1(\varphi_1+\gamma^2\varphi_2)+\frac{1}{3}F_0^2(\varphi_1+\gamma\varphi_2)+(\frac{1}{3}\varphi_1+\frac{1}{3}\varphi_2+\nu+\beta)C+\eta Q]+E_{-1}, \tag{12}$$

where F_0, F_1, E_1 and C are generators of the $sl(3)$ Kac-Moody algebra [8]. When $A_\pm$ are evaluated from vacuum solutions they become

$$A_+^{vac} = -E_1 \tag{13}$$

$$A_-^{vac} = E_1 - \partial_-\beta^{vac}C. \tag{14}$$

with $\beta^{vac} = -x_+x_-$. If the potentials are flat, we can write

$$A_\mu^{vac} = -\partial_\mu\psi_{vac}\psi_{vac}^{-1} \tag{15}$$

with

$$\psi_{vac} = e^{x_+E_1}e^{-x_-E_{-1}}. \tag{16}$$

The dressing transformation is done when we consider a constant h of the group, written in terms of exponentiating the generators of the Kac-Moody algebra $sl(3)$ and we write the Gauss decomposition (6). The potential A_μ^h defined in 9 becomes

$$A_\mu^h = G_+A_\mu^{vac}G_+^{-1} - \partial_\mu G_+G_+^{-1} \tag{17}$$

$$= G_0(G_-A_\mu^{vac}G_-^{-1} - \partial_\mu G_-G_-^{-1})G_0^{-1} - \partial_\mu G_0G_0^{-1}. \tag{18}$$

When we compare the grade zero component ($A_-^h = -\partial_-G_0G_0^{-1} - \partial_-\beta C$) with the potential (12) with $\eta = 0$, we find G_0

$$G_0 = e^{\frac{1}{3}(\varphi_1+\gamma^2\varphi_2)F_0^1+\frac{1}{3}(\varphi_1+\gamma\varphi_2)F_0^2+(\frac{1}{3}\varphi_1+\frac{1}{3}\varphi_2+\nu)C}. \tag{19}$$

Using relations of the Kac-Moody algebra $sl(3)$ and the highest weight states of representations of the Kac-Moody algebra $sl(3)$ that are annihilated by positive and negative grade generators operators, respectively $G_+|\lambda_i>=|\lambda_i>$, $<\lambda_i|G_-=<\lambda_i|$, $i=0,1,2$, we find Hirota's tau functions:

$$\tau_0 \equiv <\lambda_0|\psi_{vac}h\psi_{vac}^{-1}|\lambda_0>=<\lambda_0|G_0^{-1}|\lambda_0>=e^{-\nu} \tag{20}$$

$$\tau_1 \equiv <\lambda_1|\psi_{vac}h\psi_{vac}^{-1}|\lambda_1>=<\lambda_1|G_0^{-1}|\lambda_1>=e^{-(\varphi_1+\nu)} \tag{21}$$

$$\tau_2 \equiv <\lambda_2|\psi_{vac}h\psi_{vac}^{-1}|\lambda_2>=<\lambda_2|G_0^{-1}|\lambda_2>=e^{-(\varphi_2+\nu)} \tag{22}$$

and expressions for the fields: $\varphi_1 = log\frac{\tau_0}{\tau_1}$, $\varphi_2 = log\frac{\tau_0}{\tau_2}$ and $\nu = \beta - log\tau_0$. The variables τ_0, τ_1 and τ_2 are called Hirota's tau functions. Replacing the fields (φ_1, φ_2 and ν) into equations of motion of the Toda model $sl(3)$, what we get are Hirota's tau functions

$$\tau_i\partial_+\partial_-\tau_i - \partial_+\tau_i\partial_-\tau_i = \tau_{i-1}\tau_{i+1} - \tau_i^2 \tag{23}$$

with $i=0,1,2$, $\tau_{-1}=\tau_2$ e $\tau_3=\tau_0$. The group elements $G_\pm$ are written as follows:

$$g_{\pm,E} = exp[\sum_{n=1}^{\infty}(\xi_{1,3n+1}^{(\pm)}E_{3n+1}+\xi_{2,3n+2}^{(\pm)}E_{3n+2})] \tag{24}$$

$$g_{\pm,F} = exp[\sum_{n=1}^{\infty}(\zeta^{(\pm,1)}F_{\pm n}^1+\zeta^{(\pm,2)}F_{\pm n}^2)]. \tag{25}$$

Rewriting the relations (17) as

$$g_{+,F}A_\mu^h g_{+,F}^{-1} - \partial_\mu g_{+,F}g_{+,F}^{-1} = g_{+,E}A_\mu^{vac}g_{+,E}^{-1} - \partial_\mu g_{+,E}g_{+,E}^{-1} \equiv a_\mu^+ \tag{26}$$

$$g_{-,F}\bar{A}_\mu^h g_{-,F}^{-1} - \partial_\mu g_{-,F}g_{-,F}^{-1} = g_{-,E}A_\mu^{vac}g_{-,E}^{-1} - \partial_\mu g_{-,E}g_{-,E}^{-1} \equiv a_\mu^-, \tag{27}$$

which are used as definitions of the potentials a_μ^+ and a_μ^-. The potentials $\bar{A}_\mu^h$ are written as

$$\bar{A}_\mu^h \equiv G_0^{-1}A_\mu^h G_0 - \partial_\mu G_0^{-1}G_0, \tag{28}$$

where we used $G_\pm = g_{\pm,F}^{-1}g_{\pm,E}$ and the settings (24)-(25). Using the potentials obtained by gauge transformations ($\bar{A}_\mu^h$), vertex operators, and the discussion given in [7], the infinite number of conserved charges for the Toda model is derived. The conserved charges obtained for the 1-soliton are

$$\Omega_{(3n+1)}^{(\pm)} = \pm\sqrt{3}\left(\frac{1+\mathrm{v}}{1-\mathrm{v}}\right)^{\pm\left(\frac{3n+1}{2}\right)} \tag{29}$$

$$\Omega_{(3n+2)}^{(\pm)} = \pm\sqrt{3}\left(\frac{1+\mathrm{v}}{1-\mathrm{v}}\right)^{\pm\left(\frac{3n+2}{2}\right)}. \tag{30}$$

For 2-solitons the conserved charges have the form:

$$\Omega^{(\pm)}_{(3n+1)} = \pm\sqrt{3}\left[\left(\frac{1+\mathrm{v}_1}{1-\mathrm{v}_1}\right)^{\pm\left(\frac{3n+1}{2}\right)} + \left(\frac{1+\mathrm{v}_2}{1-\mathrm{v}_2}\right)^{\pm\left(\frac{3n+1}{2}\right)}\right] \quad (31)$$

$$\Omega^{(\pm)}_{(3n+2)} = \pm\sqrt{3}\left[\left(\frac{1+\mathrm{v}_1}{1-\mathrm{v}_1}\right)^{\pm\left(\frac{3n+2}{2}\right)} + \left(\frac{1+\mathrm{v}_2}{1-\mathrm{v}_2}\right)^{\pm\left(\frac{3n+2}{2}\right)}\right], \quad (32)$$

where $n = 0, 1, 2..$ and v is the velocity of the soliton.

2.2 Conserved charges for Toda model $sl(N)$

The potential for the Toda model $sl(N)$ are

$$A_+ = -B\Lambda_+ B^{-1} \quad (33)$$

$$A_- = -\partial_- BB^{-1} + \Lambda_-, \quad (34)$$

where B has the expression:

$$B = e^{\sum_{i=1}^{a} \varphi_a H_a + \nu C + \eta Q} \quad (35)$$

with $\Lambda_+ = \sum_{i=0}^{r} e_i$, $\Lambda_- = \sum_{i=0}^{r} v_i f_i$, where e_i and f_i are generators of the Lie algebra and v_i is a vector in the algebra and $\nu = -x_+ x_-$. The field B has a vacuum solution given by $\varphi = 0$ and $\eta = 0$ as

$$B^{(vac)} = e^{-x_+ x_- C}. \quad (36)$$

The vacuum potentials become

$$A_+^{(vac)} = -\Lambda_+ \quad (37)$$

$$A_-^{(vac)} = -\partial_-(e^{-x_+ x_- C})e^{x_+ x_- C} + \Lambda_- = -x_+ C + \Lambda_-. \quad (38)$$

These potentials satisfy the condition of zero curvature and then we define the potential $A_\mu^{(vac)}$:

$$A_\mu^{(vac)} = -\partial_\mu w_0 w_0^{-1} \quad (39)$$

where $w_0 = e^{x_+ \Lambda_+} e^{-x_- \Lambda_-}$.

From the decomposition of Gauss we have

$$G = w_0 h w_0^{-1} = e^{x_+ \Lambda_+} e^{-x_- \Lambda_-} h e^{x_- \Lambda_-} e^{-x_+ \Lambda_+} = G_- G_0 G_+, \quad (40)$$

which can be rewritten as

$$G = w_0 h w_0^{-1} = G_- G_0 G_+ = G_- \varepsilon^{(1)} \varepsilon^{(2)} G_+ \quad (41)$$

where we consider a decomposition of the generator of grade zero, $G_0 = \varepsilon_0^{(1)}\varepsilon_0^{(2)}$. In this case G_0 is the most general because it consists of a fragmentation between Θ_+ and Θ_-. The gauge potential transformed, ($A_\mu^h = \theta_\pm A_\mu^{vac}\theta_\pm^{-1} - \partial_\mu\theta_\pm\theta_\pm^{-1}$), with $\theta_+ = \varepsilon_0^{(1)}G_+$ and $\theta_- = \varepsilon_0^{(2)^{-1}}G_-^{-1}$ has the same structure as the original potential. With the states of highest weight representations of Kac-Moody algebra $sl(N)$, vertex operators and the discussion given in [7], we obtain the expressions for the conserved charges. The general expression for charges of 1-soliton:

$$\Omega_{a,n}^{(\pm)} = \pm\kappa_N\left(\frac{1+\mathrm{v}}{1-\mathrm{v}}\right)^{\pm\left(\frac{n+a}{2}\right)} \tag{42}$$

where $a = 1,...,r$ and r is the rank of the algebra, v is the velocity of the soliton and κ_N is a constant.

3 Conclusion

We emphasize the elegance and extent of the method to obtain conserved charges of $sl(3)$ and $sl(N)$ Toda models evaluated on the solutions to the orbit of the vacuum. The method is based in the representation of the equations of motion of the model in terms of the zero curvature and properties of the dressing method. For all details of calculations, see [7].

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On supersymmetric eigenvectors of the 5D discrete Fourier transform

M. K. Atakishiyeva and N. M. Atakishiyev

Abstract An explicit form of a discrete analogue of the quantum number operator, constructed in terms of the lowering and raising difference operators that govern eigenvectors of the 5D discrete (finite) Fourier transform $\Phi^{(5)}$ has been explored. This discrete number operator $\mathcal{N}^{(5)}$ has distinct eigenvalues which are employed to systematically classify eigenvectors of the $\Phi^{(5)}$, thus avoiding the ambiguity caused by the well-known degeneracy of the eigenvalues of the latter operator. In addition, we show that the hidden symmetry of the discrete number operator $\mathcal{N}^{(5)}$ manifests itself in the form of the unitary Lie superalgebra $psl(5|5)$.

We begin by recalling first a few well-known facts about the discrete Fourier transform (DFT). The discrete Fourier transform $\Phi^{(N)}$ is based on N points and represented by the $N \times N$ unitary symmetric matrix with elements

$$\Phi^{(N)}_{m,n} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi \mathrm{i}}{N} mn\right) \equiv \frac{1}{\sqrt{N}} q^{mn}, \tag{1}$$

where $q := e^{\frac{2\pi \mathrm{i}}{N}}$ and $m, n \in \{0, 1, \ldots, N-1\}$. Given a vector $\mathbf{v}$ with components $\{v_k\}_{k=0}^{N-1}$, one can compute another vector $\mathbf{u}$ with components

$$u_m = \sum_{n=0}^{N-1} \Phi^{(N)}_{m,n} v_n ,$$

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referred to as *the discrete (finite) Fourier transform* of the vector $\mathbf{v}$. Those vectors $\mathbf{f}_k$, which are solutions of the standard equations

$$\sum_{n=0}^{N-1} \Phi_{m,n}^{(N)} \left(\mathbf{f}_k\right)_n = \lambda_k \left(\mathbf{f}_k\right)_m, \qquad k \in \{0,1,\ldots,N-1\}, \tag{2}$$

then represent eigenvectors of the DFT operator $\Phi^{(N)}$, associated with the eigenvalues λ_k. Since the fourth power of $\Phi^{(N)}$ is the unit matrix, the only four distinct eigenvalues among λ_k's are the same as in the continuous case ± 1 and $\pm \mathrm{i}$.

Although there exists a plethora of discussion in the literature on eigenvectors of the DFT (see, for example [1]- [9] and the relevant references quoted there), the problem of deriving eigenvectors of DFT analytically still remains to be solved. Recently, we proposed in [10] a strategy for resolving this problem by constructing a *self-adjoint difference operator $\mathscr{N}^{(N)}$ (with distinct nonnegative eigenvalues) in terms of the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}_N^T$, which are defined by the intertwining relations*

$$\mathbf{b}_N\, \Phi^{(N)} = \mathrm{i}\, \Phi^{(N)}\, \mathbf{b}_N, \qquad \mathbf{b}_N^T\, \Phi^{(N)} = -\,\mathrm{i}\, \Phi^{(N)}\, \mathbf{b}_N^T. \tag{3}$$

The ability to solve a difference equation for eigenvectors of this discrete number operator $\mathscr{N}^{(N)}$, which commutes with the DFT operator $\Phi^{(N)}$, then enables one to define an analytical form of the desired set of eigenvectors for the latter operator.

This presentation contains a refined account of the particular dimension $N = 5$ for the general discrete Fourier transform $\Phi^{(N)}$ which includes new results not found in our earlier paper [11]. We hope that this study will deepen our understanding of the case with an arbitrary N-dimensional discrete Fourier transform and help us to provide some rigorous proofs, still needed for the generic dimensions $N > 5$.

The 5D lowering $\mathbf{b}_5$ and raising $\mathbf{b}_5^T$ difference operators for eigenvectors of the DFT operator $\Phi^{(5)}$ satisfy intertwining relations (2) with $N = 5$ and are explicitly given as

$$\mathbf{b}_5 := c\left[\mathbf{S} + \frac{1}{2}\left(\mathbf{T}^{(+)} - \mathbf{T}^{(-)}\right)\right], \qquad \mathbf{b}_5^T := c\left[\mathbf{S} - \frac{1}{2}\left(\mathbf{T}^{(+)} - \mathbf{T}^{(-)}\right)\right], \tag{4}$$

where $c = \sqrt{\frac{5}{4\pi}}$, the operator $\mathbf{S}$ represents the diagonal matrix with elements $S_{kl} := \sin(k\theta)\delta_{kl}$, $\theta := 2\pi/5$, $0 \le k,l \le 4$ and a pair of the shift operators $\mathbf{T}^{(\pm)}$ are defined as $T_{kl}^{(\pm)} := \delta_{k\pm 1,l}$ with $\delta_{-1,l} \equiv \delta_{4,l}$ and $\delta_{5,l} \equiv \delta_{0,l}$.

Let us draw attention here to the intertwining relations (2) with $N = 5$, which evidently imply that if a vector $\mathbf{f}_k$ is the eigenvector of the DFT operator $\Phi^{(5)}$, associated with the eigenvalue i^{k}, $0 \le k \le 3$, then the vectors $\mathbf{b}_5^T\, \mathbf{f}_k$ and $\mathbf{b}_5\, \mathbf{f}_k$ are also the eigenvectors of the same operator $\Phi^{(5)}$, associated with the eigenvalues $\mathrm{i}^{\mathrm{k}+1}$ and $\mathrm{i}^{\mathrm{k}-1}$, respectively.

It proves convenient to parametrize the operators $\mathbf{b}_5$ and $\mathbf{b}_5^T$ in terms of the golden ratio $\tau := (\sqrt{5}+1)/2 = -2\cos 2\theta$ and its conjugate $\tau^{-1} := (\sqrt{5}-1)/2 = 2\cos\theta =$

$\tau - 1:$

$$\left(\mathbf{b}_5\right)_{m,m'} = \frac{c}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & \kappa\tau^{1/2} & 1 & 0 & 0 \\ 0 & -1 & \kappa\tau^{-1/2} & 1 & 0 \\ 0 & 0 & -1 & -\kappa\tau^{-1/2} & 1 \\ 1 & 0 & 0 & -1 & -\kappa\tau^{1/2} \end{pmatrix}, \tag{5}$$

$$\left(\mathbf{b}_5^T\right)_{m,m'} = \frac{c}{2}\begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & \kappa\tau^{1/2} & -1 & 0 & 0 \\ 0 & 1 & \kappa\tau^{-1/2} & -1 & 0 \\ 0 & 0 & 1 & -\kappa\tau^{-1/2} & -1 \\ -1 & 0 & 0 & 1 & -\kappa\tau^{1/2} \end{pmatrix}, \tag{6}$$

where $\kappa := (5)^{1/4}$. The 'cyclic' properties of the operators $\mathbf{b}_5$ and $\mathbf{b}_5^T$ are revealed by the identities

$$\left(\mathbf{b}_5\right)^5 + 5\left(\frac{c}{2}\right)^4 \tau\,\mathbf{b}_5 = 0, \qquad \left(\mathbf{b}_5^T\right)^5 + 5\left(\frac{c}{2}\right)^4 \tau\,\mathbf{b}_5^T = 0. \tag{7}$$

The matrix elements of the discrete number operator $\mathscr{N}^{(5)} := \mathbf{b}_5^T\mathbf{b}_5$ are defined as

$$\left(\mathscr{N}^{(5)}\right)_{m,m'} = \frac{c^2}{4}\begin{pmatrix} 2 & -\kappa\tau^{1/2} & -1 & -1 & -\kappa\tau^{1/2} \\ -\kappa\tau^{1/2} & 4+\tau & \kappa\tau^{-3/2} & -1 & -1 \\ -1 & \kappa\tau^{-3/2} & 5-\tau & 2\kappa\tau^{-1/2} & -1 \\ -1 & -1 & 2\kappa\tau^{-1/2} & 5-\tau & \kappa\tau^{-3/2} \\ -\kappa\tau^{1/2} & -1 & -1 & \kappa\tau^{-3/2} & 4+\tau \end{pmatrix}. \tag{8}$$

As a product of a matrix and its transpose, the defining matrix in (4) is symmetric and all of its eigenvalues are nonnegative. Moreover, since the determinant of the matrix (4) is equal to zero, at least one of the eigenvalues should have zero value as well; but this lowest eigenvalue turns out to be unique and all eigenvalues of the matrix (4) are actually distinct. The explicit analytical form of the spectrum of the discrete number operator $\mathscr{N}^{(5)}$ can be represented as

$$\lambda_k = c^2\Big[5(1-\delta_{k0}) + 4\Big((\tau-1)\sin k\theta + \cos k\theta\Big)\sin 2k\theta\Big], \quad 0 \le k \le 4. \tag{9}$$

Orthonormal eigenvectors $\mathbf{f}_k$ of the number operator $\mathscr{N}^{(5)}$, associated with these eigenvalues λ_k, have the following components:

$$\left(\mathbf{f}_0\right)_{k=0}^4 = \frac{c^2}{4\sqrt{\lambda_2\lambda_4}}\left\{2\tau + \kappa\tau^{1/2},\, 1+\kappa\tau^{-1/2},\, 1,\, 1,\, 1+\kappa\tau^{-1/2}\right\},$$

$$\left(\mathbf{f}_1\right)_{k=0}^4 = \frac{c}{4}\sqrt{\frac{\tau}{\lambda_2}}\left\{0,\, \kappa+\tau^{1/2},\, \tau^{-1/2},\, -\tau^{-1/2},\, -\kappa-\tau^{1/2}\right\},$$

$$\left(\mathbf{f}_2\right)_{k=0}^4 = \frac{\sqrt{\tau}}{2\kappa}\left\{2(1-\tau),\, 1,\, 1,\, 1,\, 1\right\},$$

$$\left(\mathbf{f}_3\right)_{k=0}^{4} = \frac{c}{4}\sqrt{\frac{\tau}{\lambda_3}}\left\{0,\, \tau^{1/2}-\kappa,\, \tau^{-1/2},\, -\tau^{-1/2},\, \kappa-\tau^{1/2}\right\},$$

$$\left(\mathbf{f}_4\right)_{k=0}^{4} = \frac{c^2}{8\sqrt{\lambda_2\lambda_4}}\left\{2,\, -\left(\tau+2\kappa\tau^{1/2}\right),\, 2\kappa\tau^{1/2}+3\tau-2,\, III,\, II\right\}, \tag{10}$$

where III and II in the last line mean that the last two components of $\mathbf{f}_4$ coincide with the third and second components, respectively.

Recall that the existence of an explicit solution of the spectrum problem for all known exactly solvable models in quantum mechanics always indicates that there is some type of underlying hidden symmetry of the Hamiltonian, associated with each particular case [12, 13]. Since the discrete number operator $\mathcal{N}^{(5)}$ and its 5D eigenvectors $\mathbf{f}_n$ can be considered as a discrete exactly solvable model version of the linear harmonic oscillator in quantum mechanics, some hidden symmetry of the $\mathcal{N}^{(5)}$ must exist in this case as well. It turns out that this hidden symmetry of the discrete number operator $\mathcal{N}^{(5)}$ manifests itself in the form of the unitary Lie superalgebra $psl(5|5)$. This can be established in the following way.

The supersymmetric partner $\mathcal{N}_S^{(5)} = \mathbf{b}\,\mathbf{b}^\dagger$ of the 5D discrete number operator $\mathcal{N}^{(5)}$, obtained by reversing the order of $\mathbf{b}$ and $\mathbf{b}^\dagger$ in the definition of $\mathcal{N}^{(5)}$, is represented by a matrix

$$\frac{c^2}{4}\begin{pmatrix} 2 & \kappa\tau^{1/2} & -1 & -1 & \kappa\tau^{1/2} \\ \kappa\tau^{1/2} & 4+\tau & -\kappa\tau^{-3/2} & -1 & -1 \\ -1 & -\kappa\tau^{-3/2} & 5-\tau & -2\,\kappa\tau^{-1/2} & -1 \\ -1 & -1 & -2\,\kappa\tau^{-1/2} & 5-\tau & -\kappa\tau^{-3/2} \\ \kappa\tau^{1/2} & -1 & -1 & -\kappa\tau^{-3/2} & 4+\tau \end{pmatrix}. \tag{11}$$

Let $\mathbb{C}^5$ denote a 5-dimensional complex vector space, spanned by the eigenvectors of the $\mathcal{N}^{(5)}$. Then the 10D supersymmetric (SUSY) difference operator $\mathcal{N}^{(10)}$, which is built over two operators $\mathcal{N}^{(5)}$ and $\mathcal{N}_S^{(5)}$, and acts on 10-dimensional complex vector superspace $\mathbb{C}^5 \bigoplus \mathbb{C}^5$, can be written as a block matrix

$$\mathcal{N}^{(10)} := \begin{pmatrix} \mathcal{N}^{(5)} & 0_5 \\ 0_5 & \mathcal{N}_S^{(5)} \end{pmatrix}, \tag{12}$$

where 0_5 represents 5×5 zero matrix. The next natural step is to construct 10D discrete analogs of the SUSY generators (in supersymmetric theories they are called *SUSY charges*) of the form

$$\mathcal{Q}_1^{(10)} = \frac{1}{\sqrt{2}}\left(\mathbf{b}\,\Sigma_- + \mathbf{b}^\dagger\,\Sigma_+\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_5 & \mathbf{b}^\dagger \\ \mathbf{b} & 0_5 \end{pmatrix},$$

$$\mathcal{Q}_2^{(10)} = \frac{\mathrm{i}}{\sqrt{2}}\left(\mathbf{b}\,\Sigma_- - \mathbf{b}^\dagger\,\Sigma_+\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_5 & -\mathrm{i}\,\mathbf{b}^\dagger \\ \mathrm{i}\,\mathbf{b} & 0_5 \end{pmatrix}, \tag{13}$$

where 10×10 matrices $\Sigma_\pm$ are defined as $\Sigma_+ = \begin{pmatrix} 0_5 & I_5 \\ 0_5 & 0_5 \end{pmatrix}$ and $\Sigma_- = \begin{pmatrix} 0_5 & 0_5 \\ I_5 & 0_5 \end{pmatrix}$, whereas I_5 represents a 5×5 identity matrix. Then it is not hard to check that both of the SUSY generators $\mathscr{Q}_1^{(10)}$ and $\mathscr{Q}_2^{(10)}$ do commute with the SUSY discrete number operator (7), and they anti-commute among themselves:

$$\left\{\mathscr{Q}_1^{(10)}, \mathscr{Q}_2^{(10)}\right\} := \mathscr{Q}_1^{(10)}\,\mathscr{Q}_2^{(10)} + \mathscr{Q}_2^{(10)}\,\mathscr{Q}_1^{(10)} = 0\,. \tag{14}$$

Finally, it turns out that

$$\left(\mathscr{Q}_1^{(10)}\right)^2 = \left(\mathscr{Q}_2^{(10)}\right)^2 = \frac{1}{2}\mathscr{N}^{(10)} \tag{15}$$

and, consequently,

$$\mathscr{N}^{(10)} = \left(\mathscr{Q}_1^{(10)}\right)^2 + \left(\mathscr{Q}_2^{(10)}\right)^2, \tag{16}$$

which parallels one of the central features of globally SUSY theories: *the hamiltonian is the sum of the squares of the supersymmetric charges*. Thus, the three SUSY operators $\mathscr{N}^{(10)}$, $\mathscr{Q}_1^{(10)}$ and $\mathscr{Q}_2^{(10)}$ form an algebra, which closes under a combination of commutation and anti-commutation relations.

Perhaps it is worthwhile to recall at this point a few well-known facts about matrix realizations of the classical Lie superalgebras [14]- [16]. *The Lie superalgebra* $l(m,n)$ is spanned by matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{17}$$

where A and D are $gl(m)$ and $gl(n)$ matrices, B and C are $m\times n$ and $n\times m$ rectangular matrices. The supertrace function, denoted by str, is defined on $l(m,n)$ as

$$str(M) = tr(A) - tr(D). \tag{18}$$

The unitary superalgebra $sl(m|n)$ is then defined as the superalgebra of matrices $M \in l(m,n)$ satisfying the supertrace condition $str(M) = 0$. In the case of $m = n$, $sl(n|n)$ contains a one-dimensional ideal $\mathscr{I}$ generated by the identity operator I_{2n} and one sets $sl(n|n)/\mathscr{I} = psl(n|n)$.

It remains only to add that our case corresponds to $m = n = 5$ in (17) and all three matrices $\mathscr{N}^{(10)}$, $\mathscr{Q}_1^{(10)}$ and $\mathscr{Q}_2^{(10)}$ do satisfy the supertrace condition $str\left(\mathscr{N}^{(10)}\right) = 0$ (notice in particular that from (4) and (11) it is evident that the operators $\mathscr{N}^{(5)}$ and $\mathscr{N}_S^{(5)}$ have identical traces). Therefore the three SUSY operators $\mathscr{N}^{(10)}$, $\mathscr{Q}_1^{(10)}$ and $\mathscr{Q}_2^{(10)}$ are embedded in a matrix realization of the *unitary Lie superalgebra (or* $\mathbb{Z}_2$*-graded Lie algebra)* $psl(5|5)$.

It may be emphasized that this method of deriving an explicit form of the underlying supersymmetry can be readily extended to the generic dimensions $N > 5$.

To summarize, we have discussed in detail an explicit form of a difference analogue of the quantum number operator in terms of the raising and lowering operators that govern eigenvectors of the 5D discrete (finite) Fourier transform. The main algebraic properties of this operator have been examined. In particular, we have shown that the hidden symmetry of the 5D discrete number operator manifests itself in the form of the unitary Lie superalgebra $psl(5|5)$.

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Remarks on Berezin quantization on the Siegel-Jacobi ball

Stefan Berceanu

Abstract Using recent results on Berezin quantization of homogeneous Kähler manifolds, we emphasize some geometric aspects of Berezin quantization of the Siegel-Jacobi ball.

1 Introduction

The Jacobi group is defined as $G^J_n = \mathrm{H}_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$, where H_n denotes the $(2n+1)$-dimensional Heisenberg group [5, 6, 32]. The Jacobi group is an interesting object in mathematics [16, 22] and has many important applications in several branches of physics, see references in [8, 10].

The Siegel-Jacobi ball, denoted $\mathscr{D}^J_n$ [5], is the homogeneous manifold associated with the Jacobi group G^J_n, whose points are in $\mathbb{C}^n \times \mathscr{D}_n$, where $\mathscr{D}_n$ denotes the Siegel ball $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\mathrm{U}(n)$. The homogenous metric on the partially bounded domain $\mathscr{D}^J_n$ [3,5,6] was studied [8] as a balanced metric [1,21]. Recently there have been results obtained on Berezin quantization [13–15] on homogenous bounded domains [26] and homogeneous Kähler manifolds [27]. Using these results, we shall emphasize several geometric aspects of Berezin quantization on the Siegel-Jacobi ball. More details are given in [8, 9].

The paper is organized as follows. Section 3 sumarizes the notion of balanced metric in the context of Berezin quantization via coherent states. Section 3 contains a description of the balanced metric on the Siegel-Jacobi ball. The new results of this paper are contained in Remark 2 and Proposition 1 of Section 4.

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2 Coherent states, Berezin quantization, and balanced metric

Let ω_M be a G-invariant Kähler two-form

$$\omega_M(z) = \mathrm{i}\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(z)\,\mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta,\; h_{\alpha\bar{\beta}} = \bar{h}_{\beta\bar{\alpha}} = h_{\bar{\beta}\alpha}, \tag{1}$$

on the $2n$-dimensional homogeneous manifold $M = G/H$, derived from the Kähler potential $f(z,\bar{z})$, i.e., $h_{\alpha\bar{\beta}} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta}$.

The homogeneous hermitian metric on the Siegel-Jacobi ball determined in [4–7] is in fact a balanced metric, because it corresponds to the Kähler potential calculated as the scalar product of two coherent states (CS) vectors $e_{\bar{z}} \in \mathfrak{H}$, $z \in M$ in the Hillbert space $\mathfrak{H}$ of the representation of G [30],

$$f(z,\bar{z}) = \ln K_M(z,\bar{z}),\;\; K_M(z,\bar{z}) = (e_{\bar{z}}, e_{\bar{z}}). \tag{2}$$

We consider Berezin's approach to quantization on Kähler manifolds with the supercomplete set of vectors verifying the Parceval overcompletness identity [13–15]:

$$(\psi_1,\psi_2)_{\mathscr{F}_K} = \int_M (\psi_1, e_{\bar{z}})(e_{\bar{z}},\psi_2)\,\mathrm{d}\nu_M(z,\bar{z}), \quad \psi_1,\psi_2 \in \mathfrak{H}, \tag{3}$$

$$\mathrm{d}\nu_M(z,\bar{z}) = \frac{\Omega_M(z,\bar{z})}{(e_{\bar{z}}, e_{\bar{z}})}; \quad \Omega_M := \frac{1}{n!}\underbrace{\omega_M \wedge \ldots \wedge \omega_M}_{n \text{ times}}. \tag{4}$$

On the other side, it is introduced a weighted Hilbert space $\mathfrak{H}_f$ of square integrable holomorphic functions on M, with weight e^{-f} [23]:

$$\mathfrak{H}_f = \left\{ \phi \in \mathrm{hol}(M) | \int_M \mathrm{e}^{-f}\, |\phi|^2 \Omega_M < \infty \right\}. \tag{5}$$

In order to identify the Hilbert space $\mathfrak{H}_f$ defined by (5) with the Hilbert space $\mathscr{F}_K$ with the scalar product (3), it is considered the ε-function [17, 18, 31]:

$$\varepsilon(z) = \mathrm{e}^{-f(z)} K_M(z,\bar{z}). \tag{6}$$

If the Kähler metric on the complex manifold M is obtained from the Kähler potential via (1) and (2) is such that $\varepsilon(z)$ is a positive constant, then the metric is called *balanced* [1, 21, 26].

Berezin's quantization via coherent states was globalized and extended to non-homogeneous manifolds [31] in the context of geometric quantization [25]. To the Kähler manifold (M,ω_M), it is also attached the triple $\sigma = (\mathscr{L}, h, \nabla)$, where $\mathscr{L}$ is a holomorphic (pre-quantum) line bundle on M, h is the hermitian metric on $\mathscr{L}$ and ∇ is a connection compatible with metric and the Kähler structure [12]. The manifold is called *quantizable* if the curvature of the connection [20] $F(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ has the property that $F = -\mathrm{i}\,\omega_M$. Then ω_M is integral, i.e., $c_1[\mathscr{L}] =$

$[\omega_M]$. The reproducing (weighted Bergman) kernel admits the series expansion

$$K_M(z,\bar{w}) \equiv (e_{\bar{z}}, e_{\bar{w}}) = \sum_{i=0}^{\infty} \varphi_i(z)\bar{\varphi}_i(w). \tag{7}$$

$\Phi = (\varphi_0, \varphi_1, \ldots,)$ is an orthonormal base with respect to the scalar product (3).

We denote the *normalized Bergman kernel* of M (see also [2, 12]) by

$$\kappa_M(z,\bar{z}') := \frac{K_M(z,\bar{z}')}{\sqrt{K_M(z)K_M(z')}}. \tag{8}$$

The set $\Sigma_z := \{z' \in M | \kappa_M(z,\bar{z}') = 0\}$ was called *polar divisor* relative to $z \in M$ [2, 12], while a manifold for which $\Sigma_z = \emptyset, \forall z \in M$ was called in [7] a *Lu Qi-Keng manifold*, extending to manifolds a denomination introduced for bounded domains in $\mathbb{C}^n$ [28]. Note that for a particular class of compact homogeneous manifolds that includes the hermitian symmetric spaces, Σ_z is equal to the *cut locus* relative to $z \in M$ (see the definition of the cut locus, e.g., at p. 100 in [24]), and Σ_z is a *divisor* in the sense of algebraic geometry [2, 12].

3 Balanced metric on the Siegel-Jacobi ball

The Jacobi algebra is the the semi-direct sum $\mathfrak{g}^J_n := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}$, where the Heisenberg algebra $\mathfrak{h}_n$ is generated by the boson creation (respectively, annihilation) operators $a^{\dagger}_i$ (a_i), $i = 1, \ldots, n$, and the generators of the $\mathfrak{sp}(\mathbb{R})_{\mathbb{C}}$-algebra are $K^{\pm,0}_{ij}$ [3, 5, 6]. Perelomov's CS vectors [30], associated to the group G^J_n with the Jacobi algebra $\mathfrak{g}^J_n$, and based on the complex Siegel-Jacobi ball $\mathscr{D}^J_n$, have been defined as [5, 6],

$$e_{z,W} = \exp(X)e_0, \; X := \sqrt{\mu}\sum_{i=1}^{n} z_i a^{\dagger}_i + \sum_{i,j=1}^{n} w_{ij}K^{+}_{ij}, \; z \in \mathbb{C}^n; W \in \mathscr{D}_n. \tag{9}$$

The Siegel ball admits a matrix realization as a bounded homogeneous domain:

$$\mathscr{D}_n := \{W \in M(n,\mathbb{C}) : W = W^t, N > 0, N := \mathbb{1}_n - W\bar{W}\}. \tag{10}$$

If $\mu \in \mathbb{R}_+$ indexes the Heisenberg group and $k/4$ is an eigenvalue of K^0_{ij}, then the reproducing kernel $K(z,W) = (e_{z,W}, e_{z,W})$, $z \in \mathbb{C}^n, W \in \mathscr{D}_n$ is [5, 6]:

$$K(z,W) = \det(M)^{\frac{k}{2}} \exp \mu F, \;\; M = (\mathbb{1}_n - W\bar{W})^{-1}, \tag{11a}$$

$$2F = 2\bar{z}^t M z + z^t \bar{W} M z + \bar{z}^t M W \bar{z}, \tag{11b}$$

$$2F = 2\bar{\eta}^t \eta - \eta^t \bar{W} \eta - \bar{\eta}^t W \bar{\eta}, \tag{11c}$$

$$\eta = M(z + W\bar{z}); \; z = \eta - W\bar{\eta}. \tag{11d}$$

If $g \in \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ has the form

$$g = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \tag{12}$$

and $\alpha \in \mathbb{C}^n$, then the action $(g,\alpha)\times(W,z) = (W_1,z_1)$ of the Jacobi group G^J_n on the Siegel-Jacobi ball $\mathscr{D}^J_n$ is given by the formulae [5]:

$$W_1 = (pW+q)(\bar{q}W+\bar{p})^{-1} = (Wq^*+p^*)^{-1}(q^t+Wp^t), \tag{13a}$$

$$z_1 = (Wq^*+p^*)^{-1}(z+\alpha-W\bar{\alpha}). \tag{13b}$$

We use the following notation for the matrix of the hermitian metric on $\mathscr{D}^J_n$:

$$h = \begin{pmatrix} h_{i\bar{j}} & h_{i\bar{p}\bar{q}} \\ h_{pq\bar{i}} & h_{pq\bar{u}\bar{v}} \end{pmatrix} \in M(n(n+3)/2,\mathbb{C}),\; i,j=1,\dots,n;\; 1\le p\le q, u\le v\le n. \tag{14}$$

In [8] we have proved

Theorem 1. *The Kähler two-form $\omega_{\mathscr{D}^J_n}$, associated with the balanced metric of the Siegel-Jacobi ball $\mathscr{D}^J_n$, G^J_n-invariant to the action (13), has the expression*

$$\begin{aligned} -\mathrm{i}\,\omega_{\mathscr{D}^J_n}(z,W) &= \tfrac{k}{2}\mathrm{Tr}(\mathscr{B}\wedge\bar{\mathscr{B}})+\mu\mathrm{Tr}(\mathscr{A}^t\bar{M}\wedge\bar{\mathscr{A}}),\; \mathscr{A}=\mathrm{d}z+\mathrm{d}W\bar{\eta},\\ \mathscr{B} &= M\,\mathrm{d}W,\; M=(\mathbb{1}_n-W\bar{W})^{-1}. \end{aligned} \tag{15}$$

The matrix (14) of the hermitian metric on $\mathscr{D}^J_n$ has the matrix elements (16):

$$h_{i\bar{j}} = \mu\bar{M}_{ij}, \tag{16a}$$

$$h_{i\bar{p}\bar{q}} = \mu(\eta_q\bar{M}_{ip}+\eta_p\bar{M}_{iq})f_{pq},\; f_{pq} := 1-\frac{1}{2}\delta_{pq}; \tag{16b}$$

$$h_{pq\bar{i}} = \mu(\bar{\eta}_q\bar{M}_{pi}+\bar{\eta}_p\bar{M}_{qi})f_{pq}, \tag{16c}$$

$$h_{pq\bar{m}\bar{n}} = \frac{k}{2}h^k_{pq\bar{m}\bar{n}}+\mu h^\mu_{pq\bar{m}\bar{n}}, \tag{16d}$$

$$h^k_{pq\bar{m}\bar{n}} = 2M_{mp}M_{nq}d_{pq}+2M_{mq}M_{np}d_{mn}+M^2_{mp}\delta_{pq}\delta_{mn},\; d_{pq} := 1-\delta_{pq}; \tag{16e}$$

$$h^\mu_{pq\bar{m}\bar{n}} = [\bar{\eta}_p(\eta_n\bar{M}_{qm}+\eta_m\bar{M}_{qn})+\bar{\eta}_q(\eta_n\bar{M}_{pm}+\eta_m\bar{M}_{pn})]f_{pq}f_{mn}. \tag{16f}$$

The determinant of the metric matrix h is

$$\mathscr{G}_{\mathscr{D}^J_n}(z,W) := \det h_{\mathscr{D}^J_n}(z,W) = (\frac{k}{2})^{\frac{n(n+1)}{2}}\mu^n\det(\mathbb{1}_n-W\bar{W})^{-(n+2)}. \tag{17}$$

Remark 1. If $\varepsilon(z)$ is constant on M, then the balanced Hermitian metric on M is the pullback

$$\mathrm{d}s^2_M(z) = \iota^*_M\,\mathrm{d}s^2_{FS}(z) = \mathrm{d}s^2_{FS}(\iota_M(z)) \tag{18}$$

of the Fubini-Study metric via the embedding

$$\iota_M : M \hookrightarrow \mathbb{CP}^\infty,\;\; \iota_M(z) = [\varphi_0(z):\varphi_1(z):\dots]. \tag{19}$$

4 Quantization of the Siegel-Jacobi ball

Recently, some remarkable results [26, 27] about Berezin quantization, reproduced below, have been proved . We shall use these results in order to characterize Berezin quantization on the Siegel-Jacobi ball.

Theorem 2. *Let (M,ω) be a simply-connected homogeneous Kähler manifold such that the associated Kähler form ω is integral. Then there exists a constant $\mu_0 > 0$ such that M equipped with $\mu_0\omega$ is projectively induced.*

Theorem 3. *Let (M,ω) be a homogeneous Kähler manifold. Then the following are equivalent:*
a) M is contractible.
b) (M,ω) admits a global Kähler potential.
c) (M,ω) admits a global diastasis $D_M : M\times M \to \mathbb{R}$.
d) (M,ω) admits a Berezin quantization.

As a consequence of Theorem 2, the following can be proven.

Remark 2. Let $M = G/H$ be a simply-connected homogeneous Kähler manifold. Then the following assertions are equivalent:
A) M is a quantizable Kähler manifold.
B) M admits a balanced metric.
C) M is a CS-type manifold and G is a CS-type group.
D) M is projectively induced and we have (18), (19).

The notion of diastasis was introduced in [19]. The notion of CS-group is explained in [29].

Putting together Theorems 1, 2, 3, Remark 2, and Proposition 4 in [8], it follows in the particular case of the Jacobi group:

Proposition 1. *i) The Jacobi group G^J_n is an unimodular, non-reductive, algebraic group of Harish-Chandra type.*
ii) The Siegel-Jacobi domain $\mathscr{D}^J_n$ is a homogeneous reductive, non-symmetric manifold associated to the Jacobi group G^J_n by the generalized Harish-Chandra embedding.
iii) The homogeneous Kähler manifold $\mathscr{D}^J_n$ is contractible.
iv) The Kähler potential of the Siegel-Jacobi ball is global. $\mathscr{D}^J_n$ is a Lu Qi-Keng manifold, with nowhere vanishing diastasis.
v) The manifold $\mathscr{D}^J_n$ is a quantizable Kähler manifold.
vi) The manifold $\mathscr{D}^J_n$ is projectively induced, and the Jacobi group G^J_n is a CS-type group.
vii) The Siegel-Jacobi ball $\mathscr{D}^J_n$ is not an Einstein manifold with respect to the balanced metric attached to the Kähler two-form (15), but it is one with respect to the Bergman metric corresponding to the Bergman Kähler two-form $\mathrm{i}\,\partial\bar{\partial}\ln\mathscr{G}_{\mathscr{D}^J_n}$.
ix) The scalar curvature is constant and negative.

The Harish-Chandra embedding of the Siegel-Jacobi ball is explained in [11].

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The good, the bad and the ugly coherent states through polynomial Heisenberg algebras

Miguel Castillo-Celeita and David J. Fernández C.

Abstract Second degree polynomial Heisenberg algebras are realized through the harmonic oscillator Hamiltonian, together with two deformed ladder operators chosen as the third powers of the standard annihilation and creation operators. The corresponding solutions to the Painlevé IV equation are easily found. Moreover, three different sets of eigenstates of the deformed annihilation operator are constructed, called the good, the bad and the ugly coherent states. Some physical properties of such states will be studied as well.

1 Introduction

Polynomial Heisenberg algebras (PHA) of second degree are interesting deformations of the Heisenberg-Weyl algebra. In a differential representation they can be realized by one-dimensional Schrödinger Hamiltonians, together with a pair of third order ladder operators. In fact, when looking for the most general Hamiltonian ruled by such an algebraic structure, it turns out that the potential depends on solutions to a non-linear second-order ordinary differential equation called Painlevé IV (PIV). Reciprocally, if one has Hamiltonians with third-order differential ladder operators, then it is possible to design a simple algorithm for generating solutions to such an equation, by identifying just the associated extremal states [1, 2].

On the other hand, it is important to look for the simplest systems ruled by second degree PHA, such that the corresponding extremal states satisfy the boundary conditions for being eigenfunctions of the Hamiltonian [3, 4]. This is the main subject

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to be addressed in this work. Indeed, it will be shown that the harmonic oscillator Hamiltonian, together with deformed ladder operators which are the third powers of the standard annihilation and creation operators, will define a second degree PHA with such properties (Section 3). The three solutions of the PIV equation associated to this deformed algebra will be derived in the same section. The corresponding coherent states (CS) as well as their properties, will be studied in Section 3, while Section 4 will contain our conclusions.

2 Second degree PHA for the harmonic oscillator

There are several ways to realize the second degree PHA through the harmonic oscillator. Here, we look for realizations such that the three extremal states are eigenfunctions of H and, thus, we can generate from them three infinite ladders of eigenfunctions and eigenvalues [3]. Let us consider then the deformed ladder operators,

$$a_g = a^3, \qquad a_g^+ = (a^+)^3. \tag{1}$$

The operator set $\{H, a_g, a_g^+\}$ gives place to a second degree PHA, since

$$[H, a_g] = -3a_g, \qquad [H, a_g^+] = 3a_g^+, \qquad [a_g, a_g^+] = N(H+3) - N(H), \tag{2}$$

where the analogue of the number operator reads:

$$N(H) = a_g^+ a_g = \left(H - \tfrac{1}{2}\right)\left(H - \tfrac{3}{2}\right)\left(H - \tfrac{5}{2}\right). \tag{3}$$

Three extremal state energies are identified, $\mathscr{E}_j = E_{j-1} = j - \frac{1}{2},\ j = 1,2,3$, with eigenvectors given by

$$|\psi_{\mathscr{E}_j}\rangle \equiv |\psi_0^j\rangle = |j-1\rangle, \quad j = 1,2,3, \tag{4}$$

where $|j-1\rangle,\ j = 1,2,3$ are the first three energy eigenstates of the harmonic oscillator in Fock notation. Departing from them, by acting a_g^+ iteratively, we can construct three independent ladders of energy eigenstates. The eigenvalues associated to the j-th ladder are $\mathscr{E}_n^j = \mathscr{E}_j + 3n,\ n = 0,1,\ldots,\ j = 1,2,3$, and the corresponding eigenstates become

$$|\psi_n^j\rangle = |3n+j-1\rangle = \sqrt{\tfrac{(j-1)!}{(3n+j-1)!}}(a_g^+)^n|j-1\rangle, \quad j = 1,2,3. \tag{5}$$

The spectrum of H thus takes the form

$$\mathrm{Sp}(H) = \{\mathscr{E}_0^1, \mathscr{E}_1^1, \ldots\} \cup \{\mathscr{E}_0^2, \mathscr{E}_1^2, \ldots\} \cup \{\mathscr{E}_0^3, \mathscr{E}_1^3, \ldots\}, \tag{6}$$

which is the harmonic oscillator spectrum seen from a new viewpoint: the Hilbert space is the direct sum of three orthogonal supplementary subspaces,

$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$, each of them containing one ladder, which is represented in Figure 1.

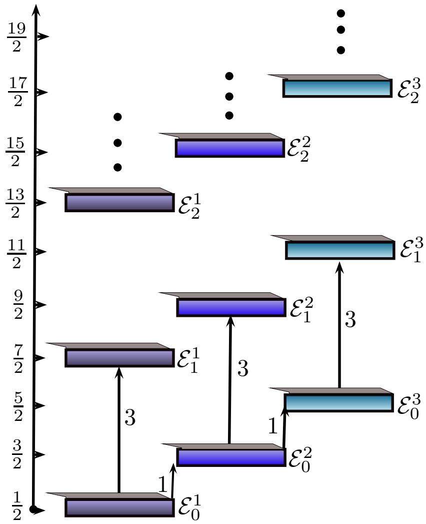


Fig. 1 The three independent ladders (with spacing $\Delta E = 3$) for the second degree PHA of Eq. (2). They produce globally the harmonic oscillator spectrum with the standard spacing $\Delta E = 1$.

Since $\{H, a_g, a_g^+\}$ generate a second degree PHA, there is a link with the PIV equation [1, 2]:

$$\frac{d^2 g}{dy^2} = \frac{1}{2g}\left(\frac{dg}{dy}\right)^2 + \frac{3}{2}g^3 + 4yg^2 + 2(y^2 - a)g + \frac{b}{g}, \tag{7}$$

which allows us to find some of its solutions. We just need to supply the three extremal states and their associated energies, in our case $\psi_{\mathscr{E}_j}(x) = \langle x|j-1\rangle,\ \mathscr{E}_j = j - 1/2,\ j = 1, 2, 3$. The PIV solution and its parameters turn out to be given by

$$g(y) = -y - \frac{d}{dy}[\ln \phi_1(y)], \quad a = \tilde{\mathscr{E}}_2 + \tilde{\mathscr{E}}_3 + 2\tilde{\mathscr{E}}_1 - 1, \quad b = -2\left(\tilde{\mathscr{E}}_2 - \tilde{\mathscr{E}}_3\right)^2, \tag{8}$$

where $\phi_1(y)$ is the first extremal state for the previous ordering (the ground state), and $y = \sqrt{3}x,\ \tilde{\mathscr{E}}_j = \mathscr{E}_j/3,\ j = 1, 2, 3$ are the changes required to fit the spacing of levels of our system ($\Delta E = 3$) with the standard spacing ($\Delta E = 1$) used in [1, 2]. Since the first label can be asigned to any extremal state, we can find indeed three PIV solutions, whose explicit expressions and corresponding parameters become

$$g(y) = -2y/3, \qquad a = 0, \qquad b = -2/9, \tag{9}$$

$$g(y) = -2y/3 - 1/y, \qquad a = -1, \qquad b = -8/9, \tag{10}$$

$$g(y) = -2y/3 - 4y/(2y^2 - 3), \qquad a = -2, \qquad b = -2/9. \tag{11}$$

3 Coherent states

Let us consider now the CS as eigenstates of the deformed annihilation operator:

$$a_g|\alpha\rangle_j = \alpha|\alpha\rangle_j, \quad j = 0,1,2, \tag{12}$$

with $|\alpha\rangle_j = \sum_{n=0}^{\infty} C_n|3n+j\rangle$. Following a standard procedure, we arrive at

$$|\alpha\rangle_j = \frac{1}{\sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n+j)!}}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(3n+j)!}}|3n+j\rangle. \tag{13}$$

Several important quantities for the CS $|\alpha\rangle_j$ can be obtained straightforwardly:

$$\langle x\rangle_j = \langle p\rangle_j = 0, \quad \langle x^2\rangle_j = \langle p^2\rangle_j = (\Delta x)_j(\Delta p)_j = \langle H\rangle_j = |a|\alpha\rangle_j|^2 + \tfrac{1}{2}, \tag{14}$$

where

$$|a|\alpha\rangle_j|^2 = \begin{cases} \frac{1}{\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r)!}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{(3n+2)!} & for \text{ j=0,} \\ \frac{1}{\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+1)!}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n)!} & for \text{ j=1,} \\ \frac{1}{\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+2)!}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n+1)!} & for \text{ j=2.} \end{cases} \tag{15}$$

Plots of the uncertainty products $(\Delta x)_j(\Delta p)_j$ for $j = 0,1,2$ are shown in Figure 2.

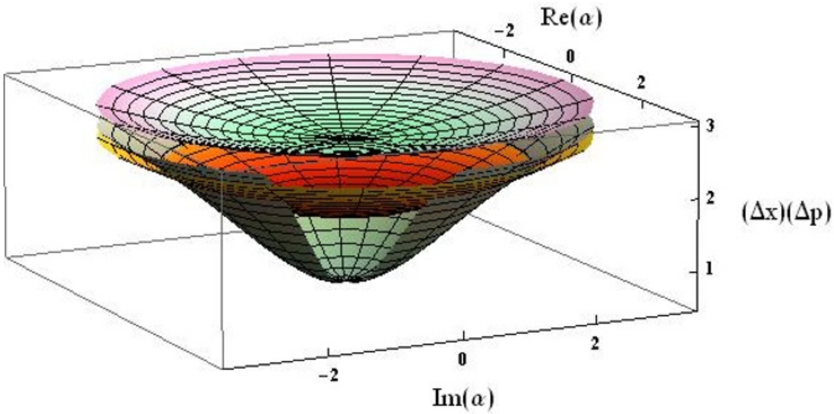


Fig. 2: Uncertainty products $(\Delta x)_j(\Delta p)_j$; the minima are $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$ for $j = 0,1,2$, respectively.

It is important to explore the completeness relation in each subspace $\mathscr{H}_j$:

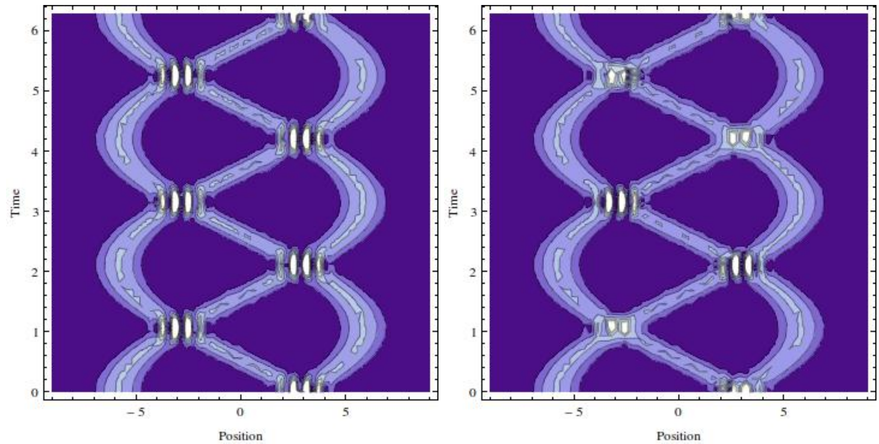
$$\int |\alpha\rangle_{j\,j}\langle\alpha| d\mu_j(\alpha) = I_j, \quad j = 0,1,2, \tag{16}$$

where I_j is the identity operator on $\mathscr{H}_j$ and

$$d\mu_j(\alpha) = \frac{1}{\pi|\alpha|}\left(\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+j)!}\right) f_j(|\alpha|^2) d|\alpha| d\varphi. \tag{17}$$

If $f_j(x)$ satisfies $\int_0^\infty x^{n-1} f_j(x)dx = \Gamma(3n+j+1)$, thus any state vector can be decomposed in terms of our CS.

Finally, the time evolution of a coherent state is quite simple, $U(t)|\alpha\rangle_j = e^{-i(j+\frac{1}{2})t}|\alpha(t)\rangle_j$, $\alpha(t) = \alpha e^{-3it}$.



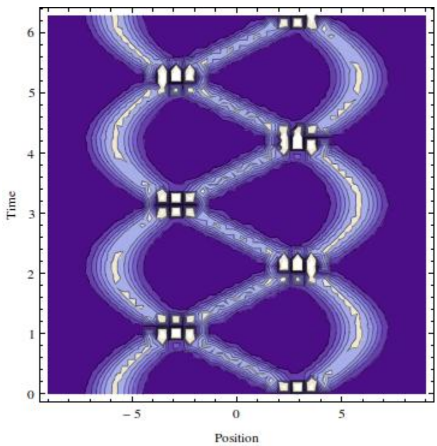


Fig. 3 Probability densities (*in position and time axis*) for the good, the bad and the ugly CS (left, right, down respectively).

Let us consider next the non-normalized coherent states:

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle, \quad |z\rangle_j = \sum_{n=0}^{\infty} \frac{z^{3n+j}}{\sqrt{(3n+j)!}}|3n+j\rangle, \quad \alpha = z^3. \tag{18}$$

The first state in Eq. (18) is a standard CS while the second one stands for the good, the bad and the ugly CS, also named three-photon CS [5]. Equation (16) ensures that $|ze^{i2\pi j/3}\rangle$ can be written in terms of $|z\rangle_j,\ j=0,1,2$ [3]. Reciprocally, we can express $|z\rangle_j$ in terms of $|ze^{i2\pi j/3}\rangle,\ j=0,1,2$:

$$|z\rangle_0 = N_0\left(|z\rangle + |e^{i2\pi/3}z\rangle + |e^{i4\pi/3}z\rangle\right), \tag{19}$$

$$|z\rangle_1 = N_1\left(|z\rangle - e^{i\pi/3}|e^{i2\pi/3}z\rangle + e^{i2\pi/3}|e^{i4\pi/3}z\rangle\right), \tag{20}$$

$$|z\rangle_2 = N_2\left(|z\rangle + e^{i2\pi/3}|e^{i2\pi/3}z\rangle + e^{i4\pi/3}|e^{i4\pi/3}z\rangle\right), \tag{21}$$

i.e., the good, the bad and the ugly CS are superpositions of standard CS with complex labels $ze^{i2\pi j/3}$ defining an equilateral triangle on the complex plane. Expressions (19-21) are used to build the wave packets associated to $|z(t)\rangle_j,\ j=0,1,2,$ whose probability densities as functions of x and t are shown in Figure 3 [3].

As we can see, the probability densities are periodic in time, with a period $(2\pi/3)$ equal to one third of the period for a classical motion for the oscillator. This implies that the good, the bad and the ugly CS cannot describe semi-classical situations, i.e., they are intrinsically quantum states. It is worth noticing the existence of some other states which are strongly quantum, e.g., the even and odd CS [4–7].

4 Conclusions

We have explored a realization of the second degree PHA in which the generators are the harmonic oscillator Hamiltonian and the ladder operators $a_g = a^3, a_g^+ = (a^+)^3$. The three associated extremal states become physical eigenstates of H, and the ladders generated from them are of infinite length. In addition, these extremal states supply some solutions to the PIV equation. The search of the eigenstates of a_g leads to three different sets, which here have been called the good, the bad and the ugly CS. Their period turns out to be a fraction $(1/3)$ of the original period (2π) for the oscillator, indicating the strong quantum nature of such states. They could be important to describe the kind of interaction matter-radiation appearing in the so-called multiphoton quantum optics [8].

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Generation and dynamics of crystallised-type states of light within the Tavis-Cummings model

O. Castaños, S. Cordero, E. Nahmad-Achar, and R. López-Peña

Abstract A generation of superpositions of photon number operator states within the generalized Tavis-Cummings model (GTC) is proposed, which is independent of the dipolar strengths and of the considered number of atoms. These are obtained by considering a linear combination of states, with total number of excitations M_1 and M_2, whose corresponding Husimi function for the electromagnetic field exhibits a cyclic point group symmetry C_n, with $n = |M_2 - M_1|$, that is, describes a crystallised-type state. Finally we establish that these superpositions under evolution with respect to the GTC Hamiltonian yields a Husimi function that preserves the cyclic point group symmetry.

1 Introduction

We know that spontaneous emission must occur if matter and radiation are to achieve thermal equilibrium. However, if the atoms are placed between mirrors in a cavity, the spontaneous emission can be controlled and manipulated. QED in cavities explores the measurement and control of atoms interacting with quantised radiation. The Dicke model studies a system of N_a non-interacting two-level atoms or molecules confined in a small container compared with the radiation wavelength. A dipolar interaction between the electromagnetic field and two-level atoms is considered in this model [1]. The $N_a = 1$ case, called the Jaynes-Cummings model (JCM) is exactly soluble with and without the rotating wave approximation (RWA) [2, 3]. The case of N_a two-level atoms or molecules in the RWA, the Tavis-Cummings model (TCM), has been also solved analytically under resonant conditions [4].

Schrödinger cat states [5, 6], even and odd coherent states [7], and squeezed states [8–11] describe non-classical states of light because they have different sta-

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tistical properties than the coherent states, which are usually called classical states of light [12–14]. More recently, the statistical properties of macroscopic superpositions of coherent states that carry irreducible representations of a finite group have been studied, together with their dynamic behaviour under evolution with respect to quadratic Hamiltonians in the quadratures of the electromagnetic field; these are called crystallised Schrödinger cats [15, 16]. The proposals to generate this type of states can be grouped as follows: (i) non-linear processes [6, 17], (ii) non-demolition measurements [18, 19], and (iii) field-atom interactions [20–22].

In this contribution we propose generating superpositions of photon-number-operator states within the GTC model for any number of particles and values of the matter-field coupling constants. Section 2 the GTC model is introduced together with the discussion of the one particle case, which can be solved analytically. In Section 3 we study the evolution of initial states with a definite value of the total number of excitations M and that of a superposition of states with M_1 and M_2 values. We determine the matter-field entanglement properties, and show that the Husimi function for the electromagnetic field exhibits a cyclic point group symmetry C_n, with $n = |M_2 - M_1|$. The conclusions of this work are presented in Section 4.

2 The generalized Tavis-Cummings model

The TCM model describes many two-level atoms or molecules interacting dipolarly with a one-mode electromagnetic field in the RWA which can be solved exactly under resonant conditions [4]. In this work, models describing many-level atoms or molecules interacting with a one-mode radiation field are called generalised Tavis-Cummings models (GTC) [23]. The Hamiltonian for the 3-level case takes the form [23, 24]

$$\boldsymbol{H}_{GTC} = \hbar\,\Omega\,\boldsymbol{a}^\dagger\boldsymbol{a} + \hbar\sum_{j=1}^{3}\omega_j\,\boldsymbol{A}_{jj} - \frac{1}{\sqrt{N_a}}\sum_{i<j=2}^{3}\mu_{ij}(\boldsymbol{a}^\dagger\,\boldsymbol{A}_{ij} + \boldsymbol{a}\boldsymbol{A}_{ji})\,, \tag{1}$$

with the convention $\omega_1 \leq \omega_2 \leq \omega_3$. $\boldsymbol{A}_{jj}$ denotes the number operator of particles in level j, and Ω the frequency of the creation and annihilation photon operators $(\boldsymbol{a}^\dagger, \boldsymbol{a})$. The raising and lowering operators of the unitary algebra in 3 dimensions can be realised in terms of bosonic operators $\boldsymbol{A}_{jk} = \boldsymbol{b}_j^\dagger\,\boldsymbol{b}_k$ and μ_{jk} is the matter-field coupling parameter between levels ω_j and ω_k. It is straightforward to check that the operator of the total number of excitations

$$\boldsymbol{M}_X = \boldsymbol{a}^\dagger\,\boldsymbol{a} + \lambda_2\,\boldsymbol{A}_{22} + \lambda_3\,\boldsymbol{A}_{33} \tag{2}$$

is a constant of motion. The parameter X indicates the atomic configuration: Ξ, V, and Λ, each with $(\lambda_2, \lambda_3) = \{(1,2), (1,1), (0,1)\}$, respectively.

Basis states are constructed in terms of the tensorial product of a Fock state $|\nu\rangle$, associated to the number of photons, and the totally symmetric Gelfand-Tsetlin state of N_a particles $|N_a\,q\,r\rangle$, i.e.,

$$|\nu;N_a\,q\,r\rangle = |\nu\rangle \otimes \frac{1}{\sqrt{(N_a-q)!\,(q-r)!\,r!}}\,\boldsymbol{A}_{31}^{N_a-q}\,\boldsymbol{A}_{21}^{q-r}|N_a,N_a,N_a\rangle\,, \tag{3}$$

where the state with all the atoms in their lowest energy level is determined by the expression $\boldsymbol{A}_{jk}|N_a,N_a,N_a\rangle = 0$, for all $k>1$. Here, r denotes the eigenvalue of $\boldsymbol{A}_{11}$, i.e., the population of the lowest level ω_1, and q denotes the sum of populations of the two lowest levels. In this basis state, one constructs the Hamiltonian matrix whose dimension d depends only on the number of particles $d=(N_a+1)(N_a+2)/2$, when $M \geq \lambda_3 N_a$ [25].

One particle case. For $N_a = 1$ one has a 3×3 Hamiltonian matrix for each value of the total number of excitations. The energy spectrum is then an infinite ladder of 3-level steps, each step determined by $E_\pm = M_X + \Delta_X/2 \pm \mathscr{E}_X$ and $E_0 = M_X$, with $\mathscr{E}_X = \sqrt{(\Delta_X/2)^2+\Omega_X^2}$. Here Δ_X denotes a detuning value Δ_{ij}, depending on the configuration and levels in question:

$$\Xi : \omega_{21} = \Omega + \Delta_{12}\,, \qquad V : \omega_{21} = \Omega + \Delta_{12}\,, \qquad \Lambda : \omega_{31} = \Omega + \Delta_{13}\,.$$

The resonant case is obtained by considering the detuning parameters Δ_{ij} equal to zero. The frequencies Ω_X are given by

$$\Omega_\Xi = \sqrt{M_\Xi\,\mu_{12}^2 + (M_\Xi-1)\,\mu_{23}^2}\,, \quad \Omega_V = \sqrt{M_V\left(\mu_{12}^2+\mu_{13}^2\right)}\,, \quad \Omega_\Lambda = \sqrt{M_\Lambda\left(\mu_{13}^2+\mu_{23}^2\right)}\,.$$

The dressed states can be determined in analytic form as they involve the diagonalisation of a 3×3 matrix for any number of total excitations. Thus they are combinations of the basis states introduced before. For the Ξ configuration these are given by

$$\begin{aligned}
|\psi_0\rangle_\Xi &= -\frac{\sqrt{M_\Xi}\,\mu_{12}}{\Omega_\Xi}|M_\Xi-2,1\,0\,0\rangle + \frac{\sqrt{M_\Xi-1}\,\mu_{23}}{\Omega_\Xi}\,|M_\Xi,1\,1\,1\rangle\,,\\
|\psi_\pm\rangle_\Xi &= \frac{1}{\mathscr{E}_\Xi\left(2\pm\frac{\Delta_{12}}{\mathscr{E}_\Xi}\right)^{1/2}}\Big\{\sqrt{M_\Xi-1}\,\mu_{23}|M_\Xi-2,1\,0\,0\rangle + \sqrt{M_\Xi}\,\mu_{12}\,|M_\Xi,1\,1\,1\rangle\\
&\quad -\left(\Delta_{12}/2\pm\mathscr{E}_\Xi\right)|M_\Xi-1,1\,1\,0\rangle\Big\}\,.
\end{aligned} \tag{4}$$

Similar expressions can be obtained for the V- and Λ-configurations.

3 Husimi function for the electromagnetic field

We first consider one atom in its ground state inside a cavity prepared in a Fock state with M photons, and study the evolution of an initial state $|M;1\,1\,1\rangle$. The state at an arbitrary time τ, in units of the frequency of the electromagnetic field, is given by

$$|\psi(\tau)\rangle_M = U_M(\tau)_{13}\,|M-\lambda_3;100\rangle + U_M(\tau)_{23}\,|M-\lambda_2;110\rangle \\ + U_M(\tau)_{33}\,|M;111\rangle\,, \tag{5}$$

where $U_M(\tau)_{ij}$ are given in the appendix for different atomic configurations.

This state yields the reduced density matrix for the radiation field

$$\rho_F^{(M)}(\tau) = P_{M-\lambda_3}(\tau)\,|M-\lambda_3\rangle\langle M-\lambda_3| + P_{M-\lambda_2}(\tau)\,|M-\lambda_2\rangle\langle M-\lambda_2| \\ + P_M(\tau)\,|M\rangle\langle M|\,, \tag{6}$$

where the time-dependent probabilities of finding $M-\lambda_3$, $M-\lambda_2$, and M photons, respectively, are

$$P_{M-\lambda_3}(\tau) = |U_M(t)_{13}|^2\,, \quad P_{M-\lambda_2}(\tau) = |U_M(\tau)_{23}|^2\,, \quad P_M(\tau) = |U_M(\tau)_{33}|^2\,. \tag{7}$$

These probabilities depend on the considered atomic configuration through the expression for the evolution matrix $U_M(t)$, which is given in the appendix. The Husimi function depends only on the magnitude of the parameter $\alpha = \rho e^{i\phi}$ of the coherent state $|\alpha\rangle$,

$$Q_H^{(M)}(\rho,\tau) = \frac{e^{-\rho^2}\rho^{2M-2\lambda_3}}{2\pi M!}\left(\frac{M!\,P_{M-\lambda_3}(\tau)}{(M-\lambda_3)!} + \frac{M!\,P_{M-\lambda_2}(\tau)}{(M-\lambda_2)!}\rho^{2(\lambda_3-\lambda_2)} + P_M(\tau)\rho^{2\lambda_3}\right).$$

The Q_H^M has a volcano shape as a function of (ρ,ϕ), whose radius at the top of the crater oscillates between $\sqrt{2(M-\lambda_3 N_a)}$ and $\sqrt{2M}$, with $N_a = 1$. This behaviour is also valid for any number of particles, and can be proved analytically and corroborated numerically.

To generate the initial state one can use the experimental result that Fock states can be prepared in a cavity. This has emerged from the interest in applications of quantum information theory, as for example secure quantum communication and quantum cryptography [26]. If instead of having the atom in the cavity, we send it through the cavity, we will have a similar behaviour for the electromagnetic sector as indicated in the Husimi function. Then we can properly select the traveling time of the atom through the cavity in order for it to leave the latter in a linear combination of two Fock states.

Without loss of generality, then one can consider a resonant cavity in a superposition of two Fock states. We then consider the evolution of a linear combination of eigenstates of two values of the total number excitations

$$|\Phi(0)\rangle = (\cos\theta|M_1\rangle + \sin\theta|M_2\rangle) \times |111\rangle\,. \tag{8}$$

It is straightforward to determine the reduced density matrix of the field and its expectation value with respect to the coherent states of light leads to

$$Q_H(\rho,\phi,\xi) = \frac{e^{-\rho^2}}{2\pi}\left(\cos^2\theta\,\frac{\rho^{2M_1}}{M_1!} + \sin^2\theta\,\frac{\rho^{2M_2}}{M_2!} + \rho^{M_1+M_2}\,\frac{\sin 2\theta}{\sqrt{M_1!M_2!}}\cos\left[(M_1-M_2)\phi\right]\right).$$

This Husimi function is an invariant under the transformation $\phi \to \phi + \frac{2\pi}{M_1-M_2}$, displaying a cyclic point symmetry $C_{|M_1-M_2|}$. This is shown in Fig. 3 for the Ξ atomic configuration with $M_1 = 5$ and $M_2 = 2$. The form of the function is qualitatively similar for different strengths of the matter-field coupling parameters.

By a similar procedure than in the previous case, its dynamics is obtained through the evolution operator; then the reduced density matrix of the field and its expectation value with respect to the coherent state are calculated as follows:

$$\begin{aligned} Q_H(\theta,\rho,\phi,t) &= \cos^2\theta\, Q_H^{M1}(\rho,t) + \sin^2\theta\, Q_H^{M_2}(\rho,t) + \sin 2\theta\,\frac{e^{-\rho^2}\rho^{M_1+M_2-2\lambda_3}}{2\pi\sqrt{M_1!M_2!}} \\ &\quad\times \cos\{(M_1-M_2)(t+\phi)\}\left(\sqrt{\frac{M_1!M_2!}{(M_1-\lambda_3)!\,(M_2-\lambda_3)!}}P_{M_1-\lambda_3}(t)P_{M_2-\lambda_3}(t)\right. \\ &\quad\left. + \rho^{\lambda_3-\lambda_2}\sqrt{\frac{M_1!M_2!}{(M_1-\lambda_2)!\,(M_2-\lambda_2)!}}P_{M_1-\lambda_2}(t)P_{M_2-\lambda_2}(t) + \rho^{2\lambda_3}\sqrt{P_{M_1}(t)P_{M_2}(t)}\right). \end{aligned}$$

The Q_H is invariant under the transformation $\phi \to \phi + \frac{2\pi}{M_1-M_2}$, proving analytically its symmetry under transformations of the cyclic group $C_{|M_1-M_2|}$, again the result is independent of the number of particles.

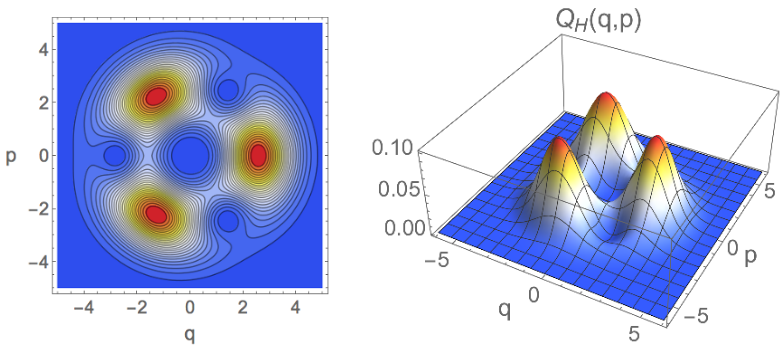

Fig. 1: Husimi function of the crystallized-type cat state with point symmetry C_3. A contour plot (left) and the corresponding 3-dimensional plot (right) are shown. We have taken $M_1 = 5$, $M_2 = 2$, $\theta = \frac{\pi}{4}$, $\mu_{12} = 1$, $\mu_{23} = \sqrt{2}$, and $q = \sqrt{2}\rho\cos\phi$ and $p = \sqrt{2}\rho\sin\phi$.

4 Conclusions

We have obtained analytic expressions for the one-particle case of the GTC model. The energy spectrum is an infinite ladder of three-level steps, each of them with a definite value for the total number of excitations M, together with the corresponding dressed states. A construction of crystallised-type Schrödinger states were exhibited for arbitrary dipolar strengths and whether or not the system is under resonant conditions with the field. The cyclic point symmetry group associated to the Husimi function of the electromagnetic field depends only on $|M_1 - M_2|$. It is important to mention that the results presented here can be extended to any number of particles, and to situations out of resonance. These extensions have been proved numerically by considering the evolution of a linear combination of states with M_1 and M_2 total number of excitations and by constructing the corresponding Husimi function [27]. Additionally, we conjecture that it can be generalised to n-level atoms.

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Appendix

The evolution operator associated to the Hamiltonian (1) can be obtained in analytic form for the different atomic configurations. It has the form $U(t) = e^{-iMt}\, U_I(t)$ with the last factor denoting the evolution operator in the interaction picture. The matrix elements needed in Eq. (7) are given by

$$U_{M_\Xi}(t)_{13} = -\sqrt{M_\Xi(M_\Xi - 1)}\, \mu_{12}\, \mu_{23}\, \frac{1 - \cos\Omega_\Xi t}{\Omega_\Xi^2}\,,$$

$$U_{M_\Xi}(t)_{23} = -i\sqrt{M_\Xi}\, \mu_{12}\, \frac{\sin\Omega_\Xi t}{\Omega_\Xi}\,, \quad U_{M_\Xi}(t)_{33} = \frac{(M_\Xi - 1)\,\mu_{23}^2 + M_\Xi\, \mu_{12}^2 \cos\Omega_\Xi t}{\Omega_\Xi^2}\,,$$

$$U_{M_V}(t)_{13} = \frac{-i\,\mu_{13}\, \sin\Omega_V t}{\sqrt{\mu_{12}^2 + \mu_{13}^2}}\,, \quad U_{M_V}(t)_{23} = \frac{-i\,\mu_{12}\, \sin\Omega_V t}{\sqrt{\mu_{12}^2 + \mu_{13}^2}}\,, \quad U_{M_V}(t)_{33} = \cos\Omega_V t\,.$$

$$U_{M_\Lambda}(t)_{13} = \frac{i\,\mu_{13}\, \sin\Omega_\Lambda t}{\sqrt{\mu_{13}^2 + \mu_{23}^2}}\,, \quad U_{M_\Lambda}(t)_{23} = \frac{-\mu_{13}\,\mu_{23}\,(1 - \cos\Omega_\Lambda t)}{\mu_{13}^2 + \mu_{23}^2}\,,$$

$$U_{M_\Lambda}(t)_{33} = \frac{\mu_{23}^2 + \mu_{13}^2 \cos\Omega_\Lambda t}{\mu_{13}^2 + \mu_{23}^2}\,.$$

Note that for the V- and Λ-configurations the dependence in the total number of excitations appears only in the argument of the trigonometric functions.

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Immanants of unitary matrices and their submatrices

Dylan Spivak and Hubert de Guise

Abstract Motivated by recent experiments, we discuss the connection between immanants of an arbitrary $m \times m$ unitary matrix U and group functions D of U. This connection also applies to submatrices of U and can be expanded with modifications to cases where U carries a representation of $SU(m)$ that is not the defining representation. Early results on the connections to twisted immanants are also included.

1 Introduction and motivation

In this paper we discuss the connection between immanants of matrices and submatrices and group functions D that occur in the representation theory of the unitary groups. Our work is motivated by recent experiments where controllable distinguishability of pulses was shown to be related to permutation properties of these pulses, and through Schur-Weyl duality to immanants of submatrices of the scattering matrix describing the interferometer in which the pulses propagate. We extend previously published work [1] to observations where multiple pulses can enter input channels of the interferometer and show that immanants of some specific non-unitary matrices are nevertheless connected with unitary group functions. Finally, we include a short discussion of twisted immanants and their connections to unitary group functions.

Littlewood [2] has defined the immanant using characters of an irreducible representation (irrep) $\{\lambda\}$ of the permutation group. For a 3×3 matrix, the relevant permutation group is S_3 and an immanant is given by

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$$\mathrm{Imm}^{\{\lambda\}}(U) = \sum_\sigma \chi_\sigma^{\{\lambda\}} U_{1\sigma(1)} U_{2\sigma(2)} U_{3\sigma(3)} = \sum_\sigma \chi_\sigma^{\{\lambda\}} P(\sigma) \left[U_{11}U_{22}U_{33}\right] . \quad (1)$$

conjugacy class	$\{1,1,1\}$	$\{2,1\}$	$\{3\}$
$\ydiagram{3}$	1	1	1
$\ydiagram{2,1}$	2	0	−1
$\ydiagram{1,1,1}$	1	−1	1

Table 1: The character table of S_3.

From the character table above and Eq.(1) one rapidly finds that in addition to the fully antisymmetric determinant $\mathrm{Det}(U) := \mathrm{Imm}^{\ydiagram{1,1,1}}(U)$, we also have a *permanent* $\mathrm{Per}(U) = \mathrm{Imm}^{\ydiagram{3}}(U)$ given by $\sum_{\sigma\in S_3} U_{1\sigma(1)}U_{2\sigma(2)}U_{3\sigma(3)}$ and a generic immanant $\mathrm{Imm}^{\ydiagram{2,1}}(U) = 2U_{11}U_{22}U_{33} - U_{12}U_{23}U_{31} - U_{13}U_{21}U_{32}$. Unlike the permanent or the determinant, which are associated with 1-dimensional irreps of the permutation group, $\mathrm{Imm}^{\ydiagram{2,1}}(U)$ does not transform into a multiple of itself under permutations of row or columns.

If the matrix U is the fundamental representation of a group, say the irrep $(1,0)$ of $SU(3)$ with Young diagram $\square$ for example, U_{ij} is the group function

$$U_{ij} = \langle i|U|j\rangle := D^{(1,0)}_{ij}(U) = D^{\square}_{ij}(U) \quad (2)$$

where $\{|j\rangle, j = 1,2,3\}$ is a basis for this irrep. It is appropriate at this point to introduce basis states for a general $SU(3)$ irrep labelled by the non-negative integers (p,q): $|(p,q)\nu_1\nu_2\nu_3;I_{23}\rangle$. These states can be conveniently realized as harmonic oscillator states [3]. The weight of the state $|(p,q)\nu_1\nu_2\nu_3;I_{23}\rangle$ is $[\nu_1-\nu_2,\nu_2-\nu_3]$ and I_{23} labels states with the same weight but transforming differently under the $SU(2) \subset SU(3)$ subgroup which mixes ν_2 and ν_3.

In this notation an immanant is then a sum of products of $D^{(1,0)}$ functions:

$$\begin{aligned}\mathrm{Imm}^{\ydiagram{2,1}}(U) &= 2D^{\square}_{11}(U)D^{\square}_{22}(U)D^{\square}_{33}(U)\\ &\quad -D^{\square}_{12}(U)D^{\square}_{23}(U)D^{\square}_{31}(U) - D^{\square}_{13}(U)D^{\square}_{21}(U)D^{\square}_{32}(U)\,, \end{aligned} \quad (3)$$

which can be rewritten in a D-function-like notation

$$\begin{aligned}\mathrm{Imm}^{\{\lambda\}}(U) &= {}_1\langle(1,0)100| \otimes {}_2\langle(1,0)100| \otimes {}_3\langle(1,0)100|\,U\\ &\times \left[\sum_\sigma \chi_\sigma^{\{\lambda\}} P_\sigma\right] |(1,0)100\rangle_1 \otimes |(1,0)010\rangle_2 \otimes |(1,0)001\rangle_3 \end{aligned} \quad (4)$$

provided we supply a recipe for the action of the permutation group. Indeed there are two such actions: a right action

$$P_{123}|1\rangle_1|2\rangle_2|3\rangle_3 = |2\rangle_1|3\rangle_2|(1\rangle_3\,, \quad (5)$$

and a left action (by the inverse element) $\bar{P}_{132}|1\rangle_1|2\rangle_2|3\rangle_3 = |1\rangle_3|2\rangle_1|3\rangle_2$.

Products like $D^{\yng(1)}_{11}(U)D^{\yng(1)}_{22}(U)D^{\yng(1)}_{33}(U)$ can be expanded in a sum of group functions in the decompositon $(1,0)\otimes(1,0)\otimes(1,0) = (3,0)\oplus 2(1,1)\oplus(0,0)$, but the operator $\hat{\Pi}^{\{\lambda\}} := \sum_\sigma \chi^{\{\lambda\}}_\sigma P_\sigma$ is nothing but a projection on the irrep $\{\lambda\} = \{\lambda_1,\lambda_2,\lambda_3\}$ of S_3 corresponding to irrep $(\lambda) := (\lambda_1-\lambda_2,\lambda_2-\lambda_3)$. In view of this one can write for instance the general expression

$$\begin{aligned}
\mathrm{Imm}^{\yng(2,1)}(U) &= \alpha_{11}D^{\yng(2,1)}_{(111)1;(111)1}(U) + \alpha_{10}D^{\yng(2,1)}_{(111)1;(111)0}(U) \\
&\quad + \alpha_{01}D^{\yng(2,1)}_{(111)0;(111)1}(U) + \alpha_{00}D^{\yng(2,1)}_{(111)0;(111)0}(U)\,, \qquad (6)\\
D^{\yng(2,1)}_{(111)I_{23};(111)I'_{23}}(U) &:= D^{(1,1)}_{(111)I_{23};(111)I'_{23}}(U) = \langle(1,1)111;I_{23}|U|(1,1)111;I'_{23}\rangle.
\end{aligned}$$

The general form of Eq.(6) is correct. Indeed, from a corollary of a theorem due to Kostant [4], we find: $\alpha_{11} = \alpha_{00} = 1$ and $\alpha_{10} = \alpha_{01} = 0$. We can extend this result to states with non-zero weights by looking at a 3×3 submatrix

$$\bar{U} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & U_{22} & U_{23} & U_{24} \\ \cdot & U_{32} & U_{33} & U_{34} \\ \cdot & U_{42} & U_{43} & U_{44} \end{pmatrix}, \qquad (7)$$

obtained from $\bar{U}$ by removing the first line and first row, to find a form very similar to the same result for the 3×3 matrix [5]:

$$\mathrm{Imm}^{\yng(2,1)}(\bar{U}) = D^{\yng(2,1)}_{(0111)1;(0111)1}(\bar{U}) + D^{\yng(2,1)}_{(0111)0;(0111)0}(\bar{U})\,. \qquad (8)$$

Symmetry restricts the terms in Eq.(6) as $\mathrm{Imm}^{\{\lambda\}}$ is invariant under conjugation by permutations: $\mathrm{Imm}^{\{\lambda\}} = P_\tau^{-1}\mathrm{Imm}^{\{\lambda\}}P_\tau$. We note that states $|(\lambda)\nu;I=0\rangle$ are $I=0$ singlets so antisymmetric w/r to P_{23}, while the states $|(\lambda)\nu;I=1\rangle$ are in $I=1$ triplets so symmetric w/r to P_{23}. Thus for instance $\langle(\lambda)\nu';1|P_{23}^{-1}U P_{23}|(\lambda)\nu;0\rangle = -D^{\{\lambda\}}_{(\nu')1;(\nu)0}(U)$, from which $\alpha_{01} = \alpha_{10} = 0$ follows.

When the submatrix is not principal diagonal, the proof of [5] or the previous line of argument does not apply but we nevertheless find a similar result:

$$\begin{aligned}
\mathrm{Imm}^{\yng(2,1)}(U_{234;134}) &= D^{\yng(2,1)}_{(0111)1;(1011)1/2}(U) + D^{\yng(2,1)}_{(0111)0;(1011)1/2}(U)\,,\\
&= \mathrm{Imm}^{\yng(2,1)}(U_{234;P_{12}(234)})\,. \qquad (9)
\end{aligned}$$

It is conjectured in [5] that the result on submatrices holds for generic submatrices with suitable minor changes.

Finally, suppose the matrix U is *not* the fundamental representation. For instance, consider the 4×4 irrep of $SU(2)$ $(J=3/2)$. The irrep $\{2,2\}$ of S_4 has dim=2 and can be expanded in terms of $SU(2)$ Wigner $D^J_{mm'}$ functions. One can verify that,

$$\mathrm{Imm}^{\yng(2,2)}(U) = \tfrac{26}{35}D^4_{00}(U) + \tfrac{6}{7}D^2_{00}(U) + \tfrac{2}{5}D^0_{00}(U)\,. \qquad (10)$$

The values $J=4,2,0$ are those that occur in the (outer) plethysm $(3/2)\otimes_{\wp}\{2,2\}$. The sum $\frac{26}{35}+\frac{6}{7}+\frac{2}{5}=2=\dim(\yng(2,2))$. More complicated cases, i.e., immanants of the 6-dimensional irrep $(2,0)$ of $SU(3)$ also have similar expressions [5]. We have no good way of analytically obtaining the coefficients in the expansion of the D's.

2 Application to interferometry

Imagine a scenario in which a 2-channel linear interferometer is injected with simultaneous three photon pulses in one port and a pair of simultaneous pulses in the other port. The relative delay between the pulses entering different input ports can be adjusted by an experimentalist. What is the rate $P(\tau)$ as a function of the relative delay τ at which triples of photon pulses come out at one of the two output ports and pairs of pulses at the other?

The rate $P(\tau)$ can be expressed [1] in a scalar product-like form $P(\tau)=\mathbf{v}^{\dagger}\cdot R(\tau)\cdot\mathbf{v}$. The vector $\mathbf{v}$ is a polynomial in the entries $U_{ij}, i,j=1,2$ of the 2×2 unitary matrix describing the scattering of individual pulses.

We approach this problem by first considering the action of S_5 on the permutations of $(1,1,1,2,2)$. There are 10 possible permutations (or words), one such word is $(1,2,1,1,2)$. If the final word is (a,b,c,d,e), we identify with it the polynomial $U_{1a}U_{1b}U_{1c}U_{2d}U_{2e}$ as one entry in the vector $\mathbf{v}$. As there are 10 distinct words to be constructed from $(1,1,1,2,2)$, we obtain a 10×10 rate matrix $R(\tau)$ which is reducible under S_5 as the partitions of 5 with at most 2 parts: $\yng(5)\oplus\yng(4,1)\oplus\yng(3,2)$. One can obtain the rates and the rate matrix starting from the the 10×10 matrix $\bar{U}$ constructed by repeating the 2×2 scattering matrix U:

$$\bar{U}=\begin{pmatrix} U & U & U & U & U \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U & U & U & U & U \end{pmatrix}. \tag{11}$$

The matrix $\bar{U}$ is neither unitary nor invertible; nevertheless $\mathrm{Per}(\bar{U})=\mathrm{Imm}^{\yng(5)}(\bar{U})=14400D^{5}_{00}(U)$ is related to an $SU(2)$ D-function for $J=5$; the coincidence rate $P(0)$ when all bosons are indistinguishable is proportional to the modulus square of this permanent.

3 Fermionic version

Suppose now we consider the interference of fermions in the simple case where three fermions enter an interferometer by different channels and output also by different channels. In the matrix $R(\tau)$ we insert a $-$ sign in row $U_{1\sigma(1)}U_{2\sigma(2)}U_{3\sigma(3)}$ and column $U_{1\sigma'(1)}U_{2\sigma'(2)}U_{3\sigma'(3)}$ if $\sigma\cdot\sigma'$ is a product of an odd number of transpositions. The final rate is an expression of the form

$$\begin{aligned} R(\tau) = & \mathbb{1}_{6\times 6} - \rho_{12}e^{-(\tau_1-\tau_2)^2} - \rho_{13}e^{-(\tau_1-\tau_3)^2} - \rho_{23}e^{-(\tau_2-\tau_3)^2} \\ & + (\rho_{123}+\rho_{132})e^{-\frac{1}{2}(\tau_1-\tau_2)^2-\frac{1}{2}(\tau_1-\tau_3)^2-\frac{1}{2}(\tau_2-\tau_3)^2}, \end{aligned} \tag{12}$$

where ρ_{ij} and ρ_{ijk} are 6×6 matrices that carry the regular representation of S_3. For bosons, $\rho_{12} = \rho_{13} = \rho_{23} = +1$ but the anticommutative nature of fermion leads to a sign that is the character of σ in alternating representation $\Gamma^{\yng(1,1,1)}$.

The regular representation of S_3 decomposes into $\Gamma^{\text{reg.}} = \Gamma^{\yng(3)} \oplus 2\Gamma^{\yng(2,1)} \oplus \Gamma^{\yng(1,1,1)}$. The effect of anticommutativity, encoded in $\Gamma^{\yng(1,1,1)}$, is to transform every irrep into its conjugate since $\yng(3)\otimes\yng(1,1,1) = \yng(1,1,1)$, $\yng(2,1)\otimes\yng(1,1,1) = \yng(2,1)$, $\yng(1,1,1)\otimes\yng(1,1,1) = \yng(3)$. As a result, what corresponds to a permanent for bosons now corresponds to a determinant for fermions etc., and some features of the landscapes are reversed, as illustrated in Fig.(1):

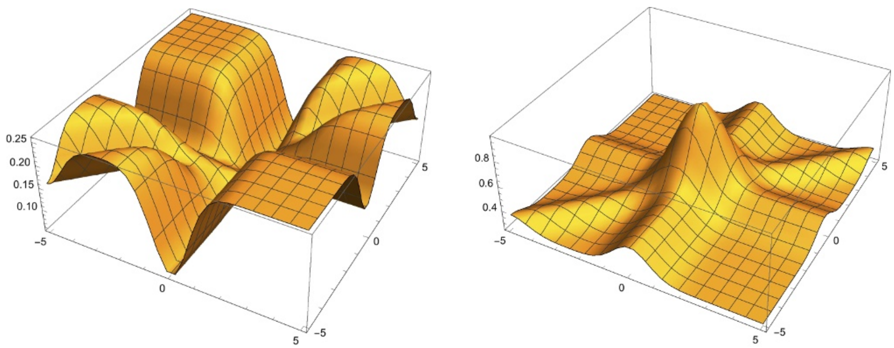

Fig. 1: Comparing a bosonic coincidence landscape (left) with a fermionic coincidence landscape (right) for the same 3×3 scattering matrix. The two axes of the landscape correspond to variable relative delays (τ_1, τ_3) between the first and second, and second and third pulses respectively.

Itoh [6] introduced the concept of *twisted immanant* for self-conjugate irreps. For instance, we have for $\yng(2,1)$:

$$\begin{aligned} \mathrm{Imm}^{\star\yng(2,1)}(U) &= i\sqrt{3}U_{12}U_{23}U_{31} - i\sqrt{3}U_{13}U_{21}U_{32}, \\ &= i\sqrt{3}\left([\text{even 3-cycle}] - [\text{odd 3-cycle}]\right). \end{aligned} \tag{13}$$

(Here, even and odd refer to the inversion number of the cycle.)

The self-conjugate irreps of S_n split in two irreps under the alternating subgroup $A_n \subset S_n$. More generally $\mathrm{Imm}^{\star\{\lambda\}}(U)$ picks up a sign under conjugation by odd elements:

$$P_{\sigma^{-1}}\,\mathrm{Imm}^{\star\{\lambda\}}(U)P_\sigma = \begin{cases} +\mathrm{Imm}^{\star\{\lambda\}}(U) & \text{if } \sigma \in A_n, \\ -\mathrm{Imm}^{\star\{\lambda\}}(U) & \text{if } \sigma \notin A_n. \end{cases} \tag{14}$$

One can express twisted immanants in terms of $SU(3)$ D-functions:

$$\mathrm{Imm}^{\star\,\yng(2,1)}(U) = -i\left(D^{\yng(2,1)}_{(111)0;(111)1}(U) - D^{\yng(2,1)}_{(111)1;(111)0}(U)\right). \tag{15}$$

A similar expression holds in $SU(4)$: $\mathrm{Imm}^{\star\,\yng(2,2)}(U) \propto D^{\yng(2,2)}_{(1111)1;(1111)0} - D^{\yng(2,2)}_{(1111)0;(1111)1}$.

The antisymmetry of Eq.(14) implies that the twisted immanants will have a D-function expansion where the input and output states are conjugate. We can visualize this by constructing states that are labelled by S_k states, and by duality, associate them with $SU(k)$ states. For the $SU(5)$ states in the self-conjugate irrep $\yng(3,1,1)$, we have

$$\yng(3,1,1) = \begin{cases} \downarrow \yng(3,1) \begin{cases} \downarrow \yng(3) \quad \downarrow \yng(2) & := |1\rangle \\ \downarrow \yng(2,1) \begin{cases} \downarrow \yng(2) & := |2\rangle \\ \downarrow \yng(1,1) & := |3\rangle \end{cases} \end{cases} \\ \downarrow \yng(2,1,1) \begin{cases} \downarrow \yng(2,1) \begin{cases} \downarrow \yng(2) & := |4\rangle \\ \downarrow \yng(1,1) & := |5\rangle \end{cases} \\ \downarrow \yng(1,1,1) \quad \downarrow \yng(1,1) & := |6\rangle \end{cases} \end{cases}$$

The requirements of antisymmetry under transposition now dictate that we consider

$$\mathrm{Imm}^{\star\,\yng(3,1,1)} = \alpha_{16} D^{\yng(3,1,1)}_{1;6} + \alpha_{61} D^{\yng(3,1,1)}_{6;1} + \alpha_{25} D^{\yng(3,1,1)}_{2;5} + \alpha_{52} D^{\yng(3,1,1)}_{5;2} + \alpha_{34} D^{\yng(3,1,1)}_{3;4} + \alpha_{43} D^{\yng(3,1,1)}_{4;3}. \tag{16}$$

Indeed we find $\alpha_{16} = \alpha_{61} = \alpha_{25} = \alpha_{52} = \alpha_{34} = \alpha_{43} = 1$.

What of rates in fermion interferometry? The boson rates are proportional to modulus squared of *linear combinations* of immanants. The fermion rates are proportional to modulus squared of *linear combinations* of twisted immanants when self-conjugate irreps occur. The linear combos of twisted immanants are also linear combos of regular immanants. As a result, there seems to be nothing fundamentally new in the fermionic rates, unless we find a scheme where only one immanant (or twisted immanant) occurs in the expression of the rates.

4 Conclusion

We note the deep connection between group functions, immanants, twisted immanants and multiphoton interferometry. This appears to be a subset of relations within the Schur-Weyl duality for irreps of $U(m)$ and the permutation group S_n. This work was supported by NSERC of Canada, and by Lakehead University.

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Group theoretical aspects of $L^2(\mathbb{R}^+)$, $L^2(\mathbb{R}^2)$ and associated Laguerre polynomials

Enrico Celeghini and Mariano A. del Olmo

Abstract A ladder algebraic structure for $L^2(\mathbb{R}^+)$ which closes the Lie algebra $h(1) \oplus h(1)$, where $h(1)$ is the Heisenberg-Weyl algebra, is presented in terms of a basis of associated Laguerre polynomials. Using the Schwinger method, the quadratic generators that span the alternative Lie algebras $so(3)$, $so(2,1)$ and $so(3,2)$ are also constructed. These families of (pseudo) orthogonal algebras also allow us to obtain unitary irreducible representations in $L^2(\mathbb{R}^2)$ similar to those in spherical harmonics.

1 Introduction

The associated Laguerre polynomials (ALP) [1], $L_n^{(\alpha)}(x)$ ($x \in [0,\infty)$, $n = 0,1,2,\ldots$ and α real fixed parameter, continuous and > -1), are defined by the 2nd order differential equation (DE)

$$\left[x\frac{d^2}{dx^2} + (1+\alpha-x)\frac{d}{dx} + n\right] L_n^{(\alpha)}(x) = 0\,. \tag{1}$$

The ALPs reduce to the Laguerre polynomials for $\alpha = 0$. From the many recurrence relations that they verify [1, 1, 2], we start from the following:

$$\left[-\frac{d}{dx} + 1\right] L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x)\,, \qquad \left[x\frac{d}{dx} + \alpha\right] L_n^{(\alpha)}(x) = (n+\alpha)\, L_n^{(\alpha-1)}(x)\,. \tag{2}$$

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For $\alpha > -1$ and fixed, the ALP $L_n^{(\alpha)}(r)$ are orthogonal in the label n with respect to the weight measure $d\mu(x) = x^\alpha\, e^{-x}\, dx$

$$\int_0^\infty dx\, x^\alpha\, e^{-x}\, L_n^{(\alpha)}(x)\, L_{n'}^{(\alpha)}(x) \;=\; \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{nn'}\,.$$

For an integer α such that $0 \leq \alpha \leq n$, we have the generalization [1] $L_n^{(-\alpha)}(x) := \frac{\Gamma(n-\alpha+1)}{\Gamma(n+1)}\,(-x)^\alpha\, L_{n-\alpha}^{(\alpha)}(x)$. Hereafter we assume here $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, $n-\alpha \in \mathbb{N}$, and we consider α as a label, like n, and not a parameter fixed at the beginning.

Following the approach of previous works [4–7], we introduce now a set of alternative functions including also the weight measure in such a way as to obtain the orthonormal bases we are used to in quantum mechanics:

$$M_n^{(\alpha)}(x) := \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}\; x^{\alpha/2}\, e^{-x/2}\, L_n^{(\alpha)}(x)\,.$$

For each fixed value of $\alpha \geq -n$ and $n \in \mathbb{N}$, the set of $M_n^{(\alpha)}(x)$, is a basis of $L^2(\mathbb{R}^+)$

$$\int_0^\infty M_n^{(\alpha)}(x)\, M_m^{(\alpha)}(x)\, dx = \delta_{nm}\,, \qquad \sum_{n=0}^{\infty} M_n^{(\alpha)}(x)\, M_n^{(\alpha)}(x') = \delta(x-x')\,.$$

2 The symmetry algebra $h(1)_n \oplus h(1)_p$

The eqs. (2) rewritten in terms of $M_n^{(\alpha)}$ take the form

$$\begin{aligned}
\left[-\sqrt{x}\frac{d}{dx} + \frac{1}{2\sqrt{x}}(\alpha+x)\right] M_n^{(\alpha)}(x) \;&=\; \sqrt{n+\alpha+1}\, M_n^{(\alpha+1)}(x)\,,\\
\left[\sqrt{x}\frac{d}{dx} + \frac{1}{2\sqrt{x}}(\alpha+x)\right] M_n^{(\alpha)}(x) \;&=\; \sqrt{n+\alpha}\, M_n^{(\alpha-1)}(x)\,,
\end{aligned} \tag{3}$$

where $p := n+\alpha$ plays, for n fixed, the role of eigenvalue of the number operator in a Heisenberg-Weyl algebra, $h(1)$, realized on the space of functions $M_n^\alpha(x)$. It is indeed a positive integer like n, so that we can define the new functions $\mathscr{M}_{n,p}(x) := M_n^{(p-n)}(x)$, which by inspection are symmetric in the interchange $n \Leftrightarrow p$, i.e., $\mathscr{M}_{n,p}(x) = (-1)^{p-n}\,\mathscr{M}_{p,n}(x)$. The previous recurrence relations (3) can thus be rewritten as

$$\begin{aligned}
\left[-\sqrt{x}\frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p-n}{2\sqrt{x}}\right] \mathscr{M}_{n,p}(x) \;&=\; \sqrt{p+1}\,\mathscr{M}_{n,p+1}(x)\,,\\
\left[\sqrt{x}\frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p-n}{2\sqrt{x}}\right] \mathscr{M}_{n,p}(x) \;&=\; \sqrt{p}\,\mathscr{M}_{n,p-1}(x)\,.
\end{aligned} \tag{4}$$

To construct the operatorial structure corresponding to the recurrence relations we define now four operators X, D_x, N and P:

$$\begin{aligned} X\,\mathscr{M}_{n,p}(x) &= x\,\mathscr{M}_{n,p}(x)\,, & D_x\,\mathscr{M}_{n,p}(x) &= \tfrac{d\mathscr{M}_{n,p}(x)}{dx}\,, \\ N\,\mathscr{M}_{n,p}(x) &= n\,\mathscr{M}_{n,p}(x)\,, & P\,\mathscr{M}_{n,p}(x) &= p\,\mathscr{M}_{n,p}(x)\,. \end{aligned}$$

Then, the second order DE (1) becomes

$$\mathbb{E}\,\mathscr{M}_{n,p}(x) = 0\,, \tag{5}$$

where

$$\mathbb{E} := XD_x^2 + D_x + \frac{N+P+1}{2} - \frac{1}{4X}(P-N)^2 - \frac{X}{4}\,.$$

Moreover from (4) we get the differential operators (DOs),

$$\mathbf{b}^{\pm} := \mp\sqrt{X}D_x + \frac{\sqrt{X}}{2} + \frac{1}{2\sqrt{X}}(P-N)\,, \tag{6}$$

that act on the functions $\mathscr{M}_{n,p}(x)$ in such a way that $\Delta n = 0$ and $\Delta p = \pm 1$. Since $[\mathbf{b}^-,\mathbf{b}^+] = \mathbb{I}$, they close an $h(1)$ algebra, $(h(1)_p)$ with quadratic Casimir $\mathscr{C}_p = \{\mathbf{b}^-,\mathbf{b}^+\} - 2(P+1/2)$ verifying $\mathscr{C}_p\,\mathscr{M}_{n,p}(x) = -2\,\mathbb{E}\,\mathscr{M}_{n,p}(x) = 0$.

Now taking into account the symmetry under the interchange $n \Leftrightarrow p$ of $\mathscr{M}_{n,p}(x)$, we can define the operators $\mathbf{a}^{\pm}(N,P) := -\mathbf{b}^{\pm}(P,N)$ that change the labels of $\mathscr{M}_{n,p}(x)$ as $\Delta p = 0$ and $\Delta n = \pm 1$. Their explicit action on $\mathscr{M}_{n,p}(x)$ is indeed

$$\mathbf{a}^+\,\mathscr{M}_{n,p}(x) = \sqrt{n+1}\,\mathscr{M}_{n+1,p}(x)\,, \qquad \mathbf{a}^-\,\mathscr{M}_{n,p}(x) = \sqrt{n}\,\mathscr{M}_{n-1,p}(x)\,.$$

The two operators $\mathbf{a}^{\pm}$ determine thus another HW algebra, $h(1)_n$. Since these bosonic operators $\mathbf{a}^{\pm}$ and $\mathbf{b}^{\pm}$ commute we have obtained in this way the global algebra $h(1)_n \oplus h(1)_p$.

Moreover inside the Universal Enveloping Algebra $UEA\,[h(1)_n \oplus h(1)_p]$ other algebras preserving the parity of $n+p$ can be found by the Schwinger procedure [8] as we will do in the next section.

3 $so(3)$, $so(2,1)$ and $so(3,2)$ symmetries

$so(3)$ symmetry

We start from $J_{\pm} := \mathbf{a}_{\pm}\,\mathbf{b}_{\mp}$, obtaining second order DOs which, taking into account eq. (5), can be rewritten in the space $\{\mathscr{M}_{n,p}(x)\}$ as first order DOs:

$$J_{\pm} = \mp D_x\,(N-P\pm 1) + \frac{1}{2X}\,(N-P\pm 1)(N-P) - \frac{1}{2}(N+P+1)\,. \tag{7}$$

Defining $J_3 := (\mathbf{a}_-\,\mathbf{a}_+ - \mathbf{b}_-\,\mathbf{b}_+)/2 \,\equiv\, (N-P)/2$ we see that $\{J_\pm, J_3\}$ closes a $su(2)$ algebra in the space $\{\mathcal{M}_{n,p}(x)\}$ since $[J_+,J_-] = 2J_3 - \frac{8}{X}\,J_3\,\mathbb{E}$. The action of $J_\pm$ is

$$J_+\,\mathcal{M}_{n,p}(x) = \sqrt{(n+1)\,p}\,\mathcal{M}_{n+1,p-1}(x)\,, \quad J_-\,\mathcal{M}_{n,p}(x) = \sqrt{n\,(p+1)}\,\mathcal{M}_{n-1,p+1}(x)\,.$$

Also the Casimir of $su(2)$, $\mathcal{C}_{su(2)} = J_3^2 + \frac{1}{2}\{J_+,J_-\}$ is closely related to eq. (5) as $\mathcal{C}_{su(2)} = J(J+1) + \frac{1}{X}\,(4J_3^2+1)\,\mathbb{E}$, where J is the diagonal operator $J := (N+P)/2$.

$so(2,1)$ symmetry

In a similar way we can define the operators $K_\pm := \mathbf{a}_\pm\,\mathbf{b}_\pm$, such as in the case of the operators $J_\pm$, we find in the space $\{\mathcal{M}_{n,p}(x)\}$

$$K_+ = X\,D_x + \frac{1}{2}(N+P+2-X)\,, \qquad K_- = -X\,D_x + \frac{1}{2}(N+P-X)\,. \tag{8}$$

Both operators together with $K_3 := (\mathbf{a}_-\,\mathbf{a}_+ + \mathbf{b}_+\,\mathbf{b}_-)/2 \,\equiv\, (N+P+1)/2$ determine a $su(1,1)$ algebra

$$[K_3,K_\pm] = \pm K_\pm\,, \qquad [K_+,K_-] = -2K_3\,,$$

since the action on the functions $\mathcal{M}_{n,p}(x)$ is

$$K_+\,\mathcal{M}_{n,p}(x) = \sqrt{(n+1)(p+1)}\,\mathcal{M}_{n+1,p+1}(x)\,, \quad K_-\,\mathcal{M}_{n,p}(x) = \sqrt{n\,p}\,\mathcal{M}_{n-1,p-1}(x)\,.$$

The Casimir of $su(1,1)$, $\mathcal{C}_{su(1,1)} = K_3^2 - \frac{1}{2}\{K_+,K_-\}$, is also connected with eq. (5) as $\mathcal{C}_{su(1,1)} = (M^2 - \frac{1}{4}) + X\,\mathbb{E}$, where $M = J_3 := (N-P)/2$.

More $so(2,1)$ symmetries

The commutators of $J_\pm$ and $K_\pm$ give the new operators

$$R_\pm := \pm[J_\pm,K_\pm]\,, \qquad S_\pm := \pm[J_\mp,K_\pm]\,.$$

Provided that we define $R_3 := J+M+1/2$ and $S_3 := J-M+1/2$, they close two $so(2,1)$ algebras with commutators

$$[R_+,R_-] = -4R_3\,, \qquad [R_3,R_\pm] = \pm 2R_\pm\,,$$

and Casimir $\mathcal{C}_R = R_3^2 - \frac{1}{2}\{R_+,R_-\} \;=\; -\frac{3}{4} + \frac{1}{X}\,(1+(X+2M)^2)\,\mathbb{E}$, similarly for $\{S_\pm,S_3\}$. Note that under the interchange $m \leftrightarrow -m$ we have $\{R_\pm,R_3\} \leftrightarrow \{S_\pm,S_3\}$.

$so(3,2)$ **symmetry**

All the operators $\{K_\pm, L_\pm, R_\pm, S_\pm, J, M\}$ can be written on the space $\{\mathscr{M}_{n,p}(x)\}$ as first order DOs. All together they determine on $\{\mathscr{M}_{n,p}(x)\}$ the representation of the Lie algebra $so(3,2)$ with $C_2^{so(3,2)} = -5/4$.

4 Representations of $so(3)$, $so(2,1)$ and $so(3,2)$ on the plane

We introduce now the operators directly related to $so(3)$, $J := (N+P)/2$ and $J_3 \equiv M := (N-P)/2$, and we define

$$\mathscr{L}_j^m(x) := \mathscr{M}_{j+m,j-m}(x) = \sqrt{\frac{(j+m)!}{(j-m)!}}\, x^{-m}\, e^{-x/2}\, L_{j+m}^{(-2m)}(x)\,.$$

The operators J_3 and $J_\pm$ (7), rewritten in terms of J and M, act on $\{\mathscr{L}_j^m(x)\}$ as

$$J_3\, \mathscr{L}_j^m(x) = m\, \mathscr{L}_j^m(x)\,, \quad J_\pm\, \mathscr{L}_j^m(x) = \sqrt{(j\mp m)(j\pm m+1)}\, \mathscr{L}_j^{m\pm 1}(x)\,.$$

So, $\{\mathscr{L}_j^m(x)\}$ with $j \in \mathbb{N}$ and $|m| \leq j$ supports the representation $\mathscr{D}_j$ of $so(3)$.

Similar results can be obtained for the other algebras $so(2,1)$ and $so(3,2)$. For instance, for the $so(2,1)$ spanned by $\{K_\pm, K_3\}$, $\{\mathscr{L}_j^m(x)\}$ supports the irreducible representation of the discrete series with Casimir $\mathscr{C}_{su(1,1)} := m^2 - \frac{1}{4}$ with m fixed and $j \geq |m|$.

On the other hand, in general these representations are not faithful because $\mathscr{L}_j^m(x) = \mathscr{L}_j^{-m}(x)$. The same difficulty is also present in the spherical harmonic where the associated Legendre polynomial P_l^m is related to P_l^{-m}. There the degeneration was removed by introducing an angle variable. Here we follow the same procedure by considering the new functions,

$$\mathscr{Z}_j^m(r,\phi) := e^{im\phi}\, \mathscr{L}_j^m(r^2)\,, \qquad \phi \in \mathbb{R}\,, -\pi \leq \phi < \pi\,.$$

Under the change of variable $x \to r^2$ the DE (5) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4m^2}{r^2} - r^2 + 4(j+\frac{1}{2})\right]\, \mathscr{Z}_j^m(r,\phi) = 0\,.$$

Normalization and orthogonality of the $\mathscr{Z}_j^m(r,\phi)$ are similar to those of $Y_j^m(\theta,\phi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \int_0^\infty 2r\, dr\;\; \mathscr{Z}_j^m(r,\phi)^*\; \mathscr{Z}_{j'}^{m'}(r,\phi) \;=\; \delta_{j,j'}\;\; \delta_{m,m'}\,,$$

$$\sum_{j,m}\; \mathscr{Z}_j^m(r,\phi)^*\; \mathscr{Z}_j^m(r',\phi') \;=\; \frac{\pi}{r}\,\delta(r-r')\,\delta(\phi-\phi')\,.$$

This means that the set $\{\mathscr{Z}_j^m(r,\phi)\}$ is a basis in the space of square integrable functions defined on the plane $L^2(\mathbb{R}^2)$, as $\{Y_j^m(\Omega)\}$ is a basis of $L^2(\mathbb{S}^2)$.

Moreover, with a convenient introduction of phases we can define the operators $\mathbb{J}_\pm := e^{\pm i\phi} J_\pm$ and $\mathbb{J}_3 := J_3$, in the finite dimensional space $\{\mathscr{Z}_j^m(r,\phi)\}$ wih fixed j

$$\mathbb{J}_\pm \; \mathscr{Z}_j^m(r,\phi) = \sqrt{(j \mp m)(j \pm m+1)} \; \mathscr{Z}_j^{m\pm 1}(r,\phi)\,, \quad \mathbb{J}_3 \; \mathscr{Z}_j^m(r,\phi) = m\, \mathscr{Z}_j^m(r,\phi)\,,$$

and analogously for the remaining operators. So $\{\mathscr{Z}_j^m(r,\phi)\}$ support irreducible representations of $so(3)$, $so(2,1)$ and $so(3,2)$ on the plane as $\{Y_j^m(\theta,\phi)\}$ are on the sphere. For more details see [7, 9, 10].

From the physical point of view, in spite of the analogy with the angular momentum, $\mathbb{J}_\pm$ and $\mathbb{J}_3$ can be related to a one-dimensional Morse system, where m and j are connected with the potential [9].

Conclusions

A relationship between Lie algebras and square integrable functions has been found. Indeed we need to restrict ourselves to $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^2)$, where $\mathbb{E}$ is identically zero, to obtain differential representations of the Lie algebras in the spaces of functions defined in $\mathbb{R}^+$ and $\mathbb{R}^2$.

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Galilean complex Sine-Gordon equation: symmetries, soliton solutions and gauge coupling

Genilson de Melo, Marc de Montigny, James Pinfold, Jack Tuszynski

Abstract We use the Galilean covariance formalism to obtain the Galilean complex Sine-Gordon equation in $1+1$ dimensions, $\Psi_{xx}(1-\Psi^*\Psi)+2im\Psi_t+\Psi^*\Psi_x^2-\Psi(1-\Psi^*\Psi)^2=0$. We determine its Lie point symmetries, discuss some group-invariant solutions, and examine some soliton solutions. We also discuss the coupling of this field with Galilean electromagnetism. This work is motivated in part by recent applications of the relativistic complex Sine-Gordon equation to the dynamics of Q-balls.

1 Galilean covariance

The objectives of the presentation given at the Group-31 conference were to summarize our recent paper [5], in which a Galilean complex Sine-Gordon (GCSG) equation [7, 10] was formulated with Galilean covariance [12, 13], and to extend it by adding couplings of the GCSG field to the Galilean electromagnetic field investigated in Ref. [6, 8].

The Galilean covariant approach is based on Galilean 5-vectors, such as [12, 13]

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$$x^\mu = (x^1, x^2, x^3, x^4, x^5) = (\mathbf{r}, t, s)$$

which transforms under a Galilean boost as

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t, \\ t' &= t, \\ s' &= s - \mathbf{r}\cdot\mathbf{v} + \tfrac{1}{2}\mathbf{v}^2 t. \end{aligned}$$

This transformation leaves invariant $(\mathbf{r}, t, s)\cdot(\mathbf{r}', t', s') \equiv \mathbf{r}\cdot\mathbf{r}' - ts' - t's$, which suggests the introduction of a 'Galilean metric':

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

If, for instance, we consider a wave-function such that $\Psi(x) = e^{-\mathrm{i}ms}\psi(\mathbf{x}, t)$ so that $\partial_s \Psi = -\mathrm{i}m\Psi$, then the Klein-Gordon equation in 5D, $\partial_\mu \partial^\mu \Psi(x) = 0$ implies that $\nabla^2\Psi - 2\partial_t\partial_s\Psi = e^{-\mathrm{i}ms}(\nabla^2\psi + 2m\mathrm{i}\partial_t\psi) = 0$ is reduced to the free Schrödinger equation in 4D: $\mathrm{i}\partial_t\psi = -\frac{1}{2m}\nabla^2\psi$. Ref. [6] shows that this 5-dimensional approach allows one to obtain the Lévy-Leblond equation [9] which is a Galilean version of the Dirac equation that was discussed at Group-31 [2], as well as the equations of Galilean electromagnetism [8], to which we shall return in the last section.

2 Q-balls and the complex Sine-Gordon equation

The MoEDAL experiment will investigate highly-ionizing electrically charges particles, such as the multiparticle excitations called 'Q-balls' [1]. A Q-ball refers to a type of non-topological soliton, thus a stable localized field configuration that carries a conserved Noether charge [4]. Bowcock et al performed an analytical study of the dynamics and interactions of relativistic Q-balls in $1+1$ dimensions [3]. They described the interactions and perturbations of Q-balls in non-integrable theories by using an integrable model: the (relativistic) complex Sine-Gordon equation [7, 10]: $\mathscr{L} = \frac{1}{1-|\Psi|^2}\partial_\mu\Psi^*\partial^\mu\Psi - U\left(|\Psi|\right)$. Hereafter we consider

$$\mathscr{L} = \frac{\partial_\mu\Psi^*\partial^\mu\Psi}{1-|\Psi|^2} + |\Psi|^2,$$

which leads to the equation of motion

$$(1-\Psi^*\Psi)\,\partial^\mu\partial_\mu\Psi + \Psi^*\partial_\mu\Psi\partial^\mu\Psi - \Psi\,(1-\Psi^*\Psi)^2 = 0.$$

With the Galilean covariance prescription, an equation in $1+1$ dimensions is obtained from a model formulated with the Galilean $2+1$ metric,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

which is then projected onto $1+1$ dimensions with the restriction:

$$\Psi(x) = e^{-\mathrm{i}ms}\psi(\mathbf{x},t) \rightarrow \partial_s\Psi = -\mathrm{i}m\Psi.$$

The equation of motion becomes

$$(\Psi_{xx} - 2\Psi_{st})(1-\Psi^*\Psi) + \Psi^*\left(\Psi_x^2 - 2\Psi_s\Psi_t\right) - \Psi(1-\Psi^*\Psi)^2 = 0,$$

which reduces to the central equation of this paper, the GCSG equation:

$$\psi_{xx}(1-\psi^*\psi) + 2\mathrm{i}m\psi_t + \psi^*\psi_x^2 - \psi(1-\psi^*\psi)^2 = 0. \qquad (1)$$

Hereafter, we will find the Lie point symmetries and determine one-soliton solutions of this equation.

If we express the function $\psi(x,t)$ as

$$\psi(x,t) = \rho(x,t)\exp(\mathrm{i}\phi(x,t)),$$

then the CGSG equation (1) leads to

$$\rho_{xx}\left(1-\rho^2\right) - \rho\phi_x^2 + \rho\rho_x^2 - 2m\rho\phi_t - \rho\left(1-\rho^2\right)^2 = 0 \qquad (2)$$

and

$$2\rho_x\phi_x + \rho\left(1-\rho^2\right)\phi_{xx} + 2m\rho_t = 0 \qquad (3)$$

We find that the Lie-point symmetries (see Ref. [11] for an introduction) generated by four vector fields: space-translations $\mathbf{v}_1 = \partial_x$, time-translation $\mathbf{v}_2 = \partial_t$, field-shift $\mathbf{v}_3 = \partial_\phi$ and a Galilean-like boost $\mathbf{v}_4 = mx\partial_\phi + t\partial_x$.

Let us consider the group-invariant solutions for the subgroup generated by $\partial_t + c\partial_x$, which admits the invariant $w = x - ct$. Then the equations (2) and (3) respectively become

$$\rho_{ww}\left(1-\rho^2\right) - \rho\phi_w^2 + \rho\rho_w^2 + 2mc\rho\phi_w - \rho\left(1-\rho^2\right)^2 = 0$$

and

$$2\rho_w\phi_w + \rho\left(1-\rho^2\right)\phi_{ww} - 2mc\rho_w = 0.$$

We impose an ansatz suggested by the two terms dependent of ϕ in the first equation:

$$-\phi_w^2 + 2mc\phi_w = \text{constant}, \qquad \phi_w = mc.$$

By substitution, multiplying by $\frac{\rho_w}{1-\rho^2}$ and integrating, we find that

$$\rho_w^2 = -\rho^4 + \rho^2 (1-k) + k - m^2 c^2. \tag{4}$$

The discriminant of the quartic polynomial in this equation is

$$(k+1)^2 - (2mc)^2 = (k+1+2mc)(k+1-2mc).$$

We find localized solutions by setting the discriminant of this equation equal to zero, so that $k = \pm 2mc - 1$. If we keep the $+$ sign, we find that

$$\rho_w^2 = -\rho^4 + 2\rho^2 (1-mc) + 2mc - 1 - m^2c^2 = -\left[\rho^2 + (mc-1)^2\right]^2,$$

which leads to $\frac{d\rho}{dw} = \pm \mathrm{i}\left(\rho^2 + \lambda^2\right)$ with $\lambda^2 = mc - 1$. If we restrict ourselves again to the $+$ sign, we find that $\rho(w) = \mathrm{i}\sqrt{mc-1}\ \tanh\left[\sqrt{mc-1}(w-w_0)\right]$, so that the probability density is

$$|\Psi(x,t)|^2 = (mc-1)\tanh^2\left[\sqrt{mc-1}(x-ct-w_0)\right].$$

This shows that one must have $mc > 1$, otherwise the density function contains infinite singularities; i.e. the speed of propagation admits a minimal value $c > \frac{1}{m}$.

Eq. (4) also leads to a soliton solution if we choose $k = m^2c^2$. Then Eq. (4) reduces to $\rho_w^2 = \rho^2\left(1-\rho^2-m^2c^2\right)$ with roots $\rho = 0$ and $\rho^2 = 1-m^2c^2$, which leads to $w - w_0 = \int \frac{d\rho}{\rho\sqrt{1-\rho^2-m^2c^2}}$. If we define $\rho = a\cos\theta$,where $a^2 = 1-m^2c^2$, then this integral becomes $\int \frac{d\theta}{\cos\theta} = 2\tanh^{-1}\left(\tan\frac{\theta}{2}\right)$. From the trig identity $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \sqrt{\frac{a-\rho}{a+\rho}}$, we find that $\tanh^2\left[\frac{a}{2}(w_0-w)\right] = \frac{a-\rho}{a+\rho}$. If we solve for ρ, we obtain

$$\rho(x,t) = \sqrt{1-m^2c^2}\ \frac{1-\tanh^2\left[\frac{1}{2}\sqrt{1-m^2c^2}(w_0-x+ct)\right]}{1+\tanh^2\left[\frac{1}{2}\sqrt{1-m^2c^2}(w_0-x+ct)\right]},$$

which can be simplified as

$$\rho(x,t) = \sqrt{1-m^2c^2}\,\mathrm{sech}(\sqrt{1-m^2c^2}(x-ct)).$$

Then the complete solution is of soliton-type,

$$\Psi(x,t) = \sqrt{1-m^2c^2}\,\mathrm{sech}(\sqrt{1-m^2c^2}(x-ct))e^{imc(x-ct)},$$

and the density of probability is given by

$$|\Psi(x,t)|^2 = (1-m^2c^2)\,\mathrm{sech}^2(\sqrt{1-m^2c^2}(x-ct)), \tag{5}$$

which is shown in Fig. 2 for $0 \le t \le 10$, with parameters $m = 1$ and $c = 0.5$.

3 Coupling with the Galilean electromagnetic field

The content of this section was not in Ref. [5]. We couple the field ψ to a Galilean electromagnetic field via the covariant derivative $\partial_\mu \to \mathscr{D}_\mu \equiv \partial_\mu + \mathrm{i}qA_\mu$. As discussed in Ref. [6], we must consider two Galilean limits, called the 'electric limit' and the 'magnetic limit'. The Lagrangian becomes

$$\begin{aligned}\mathscr{L} &= \frac{\mathscr{D}_\mu \Psi^* \mathscr{D}^\mu \Psi}{1-\Psi^*\Psi} + \Psi^*\Psi + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{1-\Psi^*\Psi}\left(\partial_\mu \Psi^* - \mathrm{i}qA_\mu \Psi^*\right)\left(\partial^\mu \Psi + \mathrm{i}qA^\mu \Psi\right) + \Psi^*\Psi + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$

The Euler-Lagrange equation with respect to Ψ^* leads to

$$\begin{aligned}(1-\Psi^*\Psi)\,\partial^\mu\partial_\mu\Psi + \Psi^*\partial_\mu\Psi\partial^\mu\Psi - \Psi\,(1-\Psi^*\Psi)^2 + 2\mathrm{i}qA^\mu\partial_\mu\Psi \\ -q^2A_\mu A^\mu\Psi + \mathrm{i}q\left(\partial_\mu A^\mu\right)\Psi - \mathrm{i}q\left(\partial_\mu A^\mu\right)\Psi^2\Psi^* = 0,\end{aligned}$$

and with respect to A_μ, we obtain

$$\partial_\nu\partial^\nu A_\mu - \partial_\mu\partial_\nu A^\nu - \mathrm{i}q\left(\Psi^*\partial_\mu\Psi - \Psi\partial_\mu\Psi^*\right) + 2q^2A_\mu\Psi^*\Psi = 0.$$

The two Galilean limits are found by setting $A_t = -\phi_m$ and $A_s = -\phi_e$ alternately equal to zero [6]. As for the Sine-Gordon field, we use $\partial_s\Psi = -\mathrm{i}m\Psi$, as before.

Magnetic limit: We obtain this limit by setting $\phi_e = 0$ so that ϕ_m is the scalar potential. Then the equation for ψ reads

$$\begin{aligned}(1-\psi^*\psi)\left(\nabla^2\psi + 2m\mathrm{i}\partial_t\psi\right) + \psi^*\left(\nabla\psi\cdot\nabla\psi + 2m\mathrm{i}\psi\partial_t\psi\right) - \psi\,(1-\psi^*\psi)^2 \\ +2\mathrm{i}q\mathbf{A}\cdot\nabla\psi + 2mq\phi_m\psi - q^2\mathbf{A}\cdot\mathbf{A}\psi + \mathrm{i}q\psi\,(\nabla\cdot\mathbf{A} + \partial_s\phi_m)\,(1-\psi^*\psi) = 0.\end{aligned}$$

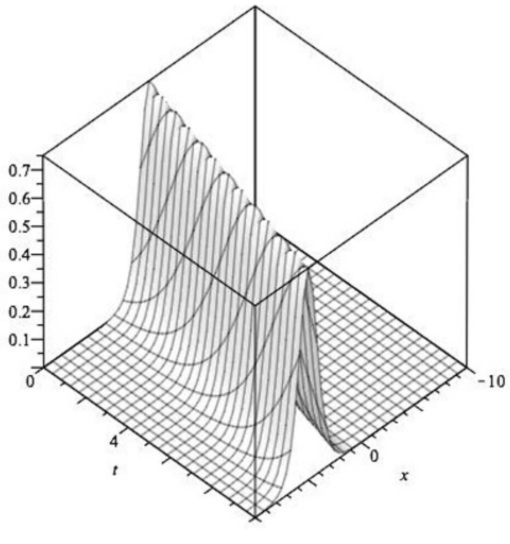

Fig. 1 Graph of $|\Psi(x,t)|^2$ in Eq. (5) with $m = 1$ and $c = 0.5$, for $0 \le t \le 10$.

The equations of motion for the gauge field read

$$\begin{aligned}
&\nabla^2\mathbf{A} - 2\partial_s\partial_t\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \partial_s\phi_m\right) + \mathrm{i}q\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - 2q^2\mathbf{A}\psi^*\psi = 0,\\
&2\partial_s\partial_t\phi_m - \nabla^2\phi_m - \partial_t\left(\nabla\cdot\mathbf{A} + \partial_s\phi_m\right) + \mathrm{i}q\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) + 2q^2\phi_m\psi^*\psi = 0,\\
&\partial_s\nabla\cdot\mathbf{A} + \partial_{ss}\phi_m - 2mq\psi^*\psi = 0.
\end{aligned}$$

Electric limit This limit corresponds to $\phi_m = 0$ and ϕ_e becomes the scalar potential. The equation for ψ reads

$$\begin{aligned}
&(1-\psi^*\psi)\left(\nabla^2\psi + 2m\mathrm{i}\partial_t\psi\right) + \psi^*\left(\nabla\psi\cdot\nabla\psi + 2m\mathrm{i}\psi\partial_t\psi\right)\psi(1-\psi^*\psi)^2\\
&+2\mathrm{i}q\mathbf{A}\cdot\nabla\psi + 2\mathrm{i}q\phi_e\partial_t\psi - q^2\mathbf{A}\cdot\mathbf{A}\psi + \mathrm{i}q\psi\left(\nabla\cdot\mathbf{A} + \partial_t\phi_e\right)(1-\psi^*\psi) = 0.
\end{aligned}$$

The equations for the gauge field are

$$\begin{aligned}
&\nabla^2\mathbf{A} - 2\partial_s\partial_t\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \partial_t\phi_e\right) + \mathrm{i}q\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - 2q^2\mathbf{A}\psi^*\psi = 0,\\
&-\partial_t\left(\nabla\cdot\mathbf{A} + \partial_t\phi_e\right) + \mathrm{i}q\left(\psi^*\partial_t\psi - (\partial_t\psi^*)\psi\right) = 0,\\
&2\partial_s\partial_t\phi_e - \nabla^2\phi_e - \partial_s\left(\nabla\cdot\mathbf{A} + \partial_t\phi_e\right) + 2mq\psi^*\psi + 2q^2\phi_e\psi^*\psi = 0.
\end{aligned}$$

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On completeness of coherent states in noncommutative spaces with the generalised uncertainty principle

Sanjib Dey

Abstract Coherent states are required to form a complete set of vectors in the Hilbert space by providing the resolution of identity. We study the completeness of coherent states for two different models in a noncommutative space associated with the generalised uncertainty relation by finding the resolution of unity with a positive definite weight function. The weight function, which is sometimes known as the Borel measure, is obtained through explicit analytic solutions of the Stieltjes and Hausdorff moment problem with the help of the standard techniques of the inverse Mellin transform.

1 Introduction

It is well known that the coherent states are useful in different areas of modern science including quantum optics, atomic and molecular physics, mathematical physics, quantum gravity, quantum cosmology, etc, for further informations; see, for instance [1, 2]. Various generalisations of the Glauber coherent states have also become very popular in recent days giving rise to the possibility of constructing many new coherent states arising from various sophisticated mathematical backgrounds [3–7]. One of such prominent examples is the noncommutative space-time structure in the framework of generalised uncertainty principle, from which the existence of minimal length appears naturally [8–11]. There have been plenty of investigations behind the applications and usefulness of coherent states emerging out of the models on the noncommutative space [12–14]. Furthermore, based on these coherent states, various nonclassical states have been constructed; such as, squeezed states [15], Schrödinger cat states [16, 17], photon added coherent states [18] and their squeezing and entanglement properties have been studied. However, the math-

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ematical completeness of such coherent states have not been studied before, which is important particularly to understand whether the coherent states are mathematically well-defined and can be utilized for describing concrete physical systems.

The purpose of this article is to fill in this gap by finding the exact analytical expression for the positive definite Borel measure, such that the coherent states satisfy the required condition of resolution of identity. For this purpose, we mainly follow [19–21] to associate our problem with the existing techniques of Stieltjes and Hausdorff moment problem, and compute the inverse Mellin transforms corresponding to our systems, which yield the precise expressions of the Borel measure. The article is organised as follows: In Sect. 3, we introduce basic notions of the generalised and nonlinear coherent states, as well as the moment problem associated with them to identify the resolution of identity. In Sect. 3, we implement the existing framework as described in Sect. 3 to study the completeness relation for coherent states in the noncommutative space for two different models, namely, the harmonic oscillator and the Pöschl-Teller. Our conclusions are stated in Sect. 4.

2 Nonlinear coherent states and resolution of identity

We commence by revisiting the basic notions of nonlinear coherent states for the purpose of referencing. Nonlinear coherent states for Hamiltonians H with discrete bounded below and nondegenerate eigenvalues, $E_n = \hbar\omega e_n = \hbar\omega n f^2(\hat{n})$, are defined as follows [5,6,22]:

$$|\alpha, f\rangle = \frac{1}{\sqrt{\mathcal{N}(\alpha, f)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{\rho_n}} |n\rangle, \quad \rho_n = \prod_{k=1}^{n} e_k = n! f^2(\hat{n})!, \quad \rho_0 = 1, \tag{1}$$

where $\alpha \in \mathbb{C}$ and the normalisation constant can be computed from the requirement $\langle \alpha, f | \alpha, f \rangle = 1$ as given by

$$\mathcal{N}(\alpha, f) = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\rho_n}. \tag{2}$$

The terminology *nonlinear* is not associated with the mathematical nonlinearity anyway, but it follows from the convention introduced in the articles [5,6,22]. More precisely, such coherent states (1) are one of the generalised versions of the Glauber coherent states [23] for models corresponding to the generic function of the number operator $f(\hat{n})$. In [24], the authors have introduced an interesting alternative to the so-called *nonlinear* generalisation by considering the coherent states to be eigenfunctions of a generalised exponential function. Nevertheless, the vectors $|\alpha, f\rangle$ in (1) are mathematically well defined in the domain $\mathcal{D}$ of allowed $|\alpha|^2$ for which the series (2) converges. The range of $|\alpha|^2$, $0 \leq |\alpha|^2 < R$, is determined by the radius of convergence $R = \lim_{n\to\infty} \sqrt{\rho_n}$, which may be finite or infinite depending on the behaviour of ρ_n for large n. Therefore, a family of such coherent states (1) is an *over-*

complete set of vectors in a Hilbert space $\mathscr{H}$, labelled by a continuous parameter α which belongs to a complex domain $\mathscr{D}$ (some domain in $\mathbb{C}$. For $R=\infty$, $\mathscr{D}=\mathbb{C}$). To be more precise, since $|n\rangle$ forms an orthonormal basis in the Hilbert space $\mathscr{H}$ and, letting e_n be an infinite sequence of positive numbers, with $e_0=0$, then the vectors $|\alpha,f\rangle$ must satisfy the resolution of identity (completeness relation) with a weight function Ω

$$\int\int_{\mathscr{D}} \frac{\mathscr{N}(\alpha,f)}{\pi}|\alpha,f\rangle\langle\alpha,f|\;\Omega(|\alpha|^2)\;d^2\alpha=\mathbb{I}_{\mathscr{H}}. \tag{3}$$

By considering $\alpha=re^{i\theta}$, the left-hand side of (3) turns out to be

$$\sum_{m,n=0}^{\infty}\frac{1}{2\pi\sqrt{\rho_m\rho_n}}\int_0^R r^{m+n}\Omega(r^2)d(r^2)\int_0^{2\pi}e^{i\theta(m-n)}d\theta\;|m\rangle\langle n| \tag{4}$$

$$=\sum_{n=0}^{\infty}\frac{1}{\rho_n}\int_0^R t^n\Omega(t)dt\;|n\rangle\langle n|, \tag{5}$$

such that one ends up with an infinite set of constraints

$$\int_0^R t^n\Omega(t)dt=\rho_n, \qquad 0<R\leq\infty, \tag{6}$$

for which the completeness relation (3) holds. Therefore, one can construct the coherent states (1) for any models corresponding to a known $f(n)$, provided that there exists a measure $\Omega(t)$ which satisfies (6). The explicit expression of the measure can be found first by associating (6) with the classical moment problem, where $\rho(n)>0$ are the power moments of the unknown function $\Omega(t)>0$ and, subsequently, by carrying out the integration using the standard techniques of the Mellin transforms [25]. For more details in this context, we refer the readers to [19, 21, 26–28]. For Glauber coherent states, i.e., for $f(n)=1$, $\rho_n=n!$, the moment problem (6) becomes

$$\int_0^{\infty} t^n\Omega(t)dt=n!, \quad n=0,1,2,....., \tag{7}$$

so that one can easily identify the measure, $\Omega(t)=e^{-t}$. For $SU(1,1)$ discrete series coherent states [3], $\rho(n)=n!\Gamma(2j)/\Gamma(2j+n)$ and, the corresponding measure is given by

$$\Omega(t)=(2j-1)(1-t)^{2j-2}, \tag{8}$$

where $\Omega(t)$ is supported in the range $(0,1)$, with $j=1,1/2,2,3/2,3....$ In the case of the Barut Girardello coherent states [29], $\rho(n)=n!\Gamma(2j+n)/\Gamma(2j)$ and, the associated measure is given by the modified Bessel function of the second kind as follows

$$\Omega(t)=\frac{2}{\Gamma(2j)}t^{\frac{2j-1}{2}}K_{2j-1}(2\sqrt{t}), \tag{9}$$

where $\Omega(t)$ is supported in the interval $(0,\infty)$. For more examples of different types of coherent states, see, [19, 30].

3 Resolution of unity for coherent states in noncommutative space

In this section, we will construct the coherent states arising from the noncommutative space [9, 10], in which the standard set of commutation relations for the canonical coordinates are replaced by noncommutative versions, such as

$$[X,P] = i\hbar(1+\check{\tau}P^2), \quad X = (1+\check{\tau}p^2)x, \quad P = p, \tag{10}$$

where the noncommutative observables X,P are represented in terms of the standard canonical variables x,p satisfying $[x,p] = i\hbar$. Here, $\check{\tau} = \tau/(m\omega\hbar)$ has the dimension of inverse squared momentum and τ is dimensionless. Since here we study a one-dimensional problem, it may not be so obvious to the readers how commutation relation (10) becomes a part of noncommutative systems. Actually, it is a reduced version of a three-dimensional noncommutative space originating from a q-deformed oscillator algebra, which was studied in [10] by the author and his collaborators. The given framework (10) is fascinating by itself, because it leads to the generalised version of Heisenberg's uncertainty relation [8] followed by the existence of minimal lengths [9, 10], which are one of the major findings of string theory. Let us now discuss some concrete models in the given structure.

3.1 *Noncommutative harmonic oscillator*

We consider a one-dimensional harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 - \hbar\omega\left(\frac{1}{2}+\frac{\tau}{4}\right), \tag{11}$$

defined on the noncommutative space satisfying (10). Here, ground state energy is conventionally shifted to allow for a factorisation of the energy. Obviously, the Hamiltonian H is non-Hermitian with respect to the standard inner product. However, we consider the Hamiltonian H to be pseudo-Hermitian and, thus by following the standard results in the literature [31, 32] we transform the non-Hermitian Hamiltonian H to a Hermitian Hamiltonian h by taking a similarity transformation $h = \eta H \eta^{-1}$ with respect to a positive definite metric η. Consequently, the energy eigenvalues of H and h turn out to be real, as computed in [8, 10] by following the standard techniques of Rayleigh-Schrödinger perturbation theory to the lowest order as follows:

$$E_n = \hbar\omega n f^2(n) = \hbar\omega n\left[1+\frac{\tau}{2}(1+n)\right]+\mathscr{O}(\tau^2). \tag{12}$$

Correspondingly, by following (1) the nonlinear coherent states are computed as

$$|\alpha,f\rangle_{\text{ncho}} = \frac{1}{\sqrt{\mathscr{N}(\alpha,f)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho_n}}|n\rangle, \quad \rho_n = n!f^2(n)! = \left(\frac{\tau}{2}\right)^n\frac{n!(n+\frac{2}{\tau}+1)!}{(1+\frac{2}{\tau})!}. \tag{13}$$

In order to verify that the states (13) are mathematically complete and well defined in the Hilbert space, one needs to find out the existence of the positive definite Borel measure $\Omega(t)$ satisfying constraint (6) as follows:

$$\int_0^\infty t^n\Omega(t)dt = \rho_n = \left(\frac{\tau}{2}\right)^n\frac{\Gamma(n+\mu+1)\Gamma(n+\beta+1)}{\Gamma(1+\beta)} = \frac{A(n)B(n)}{\Gamma(1+\beta)}, \tag{14}$$

with $A(n)=(\tau/2)^n, B(n)=\Gamma(n+\mu+1)\Gamma(n+\beta+1)$ and $\mu=0,\beta=1+2/\tau$. For the purpose of computing $\Omega(t)$ from (14) let us now briefly discuss some technicalities. First, we find the inverse Mellin transforms of the two functions $A(n)$ and $B(n)$ separately, i.e.,

$$A(n) = \left(\frac{\tau}{2}\right)^n = \int_0^\infty\left(\frac{t}{2}\right)^n\delta(t-\tau)dt = \int_0^\infty x^nC(x)dx, \tag{15}$$

$$B(n) = \Gamma(n+\mu+1)\Gamma(n+\beta+1) = \int_0^\infty x^nD(x)dx, \tag{16}$$

with

$$C(x) = 2\delta(2x-\tau), \qquad D(x) = 2x^{\frac{\mu+\beta}{2}}K_{\mu-\beta}(2\sqrt{x}), \tag{17}$$

where $K_\mu(x)$ denotes the modified Bessel function of the second kind. Then, we utilize the composition formula [20] to find the inverse Mellin transform for the composite system as given by

$$A(n)B(n) = \int_0^\infty x^n\lambda(x)dx, \quad \lambda(x) = \int_0^\infty C(u)D\left(\frac{x}{u}\right)\frac{du}{u}, \tag{18}$$

which when replaced in (14), we obtain the accurate expression of the Borel measure $\Omega(t)$ as follows:

$$\Omega(t) = \frac{1}{\Gamma(1+\beta)}\int_0^\infty C(t)D\left(\frac{t}{u}\right)\frac{du}{u} \tag{19}$$

$$= \frac{1}{\Gamma(1+\beta)}\int_0^\infty\frac{4}{u}\left(\frac{t}{u}\right)^{\frac{\mu+\beta}{2}}K_{\mu-\beta}(2\sqrt{\frac{t}{u}})\delta(2u-\tau)du \tag{20}$$

$$= \frac{2^{\frac{1}{2}(4+\mu+\beta)}}{\tau\Gamma(1+\beta)}\left(\frac{t}{\tau}\right)^{\frac{\mu+\beta}{2}}K_{\mu-\beta}(2\sqrt{\frac{2t}{\tau}}). \tag{21}$$

3.2 A Pöschl-Teller model in noncommutative space

Let us now consider another interesting Hamiltonian based on the noncommutative space satisfying (10):

$$H_{\mathrm{PT}} = \frac{\varepsilon}{2m}P^2 + \frac{\hbar\omega\gamma}{2\check{\tau}}P^{-2} + \frac{m\omega^2}{2}X^2 + \frac{\hbar\omega\gamma}{2} + \frac{\varepsilon}{2m\check{\tau}}, \qquad \gamma,\varepsilon \in \mathbb{R}. \tag{22}$$

Although the model (22) does not belong to a familiar class of models, however, it is very interesting as it leads to the well-known Pöschl-Teller potential when the noncommutative observables are represented in terms of the standard canonical variables by using (10). Therefore, one can describe the model as a noncommutative version of the Pöschl-Teller model, although the Hamiltonian (22) cannot be viewed as a deformation of a model on standard commutative space in the sense that it does not reduce to the usual Pöschl-Teller model in the commutative limit $\tau \to 0$. For further details on the model, see [11], where the eigenvalues of the corresponding Hamiltonian were computed in an exact manner as given below:

$$E_n = \frac{\hbar\omega\tau}{2}(1+2n+a+b)^2, \quad a = \frac{1}{2}\sqrt{1+\frac{4\gamma}{\tau}},\ b = \frac{1}{2}\sqrt{1+\frac{4\varepsilon}{\tau}}. \tag{23}$$

The corresponding nonlinear coherent states can be computed by following (1) with

$$\rho_n = 2^n\tau^n\frac{\Gamma(n+\nu)^2}{\Gamma(\nu)^2}, \qquad \nu = \frac{3+a+b}{2}. \tag{24}$$

Note that the form of ρ_n in (24) is very similar to the case of the harmonic oscillator as in (13) and, thus we follow the similar procedure as discussed in the previous section to calculate the explicit expression of the measure

$$\Omega(t) = \frac{\tau^{-\nu}}{\Gamma(\nu)^2}\left(\frac{t}{2}\right)^{\nu-1}K_0\left(\sqrt{\frac{2t}{\tau}}\right). \tag{25}$$

Numerically, we check that the measures (19) and (25) are positive definite for $\tau \in \mathbb{R}^+$. However, the exact analytical treatment of this problem may be more involved. The interesting aspect of the moment problem in our case is that the moments (14) and (24) satisfy the Carleman condition

$$\sum_{n=1}^{\infty}\frac{1}{\rho_n^{1/2n}} = +\infty, \tag{26}$$

which implies that the obtained measures in both of the cases are unique.

4 Conclusions

Coherent states in noncommutative spaces related to the generalised uncertainty relation have been found to be interesting and useful for many different purposes [12–18]. However, it was necessary to find out their resolution of identity by computing the weight functions associated with them to show the mathematical completeness of the corresponding models. In this article we study the missing link by finding the completeness relations of coherent states for the harmonic oscillator and the Pöschl-Teller model based on the noncommutative structure. We compute exact analytical expressions for the weight functions through good solutions of Stieltjes and Hausdorff moment problem for the two cases and show that the coherent states in noncommutative space are indeed mathematically well defined and form a complete set of vectors in the Hilbert space.

Evidently, there are many interesting open challenges left which will directly follow our results. By utilizing the Hankel determinant method, or any other existing mechanism in the literature [27,30,33,34], one may study the orthogonal polynomials, that are associated with our coherent states. The investigation may end up with some known orthogonal polynomials, or some fascinating q-orthogonal polynomials. However, because of our sophisticated structure, the outcome may bring more exciting possibilities for constructing some new orthogonal polynomials out of our systems.

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Majorana neutrinos in an effective field theory approach

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Abstract The discovery of neutrino oscillations and non-vanishing neutrino masses is one of the key recent advances made in particle physics. Tiny neutrino masses are very difficult to generate in a natural way in the Standard Model, based in the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. Standard Yukawa interactions cannot explain the huge mass difference between the neutrinos and other fermions, and a very attractive scheme is the seesaw mechanism, incorporating right-handed Majorana neutrino species that allow for the lepton number violation. However, in typical seesaw scenarios, the couplings between the Majorana neutrinos and the light neutrinos must be vanishingly small in order to obtain tiny observed masses, leading to the decoupling of the former, and thus their detection (e.g., via lepton number violating processes) would be a signal of physics beyond the minimal seesaw framework. Here we consider a model independent scenario with one Majorana neutrino N with negligible mixing with the standard ν_L, introducing its interactions via an effective Lagrangian involving N and the standard fields and preserving SM symmetry. This leads to a very rich N phenomenology, and we have studied its decay, production, and detection mechanisms in present and future collider experiments.

1 Introduction

As is well known, in the Standard Model, the charged leptons acquire their masses after the electroweak symmetry breaking (EWSB) by the vacuum expectation value

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of the Higgs field $\langle \Phi^T \rangle = (0, v/\sqrt{2})^T$ leading to the Yukawa lagrangian term, where the left-handed $SU(2)_L$ doublet of flavor i $L^{i\,T} = (\nu^i, \ell^i)_L^T$ combines with the right-handed singlet ℓ_R^j: $\mathscr{L}_{Yukawa} \supset Y_\ell^{ij}\overline{L^i}\Phi\ell_R^j \to Y_\ell^{ij} v/\sqrt{2}\ell_L^i\ell_R^j$. A right-handed neutrino component could be added 'by hand' to have a Yukawa term generating a Dirac mass for the neutrinos. However, to get sub-eV left-handed neutrino masses (as known from current neutrino data [16]) one would need extremely tiny Yukawa couplings of order $\lesssim 10^{-12}$. As there is no theoretical justification for such small neutrino Yukawa couplings, a new viewpoint is taken considering other new physics to be responsible for the observed smallness of neutrino masses.

It seems natural to invoke some new physics beyond the SM at a higher scale Λ to explain the neutrino masses: this is done by means of the dimension 5 Weinberg operator [18]: $\mathscr{L}_{\nu SM} \supset -\frac{\lambda_{ij}}{2\Lambda} L^i L^j \Phi\Phi$, leading to neutrino masses after the EWSB $\mathscr{L}_{\nu SM} \supset -\frac{m_{ij}}{2}\nu_i\nu_j$ with $m_{ij}/2 = \lambda_{ij}\frac{v^2}{\Lambda}$. If $\Lambda \gg v$, then neutrino masses can be made much smaller than those of the charged leptons. This operator has only 3 renormalizable tree-level realizations: the most popular is the Type-I seesaw mechanism, where Φ and L^i combine into an $SU(2)_L$ scalar. This needs the addition of new 'sterile' $SU(2)_L$ singlet neutrinos N_R as intermediate particles, identified as right-handed Majorana neutrinos. The origin of the bare Majorana mass terms responsible for the explicit violation of the $B-L$ (Baryon-Lepton numbers) symmetry can be understood from natural implementations of the seesaw mechanism in specific ultra-violet complete theories. In the conventional seesaw scenarios, the light neutrino masses are inversely proportional to a large lepton-number breaking scale (and hence the name seesaw) [13].

In this context the 'vanilla' seesaw is implemented adding 3 SM singlets N_R, and writing the most general renormalizable lagrangian:

$$\mathscr{L}_\nu = \mathscr{L}_{SM} - Y_{\alpha i}\overline{L^\alpha}\tilde{\Phi}N_{Ri} - \sum_{i,j=1}^{3}\frac{M_{N_{ij}}}{2}\overline{N_{iL}^c}N_{jR} + h.c.,$$

with Dirac and Majorana mass terms for the neutrinos. In the flavor basis these terms give the following mass matrix, diagonalized by a unitary matrix U:

$$\frac{1}{2}(\overline{\nu_L}\,\overline{N_L^c})\begin{pmatrix} O & m_D \\ m_D^T & M_N \end{pmatrix}\begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix}, \qquad \mathrm{U}^{\mathrm{T}}\begin{pmatrix} O & m_D \\ m_D^T & M_N \end{pmatrix}\mathrm{U} \simeq \begin{pmatrix} -m_D M_N^{-1} m_D^T & 0 \\ 0 & M_N \end{pmatrix}$$

with entries $\mathrm{U} \simeq \begin{pmatrix} 1 & A \\ -A^\dagger & 1 \end{pmatrix}$, and $A^\dagger = M_N^{-1}m_D^T$.

In this way, left-handed neutrinos of flavor ℓ $\nu_{\ell L}$ can be written in terms of mass states as $\nu_{\ell L} = U_{\ell m}\nu_m + U_{\ell N}N$: the $\nu_L - N$ mixings $U_{\ell N}$ take values $U_{\ell N} \simeq \frac{m_D}{M_N} = \sqrt{\frac{m_\nu}{M_N}}$, and the $\nu_L - \nu_m$ mixing describes the oscillation phenomena. This gives light neutrinos ν with masses $m_\nu = m_D M_N^{-1} m_D^T$. As the Dirac mass m_D comes from the Yukawa term $Y_{\alpha i}\overline{L^\alpha}\tilde{\Phi}N_{Ri}$ and takes values $m_D = Y\frac{v}{\sqrt{2}}$, in order to accommodate neutrino masses $m_\nu \simeq 0.001 eV \simeq Y^2\frac{v^2}{2M_N}$ the Yukawa couplings need to be $Y \simeq 10^{-7}$

for $M_N \simeq 100 GeV$. Then the $\nu_L - N$ mixing would be $U_{\ell N} \simeq \frac{m_D}{M_N} \lessapprox 10^{-6}\sqrt{\frac{100GeV}{M_N}}$. As this mixing weighs the coupling between Majorana neutrinos and the standard bosons in the lagrangian terms:

$$\mathscr{L} \supset -\frac{g}{\sqrt{2}} U_{\ell N} \overline{N}^c \gamma^\mu P_L l W_\mu^+ - \frac{g}{2c\theta_W} \overline{\nu_\ell} \gamma^\mu U_{\ell N} P_L N Z_\mu + h.c.,$$

the experimental observation of the lepton number violation in this scenario depends only on the tiny $\nu_L - N$ mixing, making it very difficult to find in current collider searches.

2 Effective field theory with Majorana neutrinos

While most of the recent work has been focused on the study of heavy Majorana neutrinos that mix with *SM* light neutrinos in the framework of Type-I seesaw scenarios [3, 4], the aim of our approach is to investigate the possible contribution of a heavy Majorana neutrino with negligible mixing to the *SM* ν_L [6].

Thus we consider an effective lagrangian in which we include a relatively light right-handed Majorana neutrino N as one of the observable degrees of freedom. The effects of the new physics involving one heavy sterile neutrino and the *SM* fields are parameterized by a set of effective operators $\mathscr{O}_J$ constructed with the standard model and the Majorana neutrino fields and satisfying $SU(2)_L \otimes U(1)_Y$ gauge symmetry.

The effect of these operators is suppressed by inverse powers of the new physics scale Λ, for which we take the value $\Lambda = 1\ TeV$. The total lagrangian is organized as follows:

$$\mathscr{L} = \mathscr{L}_{SM} + \sum_{n=6}^{\infty} \frac{1}{\Lambda^{n-4}} \sum_J \alpha_J \mathscr{O}_J^{(n)}, \tag{1}$$

where n is the mass dimension of $\mathscr{O}_J^{(n)}$.

The dominating effects come from dimension 6 operators that can be generated at tree level in the unknown underlying renormalizable theory. Following [6], we start with a rather general effective lagrangian density for the interaction of right-handed Majorana neutrinos N with leptons and quarks, including dimension 6 operators. The first subset includes operators with scalar and vector bosons (SVB),

$$\mathscr{O}_{LN\phi} = (\phi^\dagger \phi)(\bar{L}_i N \tilde{\phi}), \mathscr{O}_{NN\phi} = i(\phi^\dagger D_\mu \phi)(\bar{N}\gamma^\mu N), \mathscr{O}_{Ne\phi} = i(\phi^T \varepsilon D_\mu \phi)(\bar{N}\gamma^\mu e_i), \tag{2}$$

and a second subset includes the baryon-number conserving four fermion contact terms:

$$\mathcal{O}_{duNe} = (\bar{d}\gamma^{\mu}u)(\bar{N}\gamma_{\mu}l), \;\; \mathcal{O}_{fNN} = (\bar{f}\gamma^{\mu}f)(\bar{N}\gamma_{\mu}N), \;\; \mathcal{O}_{LNLe} = (\bar{L}N)\varepsilon(\bar{L}l),$$
$$\mathcal{O}_{LNQd} = (\bar{L}N)\varepsilon(\bar{Q}d), \;\; \mathcal{O}_{QuNL} = (\bar{Q}u)(\bar{N}L), \;\; \mathcal{O}_{QNLd} = (\bar{Q}N)\varepsilon(\bar{L}d),$$
$$\mathcal{O}_{LN} = |\bar{N}L|^2 \quad and \quad \mathcal{O}_{QN} = |\bar{Q}N|^2, \tag{3}$$

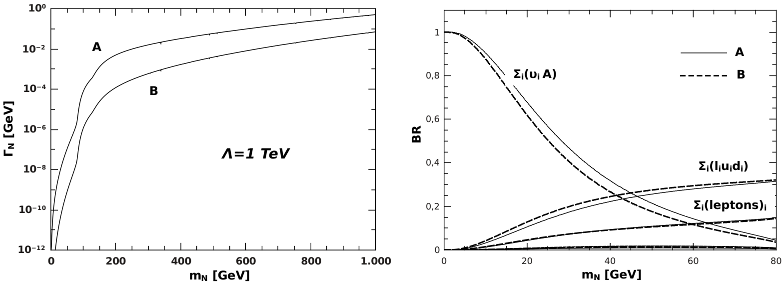
where e_i, u_i, d_i and L_i, Q_i denote, for the family labeled i, the right-handed $SU(2)$ singlet and the left-handed $SU(2)$ doublets, respectively.

One can also consider operators generated at one-loop level in the underlying full theory, whose coefficients are naturally suppressed by a factor of $1/16\pi^2$:

$$\mathcal{O}^{(5)}_{NNB} = \bar{N}\sigma^{\mu\nu}N^c B_{\mu\nu},$$
$$\mathcal{O}_{NB} = (\bar{L}\sigma^{\mu\nu}N)\tilde{\phi}B_{\mu\nu}, \qquad \mathcal{O}_{NW} = (\bar{L}\sigma^{\mu\nu}\tau^I N)\tilde{\phi}W^I_{\mu\nu},$$
$$\mathcal{O}_{DN} = (\bar{L}D_{\mu}N)D^{\mu}\tilde{\phi}, \qquad \mathcal{O}_{\bar{D}N} = (D_{\mu}\bar{L}N)D^{\mu}\tilde{\phi} \; . \tag{4}$$

The effective couplings α_J in (1) can be bounded exploiting the existing constraints on the different processes mediated by their associated operators. In general phenomenological approaches, recent reviews [7, 8] summarize the existing experimental bounds considering low scale minimal seesaw models parameterized by a single heavy neutrino mass scale M_N and a light-heavy mixing U_{lN}. In previous works [10, 11], we have presented in detail the way in which we take into account existing constraints on processes like neutrinoless double beta decay ($0\nu\beta\beta$), electroweak precision data (EWPD), LNV rare meson decays, as well as direct collider searches, including Z decays. We consider the existing experimental constraints on sterile-active neutrino mixings, relating the U_{lN} mixings in Type-I seesaw models [3, 4] with our effective couplings by the relation $U^2_{lN} \simeq \left(\frac{\alpha v^2}{2\Lambda^2}\right)^2$. For the couplings involving the first fermion family, the most stringent are the $0\nu\beta\beta$-decay bounds obtained by the KamLAND-Zen collaboration [12]. Following the treatment made in [7, 15], they give us an upper limit $\alpha^{bound}_{0\nu\beta\beta} \leq 3.2\times10^{-2}\left(\frac{m_N}{100\ GeV}\right)^{1/2}$. Concerning the second fermion family, for sterile neutrino masses $2\ GeV \lesssim m_N \lesssim 10\ GeV$ the upper limits come from the DELPHI collaboration [2]. Considering $\Omega_{ll'} = U_{lN}U_{l'N}$ as in [4], we obtain the bound $\alpha^{bound}_{DELPHI} \lesssim 2.3$. The Belle [14] and LHCb [1] collaborations also find competitive upper limits in the $2\ GeV \lesssim m_N \lesssim 5\ GeV$ region. The bound from Belle is still the most stringent, giving a value $\alpha^{bound}_{Belle} \lesssim 0.3$. For higher masses, in the range $m_W \lesssim m_N$, the upper limits come from EWPD as the radiative lepton flavor violating (LFV) decays as $\mu \to e\gamma$ [5, 17] giving a bound $\alpha^{bound}_{EWPD} \leq 0.32$.

In order to simplify the discussion, for numerical calculations we only consider the most stringent bounds, taking the couplings associated to the operators that contribute to the $0\nu\beta\beta$-decay for the first family as restricted by the $\alpha^{bound}_{0\nu\beta\beta}$, and the other couplings to the value $\alpha^{bound} \leq 0.3$.



(a) Total N decay width (b) Low m_N region branching ratios

Fig. 1: Decay width [11] and branching ratios [10] for the Majorana neutrino N.

3 Majorana neutrino phenomenology

We have studied the Majorana neutrino phenomenology, searching for its production in ep colliders [9] and calculated its decay channels for low $m_N < m_W$ [10] and high masses $m_N < 1TeV$ [11].

For the low m_N region, the neutrino plus photon decay channel is found to be dominant respective to the pure-leptonic or the $N \to l\bar{u}d$ modes. This decay mode is driven by the tensorial one-loop generated operators $\mathcal{O}_{NB}$ and $\mathcal{O}_{NW}$ in (4), and leads to an interesting phenomenology regarding different neutrino anomalies, as discussed in [10]. In Figure 3 we show the total decay width and the branching ratios for the N.

In our work [9] we studied the N production in the $ep \to l^+N \to l^+ + 3jets$ channel, which is a clear signature of the Majorana character of the intermediate N due to the lepton number violation. We found the future Large Hadron electron Collider (LHeC) could be able to discover Majorana neutrinos with $m_N < 700$ and $m_N < 1300\ GeV$ for electron beams settings of $E_e = 50$ and $E_e = 150\ GeV$ respectively, as shown in Figure 2.

We are currently investigating the N production and decay in the LHC, exploiting the fact that for low $m_N \simeq$ a few GeV the Majorana neutrinos behave as long-lived neutral particles that can be searched using displaced vertices techniques.

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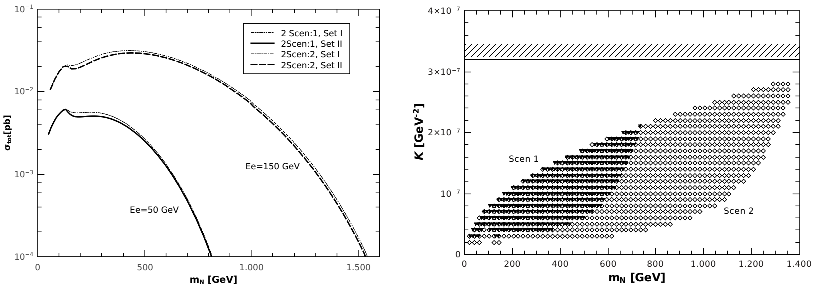

(a) Total cross section (b) Discovery regions

Fig. 2: Total cross section and discovery regions for the Majorana neutrino production in the LHeC [9].

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Feynman-Dyson propagators for neutral particles (local or non-local?)

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Abstract An analog of the $S = 1/2$ Feynman-Dyson propagator is presented in the framework of $S = 1$ Weinberg's theory. The basis for this construction is the concept of the Weinberg field as a system of four field functions differing by parity and dual transformations. Next, we analyze the controversy in the definitions of the Feynman-Dyson propagator for the field operator containing the $S = 1/2$ self/anti-self charge conjugate states in the papers by D. Ahluwalia et al. and by W. Rodrigues Jr. et al. The solution of this mathematical controversy is obvious. It is related to the necessary doubling of the Fock Space (as in the Barut and Ziino works), thus extending the corresponding Clifford algebra. However, the logical interrelations of different mathematical foundations with the physical interpretations are not so obvious (Physics should choose only one correct formalism: it is not clear, why two correct mathematical formalisms , which are based on the same postulates, lead to different physical results.)

1 Weinberg propagators

According to the Feynman-Dyson-Stueckelberg ideas, a causal propagator has to be constructed using the formula in Ref. [1, p.91]. In the $S = 1/2$ Dirac theory it results in

$$S_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{\hat{k} + m}{k^2 - m^2 + i\varepsilon}\,, \tag{1}$$

provided that $\hat{k} = k_\mu \gamma^\mu$, the constant a and b are determined by imposing $(i\hat{\partial}_2 - m)S_F(x_2, x_1) = \delta^{(4)}(x_2 - x_1)$ in [1, p.91], $\hat{\partial}_2 = \frac{\partial}{\partial x_2^\mu}\gamma^\mu$, namely, $a = -b = 1/i$.

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However, attempts to construct the covariant propagator in this way have failed in the framework of Weinberg's theory, Ref. [2]. It is a generalization of the Dirac ideas to higher spins. For instance, on page B1324 of Ref. [2] Weinberg writes: "*Unfortunately, the propagator arising from Wick's theorem is NOT equal to the covariant propagator except for* $S=0$ *and* $S=1/2$. *The trouble is that the derivatives act on the* $\varepsilon(x)=\theta(x)-\theta(-x)$ *in* $\Delta^C(x)$ *as well as on the functions*[1] Δ *and* Δ_1. *This gives rise to extra terms proportional to equal-time* δ *functions and their derivatives... The cure is well known: ... compute the vertex factors using only the original covariant part of the Hamiltonian* $\mathscr{H}$*; do not use the Wick propagator for internal lines; instead use the covariant propagator.*" The propagator proposed in Ref. [3] is the causal propagator. However, the old problem persists: the Feynman-Dyson propagator is not the Green function of the Weinberg equation. As mentioned, the covariant propagator proposed by Weinberg propagates kinematically spurious solutions [3].

The aim of my paper is to consider the problem of constructing the propagator in the framework of the model given in [4]. The concept of the Weinberg field 'doubles' has been proposed there. It is based on the equivalence between the Weinberg field and the antisymmetric tensor field, which can be described by both $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$. These field operators may be used to form a parity doublet. An essential ingredient of my consideration is the idea of combining the Lorentz and the dual transformation. The set of four equations has been proposed in Ref. [4].

The simple calculations give

$$u_1^{(1)}\overline{u}_1^{(1)}=\frac{1}{2}\begin{pmatrix} m^2 & S_p\otimes S_p\\ \overline{S}_p\otimes\overline{S}_p & m^2\end{pmatrix}, u_2^{(1)}\overline{u}_2^{(1)}=\frac{1}{2}\begin{pmatrix} -m^2 & S_p\otimes S_p\\ \overline{S}_p\otimes\overline{S}_p & -m^2\end{pmatrix}, \tag{2}$$

$$u_1^{(2)}\overline{u}_1^{(2)}=\frac{1}{2}\begin{pmatrix} -m^2 & \overline{S}_p\otimes\overline{S}_p\\ S_p\otimes S_p & -m^2\end{pmatrix}, u_2^{(2)}\overline{u}_2^{(2)}=\frac{1}{2}\begin{pmatrix} m^2 & \overline{S}_p\otimes\overline{S}_p\\ S_p\otimes S_p & m^2\end{pmatrix}, \tag{3}$$

where

$$S_p=m+(\mathbf{S}\cdot\mathbf{p})+\frac{(\mathbf{S}\cdot\mathbf{p})^2}{E+m},\quad \overline{S}_p=m-(\mathbf{S}\cdot\mathbf{p})+\frac{(\mathbf{S}\cdot\mathbf{p})^2}{E+m}, \tag{4}$$

and $u-$ are 6-component objects for spin 1, which are solutions of the Weinberg "double" equations in the momentum space. One can conclude: the generalization of the notion of causal propagators is admitted using 'Wick's formula' for the time-ordered particle operators provided that $a=b=1/4im^2$. It is necessary to consider all four equations. Obviously, this is related to the 12-component formalism, which I presented in [4].

The $S=1$ analogues of the formula (1) for the Weinberg propagators follow immediately. In the Euclidean metrics they are

[1] In the cited paper $\Delta_1(x)\equiv i[\Delta_+(x)+\Delta_+(-x)]$ and $\Delta(x)\equiv\Delta_+(x)-\Delta_+(-x)$ have been used. $i\Delta_+(x)\equiv\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_p}\exp(ipx)$ is the particle Green function.

$$S_F^{(1)}(p) \sim -\frac{1}{i(2\pi)^4(p^2+m^2-i\varepsilon)}\left[\gamma_{\mu\nu}p_\mu p_\nu - m^2\right], \tag{5}$$

$$S_F^{(2)}(p) \sim -\frac{1}{i(2\pi)^4(p^2+m^2-i\varepsilon)}\left[\gamma_{\mu\nu}p_\mu p_\nu + m^2\right], \tag{6}$$

$$S_F^{(3)}(p) \sim -\frac{1}{i(2\pi)^4(p^2+m^2-i\varepsilon)}\left[\widetilde{\gamma}_{\mu\nu}p_\mu p_\nu + m^2\right], \tag{7}$$

$$S_F^{(4)}(p) \sim -\frac{1}{i(2\pi)^4(p^2+m^2-i\varepsilon)}\left[\widetilde{\gamma}_{\mu\nu}p_\mu p_\nu - m^2\right]. \tag{8}$$

$\gamma_{\mu\nu}$ are the covariantly defined 6×6 matrices of the $(1,0)\oplus(0,1)$ representation, $\widetilde{\gamma}_{\mu\nu} = \gamma_{44}\gamma_{\mu\nu}\gamma_{44}$.

We should use the obtained set of Weinberg propagators (5,6,7,8) in the perturbation calculus of scattering amplitudes. In Ref. [6] the amplitude for the interaction of two $2(2S+1)$ bosons has been obtained on the basis of the use of one field only and it is obviously incomplete, see also Ref. [5]. But, it is interesting that the spin structure was proved there to be the same, regardless of whether we consider the two-Dirac-fermion interaction or the two-Weinberg($S=1$)-boson interaction. However, the denominator slightly differs in the cited papers [6] from fermion-fermion case. More accurate considerations of the fermion-boson and boson-boson interactions in the framework of the Weinberg theory has been reported elsewhere [7].

2 The self/anti-self charge conjugate construct in the (1/2,0)⊕(0,1/2) representation

The first formulations with doubling solutions of the Dirac equations have been presented in Refs. [11], and [12]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [13], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called 'the Bargmann-Wightman-Wigner-type quantum field theory' in 1993). M. Markov wrote long ago *two* Dirac equations with the opposite signs at the mass term [11]:

$$\left[i\gamma^\mu\partial_\mu - m\right]\Psi_1(x) = 0, \tag{9}$$
$$\left[i\gamma^\mu\partial_\mu + m\right]\Psi_2(x) = 0. \tag{10}$$

In fact, he studied all properties of this relativistic quantum model (while he did not yet know the quantum field theory in 1937). Next, he added and subtracted these equations. What did he obtain?

$$i\gamma^\mu\partial_\mu\varphi(x) - m\chi(x) = 0, \quad i\gamma^\mu\partial_\mu\chi(x) - m\varphi(x) = 0. \tag{11}$$

Thus, φ and χ solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with equations for the Majorana-like $\lambda-$ and $\rho-$ spinors, which we presented in Ref. [8, 9]. The equations can be written in the 8-component form as follows:

$$\left[i\Gamma^\mu \partial_\mu - m\right] \Psi_{(+)}(x) = 0\,,, \quad \left[i\Gamma^\mu \partial_\mu + m\right] \Psi_{(-)}(x) = 0\,. \tag{12}$$

The signs at the mass terms depend on how do we choose the "positive"- and "negative"- energy solutions. For instance,

$$\Psi_{(+)}(x) = \begin{pmatrix} \rho^A(x) \\ \lambda^S(x) \end{pmatrix}, \Psi_{(-)}(x) = \begin{pmatrix} \rho^S(x) \\ \lambda^A(x) \end{pmatrix}, \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}. \tag{13}$$

It is easy to find the corresponding projection operators, and the Feynman-Dyson-Stueckelberg propagator.

You may say that all of this is just related to the spin-parity basis rotation (unitary transformations). However, in previous papers I explained: the connection with the Dirac spinors has been found [9, 13], provided that the 4-spinors have the same physical dimension. Thus, we can see that the two 4-spinor systems are connected by the unitary transformations, and this represents itself the rotation of the spin-parity basis. However, it is usually assumed that the $\lambda-$ and $\rho-$ spinors describe the neutral particles, and meanwhile $u-$ and $v-$ spinors describe the charged particles. Kirchbach [13] found the amplitudes for neutrinoless double beta decay ($00\nu\beta$) in this scheme. It is obvious from that connections that there are some additional terms comparing with the standard formulation.

Barut and Ziino [12] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [4, 13]) in the framework of quantum field theory. In their case the self/anti-self charge conjugate states are simultaneously the eigenstates of the chirality. Next, our formulation with the $\lambda-$ and $\rho-$ spinors naturally lead to the Ziino-Barut scheme of massive chiral fields.

3 The controversy

I cite Ahluwalia et al., Ref. [14]: "*To study the locality structure of the fields $\Lambda(x)$ and $\lambda(x)$, we observe that field momenta are*

$$\Pi(x) = \frac{\partial \mathscr{L}^\Lambda}{\partial \dot{\Lambda}} = \frac{\partial}{\partial t} \overset{\neg}{\Lambda}(x), \tag{14}$$

and similarly $\pi(x) = \frac{\partial}{\partial t} \overset{\neg}{\lambda}(x)$. The calculational details for the two fields now differ significantly. We begin with the evaluation of the equal time anticommutator for $\Lambda(x)$ and its conjugate momentum

$$\{\Lambda(\mathbf{x},t),\, \Pi(\mathbf{x}',t)\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}$$
$$\times \underbrace{\sum_\alpha \left[\xi_\alpha(\mathbf{p})\, \overset{\neg}{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\, \overset{\neg}{\zeta}_\alpha(-\mathbf{p})\right]}_{=2m[I+\mathscr{G}(\mathbf{p})]}.$$

The term containing $\mathscr{G}(\mathbf{p})$ vanishes only when $\mathbf{x}-\mathbf{x}'$ lies along the z_e axis (see Eq. (24) [therein], and discussion of this integral in Ref. [15])

$$\mathbf{x}-\mathbf{x}' \textit{ along } z_e: \quad \{\Lambda(\mathbf{x},t),\, \Pi(\mathbf{x}',t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')I \tag{15}$$

The anticommutators for the particle/antiparticle annihilation and creation operators suffice to yield the remaining locality conditions,

$$\{\Lambda(\mathbf{x},t),\, \Lambda(\mathbf{x}',t)\} = O, \quad \{\Pi(\mathbf{x},t),\, \Pi(\mathbf{x}',t)\} = O. \tag{16}$$

The set of anticommutators contained in Eqs. (15) and (16) establish that $\Lambda(x)$ becomes local along the z_e axis. For this reason we call z_e as the dark axis of locality."

Next, I cite Rodrigues et al., Ref. [16]: "*We have shown through explicitly and detailed calculation that the integral of $\mathscr{G}(\mathbf{p})$ appearing in Eq.(42) of [14] is null for $\mathbf{x}-\mathbf{x}'$ lying in three orthonormal spatial directions in the rest frame of an arbitrary inertial frame $\mathbf{e}_0 = \partial/\partial t$.*

This shows that the existence of elko spinor fields does not imply any breakdown of locality concerning the anticommutator of $\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x}',t\}$ and moreover does not imply any preferred spacelike direction field in Minkowski spacetime."

Who is right? In 2013 W. Rodrigues [17] changed a bit his opinion. He wrote: "*When $\Delta_z \neq 0$, $\hat{\mathscr{G}}(\mathbf{x}-\mathbf{x}')$ is null the anticommutator is* local *and thus there exists in the elko theory as constructed in [14] an infinity number of 'locality directions'. On the other hand $\hat{\mathscr{G}}(\mathbf{x}-\mathbf{x}')$ is a distribution with support in $\Delta_z = 0$. So, the directions $\Delta = (\Delta_x, \Delta_y, 0)$ are nonlocal in each arbitrary inertial reference frame $\mathbf{e}_0$ chosen to evaluate $\hat{\mathscr{G}}(\mathbf{x}-\mathbf{x}')$*", thus accepting the Ahluwalia et al. viewpoint. See the cited papers for the notation.

Meanwhile, I suggest using the 8-component formalism (see Section 2) in the similarity with the 12-component formalism of Section 1. If we calculate

$$S_F^{(+,-)}(x_2,x_1) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{E_k}\sum_\sigma \Big[\theta(t_2-t_1)\, a\, \Psi_+^\sigma(k)\otimes \overline{\Psi}_+^\sigma(k)e^{-ikx} +$$
$$+\, \theta(t_1-t_2)\, b\, \Psi_-^\sigma(k)\otimes \overline{\Psi}_-^\sigma(k)e^{ikx}\Big] =$$
$$= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{(\hat{k}\pm m)\otimes I_2}{k^2-m^2+i\varepsilon}, \tag{17}$$

we easily come to the result that the corresponding Feynman-Dyson propagators are local in the sense: $[i\Gamma_\mu \partial_2^\mu \mp m]S_F^{(+,-)}(x_2-x_1) = \delta^{(4)}(x_2-x_1)$. The constants a and b are defined as in Ref. [1]. However, again: Physics should choose only one correct formalism. It is not clear, why two correct mathematical formalisms lead to different physical results?

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Generalized equations and their solutions in the (S,0)⊕(0,S) representations of the Lorentz group

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Abstract In this paper I present three explicit examples of generalizations in relativistic quantum mechanics. First of all, I discuss the generalized spin-1/2 equations for neutrinos. They have been obtained by means of the Gersten-Sakurai method for derivations of arbitrary-spin relativistic equations. Possible physical consequences are discussed. Next, it is easy to check that both Dirac algebraic equation $Det(\hat{p}-m)=0$ and $Det(\hat{p}+m)=0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0=\pm E_p=\pm\sqrt{\mathbf{p}^2+m^2}$. The same is true for higher-spin equations. Meanwhile, every book considers the equality $p_0=E_p$ for both $u-$ and $v-$ spinors of the $(1/2,0)\oplus(0,1/2))$ representation only, thus applying the Dirac-Feynman-Stueckelberg procedure for eliminating negative-energy solutions. The recent Ziino works (and, independently, the articles of several others) show that the Fock space can be doubled. We re-consider this possibility on the quantum field level for both $S=1/2$ and higher spin particles. The third example is: we postulate the non-commutativity of 4-momenta, and we derive the mass splitting in the Dirac equation. The applications are discussed.

1 Generalized neutrino equations

A. Gersten [1] proposed a method for derivations of massless equations of arbitrary-spin particles. In fact, his method is related to the van der Waerden-Sakurai [2] procedure for the derivation of the massive Dirac equation. I commented on the derivation of the Maxwell equations in [3]. Then, I showed that the method is rather ambigious because instead of free-space Maxwell equations, one can obtain *generalized* $S=1$ equations, which connect the antisymmetric tensor field with additional scalar

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fields. The problem of physical significance of additional scalar chi-fields should be solved of course by experiment.

In the present paper I apply the van der Waerden-Sakurai-Gersten procedure to spin-1/2 fields. As a result one obtains equations which *generalize* the well-known Weyl equations. However, these equations are known for a long time [4]. Raspini [5, 6] analyzed them again in detail. I add some comments on physical contents of the generalized spin-1/2 equations. The generalized equation can be written in the covariant form.

$$\left[i\gamma^\mu\partial_\mu - \frac{m_2^2 c}{m_1\hbar}\frac{(1-\gamma^5)}{2} - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \tag{1}$$

The standard representation of γ^μ matrices has been used here. If $m_1 = m_2$ we can recover the standard Dirac equation. As noted in [4b] this procedure can be viewed as the simple change of the representation of γ^μ matrices. However, this is valid unless $m_2 \neq 0$. Otherwise, entries in the transformation matrix become singular. Furthermore, one can either repeat a similar procedure (the modified Sakurai procedure) starting from the *massless* equation (4) of [1a] or put $m_2 = 0$ in eq. (1). The *massless equation* is

$$\left[i\gamma^\mu\partial_\mu - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \tag{2}$$

It is necesary to stress that the term *'massless'* is used in the sense that $p_\mu p^\mu = 0$. Then we may have different physical consequences following from (2) comparing with those which follow from the Weyl equation. The mathematical reason of such a possibility of different massless limits is that the corresponding change of representation of γ^μ matrices involves mass parameters m_1 and m_2 themselves. It is interesting to note that we can also repeat this procedure for other definitions, which gives us yet another equation in the massless limit ($m_4 \to 0$):

$$\left[i\gamma^\mu\partial_\mu - \frac{m_3 c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde{\Psi} = 0\,, \tag{3}$$

differing in the sign at the γ_5 term.

The above procedure can be generalized to *any* Lorentz group representations, i.e., to any spins. Is the physical content of the generalized $S = 1/2$ *massless* equations the same as that of the Weyl equation? Our answer is 'no'. The excellent discussion can be found in [4a,b]. First of all, the theory does *not* have chiral invariance. Those authors call the additional parameters as measures of the degree of chirality. Apart from this, Tokuoka introduced the concept of gauge transformations (not to be confuses with phase transformations) for the 4-spinor fields. He also found some strange properties of the anti-commutation relations (see Sec. 3 in [4a] and cf. [8]). And finally, the equation (2) describes *four* states, two of which answer for the positive energy $E = |\mathbf{p}|$, and two others answer for the negative energy $E = -|\mathbf{p}|$.

I just want to add the following to the discussion. The operator of the *chiral-helicity* $\hat{\eta} = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})$ (in the spinorial representation) used in [4b] does *not* commute, e.g., with the Hamiltonian of the equation (2). Do not confuse with the Dirac Hamiltonian!

$$[\mathscr{H}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}]_- = 2\frac{m_1 c}{\hbar}\frac{1-\gamma^5}{2}(\gamma \cdot \hat{\mathbf{p}})\,. \tag{4}$$

For eigenstates of the *chiral-helicity* the system of corresponding equations can be read ($\eta =\uparrow,\downarrow$)

$$i\gamma^\mu \partial_\mu \Psi_\eta - \frac{m_1 c}{\hbar}\frac{1+\gamma^5}{2}\Psi_{-\eta} = 0\,. \tag{5}$$

The conjugated eigenstates of the Hamiltonian $|\Psi_\uparrow + \Psi_\downarrow >$ and $|\Psi_\uparrow - \Psi_\downarrow >$ are connected, in fact, by γ^5 transformation $\Psi \to \gamma^5 \Psi \sim (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\Psi$ (or $m_1 \to -m_1$). However, the γ^5 transformation is related to the PT ($t \to -t$ only) transformation [4b], which, in its turn, can be interpreted as $E \to -E$, if one accepts the Stueckelberg idea about antiparticles. We associate $|\Psi_\uparrow + \Psi_\downarrow >$ with the positive-energy eigenvalue of the Hamiltonian $E = |\mathbf{p}|$ and $|\Psi_\uparrow - \Psi_\downarrow >$, with the negative-energy eigenvalue of the Hamiltonian ($E = -|\mathbf{p}|$). Thus, the free chiral-helicity massless eigenstates may oscillate to one another with the frequency $\omega = E/\hbar$ (as the massive chiral-helicity eigenstates, see [7a] for details). Moreover, a special kind of interaction which is not symmetric with respect to the chiral-helicity states (for instance, if the left chiral-helicity eigenstates interact with the matter only) may induce changes in the oscillation frequency, like in the Wolfenstein (MSW) formalism.

2 Negative energies in the Dirac equation

The general scheme for constructing the field operator has been presented in [9]. During the calculations above we had to represent $1 = \theta(p_0) + \theta(-p_0)$ in order to get positive- and negative-frequency parts. Moreover, during these calculations we did not yet assume which equation this field operator (namely, the $u-$ spinor) does satisfy, with negative- or positive- mass? In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following; see [10]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p) b_h^\dagger(p) = \sum_h u_h(-p) a_h(-p)\,. \tag{6}$$

By direct calculations, we find that

$$-mb_{(\mu)}^\dagger(p) = \sum_\lambda \Lambda_{(\mu)(\lambda)}(p) a_{(\lambda)}(-p)\,. \tag{7}$$

Hence, $\Lambda_{(\mu)(\lambda)} = -im(\boldsymbol{\sigma} \cdot \mathbf{n})_{(\mu)(\lambda)}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and

$$b^{\dagger}_{(\mu)}(p) = i\sum_{\lambda}(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{8}$$

However, other ways of thinking are possible. Unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ satisfy the accustomed "positive-energy" Dirac equation. We should then expect the same physical content. Their explicite forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. They are the following

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p^+ + m\\ -p_r\\ p^- - m\\ -p_r\end{pmatrix}, \tag{9}$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p_l\\ -p^- + m\\ -p_l\\ p^+ - m\end{pmatrix}. \tag{10}$$

$E_p = \sqrt{\mathbf{p}^2+m^2} > 0$, $p_0 = \pm E_p$, $p^{\pm} = E \pm p_z$, $p_{r,l} = p_x \pm i p_y$. Their normalization is to $(-2N^2)$. Similar formulations have been presented in Refs. [11], and [12]. The group-theoretical basis for such doubling has been given in the papers of Gelfand, Tsetlin and Sokolik [13], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called as "the Bargmann-Wightman-Wigner-type quantum field theory" in 1993). The Markov equations, of course, can be identified with equations for the Majorana-like $\lambda-$ and $\rho-$, which we presented in Ref. [7]. Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$\left[i\Gamma^\mu\partial_\mu - m\right]\Psi_{(+)}(x) = 0\,, \tag{11}$$

$$\left[i\Gamma^\mu\partial_\mu + m\right]\Psi_{(-)}(x) = 0\,. \tag{12}$$

One can also re-write the above equations into two-component forms. Thus, one obtains the Feynman-Gell-Mann [14] equations. As Markov wrote himself, he was expecting "new physics" from these equations. Barut and Ziino [12] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has a different sign in comparison with the ordinary formulation, and the so-defined charge conjugation applies to the whole system, fermion + electromagnetic field, $e \to -e$ in the covariant derivative. Superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [13, 15]) in the framework of quantum field theory. In their case the self/anti-self charge conjugate states are simultaneously the eigenstates of the chirality. Finally, I would like to mention that in general, in the Weyl basis, the $\gamma-$ matrices are *not* Hermitian, $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$. So, $\gamma^{i^\dagger} = -\gamma^i$, $i = 1,2,3$, the pseudo-Hermitian matrix. The energy-momentum operator $i\partial_\mu$ is obviously Hermitian. So, the question is whether the eigenvalues of the Dirac operator $i\gamma^\mu \partial_\mu$ (the mass, in fact) would be always real? The question of the complete system of the eigenvectors of the *non*-Hermitian operator deserve careful consideration [16]. Bogoliubov and Shirkov [9, p.55-56] used the scheme to construct a complete set of solutions of the relativistic equations, fixing the sign of $p_0 = +E_p$.

The main points of this section are: there are "negative-energy solutions" in that is previously considered as "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of spin-1/2. Next, relations to previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases. Similar conclusion can be deduced for higher-spin equations.

3 Non-commutativity in the Dirac equation

The non-commutativity [17, 18] exibits interesting peculiarities in the Dirac case. We analyzed Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [19]. We can start from

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0\,. \tag{13}$$

Obviously, the inverse operators of the Dirac operators of the positive- and negative-masses exist in the non-commutative case. We postulate the non-commutativity relations for the components of 4-momenta: $[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i$ as usual. Thus, we come to

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \tag{14}$$

However, let us apply the unitary transformation. It is known [7, 20] that one can

$$U_1(\boldsymbol{\sigma}\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|\,. \tag{15}$$

The explicit form of the U_1 matrix can be found in [19, 20].

Let us apply the second unitary transformation:

$$\mathscr{U}_2\alpha_3\mathscr{U}_2^\dagger = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\alpha_3\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{16}$$

The final equation is

$$[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral}|\boldsymbol{\theta}|]\Psi'_{(4)} = 0\,. \tag{17}$$

In the physical sense this implies the mass splitting for a Dirac particle over the non-commutative space $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive for explaining the mass creation and mass splitting for fermions.

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Quantum cosmology with k-Essence theory

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Abstract A class of k-Essence cosmological models, with a power law kinetic term, is quantised in the mini-superspace. It is shown that for a specific configuration, corresponding to a pressureless fluid, a Schrödinger-type equation is obtained with the scalar field playing the role of time. The resulting quantum scenario reveals a bounce, the classical behaviour being recovered asymptotically.

1 Introduction

One of the main problems in the canonical quantisation of the Einstein-Hilbert Lagrangian [1–4] is the absence of a clear time coordinate [5, 6]. There are many approaches to deal with this problem. One of them is to identify an internal parameter that can play the role of time, a procedure called *deparametrisation* [7]. Another one, is to introduce matter fields such that they can be identified with the time coordinate. One example of the last procedure is to introduce a fluid with internal degrees of freedom using, e.g., Schutz's variables [8, 9]. In this case, the

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quantisation of the Einstein-Hilbert action coupled to a fluid in the mini-superspace leads to a Schrödinger-like equation, where the time coordinate is related to the conjugated momentum of the fluid variables, which appears linearly in the Hamiltonian [10–13]. The connection of the fluid variables with a time coordinate through Schutz's variable has been studied extensively in the literature. One interesting result is that the initial cosmological singularity is replaced by a bounce, and classical solutions are recovered asymptotically [12, 13]. This scenario is consistent with the general belief that quantum effects must be important in the primordial universe, while our present universe is essentially classical.

Among the different proposals found in the literature to describe an accelerated phase of expansion of the universe, the k-Essence theories [14–16] have a very particular position. Conceived initially to describe the inflationary universe, the k-Essence theories have been used also to describe the present phase of accelerated expansion. This class of theories considers a non-canonical kinetic term instead of a self-interacting scalar field. In some cases, the k-essence behaviour can be recovered from an effective string action, as it happens with the DBI action [17]. In a cosmological context, one of the characteristics of these theories is that, under some hypothesis, they can reproduce a fluid dynamics at the background and perturbative levels [18, 19]. This is particularly true for the a kinetic power law expression, which can reproduce a linear relation between pressure and density $p = \omega\rho$, and the speed of sound for the adiabatic perturbations of the fluid.

In this paper we will investigate the possibility of obtaining a time variable, in a way similar to the employment of Schutz's variables, using a power law non-canonical kinetic term. We will show that this is possible in a very special circumstance, which corresponds to a pressureless fluid. We will obtain a Schrödinger type equation, which will allow us to compute the expectation value for the scale factor, which reveals a bouncing universe in the same way as it occurs using the Schutz variables.

2 A k-Essence quantum model

The general k-Essence action can be written as[1]

$$\mathscr{S} = \int dx^4 \sqrt{-g}\Big\{R - f(X) + V(\phi)\Big\}, \tag{1}$$

where $g = \det g_{\mu\nu}$, and $f(X)$ is an arbitrary function of the kinetic term $X = \phi_{;\rho}\phi^{;\rho}$ and $V(\phi)$ is a potential term. If $f(X) = X$, the usual minimally coupled system gravity-self interacting scalar field is recovered.

In what follows we will concentrate on the power law k-Essence model, for which $f(X) = \varepsilon X^n$, where n is a real number, and $\varepsilon = \pm 1$. With the introduction of ε the

[1] We use the signature $(+---)$ and the following convention for the Ricci tensor: $R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}$.

possibility of a phantom configuration is taken into account. The usual gravity-scalar field system corresponds to $n = 1$, $\varepsilon = 1$. Moreover, we will consider $V(\phi) = 0$. In this case, a cosmological fluid scenario with $p = \omega\rho$ and $\omega =$ constant is reproduced by the k-Essence model provided that

$$\omega \; = \; \frac{1}{2n-1}. \tag{2}$$

This particular k-Essence class of theories has been recently investigated in the context of static spherically symmetric configurations, revealing some very peculiar new structures [20].

Let us consider the flat, homogenous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric defined by

$$ds^2 = N(t)^2 dt^2 - a(t)^2[dx^2 + dy^2 + dz^2], \tag{3}$$

where $N(t)$ is the lapse function. With this metric, the action (1), after integrating by parts and discarding total derivatives, reduces to

$$\mathscr{S} = \int dt \left\{ \frac{6}{N}\dot{a}^2 a - \varepsilon a^3 N^{1-2n} \dot{\phi}^{2n} \right\}. \tag{4}$$

In order to have analyticity, we will consider $\dot{\phi}$ positive, but it is possible to extend the results for the whole real line. The corresponding conjugate momenta for the scale factor a and the field ϕ are

$$\pi_a = \frac{12}{N} a\dot{a} \quad , \quad \pi_\phi = -2n\varepsilon a^3 N^{1-2n} \dot{\phi}^{2n-1}. \tag{5}$$

In expressing $\dot{\phi}$ in terms of π_ϕ we must invert the relation above. For $n = 2k$, k is a natural number such that $k \neq 0$, the radicand must be positive ($\varepsilon = -1$); for $n = 2k+1$, the radicand does not need to be positive, but analyticity is lost at the origin $\pi_\phi = 0$. In spite of this, we will proceed in a general way since the configurations that interest us imply different conditions on n. The Hamiltonian reads $H \; = \; N\mathscr{H}$, where

$$\mathscr{H} \; = \; \frac{1}{24}\frac{\pi^2}{a} + (2n-1)\,(-\varepsilon a^3)^{-\frac{1}{2n-1}} \left(\frac{\pi_\phi}{2n}\right)^{\frac{2n}{2n-1}}. \tag{6}$$

If $n \to \infty$, the conjugated momentum associated to ϕ appears linearly in the Hamiltonian, so that ϕ can play the role of time.

3 The case $n \to \infty$

In the limit $n \to \infty$, the Hamiltonian takes the form

$$\mathscr{H} = \frac{1}{24}\frac{\pi_a^2}{a} + \pi_\phi. \tag{7}$$

A very important remark is that even if the Hamiltonian is well defined in the limit $n \to \infty$, the Lagrangian is not well defined. After the redefinition $\frac{\phi}{24} \to \phi$, the corresponding Schrödinger equation, with $\hbar = 1$, reads

$$-\partial_a^2\Psi - \frac{q}{a}\partial_a\Psi = a\,i\partial_\phi\Psi, \tag{8}$$

where we have introduced a factor ordering q. This is essentially the same equation found in reference [13] with the Schutz formalism. In what follows we will consider $q = 1$. In this case, it is possible to show that the effective Hamiltonian is self-adjoint [21]. Other choices for q could be used without changing in an essential way the final results.

The effective Hamiltonian represented by the terms on the left-hand side of (8) is symmetric (or, hermitian) if a non-trivial measure is introduced in computing the scalar product:

$$(\phi, \psi) = \int_0^\infty \phi^*\,\psi\,a^2\,da. \tag{9}$$

Let us consider a stationary state, such that $\Psi(a,\phi) = \Phi(a)\,e^{-iE\phi}$. Then, the Schrödinger equation (8) takes the form

$$\partial_a^2\Phi + \frac{1}{a}\partial_a\Phi + aE\Phi = 0. \tag{10}$$

It is not difficult to show, using a non-trivial measure of the scalar product, that the energy is positive, $E > 0$, which is important for the stability of the system. Changing to the variable $x = a^{\frac{3}{2}}$ and identifying $\frac{4}{9}E \to E$, we end up with Bessel's equation, with the solution

$$\Psi(a,\phi) = A(E)J_0(E\,a^{\frac{3}{2}})e^{-iE\phi}, \tag{11}$$

where $A(E)$ is a weight factor, and we have discarded the second solution of the Bessel equation, corresponding to the Neumann function, since it is divergent at the origin.

The solution (11) may lead to a non-singular cosmological scenario as in reference [13]. In fact, let us consider the wavepacket constructed with the following superposition [22]:

$$\Psi_\phi(a) = \int_0^\infty y e^{-\alpha y^2} J_0(y a^{\frac{3}{2}})dy = \frac{1}{2(\gamma + i\phi)} e^{-\frac{a^3}{4(\gamma+i\phi)}}, \tag{12}$$

where $y = \sqrt{E}$ and $\alpha = \gamma + i\phi$, with $\gamma > 0$. Now, we can calculate the expectation value for the scale factor, considering ϕ as the corresponding time variable. The expectation value is

$$\langle a\rangle_\phi = \int_0^\infty \Psi^* a \Psi a^2\, da = C(\gamma^2+\phi^2)^{\frac{1}{3}}, \tag{13}$$

where $C > 0$ is a constant. This implies a bouncing universe, with no singularity, since $\langle a\rangle_\phi \geq C\gamma^{2/3}$. Furthermore, asymptotically (that is when $\phi \to \infty$) we have $\langle a\rangle_\phi \propto \phi^{\frac{2}{3}}$.

We can easily verify that the corresponding classical cosmological scenario is recovered asymptotically. Using the FLRW, we find the differential equations (by fixing the cosmic time, such that $N = 1$):

$$\left(\frac{\dot a}{a}\right)^2 = \frac{(2n-1)}{6}\varepsilon\dot\phi^{2n} \quad , \quad \dot\phi^{2n-1} = Ka^{-3}, \tag{14}$$

where K is an integration constant. Hence, we have the following equation for the scale factor:

$$3\left(\frac{\dot a}{a}\right)^2 = \bar K a^{-\frac{6n}{2n-1}} =: \rho_\phi, \tag{15}$$

where $\bar K$ is a combination of the previous constants. A general solution can be easily obtained:

$$a \propto t^{\frac{2n-1}{3n}} \quad , \quad \phi \propto t^{\frac{n-1}{n}}. \tag{16}$$

In the limit $n \to \infty$, the solutions read

$$a \propto t^{\frac{2}{3}} \quad , \quad \phi \propto t. \tag{17}$$

The last relation confirms the previous statement that ϕ plays essentially the role of time in the limit $n \to \infty$. Moreover, in this limit, the scale factor behaves as in a dust dominated universe. We have classically the relation $a \propto \phi^{\frac{2}{3}}$, which agrees with the relation found asymptotically in the quantum model.

4 Conclusions

In this paper we have studied a quantum model in the mini-superspace from a class of k-Essence cosmology based on a power law kinetic term X^n, where X is the usual expression for the kinetic term of a scalar field. We found that the momentum for the scalar field appears linearly in the Hamiltonian in the limit $n \to \infty$. In this case, the scalar field may play the role of a time variable. The corresponding quantum scenario has been worked out, leading to a bounce universe, which recovers classical behaviour asymptotically. The case $n \to \infty$ leads, at the classical level, to a cosmological model equivalent to that obtained by a pressureless fluid matter component, with $a \propto t^{\frac{2}{3}}$. A clear identification of the scalar field as the time component is possible only in this special case. The canonical transformation allowing the identification of scalar field as a time component seems only well defined for that limit, otherwise we must face problems with fractional derivatives which may imply loosing the notion of locality. The fact that only the case corresponding to a pressureless fluid leads to a possible identification of the scalar field with a time variable evokes previous proposals that a pressureless fluid may allow recovery of the notion of a time variable in cosmology [23, 24].

It must be remarked, however, that strictly speaking, a pressureless fluid is an idealisation, since no real fluid has zero pressure exactly. In some sense, maybe such an aspect of the problem is related to the curious properties of the original k-Essence model developed here in the limit $n \to \infty$, with a well-defined Hamiltonian, but with no Lagrangian. The possible deep meaning of such a limit process remains an open problem.

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Troubles with the radiation reaction in electrodynamics

Sofiane Faci, José A. Helayel-Neto and V. H. Satheeshkumar

Abstract The dynamics of a radiating charge is one of the oldest unsettled problems in classical physics. The standard Lorentz-Abraham-Dirac (LAD) equation of motion is known to suffer from several pathologies and ambiguities. This paper briefly reviews these issues, and reports on a new model that fixes these difficulties in a natural way. This model is based on a hypothesis that there is an infinitesimal time delay between action and reaction. This can be related to Feynman's regularization scheme, leading to a quasi-local QED with a natural UV cutoff, hence without the need for renormalization as the divergences are absent. Besides leading to a pathology-free equation of motion, the new model predicts a modification of the Larmor formula that is testable with current and near future ultra-intense lasers.

1 Introduction

The problem of electromagnetic radiation reaction goes back to the end of the nineteenth century [1]. This history is long, rich and also particularly surprising given the simplicity of the problem at first sight. The standard Lorentz-invariant equation of motion of a radiating charged particle is given by the LAD equation. It is well-known that this equation is plagued by several pathologies and ambiguities. Although these have cast doubt on the foundations of classical electrodynamics, they were long considered harmless for all practical purposes. However, the recent

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advances in ultra-intense laser technology [2–4] and related sophisticated numerical simulations [5,6] have renewed interest in this problem.

The LAD equation reads

$$m\ddot{z}^{\mu} = F^{\mu}_{ext} + F^{\mu}_{rad}, \tag{1}$$

where F^{μ}_{ext} is the exterior Lorentz force, and F^{μ}_{rad} is the radiation damping force given by

$$F^{\mu}_{rad} = m\varepsilon(\dddot{z}^{\mu} + \ddot{z}^{2}\,\dot{z}^{\mu}), \tag{2}$$

with $\varepsilon = \frac{2e^2}{3m}$ and $\ddot{z}^{\mu} = \frac{d^2}{d\tau^2}z^{\mu}$ being the acceleration; $z^{\mu} = z^{\mu}(\tau)$ are the coordinates of the charge given as functions of the proper time τ. We use units $c = k = \hbar = 1$ (k is Coulomb's constant) and the spacetime is flat with signature $(+,---)$. The first term on the r.h.s. of the Eq. (2) is the so-called Schott term, and the second is the Larmor term. This is because Larmor's formula for the radiated four-momentum is given by

$$\delta P^{\mu}_{Larmor} = -m\,\varepsilon\,\ddot{z}^{2}\,\dot{z}^{\mu}. \tag{3}$$

Up to the current experimental precision, this formula correctly describes the observed radiated energy not only in everyday devices like cellphones and WiFi spots, but also in sophisticated cyclotrons and synchrotrons.

2 LAD equation: pathologies and ambiguities

In this section, we give a brief review of the two pathologies and three ambiguities of the LAD equation.

Self-acceleration or *runaway*. This pathology can be inferred from the non-relativistic limit of the LAD equation, $m\mathbf{a} = \mathbf{f} + m\varepsilon\,\dot{\mathbf{a}}$. For simplicity let us consider $\mathbf{f} = 0$; the solution reads $\mathbf{a}(t) = \mathbf{a}_o \exp(\varepsilon t)$, which is divergent for non-vanishing initial acceleration. There have been several attempts to fix this pathology, among which the most notable is certainly the Landau-Lifshitz equation [7]. This involves rewriting LAD equation (1) in a perturbative way and linking explicitly the radiation force (2) to the external forces; this is known as order reduction. Its non-relativistic limit reads $m\mathbf{a} = \mathbf{f_{ext}} + \varepsilon\ \dot{\mathbf{f}}_{\mathbf{ext}} + \textit{higher orders}$. This equation is obviously free of runaway solutions but suffers from the remaining problems of the LAD equation. Moreover, since the perturbation parameter is given by $\dot{f}/f$, the Landau-Lifshitz model is limited to slowly varying external forces. One can also mention the similar and familiar equation of Ford and O'Connell where no divergencies appear [8]. Another attempt came from Rohrlich whose solution has the peculiarity of worsening the pre-acceleration behaviour since the charge needs to know the whole future history of the external force to adapt its acceleration [9].

Pre-acceleration. The charge's acceleration always precedes the external force, $m\mathbf{a}(t-\varepsilon) \approx \mathbf{f}(t)$, leading to causality violation. There have been not many attempts at fixing this pathology. Since it is characterised by the infinitesimal time

$\varepsilon \approx 10^{-23}s$, it is believed that there could be no classical resolution. Quantum mechanics is required to go further even though it is not well suited to describe motion[1]. The first to implement this program were Moniz and Sharp who used the Heisenberg picture and standard perturbative theory [11]. The pre-acceleration pathology is avoided by introducing a cutoff that corresponds to the Compton scale, $\lambda = 137\varepsilon$ (recall that $c = 1$). This comes as no surprise since the cutoff is much bigger than the pathology typical scale. More recent developments include the work of Higuchi and Martin who take into consideration the full relativistic QED [12]. Unfortunately they recover the LAD equation and its associated pathologies in the classical limit.

Time-reversible or not. One might argue that the time-irreversible character of the LAD equation is obvious due to the presence of the Schott term, $\propto \dddot{z}^{\mu}$, indeed, every odd-order time-derivative of the position being irreversible. However, some authors believe that classical electrodynamics should be time reversible and sometimes prefer to rewrite the radiation force (2) in an integro-differential form to *hide* the Schott term [13]. Rohrlich has argued that the LAD equation is reversible provided that the retarded fields are replaced with advanced ones [14], but the radiation process, as a whole, is irreversible for nature preferring retarded instead of advanced fields [15]. Rovelli refuted the argument stating that time reversal should also interchange cause and effect [16].

Uniform acceleration. The problems with the LAD equation become evident when considering uniform acceleration. Instead of leading to trivial results, as one would expect, it raises more questions. Indeed it is not clear why there is no radiation damping and the very origin of the radiated energy is mysterious in this case [17]. In addition, this might give rise to a conflict with the Equivalence Principle which locally equates acceleration and gravitational field. A free (unbound) charge on earth would emit energy forever, which does not seem to happen. This is so troublesome that Feynman claimed there could be no radiation in this case and commented that the dependence of Larmor's formula on the acceleration (instead of its variation) *has led us astray* [18]. Since then an intense work has been devoted to this problem, see [19] and references therein. The accepted resolution, due to Boulware [20], asserts that a uniformly accelerated charge does radiate, but such a radiation cannot be detected by a comoving observer because it falls outside of his (or her) future cone.

Energy balance paradox. There is a systematic energy balance discrepancy in the LAD equation. Indeed, it is not possible to relate the work done against the radiation reaction force and the radiated energy-momentum. In other words, the Larmor formula cannot be recovered from the LAD equation. This is evident for uniform acceleration, as discussed in the previous paragraph, but is not limited to this particular case. This energy balance paradox was recently revealed in [21] where it was also shown that the widely accepted treatment based on the bound field technique cannot fix this discrepancy. The underlying reason is that the momentum defined by Schott and later by Teitelboim is not a legitimate four-momentum for being indefinite and non-conserved.

[1] It is possible to infer the equation of motion from non-relativistic QM as a limit for averaged operators using the Ehrenfest theorem but we do not know exactly how to describe the motion of radiating charges in this framework [10].

3 Time-delayed electrodynamics

In this section, we discuss a recently proposed model for the motion of a classical charge which appears to fix the above difficulties [25]. This model is based on the hypothesis of an infinitesimal time-delay between the action of an external electromagnetic field and the inertial reaction of elementary charges. The time-delay is given by $\varepsilon = 2e^2/(3m)$ which is of order $10^{-23}s$ for an electron[2]. This corresponds to $2/3$ the time that it takes light to cross the classical radius of the electron. The infinitesimal delay parameter ε should be seen as a scalar with dimension of time (or distance if multiplied by c). Hence ε is Lorentz-invariant and is thus observer-independent. Note that no particular assumptions are required with respect to the structure, shape or size of the electron. In particular the problems related to the rigid spherical electron do not apply to this model. The new equation of motion reads

$$f_\mu(\tau) - m\ddot{z}_\mu(\tau) = m\left|\delta\ddot{z}(\tau,\varepsilon)^\perp_\mu\right| = m\left|\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\,z_\mu^{(n+2)\perp}(\tau)\right|, \tag{4}$$

where $\delta\ddot{z}(\tau,\varepsilon) = \ddot{z}(\tau+\varepsilon) - \ddot{z}(\tau)$, $\left|\delta\ddot{z}(\tau,\varepsilon)^\perp_\mu\right| = s\,\delta\ddot{z}(\tau,\varepsilon)^\perp_\mu$, with $s = sign(\delta\ddot{z}(\tau,\varepsilon)^\perp_o)$. This guarantees that energy flux goes from the external force f to the kinetic sector $\ddot{z}$ when the acceleration is positive and the opposite for negative acceleration. The projector on the hyperplane $\Sigma(\tau)$ orthogonal to the charge worldline (i.e., to $\dot{z}^\mu$) at instant τ is denoted $\perp_{\mu\nu} = \eta_{\mu\nu} - \|_{\mu\nu}$ with $\|_{\mu\nu} = \dot{z}_\mu\,\dot{z}_\nu$ being the parallel projector on the worldline. This is needed for consistency since $f_\mu(\tau) \in \Sigma(\tau)$ while $\ddot{z}_\mu(\tau+\varepsilon) \in \Sigma(\tau+\varepsilon)$, the two hyperplanes being not parallel, except for inertial motion. It is important to remark that $\delta\ddot{z}(\tau,\varepsilon)$ can be equivalently replaced by $\delta f(\tau,\varepsilon) = f(\tau) - f(\tau-\varepsilon)$ in this equation (and throughout the text) provided the external field is far below the Schwinger critical limit, $E_c = \frac{m^2}{e}$ (linear electrodynamics) and the frequency under the limit ε^{-1} (electron-positron pair creation). Both limits are far above current experimental capabilities [24]. Within these limits, and up to the first-order expansion in terms of ε, equation (4) reduces to

$$m\ddot{z}_\mu(\tau) = f_\mu(\tau) - sm\varepsilon\,\dddot{z}^\perp_\mu + o(\varepsilon^2), \tag{5}$$

with now $s = sign(\dddot{z}^\perp_o)$. This is the LAD equation (1) when $\dddot{z}^\perp_o < 0$, implying $s = -1$, which corresponds for example to circular motion (cyclotron and synchrotron). For $\dddot{z}^\perp_o > 0$ the radiation force has an opposite sign in comparison with the LAD equation and this, in principle, is experimentally testable. That is, the pre-acceleration behaviour appears only when $\dddot{z}^\perp_o < 0$, and one has a post-acceleration

[2] This is comparable to the observed time delay in the photoelectric effect by atoms and molecules. Indeed, the recent advances in the so-called attosecond chronoscopy have raised fundamental questions and generated an intense theoretical and experimental activity. This was predicted by Wigner [23] and confirmed by direct observations. Time scales vary around $10^{-18}s$ for small atoms and molecules. A recent proposal has demonstrated the technical possibility of reaching precision of $10^{-21}s$ by using high harmonic x-ray pulses generated with midinfrared lasers [22]. Hence the time shift attributed to the electron will be soon within the range of experimental capabilities.

for $\dddot{z}_o^{\perp} > 0$. Hence pre-acceleration is not systematic and consequently not problematic. Note also that the time-irreversal character of equation (4) is evident since even and odd high-order terms in the expansion series do not transform equally under time reversal. As for the radiated energy-momentum, it is given by the parallel projection

$$\delta P^{\mu}_{rad} = m\,\delta\ddot{z}(\tau,\varepsilon)^{\parallel} = m\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\,z_{\mu}^{(n+2)\parallel}(\tau). \tag{6}$$

Like the equation of motion (1), this formula is clearly time-irreversible. The first term of the expansion corresponds to Larmor formula (3). The higher-order terms are new and might drastically change the behaviour of radiating charges in the case of rapidly changing external forces, as in high-frequency lasers experiments. The acceleration vector being spacelike, the Larmor term is evidently positive. The odd higher-order terms are shown to be positive in [25]. The even derivative terms have an indefinite sign and are time-reversible. However, within the validity limit of the model, the dominant term is the Larmor term and so the radiated momentum is always positive and forward oriented. In addition, performing a motion back and forth results in a null momentum coming from even terms. Furthermore, using the identity $\delta\ddot{z}(\tau,\varepsilon)^2 = (\delta\ddot{z}(\tau,\varepsilon)^{\perp})^2 + (\delta\ddot{z}(\tau,\varepsilon)^{\parallel})^2$, together with equations (4) and (6), defining the total momentum flux (between the instants τ and $\tau+\varepsilon$) as $\delta P^{\mu}_{tot}(\tau) = m\,\delta\ddot{z}(\tau,\varepsilon)$ and the internal momentum flux as $\delta P^{\mu}_{int}(\tau) = f^{\mu}(\tau) - m\ddot{z}^{\mu}(\tau)$, one obtains

$$\delta P^2_{tot} = \delta P^2_{int} + \delta P^2_{rad}. \tag{7}$$

This formula stands for energy-momentum conservation. It says that the total momentum, δP_{tot} is split into an internal flux δP_{int}, which flows between the kinetic and potential sectors, and an external flux δP_{rad}, which is dissipated. Moreover since it involves scalar quantities, the relation (7) is frame-independent.

Let us now apply the above formula to a simple and testable example related to the ultra-high intense laser experiments. In particular, we consider a nonrelativistic electron interacting with a monochromatic plane wave laser of frequency ω and intensity $I = \frac{1}{4\pi}E_o^2$, where E_o stands for the mean value of the electric field. The equation of motion is given by (5) with $s=-1$ (this is a cyclic motion) while the radiated power (6) yields

$$\delta P_{rad} = \delta P_{Larmor}[1 + \frac{1}{6}(\varepsilon\omega)^2 + o(\varepsilon\omega)^4], \tag{8}$$

where $\delta P_{Larmor} = m\varepsilon\mathbf{a}^2 = \frac{4\pi e^2}{m}I\varepsilon$ comes out of Larmor's formula (3). Hence the new formula predicts a higher amount of radiated energy. The excess radiated power depends linearly on the intensity of the laser I, and non-linearly on its frequency, ω. Consequently one can remain well below the Schwinger and ε^{-1} limits (which limit the validity of the present model) while the experimental conditions for testing the predicted deviation from Larmor's formula are guaranteed, which is more easily attained by increasing the laser frequency.

4 Summary

In this paper, we have attempted to fix many pathologies of the LAD equation describing the motion of a radiating charge. Our model is based on a hypothesis of an infinitesimal time delay between action and reaction. Accordingly, the force and acceleration vectors do not live on the same hyperplane orthogonal to the worldline. The orthogonal projection of the delayed force leads to the equation of motion, a discrete delay differential equation [26] whose expansion reduces to the LAD equation at the first-order and for cyclic motion. The radiated four-momentum is extracted from the parallel projection on the worldline of the charge, which exactly reduces to Larmor formula at the first-order. The higher-order terms are new and experimentally testable, thanks to recent advances in laser technology. One practical example we have outlined has precise and explicit predictions. Finally, we would like to mention that the time-delay ε yields a quasi-local QED exhibiting a natural UV cutoff. This might be related to Feynman's regularization scheme [27] but with no need for renormalization since no divergencies need to be cured.

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Gravitational “seesaw” and light bending in higher-derivative gravity

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Abstract Local gravitational theories with more than four derivatives have remarkable quantum properties, e.g., they are super-renormalizable and may be unitary in the Lee-Wick sense. Therefore, it is important to explore also the IR limit of these theories and identify observable signatures of the higher derivatives. In the present work we study the scattering of a photon by a classical external gravitational field in the sixth-derivative model whose propagator contains only real, simple poles. Also, we discuss the possibility of a gravitational seesaw-like mechanism, which could allow the makeup of a relatively small physical mass from the huge massive parameters of the action. If possible, this mechanism would be a way out of the Planck suppression, affecting the gravitational deflection of low energy photons. It turns out that the mechanism which actually occurs works only to shift heavier masses to the further UV region. This fact may be favourable for protecting the theory from instabilities, but makes experimental detection of higher derivatives more difficult.

1 Introduction

The idea of including higher-derivative terms in the Einstein-Hilbert action was proposed still in the early years of general relativity, and was considered more seriously during the 1960’s and 1970’s driven by quantum theoretical considerations. Indeed, the renormalization of quantum fields on curved space-time requires the introduction of curvature-squared terms [12]; also, it was shown that the fourth-derivative

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gravity is renormalizable, in opposition to the Einsteinian theory [11]. As it is widely known, this type of theory usually suffers from Ostrogradsky instabilities at the classical level and have negative-norm states when quantized; notwithstanding, in absence of a straight road to quantum gravity, the role played by higher-derivative terms should be investigated. In this spirit, it was recently shown that gravity theories with more than four derivatives are super-renormalizable [4], and may yield a unitary S-matrix in the Lee-Wick sense if all the massive poles in the propagator are complex [9]. Some other recent studies on general super-renormalizable theories can be found in Refs. [1,2,5,8].

In the present work we study the bending of light in the most simple super-renormalizable gravity theory, i.e., the sixth-derivative model described by the action

$$S = S_{\text{grav}} + \int d^4x\sqrt{-g}\,\mathscr{L}_{\text{m}}\,, \tag{1}$$

$$S_{\text{grav}} = \int d^4x\sqrt{-g}\left\{\frac{2}{\kappa^2}R + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R_{\mu\nu}^2 + \frac{A}{2}R\Box R + \frac{B}{2}R_{\mu\nu}\Box R^{\mu\nu}\right\}, \tag{2}$$

where an additional matter action was introduced. Here α, β, A and B are free parameters; the first two are dimensionless while A and B carry dimension of $(\text{mass})^{-2}$. The notation $\kappa^2/2 = 16\pi G = M_P^{-2}$ is conventional in the quantum gravity literature; here M_P is the Planck mass.

In Section 2 we discuss the deflection of light caused by a static massive body within the semi-classical framework, while in Section 3 we analyse the possibility of avoiding Planck suppression effects to this phenomenon due to a specific seesaw-like mechanism. Our conclusions are summarized in Section 4. We note that further consideration of the issues presented in this work can be found in [1,2].

Our sign convention follows from the definitions $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, $R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \cdots$ and $R_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$. Also, we set $\hbar = c = 1$.

2 Light bending in the sixth-order gravity

In the weak field regime we consider the metric to be a fluctuation around the flat-space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $|\kappa h_{\mu\nu}| \ll 1$. Then, it is possible to show that the field generated by a static point-like mass, has non-zero components given by [2]:

$$\begin{aligned} h_{00} &= \frac{M\kappa}{16\pi}\left(-\frac{1}{r} + \frac{4}{3}F_2 - \frac{1}{3}F_0\right), \\ h_{11} = h_{22} = h_{33} &= \frac{M\kappa}{16\pi}\left(-\frac{1}{r} + \frac{2}{3}F_2 + \frac{1}{3}F_0\right), \end{aligned} \tag{3}$$

where

$$F_k = \frac{m_{k+}^2}{m_{k+}^2 - m_{k-}^2}\frac{e^{-m_{k-}r}}{r} + \frac{m_{k-}^2}{m_{k-}^2 - m_{k+}^2}\frac{e^{-m_{k+}r}}{r}\,.$$

Here $k = 0, 2$ labels the spin of the particles, whose masses are defined by the positions of the poles of the propagator,

$$m_{2\pm}^2 = \frac{\beta \pm \sqrt{\beta^2 + \frac{16}{\kappa^2} B}}{2B}, \quad m_{0\pm}^2 = \frac{\sigma_1 \pm \sqrt{\sigma_1^2 - \frac{8\sigma_2}{\kappa^2}}}{2\sigma_2}, \quad \sigma_1 = 3\alpha + \beta, \; \sigma_2 = 3A + B. \tag{4}$$

As mentioned above, in this work we assume that the parameters α, β, A and B are such that $m_{k\pm} \in \mathbb{R}$ and $m_{k+} \neq m_{k-}$ (for the most general scenario see [2]). In particular, it must hold that $\beta, B < 0$. It is possible to show that m_{2+} and m_{0+} are ghost modes, while the others are healthy excitations [8].

The deflection of light due to a weak gravitational field can be evaluated within the semi-classical approach by considering the photon to be a quantum particle which interacts with the classical external field (3). At tree-level the only diagram contributing to the scattering is the one depicted in Fig. 1, which produces the vertex function

$$\begin{aligned} V_{\mu\nu}(p,p') = \frac{\kappa}{2} h^{\lambda\rho}_{\text{ext}}(\mathbf{k}) \big[&- \eta_{\mu\nu}\eta_{\lambda\rho}\, p \cdot p' + \eta_{\lambda\rho} p'_\mu p_\nu \\ &+ 2 \left(\eta_{\mu\nu} p_\lambda p'_\rho - \eta_{\nu\rho} p_\lambda p'_\mu - \eta_{\mu\lambda} p_\nu p'_\rho + \eta_{\mu\lambda}\eta_{\nu\rho}\, p \cdot p' \right) \big]. \end{aligned} \tag{5}$$

Neglecting energy exchange between the photon and gravitational field and assuming that the bending angle is small, it is possible to show that the unpolarized cross section for this process reads [1]:

$$\frac{d\sigma}{d\Omega} = 16 G^2 M^2 \left[\frac{1}{\theta^2} + \frac{E^2}{m_{2-}^2 - m_{2+}^2} \left(\frac{m_{2+}^2}{E^2\theta^2 + m_{2-}^2} - \frac{m_{2-}^2}{E^2\theta^2 + m_{2+}^2} \right) \right]^2, \tag{6}$$

where $E = E'$ is the energy of the photon and θ is the deflection angle, i.e., the angle encompassed by $\mathbf{p}$ and $\mathbf{p}'$.

From the previous expression it is possible to conclude that

i. light deflection does not depend on $m_{0\pm}$, and thus on the sectors R^2 and $R\Box R$. This happens because these sectors can be regarded as conformal transformations on the metric [2].

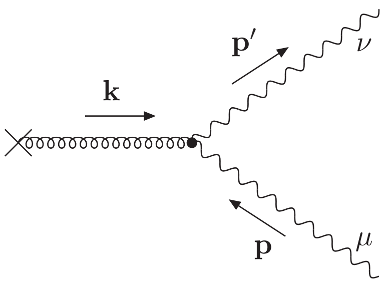

Fig. 1 Photon scattering by an external gravitational field. Here $|\mathbf{p}| \approx |\mathbf{p}'|$.

ii. Light deflects less than in general relativity. In fact, the ghost m_{2+} gives opposite-sign effect compared to the healthy massive mode m_{2-} and the graviton, and

$$m_{2-} > m_{2+} \implies \frac{m_{2-}^2 E^2}{E^2\theta^2 + m_{2+}^2} > \frac{m_{2+}^2 E^2}{E^2\theta^2 + m_{2-}^2} \implies \left(\frac{d\sigma}{d\Omega}\right)_E > \frac{d\sigma}{d\Omega} > 0, \tag{7}$$

where $(d\sigma/d\Omega)_E = (4GM/\theta^2)^2$ is the cross-section for general relativity.

iii. The scattering is dispersive – more energetic photons undergo less deflection. In fact, the second relation in (7) shows that among the dispersive interactions, the repulsive one is stronger. Therefore, since all the photons are equally attracted by the R-sector, the more energetic ones are more repelled and thus less scattered.

In order to evaluate the deflection undergone by a photon with energy E and impact parameter b, we can compare the previous expression to the classical cross-section formula $d\sigma/d\Omega = -b\,\theta^{-1}db/d\theta$, which yields

$$\begin{aligned}\frac{1}{\theta_E^2} = \frac{1}{\theta^2} &+ \frac{E^2}{(m_{2-}^2 - m_{2+}^2)^2}\left(\frac{m_{2-}^4}{E^2\theta^2 + m_{2+}^2} + \frac{m_{2+}^4}{E^2\theta^2 + m_{2-}^2}\right) \\ &+ \frac{2E^2}{m_{2-}^2 - m_{2+}^2}\left[\frac{m_{2-}^2}{m_{2+}^2}\ln\left(\frac{E^2\theta^2}{E^2\theta^2 + m_{2+}^2}\right) - \frac{m_{2+}^2}{m_{2-}^2}\ln\left(\frac{E^2\theta^2}{E^2\theta^2 + m_{2-}^2}\right)\right. \\ &- \left.\frac{m_{2-}^2 m_{2+}^2}{(m_{2-}^2 - m_{2+}^2)^2}\ln\left(\frac{E^2\theta^2 + m_{2-}^2}{E^2\theta^2 + m_{2+}^2}\right)\right],\end{aligned} \tag{8}$$

where $\theta_E = 4GM/b$ is the scattering angle in Einstein's gravity.

The effect of both massive modes is related to the ratio $E/m_{2\pm}$, in such a manner that photons with transplanckian energies would not be deflected at all,[1] while sufficiently low-energetic photons are scattered just like as in general relativity. Only at an intermediate scale of energy is there a non-trivial scattering.

In particular, it is possible to conceive a scenario in which the hierarchy between the masses is so strong, i.e., $m_{2-} \sim M_P \gg m_{2+}$, that the effect of higher derivatives could be perceived even for the energy scale currently measured, emitted by astrophysical sources. (At the same time, the influence of the healthy massive mode is negligible.) Due to the analogy with the seesaw mechanism of neutrino physics – in which large-mass parameters combine to yield physical masses with strong hierarchy, we shall call this possibility the gravitational seesaw. Under these circumstances, the equation for the deflection angle (8) reduces to

$$\frac{1}{\theta_E^2} = \frac{1}{\theta^2} + \frac{E^2}{E^2\theta^2 + m_{2+}^2} + \frac{2E^2}{m_{2+}^2}\ln\frac{E^2\theta^2}{E^2\theta^2 + m_{2+}^2}, \tag{9}$$

which is the same expression that occurs in the fourth-derivative gravity [3].

[1] This conclusion follows, for example, from the cross-section formula (6), which tends to zero as $E/m_{2\pm} \longrightarrow \infty$.

3 On the gravitational seesaw

From Eq. (4) it is straightforward to derive the seesaw condition for the masses $m_{2\pm}$:

$$16|B| \ll \kappa^2\beta^2. \tag{10}$$

If this condition is satisfied, the masses $m_{2\pm}$ can be approximated by

$$m_{2+}^2 \approx \frac{4}{\kappa^2|\beta|} \ll m_{2-}^2 \approx \frac{\beta}{B}. \tag{11}$$

As in the original neutrino seesaw mechanism one of the masses depends, roughly, on only one parameter, while the other depends on both. There is, however, a remarkable difference with respect to the neutrino case: while there it works to make the lightest mass even lighter, in gravity the effect is to shift the largest mass further to the UV region, according to Eq. (4). In fact, if the lighter mass is reduced, then the larger mass is augmented. This happens because of the parameter B which occurs in the denominator of Eq. (4); indeed, it easy to verify that m_{2+} is a decreasing function on B. Thus, the only form of reducing the lightest mass by changing the sixth-derivative parameter is to make it tend to zero (remember that $B < 0$); this procedure makes the ghost mode to approach the mass of the fourth-derivative gravity tensor excitation [11] as shown in Eq. (4). As a consequence, in order to have $m_{2+} \ll m_{2-} \sim M_P$ one must have $\beta \gg 1$.

In this spirit, now focusing our attention on the healthy mode, there are two possible ways of having m_{2-} of the order of the Planck mass: to have a small $|B|$ or a large $|\beta|$. The former is the standard choice, since it prescribes that $\beta \sim 1$ and $B \sim M_P^{-2}$ so as to have all the masses of the order of M_P. The latter relies on the seesaw mechanism, allowing one to have $|B| \gg M_P^{-2}$ and still have $m_{2-} \sim M_P$. Of course, having a large $|B|$ which yields a large mass can only be achieved through the ghost mass reduction with a parameter $\beta \gg 1$.

Therefore, the much lighter mass of the first ghost depends only on the second- and fourth-derivative terms; and the higher-order ones cannot produce an efficient seesaw mechanism working as in the case of the neutrino mass. Only a "weak seesaw" is possible, i.e., the reduction of the lightest mass by having a huge dimensionless parameter β. (See [2] for a discussion of this result in the complex poles case; and [1] for the generalization to the case of arbitrary-order local models.)

Let us now return to the deflection angle equation (9) in the presence of the "weak seesaw". We notice that the energy of the photon and the quantity m_{2+} always appear through the ratio m_{2+}/E. Thus, one can fix the scattering angle at a slightly different figure from that of general relativity – this could be, e.g., the experimental accuracy of a set of detectors, say $\theta = \theta_E - \Delta\theta$ – and solve the equation for the aforementioned ratio. For example, if we set $\theta = 1.65" = \theta_E - 0.10"$ for a photon just grazing the sun, then Eq. (9) yields

$$\frac{m_{2+}^2}{E^2} = 4.30 \times 10^{-9}, \tag{12}$$

which relates the energy of the photon and the mass of the particle necessary to cause a shift of 0.1” from general relativity’s prediction. Considering, e.g., that this is the accuracy of the measurements done in the visible spectrum during solar eclipses [7, 10], it follows the bound $m_{2+} \gtrsim 10^{-13}$ GeV. (See [6] for the comparison to other experimental bounds.) This limit is still very far from the Planck scale, and only with much higher frequencies is it expected that the massive modes could be detected.

4 Conclusions

We have described the bending of light in the sixth-derivative super-renormalizable gravity theory, in the particular case that the propagator has only real, simple poles. Among the main conclusions of this semi-classical analysis we mention the fact that light is less scattered than in general relativity, and that more energetic photons undergo less deflection. A seesaw-like mechanism which could, in principle, avoid the Planck suppression to one of the masses was also proposed. We showed, however, that differently from the neutrino, the gravitational seesaw can only work to make the largest mass even larger, on account of the reduction of the smallest one. Therefore, the only possibility of having a small physical mass (while the other is of the order of M_P) is to have a huge β. The impossibility of an efficient seesaw mechanism makes the experimental detection of higher-derivatives more difficult, but is favourable for protecting the theory from instabilities related to a much lighter ghost.

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History of particles in the early universe from contracting the Standard Model

N. A. Gromov

Abstract The high-temperature limit of the Standard Model generated by the contractions of gauge groups is discussed. Contraction parameters of gauge groups $SU(2)$ of the Electroweak Model and $SU(3)$ of Quantum Chromodynamics are taken to be identical and tending to zero when temperature increase. Properties of the elementary particles change drastically at the infinite temperature limit: all particles lose masses, all quarks are monochromatic. Electroweak interactions become long-range and are mediated by the neutral currents. Particles of different kind do not interact. It looks like some stratification with only one type of particle in each stratum. The Standard Model passes in this limit through several stages, which are distinguished by the powers of contraction parameters. The developed approach describes the evolution of the Standard Model in the early universe from the Big Bang up to the end of several nanoseconds.

1 Introduction

Modern knowledge of the particle world is concentrated in the Standard Model (SM). This theory consist of the Electroweak Model (EWM), which unified electromagnetic and weak interactions, as well as Quantum Chromodynamics (QCD), describing their strong interactions. The Standard Model is a gauge theory with $SU(3) \times SU(2) \times U(1)$ gauge group, which is the direct product of a simple groups. The operation of group contraction [4] transforms a simple group to a non-semisimple one. For a symmetric physical system the contraction of its symmetry group means a transition to some limit state. In the case of a complicated physical system the investigation of its limit states under the limit values of some of its parameters enables to better understand the system behavior.

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In this paper we investigate the high-temperature limit of the Standard Model generated by contraction of the gauge groups $SU(2)$ and $SU(3)$. Similar very high temperatures can exist in the early universe after inflation and reheating during the first stages of the Hot Big Bang [2]. At these times the elementary particles demonstrate rather unusual properties. As far as the temperature in the hot universe is connected with its age, then moving forward in time, i.e., back to high-temperature contraction, we conclude that after the universe creation elementary particles and their interactions pass a number of stages in their evolution from the Planck temperature state up to the SM state.

2 EWM at high temperature

The Electroweak Model is gauge theory with the gauge group $SU(2)\times U(1)$ acting in boson, lepton and quark sectors [5, 6]. Its Lagrangian L is taken to be invariant with respect to the action of the gauge group in the space of the fundamental representation $\mathbf{C}_2$. Leptons and quarks are described by $SU(2)$-doublets (or vectors), whereas gauge bosons are $SU(2)$-singlets (or scalars).

We consider a model where the contracted gauge group $SU(2;j)\times U(1)$. The contracted group $SU(2;j)$ is obtained [3] by **the consistent rescaling** of the fundamental representation of $SU(2)$ and the space $\mathbf{C}_2$:

$$z'(\varepsilon)=\begin{pmatrix} z_1' \\ \varepsilon z_2' \end{pmatrix}=\begin{pmatrix} \alpha & \varepsilon\beta \\ -\varepsilon\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} z_1 \\ \varepsilon z_2 \end{pmatrix}=u(\varepsilon)z(\varepsilon),$$

$$\det u(\varepsilon)=|\alpha|^2+\varepsilon^2|\beta|^2=1, \quad u(\varepsilon)u^{\dagger}(\varepsilon)=1, \tag{1}$$

when the real contraction parameter tends to zero $\varepsilon\to 0$. In the contraction scheme (1) the standard boson fields and left lepton and quark fields are transformed as follows:

$$W_\mu^{\pm}\to\varepsilon W_\mu^{\pm},\ Z_\mu\to Z_\mu,\ A_\mu\to A_\mu.$$

$$e_l\to\varepsilon e_l, \quad d_l\to\varepsilon d_l, \quad \nu_l\to\nu_l, \quad u_l\to u_l. \tag{2}$$

The next reason for inequality of the first and second doublet components is the special mechanism of spontaneous symmetry breaking, which is used to generate mass of vector bosons and other elementary particles of the model. In this mechanism one of Lagrangian ground states $\phi^{vac}=\begin{pmatrix} 0 \\ v \end{pmatrix}$ is taken as the vacuum of the model and then small field excitations $v+\chi(x)$ with respect to this vacuum are regarded. So the Higgs boson field χ and constant v are multiplied by ε. As far as the masses of all particles being proportionate to v, we obtain the following transformation rule:

$$\chi\to\varepsilon\chi, \quad v\to\varepsilon v, \quad m_p\to\varepsilon m_p, \quad p=\chi,W,Z,e,u,d. \tag{3}$$

After transformations (5),(3) the complete Lagrangian of the modified model can be written in the form

$$L(\varepsilon) = L_\infty + \varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3 + \varepsilon^4 L_4 \tag{4}$$

with the exact expressions for each L_k. The contraction parameter is the monotonous function $\varepsilon(T)$ of the temperature with the property $\varepsilon(T) \to 0$ for $T \to \infty$. Similarly, very high temperatures can have existed in the very early universe.

In the infinite temperature limit ($\varepsilon = 0$) Lagrangian (4) is equal to

$$L_\infty = -\frac{1}{4}\mathscr{Z}_{\mu\nu}^2 - \frac{1}{4}\mathscr{F}_{\mu\nu}^2 + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l +$$

$$+ e_r^\dagger i\tau_\mu \partial_\mu e_r + d_r^\dagger i\tau_\mu \partial_\mu d_r + u_r^\dagger i\tau_\mu \partial_\mu u_r + L_\infty^{int}(A_\mu, Z_\mu), \tag{5}$$

where

$$L_\infty^{int}(A_\mu, Z_\mu) = \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l + \frac{2e}{3}u_l^\dagger \tilde{\tau}_\mu A_\mu u_l +$$

$$+g'\sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l - g'\cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r -$$

$$-\frac{1}{3}g'\cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r + \frac{1}{3}g'\sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r +$$

$$+ \frac{2}{3}g'\cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3}g'\sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r. \tag{6}$$

We can conclude that the limit model includes only *massless particles*: photons A_μ and neutral bosons Z_μ, left quarks u_l and neutrinos ν_l, right electrons e_r and quarks u_r, d_r. This phenomenon has a simple physical explanation: the temperature is so high that the particle mass becomes a negligible quantity as compared to its kinetic energy. The electroweak interactions become long-range because they are mediated by the massless neutral Z-bosons and photons. Let us note that $W_\mu^\pm$-boson fields are absent in the limit Lagrangian L_∞ (5).

From the explicit form of the interaction part $L_\infty^{int}(A_\mu, Z_\mu)$ it follows that there are no interactions between particles of different kinds; for example neutrinos interact with each other by neutral currents. All other particles are charged and do only interact with particles of the same kind by massless Z_μ-bosons and photons. Particles of different kinds do not interact. It looks like some stratification of the Electroweak Model with only one kind of particles in each stratum.

3 QCD with contracted gauge group

Strong interactions of quarks are described by QCD. Like the Electroweak Model, QCD is a gauge theory based on the local color degrees of freedom [1]. The QCD gauge group is $SU(3)$, acting in three-dimensional complex space $\mathbf{C}_3$ of color quark states. The $SU(3)$ gauge bosons are called gluons. There are eight gluons in total, which are the force carrier of the theory between quarks. The QCD Lagrangian is taken in the form

$$\mathscr{L} = \sum_q \bar{q}^i (i\gamma^\mu)(D_\mu)_{ij} q^j - \frac{1}{4} \sum_{\alpha=1}^{8} F^\alpha_{\mu\nu} F^{\mu\nu\,\alpha}, \tag{7}$$

where $D_\mu q$ are covariant derivatives of the quark fields $q = u, d, s, c, b, t$,

$$D_\mu q = \left(\partial_\mu - i g_s \left(\frac{\lambda^\alpha}{2}\right) A^\alpha_\mu\right) q, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \equiv \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix} \in \mathbf{C}_3, \tag{8}$$

g_S is the strong coupling constant, $t^a = \lambda^a/2$ are generators of $SU(3)$, λ^a are Gell-Mann's matrices and $F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g_s f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu$ is the gluon stress tensor.

The contracted special unitary group $SU(3;\varepsilon)$ is defined by the action

$$q'(\varepsilon) = \begin{pmatrix} q'_1 \\ \varepsilon q'_2 \\ \varepsilon^2 q'_3 \end{pmatrix} = \begin{pmatrix} u_{11} & \varepsilon u_{12} & \varepsilon^2 u_{13} \\ \varepsilon u_{21} & u_{22} & \varepsilon u_{23} \\ \varepsilon^2 u_{31} & \varepsilon u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} q_1 \\ \varepsilon q_2 \\ \varepsilon^2 q_3 \end{pmatrix} =$$

$$= U(\varepsilon) q(\varepsilon), \quad \det U(\varepsilon) = 1, \quad U(\varepsilon) U^\dagger(\varepsilon) = 1 \tag{9}$$

on the complex space $\mathbf{C}_3(\varepsilon)$ when the contraction parameter tends to zero: $\varepsilon \to 0$. Transition from the classical group $SU(3)$ and space $\mathbf{C}_3$ to the group $SU(3;\varepsilon)$ and space $\mathbf{C}_3(\varepsilon)$ is given by the substitution

$$q_1 \to q_1, \quad q_2 \to \varepsilon q_2, \quad q_3 \to \varepsilon^2 q_3,$$

$$A^{GR}_\mu \to \varepsilon A^{GR}_\mu, \quad A^{BG}_\mu \to \varepsilon A^{BG}_\mu, \quad A^{BR}_\mu \to \varepsilon^2 A^{BR}_\mu, \tag{10}$$

the diagonal gauge fields $A^{RR}_\mu, A^{GG}_\mu, A^{BB}_\mu$ remain unchanged.

Substituting (10) in Lagrangian (7), we obtain the quark part $\mathscr{L}_q(\varepsilon)$ and gluon part $\mathscr{L}_{gl}(\varepsilon)$ in the form

$$\mathscr{L}_q(\varepsilon) = L^\infty_q + \varepsilon^2 L^{(2)}_q + \varepsilon^4 L^{(4)}_q,$$

$$\mathscr{L}_{gl}(\varepsilon) = L^\infty_{gl} + \varepsilon^2 L^{(2)}_{gl} + \varepsilon^4 L^{(4)}_{gl} + \varepsilon^6 L^{(6)}_{gl} + \varepsilon^8 L^{(8)}_{gl}. \tag{11}$$

In the infinite temperature limit $\varepsilon \to 0$, we can write out the QCD Lagrangian explicitly

$$\mathscr{L}_\infty = L^\infty_q + L^\infty_{gl} = \sum_q \left\{ i\bar{q}_R \gamma^\mu \partial_\mu q_R + \frac{g_s}{2} |q_R|^2 \gamma^\mu A^{RR}_\mu \right\}$$

$$- \frac{1}{4} \left(F^{RR}_{\mu\nu}\right)^2 - \frac{1}{4} \left(F^{GG}_{\mu\nu}\right)^2 - \frac{1}{4} F^{RR}_{\mu\nu} F^{GG}_{\mu\nu}. \tag{12}$$

From $\mathscr{L}_\infty$ we conclude that only the dynamic terms for the first color component of massless quarks survive under infinite temperature, which means that the quarks are monochromatic, and the terms also survive, which describe the interactions of these components with R-gluons.

4 Estimation of boundary temperatures

The contraction of the QCD gauge group gives us an opportunity to order in time different stages of its development, but does not make it possible to bear their absolute date. Let us try to estimate this date with the help of additional assumptions. The equality of the contraction parameters for QCD and the EWM is one of these assumptions.

Then we use the fact that the electroweak epoch starts at the temperature $T_4 = 100\,GeV \quad (1\,GeV = 10^{13}K)$ and the QCD epoch begins at $T_8 = 0,2\,GeV$. Let us denote by Δ the cutoff level for ε^k, $k = 1,2,4,6,8$, i.e., for $\varepsilon^k < \Delta$ all the terms proportionate to ε^k are negligible quantities in the Lagrangian. At last we suppose that the contraction parameter depends inversely on the temperature

$$\varepsilon(T) = \frac{A}{T}, \tag{13}$$

where A is constant.

As far as the minimal terms in the QCD Lagrangian are proportional to ε^8 and QCD is completely reconstructed at $T_8 = 0,2\,GeV$, we have the equation $\varepsilon^8(T_8) = A^8 T_8^{-8} = \Delta$ and obtain $A = T_8\Delta^{1/8} = 0,2\Delta^{1/8}\,GeV$. The minimal terms in the EWM Lagrangian are proportional to ε^4 and it is reconstructed at $T_4 = 100\,GeV$, so we have $\varepsilon^4(T_4) = A^4 T_4^{-4} = \Delta$, i.e., $T_4 = A\Delta^{-1/4} = T_8\Delta^{1/8}\Delta^{-1/4} = T_8\Delta^{-1/8}$ and we obtain the cutoff level $\Delta = (T_8 T_4^{-1})^8 = (0,2\cdot 10^{-2})^8 \approx 10^{-22}$, which is consistent with the typical energies of the Standard Model. From the equation $\varepsilon^k(T_k) = A^k T_k^{-k} = \Delta$ we obtain

$$T_k = \frac{A}{\Delta^{1/k}} = \frac{T_8\Delta^{1/8}}{\Delta^{1/k}} = T_8\Delta^{\frac{k-8}{8k}} \approx 10^{\frac{88-15k}{4k}}\,GeV. \tag{14}$$

Simple calculations give the following estimations for the boundary values of the temperature in the early universe (GeV): $T_1 = 10^{18}$, $T_2 = 10^7$, $T_3 = 10^3$, $T_4 = 10^2$, $T_6 = 1$, $T_8 = 2\cdot 10^{-1}$. The resulting estimation for the temperature at ”infinity” $T_1 \approx 10^{18}\,GeV$ is comparable with the Planck energy $\approx 10^{19}\,GeV$, at which scale the gravitation effects are important. So the developed evolution of the elementary particles does not exceed the range of the problems described by electroweak and strong interactions.

5 Conclusion

We have investigated the high-temperature limit of the SM which was obtained from the first principles of the gauge theory as contraction of its gauge group. The SM passes in this limit through several stages, which are distinguished by the powers of the contraction parameters, which gives us the opportunity to classify them in time from earlier to later. To determine the absolute date of these stages, additional assumptions were used, namely, the inverse dependence of ε on the temperature and the cutoff level Δ on ε^k. Unknown parameters are determined with the help of the QCD and EWM typical energies.

At the infinite temperature limit ($T > 10^{18}\,GeV$) all particles including vector bosons lose their masses and electroweak interactions become long-range.

From exact expressions for the respective Lagrangians at any stage in the SM evolution [3] it is possible already at the level of classical gauge fields to give some conclusions on the appearance of elementary particle masses at different stages of evolution of the universe. In particular we can conclude that half of the quarks $(\approx \varepsilon,\ 10^{18}\,GeV > T > 10^{7}\,GeV)$ restore their masses first. Then the Z-bosons, electrons and other quarks become massive $(\approx \varepsilon^2,\ 10^{7}\,GeV > T > 10^{3}\,GeV)$. Finally the Higgs boson χ and the charged $W^{\pm}$-bosons are the last to restore their masses because these are multiplied by ε^4 $(T < 10^{2}\,GeV)$. In a similar way it is possible to describe the evolution of particle interactions.

The evolution of elementary particles and their interactions in the early universe obtained with the help of contractions of gauge groups of the SM does not contradict the canonical one [2], according to which the QCD phase transitions take place later than the electroweak phase transitions. The developed evolution of the SM present the basis for a more detailed analysis of different phases in the formation of leptons and quark-gluon plasma.

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Ternary Z_6-graded algebras

Richard Kerner

Abstract We investigate the possibility of combining the usual Grassmann algebras with their ternary Z_3-graded counterpart, thus creating a more general algebra with coexisting quadratic and cubic constitutive relations. We study a particular case of algebras generated by two types of variables, ξ^a and θ^A, satisfying quadratic and cubic relations respectively, $\xi^a \, \xi^b = -\xi^b \xi^a$ and $\theta^A \theta^B \theta^C = j \; \theta^B \theta^C \theta^A$ and $\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \; \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}}$, with $j = e^{\frac{2\pi i}{3}}$. We show how one can combine the Z_2 and the Z_3 gradings of those *binary* and *ternary* algebras and merge them into a common Z_6-graded algebra.

1 Generalized $Z_2 \times Z_3$-graded ternary algebra

Ternary analogs of the Grassman algebras, satisfying cubic constitutive relations, were first proposed in ([1], [2]) and in ([3]), and developed later generalizing the supersymmetric spaces and algebras in ([4]), [5]). These papers explored the consequences of replacing the usual Z_2 grading by the Z_3 grading, more suited to be applied in the case of ternary algebras with cubic constitutive relations.

Here we consider agebras on which both gradings can operate at the same time, and how these gradings can be merged into one common Z_6 grading.

Let us suppose that we have binary and ternary skew-symmetric products defined by corresponding structure constants:

$$\xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha \tag{1}$$

$$\theta^A \theta^B \theta^C = j \; \theta^B \theta^C \theta^A \quad \text{and} \quad \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \; \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}}, \;\; \theta^A \bar{\theta}^{\dot{B}} = -j \; \bar{\theta}^{\dot{B}} \theta^A. \tag{2}$$

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The unifying ternary relation should take into account the full S_3 permutation group:

$$X^i X^m X^k + X^m X^k X^i + X^k X^i X^m + X^k X^m X^i + X^m X^i X^k + X^i X^k X^m = 0. \qquad (3)$$

It is obviously satisfied by both types of variables; the θ^A's and $\bar{\theta}^{\dot{B}}$'s by definition of the product, for which at this stage the associativity property can be yet undecided; on the contrary, the product of grassmannian ξ^α variables (1) should be associative in order to make the formula (3) applicable.

It can be added that the cubic constitutive relation (2) satisfies a simpler condition with cyclic permutations only,

$$\theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B = 0,$$

but the cubic products of grassmannian variables are invariant under even (cyclic) permutations, so that only the combination of all six permutations of $\xi^\alpha \xi^\beta \xi^\gamma$, as in (3) does vanish.

Now, if we want to merge the two algebras into a common one, we must impose a general condition (3) on the mixed cubic products. These are of two types: $\theta^A \xi^\alpha \theta^B$ and $\xi^\alpha \theta^B \xi^\beta$, with two θ's and one ξ, or with two ξ's and one θ. These identities, all like (3) should follow from *binary* constitutive relations imposed on the *associative* products between one θ and one ξ variable. Let us suppose that one has

$$\xi^\alpha \theta^B = \omega\, \theta^B \xi^\alpha \quad \text{and consequently} \quad \theta^A \xi^\beta = \omega^{-1}\, \xi^\beta \theta^A. \qquad (4)$$

A simple exercise leads to the conclusion that in order to satisfy the general condition (3), the unknown factor ω must verify the equation $\omega + \omega^{-1} + 1 = 0$, or equivalently, $\omega + \omega^2 + \omega^3 = 0$. Indeed, we have, assuming the associativity,

$$\theta^A \xi^\alpha \theta^B = \omega^{-1}\, \xi^\alpha \theta^A \theta^B = \omega\, \theta^A \theta^B \xi^\alpha,$$

$$\theta^B \xi^\alpha \theta^A = \omega^{-1}\, \xi^\alpha \theta^B \theta^A = \omega\, \theta^B \theta^A \xi^\alpha.$$

From this, by transforming all the six products so that ξ^α should appear always in front of the monomials, we get:

$$\theta^A \xi^\alpha \theta^B = \omega^{-1}\, \xi^\alpha \theta^A \theta^B, \;\; \theta^A \theta^B \xi^\alpha = \omega^{-2}\, \xi^\alpha \theta^A \theta^B,$$

$$\theta^B \xi^\alpha \theta^A = \omega^{-1}\, \xi^\alpha \theta^B \theta^A, \;\; \theta^B \theta^A \xi^\alpha = \omega^{-2}\, \xi^\alpha \theta^B \theta^A.$$

Adding up all permutations, even (cyclic) and odd alike, we get the following result:

$$\theta^A \xi^\alpha \theta^B + \xi^\alpha \theta^B \theta^A + \theta^B \theta^A \xi^\alpha + \theta^B \xi^\alpha \theta^A + \xi^\alpha \theta^A \theta^B + \theta^A \theta^B \xi^\alpha$$

$$= (1 + \omega + \omega^{-1})\, \xi^\alpha \theta^A \theta^B + (1 + \omega + \omega^{-1})\, \xi^\alpha \theta^B \theta^A. \qquad (5)$$

The expression in (5) will identically vanish if $\omega = j = e^{\frac{2\pi i}{3}}$ (or j^2, which satisfies the same relation $j + j^2 + 1 = 0$.

The second type of cubic monomials $\xi^\alpha \theta^B \xi^\beta$ satisfies the identity

$$\xi^{\alpha}\theta^{B}\xi^{\delta}+\theta^{B}\xi^{\delta}\xi^{\alpha}+\xi^{\delta}\xi^{\alpha}\theta^{B}+\xi^{\delta}\theta^{B}\xi^{\alpha}+\theta^{B}\xi^{\alpha}\xi^{\delta}+\xi^{\alpha}\xi^{\delta}\theta^{B}=0 \qquad (6)$$

no matter what the value of ω is chosen in the constitutive relation (4), the antisymmetry of the product of two ξ's suffices. Assuming $\xi^{\alpha}\xi^{\delta}=-\xi^{\delta}\xi^{\alpha}$ in the formula (6), the second term cancels the fifth term, and the third term is cancelled by the sixth one. What remains is the sum of the first and fourth terms:

$$\xi^{\alpha}\theta^{B}\xi^{\delta}+\xi^{\delta}\theta^{B}\xi^{\alpha}.$$

Now we can transform both terms so as to put the factor θ in front; this will give

$$\xi^{\alpha}\theta^{B}\xi^{\delta}+\xi^{\delta}\theta^{B}\xi^{\alpha}=\omega\theta^{B}\xi^{\alpha}\xi^{\delta}+\omega\theta^{B}\xi^{\delta}\xi^{\alpha}=0 \qquad (7)$$

because of the anti-symmetry of the product between the two ξ's. This completes the construction of the $Z_3\times Z_2$-graded extension of the Grassman algebra.

The existence of *two* cubic roots of unity j and j^2 suggests that one can extend the above algebraic construction by introducing a set of *conjugate* generators, denoted for convenience with a bar and with dotted indices, satisfying conjugate ternary constitutive relations. The unifying condition, that is the vanishing of the sum of all permutations will be automatically satisfied.

But now we have to extend this condition to the triple products of the type $\theta^{A}\bar{\theta}^{\dot{B}}\theta^{C}$ and $\bar{\theta}^{\dot{A}}\theta^{B}\bar{\theta}^{\dot{C}}$. This will be achieved if we impose the obvious condition, similar to the one proposed already for binary combinations $\xi\theta$:

$$\theta^{A}\bar{\theta}^{\dot{B}}=j\bar{\theta}^{\dot{B}}\theta^{A},\quad \bar{\theta}^{\dot{B}}\theta^{A}=j^{2}\,\theta^{A}\bar{\theta}^{\dot{B}}. \qquad (8)$$

The proof of the validity of the condition (3) for the above combinations is exactly the same as for the triple products $\xi^{\alpha}\theta^{B}\xi^{\gamma}$ and $\theta^{A}\xi^{\delta}\theta^{B}$.

We have also to impose commutation relations on the mixed products of the type (see [8]):

$$\xi^{\alpha}\bar{\theta}^{\dot{B}}\xi^{\beta}\quad\text{and}\quad\bar{\theta}^{\dot{B}}\xi^{\beta}\bar{\theta}^{\dot{C}}.$$

It is easy to see that as in the former case, it is enough to impose the commutation rule similar to the former one with θ's, namely,

$$\xi^{\alpha}\bar{\theta}^{\dot{B}}=j^{2}\,\bar{\theta}^{\dot{B}}\xi^{\alpha}. \qquad (9)$$

Although we could stop at this point the extension of our algebra, for the sake of symmetry it seems useful to introduce a new set of conjugate variables $\bar{\xi}^{\dot{\alpha}}$ of the Z_2-graded type. We shall suppose that they anti-commute, like the ξ^{β}'s, and not only between themselves, but also with their conjugates, which means that we assume

$$\bar{\xi}^{\dot{\alpha}}\bar{\xi}^{\dot{\beta}}=-\bar{\xi}^{\dot{\beta}}\bar{\xi}^{\dot{\alpha}},\quad \xi^{\alpha}\bar{\xi}^{\dot{\beta}}=-\bar{\xi}^{\dot{\beta}}\xi^{\alpha}. \qquad (10)$$

This ensures that condition (3) will be satisfied by any ternary combination of the Z_2-graded generators, including the mixed ones like

$$\bar{\xi}^{\dot{\alpha}}\xi^{\beta}\bar{\xi}^{\dot{\delta}} \quad \text{or} \quad \xi^{\beta}\bar{\xi}^{\dot{\alpha}}\xi^{\gamma}.$$

The dimensions of classical Grassmann algebras with n generators are well known: they are equal to 2^n, with subspaces spanned by the products of k generators having the dimension $C^n_k = n!/(n-k)!k!$. With $2n$ anticommuting generators ξ^α and $\bar{\xi}^{\dot{\beta}}$, we shall have the dimension of the corresponding Grassmann algebra equal to 2^{2n}.

It is also quite easy to determine the dimension of the Z_3-graded generalizations of Grassmann algebras constructed above (see, e.g., in [2], [4], [5]). The Z_3-graded algebra with N generators θ^A has the total dimension $N+N^2+(N^3-N)/3 = (N^3+N^2+2N)/3$. The conjugate algebra, with the same number of generators, has the identical dimension. However, the dimension of the extended algebra unifying both these algebras is not equal to the square of the dimension of one of them because of the extra conditions on the mixed products between the generators and their conjugates, $\theta^A\bar{\theta}^{\dot{B}} = \bar{\theta}^{\dot{B}}\theta^A$.

2 Two distinct gradings: $Z_3 \times Z_2$ versus Z_6

In the case of ternary algebras presented above, the grade 1 is attributed to the generators θ^A and the grade 2 to the conjugate generators $\bar{\theta}^{\dot{B}}$. Consequently, their products acquire the grade which is the sum of grades of the factors modulo 3. When we consider an algebra including a ternary Z_3-graded subalgebra and a binary Z_2-graded one, we can introduce a combination of the two gradings considered as a pair of two numbers, say (a,λ), with $a = 0,1,2$ representing the Z_3-grade, and $\lambda = 0,1$ representing the Z_2 grade, $\lambda = 0,1$. The first grades add up modulo 3, the second grades add up modulo 2. The six possible combined grades are then

$$(0,0), \quad (1,0), \quad (2,0), \quad (0,1), \quad (1,1) \quad \text{and} \quad (2,1). \tag{11}$$

To add up two of the combined grades amounts to adding up their first entries modulo 3, and their second entries modulo 2. Thus, we have

$$(2,1)+(1,1) = (3,2) \simeq (0,0), \quad \text{or} \quad ((2,1)+(1,0) = (3,1) \simeq (0,1), \quad \text{and so forth.}$$

It is well known that the cartesian product of two cyclic groups $Z_N \times Z_n$, N and n being two prime numbers is the cyclic group Z_{Nn} corresponding to the product of those prime numbers. This means that there is an isomorphism between the cyclic group Z_6, generated by the *sixth* primitive root of unity $q^6 = 1$, satisfying

$$q+q^2+q^3+q^4+q^5+q^6 = 0.$$

The elements of the group Z_6 represented by complex numbers multiply modulo 6, e. g. $q^4 \cdot q^5 = q^9 \simeq q^3$, etc. The six elements of Z_6 can be put in one-to-one correspondence with the pairs defining six elements of $Z_3 \times Z_2$ according to the following scheme:

$$(0,0) \simeq q^0 = 1,\ (2,1) \simeq q,\ (1,0) \simeq q^2,\ (0,1) \simeq q^3,\ (2,0) \simeq q^4,\ (1,1) \simeq q^5. \quad (12)$$

The same result can be obtained directly using the representations of Z_3 and Z_2 in the complex plane. Taken separately, each of these cyclic groups is generated by one non-trivial element, the third root of unity $j = e^{\frac{2\pi i}{3}}$ for Z_3 and $-1 = e^{\pi i}$ for Z_2. It is enough to multiply these complex numbers and take their different powers in order to get all the six elements of the cyclic group Z_6. One then easily identifies

$$-j^2 = q,\ \ j = q^2,\ \ -1 = q^3,\ \ j^2 = q^4,\ \ -j = q^5,\ \ 1 = q^6.$$

This group can be represented on the complex plane, with $q = e^{\frac{2\pi i}{6}}$, as shown on the diagram below:

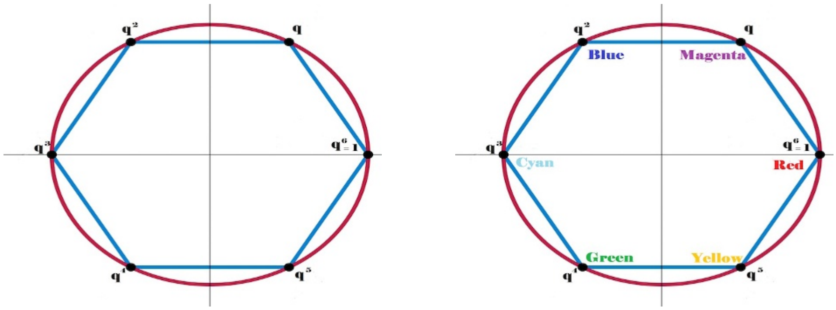

Fig. 1: Representation of the cyclic group Z_6 in the complex plane with three colors and three "anti-colors" attributed to even and odd powers of q, accordingly with colors attributed in Quantum Chromodynamics to quarks and anti-quarks.

The colors attributed to the powers of the complex generator q can be used to model the exclusion principle used in Quantum Chromodynamics, where exclusively "white" combinations of three quarks and three anti-quarks, as well as "white" quark-anti-quark pairs are declared observable. Replacing the word "white" by 0, we see that there are *two* vanishing linear combinations of *three* powers of q, and *three* pairs of powers of q that are also equal to zero. Indeed, we have

$$q^2 + q^4 + q^6 = j + j^2 + 1 = 0, \text{ and } q + q^3 + q^5 = -j^2 - 1 - j = 0, \quad (13)$$

$$\text{as well as} \quad q + q^4 = 0,\ \ q^2 + q^5 = 0,\ \ q^3 + q^6 = 0. \quad (14)$$

The Z_6-grading should unite both Z_2 and Z_3 gradings, reproducing their essential properties. Obviously, the Z_3 subgroup is formed by the elements 1, q^2 and q^4, while the Z_2 subgroup is formed by the elements 1 and $q^3 = -1$. In what follows, we shall see that the associativity imposes many restrictions which can be postponed in the case of non-associative ternary structures ([9]).

For the Z_3-graded algebra with cubic relations we attribute grade 1 to the generators θ^A, and grade 2 to their conjugates $\bar{\theta}^{\dot{B}}$. All other expressions formed by products and powers of those got the well-defined grade, the sum of the grades of factors modulo 3. In a simple Cartesian product of two algebras, a Z_3-graded with a Z_2-graded one, the generators of the latter will be given grade 1, and their products will get automatically the grade which is the sum of the grades of factors modulo 2, which means that all products and powers of generators ξ^α will acquire grade 1 or 0 according to the number and character of factors involved. The mixed products of the type $\theta^A \xi^\beta$, $\xi^\beta \theta^B \theta^C$, etc. can be given the double $Z_3 \times Z_2$ grade according to (11). According to the isomorphism defined by (12), this is equivalent to a Z_6-grading of the product algebra.

As long as the algebra is supposed to be *homogeneous* in the sense that all the constitutive relations contain exclusively terms of *one and the same type*, as in the extension of the Grassmann algebra discussed above, the supposed associativity does not impose any particular restrictions. However, this is not the case if we consider the possibility of *non-homogeneous* constitutive equations, including terms of a different nature, but with the same Z_6-grade. The grading defined by (12) suggests the possibility of extending the constitutive relations by comparing terms of the type $\theta^A \theta^B \theta^C$, whose Z_6-grade is 3, to the generators ξ^α having the same Z_6-grade. This will lead to the following constitutive relations:

$$\theta^A \theta^B \theta^C = \rho^{ABC}{}_\alpha \, \xi^\alpha \quad \text{and} \quad \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = \bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}} \, \bar{\xi}^{\dot{\alpha}} \tag{15}$$

with the coefficients (structure constants) $\rho^{ABC}{}_\alpha$ and $\bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}}$ displaying obvious symmetry properties mimicking the properties of ternary products of θ-generators with respect to cyclic permutations:

$$\rho^{ABC}{}_\alpha = j\,\rho^{BCA}{}_\alpha = j^2\,\rho^{CAB}{}_\alpha \quad \text{and} \quad \bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}} = j^2\,\bar{\rho}^{\dot{B}\dot{C}\dot{A}}{}_{\dot{\alpha}} = j\,\bar{\rho}^{\dot{C}\dot{A}\dot{B}}{}_{\dot{\alpha}}. \tag{16}$$

If all products are supposed to be associative, then we see immediately that the products between θ and ξ generators, as well as those between $\bar{\theta}$ and $\bar{\xi}$ generators must vanish identically, because of the vanishing of quartic products $\theta\theta\theta\theta = 0$ and $\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta} = 0$. This means that we must set

$$\theta^A\,\xi^\beta = 0, \quad \xi^\beta \theta^A = 0, \quad \text{as well as} \quad \bar{\theta}^{\dot{B}} \bar{\xi}^{\dot{\alpha}} = 0, \quad \bar{\xi}^{\dot{\alpha}} \bar{\theta}^{\dot{B}} = 0. \tag{17}$$

But now we want to unite the two gradings into a unique common one. Let us start by defining a ternary product of generators, not necessarily derived from an ordinary associative algebra. We shall just suppose the existence of a ternary product of generators, displaying the j-skew symmetry property:

$$\{\theta^A, \theta^B, \theta^C\} = j\{\theta^B, \theta^C, \theta^A\} = j^2\{\theta^C, \theta^A, \theta^B\}. \tag{18}$$

and similarly, for the conjugate generators,

$$\{\bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}, \bar{\theta}^{\dot{C}}\} = j^2\,\{\bar{\theta}^{\dot{B}}, \bar{\theta}^{\dot{C}}, \bar{\theta}^{\dot{A}}\} = j\,\{\bar{\theta}^{\dot{C}}, \bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}\}. \tag{19}$$

Let us attribute the Z_6-grade 1 to the generators θ^A. Then it is logical to attribute the Z_6 grade 5 to the conjugate generators $\bar{\theta}^{\dot{B}}$, so that mixed products $\theta^A \bar{\theta}^{\dot{B}}$ would be of Z_6 grade 0. Ternary products (18) are of grade 3, and ternary products of conjugate generators (19) are also of grade 3, because $5+5+5=15$, and 15 *modulo* $6=3$. But we have also $q^3=-1$, which is the generator of the Z_2-subalgebra of Z_6. Therefore we should attribute the Z_6-grade 3 to both types of anti-commuting variables, ξ^α and $\bar{\xi}^{\dot{\beta}}$, because we can write their constitutive relations using the root q as follows:

$$\xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha = q^3\, \xi^\beta \xi^\alpha, \quad \bar{\xi}^{\dot{\alpha}} \bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}} \bar{\xi}^{\dot{\alpha}}, \quad \xi^\alpha \bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}} \xi^\alpha. \tag{20}$$

On the other hand, the expressions containing products of θ with $\bar{\xi}$ and $\bar{\theta}$ with ξ are

$$\theta^A \bar{\xi}^{\dot{\alpha}} \quad \text{and} \quad \bar{\theta}^{\dot{B}} \xi^\beta$$

The first expression has the Z_6-grade $1+3=4$, and the second product has the Z_6-grade $5+3=8$ *modulo* $6=2$. Other products endowed with the same grade in our associative Z_6-graded algebra are $\bar{\theta}^{\dot{A}}\, \bar{\theta}^{\dot{B}}$ This suggests that the following non-homogeneous constitutive relations can be proposed:

$$\theta^A \bar{\xi}^{\dot{\alpha}} = f^{A\dot{\alpha}}{}_{\dot{C}\dot{D}}\, \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{D}}, \quad \text{and} \quad \bar{\theta}^{\dot{A}} \xi^\alpha = \bar{f}^{\dot{A}\alpha}{}_{CD}\, \theta^C \theta^D, \tag{21}$$

where the coefficients should display the symmetry properties contravariant to those of the generators themselves, which means that we should have

$$f^{A\dot{\alpha}}{}_{\dot{C}\dot{D}} = j^2\, f^{\dot{\alpha}A}{}_{\dot{C}\dot{D}} \quad \text{and} \quad \bar{f}^{\dot{A}\alpha}{}_{CD} = j\, \bar{f}^{\alpha\dot{A}}{}_{CD}. \tag{22}$$

More details can be found in ([6], [7]).

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The zeta function approach applied to Casimir effects in a stack of conductive planes

Nail Khusnutdinov, Rashid Kashapov and Lilia M. Woods

Abstract We consider the zeta-regularization approach for vacuum energy and calculate Casimir and Casimir-Polder effects in systems with stacked equally spaced conductive planes. In the framework of the zeta approach, we obtained explicit forms of the Casimir energy of the stack and the Van der Waals/Casimir-Polder energy of a microparticle interacting with this stack at zero as well as non-zero temperatures.

1 Zeta-regularization approach

The zeta function of an operator is a vast subject in mathematical physics (see, for example books [1–4] and review [5]). It closely connects with the electromagnetic eigenvalue problem with appropriate boundary conditions. The inverse problem, identified by M. Kac [6] as ”Can one hear the shape of a drum?”, tries to find the shape of the boundary, taking into account the spectrum of an operator. The powerful application of the zeta function is related to the Atiyah-Patodi-Singer index theorem [7], which establishes a relation between the index of an operator and some local variables (such as heat kernel coefficients). The index of an operator gives us local expressions for the topological invariants of manifolds. Topological insulators have

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a very close relation with the index theorem and different topological invariants of the manifold, and they have become a popular and interesting field of research [8].

The zeta-regularization approach in quantum field theory was first suggested by Dowker and Critchley [9] and independently by Hawking [10]. In the framework of this approach, the energy is represented as an analytical function which is renormalized using three (in $D = 3+1$) first heat kernel coefficients of the corresponding operator. This approach was applied in calculating the Casimir effect by Blau, Visser, and Wipf in Ref. [11], who have shown that the main problem is obtaining the spectrum of the underlying operator for appropriate boundary conditions in closed form. Useful modifications of the zeta-regularization approach were suggested in Ref. [12] and the subsequent developments of this approach are collected in books as well [4, 13]. In this approach the regularized energy per unit are of a system with planar symmetry reads as

$$\mathscr{E}(s) = -\hbar c \mu^{2s} \frac{\cos \pi s}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \int_0^\infty d\lambda \lambda^{1-2s} \frac{\partial}{\partial \lambda} \ln \Psi(i\lambda c), \tag{1}$$

where the relation $\Psi(\omega) = 0$ defines the spectrum of the energy.

The zeta-regularization method has broadened its applications in light of the importance of Casimir phenomena and Van der Waals/Casimir-Polder attraction (adhesion) effects at the nanoscale (see, for example, book [13] and the recent review [14]). Such fluctuation induced phenomena are also important in chemistry and biology [15], with the Gecko effect being one of the most striking examples. Graphene related systems also have unusual Casimir/Van der Waals effects which need to be considered for possible applications (see, for example, [16]). The graphene property of relevance here is the existence of a universal conductivity for a relatively large domain of frequencies. Furthermore, Van der Waals/Casimir interactions in multilayer structures are also of great interest [15]. Recent progress in this field was made by Sernelius in Ref. [17], who obtained recurrent relations for these systems. An outstanding problem for Casimir interactions in multilayered systems is an effective way to treat the number of layers explicitly. Here we demonstrate a solution to this problem using the zeta function formalism for Casimir and Casimir-Polder energies at zero and non-zero temperatures.

2 Layered system at zero temperature

Let us consider a layered system consisting of a stack of $\mathscr{N}$ conductive equally spaced parallel and infinitely thin planes with surface conductivity $\sigma(\omega)$. The $\mathscr{N}$ parallel planes are located at points $z = a, a+d, a+2d, a+3d, \dots, a+(\mathscr{N}-1)d$. The half-space $z \leq 0$ is filled by dielectric media with permeability $\varepsilon(\omega)$. To calculate the Casimir energy, we set $\varepsilon = 1$. To calculate the Casimir-Polder interaction, we take advantage of the idea developed by Lifshitz [18] relying on media rarefication. Specifically, we take the half space at $z < 0$ to be described as $\varepsilon(\omega) = 1 + 4\pi N \alpha(\omega)$, where N is the amount of atoms per unit volume and α is the polarizability of single atom in this material. In the limit of $N \to 0$, we obtain the energy $E^{(\mathscr{N})}$ per atom at a distance a:

$$E^{(\mathscr{N})}(s) = -\lim_{N\to 0}\frac{1}{N}\frac{\partial \mathscr{E}^{(\mathscr{N})}(s)}{\partial a}, \tag{2}$$

where $\mathscr{E}^{(\mathscr{N})}(s)$ is the zeta-regularized energy for configuring $\mathscr{N}$ planes and the dielectric medium.

The electromagnetic field excitations supported by a system with planar symmetry can be separated into two modes: transverse magnetic (TM) and transvesre electric (TE) modes. The respective boundary conditions are

$$\begin{aligned}
\mathsf{TM}:&\ \left[e_z'\right]_{z=a+jd}=0,\ [e_z]_{z=a+jd}=-\frac{4\pi i\sigma}{\omega}e_z',\ \left[e_z'\right]_{z=0}=0,\ [\varepsilon e_z]_{z=0}=0,\\
\mathsf{TE}:&\ [h_z]_{z=a+jd}=0,\ \left[h_z'\right]_{z=a+jd}=4\pi i\sigma\omega h_z,\ \left[h_z'\right]_{z=0}=0,\ [\mu h_z]_{z=0}=0,
\end{aligned}$$

where $[f]_z = f(z-0)-f(z+0)$ and $j=\overline{0,\mathscr{N}-1}$.

Thus, there are $2\mathscr{N}+2$ coupled equations, whose main determinant can be written on the imaginary axis $\omega = i\lambda$ in the following form:

$$\triangle_{\mathsf{TM}} = \begin{vmatrix} Z_1 & B_0 & 0 & \dots & 0 & 0\\ 0 & A_0 & B_1 & \dots & 0 & 0\\ 0 & 0 & A_1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & A_{\mathscr{N}-2} & B_{\mathscr{N}-1}\\ Z_2 & 0 & 0 & \dots & 0 & A_{\mathscr{N}-1}\end{vmatrix}.$$

This $(\mathscr{N}+1)\times(\mathscr{N}+1)$ determinant is in a block-diagonal form with elements

$$\begin{aligned}
B_l &= \begin{pmatrix} e^{(l-1)p} & -e^{-(l-1)p}\\ e^{(l-1)p} & e^{-(l-1)p}\end{pmatrix}, B_0 = \begin{pmatrix} e^{-\bar{a}p} & -e^{\bar{a}p}\\ e^{-\bar{a}p} & e^{\bar{a}p}\end{pmatrix},\ Z_1 = \begin{pmatrix} -\frac{\kappa_\varepsilon}{\kappa} & 0\\ -\varepsilon & 0\end{pmatrix},\\
A_j &= \begin{pmatrix} -e^{jp} & e^{-jp}\\ (-\frac{2\pi\sigma}{c}-1)e^{jp} & (\frac{2\pi\sigma}{c}-1)e^{-jp}\end{pmatrix}, Z_2 = \begin{pmatrix} 0 & -e^{-(\mathscr{N}-1)p}\\ 0 & e^{-(\mathscr{N}-1)p}\end{pmatrix},
\end{aligned}$$

where $l=\overline{1,\mathscr{N}-1}$, $p=d\kappa$, $\bar{a}=\frac{a}{d}$, $\kappa_\varepsilon=\sqrt{k_\perp^2+\varepsilon\mu\lambda^2}$, and $\kappa=\sqrt{k_\perp^2+\lambda^2}$.

The Casimir-Polder energy

Taking into account Eqs. (1) and (2), we obtain the following expressions for the Casimir-Polder energy (see details in Ref. [19])

$$\begin{aligned}
E_{\mathsf{TM}}^{(\mathscr{N})} &= \int_0^\infty dy\int_0^1 dx\alpha(\frac{xy}{d})\Gamma_{\mathscr{N}}\left(\frac{1}{x}\eta(\frac{xy}{d})\right)(2-x^2),\\
E_{\mathsf{TE}}^{(\mathscr{N})} &= \int_0^\infty dy\int_0^1 dx\alpha(\frac{xy}{d})\Gamma_{\mathscr{N}}\left(x\eta(\frac{xy}{d})\right)x^2.
\end{aligned}$$

The two contributions have a common function

$$\Gamma_{\mathcal{N}}(t) = -\frac{\hbar c}{2\pi d^4}\frac{y^3 t e^{-\frac{2a}{d}y}}{1+t-e^{-y}\Pi_{\mathcal{N}}^{\mathcal{N}-1}},$$
$$f = q+\sqrt{q^2-1},\ q=\cosh y + t\sinh y,$$

where function $\eta = 2\pi\sigma/c$ is the dimensionless conductivity and the arguments of α and η are xy/d. Here

$$\Pi_{\mathcal{N}}^{\mathcal{N}-1} = f\frac{1-f^{2(\mathcal{N}-1)}}{1-f^{2\mathcal{N}}} = \frac{\sinh((\mathcal{N}-1)u)}{\sinh(\mathcal{N}u)}, \tag{3}$$

where $u = \operatorname{arccosh}(q)$. The function $\Pi_{\mathcal{N}}^{\mathcal{N}-1}$ is the ratio $\frac{P_{\mathcal{N}-2}(q)}{Q_{\mathcal{N}-1}(q)}$ of two polynomials of order $\mathcal{N}-2$ and $\mathcal{N}-1$ and for an infinite number of planes it becomes irrational. Particular cases of function (3) are given in Eq. (4).

$$\Pi_1^0 = 0,\ \Pi_1^2 = \frac{1}{2q},\ \Pi_2^3 = \frac{2q}{4q^2-1}, \Pi_3^4 = \frac{4q^2-1}{4q(2q^2-1)},\ \Pi_\infty = \frac{1}{\sqrt{q^2-1}+q}. \tag{4}$$

The Casimir energy

In the same way (see details in Refs. [20, 21]), the Casimir energy $\mathcal{E}^{(N)}$ per unit area stored in the stack can be represented as

$$\mathcal{E}_{\mathsf{TM}}^{(N)} = \int_0^\infty y^2 dy \int_0^1 dx \ln \Phi_{\mathcal{N}}\left(\frac{1}{x}\eta\right),\ \mathcal{E}_{\mathsf{TE}}^{(N)} = \int_0^\infty y^2 dy \int_0^1 dx \ln \Phi_{\mathcal{N}}(x\eta),$$

where the integrand function reads as

$$\Phi_{\mathcal{N}}(t) = \frac{\hbar c}{2\pi^2 d^3}\frac{e^{-y(\mathcal{N}-1)}\Pi_1^{\mathcal{N}}}{(1+t)^{\mathcal{N}}}\left(1+t-e^{-y}\Pi_{\mathcal{N}}^{\mathcal{N}-1}\right).$$

In the framework of this approach we obtain the force, $\mathscr{A}^{(m,\mathcal{N})}$, acting on the plane m in the stack of $\mathcal{N}$ planes:

$$\mathscr{A}_{\mathsf{TM}}^{(m,\mathcal{N})} = \int_0^\infty y^2 dy \int_0^1 dx \Theta_{m,\mathcal{N}}\left(\frac{1}{x}\eta\right),\ \mathscr{A}_{\mathsf{TE}}^{(m,\mathcal{N})} = \int_0^\infty y^2 dy \int_0^1 dx \Theta_{m,\mathcal{N}}(x\eta),$$

where

$$\Theta_{m,\mathcal{N}}(t) = \frac{\hbar c}{\pi^2 d^4}\frac{e^{-y} y t^2 \Pi_{\mathcal{N}}^{\mathcal{N}-2m-1}}{1+t-e^{-y}\Pi_{\mathcal{N}}^{\mathcal{N}-1}}.$$

3 Layered system at non-zero temperature

In the case of non-zero temperature, the following replacement is made

$$\int_0^\infty g(\lambda)d\lambda \Rightarrow \xi_1 \sum_{n=0}^{\infty}{}' g(\xi_n),\ \xi_n = \frac{2\pi n k_B T}{\hbar c} = n\xi_1,$$

where the prime means that a factor $\frac{1}{2}$ multiplies the zero term ($n = 0$) and ξ_n are the Matsubara wavelengths.

The Casimir-Polder free energy

Taking into account Eq. (2), we obtain the following expressions for the Casimir-Polder free energy:

$$F_{\mathsf{TM}} = \sum_{n=0}^{\infty}{}' \alpha(\xi_n) \int_1^\infty \Psi_{n,\mathscr{N}}\left(x\eta\left(\xi_n\right)\right)\left(2x^2-1\right)dx,$$
$$F_{\mathsf{TE}} = \sum_{n=0}^{\infty}{}' \alpha(\xi_n) \int_1^\infty \Psi_{n,\mathscr{N}}\left(\frac{1}{x}\eta\left(\xi_n\right)\right)dx,$$

where

$$\Psi_{n,\mathscr{N}}(t) = -\frac{k_B T \chi^3}{a^3}\frac{n^3 t e^{-2n\chi x}}{1+t-e^{-n\tau x}\Pi_{\mathscr{N}}^{\mathscr{N}-1}},$$

and $\tau = \xi_1 d, \chi = \xi_1 a$, and $\Pi_{\mathscr{N}}^{\mathscr{N}-1} = \Pi_{\mathscr{N}}^{\mathscr{N}-1}(t, n\tau x)$.

The Casimir free energy

The temperature dependent Casimir free energy can also be obtained via the following expressions:

$$\mathscr{F}_{\mathsf{TM}} = \sum_{n=0}^{\infty}{}' \int_1^\infty \Sigma_{n,\mathscr{N}}\left(x\eta\left(\xi_n\right)\right)dx,$$
$$\mathscr{F}_{\mathsf{TE}} = \sum_{n=0}^{\infty}{}' \int_1^\infty \Sigma_{n,\mathscr{N}}\left(\frac{1}{x}\eta\left(\xi_n\right)\right)dx,$$

where

$$\Sigma_{n,\mathscr{N}}(t) = \frac{k_B T \chi^2}{2\pi a^2}\frac{n^2 e^{-2n\chi x}\Pi_1^{\mathscr{N}-1}}{(1+t)^{\mathscr{N}}}\left(1+t-e^{-n\tau x}\Pi_{\mathscr{N}}^{\mathscr{N}-1}\right),$$

and $\tau = \xi_1 d, \chi = \xi_1 a$, and $\Pi_{\mathscr{N}}^{\mathscr{N}-1} = \Pi_{\mathscr{N}}^{\mathscr{N}-1}(t, n\tau x)$.

4 Conclusion

In the paper we considered the Casimir and Casimir-Polder energies for a layered system in the framework of the zeta function approach. The temperature effects are also taken into account via the Matsubara wavelengths. The expressions obtained can be applied to arbitrary models of the layers conductivity and arbitrary microparticles.

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Rephasing invariant monomials of the Cabibbo-Kobayashi-Maskawa matrix

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Abstract We consider monomials, built from elements of the Cabibbo-Kobayashi-Maskawa matrix, that are invariant upon phase transformation of the quark fields. We obtain a general form of such monomials and demonstrate that they can be expressed as a product of a small number of *fundamental rephasing invariant monomials*. The results of this paper can lead to a simplification of the renormalization group equations and be helpful in calculations of cross sections and Feynman diagrams in the Standard Model.

1 Introduction

The Standard Model (SM) [2] of Elementary Particles, based on gauge invariant field theory with the gauge group $U(1) \times SU(2)_L \times SU(3)$, has had an impressive phenomenological success [6], leading to the discovery of all the particles that were predicted by the model. Despite this success SM has some drawbacks, like, e.g., the massive neutrino problem or relatively large number of free parameters. For these reasons the SM and its extensions are extensively studied in order to remove inconsistencies and to eliminate deficiencies in its description of elementary particles.

The generation of masses in the fermion sector of SM (quarks and leptons) arises through the Yukawa interactions with the Higgs doublet using the mechanism of spontaneous symmetry breaking. The description of Yukawa interactions requires

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13 parameters out of a total 18 parameters of the SM. A good understanding of this sector is crucial for progress in the theory of elementary particles.

Yukawa interactions are described by three complex 3×3 matrices for the up and down quarks and charged leptons. The charged lepton matrix is diagonal with matrix elements proportional to the lepton masses. The matrices of the quark Yukawa couplings are not diagonal and 18 complex matrix elements are described by only 10 phenomenological parameters: 6 quark masses and 4 parameters of the Cabibbo-Kobayashi-Maskawa matrix [4]. One of the reasons for such a reduction in the number of parameters is the freedom of choice of the phases of the quark fields (rephasing freedom); therefore all the observables related to the Cabibbo-Kobayashi-Maskawa matrix must be rephasing invariant. In this paper we discuss and obtain general monomials constructed from the matrix elements of the Cabibbo-Kobayashi-Maskawa matrix.

2 Yukawa couplings and rephasing invariants

The SM is defined by its Lagrangian. The part of the Lagrangian that corresponds to the quark Yukawa interactions has the following form:

$$y_u \overline{u}_R(\phi^+ u_L) + y_d \overline{d}_R(\phi d_L) + \text{h.c.} \tag{1}$$

Here y_u and y_d are the Yukawa couplings for up and down quarks, ϕ is the Higgs field, and $u_{L,R}$, $d_{L,R}$ are the quark fields. The y_u and y_d are 3×3 complex matrices and they are not observables in SM. The y_u and y_d can be diagonalized by biunitary transformations ($Y_i^{u,d}$ being their eigenvalues),

$$\text{diag}(Y_1^u, Y_2^u, Y_3^u) = U_R^u y_u U_L^{u\dagger}, \quad \text{diag}(Y_1^d, Y_2^d, Y_3^d) = U_R^d y_d U_L^{d\dagger} \tag{2}$$

and the quark fields are transformed by the unitary transformations $U_{L,R}^{u,d}$. Upon this transformation and after spontaneous symmetry breaking, the terms of the Yukawa Lagrangian (1) are transformed into the quark mass terms with the quark masses equal to $m_i^{u,d} = Y_i^{u,d} v/\sqrt{2}$ [1] and the charged current ceases to be diagonal; instead it is described by the Cabibbo-Kobayashi-Maskawa (CKM) V,

$$V = U_L^u U_L^{d\dagger}. \tag{3}$$

The matrices $U_{L,R}^{u,d}$ in Eq. (2) are different for the up and down quarks and, because of that, there is a rephasing freedom for the CKM matrix [1, 5]. The quark masses (or $Y_i^{u,d}$) and the elements of the CKM matrix are measured observables in the SM.

In calculating cross sections in the SM, the typical terms related to the Yukawa couplings have the following form:

[1] v is the Higgs field vacuum expectation value.

$$\mathrm{Tr}((y_u^\dagger y_u)^{l_1}(y_d^\dagger y_d)^{l_2}(y_u^\dagger y_u)^{l_3}\cdots(y_d^\dagger y_d)^{l_4}) \tag{4}$$

and consequently they are expressed by the observables built from the CKM matrix and the eigenvalues $Y_i^{u,d}$. For example, we have

$$\begin{aligned}
&\mathrm{Tr}(y_u^\dagger y_u) = \sum_{i=1}^{3}(Y_i^u)^2, \quad \mathrm{Tr}(y_u^\dagger y_u y_d^\dagger y_d) = \sum_{i,j=1}^{3}(Y_i^u Y_j^d)^2|V_{ij}|^2, \\
&\mathrm{Tr}((y_u^\dagger y_u)^2(y_d^\dagger y_d)^2 y_u^\dagger y_u y_d^\dagger y_d) = \sum_{i,j,k,l=1}^{3}((Y_i^u)^2(Y_j^d)^2 Y_k^u Y_l^d)^2 V_{ik}V_{jl}V_{il}V_{jk}.
\end{aligned} \tag{5}$$

All the terms on the right-hand side of Eqs. (5) are rephasing invariant. The eigenvalues $Y_i^{u,d}$ are clearly rephasing invariant, but more interesting are the terms built from the CKM matrix elements V_{ij}. We will discuss the general form of the rephasing invariant monomials that are built from the elements of the CKM matrix V_{ij}.

3 Rephasing invariant monomials

Let us consider the most general monomial built from the matrix elements of the CKM matrix V_{ij} and its conjugates V_{kl}^*,

$$P(m,n) = \prod_{i,j}(V_{ij})^{m_{ij}}\prod_{k,l}(V_{kl}^*)^{n_{kl}}. \tag{6}$$

Here m and n are 3×3 matrices[2]. It is easy to show the following:

Theorem 1. *The monomial in Eq. (6) is rephasing invariant if the matrices m and n fulfill the following conditions:*

$$\sum_{j=1}^{3} m_{ij} = \sum_{j=1}^{3} n_{ij}, \quad \sum_{i=1}^{3} m_{ij} = \sum_{i=1}^{3} n_{ij}, \;\; i,j = 1,2,3. \tag{7}$$

We thus see that the general rephasing invariant monomials are parameterised by two 3×3 matrices. For example, the matrices m and n for monomial $V_{11}^2 V_{22} V_{11}^* V_{12}^* V_{21}^*$ are equal:

$$m = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

[2] We will assume the matrix elements of m and n are positive integers, but this is not necessary for the rephasing invariance.

The monomial in Eq. (8) is equal to

$$V_{11}^2 V_{22} V_{11}^* V_{12}^* V_{21}^* = |V_{11}|^2 V_{11} V_{22} V_{12}^* V_{21}^*, \tag{9}$$

and we see that it factorizes into $|V_{11}|^2$ and the remaining part. The term $|V_{11}|^2$ does not carry any information about the phases of the CKM matrix elements and is clearly rephasing invariant; for this reason we consider rephasing invariants monomials that do not contain terms of type $|V_{ij}|^2$ and introduce the notion of the *pure rephasing invariant monomials*.

Definition 1. The rephasing invariant monomial of the CKM matrix which cannot be factored out into the product of the absolute values of the elements of the CKM matrix and other invariant is called the *Pure Rephasing Invariant Monomial* (PRIM).

General rephasing invariant monomials in Eq. (1) were parameterised by two matrices m and n and PRIMs can also be parameterised by two matrices, but there exists parameterisation by one 3×3 matrix p with elements which are positive and negative and the sum of the elements of each row and column is equal to zero:

$$p \to \Pi_{p_{ij}>0} (V_{ij})^{p_{ij}} \cdot \Pi_{p_{kl}<0} (V_{kl}^*)^{-p_{kl}}. \tag{10}$$

One can show that the monomial defined in Eq. (10) is indeed a pure rephasing invariant monomial.

Following the definition of PRIMs it is easy to see that there is an infinite number of such monomials. However, it can be demonstrated that any PRIM can be expressed by a finite number of *Fundamental Rephasing Invariant Monomials*, which are defined as follows.

Definition 2. The *Fundamental Rephasing Invariant Monomial* (FRIM) is such a pure rephasing invariant monomial that is the product of 4 or 6 CKM matrix elements and its complex conjugates.

There are 18 FRIMs of 4-th order,

$$
\begin{aligned}
&J_1 = V_{11}V_{22}V_{12}^*V_{21}^*, \quad J_5 = V_{11}V_{33}V_{13}^*V_{31}^*,\\
&J_2 = V_{11}V_{23}V_{13}^*V_{21}^*, \quad J_6 = V_{12}V_{33}V_{13}^*V_{32}^*,\\
&J_3 = V_{12}V_{23}V_{13}^*V_{22}^*, \quad J_7 = V_{21}V_{32}V_{22}^*V_{31}^*,\\
&J_4 = V_{11}V_{32}V_{12}^*V_{31}^*, \quad J_8 = V_{21}V_{33}V_{23}^*V_{31}^*,\\
&J_9 = V_{22}V_{33}V_{23}^*V_{32}^*\\
&J_{9+i} = (J_i)^*, \quad i = 1,\ldots,9.
\end{aligned} \tag{11}
$$

and 12 FRIMs of 6-th order,

$$
\begin{aligned}
&I_1 = V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^*, \quad I_4 = V_{11}V_{23}V_{32}V_{13}^*V_{22}^*V_{31}^*,\\
&I_2 = V_{11}V_{22}V_{33}V_{12}^*V_{23}^*V_{31}^*, \quad I_5 = V_{12}V_{23}V_{31}V_{13}^*V_{21}^*V_{32}^*,\\
&I_3 = V_{11}V_{23}V_{32}V_{12}^*V_{21}^*V_{33}^*, \quad I_6 = V_{12}V_{21}V_{33}V_{13}^*V_{22}^*V_{31}^*\\
&I_{6+i} = (I_i)^*, \quad i = 1,\ldots,6.
\end{aligned} \tag{12}
$$

4 General form of the rephasing invariant monomials

The notions introduced of PRIMs and FRIMs allow us to demonstrate the following two theorems:

Theorem 2. *Any pure rephasing invariant monomial for 3 generations can be expressed in a unique way as the product of positive powers of at most 4 fundamental rephasing invariants monomials. Not more than one of these invariants can be from the set* (11) *and the remaining are from the set* (10).

Theorem 3 (Main Theorem for the Rephasing Invariants). *Any rephasing invariant monomial of the CKM matrix for 3 generations can be expressed in a unique way as the product of no more than 5 factors: 4 fundamental rephasing invariants taken to positive powers and the product of the squares of the absolute values of the CKM matrix elements also taken to positive powers. Not more than one fundamental invariant can be from the set* (11).

The proof of these theorems is algebraic [7]. One first demonstrates Theorem 2, using the parameterisation given in Eq. (10) and show that only 7 types of matrix p are possible. Next, for each type one shows that Theorem 2 is fulfilled. Theorem 3 follows from Theorem 2, because a general rephasing invariant monomial is equal to a pure rephasing invariant monomial multiplied by the product of the squares of the absolute values of elements of the CKM matrix.

Until now we were assuming only rephasing invariance, but the CKM matrix is unitary, and this fact has important consequences for properties of the fundamental rephasing invariant monomials. For the FRIMs of the 4-th order given in Eq. (10), one can show that the imaginary part of all J_i's are related to Jarlskog invariant [3]:

$$|\operatorname{Im}(V_{ik}V_{jl}V_{il}^*V_{jk}^*)| = J. \tag{13}$$

One can also show that the 6-th order FRIMs can be expressed by the 4-th order FRIMs multiplied by the squares of the absolute vales of the CKM matrix elements, e.g.,

$$V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^* = -|V_{22}|^2V_{12}V_{33}V_{13}^*V_{32}^* - |V_{13}|^2V_{22}V_{33}V_{23}^*V_{32}^*. \tag{14}$$

From Eqs. (12) and (13) it follows that the imaginary part of any rephasing invariant monomial is proportional to the Jarlskog invariant J (12) or is equal to 0.

5 Conclusions

We have analyzed and classified the rephasing invariant monomials. The most important results are given in Theorems 2 and 3. This result is mathematically interesting, and physically it can simplify calculations of the cross sections and Feynman diagrams containing loops in the Standard Model by expressing the final result in terms of a relatively small number of rephasing invariant monomials. The results of this paper may also lead to significant simplifications of the renormalization group equations in the Standard Model.

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The Schrödinger equation in rotating frames by using the stochastic variational method

Tomoi Koide, Kazuo Tsushima and Takeshi Kodama

Abstract We give a pedagogical introduction of the stochastic variational method by considering the quantization of a non-inertial particle system. We show that the effects of fictitious forces are represented in the forms of vector fields which behave analogously to gauge fields in the electromagnetic interaction. We further discuss that the operator expressions for observables can be defined by applying the stochastic Noether theorem.

1 Introduction

The variational approach conceptually plays a fundamental role in elucidating the structure of classical mechanics, clarifying the origin of dynamics and the relation between symmetries and conservation laws. On the other hand, its operations in classical and quantum systems lack coherence. In fact, in classical mechanics the Lagrangian is usually given by $T - V$, where T and V are kinetic and potential terms, respectively, but in quantum mechanics the Lagrangian which is needed to derive Schrödinger's equation does not have such structure. That is, any clear and direct correspondence between classical and quantum mechanics does not seem to exist in the variational point of view.

However, if we extend the idea of the variational principle to the stochastic variable, it can describe classical and quantum behaviors in a unified way. This

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method is called the stochastic variational method (SVM), and was first proposed by Yasue [1–5] in order to reformulate Nelson's stochastic quantization [6]. This framework is, however, based on special techniques attributed to stochastic calculus, which is not familiar to physicists.

In this paper, we introduce this method by applying it to the quantization of a non-inertial particle system, which is still controversial. The appearance of the nontrivial interference effect of wave functions on a rotating non-inertial frame was experimentally observed in 1979 [7]. Later Sakurai pointed out that such an effect can be understood in thinking of the similarity between the Coriolis force and the Lorentz force [8]. So far, there are various approaches to derive the Schrödinger equation in a non-inertial frame [9–13].

2 Classical equations in non-inertial frames

Let us introduce a non-inertial frame in which the position is denoted by $\mathbf{q}$. Expressing the position in an inertial frame by $\mathbf{r}$. The transformation of these vectors is defined by

$$\mathbf{q} = \mathbf{R}(t)\,\mathbf{r} + \mathbf{c}(t), \tag{1}$$

where $\mathbf{c}(t)$ is a time-dependent translation, and $\mathbf{R}(t)$ is a general 3×3 rotation matrix satisfying $\mathbf{R}^T(t)\,\mathbf{R}(t) = 1$. Both of $\mathbf{r}$ and $\mathbf{q}$ are given by the Cartesian coordinate.

We usually consider a one particle system in the inertial frame. Applying the coordinate transformation (1), the same system observed in the non-inertial frame is characterized by the following Lagrangian:

$$L = \frac{M}{2}(\dot{\mathbf{q}} + \mathbf{A}(\mathbf{q},t) + \mathbf{B}(t))^2 - V(\mathbf{q}), \tag{2}$$

where V is the potential and

$$\mathbf{A}(\mathbf{q},t) = \mathbf{R}\,\dot{\mathbf{R}}^T(\mathbf{q} - \mathbf{c}), \quad \mathbf{B}(t) = -\dot{\mathbf{c}} \tag{3}$$

are vector fields we have introduced. The equations of motion obtained from this Lagrangian are given by

$$\mathbf{p} = M(\dot{\mathbf{q}} + \mathbf{A}(\mathbf{q},t) + \mathbf{B}(t)), \tag{4}$$

$$\partial_t \mathbf{p}^i = (\mathbf{R}\dot{\mathbf{R}}^T)^{ji}\mathbf{p}^j - \partial_i V(\mathbf{q}). \tag{5}$$

3 The stochastic variational method

The discussion in this section follows the pedagogical introduction of SVM given by Ref. [14]. For a review on SVM with an alternative quantization scheme, see Ref. [15].

In the variational principle for stochastic variables, a particle trajectory is no longer smooth and is seen as given by a zig-zag path in general. Consequently, the evolution of a particle trajectory is defined by the following forward stochastic differential equation (SDE),

$$d\mathbf{q}(t) = \left(\frac{\mathbf{p}(\mathbf{q}(t),t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t) \right) dt + \sqrt{2\nu} d\mathbf{W}_t \quad (dt > 0). \tag{6}$$

Here $\mathbf{p}(\mathbf{x},t)$ is an unknown field determined by the stochastic variation. Note that in what follows $\mathbf{x}$ is used to denote the spatial position in a non-inertial frame. The last term in Eq. (6) is the origin of the zig-zag motion and is called the noise term. The parameter ν characterizes the strength of this noise term. The property of $\mathbf{W}_t$ is given by the standard Wiener process, which is characterized by the following correlation properties:

$$E[d\mathbf{W}_t] = 0, \quad E[(dW_t^i)(dW_t^j)] = |dt|\delta^{ij}, \quad (i,j = x,y,z), \tag{7}$$

$$E[W_t^i dW_{t'}^j] = 0 \ \text{ for } \ (t \le t'), \tag{8}$$

where $E[\ \ ]$ indicates the average of stochastic events.

The probabilistic nature of the particle distribution described by Eq. (6) is easily characterized by introducing the probability distribution defined by $\rho(\mathbf{q},t) = \int d^3\mathbf{q}_i \ \rho_I(\mathbf{q}_i) E[\delta^{(3)}(\mathbf{q} - \mathbf{q}(t))]$, where $\mathbf{q}(t)$ (more exactly $\mathbf{q}(t;\mathbf{q}_i)$) is the solution of Eq. (6) and $\rho_I(\mathbf{q}_i)$ is the initial particle distribution at an initial time t_i. As is well-known, the evolution equation of $\rho(\mathbf{q},t)$ is derived from the SDE (6) and is called the Fokker-Planck equation,

$$\partial_t \rho(\mathbf{x},t) = \nabla \cdot \left\{ - \left(\frac{\mathbf{p}(\mathbf{x},t)}{M} - \mathbf{A}(\mathbf{x},t) - \mathbf{B}(t) \right) + \nu \nabla \right\} \rho(\mathbf{x},t). \tag{9}$$

If the probability distribution evolves from $\rho_I(\mathbf{q})$ to $\rho_F(\mathbf{q}) \equiv \rho(\mathbf{q}(t_f),t_f)$ at a final time t_f following Eq. (9), the corresponding time-reversed process should describe the evolution from ρ_F to ρ_I. Suppose that this process is described by the backward SDE,

$$d\mathbf{q}(t) = \left(\frac{\tilde{\mathbf{p}}(\mathbf{q}(t),t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t) \right) dt + \sqrt{2\nu} d\mathbf{W}_t, \quad (dt < 0). \tag{10}$$

To reproduce Eq. (9) from the backward SDE, we find that the following consistency condition should be satisfied, $\mathbf{p}(\mathbf{x},t) = \tilde{\mathbf{p}}(\mathbf{x},t) + 2\nu\nabla \ln \rho(\mathbf{x},t)$.

We should stress that the usual definition of the particle velocity is not applicable, because $d\hat{\mathbf{r}}/dt$ is not well defined in the vanishing limit of dt due to the singular behavior of $\mathbf{W}_t$. The possible time differential in such a case was studied by Nelson [6] and it is known that there are two possibilities: One is the mean forward derivative

$$D\mathbf{q}(t) = \lim_{dt \to 0+} E\left[\frac{\mathbf{q}(t+dt) - \mathbf{q}(t)}{dt} \Big| \mathscr{P}_t \right], \tag{11}$$

and the other is the mean backward derivative,

$$\tilde{D}\mathbf{q}(t) = \lim_{dt\to 0-} E\left[\frac{\mathbf{q}(t+dt)-\mathbf{q}(t)}{dt}\Big|\mathscr{F}_t\right]. \tag{12}$$

These expectations are conditional averages, where $\mathscr{P}_t$ (resp. $\mathscr{F}_t$) indicates fixing the values of $\mathbf{r}(t')$ for $t' \le t \ \ (resp.\ t' \ge t)$. For the σ-algebra of all measurable events of $\mathbf{r}(t)$, $\{\mathscr{P}_t\}$ and $\{\mathscr{F}_t\}$ represent, respectively, increasing and decreasing families of sub-σ-algebras. Using these derivatives in Eqs. (6) and (10), we obtain, respectively,

$$D\mathbf{q}(t) = \frac{\mathbf{p}(\mathbf{q},t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t), \quad \tilde{D}\mathbf{q}(t) = \frac{\tilde{\mathbf{p}}(\mathbf{q},t)}{M} - \mathbf{A}(\mathbf{q},t) - \mathbf{B}(t). \tag{13}$$

4 Quantization in non-inertial frames

Let us apply the stochastic variation to the system given by the Lagrangian (2). Then the particle trajectory in Eq. (2) should be replaced by the stochastic one, as was discussed in the previous section. Due to the existence of two different time-derivatives D and $\tilde{D}$, there is an ambiguity when replacing the kinetic term. In this work, we adopt the following replacement,

$$L(\mathbf{q}, D\mathbf{q}, \tilde{D}\mathbf{q}) = \frac{m}{2}\left[\frac{(D\mathbf{q}(t)+\mathbf{A}+\mathbf{B})^2 + (\tilde{D}\mathbf{q}(t)+\mathbf{A}+\mathbf{B})^2}{2}\right] - V(\mathbf{q}(t)). \tag{14}$$

See Ref. [16] for a more precise discussion of this replacement.

The stochastic variation of the particle Lagrangian leads to the stochastic Euler-Lagrange equation

$$\tilde{D}\frac{\partial L}{\partial (D\mathbf{q}(t))} + D\frac{\partial L}{\partial (\tilde{D}\mathbf{q}(t))} - \frac{\partial L}{\partial \mathbf{q}(t)}\bigg|_{\mathbf{q}(t)=\mathbf{x}} = 0. \tag{15}$$

Here $\mathbf{q}(t)$ is replaced by the position parameter $\mathbf{x}$ at the last step of the calculation. Substituting Eq. (14), we obtain

$$\left(\partial_t + \left(\frac{\mathbf{p}_m}{M} - \mathbf{A} - \mathbf{B}\right)\cdot\nabla\right)\mathbf{p}_m - 2M\nu^2\nabla\rho^{-1/2}\Delta\sqrt{\rho} = \mathbf{p}_m\cdot\nabla_i\mathbf{A} - \nabla_i V, \tag{16}$$

where $\mathbf{p}_m = (\mathbf{p}+\tilde{\mathbf{p}})/2$.

The result of this variation can be re-expressed in the form of the Schrödinger equation by introducing the wave function defined by $\Psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\theta(\mathbf{x},t)}$. Here $\rho(\mathbf{x},t)$ is the probability distribution introduced above Eq. (9), and the phase $\theta(\mathbf{x},t)$ is defined by $\mathbf{p}_m = 2M\nu\nabla\theta(\mathbf{x},t)$. Then we find that the evolution equation of the wave function is given by the following Schrödinger equation

$$i\hbar\partial_t\Psi(\mathbf{x},t) = \left[\frac{1}{2M}\Big(-i\hbar\nabla - M\big(\mathbf{A}(\mathbf{x},t)+\mathbf{B}(t)\big)\Big)^2 - \frac{M}{2}\Big(\mathbf{A}(\mathbf{x},t)+\mathbf{B}(t)\Big)^2 + V(\mathbf{x})\right]\Psi(\mathbf{x},t). \tag{17}$$

Here we choose $\nu = \hbar/(2M)$. One can see that the effect of the non-inertial forces appears in the vector fields $\mathbf{A}(\mathbf{x},t)$ and $\mathbf{B}(t)$ which behave like the gauge field in the electromagnetic interaction.

5 Observables

The dynamics described by the above Schrödinger equation satisfies Eherenfest's theorem. In fact, the time evolution of the expectation value of the operator $-\hbar\nabla$ is given by

$$\partial_t\langle -i\hbar\partial_i\rangle = \langle(\mathbf{R}\dot{\mathbf{R}}^T)^{ji}(-i\hbar\partial_j)\rangle - \langle\partial_i V\rangle. \tag{18}$$

One can see that if we can interpret $\hat{p} = -i\hbar\nabla$, the above equation corresponds to Eq. (5).

However, to be precise, it is non-trivial as to whether we can interpret $-\hbar\nabla$ as the momentum operator even in the non-inertial frame. In SVM, the operator representations of observables are defined through the conservation laws obtained from the stochastic Lagrangian (14).

For the sake of simplicity, let us consider the rotation around the z-axis, where

$$\mathbf{R}(t) = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) & 0 \\ -\sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{c}(t) = 0. \tag{19}$$

This non-inertial system still holds the invariance for the rotation if $V(\mathbf{x}) = V(|\mathbf{x}|)$. Then from the invariance of the stochastic action, we can obtain the angular momentum conservation of the present non-inertial system. For the infinitesimal rotation, $\mathbf{q}(t)$ is transformed as $\mathbf{q}(t) \longrightarrow \mathbf{q}(t) + \mathbf{A}(\phi(t))$, where $\mathbf{A}(\phi(t)) = \delta\dot{\phi}(-y,x,0)$.

On the other hand, if the action is invariant for the above rotation, we can show that the following quantity is conserved by applying the stochastic Noether theorem [17,18],

$$Q = E\left[\mathbf{q}(t)\times\left(\frac{\partial L}{\partial(D\mathbf{q}(t))} + \frac{\partial L}{\partial(\tilde{D}\mathbf{q}(t))}\right)\right]. \tag{20}$$

Here $\times$ denotes the vector product. Substituting the result of the stochastic variation, the above equation is now expressed as

$$Q = \int d^3\mathbf{x}\ \Psi(\mathbf{x},t)L_z\Psi(\mathbf{x},t), \tag{21}$$

where the angular momentum operator is introduced, $L_z = -i\hbar(x\partial_y - y\partial_x)$. This result means that $-i\hbar\nabla$ can be interpreted as the momentum operator even in the non-inertial system.

6 Concluding remarks

We gave a brief summary of the stochastic variational method and showed that this is applicable to the quantization of the non-inertial particle system. Then we found that the Eherenfest's theorem is still satisfied even for the Schrödinger equation in the non-inertial frame, and thus the result is consistent with those in Refs. [9–11], but different from Refs. [12, 13].

The advantage of the present approach compared to Refs. [9–11] is that the operator representations for observables are systematically obtained by applying the stochastic Noether theorem.

Although the framework of SVM was originally proposed to reformulate Nelson's stochastic quantization, its applicability is not restricted to quantization. The derivation of the classical dissipative dynamics can be cast into the form of SVM: the Navier-Stokes-Fourier equation is obtained by employing the stochastic variation to the classical action of the Euler (ideal fluid) equation. See Refs. [18, 19] for details.

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Perturbation of the Malliavin Calculus of Bismut type for a large order on a Lie group

Rémi Léandre

Abstract Roughly speaking, in the qualitative theory of an elliptic operator, only the main term (which is given by its principal symbol) plays a role. We show that this statement is true for the Malliavin Calculus of Bismut type for a large order on a Lie group.

1 Introduction

Let us consider an elliptic operator of order r on a compact manifold M. If we perturb it by a strictly lower order operator L_p, by the theory of a pseudo-differential operator (which is given by the role of the principal symbol of an elliptic operator) the result is that the qualitative behaviour (hypoellipticity) is the same as that of $L + L_p$. See [3-6] for various textbooks in analysis about this problem.

Recently, we introduced an elliptic operator of order 4 $L_0 = \sum e_i^4$, where e_i is an orthonormal basis of the Lie algebra of a compact Lie group G of dimension m with generic element g. We have established the Malliavin Calculus [11] of Bismut type for L [1]. For a semigroup P_t on a manifold M, a natural question is to know if the semigroup has a heat kernel $P_t f(x) = \int_M f(y) p_t(x,y) dy$ for any test function f. There are 3 approaches for this:

- Harmonic analysis which uses functional inequalities,
- Micro-local analysis which uses Fourier transform as a tool,
- The Malliavin Calculus, which is valid only for Markov semigroups represented by stochastic processes.

Bismut's approach to the Malliavin Calculus allows for an integration by parts using a cascade system of stochastic differential equations, it is therefore a system

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of Markovian semigroup in cascade. For non-Markovian semigroups (i.e., with no associated stochastic process), it is possible to adapt this system of stochastic differential equations in cascade in order to get the integration by parts of Malliavin type for a system of enlarged (non-Markovian) semigroups, in which the Malliavin matrix plays a key role. We begin by a simple case.

The goal of this paper is to define an abstract version of the Malliavin Calculus that would be valid for a large class of elliptic operators, where a system of enlarged (non-Markovian) semigroups would play a big role. If we consider an elliptic operator and if we perturb it by a lower order operator, the behaviour of the two semigroups are more or less the same. We would like to extend this strategy for the Malliavin Calculus.

We consider a polynomial Q of degree strictly smaller than 4 in the vector fields e_i with constant components. For the perturbed semigroup and the non-perturbed semigroup, abstract formulas are the same and the formulas for the enlarged semigroups are the same. We consider the total operator

$$L = L_0 + Q. \tag{1}$$

We also aim to prove the following theorem, using a small interpretation of [9][1].

Theorem 1. *The semigroup generated by L has a heat kernel.*

Remark It should be possible to establish the same theorem using Malliavin Calculus techniques for the operator

$$L = \sum e_i^{2k} + Q, \tag{2}$$

where Q is a polynomial with constant components in the e_i of degree strictly smaller than the integer $2k$.

Remark It should be possible in this situation to adapt the result of [10] getting rough logarithmic estimates of the density of the generator where we consider a small parameter in its definition. Let us stress that we have examined the stochastic analysis for a non-Markovian generator in [8].

2 The theorem of Malliavin for a perturbated operator of order four

We consider the elliptic operator on $G \times R$

$$Q + \sum_i^m e_i^4 + \sum h_{i,t} e_i \frac{\partial}{\partial u} + \frac{\partial^4}{\partial u^4} = \tilde{L}_t^h. \tag{3}$$

[1] Note that we have translated the Malliavin Calculus for diffusions into the language of semigroup theory, [7].

It generates by the elliptic theory a semigroup $\tilde{P}_t^h$ on $C_b(G \times R)$, the space of a bounded continuous function on $G \times R$ endowed with the uniform norm.

Theorem 2. (Elementary integration by parts formula).*We have if f is smooth with compact support*

$$\int_0^t P_{t-s} \sum h_{s,i} e_i P_s[f] ds = \tilde{P}_t^h[uf](.,0). \tag{4}$$

Proof. It is the same proof as that of Theorem 3 of [9]

Remark It is the same abstract integration by parts formula as in the above formula if we remove Q in (3) and it is the same enlargement mechanism.

Let $V = G \times M_m$. M_m is the space of symmetric matrices on $Lie\,G$ ($(x,v) \in V$), the Lie algebra of G. Here the Lie algebra of G is considered as the tangent space of G at the identity, unlike the definition of L where the elements of the Lie algebra of G are considered as vector fields. v is called the Malliavin matrix. We consider

$$\hat{X}_0 = (0, \sum < g^{-1} e_i, . >^2). \tag{5}$$

We consider the Malliavin generator

$$\hat{L} = \sum e_i^4 - \hat{X}_0 + Q. \tag{6}$$

Theorem 3. *$\hat{L}$ spans a semigroup $\hat{P}_t$ called the Malliavin semigroup on $C_b(V)$.*

Proof. It is the same proof of Theorem 4 of [9] since Q is a polynomial with constant components in the e_i and L generates a $C_b(G)$ semigroup.

Remark The formula for the Malliavin matrix is the same as for L_0 and depends only on the main part of L. Formulas in the proof, where we use Volterra expansion, are abstractly the same as for L_0. Estimates in the Volterra expansion are the same, because the global behaviour for the semigroup P_t for bounded time is the same as the non-perturbed semigroup generated by L_0.

We consider the generator

$$L_\lambda = \sum e_i^4 - \lambda \sum < \phi(g), h_t >^i e_i + Q. \tag{7}$$

It generates by the elliptic theory a semigroup on $C_b(G)$ which depends smoothly on λ. We denote it by P_t^λ. We suppose that $g \to < \phi(g), h_t >$ is a smooth function in $g \in G$ with values in R^m which depends continuously on λ.

We consider

$$\tilde{X}_0 = (0, \sum g^{-1} e_i < \phi(g), h_t >^i). \tag{8}$$

We consider the operator on $G \times T_e G$

$$\tilde{L} = L - \tilde{X}_0. \tag{9}$$

As the Malliavin operator, it is not the perturbation of an elliptic operator on $G \times T_eG$. But we can redefine it as we did for the Malliavin semigroup $\hat{P}_t$, redefining it by its associated semigroup $\tilde{P}_t$.

This allows to show

Proposition 1. *We have if f is smooth with compact support*

$$\frac{\partial}{\partial \lambda} P_t^0[f](g_0) = \tilde{P}_t[< Df, gu >](g_0, 0). \quad (10)$$

Proof. It is the same proof as for the Proposition 5 of [9]. The only problem is that u is not bounded. We perform the Davies gauge transform [2] associated with $h(u) = |u^2| + 1$. We get

$$\tilde{L}_d = h^{-1}\tilde{L}(h.) = \tilde{L} + h^{-1} < \tilde{X}_0, h >= \tilde{L} + V. \quad (11)$$

But the potential V is bounded. By Volterra expansion, $\tilde{L}_d$ generates a semigroup on $C_b(G \times T_eG)$ wich is equal to $h^{-1}\tilde{P}_t[h.]$. The results arise because $\tilde{L}$ generates a continuous semigroup on $C_b(G \times T_eG)$.

We have the main theorem of this paper:

Theorem 4. (Malliavin) *If the Malliavin condition holds, then*

$$|\hat{P}_t][v^{-p}](g, 0) < \infty \quad (12)$$

for all positive integers p, P_t has a heat kernel.

Proof. It is the same proof as in the beginning of the proof of Theorem 6 of [9]. Under the Malliavin assumption, we can optimize the elementary integration by part of Theorem 2, in order to get, according the framework of the Malliavin Calculus, the inequality for any smooth function f on G:

$$|P_t[< df, e_i >]| \leq C\|f\|_\infty. \quad (13)$$

By this inequality, we deduce according to the framework of the Malliavin Calculus that

$$P_t[f](g) = \int_G f(g')p_t(g, g')dg' \quad (14)$$

for a non strictly positive heat kernel p_t (dg' denotes the normalized Haar mesure on G).

3 Inversion of the Malliavin matrix

Theorem 5. *Under the previous elliptic assumptions,*

$$|\hat{P}_t|[|v^{-p}|]](g_0, 0) < \infty \quad \text{if} \quad t > 0. \quad (15)$$

Proof. It is the same proof as the proof of Theorem 8 of [6].

Remark For diffusion, this means for operators governed by the second order operator $\sum_{i=1}^{m} X_i^2 + X_0$ and if we are in the elliptic case (the vector fields X_i, $i > 0$ span the whole space), the strategy to invert the Malliavin matrix is the same whether the drift appears or not. It is the same in this case.

Proof of theorem 1 This comes from Theorem 4 and Theorem 5.

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Shift operators and recurrence relations for individual Lamé polynomials

Eugenio Ley-Koo and Ricardo Méndez Fragoso

Abstract Our contribution to Gr31 "Review of the development and application of spheroconal theory of angular momentum" included a mention of our current work on individual Lamé polynomials. This written version reports original results on shift operators and recurrence relations connecting a Lamé polynomial with angular momentum ℓ, species $[A]$ and excitation n, with neighbouring polynomials with $\ell' = \ell \pm 1$, species $[A']$ where the prime indicates a derivative, and excitations n' in their different possibilities, for the derivative operator. Other operators involve multiplication by another Lamé polynomial, starting with the singularity removing factors $A_i = 1$, sn, cn, dn, sn cn, sn dn, cn dn, sn cn dn as monomials. The successive and complementary use of these operators for Lamé polynomials in their two respective degrees of freedom connects with the ladder and shift operator actions of cartesian components of the angular momentum and linear momentum on the product of those polynomials as rotational eigenstates. The identification of the operators for individual Lamé polynomials fills a gap in the study of their properties and connections.

1 Introduction

This section contains a written enumeration of specific results in our contributions to the development of the spheroconal theory of angular momentum, complementary to those presented orally and visually at Gr31. Reference [9] includes orig-

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inal works on which we comment next. In [2], matrix evaluation of the spheroconal harmonics in the basis of spherical harmonics provides the transformation coefficients between both sets for any asymmetry of the molecules. Both sets share the same generating function, the inverse of the distance between two points, and its expansion in the spheroconal basis was obtained. The analysis and evaluations in [3] were implemented using only spheroconal coordinates in their Jacobi elliptic integral representations, motivating us to develop the alternate theory of angular momentum. The plane wave as a generating function of spherical waves, via its Rayleigh expansion, also has its alternate expansion in spheroconal waves. The superintegrability of the hydrogen atom allowed us to identify and construct a generating function for its eigenfunctions in spherical, parabolic, prolate spheroidal, and spheroconal coordinates, and its respective expansions [1]. Three sets of ladder operators connecting spheroconal harmonics were also identified [8]: the cartesian components of the linear momentum connect states of neighbouring angular momentum $\ell' = \ell \pm 1$, different parities and different species; the cartesian components of the angular momentum connects states of the same angular momentum and parity, and different species; the rotational eigenstates as products of Lamé polynomials in the respective degrees of freedom for a given angular momentum and species $[AB]$ are matched by their complementary increasing and decreasing excitations and eigenvalues: $n_1 + n_2 = \ell - n^{AB}$ and $h_{n_1} + h_{n_2} = \ell(\ell+1)$, where n^{AB} counts the number of cartesian plane nodes and n_1 and n_2 the respective number of elliptical cone nodes. Spheroconal harmonics are alternatives to the spherical harmonics: for the Schrödinger equation with isotropic potentials, as applied to the free particle confined in a sphere and the isotropic harmonic oscillator [6], and for the equations of magnetostatics [4]. The rotations of the hydrogen atom [5] and asymmetric molecules [6, 7] confined in elliptical cones have also been investigated.

The main body of this contribution contains in Section 2, the ordinary Lamé differential equation and its eigenfunctions, in terms of Jacobi elliptic functions; in Section 3 the identification of new shift operators in the form of derivatives and multiplications of such functions, leading to linear superposition of other eigenfunctions with other values of angular momentum, different species and different excitations; and in Section 4, a discussion of these results and their applications.

2 Lamé Ordinary Differential Equation and its Eigenfunctions

The Appendix in [9] contains details about the spheroconal coordinates, $x = r\mathrm{dn}(\chi_1|k_1^2)\mathrm{sn}(\chi_2|k_2^2)$, $y = r\mathrm{cn}(\chi_1|k_1^2)\mathrm{cn}(\chi_2|k_2^2)$, $z = r\mathrm{sn}(\chi_1|k_1^2)\mathrm{dn}(\chi_2|k_2^2)$, and the simultaneous solution of eigenvalue problems for the square of the angular momentum and asymmetry-distribution Hamiltonians, leading to Lamé ordinary differential equation [10], in the respective variables and parameters, for $i = 1, 2$:

$$\left(\frac{d^2}{d\chi_i^2} - \ell(\ell+1)k_i^2 \mathrm{sn}^2(\chi_i|k_i^2) + h_i \right) \Lambda(\chi_i) = 0 \,. \tag{1}$$

Here we recall linear relationships among the squares of the Jacobi elliptic functions:

$$\mathrm{cn}^2(\chi_i|k_i^2) = 1 - \mathrm{sn}^2(\chi_i|k_i^2) \quad , \qquad \mathrm{dn}^2(\chi_i|k_i^2) = 1 - k_i^2\mathrm{sn}^2(\chi_i|k_i^2) \tag{2}$$

as well as their derivatives:

$$\frac{d}{d\chi_i}\mathrm{sn}(\chi_i|k_i^2) = \mathrm{cn}(\chi_i|k_i^2)\mathrm{dn}(\chi_i|k_i^2)\ , \quad \frac{d}{d\chi_i}\mathrm{cn}(\chi_i|k_i^2) = -\mathrm{sn}(\chi_i|k_i^2)\mathrm{dn}(\chi_i|k_i^2) \tag{3}$$

$$\frac{d}{d\chi_i}\mathrm{dn}(\chi_i|k_i^2) = -k_i^2\mathrm{sn}(\chi_i|k_i^2)\mathrm{cn}(\chi_i|k_i^2)\ . \tag{4}$$

The latter allows for identifying the singularity removing factors in the solutions of Eq. (1), $A(\chi_i) = [1, s, c, d, cd, sd, cs, scd]$.

Also notice the invariance of the differential equation under $\chi \to -\chi$ which implies eigenfunctions of a definite parity, determined by the respective factors $A(\chi_i)$. Correspondingly, the other factor can be written as a series in the squares of $\mathrm{sn}^2(\chi_i|k_i^2)$, so that the solution of Eq. (1) takes the form

$$\Lambda_{n_i}^{\ell[A]}(\chi_i) = A(\chi_i)\sum_{t=0}^{N_{max}} a_t \mathrm{sn}^{2t}(\chi_i|k_i^2)\ . \tag{5}$$

The series become polynomials of degrees $\ell - n^A$, where $n^A = 0, 1, 2, 3$ according to the number of Jacobi elliptic factors in the respective singularity removing factors. Going back to the cartesian coordinates, products of the matching removing factor for each degree of freedom $[A(\chi_1)][B(\chi_2)] = [1, x, y, z, xy, xz, yz, xyz]$ allows for the identification and classification of the complete spheroconal harmonics according to their parities and number of nodal planes. The index n_i in Eq. (5) counts the number of nodal elliptical cones, coming in pairs. For the matching Lamé polynomials in the respective variables to be multiplied to form the spheroconal harmonics, their respective number of nodes and eigenvalues are restricted by $n_1 + n_2 + n^{AB} = \ell$ and $h^{\ell[A]n_1} + h^{\ell[B]n_2} = \ell(\ell+1)$, counting the common total number of nodes, and also the common square of the angular momentum, respectively. Notice that all the polynomials in Eq. (5) are of degree $\ell - n^A$, and $n_i = 0, 2, ..., \ell - n^A$.

3 Shift operators and recurrence relations for individual Lamé polynomials

In this section, the shifting action of the derivative of a Lamé polynomial connecting it with its neighbors with angular momentum differing by one, of different parity species and excitations, is illustrated and proven by mathematical induction. We do likewise for the product of two Lamé polynomials. The connection between monomials is direct from $\ell = 1$ to $\ell = 2$:

$$\frac{d}{d\chi}\begin{pmatrix} \Lambda_0^{1[s]} \\ \Lambda_0^{1[c]} \\ \Lambda_0^{1[d]} \end{pmatrix} = \begin{pmatrix} cd \\ -sd \\ -k^2 sc \end{pmatrix} = \begin{pmatrix} \Lambda_0^{2[cd]} \\ -\Lambda_0^{2[sd]} \\ -k^2\Lambda_0^{2[sc]} \end{pmatrix}. \tag{6}$$

The connection between the binomials $2[1]$ and the monomial $3[scd]$ is also direct.

$$\frac{d}{d\chi}\begin{pmatrix} \Lambda_n^{2[1]} \\ \Lambda_0^{2[cd]} \\ \Lambda_0^{2[sd]} \\ \Lambda_0^{2[sc]} \end{pmatrix} = \begin{pmatrix} 2a_1^n scd \\ -s(d^2+k^2c^2) \\ c(d^2-k^2s^2) \\ d(c^2-s^2) \end{pmatrix} = \begin{pmatrix} 2a_1^n\Lambda_0^{3[scd]} \\ R_0^s\Lambda_0^{3[s]} + R_2^s\Lambda_2^{3[s]} \\ R_0^c\Lambda_0^{3[c]} + R_2^c\Lambda_2^{3[c]} \\ R_0^d\Lambda_0^{3[d]} + R_2^d\Lambda_2^{3[d]} \end{pmatrix}. \tag{7}$$

However, the connections between the monomials $2[e_je_k]$ and the binomials $3[e_i]$ require the identification of the raising coefficients $R_n^{e_i}$ for $n = 0, 2$ by comparing the respective coefficients of s^{2t} in the last equalities. There are also superpositions of the lowering and raising combinations:

$$L_0^{e_i}\Lambda_0^{1[e_i]} + R_n^{e_i}\Lambda_2^{3[e_i]} \text{ for } n = 0, 2. \tag{8}$$

The generalization and proof by mathematical induction of the shifting action of the derivative operator on the Lamé polynomials from $\ell = 2N$ to $\ell = 2N+1$ and from $\ell = 2N+1$ to $\ell = 2N+2$ is sketched as follows:

$$\frac{d}{d\chi}\begin{pmatrix} \Lambda_n^{2N[1]} \\ \Lambda_n^{2N[cd]} \\ \Lambda_n^{2N[sd]} \\ \Lambda_n^{2N[cs]} \end{pmatrix} = \begin{pmatrix} scd\sum_t 2ta_t s^{2(t-1)} \\ sc^2d^2\sum_t 2ta_t s^{2(t-1)} - s(d^2+k^2c^2)\sum_t a_t s^{2t} \\ s^2cd^2\sum_t 2ta_t s^{2(t-1)} + c(d^2-k^2s^2)\sum_t a_t s^{2t} \\ s^2c^2d\sum_t 2ta_t s^{2(t-1)} + d(c^2-s^2)\sum_t a_t s^{2t} \end{pmatrix} = \begin{pmatrix} \sum_{n'} R_{n'}\Lambda_{n'}^{2N+1[scd]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+1[s]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+1[c]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+1[d]} \end{pmatrix}$$

$$\frac{d}{d\chi}\begin{pmatrix} \Lambda_n^{2N+1[scd]} \\ \Lambda_n^{2N+1[s]} \\ \Lambda_n^{2N+1[c]} \\ \Lambda_n^{2N+1[d]} \end{pmatrix} = \begin{pmatrix} (c^2d^2 - s^2d^2 - k^2s^2c^2)\sum_t a_t s^{2(t-1)} \\ s^2cd\sum_t 2ta_t s^{2(t-1)} + cd\sum_t a_t s^{2t} \\ sc^2d\sum_t 2ta_t s^{2(t-1)} - sd\sum_t a_t s^{2t} \\ scd^2\sum_t 2ta_t s^{2(t-1)} - k^2sc\sum_t a_t s^{2t} \end{pmatrix} = \begin{pmatrix} \sum_{n'} R_{n'}\Lambda_{n'}^{2N+2[1]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+2[cd]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+2[sd]} \\ \sum_{n'} R_{n'}\Lambda_{n'}^{2N+2[sc]} \end{pmatrix}.$$

In the first case, the polynomial of species $[1]$ is of degree $\frac{\ell}{2} = N$ in sn^2, while its companions $[e_ie_j]$ have polynomials of degree $\frac{\ell}{2} - 1$. Their respective derivatives of species $[scd]$ and $[e_k]$ have polynomials of degrees $\frac{\ell}{2} - 1 = \frac{(2N+1)-3}{2}$ and $\frac{\ell}{2} = \frac{(2N+1)-1}{2}$, allowing for identifying the superposition of the corresponding Lamé polynomials with the value $\ell = 2N+1$. In the second case, the polynomials of species $[scd]$ are of degree $\frac{(2N+1)-3}{2}$, and their companions of species $[e_i]$ have polynomials of degree $\frac{(2N+1)-1}{2}$. Their respective derivatives of species $[1]$ and $[e_ie_j]$ have polynomials of degree $\frac{(2N+1)-3}{2}+2=\frac{2N+2}{2}$ and $\frac{2N}{2}=\frac{(2N+2)-2}{2}$, allowing for identifying the superposition of Lamé polynomials with the common value $\ell = 2N+2$ for all species.

As for multiplying two Lamé polynomials, we start with those of lower degrees:

$$\Lambda_0^{1[e_i]}\Lambda_0^{1[e_j]} = \Lambda_0^{2[e_ie_j]}, \tag{9}$$

$$\Lambda_0^{1[e_i]}\Lambda_0^{1[e_i]} = e_i^2 = R_0^{e_i}\Lambda_0^{2[1]} + R_2^{e_i}\Lambda_2^{2[1]} = L_0^{e_i}\Lambda_0^{0[1]} + R_n^{e_i}\Lambda_n^{2[1]}, \tag{10}$$

$$\Lambda_0^{2[e_ie_j]}\Lambda_0^{2[e_ie_j]} = e_i^2e_j^2 = \sum_{n'=0,2,4} R_{n'}\Lambda_{n'}^{4[1]}, \tag{11}$$

$$\Lambda_0^{1[e_i]}\Lambda_0^{3[e_ie_je_k]} = e_i^2e_je_k = \sum_{n'=0,2} R_{n'}\Lambda_{n'}^{4[e_je_k]}, \tag{12}$$

$$\Lambda_0^{2[e_ie_j]}\Lambda_0^{3[e_ie_je_k]} = e_i^2e_j^2e_k = \sum_{n'} R_{n'}\Lambda_{n'}^{5[e_k]}, \tag{13}$$

$$\Lambda_0^{1[e_i]}\Lambda_n^{4[1]} = \sum_{n'=0,2,4} R_{n'}\Lambda_{n'}^{5[e_i]}, \tag{14}$$

$$\Lambda_0^{3[scd]}\Lambda_0^{3[scd]} = s^2c^2d^2 = \sum_{n'=0,2,4,6} R_{n'}\Lambda_{n'}^{6[1]}. \tag{15}$$

The last example is interpreted in terms of a superposition of Lamé polynomials of angular momentum $\ell = 6$, species $[1]$, and the excitations $n = 0,2,4,6$. However, just as we did in going from Eqs. (7) to (8), we recognize other alternatives involving superpositions of one or more of the above examples with other states of the same species and lower values of $\ell = 0,2,4$. By using the ket notation, the orthogonality and completeness of the individual Lamé polynomials, we write the product of any two of them as a superposition of the connected states allowed by the addition of their angular momenta:

$$|\ell_1[A_1]n_1\rangle\,|\ell_2[A_2]n_2\rangle = \sum_{\ell_3,[A_3],n_3} |\ell_3[A_3]n_3\rangle\,\langle \ell_3[A_3]n_3 \mid \ell_1[A_1]n_1\ell_2[A_2]n_2\rangle\,, \tag{16}$$

where the overlap integrals are the coefficients of the superposition, directly identifiable by comparing their singularity removing factors and coefficients of equal powers of sn^2.

The same idea is applicable for the derivative operation involving the addition $\vec{\ell}_i + \vec{1} = \vec{\ell}_f$.

4 Discussion and conclusions

In the previous sections we provided a succinct review of our recent contributions to the development and application of the spheroconal theory of angular momentum; a description of the Lamé ordinary differential equation with the classification and characterization of its solutions and eigenvalues on the basis of their parities and species; and the identification of the derivative and multiplication of Lamé polynomials as operations changing their angular momentum, species and excitations. Now, we proceed to make some concluding remarks about the latter in (i) the domain of the individual Lamépolynomials, and (ii) the domain of the rotational eigenstates of asymmetric molecules, constructed as the product of individual Lamé polynomials in the two different and complementary degrees of freedom.

An immediate application of the general result expressed in Eq. (16) is the evaluation of the integrals of the product of three Lamé polynomials, including the selection rules that limit the third factor on the left-hand side to those in the superposition on the right-hand side. This is the counterpart of the familiar result for Legendre polynomials.

The cartesian components of the linear and angular momentum operators involve derivatives with respect to the spheroconal coordinates χ_1 and χ_2, multiplied by the components of their unit vectors along the respective cartesian axes, as described in the appendix in [9]. We have started to reexamine their actions on the rotational states, as the products of complementary Lamépolynomials in the respective spheroconal coordinates, in light of the results of the derivative and multiplication operations on the individual polynomials. Equation (16) also suggests that we are ready to tackle the problem of the addition of angular momenta for the complete spheroconal eigenstates.

In conclusion, the availability of these operators and their related recurrence relations place the Lamé polynomials more on a par with their spherical counterparts.

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Shift operators and recurrence relations for Legendre polynomials with noninteger associativity and definite parity

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Abstract This is the written version of new results reported at Gr31 in our contribution "O(2) Symmetry Breaking in Dihedrally Confined Atoms and Consequent Modifications of the Periodic Table". It is focused on identifying new shift operators and recurrence relations for Legendre polynomials with noninteger asociativities and definite parities, due to the dihedral confinement and its O(2) symmetry breaking. Recurrence relations play a key role in evaluating the two-electron matrix elements of their Coulomb repulsion multipole components in the Hartree-Fock calculations for the confined atoms.

1 Introduction

Our previous work on the hydrogen atom confined by dihedral angles [3] led us to recognize its O(2) symmetry breaking as manifested by the noninteger values, $\mu = n_\phi \pi/\phi_0$, of the eigenvalues of the component of the orbital angular momentum along the edge of the dihedral angle, of magnitude ϕ_0, for $n_\phi = 1, 2, 3, \ldots$. Figure 1.1 in that reference illustrates the atomic orbitals $\sin \mu\phi$ for the ground state as polar graphs vanishing at the planes defining the dihedral angle, to be compared with circles for the free atom. Some of the consequences of the symmetry breaking consists of the atom: (1) acquiring an electric dipole moment (Sect. 1.3.2), because the center of charge does not coincide with the position of the nucleus; (2) the Fermi contact interaction in the hyperfine structure vanishes because the probability of finding the electron at the position of the nucleus also vanishes (Sect. 1.3.4); (3)

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the Zeeman effect is suppressed in first order perturbation theory because matrix elements of the z-component of angular momentum $\langle n'_\phi | \hat{l}_z | n_\phi \rangle$ vanish for $n'_\phi \pm n_\phi$ even, and are nonvanishing for $n'_\phi \pm n_\phi$ odd. On the other hand, (4) Table 1.1 illustrates the order of the energy levels $E_{n_r n_\theta n_\phi}(\phi_0)$ and their degeneracies D for different openings of the confining angle. They are definitely different from those of the free atom. Additionally, (5) in Section 1.4.2 the filling of shells in multielectron atoms in the same situation of confinement based on those degeneracies suggest substantial changes in the periodicities of the elements in the periodic table.

The quantitative answer to point (5) requires Hartree-Fock calculations for atoms with nuclear charge Z, and Z and $(Z-1)$ electrons. The key ingredient for such calculations is the matrix elements of the electron-electron Coulomb repulsion multipole components in the basis of two-electron hydrogen-like confined orbitals, to be explained in Section 2. This led us to identifying new shift operators and recurrence relations for the Legendre polynomials with non-integer associativities and definite parities, to be presented in Section 3; the recursive use of these relations is also needed for evaluating the polar angle matrix elements. Section 4 summarizes the new results and their application to implement the Hartree-Fock calculations.

2 Evaluation of matrix elements

The dihedrally confined hydrogen orbitals are the novelty elements in the matrix elements to be evaluated.

The azimuthal eigenfunctions of the square of the z-component of the angular momentum vanishing at the angles $\phi = \mp\phi_0/2$ and noninteger values of μ:

$$|\mu\rangle = \sqrt{\frac{2}{\phi_0}} \begin{cases} \sin\mu\phi & (\mu = 2n_\phi \pi/\phi_0\,;\, n_\phi = 1,2,3,\ldots) \\ \cos\mu\phi & (\mu = (2n_\phi + 1)\pi/\phi_0\,;\, n_\phi = 0,1,2,\ldots). \end{cases} \tag{1}$$

The polar angle eigenfunctions in the variable $\eta = \cos\theta$ with noninteger associativity and definite parity are

$$|n_\theta, \mu\rangle = N_{n_\theta,\mu}(1-\eta^2)^{\mu/2} F^{\pm}_{n_\theta,\mu}(\eta) \tag{2}$$

where

$$F^{\pm}_{n_\theta,\mu}(\eta) = \frac{1}{2}\left[{}_2F_1\left(-n_\theta, n_\theta + 2\mu + 1; \mu + 1; \frac{1-\eta}{2}\right) + \right. \\ \left. +(-1)^{n_\theta} {}_2F_1\left(-n_\theta, n_\theta + 2\mu + 1; \mu + 1; \frac{1+\eta}{2}\right)\right]. \tag{3}$$

Notice the difference from the familiar Legendre polynomials $P^m_{n_\theta+m}(\eta)$ which in their usual hypergeometric representations have a definite parity only for integer values of m. The O(2) symmetry breaking for non integer values μ is manifested by $P^{\mu}_{n_\theta+\mu}(\eta)$ containing even and odd powers of η. The combination in $F^{\pm}_{n_\theta,\mu}(\eta)$ restores the parity as originally discussed in [4] and [3].

The electron-electron Coulomb repulsion in its multipole expansion is written in a form appropriate to connect the confined orbitals with definite parities:

$$\frac{1}{r_{12}} = 4\pi \sum_{\ell,m} \frac{N^2_{lm}}{2l+1} \frac{r^{\ell}_{<}}{r^{\ell+1}_{>}} P^m_l(\eta_1) P^m_l(\eta_2)(2-\delta_{m,0}) \cos m(\phi_2-\phi_1). \tag{4}$$

The general form of its matrix elements is

$$\begin{aligned} &\left\langle c\, n'_{1\theta}\, \mu'_1 \right| \left\langle d\, n'_{2\theta}\, \mu'_2 \right| \frac{1}{r_{12}} \left| a\, n_{1\theta}\, \mu_1 \right\rangle \left| b\, n_{2\theta}\, \mu_2 \right\rangle \\ &\qquad = 4\pi \sum_{\ell,m} \frac{N^2_{\ell m}(2-\delta_{m,0})}{2\ell+1} I^{\ell}_{12}(cd;ab)\, I^{\ell m}_{1\theta}(n'_{1\theta},\mu'_1;n_{1\theta},\mu_1) \times \\ &\qquad\qquad \times I^{\ell m}_{2\theta}(n'_{2\theta},\mu'_2;n_{2\theta},\mu_2)\, I^m_{1\phi}(\mu'_1,\mu_1) \cdot I^m_{2\phi}(\mu'_2,\mu_2) \end{aligned} \tag{5}$$

The factorizability of the orbitals and the Coulomb repulsion components into their radial, polar angle and azimuthal parts for each electron lead to integrals of the same type for each degree of freedom, differing in their respective quantum labels. Accordingly, the radial, azimuthal and polar matrix elements are defined by

$$I^{\ell}_{12} = \int\int R_c(r_1) R_d(r_2) \frac{r^{\ell}_{<}}{r^{\ell+1}_{>}} R_a(r_1) R_b(r_2) r_1^2 dr_1 r_2^2 dr_2 \tag{6}$$

$$I^m_{j\phi} = \left\{ I^{m+}_{j\phi} = \left\langle \mu'_j \right| \cos m\phi_j \left| \mu_j \right\rangle, I^{m-}_{j\phi} = \left\langle \mu'_j \right| \sin m\phi_j \left| \mu_j \right\rangle \right\} \tag{7}$$

$$I^{\ell m}_{j\theta} = \left\langle n'_{j\theta} \mu'_j \right| P^m_l(\eta_j) \left| n_{j\theta} \mu_j \right\rangle \tag{8}$$

where $j = 1,2$ in the last two expressions denote electron 1 and 2, respectively.

While for the free atoms, the evaluation of the ϕ dependent integrals is reduced to the selection rules $\delta_{m_2,m+m_1}$, in the present case only the integrals with an even number of sine functions survive:

$$\left\langle \cos \mu'\phi_j \right| \cos m\phi_j \left| \cos \mu\phi_j \right\rangle = A^+_{\mu+\mu'} + A^-_{\mu+\mu'} + A^+_{\mu'-\mu} + A^-_{\mu'-\mu}\,, \tag{9}$$

$$\left\langle \sin \mu'\phi_j \right| \sin m\phi_j \left| \cos \mu\phi_j \right\rangle = A^-_{\mu+\mu'} - A^+_{\mu+\mu'} + A^-_{\mu'-\mu} - A^+_{\mu'-\mu}\,, \tag{10}$$

$$\left\langle \cos \mu'\phi_j \right| \sin m\phi_j \left| \sin \mu\phi_j \right\rangle = A^-_{\mu+\mu'} - A^+_{\mu+\mu'} - A^-_{\mu'-\mu} + A^+_{\mu'-\mu}\,, \tag{11}$$

$$\left\langle \sin \mu'\phi_j \right| \cos m\phi_j \left| \sin \mu\phi_j \right\rangle = A^+_{\mu'-\mu} + A^-_{\mu'-\mu} - A^+_{\mu+\mu'} - A^-_{\mu+\mu'}\,, \tag{12}$$

where the following definition holds :

$$A^{\pm}_{\gamma} = \frac{\sin[(\gamma \pm m)\phi_0/2]}{2(\gamma \pm m)}\,. \tag{13}$$

The polar angle integrals can be written using the explicit polynomial form of the associated Legendre polynomials,

$$I_{j\theta}^{\ell m} = \left\langle n'_{j\theta}\mu'_j \right| P_\ell^m(\eta) \left| n_{j\theta}\,\mu_j \right\rangle = \sum_{s=0(1)}^{\ell-m} a_s \left\langle n'_{j\theta}\mu'_j \right| (1-\eta^2)^{m/2}\eta^s \left| n_{j\theta}\mu_j \right\rangle, \quad (14)$$

recognizing that we need to evaluate the matrix elements of the product of powers of the removing singularity factor and of the η-variable: $(1-\eta^2)^{m/2}\eta^s$. Section 3 addresses this evaluation.

The radial integrals I_{12}^{ℓ}, involving variational radial functions $R_a(r_j)$, are of the same type as those for free atoms.

3 Shift operators and recurrence relations for Legendre polynomials with noninteger associativity and definite parity

Our work in Ref. [4] identified the recurrence relations of Eqs. (20-30) using multiplications by η, η^2 and derivative operators to connect neighbouring Legendre polynomials of the same associativity and opposite parities as the counterparts of Eqs. (8.5.2) and (8.5.4) of the Legendre functions in [1]. The reader should note the differences in the coefficients of the respective relations, especially those in Eqs. (31-32) of [4]. The identification was made by mathematical induction using the polynomials in Table 4 of [4]. Eliminating the derivative terms in Eqs. (29-30) lead to the simple relationship between polynomials of the same associativity:

$$\frac{\eta\,|n_\theta\,\mu\rangle}{N_{n_\theta,\mu}} = \frac{n_\theta}{2n_\theta+2\mu+1}\frac{|n_\theta-1\,\mu\rangle}{N_{n_\theta-1,\mu}} + \frac{n_\theta+2\mu+1}{2n_\theta+2\mu+1}\frac{|n_\theta+1\,\mu\rangle}{N_{n_\theta+1,\mu}} \quad (15)$$

involving the factor only. This recurrence relation and its recursive use proved useful for us in our investigation of the hydrogen molecular ion confined in a dihedral angle [2].

Evaluating the polar angle integrals $I_{j\theta}^{\ell m}$ for $m \neq 0$ requires constructing shift operators and recurrence relations connecting polynomials with different associativities $\mu' \neq \mu$. Taking the same structure of Eqs. (8.5.1) and (8.5.5) of [1] and using mathematical induction, we have identified the following relationships:

$$(1-\eta^2)^{1/2}\frac{|n_\theta-1\,\mu+1\rangle}{N_{n_\theta-1,\mu+1}} = \frac{2(\mu+1)}{2\mu+n_\theta+1}\left[\frac{|n_\theta-1\,\mu\rangle}{N_{n_\theta-1,\mu}} - \eta\frac{|n_\theta\,\mu\rangle}{N_{n_\theta,\mu}}\right] \quad (16)$$

$$(1-\eta^2)^{1/2}\frac{|n_\theta+1\,\mu-1\rangle}{N_{n_\theta+1,\mu-1}} = \frac{(2\mu+n_\theta+1)(2\mu+n_\theta)}{2\mu\,(2\mu+2n_\theta+1)}\frac{|n_\theta+1\,\mu\rangle}{N_{n_\theta+1,\mu}} - \\ - \frac{n_\theta(n_\theta+1)}{2\mu\,(2\mu+2n_\theta+1)}\frac{|n_\theta-1\,\mu\rangle}{N_{n_\theta-1,\mu}}. \quad (17)$$

We also include their initial recursions.

$$\eta^2 \frac{|n_\theta\,\mu\rangle}{N_{n_\theta,\mu}} = \frac{n_\theta(n_\theta-1)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu-1)}\frac{|n_\theta-2\,\mu\rangle}{N_{n_\theta-2,\mu}} + \\ + \left[\frac{n_\theta(n_\theta+2\mu)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu-1)} + \frac{(n_\theta+2\mu+1)(n_\theta+1)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu+3)}\right]\frac{|n_\theta\,\mu\rangle}{N_{n_\theta,\mu}} + \\ + \frac{(n_\theta+2\mu+2)(n_\theta+2\mu+1)}{(2n_\theta+2\mu+3)(2n_\theta+2\mu+1)}\frac{|n_\theta+2\,\mu\rangle}{N_{n_\theta+2,\mu}} \tag{18}$$

$$\eta^3 \frac{|n_\theta\,\mu\rangle}{N_{n_\theta,\mu}} = \frac{n_\theta(n_\theta-1)(n_\theta-2)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu-1)(2n_\theta+2\mu-3)}\frac{|n_\theta-3\,\mu\rangle}{N_{n_\theta-3,\mu}} + \\ + \left[\frac{(n_\theta+2\mu-1)n_\theta(n_\theta-1)}{2n_\theta+2\mu-3)(2n_\theta+2\mu-1)} + \frac{n_\theta^2(n_\theta+2\mu)}{(2n_\theta+2\mu-1)(2n_\theta+2\mu+1)^2} + \right. \\ \left. + \frac{(n_\theta+2\mu+1)n_\theta(n_\theta+1)}{(2n_\theta+2\mu+1)^2(2n_\theta+2\mu+3)}\right]\frac{|n_\theta-1\,\mu\rangle}{N_{n_\theta-1,\mu}} + \\ + \left[\frac{n_\theta(n_\theta+2\mu)(n_\theta+2\mu+1)}{(2n_\theta+2\mu-1)(2n_\theta+2\mu+1)^2} + \frac{(n_\theta+2\mu+1)^2(n_\theta+1)}{(2n_\theta+2\mu+3)(2n_\theta+2\mu+1)^2} + \right. \\ \left. + \frac{(n_\theta+2\mu+1)(n_\theta+2\mu+2)(n_\theta+2)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu+3)(2n_\theta+2\mu+5)}\right]\frac{|n_\theta+1\,\mu\rangle}{N_{n_\theta+1,\mu}} + \\ + \frac{(n_\theta+2\mu+1)(n_\theta+2\mu+2)(n_\theta+2\mu+3)}{(2n_\theta+2\mu+1)(2n_\theta+2\mu+3)(2n_\theta+2\mu+5)}\frac{|n_\theta+3\,\mu\rangle}{N_{n_\theta+3,\mu}} \tag{19}$$

$$(1-\eta^2)\frac{|n_\theta-1\,\mu+1\rangle}{N_{n_\theta-1,\mu+1}} = \frac{2\mu(2\mu+1)}{2\mu+n_\theta+1}\left[\frac{1}{2\mu+n_\theta-1}\frac{|n_\theta-1\,\mu-1\rangle}{N_{n_\theta-1,\mu-1}} - \right. \\ \left. -\eta\left(\frac{1}{2\mu+n_\theta-1} + \frac{1}{2\mu+n_\theta}\right)\frac{|n_\theta\,\mu-1\rangle}{N_{n_\theta,\mu-1}} + \eta^2\frac{1}{2\mu+n_\theta}\frac{|n_\theta+1\,\mu-1\rangle}{N_{n_\theta+1,\mu-1}}\right] \tag{20}$$

$$(1-\eta^2)\frac{|n_\theta+1\,\mu-1\rangle}{N_{n_\theta+1,\mu-1}} = \\ \frac{(2\mu+n_\theta+3)(2\mu+n_\theta+2)(2\mu+n_\theta+1)(2\mu+n_\theta)}{2(\mu+1)(2\mu)(2\mu+2n_\theta+3)(2\mu+2n_\theta+1)}\frac{|n_\theta+1\,\mu+1\rangle}{N_{n_\theta+1,\mu+1}} - \\ - \frac{(2\mu+n_\theta+1)(2\mu+n_\theta)n_\theta(n_\theta+1)}{2(\mu+1)(2\mu)(2\mu+2n_\theta+1)}\left(\frac{1}{2\mu+2n_\theta+3} + \frac{1}{2\mu+2n_\theta-1}\right)\frac{|n_\theta-1\,\mu+1\rangle}{N_{n_\theta-1,\mu+1}} + \\ + \frac{(n_\theta+1)n_\theta(n_\theta-1)(n_\theta-2)}{2(\mu+1)(2\mu)(2\mu+2n_\theta+1)(2\mu+2n_\theta-1)}\frac{|n_\theta-3\,\mu+1\rangle}{N_{n_\theta-3,\mu+1}}. \tag{21}$$

4 Discussion

Some comparative remarks on Eqs. (15–17) are appropriate. In (15), the operation of multiplying the ket on the lhs by η leads to the superposition of the kets on the rhs, with the same associativity, and polar excitations one unit below and above, respectively. In the other two cases, multiplication by $(1-\eta)^{1/2}$ leads in (16) to the superposition of kets with associativities one unit below the original one, and the polar excitation is the same for the first term, and one unit above for the second one which is additionally multiplied by η; and in (17) the associativities are one unit above the original one, and their polar excitations are the same and two units below, respectively.

The initial recursive forms of Eqs. (19–21) illustrate the effects of multiplying twice by the respective factors, obtaining the superpositions of three sets of kets with increased differences in their associativities and polar excitations; notice the two terms for the intermediate ket. The addition of another factor leads to the superposition of four kets with one, three, three and one term coefficients. The multiplication by N factors leads to the superposition of $N+1$ kets with coefficients involving numbers of terms counted by the binomial coefficients, associated with the number of paths connecting the initial ket with the ones in the superposition.

The recursive relations (15–17) may also be used recursively in different combinations, leading to other linear of superpositions of other kets. For those using (15) and/or (16) some of the kets are also multiplied by successive powers of η.

In short, these extended recurrence relations allow for the evaluation of the polar matrix elements described in Section 2 for any associativity m and power s in the Coulomb electron repulsion multipole components, via the multiplication with the bra $\langle n'_\theta \mu'|$, projecting the states in the linear superpositions, compatible with the conservation of angular momentum and parity, taking into account O(2) symmetry breaking.

The coefficients in the recurrence relations and their combinations in their recursive combinations lend themselves to computational programming. This is an important tool for the Hartree-Fock calculations of dihedrally confined atoms.

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On completeness of Bethe Ansatz solutions for $sl(2)$ Richardson–Gaudin systems

Jon Links

Abstract The Bethe Ansatz solution for the class of rational, $sl(2)$ Richardson–Gaudin systems is presented. Completeness of this solution is discussed for the case where all operators are realised in terms of the spin-1/2 representation. This discussion is based on a set of operator identities. Next, a generalised system with broken $u(1)$-symmetry is introduced, which admits an analogous set of operator identities. Analysis of this generalised system shows that the Bethe Ansatz solution for it is also complete. The prospects for extending this approach to higher spin systems are mentioned.

1 Introduction

The names of Richardson and Gaudin have become associated with a class of integrable quantum systems affording applications to a wide range of physical problems. See e.g., [1–4]. These systems possess an exact solution in the sense of the Bethe Ansatz method. As with other Bethe Ansatz solved models, it is natural to ask whether the exact solution of a given model is complete [5–8]. Recently, this problem was investigated for the case of the spin-1/2, rational, Richardson–Gaudin system [9]. The method adopted there relied on the existence of a set of operator identities. This example will be summarised below. Then an extended analysis will be presented for a generalised system with broken $u(1)$-symmetry. This system also admits a set of operator identities. Following the approach of [9], it will be shown that the Bethe Ansatz solution for the generalised system is complete. The route forward for extensions to higher spin systems will be discussed in the concluding remarks.

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2 The rational, $sl(2)$ Richardson-Gaudin system

This integrable system is described in terms of a collection of $sl(2)$ spins, labelled by $j = 1,...,L$, satisfying the canonical commutation relations

$$[S^z_j, S^{\pm}_k] = \pm\delta_{jk}S^{\pm}_j, \qquad [S^+_j, S^-_k] = 2\delta_{jk}S^z_j.$$

Each spin is assigned a representation with highest weight s_j. Let α be a real parameter, and let $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_L)$ denote an L-tuple of pairwise-distinct real parameters. The operators

$$T_j = 2\alpha S^z_j + \sum_{k\neq j}^{L} \frac{\theta_{jk} - 2s_js_kI}{\varepsilon_j - \varepsilon_k}, \tag{1}$$

where I denotes the identity operator and $\theta_{jk} = S^+_j S^-_k + S^-_j S^+_k + 2S^z_j S^z_k$, are mutually commuting, i.e.,

$$[T_j, T_k] = 0, \qquad \forall j,k = 1,...,L.$$

The spectrum of these operators is given by way of a Bethe Ansatz solution. The eigenvalues are expressed as

$$\mu_j = \left(2\alpha - \sum_{m=1}^{M} \frac{2}{\varepsilon_j - w_m}\right) s_j, \qquad j = 1,...,L \tag{2}$$

where

$$2\alpha + \sum_{j=1}^{L} \frac{2s_j}{w_m - \varepsilon_j} = \sum_{n\neq m}^{M} \frac{2}{w_m - w_n}, \qquad m = 1,...,M. \tag{3}$$

The eigenstate associated with each such solution is one with a z-component of spin s^z given by

$$s^z = \sum_{j=1}^{L} s_j - M$$

which is the eigenvalue of

$$S^z = \sum_{j=1}^{L} S^z_j = \frac{1}{2\alpha}\sum_{j=1}^{L} T_j.$$

As a result, the system is seen to have $u(1)$-symmetry with invariant sectors determine by s^z. Defining the polynomials

$$P(u) = \prod_{j=1}^{L} (u - \varepsilon_j)^{2s_j},$$

$$Q(u) = \prod_{m=1}^{M} (u - w_m),$$

the previous relations (2,3) may be equivalently expressed as

$$\mu_j = 2\left(\alpha - \frac{Q'(\varepsilon_j)}{Q(\varepsilon_j)}\right)s_j, \tag{4}$$

$$Q''(w_m)P(w_m) = 2\alpha Q'(w_m)P(w_m) + Q'(w_m)P'(w_m). \tag{5}$$

To show that the solution is complete, it is necessary to show that it is possible to parametrise the μ_j as (1) subject to (2). In the case where all representations are spin-1/2, it was shown in [9] that the conserved operators satisfy the quadratic identities

$$T_j^2 = \alpha^2 I - \sum_{k\neq j}^{L} \frac{T_j - T_k}{\varepsilon_j - \varepsilon_k}, \qquad j = 1,...,L. \tag{6}$$

It follows that the eigenvalues observe analogous relations

$$\mu_j^2 = \alpha^2 - \sum_{k\neq j}^{L} \frac{\mu_j - \mu_k}{\varepsilon_j - \varepsilon_k}, \qquad j = 1,...,L \tag{7}$$

and that these are necessarily complete. The relations (7) first appeared in the studies of Babelon and Talalaev [10]. In that work it was shown that (7) follows from (2). The objective of [9] was to construct the inverse mapping. This led to the conclusion that the Bethe Ansatz equations could be considered to be complete, except the method did not include instances where root multiplicities occur in the Bethe Ansatz equations. Below, an extension of this approach will be described for a generalised system in a manner which accommodates root multiplicities.

3 A generalised system with broken $u(1)$-symmetry

Maintaining the assignment $s_j = 1/2, \forall j = 1,...,L$, the generalised operators depend on an additional parameter γ which leads to the breaking of $u(1)$-symmetry. In terms of a real variable α and an L-tuple of pairwise-distinct, non-zero, real parameters $(z_1, z_2, ..., z_L)$, the conserved operators read [11]:

$$\mathscr{T}_j = 2\alpha S_j^z + 2\gamma z_j S_j^x + \sum_{k\neq j}^{L}\left(\frac{z_j^2}{z_j^2 - z_k^2}(4S_j^z S_k^z - I) + \frac{2z_j z_k}{z_j^2 - z_k^2}(S_j^+ S_k^- + S_j^- S_k^+)\right) \tag{8}$$

which are again mutually commuting. However, the operators (8) do not commute with S^z. The quadratic identities

$$\mathscr{T}_j^2 = \alpha^2 I + \gamma^2 z_j^2 I - 2z_j^2 \sum_{k\neq j}^{L} \frac{1}{z_j^2 - z_k^2}(\mathscr{T}_j - \mathscr{T}_k), \tag{9}$$

satisfied by the generalised set of operators were given in [12].

To show the connection to the operators (1), introduce a real parameter η and make the following replacements in (8):

$$\alpha \mapsto \alpha/\eta,$$
$$\gamma \mapsto 0,$$
$$z_j \mapsto \exp(\eta \varepsilon_j).$$

Then

$$T_j = \lim_{\eta \to 0} \eta \mathscr{T}_j.$$

Letting λ_j denote the eigenvalues of $\mathscr{T}_j$, it follows from (9) that

$$\lambda_j^2 = \alpha^2 + \gamma^2 z_j^2 - 2z_j^2 \sum_{k \neq j}^{L} \frac{\lambda_j - \lambda_k}{z_j^2 - z_k^2} \tag{10}$$

which are necessarily complete. For each λ_j define $Q(u)$ to be a polynomial of order not greater than L, satisfying

$$2z_j^2 Q'(z_j^2) + (\lambda_j - \alpha) Q(z_j^2) = 0, \qquad j = 1, \ldots, L. \tag{11}$$

This linear system admits a non-trivial solution. Set

$$Q(u) = \prod_{k=1}^{M} (u - v_k)^{m_k},$$

with m_k denoting the multiplicity of the root v_k, such that

$$\sum_{k=1}^{M} m_k \leq L.$$

Then for $k = 1, \ldots, M$, the following generalisation of (2) holds [11]

$$(v_k Q''(v_k) + (1-\alpha) Q'(v_k)) P(v_k) - v_k Q'(v_k) P'(v_k) = -\frac{\gamma^2}{4} \left[P(v_k) \right]^2, \tag{12}$$

where now

$$P(u) = \prod_{j=1}^{L} (u - z_j^2).$$

That is, associated to every eigenstate of the system, there exists a solution of the system of equations (12). A derivation of (12) will be outlined below, following the techniques described in [13].

Note that if $Q(z_j^2) = 0$, it follows from (11) that $Q'(z_j^2) = 0$. In such a case, (12) holds for $v_k = z_j^2$. Now, provided $Q(z_j^2) \neq 0$ for all $j = 1, \ldots, L$, then from (10)

$$
\begin{aligned}
\lambda_j^2 &= \alpha^2 + \gamma^2 z_j^2 - 2z_j^2 \sum_{k\neq j}^{L} \frac{\lambda_j - \lambda_k}{z_j^2 - z_k^2} \\
&= \alpha^2 + \gamma^2 z_j^2 + 4z_j^2 \sum_{l=1}^{M} \sum_{k\neq j}^{L} \frac{m_l v_l}{(z_k^2 - v_l)(v_l - z_j^2)} \\
&= \alpha^2 + \gamma^2 z_j^2 + 4z_j^2 \sum_{l=1}^{M} \frac{m_l v_l}{z_j^2 - v_l} \frac{P'(v_l)}{P(v_l)} + 4z_j^2 \sum_{l=1}^{M} \frac{m_l z_j^2}{(v_l - z_j^2)^2} + 4z_j^2 \sum_{l=1}^{M} \frac{m_l}{v_l - z_j^2} \\
&= \alpha^2 + \gamma^2 z_j^2 + 4z_j^2 \sum_{l=1}^{M} \frac{m_l v_l}{z_j^2 - v_l} \frac{P'(v_l)}{P(v_l)} + 4z_j^4 \sum_{l=1}^{M} \frac{m_l}{(v_l - z_j^2)^2} - 4z_j^2 \frac{Q'(z_j^2)}{Q(z_j^2)}. \qquad (13)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\lambda_j^2 &= \alpha^2 - 4\alpha z_j^2 \frac{Q'(z_j^2)}{Q(z_j^2)} + 4z_j^4 \left(\frac{Q'(z_j^2)}{Q(z_j^2)} \right)^2 \\
&= \alpha^2 - 4\alpha z_j^2 \frac{Q'(z_j^2)}{Q(z_j^2)} + 4z_j^4 \sum_{k=1}^{M} \frac{m_k}{(z_j^2 - v_k)^2} + 4z_j^4 \frac{Q''(z_j^2)}{Q(z_j^2)}. \qquad (14)
\end{aligned}
$$

For equality of expressions (13) and (14) it is required that

$$
\frac{\gamma^2}{4} + \sum_{l=1}^{L} \frac{m_l v_l}{z_j^2 - v_l} \frac{P'(v_l)}{P(v_l)} = (1-\alpha) \frac{Q'(z_j^2)}{Q(z_j^2)} + z_j^2 \frac{Q''(z_j^2)}{Q(z_j^2)}. \qquad (15)
$$

Set

$$
S(u) = uQ''(u) + (1-\alpha)Q'(u) - \left(\frac{\gamma^2}{4} + \sum_{j=1}^{L} \frac{m_j v_j}{u - v_j} \frac{P'(v_j)}{P(v_j)} \right) Q(u) \qquad (16)
$$

which is a polynomial of order M. It follows from (15) that

$$
S(z_j^2) = 0, \qquad j = 1, \dots, L
$$

which, along with the consideration of the asymptotic behaviour of (16) as $u \to \infty$, establishes that

$$
S(u) = -\frac{\gamma^2}{4} P(u) \qquad (17)
$$

and $M = L$ for $\gamma \neq 0$. Evaluating $S(v_k)$ through (16) and (17) and equating these expressions then yields the Bethe Ansatz equations (12). Note that if $m_k \geq 1$, then $Q'(v_k) = 0$. In this case the associated Bethe Ansatz equation from (16) and (17) becomes

$$
v_k Q''(v_k) = -\frac{\gamma^2}{4} P(v_k),
$$

which is consistent with (12). When $\gamma = 0$ the above equation admits $v_k = 0$ as a solution. This is a known example for which multiplicities do occur [14].

4 Conclusion

This work examined some properties of the Bethe Ansatz solution for a generalisation of the rational Richardson-Gaudin system in the spin-1/2 case. Based on a set of quadratic operator identities reported in [12], a set of Bethe Ansatz equations were derived. The same equations were previously obtained in [11, 12] using different methods. It was argued that the Bethe Ansatz solution is complete, supported by calculations accommodating instances where root multiplicities occur.

Extending these results to higher-spin models requires the identification of higher-order polynomial operator identities generalising (6) and (9). In the spin-1 case, the required identities are cubic. The identities are obtained through a tensor product procedure following [13]. These results will be reported at a later date [15].

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Nonadiabatic bounce in quantum cosmology

Przemysław Małkiewicz

Abstract We quantize and analyze the dynamics of the closed homogeneous and anisotropic universe, the so-called Bianchi type IX model. The isotropic part of the geometry is encoded in the phase space which is the half-plane, and its underlying symmetry is the affine group rather than the Weyl-Heisenberg group. We make use of affine coherent states first to quantize the half-plane and next to give a semi-classical portrait to the respective quantum dynamics. The anisotropic part of the geometry is encoded in the usual $\mathbb{R}^2$-phase space and is quantized canonically. In order to solve the quantum dynamics we employ both adiabatic and nonadiabatic methods known from molecular physics. We find that the big bang singularity of the classical dynamics is replaced by a smooth bounce at the quantum level. Moreover, in the adiabatic regime, the oscillations of the anisotropic geometry are suppressed and the universe contracts smoothly. In the nonadiabatic regime, the bounce breaks the adiabatic evolution and triggers an extended post-bounce inflationary phase accompanied by production of quanta of the anisotropic geometry.

1 Introduction

General relativity notoriously suffers from singularities. The most relevant ones are the spacelike (or, cosmological) singularities which reside inside black holes or in the beginning of the universe. The understanding of a generic dynamics of the gravitational field approaching the singularity is due to Belinskii, Khalatnikov and Lifshitz. They discovered that asymptotically the dynamics becomes ultralocal and that at each spatial point, it converges to the dynamics of a spatially homogenous configuration of the field. Among the homogenous configurations, the so-called Bianchi

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type IX model is the most generic one and it is believed to play a pivotal role in generic spacelike singularities.

The Bianchi type IX model with regular matter describes a spherical universe which contracts to the singular state of vanishing volume and blowing-up contraction rate. On the way to the singularity it undergoes an infinite number of oscillations of aspherical deformations. The oscillatory part of its geometry can be viewed as two modes of a non-linear gravitational wave whose energy grows rapidly as the universe contracts and it eventually dominates all regular forms of matter fueling a strong singularity.

The most commonly studied closed Friedmann-Robertson-Walker (FRW) universe is obtained in the limit of vanishing of the gravitational wave, in which the evolution becomes purely isotropic and terminates in a significantly weaker singularity. Thus, extending the configuration space of the gravitational field by the anisotropic degrees of freedom brings a substantial change in the asymptotic dynamics at the classical level and, as we will shortly see, at the quantum level as well.

2 Classical model

The line element of the diagonal Bianchi type IX universe reads:

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + \sum_i q_i(t)\omega^i \otimes \omega^i,$$

where $\mathrm{d}\omega^i = \frac{1}{2}\varepsilon_{ijk}\omega^j \wedge \omega^k$ are $SO(3)$-invariant basis dual vectors, N is the lapse function. We assume that the evolving metric remains diagonal with respect to the one-forms ω^i. We follow closely the Arnowitt-Deser-Misner formulation of the Hamiltonian formalism in which the dynamics of the gravitational field resembles the dynamics of a particle on a certain spacetime and in a potential. The potential is due to the non-vanishing intrinsic curvature. In our case, the dynamical variables can be divided into isotropic and anisotropic ones. The isotropic variables describe the evolution of the volume of the universe and they form the half-plane, $(q,p) \in \mathbb{R}_+^* \times \mathbb{R}$. The anisotropic variables describe aspherical, volume-preserving deformations of the universe and they form two planes, $(\beta_\pm, p_\pm) \in \mathbb{R}^2$. The dynamics is governed by the following Hamiltonian constraint [3]:

$$H = p^2 + Lq^{2/3} - H_q^{ani}$$

where p^2 is the kinetic energy of the isotropic motion, the potential $Lq^{2/3}$ (L is a constant) comes from the isotropic part of the intrinsic curvature and H_q^{ani} is the anisotropic Hamiltonian:

$$H_q^{ani} := \frac{p_+^2 + p_-^2}{q^2} + q^{2/3}V(\beta_\pm)$$

in which $q^{-2}p_{\pm}^2$ describes the kinetic energy of the anisotropic motion and the potential $V(\beta_{\pm})$ represent the anisotropic part of the intrinsic curvature. It is confining and leads to the oscillatory behaviour of the anisotropic deformations (see Fig. 1).

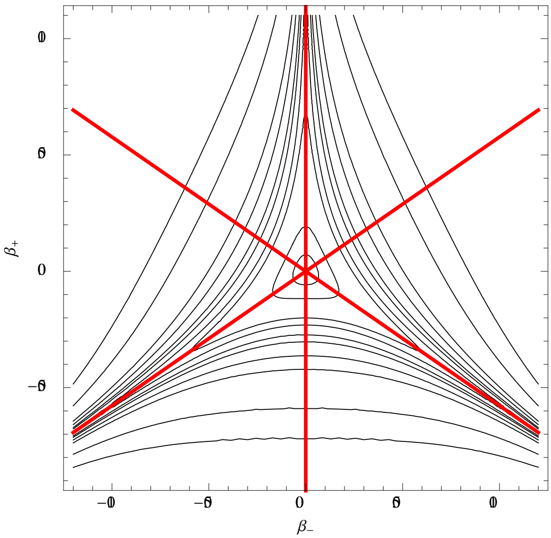

Fig. 1 The anisotropy potential $V(\beta_{\pm})$ of the Bianchi IX model. Around its visible minimum the harmonic approximation can be assumed. Away from the minimum the potential is usually approximated by an infinite triangular wall [3].

3 Quantization

We first introduce the affine coherent states. The term *coherent states* was originally assigned to the eigenvectors of the annihilation operator whose spectrum is the complex plane: $\hat{a}|z\rangle = z|z\rangle$. They can be constructed with the Weyl-Heisenberg representation as $|z\rangle = D(z)|0\rangle$, where $D(z)$ is the displacement operator and $|0\rangle$ is the ground state of the harmonic oscillator. This construction can be generalised by (i) replacing the Weyl-Heisenberg representation by any unitary irreducible and integrable representation of a phase space symmetry and (ii) replacing the ground state $|0\rangle$ by an (almost) arbitrary normalized vector in a given Hilbert space. We note that the isotropic sector of the phase space is the half-plane which is symmetric with respect to the action of the affine group defined as $(q',p')\cdot(q,p) = (q'q, q'^{-1}p + p')$. This group has a unique (up to irrelevant sign) integrable UIR, which we denote as $U(q,p)$ in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$. Then the affine coherent states are defined as [5,6]:

$$|q,p\rangle = U(q,p)|\psi_0\rangle,$$

where $|\psi_0\rangle$ is the so-called fiducial vector that satisfies $\int_{\mathbb{R}_+} |\psi_0(x)|^2 x^{-1}\mathrm{d}x < \infty$.

We quantize the classical theory by combining canonical quantization of anisotropic variables $(\beta_{\pm}, p_{\pm})$ and the so-called coherent state (CS) quantization (a generalisation of the Berezin-Klauder-Toeplitz quantization) of the isotropic pair (q,p). The latter is defined as follows:

$$f(q,p)\mapsto A_f:=\int \frac{\mathrm{d}q\mathrm{d}p}{2\pi c} f(q,p)|q,p\rangle\langle q,p|,$$

where c is a normalization constant. We find [3,4]:

$$\hat{H}=\hat{p}^2+\frac{\hbar^2\mathfrak{K}_1}{\hat{q}^2}+L\mathfrak{K}_3\hat{q}^{2/3}-\hat{H}^{(\mathrm{ani})}_{\hat{q}}\ ,\quad \hat{H}^{(\mathrm{ani})}_{\hat{q}}=\mathfrak{K}_2\frac{\hat{p}_+^2+\hat{p}_-^2}{\hat{q}^2}+\mathfrak{K}_3\hat{q}^{2/3}V(\hat{\beta}_\pm)$$

where the values of the constants $\mathfrak{K}_i$ depend on the particular choice of the fiducial vector. We note the isotropic repulsive potential, $\frac{\hbar^2\mathfrak{K}_1}{\hat{q}^2}$, which is issued from the CS quantization and which *regularizes* the singularity $q=0$.

For the sake of simplicity we make the harmonic approximation to $V(\beta_\pm)\simeq$ *const* $\cdot(\beta_+^2+\beta_-^2)+o(\beta_\pm^2)$. We note that the anisotropic Hamiltonian describes now a $\hat{q}$-dependent harmonic oscillator and can be analytically decomposed with respect to its $\hat{q}$-dependent eigenvectors, i.e., $\hat{H}^{(\mathrm{ani})}_{\hat{q}}=\sum_n E_n(\hat{q})|e_n(\hat{q})\rangle\langle e_n(\hat{q})|$. The quantum evolution of the isotropic geometry will be approximated by means of semiclassical trajectories in the phase space that capture the most essential quantum corrections to the classical motion. The trajectories in (q,p) are generated by the semiclassical Hamiltonian, $\check{H}$:

$$\check{H}=\langle q,p|\hat{H}|q,p\rangle$$

which is still a linear operator acting in the Hilbert space of the anisotropic geometry and depends on classical variables (q,p).

4 Adiabatic approximation

The Born-Oppenheimer (BO) approximation applies to the case when the isotropic evolution is very slow relative to the anisotropic oscillations. It assumes that the wave remains in a fixed eigenstate and the isotropic motion couples only to the energy of the wave. The wavefunction is approximated by $|\Psi\rangle\approx|q,p\rangle\otimes|e_n(q)\rangle$ for a fixed n. The Hamiltonian which now reads [3]:

$$\hat{H}=\hat{p}^2+\frac{\hbar^2\mathfrak{K}_1}{\hat{q}^2}+L\mathfrak{K}_3\hat{q}^{2/3}-E_n(\hat{q})\ ,$$

after contracting with the affine coherent states becomes completely semiclassical:

$$\check{H}_n(q,p)=p^2+\frac{K\hbar^2}{q^2}+Lq^{\frac{2}{3}}-E_n(q),$$

where K is a constant (see Fig. 2 for the respective dynamics).

The Born-Huang approximation additionally assumes that the isotropic evolution can be affected by the *motion* of the anisotropic geometry apart from its energy. The gravitational wave remains in a fixed eigenstate. It is convenient to work in a unitarily transformed Hamiltonian in which an extra correction to the BO scheme becomes apparent:

$$U(\hat{q})^{\dagger}\hat{H}U(\hat{q}) = (\hat{p} - \hat{\mathrm{A}}(\hat{q}))^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2} + L\mathfrak{K}_3 \hat{q}^{2/3} - E_n(\hat{q}),$$

where $U(\hat{q}) := |e_n(\hat{q})\rangle\langle e_n|$ is a unitary operator which relates the $\hat{q}$-dependent anisotropy eigenstates with $\hat{q}$-independent elements of any fixed orthonormal basis in $\mathscr{H}$, and $\hat{\mathrm{A}}(\hat{q}) = i\hbar \frac{\mathrm{d}U}{\mathrm{d}\hat{q}} U^{\dagger}$ is the corrective term coupling the isotropic and anisotropic motions. In this scheme, the total wavefunction of the transformed Hamiltonian is $|\Psi\rangle \approx |q,p\rangle \otimes |e_n\rangle$.

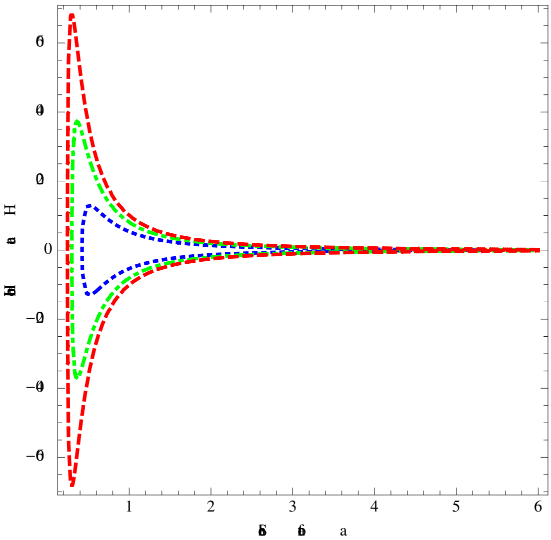

Fig. 2 Semiclassical portrait of the quantum *adiabatic* evolution of the Bianchi IX universe corresponds to the closed Friedmann universe with an extra source term due to the non-vanishing vacuum energy of the gravitational wave and with a repulsive potential that removes the singularity [4].

5 Nonadiabatic approximation

In the regime where the adiabatic approximation breaks down, one has at his disposal a powerful method called the vibronic approach. The basic feature of the vibronic approach is that it allows for transitions between the anisotropy eigenstates and therefore, the anisotropy is assumed in a general state denoted by $|\phi^{(\mathrm{ani})}\rangle$. The isotropic evolution is described with semiclassical variables and thus, the total wavefunction (of the transformed Hamiltonian) is assumed $|\Psi\rangle \approx |q,p\rangle \otimes |\phi^{(\mathrm{ani})}\rangle$. Let us introduce $H^{sem}(q,p) := \langle q,p|U(q)^{\dagger}\hat{H}U(q)|q,p\rangle$, which is a (q,p)-dependent operator acting in the anisotropic sector of $\mathscr{H}$. Based on it, we derive the following set of coupled equations for the semiclassical motion of the isotropic geometry, the quantum motion of the anisotropic geometry and the semiclassical constraint equation [2]:

$$\dot{q} = N\langle\phi^{(\mathrm{ani})}|\,\partial_p H^{sem}\,|\phi^{(\mathrm{ani})}\rangle, \quad \dot{p} = -N\langle\phi^{(\mathrm{ani})}|\,\partial_q H^{sem}\,|\phi^{(\mathrm{ani})}\rangle$$

$$-i\hbar\,\partial_t\,|\phi^{(\mathrm{ani})}\rangle = N H^{sem}\,|\phi^{(\mathrm{ani})}\rangle, \quad \langle\phi^{(\mathrm{ani})}|\,H^{sem}\,|\phi^{(\mathrm{ani})}\rangle = 0.$$

The above set of equations is self-consistent and time-reparametrization invariant. It leads to, at least in all the numerically studied examples, a nonsingular dynamics with a bounce. If the initial state $|\phi_0^{(\mathrm{ani})}\rangle$ is a fixed anisotropy eigenstate, then the universe follows the adiabatic trajectory at least till the bounce. At the bounce, in general the adiabatic condition breaks down and the initial eigenstate gets excited, and the produced anisotropic energy in turn *backreacts* on the isotropic post-bounce expansion by accelerating it (see Fig. 3). A rough estimate suggests that in a realistic cosmological scenario, one should expect a huge production of the anisotropic energy and a lasting extended phase of accelerated expansion should follow the bounce [1].

6 Conclusions

We have studied the Bianchi type IX dynamics. A crucial role in the quantization and the subsequent analysis was played by the affine coherent states. We obtained significant insights into the quantum dynamics thanks to molecular physics approximations. Specifically, we have found that the quantum model contains both FRW-like and less symmetric solutions. The former and the latter are associated with respectively the adiabatic and nonadiabatic dynamical regimes. The possibility of the breakdown of FRW-like dynamics due to a nonadiabatic bounce is contrary to the classical dynamics in which the symmetries once imposed hold forever. The nonadiabatic bounce induces a subsequent extended inflationary phase. Our next steps include removing the harmonic approximation, exploring the dynamics for states of large anisotropy and incorporating linear perturbations to homogeneity.

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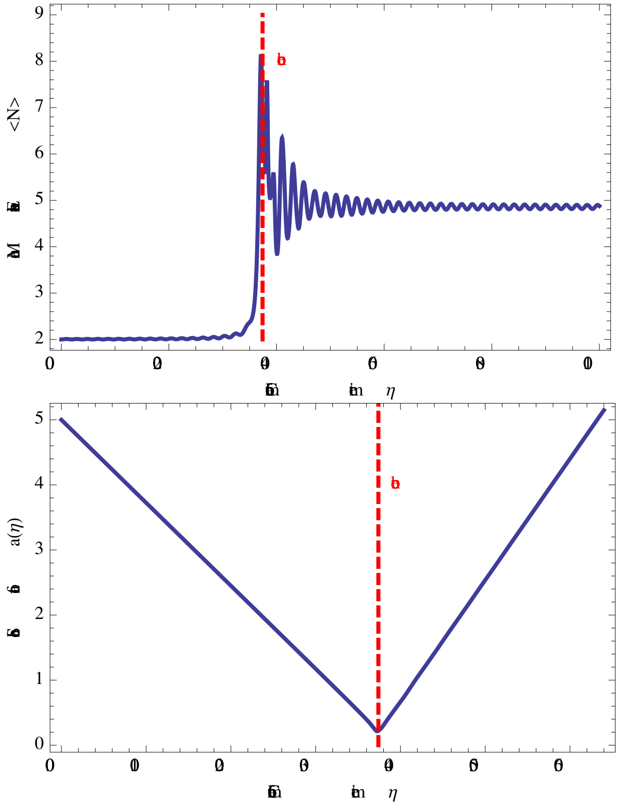

Fig. 3: Slightly *nonadiabatic* evolution of coupled isotropic and anisotropic variables in conformal time starting from $|\phi_0^{(\mathrm{ani})}\rangle = |e_2\rangle$ away from the bounce. The top panel shows the increase of the initial population, $n = 2$, to the final one, $n \approx 5$. The bottom panel shows the backreaction of the excited anisotropy state on the isotropic geometry which results in a slightly non-symmetric bounce [2].

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Two-dimensional massless light-front fields and conformal field theory

L'ubomír Martinovič

Abstract A consistent quantization of two-dimensional (2D) massless light-front fields (scalar and fermion) is formulated. Their two-point functions exactly reproduce the massless limit of the two-point functions of the corresponding massive fields. The novel formalism incorporates bosonization in a natural way and also provides us with elements needed for an independent light-front (LF) study of the exactly solvable models (the Thirring or Thirring-Wess model, e.g.). Moreover, it displays closeness of the 2D massless LF quantum fields to conformal field theory (CFT). We calculate a few correlators including those between the components of the LF energy-momentum tensor and derive the Virasoro algebra in the LF operator form. Going over to the euclidean time, we can direcly transform all calculated quantities to the (anti)holomorphic form, in agreement with those from CFT.

1 Introduction

The light front (LF) form of quantum field theory (QFT) has been praised for its potential for decades. Its features that are superior to the conventional ("space-like" - SL) form of QFT, include the minimal number (3) of dynamical Poincaré generators [1], the status of the vacuum state, and a reduced number of independent field components. The most fundamental aspect is the equality of the physical vacuum state (= the lowest energy eigenstate of the full generic Hamiltonian) to the Fock vacuum (= state without field quanta). This property follows from the positivity and conservation of the LF momentum p^+. Only the field zero modes, carrying $p^+ = 0$, and a narrow set of (symmetry) operators [2,3], depending on the details of the specific dynamics, can transform the LF Fock vacuum into a more complex object. The

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latter will however be much simpler than its SL counterpart, which in principle has to be obtained by (unrealistically complicated) dynamical calculations.

Availability of the consistent Fock expansion based on the LF vacuum, with the amplitudes having direct probabilistic interpretaion makes the LF approach attractive from the point of view of phenomenological applications. On the other hand, proliferation of the non-dynamical field variables complicates the theory by the need to invert operator constraint equations. There still exist some concerns pertaining to the validity of the LF theory. The typical question is how the LF scheme can cope with the isue of vacuum condensates and the symmetry breaking with underlying vacuum degeneracies, given its greatly simplified, structureless, ground state.

What is then the relation between the SL and LF theory? Could it be that the LF version conceptually as well as technically simplifies the structure of QFT while still maintaining potential for reliable predictions? The area of 2D solvable relativistic models represents a very suitable environment to study these questions [4].

Surprisingly however, the 2D massless LF fields, being the essential elements for exact operator solutions of the models, have not been understood and correctly quantized until nowadays. Not even the simplest (and prototypic) gauge theory, the massless Schwinger model, has been solved in the LF version of the theory [5].

Recently, a simple and natural way of quantizing the two-dimensional massless LF fields has been suggested [6]. In our contribution, we shall first give a brief exposition of this quantization scheme. Its validity will be demonstrated by by the LF bosonization of the massless fermion field. In the second part, the closeness of the massless LF quantum fields to conformal field theory (CFT) will be demonstrated by calculating several correlation functions of elementary and composite operators. Going over to the euclidean time, one immediately reproduces the CFT results. Virasoro algebra is also obtained directly in the LF operator formalism.

Throughout this paper, we will use the following LF notation: $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$. The momentum is designed as k^μ (or p^μ), $k^\mu = (k^+, k^-)$,

$$\partial_\pm = \frac{\partial}{\partial x^\pm}, \;\; \hat{k}\cdot x = \frac{1}{2}k^+x^- + \frac{1}{2}\hat{k}^-x^+, \;\; k^2 = \mu^2 \Rightarrow \hat{k}^- = \frac{\mu^2}{k^+}. \tag{1}$$

$\hat{k}^-$ is the on-shell LF energy. in the LF form. Both k^+, k^- can be taken positive.

2 Quantization of free massless light-front fields in 2D

Our quantization of the massless **LF scalar field** starts from the massive field. Its Lagrangian and the field equation takes in terms of the LF variables the form

$$\mathscr{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2, \quad \big(4\partial_+\partial_- + \mu^2\big)\phi(x) = 0. \tag{2}$$

The solution of the field equation (2) is expressed in terms of Fock operators as

$$\phi(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+)e^{-\frac{i}{2}k^+x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+}\right], \qquad (3)$$

$\left[a(k^+), a^\dagger(l^+)\right] = \delta(k^+ - l^+)$, $a(k^+)|0\rangle = 0$. The LF Hamiltonian and momentum operator is given in terms of densities $T^{++} = 4 : \partial_-\phi\partial_-\phi :$, $T^{+-} = \mu^2 : \phi^2 :$,

$$P^\nu = \frac{1}{2}\int_{-\infty}^{+\infty} dx^- T^{+\nu}(x) = \int_0^{+\infty} dk^+ \hat{k}^\nu a^\dagger(k^+)a(k^+), \;\; \hat{k}^\nu = (\frac{\mu^2}{k^+}, k^+). \qquad (4)$$

From (3) we calculate the conjugate momentum $\pi(x) = 2\partial_-\phi(x)$ and the time derivative $\theta(x) = 2\partial_+\phi(x)$. In the following, we shall need the correlation functions

$$D_0^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle, \; D_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle, \; D_2^{(+)}(z) = \langle 0|\phi(x)\theta(y)|0\rangle, \qquad (5)$$

$$D_i^{(+)}(z) = i\int_0^{\infty} \frac{dk^+}{4\pi} f_i(k^+) e^{-\frac{i}{2}k^+(z^- - i\varepsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\varepsilon^+)}, \;\; z = x - y. \qquad (6)$$

Here $f_0(k^+) = -\frac{i}{k^+}$, $f_1(k^+) = 1$, $f_2(k^+) = \frac{\mu^2}{k^{+2}}$. The small imaginary parts in the exponents are necessary for the existence of the integrals, which are evaluated in terms of the (modified) Bessel functions $J_\nu(z), N_\nu(z), K_\nu(z), \nu = 0, 1$:

$$D_1^{(+)}(z) = -\theta\left(z^2\right)\frac{\mu}{4}\sqrt{\frac{z^+}{z^-}}i\left[J_1\left(\mu\sqrt{z^2}\right) - i\,\mathrm{sgn}(z^+)N_1\left(\mu\sqrt{z^2}\right)\right] + \qquad (7)$$

$$- \;\theta\left(-z^2\right)\mathrm{sgn}(z^+)\frac{\mu}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\left(\mu\sqrt{-z^2}\right), \;\; D_2^{(+)} = D_1^{(+)}(x^+ \leftrightarrow x^-).$$

Now, one observes that both $D_1^{(+)}$ and $D_2^{(+)}$ have a non-vanishing massless limit,

$$D_1^{(+)}(z; \mu^2 = 0) = \frac{1}{2\pi}\frac{1}{(z^- - i\varepsilon^-)}, \; D_2^{(+)}(z; \mu^2 = 0) = \frac{1}{2\pi}\frac{1}{(z^+ - i\varepsilon^+)}. \qquad (8)$$

Technically, this is due to the behaviour of the Bessel function $K_1(z) \sim \frac{1}{z}$ for the small value of z. These results suggest that there must exist massless analogs of the fields $\phi(x), \pi(x), \theta(x)$ reproducing (8). Indeed, from the LF massless Klein-Gordon equation $\partial_+\partial_-\tilde{\phi}(x) = 0$, one expects a general solution of the form

$$\tilde{\phi}(x) = \tilde{\phi}(x^+) + \tilde{\phi}(x^-). \qquad (9)$$

Since the integration measure of the LF field is mass-independent [7], the massless limit ($\mu = 0$ in the plane-wave factors) of the massive solution (3) gives just $\tilde{\phi}(x^-)$. The piece $\tilde{\phi}(x^+)$ can be recovered from (3) by the change of variables (done more correctly at the classical level) $k^+ = \frac{\mu^2}{k^-}$. x^+ and x^- interchange their places in (3), and the Fock operators in terms of the new variable should satisfy [6]

$$\Big[\frac{\mu}{k^-}a\big(\frac{\mu^2}{k^-}\big),\frac{\mu}{l^-}a^\dagger\big(\frac{\mu^2}{l^-}\big)\Big]=\frac{\mu^2}{k^-l^-}\delta\big(\frac{\mu^2}{k^-}-\frac{\mu^2}{l^-}\big)=\delta(k^--l^-). \tag{10}$$

The rhs of (10) survives the massless limit, hence $\lim_{\mu\to 0}\frac{\mu}{k^-}a\big(\frac{\mu^2}{k^-}\big)\equiv\tilde{a}(k^-)\neq 0$, with the commutators $\big[\tilde{a}(k^-),\tilde{a}^\dagger(l^-)\big]=\delta(k^--l^-)$, $\big[\tilde{a}(k^+),\tilde{a}^\dagger(l^-)\big]=0$. After the change of variables, the massless limit in (3) yields

$$\tilde{\phi}(x^+)=\int_0^{+\infty}\frac{dk^-}{\sqrt{4\pi k^-}}\big[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+}+\tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\big], \tag{11}$$

and similarly for $\theta(x^+)$ and $\pi(x^-)$. The basic field commutators are consequently

$$\big[\tilde{\phi}(x^-),\tilde{\phi}(y^-)\big]=-\frac{i}{4}\varepsilon(x^--y^-),\ \ \big[\tilde{\phi}(x^+),\tilde{\phi}(y^+)\big]=-\frac{i}{4}\varepsilon(x^+-y^+). \tag{12}$$

The variables k^+ and k^- actually coincide, in complete analogy with the SL case $k^0=|k^1|$. Also, one verifies that the two-point functions calculated from the massless fields coincide with the massless limits (8) of the massive functions. Using similar reasoning and the above Fock commutators, the operators

$$P^+=\int_0^{+\infty}dk^+k^+a^\dagger(k^+)a(k^+),\quad P^-=\int_0^{+\infty}dk^+k^-a^\dagger(k^-)a(k^-) \tag{13}$$

are shown to generate the correct Heisenberg equations $2i\partial_\pm\phi(x^\pm)=-[P^\mp,\phi(x^\pm)]$.

The same procedure can be applied to the **light front fermion field**. The massive (two-dimensional Dirac) field equation $i\gg^\mu\ \partial_\mu\psi(x)=m\psi(x)$ decomposes as

$$2i\partial_+\psi_2(x)=m\psi_1(x),\ 2i\partial_-\psi_1(x)=m\psi_2(x) \tag{14}$$

$$\Rightarrow\psi_2(x)=\tilde{\psi}_2(x^-),\ \psi_1(x)=\tilde{\psi}_1(x^+),\ \ \text{if } m=0. \tag{15}$$

For the correct quantization, we again start from the two components of the massive field in the momentum representation that solve the field equations (14):

$$\psi_2(x)=\int_0^{+\infty}\frac{dp^+}{4\pi}\big[b(p^+)e^{-\frac{i}{2}p^+x^--\frac{i}{2}\frac{m^2}{p^+}x^+}+d^\dagger(p^+)e^{\frac{i}{2}p^+x^-+\frac{i}{2}\frac{m^2}{p^+}x^+}\big], \tag{16}$$

$$\psi_1(x)=\int_0^{+\infty}\frac{dp^+}{4\pi}\frac{m}{p^+}\Big[b(p^+)e^{-\frac{i}{2}p^+x^--\frac{i}{2}\frac{m^2}{p^+}x^+}-d^\dagger(p^+)e^{\frac{i}{2}p^+x^-+\frac{i}{2}\frac{m^2}{p^+}x^+}\Big], \tag{17}$$

where $\{b(p^+),b^\dagger(q^+)\}=\{d(p^+),d^\dagger(q^+)\}=\delta(p^+-q^+)$, and study their massless limit. For ψ_2, this again is straightforward. The limits of the fermion two-point functions $S_{11}(z), S_{22}(z)$ coincide up to the factor $(-i)$ with that of $D_1^{(+)}$ and $D_2^{(+)}$. Hence we change the variables for $\psi_1(x)$ and repeat all the steps from the scalar-field case. This results in the massless field expansions and their Fock algebra:

$$\tilde{\psi}_2(x^-) = \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}} \left[\tilde{b}(p^+)e^{-\frac{i}{2}p^+x^-} + \tilde{d}^\dagger(p^+)e^{\frac{i}{2}p^+x^-}\right],$$

$$\tilde{\psi}_1(x^+) = \int_0^{+\infty} \frac{dp^-}{\sqrt{4\pi}} \left[\tilde{b}(p^-)e^{-\frac{i}{2}p^-x^+} - \tilde{d}^\dagger(p^-)e^{\frac{i}{2}p^-x^+}\right], \tag{18}$$

$$\{\tilde{b}(p^+), \tilde{d}^\dagger(q^+)\} = \delta(p^+ - q^+),\ \{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \delta(p^- - q^-). \tag{19}$$

The $\tilde{d}$-operators satisfy the same anticommutation relations. As a consequence,

$$\left\{\tilde{\psi}_1(x^+), \tilde{\psi}_1{}^\dagger(y^+)\right\} = \delta(x^+ - y^+), \quad \left\{\tilde{\psi}_2(x^-), \tilde{\psi}_2{}^\dagger(y^-)\right\} = \delta(x^- - y^-). \tag{20}$$

The two kinds of modes decouple: $\{\tilde{b}(p^-), \tilde{b}^\dagger(q^+)\} = \{\tilde{b} \to \tilde{d}\} = 0$. The two-point function of the massless $\tilde{\psi}_1(x^+)$ coincides with the massless limit of the massive 2-point function. From the expansions (18), one constructs the bilinear operators (the current $j^\mu = (: \tilde{\psi}_1{}^\dagger \tilde{\psi}_1 :, : \tilde{\psi}_2{}^\dagger \tilde{\psi}_2 :)$ and the scalar densities $\tilde{\psi}_2{}^\dagger \tilde{\psi}_1 \pm \tilde{\psi}_1{}^\dagger \tilde{\psi}_2$).

Thus, the quantum theory of the massless LF fermion field has been established. The necessary information is contained in the original massive solutions. Since solvable models are based on free Heisenberg fields, the above derivation opens the road to the genuine LF solution of the class of models with massless fermions [8].

Consistency of the scheme is further confirmed by **LF bosonization**. Bosonization is a remarkable property of the 2D field theory: fermion fields can be represented in terms of boson variables [9, 10]. Our derivation of its LF version is based on the natural decomposition of the massless $\phi(x)$ and $\psi(x)$ fields (9),(15).

Consider first $\tilde{\psi}_2(x^-)$. Assume that it can be represented as

$$\varphi_2(x^-) = C : e^{i\alpha\phi(x^-)} : \ = Ce^{i\alpha\phi^{(-)}(x^-)}e^{i\alpha\phi^{(+)}(x^-)}. \tag{21}$$

The constants C and α can be adjusted in such a way that two φ_2 with different arguments anticommute and $\varphi_2(x^-)$, $\varphi_2^\dagger(y^-)$ satisfy the anticommutation relation (20). The first condition fixes α to the value $\hat{\alpha} = 2\sqrt{\pi}$. The second determines the constant C as $\hat{C} = \left(\frac{\lambda e^{\gg_E}}{4\pi}\right)^{1/2}$ (λ is the infrared cutoff associated with the massless $D_0^{(+)}$ function [6] and $\gg_E$ is the Euler's constant). It follows that the operators $\hat{\varphi}(x^-)$ and the analogously obtained $\hat{\varphi}(x^+)$ represent the bosonized form of the fields $\tilde{\psi}_2(x^-)$ and $\tilde{\psi}_1(x^+)$. Forming their appropriate point-split products, the bosonized vector current is found to be $\hat{j}^+(x^-) = 2\pi^{-1/2}\partial_-\phi(x^-)$, $\hat{j}^-(x^+) = 2\pi^{-1/2}\partial_+\phi(x^+)$. It correctly reproduces the Schwinger term in the current-current commutators, $\left[\hat{j}^\mp(x^\pm), \hat{j}^\mp(y^\pm)\right] = i\pi^{-1}\partial_x\delta(x^\pm - y^\pm)$. Similarly, for the scalar densities, one gets

$$\overline{\psi}(x)\psi(x) = \frac{\lambda e^{\gg_E}}{4\pi}\cos\left(2\sqrt{\pi}\phi(x)\right),\ \overline{\psi}(x) \gg^5 \psi(x) = i\frac{\lambda e^{\gg_E}}{4\pi}\sin\left(2\sqrt{\pi}\phi(x)\right). \tag{22}$$

Thus the LF version of bosonization yields the results known from the SL theory.

3 Conformal properties of the 2D massless LF fields

The massless 2D fields exhibit conformal symmetry, whose (anti)holomorphic formulation was developed in [11]. Here we shall show that after switching to the euclidean time, our formalism generates results in agreement with CFT.

The Hamiltonian density $T^{+-}(x)$ of the free massless scalar field vanishes, as required by conformal symmetry (the massless limit (13) of the massive $P^- \neq 0$, however). The other components of the energy-momentum tensor are nonvanishing:

$$T^{++}(x^-) =: \pi(x^-)\pi(x^-):,\ T^{--}(x^+) =: \theta(x^+)\theta(x^+): . \tag{23}$$

Note that the LF Hamiltonian (13) can also be obtained as the x^+-integral of the density $T^{--}(x^+)$, analogously to P^+ which is the x^--integral of $T^{++}(x^-)$.

We compute a few additional correlation functions ($z^\pm = x^\pm - y^\pm$),

$$\langle 0|\theta(x^+)\theta(y^+)|0\rangle = \frac{\pi^{-1}}{(z^+ - i\delta^+)^2}, \langle 0|\pi(x^-)\pi(y^-)|0\rangle = \frac{\pi^{-1}}{(z^- - i\delta^-)^2}, \tag{24}$$

as well as those between components of the energy-momentum tensor,

$$\langle 0|T^{\pm\pm}(x^\mp)T^{\pm\pm}(y^\mp)|0\rangle = \frac{2}{\pi^2}\frac{1}{(x^\mp - y^\mp - i\delta^\mp)^4}. \tag{25}$$

In the holomorphic form of 2D CFT [11,12], the Laurent expansion in the variables

$$z = e^{\frac{2\pi}{L}\zeta},\ \ \bar{z} = e^{\frac{2\pi}{L}\bar{\zeta}},\ \ \text{where}\ \ \zeta = \tau - ix,\ \ \bar{\zeta} = \tau + ix, \tag{26}$$

is commonly used. It is based on radial quantization with the euclidean time τ, $t \to -i\tau$. We need to reformulate our results for $\phi(x)$ in the form of infinite series to conform with the discrete picture of [11]. Thus, we consider the massive field in a finite box of length $2L$ in x^- or $2T$ in x^+ with periodic boundary conditions $\phi(x^+, x^- = -L) = \phi(x^+, x^- = L)$, $\phi(x^+ = -T, x^-) = \phi(x^+ = T, x^-)$. Performing the change of variables and the massless limit as before, we arrive at

$$\phi(x^-) = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2Lk_n^+}}\left[a_n e^{-\frac{i}{2}k_n^+ x^-} + H.c.\right] = \frac{1}{\sqrt{4\pi}}\sum_{n=\pm 1,\ldots}\frac{1}{\sqrt{|n|}}a_n e^{-i\frac{\pi}{L}nx^-}, \tag{27}$$

$$\phi(x^+) = \phi_0 + \sum_{n=1}^{\infty}\frac{1}{\sqrt{2Lk_n^-}}\left[\bar{a}_n e^{-\frac{i}{2}k_n^- x^+} + H.c.\right] = \frac{1}{\sqrt{4\pi}}\sum_{n=\pm 1,\ldots}\frac{1}{\sqrt{|n|}}\bar{a}_n e^{-i\frac{\pi}{L}nx^+} \tag{28}$$

with $\left[a_m, a_n^\dagger\right] = \left[\bar{a}_m, \bar{a}_n^\dagger\right] = \delta_{m,n},\ \ \left[\bar{a}_m, a_n^\dagger\right] = 0$ and $a_{-n} \equiv a_n^\dagger,\ \bar{a}_{-n} \equiv \bar{a}_n^\dagger$.

Since $\mu = 0$, ϕ_0 can be non-zero. It is however just a constant whose conjugate momentums vanishes. The 2-point functions $D_0^{(+)}$ are evaluated for $L >> 1$ as

$$D_0^{(+)}(z^{\pm}) = \langle 0|\phi(x^{\pm})\phi(y^{\pm})|0\rangle = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-i\frac{\pi}{L}n(z^{-}-i\varepsilon)} \approx \frac{1}{4\pi}\ln\left[\frac{i\pi}{L}(z^{-}-i\varepsilon)\right]. \tag{29}$$

L plays the role of the infared regularization parameter. It also introduces the necessary dimension to (29). L drops out of all the other correlation functions (due to the derivatives present). The results match the continuum results (24–25).

The components of the energy-momentum tensor in the discrete form read

$$T^{++}(x^{-}) = K\sum_{m,n}\varepsilon(m)\varepsilon(n)\sqrt{|m||n|} : a_m a_n : e^{-i\frac{\pi}{L}(n+m)x^{-}}, \tag{30}$$

$$T^{--}(x^{+}) = K\sum_{m,n}\varepsilon(m)\varepsilon(n)\sqrt{|m||n|} : \bar{a}_m \bar{a}_n : e^{-i\frac{\pi}{L}(n+m)x^{+}},\ K = -\frac{\pi}{L^2}.$$

They can be transformed to a “Virasoro form” by simply taking a Fourier transform. Indeed, assume that $T^{++}(x^{-})$ can be represented as

$$T^{++}(x^{-}) = \frac{1}{4L^2}\sum_{l=0,\pm1,\ldots} L_l e^{-i\frac{\pi}{L}lx^{-}}, L_l = 2L\int_{-L}^{+L} dx^{-} e^{i\frac{\pi}{L}lx^{-}}T^{++}(x^{-}). \tag{31}$$

Inserting $T^{++}(x^{-})$ in the Fock form (30) into (31) gives ($L_0 = 4LP^{+}$),

$$L_n = -4\pi\sum_{k=\pm1,\ldots}\varepsilon(k)\varepsilon(n-k)\sqrt{|k||n-k|}a_k a_{n-k}. \tag{32}$$

A calculation based on the commutators below Eq.(28) yields the LF version of the Virasoro algebra, including the c-number term, not present at the classical level:

$$\left[L_n, L_m\right] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0},\ c = 1, \tag{33}$$

where c is the “central charge”. Taking $T^{--}(x^{+})$ in (31) instead of T^{++} generates the algebra (33) with $L_n \to \bar{L}_n$. It follows from $\left[a_n, \bar{a}_m\right] = 0$ that $\left[L_n, \bar{L}_m\right] = 0$.

To give a few details of these calculations, we switch back to the “$a, a^{\dagger}$” picture:

$$L_n = -\sum_{k=1}^{n-1}\sqrt{k(n-k)}a_k a_{n-k} + 2\sum_{k=n+1}^{\infty}\sqrt{k(k-n)}a^{\dagger}_{k-n}a_k,\ L^{\dagger}_n = L_{-|n|}. \tag{34}$$

The “anomaly” comes from the commutator between the first terms:

$$\Big[\sum_{l=1}^{m-1}\sqrt{l(m-l)}a_l a_{m-l},\sum_{k=1}^{n-1}\sqrt{k(n-k)}a_k^\dagger a_{n-k}^\dagger\Big]=\sum_{l=1}^{m-1}\sqrt{l(m-l)}\sum_{k=1}^{n-1}\sqrt{k(n-k)}\times$$

$$\times\big\{\delta_{m-l,k}\delta_{l,n-k}+\delta_{l,k}\delta_{m-l,n-k}\big\}=2\delta_{m,n}\sum_{l=1}^{m-1}l(m-l)=\frac{1}{3}m(m^2-1)\delta_{m,n}. \qquad (35)$$

This agrees with the CFT result after taking into account the different normalization.

All the LF results can be easily transformed into the conformal ((anti)holomorphic) form by switching to the euclidean time and defining the variables ζ and $\bar{\zeta}$ (26).

With the conventional CFT normalization (factor 2π in the definition of the energy-momentum tensor instead of 4 in the LF case), we get (cf. Eq.(24)):

$$\langle 0|\pi(\zeta)\pi(\zeta')|0\rangle=-\frac{1}{(\zeta-\zeta')^2},\ \langle 0|T(\zeta)T(\zeta')|0\rangle=\frac{c}{2}\frac{1}{(\zeta-\zeta')^4},\ c=1. \qquad (36)$$

Our field expansions (28,27) read $(\phi(\bar{\zeta})=\phi(\zeta)$ with $(\zeta,z,a_n)\to(\bar{\zeta},\bar{z},\bar{a}_n))$

$$\phi(\zeta)=\frac{1}{\sqrt{4\pi}}\sum_{n=\pm1,\pm2,\ldots}\frac{1}{\sqrt{|n|}}a_n z^n,\ \ [a_m,a_n]=\delta_{m+n,0}. \qquad (37)$$

It is analogous to the transition [12] to the conformal field in the conventional treatment. A completely parallel LF analysis can be given for the fermion field.

4 Conclusions

We have formulated the quantum theory of two-dimensional massless light-front fields as a unique limit of the corresponding massive fields. Its consistency is proved by the equality of the two-point functions calculated from the massless fields to the massless limit of the massive two-point functions. Our quantization scheme leads to the LF form of bosonization and to the genuine LF operator solutions of a few exactly solvable models (like the Thirring and Thirring-Wess models). The developed LF operator formalism also reproduces known results of conformal field theory.

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Pattern recognition of amino acids via a Poisson statistical approach

R.P. Mondaini and S.C. de Albuquerque Neto

Abstract A Poisson statistical approach is derived from a master equation in order to introduce a pattern recognition of amino acids in protein families. Probability distribution functions of two variables are then obtained and their level curves are associated to the occurrences of each amino acid.

1 Introduction

The statistical modelling of formation and evolution of proteins is supposed to be very useful for understanding the protein folding dynamics. The information contained fin the intermediary stagesf, which are supposed to guarantee the folding process [1]f, would be unveiled by studying the fcollection of proteins into families. We assume that nature is playing a game of allocating in families of proteins, the amino acids built at the Ribosome [2, 3]. We then start by introducing the sample space of this statistical approach with the selection of amino acid blocks of m rows (domains) and n columns (amino acids).

2 The statistical sample space

The $m \times n$ blocks are organized by specifying a number of n columns on a protein domain which is obtained from m proteins. Each domain has n_r amino acids, $r =$

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$1, 2, \ldots, m$. We then discard all domains such that $n_r < n$, and we also discard the last $(n_r - n)$ amino acids in the remaining domains. The protein families should contain at least one of these blocks. In Fig.1, an example of a $m \times n$ block from the Pfam database [4, 5] is given according to these requirements.

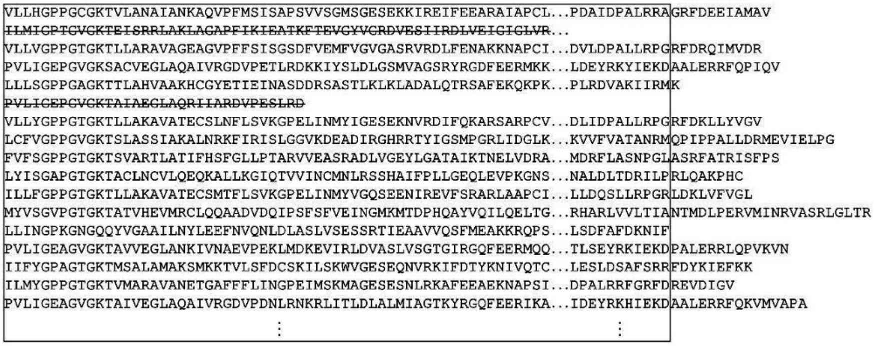

Fig. 1: A block of $m = 100 \times n = 200$ amino acids from the Pfam database.

The Pfam database version 27.0 [4, 5] has 4563 protein families classified into 515 Clans. 1441 of these families can be represented by $(m = 100 \times n = 200)$ blocks. We work with blocks of this size in the present work.

In order to describe the probabilistic space, we introduce the probability of occurrence vectors

$$\mathbf{p_j} = \big(p_j(a_1), p_j(a_2), \ldots, p_j(a_{20})\big)^{\mathrm{T}}, \quad j = 1, 2, \ldots, n, \tag{1}$$

where $a = a_1, a_2, \ldots, a_{20}$ stand for the twenty amino acids written in one letter code, $a =$ A, C, D, E, F, G, H, I, K, L, M, N, P, Q, R, S, T, V, W, Y.

The components of the vectors (7) are given by

$$p_j(a) = \frac{n_j(a)}{m} \tag{2}$$

where $n_j(a)$ is the number of occurrences of the a-amino acid in the j-th column of the $(m \times n)$ block.

We have

$$\sum_a n_j(a) = m \Rightarrow \sum_a p_j(a) = 1 \quad \forall j. \tag{3}$$

3 The master equation approach for probability evolution and Poisson distribution

A very convenient scheme for modelling the *temporal* evolution of random variables is provided by the master equation approach. We should use it here [6] for a fixed number of amino acids m on each row of the $(m \times n)$ block.

Let $p\left(n_j\big(t(a)\big)\right)$ be the probability of occurrence of the generic a-amino acid in the j-th column of the block. If $\sigma\big(t(a)\big)$ is the transition probability by unit time between the $(j-1)$-th and j-th column, the probability of occurrence of the a-amino acid at a time $t(a)+\Delta t$ will be given by

$$p\left(n_j\big(t(a)+\Delta t\big)\right) = \sigma\big(t(a)\big)\Delta t\, p\left(n_{j-1}\big(t(a)\big)\right) + \big(1-\sigma\big(t(a)\big)\big)\,\Delta t\, p\left(n_j\big(t(a)\big)\right). \quad (4)$$

We now consider that at an extra column $j=0$, there is a “universal amino acid factory”, the Ribosome, where all amino acids are already present at time $t_0(a)$, or,

$$p\left(n_0\big(t_0(a)\big)\right) = 1\,, \quad \forall a. \quad (5)$$

This also means that there are no amino acids in columns $j \neq 0$, or

$$p\left(n_j\big(t_0(a)\big)\right) = 0\,, \quad j \neq 0\,, \quad \forall a. \quad (6)$$

At the limit $\Delta t \to 0$, we get from eq. (4),

$$\frac{\partial p}{\partial t(a)}\left(n_j\big(t(a)\big)\right) = \sigma\big(t(a)\big)\left(p\left(n_{j-1}\big(t(a)\big)\right) - p\left(n_j\big(t(a)\big)\right)\right), \quad j \neq 0 \quad (7)$$

and

$$\frac{\partial p}{\partial t(a)}\left(n_0\big(t(a)\big)\right) = -\sigma\big(t(a)\big)p\left(n_0\big(t(a)\big)\right). \quad (8)$$

From eqs. (5)-(8), we can write

$$p\left(n_0\big(t(a)\big)\right) = e^{-\nu\left(t(a)\right)}, \quad (9)$$

$$p\left(n_1\big(t(a)\big)\right) = e^{-\nu\left(t(a)\right)}\nu\big(t(a)\big), \quad (10)$$

$$p\left(n_2\big(t(a)\big)\right) = e^{-\nu\left(t(a)\right)}\frac{\nu^2\big(t(a)\big)}{2}, \quad (11)$$

where

$$\nu\big(t(a)\big) = \int_{t_0(a)}^{t(a)} \sigma\big(t'(a)\big)\mathrm{d}t'(a). \quad (12)$$

The Poisson distribution will follow by finite induction as

$$p\left(n_j\big(t(a)\big)\right) = e^{-\nu\left(t(a)\right)}\frac{\nu^j\big(t(a)\big)}{j!}\,, \quad \forall j, a. \quad (13)$$

4 Two examples of statistical distributions for describing amino acid occurrences

We associate the marginal probability distribution associated to eq. (13), in order to derive distributions which are adequate to describe the occurrences of amino acids in the columns of the $(m \times n)$ blocks. We have

$$p_j\big(t(a)\big) = \int_{t_0(a)}^{t(a)} p\left(n_j\big(t'(a)\big)\right) \mathrm{d}t'(a) \tag{14}$$

From eq. (13), we can write

$$p_j\big(t(a)\big) = \frac{(-1)^j}{j!} \lim_{\alpha \to 1} \frac{\partial^j}{\partial \alpha^j} \int_{t_0(a)}^{t(a)} e^{-\alpha \nu\left(t'(a)\right)} \mathrm{d}t'(a). \tag{15}$$

Two cases should be emphasized:

1. A linear approximation [7] for $\nu\big(t(a)\big)$, $\sigma\big(t(a)\big) = \sigma(a)$:

$$\nu\big(t(a)\big) = \sigma(a)\big(t(a) - t_0(a)\big). \tag{16}$$

From eq. (15), we have

$$p_j\big(t(a)\big) = \frac{(-1)^{j+1}}{j!\,\sigma(a)} \lim_{\alpha \to 1} \frac{\partial^j}{\partial \alpha^j} \left(\frac{e^{-\alpha\,\sigma(a)(t(a)-t_0(a))} - 1}{\alpha}\right). \tag{17}$$

fWe now write

$$t_j(a) \equiv t_0(a) + j\Delta(a)\,, \quad j = 1, 2, \ldots, n, \tag{18}$$

where $t_j(a)$ is the time in which the a-amino acid is seen to occur at the j-th column of the $(m \times n)$ block and $\Delta(a)$ is the time interval for the transition of the a-amino acid between consecutive columns:

$$\Delta(a) = t_j(a) - t_{j-1}(a)\,, \quad j = 1, 2, \ldots, n, \tag{19}$$

where $\sigma(a), \Delta(a)$ is a couple of variables of the statistical distribution for this case.

We can then write

$$p_j\big(\sigma(a), \Delta(a)\big) = \tfrac{1}{\sigma(a)} \left(1 - e^{-j\,\sigma(a)\Delta(a)} \sum_{m=0}^{j} \frac{\big(j\,\sigma(a)\Delta(a)\big)^m}{m!}\right). \tag{20}$$

2. A saddle-point approximation [8,9]:
The integral of eq. (15) for $\alpha > 0$ is determined by values of $\nu\big(t(a)\big)$ on the neighbourhood of the minimum $t_m(a)$, $\nu'\big(t_m(a)\big) = 0$,

$$\nu\big(t(a)\big) \approx \nu\big(t_m(a)\big) + \frac{\nu''\big(t_m(a)\big)}{2}\big(t(a) - t_m(a)\big)^2. \tag{21}$$

We then have from eq. (15),

$$p_j\big(t(a)\big) \approx \frac{(-1)^j}{j!}\left(\frac{2\pi}{\nu''\big(t_m(a)\big)}\right)^{\frac{1}{2}} \lim_{\alpha\to 1} \frac{\partial^j}{\partial\alpha^j}\left(\alpha^{-1/2} e^{-\alpha\,\nu\left(t_m(a)\right)}\right), \tag{22}$$

and the statistical distribution for this second case will be written as

$$p_j\big(x(a),y(a)\big) \approx \tfrac{1}{j!}\left(\tfrac{2\pi}{y(a)}\right)^{\frac{1}{2}} e^{-x(a)}\left(x^j(a) + \tfrac{1}{2}\sum_{m=0}^{j-1}\tbinom{j}{m+1}(2m+1)!!\,\tfrac{x^{j-m-1}(a)}{2^m}\right) \tag{23}$$

where

$$x(a) \equiv v\big(t_m(a)\big)\,, \quad y(a) \equiv v''\big(t_m(a)\big)$$

and

$$(2m+1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2m-1)\cdot(2m+1)$$

stands for the double factorial.

5 Histograms and level curves

In this section we intend to show the advantage of the information obtained from the analysis of level curves of the surfaces $p_j\big(\sigma(a),\Delta(a)\big)$ and $p_j\big(x(a),y(a)\big)$ as compared to fthose of the corresponding histograms. First of all, we present in the 1st column of Table 1 the probabilities of occurrence for the $a =$ A amino acid of the PF03399 family obtained from a $(m = 100\times n = 200)$ block. These probability values correspond to j-value partitions of the set of j-values, $j = 1,2,\ldots,200$ and they will define the level curves, according to

$$p_{j\in J_s} = M_{j\in J_s} = \frac{n_{j\in J_s}(A)}{100} \tag{24}$$

The histogram of the data of Table 1 is represented in

The corresponding level curves of the two surfaces, eqs. (20,23), are presented in Fig. 3. The 2-dimensional domains of the variables $\sigma(a),\Delta(a)$ and $x(a),y(a)$ are obtained from an exhaustive numerical analysis of maxima and minima of the level curves [7, 10].

Table 1: The $j \in J_s$ values of the subset J_s for defining level curves of the surface $p_{j \in J_s}$. Data obtained from a (100×200) block as a representative of the PF03399 protein family $a =$ A amino acid.

$M_{j \in J_s}$	J_s
0	7, 8, 9, 14, 15, 18, 20, 21, 82, 96, 150
1/100	10, 24, 25, 38, 48, 53, 55, 102, 106, 107
2/100	11, 23, 28, 44, 47, 51, 54, 90, 98, 100, 101, 115
3/100	4, 35, 39, 40, 42, 49, 50, 65, 71, 75, 79, 91, 93, 105, 108, 117, 158, 172, 182
4/100	6, 29, 31, 43, 69, 86, 87, 103, 110, 113, 123, 125, 133, 134, 147, 149, 154, 156, 174, 175, 176, 177, 194, 197
5/100	5, 19, 22, 36, 80, 83, 92, 95, 97, 99, 104, 118, 129, 140, 141, 146, 165, 180
6/100	32, 52, 57, 73, 78, 81, 89, 126, 136, 142, 143, 153, 164, 166, 170, 171, 199
7/100	3, 27, 30, 41, 61, 84, 85, 94, 109, 119, 122, 127, 131, 132, 139
8/100	16, 26, 33, 37, 46, 56, 68, 70, 74, 88, 111, 112, 124, 145, 151, 159, 173, 188, 195, 200
9/100	1, 17, 58, 60, 62, 114, 116, 137, 138, 169, 187, 190, 191, 193
10/100	34, 45, 59, 77, 161, 185, 189
11/100	66, 72, 121, 128, 152, 155, 168, 178, 179, 198
12/100	13, 148, 162, 183, 196
13/100	120, 167
14/100	12, 67, 76, 130, 135, 163
15/100	2, 64, 181, 186
16/100	63, 160, 184, 192
17/100	144
18/100	157

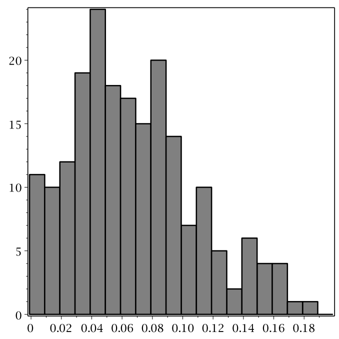

Fig. 2 The histogram corresponding to Table 1. $a =$ A amino acid, Protein family PF03399.

6 Concluding remarks

In the present work, we introduced statistical modelling of the evolution of protein families and two proposals fregarding probability distributions fthat are derived from a master equation approach. An exhaustive application of these methods to protein families of the Pfam database is now in progress. An essential step of this application is the comparison of the level curves of the surfaces $p\big(\sigma(a), \Delta(a)\big)$, $p\big(x(a), y(a)\big)$ with the corresponding histograms. A careful choice

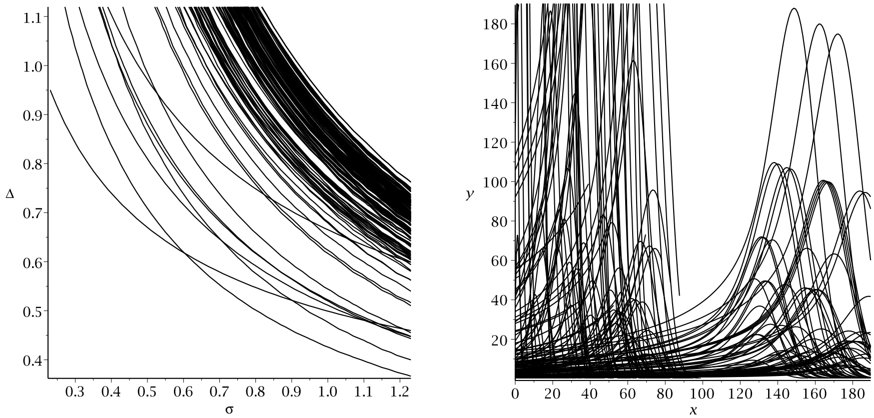

Fig. 3: $a =$ A amino acid from Protein family PF03399. The level curves of the surface $p_j\big(\sigma(a), \Delta(a)\big)$ - linear approximation (left). The level curves of the surface $p_j\big(x(a), y(a)\big)$ - saddle-point approximation (right).

of the 2-dimensional domains of variables is also essential for the success of the pattern recognition process [7, 10]. It seems that a conformal transformation of variables could provide a better layout of level curves for the saddle-point approximation case. However, a new method of representation in terms of the loci of Steiner points as arcs of circles whose centers and diameters are given by the domains of the variables will lead to a more efficient representation [11]. The fundamental part of these developments is now concluded and will be published elsewhere.

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Real pseudo-orthogonal groups and the canonical commutation relations

Patrick Moylan

Abstract Let $\mathbf{W}_n(\mathbb{R})$ be the Weyl algebra of index n over $\mathbb{R}$ and let $\widetilde{\mathfrak{D}}(\mathfrak{so}(2,1))$ be a certain extension of the skew field of fractions of $\mathfrak{U}(\mathfrak{so}(2,1))$, the universal enveloping algebra of $\mathfrak{so}(2,1)$. In a previous work we have established a skew field isomorphism between $\widetilde{\mathfrak{D}}(\mathfrak{so}(2,1))$ and $\mathfrak{D}_{(1,1)}(\mathbb{R})$ where $\mathfrak{D}_{1,1}(\mathbb{R})$ is the fraction field of $\mathbf{W}_{1,1}(\mathbb{R}) \simeq \mathbf{W}_1(\mathbb{R}) \otimes \mathbb{R}(y)$ with $\mathbb{R}(y)$ being the ring of polynomials over $\mathbb{R}$ in the indeterminate y. Using this isomorphism, we were able to construct, out of unitary and irreducible representations of the universal covering group of $SO_0(2,1)$, representations of $\mathbf{W}_1(\mathbb{R})$ with all of the desired properties required by physics, including hermicity of the momentum and position operators. Thus, we have obtained the canonical commutation relations and acceptable representations of them out of the $\mathfrak{so}(2,1)$ symmetry. In this work we investigate generalizations of the above results to higher dimensions. In particular, we describe generalizations with $\widetilde{\mathfrak{D}}(\mathfrak{so}(2,1))$ replaced by $\widetilde{\mathfrak{D}}(\mathfrak{so}(p,q))$ and $\mathfrak{D}_{1,1}(\mathbb{R})$ replaced by $\mathfrak{D}_{p+q-2,1}(\mathbb{R})$. As in the $\mathfrak{so}(2,1)$ case, we make use of our results to obtain applications to representations.

1 Introduction

Noncommutative localization is used in both mathematics and physics to relate different, i.e., non-isomorphic, algebraic structures by embedding them into larger structures which may be related to one another by isomorphism. One of the most famous examples in mathematics, and which is very much related to what we do in this paper, is the Gelfand-Kirillov conjecture [3].

A brief description of this famous conjecture is the following. Let $\mathbb{K}$ be an algebraically closed field. Given a Lie algebra L over $\mathbb{K}$, we denote by $\mathfrak{U}(L)$ the universal

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enveloping algebra of L. Since $\mathfrak{U}(L)$ is a Noetherian domain, it admits a field of fractions which we shall denote by $\mathfrak{D}(L)$. Let $\mathbf{W}_n(\mathbb{K})$ denote the Weyl algebra of index n over the field $\mathbb{K}$. It is generated over $\mathbb{K}$ by $2n$ generators $p_1,\dots,p_n,q_1,\dots,q_n$ subject to the relations $[p_i,p_j]=[q_i,q_j]=0$ and $[p_i,q_j]=\delta_{ij}$ for all $i,j\le n$. Given a collection of free variables $y_1,\dots,y_s$, we define

$$\mathbf{W}_{n,s}(\mathbb{K}) := \mathbf{W}_n(\mathbb{K})\otimes\mathbb{K}[y_1,\dots,y_s].$$

Being a Noetherian domain, the algebra $\mathbf{W}_{n,s}(\mathbb{K})$ also admits a field of fractions denoted by $\mathfrak{D}_{n,s}(\mathbb{K})$. In [3] Gelfand and Kirillov put forth the following conjecture:

Gelfand–Kirillov conjecture. *If* $\mathrm{char}(\mathbb{K})=0$ *and L is the Lie algebra of an algebraic $\mathbb{K}$-group, then* $\mathfrak{D}(L)\cong\mathfrak{D}_{n,s}(\mathbb{K})$ *for some n,s depending on L.*

A major breathrough for the case in which L is simple occurred in 2010 when Alexander Premet proved that the conjecture fails for simple Lie algebras of type B_n ($n\ge 3$), D_n ($n\ge 4$), E_6, E_7, E_8 or F_4 leaving the conjecture unsettled only for the remaining case of C_n [12]. The conjecture also makes sense over fields that are not algebraically closed. Clearly, if $\mathfrak{D}(L)\cong\mathfrak{D}_{n,s}(\mathbb{F})$ over a field $\mathbb{F}$ (where $\mathfrak{D}(L)$ now means the Lie field of the Lie algebra L over the field $\mathbb{F}$), then $\mathfrak{D}(L\otimes_{\mathbb{F}}\mathbb{K})\cong\mathfrak{D}_{n,s}(\mathbb{K})$ where $L\otimes_{\mathbb{F}}\mathbb{K}$ is a Lie algebra over a field extension $\mathbb{K}$ of $\mathbb{F}$. Thus, at least for certain cases, we can reduce the case of a non-algebraically closed field $\mathbb{F}$ to its respective algebraic closure. In particular, this implies that the Gelfand-Kirillov conjecture itself fails for any real form of a complex Lie algebra of type B_n ($n\ge 3$), D_n ($n\ge 4$), E_6, E_7, E_8 or F_4. It is interesting that the physically important cases of $SO_0(2,3)$ and $SO_0(1,4)$ are still unsettled. This is so because Premet's results for B_n hold only for $n\ge 3$ and, since $\mathrm{B}_2\simeq\mathrm{C}_2$, the validity of the conjecture for B_2 and its real forms is still unknown. For the status of the conjecture in the general case (i.e., for L not simple) we refer to [1].

One of the reasons why localization is important to physics is for understanding the structure of the observables and invariants of a quantum physical system associated with a given Lie algebra and their relationship to observables associated with the Weyl algebras, such as position and momentum. Surely such physical considerations must have played an important role in motivating Gelfand and Kirillov to formulate their conjecture. The Gelfand-Kirillov conjecture demonstrates that for Lie algebras L for which it is true, the Lie field of L is no more complicated than that of the Lie field associated with the corresponding extension of the Weyl algebra, and, in this sense, the Heisenberg canonical commutation relations follow from and exist as relations in the Lie field $\mathfrak{D}(L)$ when the two algebras have isomorphic Lie fields.

2 Line bundles over M

Let $n = p+q$ ($q > 1$) and consider $M^{(p,q)} = \mathbb{R}^n$ with the quadratic form

$$Q(x) = x_0^2 + x_1^2 + \ldots + x_{p-1}^2 - x_p^2 - \ldots - x_{p+q-1}^2 \; (x = (x_0, x_1, \ldots, x_{p+q-1}) \in \mathbb{R}^n). \quad (1)$$

Let $\mathbb{K}^*$ be the cone $\{x \in \mathbb{R}^m | Q(x) = 0, x \neq 0\}$, and let M bet the quotient of $\mathbb{K}^*$ with respect to the equivalence relation $x \sim \lambda x$. M is naturally diffeomorphic to $(S^{p-1} \times S^{q-1})/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is the product of antipodal maps on S^{p-1} and S^{q-1}. We denote $S^{p-1} \times S^{q-1}$ by $\bar{M}$.

Define the line bundle $L^\sigma(M)$ over M associated with the character $\lambda \to |\lambda|^{-\sigma}$ of $\mathbb{R}^*$ as the bundle whose fibre over $[x] \in M$ is the set of all pairs $(\lambda x, |\lambda|^\sigma) \in \mathbb{K} \times \mathbb{C}$, ($\sigma \in \mathbb{C}$). Denote by $\Gamma^\sigma(M)$ the space of smooth sections of $L^\sigma(M)$. There is a unique isomorphism between $\Gamma^\sigma(M)$ and the space of smooth functions $f : \mathbb{K}^* \longrightarrow \mathbb{C}$ that satisfy the homogeneity condition [5], [6]

$$f(\lambda x) = |\lambda|^{-\sigma} f(x) . \quad (2)$$

$\Gamma^\sigma(M)$ is an $SO_0(p,q)$-module with respect to the representation $\pi^\sigma(G)$ ($G = SO_0(p,q)$), defined by $(\pi^\sigma(g) f)(x) = f(g^{-1}x)$, where $f \in \Gamma^\sigma(M)$, $g \in SO_0(p,q)$, $x \in \mathbb{K}^*$, and $g^{-1} x$ denotes the action of g^{-1} on $x \in \mathbb{K}^*$. We denote the associated representation of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p,q)$ by $d\pi^\sigma(\mathfrak{g})$. Since any function $f \in C^\infty(\mathbb{K}^*)$ satisfying Eq. (2) is completely determined by its values on M and lifts to an even function on $\bar{M} = S^{p-1} \times S^{q-1}$, we can use the homogeneity condition, Eq. (2), to obtain the following parallelized form of the representation:

$$\pi^\sigma(g)\phi(\xi) = |(g^{-1}\xi)|^\sigma \phi(\bar{g}^{-1}\xi) \quad (3)$$

where $\phi \in C^\infty(S^{p-1} \times S^{q-1})$, $\phi(-\xi) = \phi(\xi)$, $\xi \in S^{p-1} \times S^{q-1}$ and

$$|x| = \sqrt{x_0^2 + x_1^2 + \ldots + x_{p-1}^2} = \sqrt{x_p^2 + x_{p+1}^2 + \ldots + x_{p+q-1}^2} \quad (4)$$

for $x = (x_0, x_1, \ldots + x_{p-1}, x_p, \ldots + x_{p+q-1}) \in \mathbb{K}^*$ and $\bar{g}^{-1}\xi \in S^{p-1} \times S^{q-1}$ is the image of ξ under the action for $g^{-1} \in G$ on $S^{p-1} \times S^{q-1}$. We call this parallelized form of the representation $\pi^\sigma(G)$ the "curved parallelization".

Now consider the subset of $\mathbb{K}^*$ defined by $V = \{x \in \mathbb{K}^* | x_0 + x_{p+q-1} = 1\}$. Let $S = \{x \in \mathbb{K}^* | x_0 + x_{p+q-1} = 0\}$ and let p be the map $p : \bar{M}\backslash\{S\} \to V \simeq \mathbb{R}^{p+q-2}$ with $p(\xi_0, \xi_1, \ldots \xi_p, \ldots \xi_{p+q-1}) = \left(\frac{\xi_0}{\xi_0+\xi_{p+q-1}}, \frac{\xi_1}{\xi_0+\xi_{p+q-1}}, \ldots \frac{\xi_p}{\xi_0+\xi_{p+q-1}}, \ldots \frac{\xi_{p+q-1}}{\xi_0+\xi_{p+q-1}}\right)$. Let $x_i = \frac{2\xi_i}{\xi_0+\xi_{p+q-1}}$ ($i = 1, 2, \ldots p+q-2$). The inverse map $p^{-1} : V \to \bar{M}\backslash\{S\}$ is $\xi_0 = \frac{F}{(F^2+x_1^2+\ldots+x_{p-1}^2)^{1/2}}$, $\xi_i = \frac{x_i}{(F^2+x_1^2+\ldots+x_{p-1}^2)^{1/2}}$, $\xi_{p+q-1} = \frac{D}{(F^2+x_1^2+\ldots+x_{p-1}^2)^{1/2}}$ with $F = 1 - \frac{x^2}{4}$, $D = 1 + \frac{x^2}{4}$ and $x^2 = x_1^2 + x_2^2 + \ldots + x_{p-1}^2 - x_p^2 - \ldots + x_{p+q-2}^2$. We now introduce the "flat parallelization" for the representation π^σ. It is specified by the equation

$$C^\infty(\bar{M}) \ni \phi(\xi) \to \tilde{\phi}(\xi(x_i)) = |(F^2 + x_1^2 + \,..\, + x_{p-1}^2|^{\sigma/2}\phi(\xi) \in C^\infty(\mathbb{R}^{p+q-2}). \quad (5)$$

Using this equation we determine the parallelized actions of the infinitesimal generators of $G = SO_0(p,q)$ in the representation $\pi^\sigma(G)$. Our results are compiled in Table I. (Compare the $\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$ case treated in [11] which is somewhat prototypical of the general case considered here. Definitions and notation for our infinitesmal generators are straightforward generalizations to $SO_0(p,q)$ of those given in [11] which were taken from [13].)

Table 1: Actions of infinitesimal generators of $SO_0(p,q)$ for the representation $\pi^\sigma(SO_0(p,q))$ in the flat parallization[a].

infinitesmal generator	action on $C^\infty(\mathbb{R}^{p+q-2})$
$\mathbf{L}_{0,q+p-1}$	$S+\sigma$
$\mathbf{L}_{i,q+p-1}$ $(1 \le i \le p-1)$	$-D\partial_i + \frac{1}{2}x_i(S+\sigma)$
$\mathbf{L}_{j+p-1,q+p-1}$ $(1 \le j \le q-1)$	$-D\partial_{j+p-1} - \frac{1}{2}x_{j+p-1}(S+\sigma)$
$\mathbf{L}_{0,i}$ $(1 \le i \le p-1)$	$-F\partial_i - \frac{1}{2}x_i(S+\sigma)$
$\mathbf{L}_{0,j+p-1}$ $(1 \le j \le q-1)$	$-F\partial_{j+p-1} + \frac{1}{2}x_{j+p-1}(S+\sigma)$
$\mathbf{L}_{i,j+p-1}$ $(1 \le i \le p-1, 1 \le j \le q-1)$	$-(x_i\partial_{j+p-1} + x_{j+p-1}\partial_i)$
$\mathbf{L}_{i,k}$ $(1 \le i,k \le p-1)$	$-(x_i\partial_k - x_k\partial_i)$
$\mathbf{L}_{j+p-1,\ell+p-1}$ $(1 \le j,\ell \le q-1)$	$x_{j+p-1}\partial_{\ell+p-1} - x_{\ell+p-1}\partial_{j+p-1}$

[a] $S = x_1\partial_1 + x_2\partial_2 + \,...\, + x_{p-1}\partial_{p-1} + x_p\partial_p + \,...\, + x_{p+q-2}\partial_{p+q-2}$ and $\partial_i = \frac{\partial}{\partial x^i}$. (For $p = 1$ the 2nd, 4th, 6th and 7th rows of the Table are vacuous.)

3 The isomorphism $\widetilde{\mathrm{d}(\mathfrak{so}(p,q))} \simeq \mathfrak{D}_{p+q-2,1}(\mathbb{R})$

We use the table to construct an isomorphism between $\mathrm{d}_{p+q-2,1}(\mathbb{R})$ and an algebraic extension $\widetilde{\mathrm{d}(\mathfrak{so}(p,q))}$ of the Lie field $\mathrm{d}(\mathfrak{so}(p,q))$ of $\mathfrak{so}(p,q)$, the Lie algebra of $SO_0(p,q)$. We start with $\mathbf{W}_{p+q-2,1}(\mathbb{R}) \simeq \mathbf{W}_{p+q-2} \otimes \mathbb{R}[Y]$ and let $\mathrm{d}_{p+q-2,1}(\mathbb{R})$ be its quotient field. Based on the results of the table, we define $p+q-2$ translation generators and $p+q-2$ position operators as follows:

$$\mathbf{P}_i := -\frac{1}{2}(\mathbf{L}_{0,i} + \mathbf{L}_{i,q+p-1}), \quad (6)$$

$$\mathbf{Q}_i := \left\{ \sum_{k=1}^{p+q-2} \mathbf{L}_{i,k}\mathbf{P}^k + (\mathbf{L}_{0,q+p-1} - Y)\mathbf{P}_i \right\} \Box^{-1}\,, \quad (7)$$

where i takes values from 1 to $p+q-2$, Y commutes with the $\mathbf{L}_{ij}$ and

$$\Box = \mathbf{P}_1^2 + \mathbf{P}_2^2 + \ldots + \mathbf{P}_{p-1}^2 - \mathbf{P}_p^2 - \mathbf{P}_{p+1}^2 - \,\ldots - \mathbf{P}_{p+q-2}^2\,. \quad (8)$$

The quadratic Casimir operator of $SO_0(p,q)$ is $\mathbf{C}_2 = \frac{1}{2}\sum_{i,j=0}^{p+q-1} \mathbf{L}_{ij}\,\mathbf{L}^{ji}$. The real Lie algebra $\mathfrak{so}(p,q)$ has a natural involutive automorphism given by $\mathbf{L}_{ij}^{\dagger} = -\mathbf{L}_{ij}$ which extends to an involution on the whole Lie field $\mathrm{d}(\mathfrak{so}(p,q))$.

Theorem 1. *Let Y be such that $\mathbf{C}_2 = \{Y(Y-(p+q-2)\}\cdot I$. Then the $\mathbf{P}_i$ and $\mathbf{Q}_j$ satisfy*

$$[\mathbf{P}_i,\mathbf{Q}_j] = \delta_{ij},\ [\mathbf{P}_i,\mathbf{P}_j] = 0,\ [\mathbf{Q}_i,\mathbf{Q}_j] = 0\,. \tag{9}$$

Define τ such that $\tau(p_i) = \mathbf{P}_i$, $\tau(q_i) = \mathbf{Q}_i$ and $\tau(Y) = Y$. τ can be extended by linearity to all of $\mathfrak{D}_{p+q+2,1}(\mathbb{R})$ to give an isomorphism of Lie fields $\mathfrak{D}_{p+q+2,1}(\mathbb{R}) \simeq \widetilde{\mathrm{d}(\mathfrak{so}(p,q))}$ where $\widetilde{\mathrm{d}(\mathfrak{so}(p,q))}$ is $\mathfrak{D}(\mathfrak{so}(p,q))[Y]/R[Y]$ with $R[Y]$ being the maximal ideal in $\mathfrak{D}(\mathfrak{so}(p,q))[Y]$ generated by the relation $\mathbf{C}_2 - \{Y(Y-(p+q-2)\}\cdot I$. Further let Y satisfy

$$Y + Y^{\dagger} = (p+q-2)\}\cdot I\,; \tag{10}$$

then the $\mathbf{P}_i$ and $\mathbf{Q}_j$ are skew-symmetric translation generators and symmetric position operators, respectively.

The most difficult part of the proof is in establishing the commutativity of the $\mathbf{Q}_i$ defined by Eq. (7). It involves lengthy computations for which we do not have sufficient space here, and the proof of the theorem will be published elsewhere. Note that the part of the proof establishing that τ is an isomorphism follows easily from the fact that $\mathbf{W}_{p+q-2}(\mathbb{R})$ is simple.

4 Applications

Let $\tau|_{\mathbf{W}_{p+q-2}(\mathbb{R})}$ denote the mapping τ restricted to $\mathbf{W}_{p+q-2}(\mathbb{R})$. Then $\tau|_{\mathbf{W}_{p+q-2}(\mathbb{R})}$ gives an isomorphism from $\mathbf{W}_{p+q-2}(\mathbb{R})$ onto its image in $\widetilde{\mathrm{d}(\mathfrak{so}(p,q))}$ such that $\tau(p_i) = \mathbf{P}_i$ and $\tau(q_j) = \mathbf{Q}_j$. In order to construct representations of $\mathbf{W}_{p+q-2}(\mathbb{R})$ out of representations of $\mathfrak{U}(\mathfrak{so}(p,q))$ by using τ, we need the following lemma [2]:

Lemma 1. *Suppose $f : R \longrightarrow R_1$ is a ring homomorphism and Q is a left (resp. right) quotient ring of R with respect to S. If $f(s)$ is a unit in R_1 for every $s \in S$, then there exists a (unique) ring homomorphism $g : Q \longrightarrow R_1$ which extends f.*

($f(s)$ is a unit in R_1 means that $f(s)$ is both left and right invertible i.e., $\exists c \in R_1$(resp. $b \in R_1$) such that $cf(s) = I_{R_1}$(resp.$f(s)b = I_{R_1}$).)) For us this means that for a given representation $d\pi$ of $\mathfrak{U}(\mathfrak{so}(p,q))$, zero should lie in the resolvent set of $d\pi(\Box)$. Since $-\Box = -\sum_{i=1}^{p+q-2} \mathbf{P}_i\mathbf{P}^i$ is the square of the mass operator, this implies that the spectrum of the mass operator in the representation should be one-sided and bounded from above or below by zero. Likely candidates for such representations of $\mathfrak{so}(p,q)$ are thus positive energy ones. Since positive energy representations of a Lie algebra are associated with invariant cones of positivity in the Lie algebra, we should consider $\mathfrak{so}(p,q)$'s which contain non-trivial invariant cones. A corollary of a well-known theorem of Kostant is the following: *there exists a non-trivial invariant causal cone in a simple Lie algebra* $\mathfrak{g}$ *if* $\mathfrak{g}$ *is an hermitian symmetric Lie algebra* [10]. A list of simple hermitian symmetric Lie algebras is given in Helgason [4]. They include: $\mathfrak{so}(2,n)(n>3)$, $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{su}(p,q)$ $(p \geq q \geq 1)$. Since $\mathfrak{so}(2,1) \simeq \mathfrak{sp}(1,\mathbb{R})$ and $\mathfrak{so}(2,3) \simeq \mathfrak{sp}(2,\mathbb{R})$, $\mathfrak{so}(p,q)$'s which are simple and hermitian symmetric are $\mathfrak{so}(2,1)$ and $\mathfrak{so}(2,q)$ with $q \geq 3$.

The $\mathfrak{so}(2,1)$ case is treated in detail in [9]. There we show that it is necessary to go to the four-fold covering group of $SO_0(2,1)$ in order to find representations which satisfy all of the conditions of Theorem 1 and also of Lemma 1, so that on those representation spaces we obtain representations of $\mathbf{W}_1(\mathbb{R})$ with the desired symmetricity properties for the position and momentum operators.

We now describe some analogous results for $\mathfrak{so}(2,q)$ $(q \geq 3)$ involving the representations $d\pi^\sigma$. Let V^σ denote the representation space of the representation $d\pi^\sigma(\mathfrak{so}(2,q))$ introduced in Section 2. A study of the action of $SO(2,q)$ on the $K = SO(2) \times SO(q)$-types shows that V^σ is irreducible, except when $p+q$ even, $q > p = 2$ and $\sigma = (p+q-2)/2 = q/2$ [7], [14]. If V^σ is irreducible, then zero cannot be in the resolvent set of $d\pi^\sigma(\Box)$, so we consider only V^σ for which $q > p = 2$, $p+q$ even and $\sigma = (p+q-2)/2 = q/2$. For such σ, we have that V^σ decomposes into the direct sum of three irreducible $SO_0(2,q)$ invariant subspaces [7]:

$$V^\sigma = V^+ \oplus V^0 \oplus V^-.$$

V^+ and V^- are positive and negative energy subspaces of V^σ and V^0 is the tachyonic subpace (i.e., on V^0 the spectrum of the square of the mass operator is negative). We claim that on V^+ and V^- all of the conditions of Theorem 1 and Lemma 1 are satisfied, so that as in the $\mathfrak{so}(2,1)$ case, we have on V^+ and V^- representations of $\mathbf{W}_q(\mathbb{R})$ with skew-symmetric translation generators and symmetric position operators. To see this, we first verify the condition $d\pi^\sigma(Y+Y^\dagger) = d\pi^\sigma(Y) + d\pi^\sigma(Y)^\dagger = (p+q-2)\cdot I$. ($d\pi^\sigma(Y)^\dagger$ is the adjoint of the operator $d\pi^\sigma(Y)$.) Now Y is defined by $\mathbf{C}_2 = -\{Y(Y-(p+q-2)\}\cdot I$ and for the representation $d\pi^\sigma(\mathfrak{so}(2,q))$ with $\sigma = q/2$, we have $d\pi^\sigma(\mathbf{C}_2) = q^2/4$ [14]. From this we obtain the equation $d\pi^\sigma(Y^2 - qY + \frac{q^2}{4}) = 0$ with solutions $d\pi^\sigma(Y) = \pm\frac{q}{2}\cdot I$. So, in order to satisfy the above condition on Y, we take $d\pi^\sigma(Y) = +\frac{q}{2}\cdot I = d\pi^\sigma(Y)^\dagger$.

Finally we must show that on V^+ and V^- the spectrum for the mass operator in $d\pi^{\frac{q}{2}}(\mathfrak{so}(2,q))$ does not include zero and is bounded from above and below by zero, respectively. For simplicity, we consider the case of $q=4$ which is completely representative of the general case. For $q=4$, we have $K = SO(2) \times SO(4)$ and the decomposition of $V^{\sigma=2}$ into K irreducibles is as follows. Working in the curved parallelization, we introduce a basis of C^∞ functions on $\bar{M}$, called the K-finite basis. Denote elements of this basis by $|k\ell mn>$ with k,ℓ,m and n all integers such that $k,\ell \geq 0$ and $-\ell \leq m \leq \ell$. Let $\mathscr{H}_{n,p}$ = linear span all of $|k\ell mn>$ with $k+\ell = p$ and where $n = p$ mod 2 must be satisfied [11]. Then $V^{\sigma=2} = V^+ \oplus V^0 \oplus V^-$ with [11]

$$V^+ = \sum_{n>p} \mathscr{H}_{n,p}, \quad V^0 = \sum_{|n|\leq p} \mathscr{H}_{n,p}, \quad V^- = \sum_{n<-p} \mathscr{H}_{n,p}.$$

Since the spectrum of $i\partial_\tau$, where ∂_τ is the $SO(2)$ generator, is n, we see that the spectrum of $i\partial_\tau$ is strictly greater than one for V^+ and strictly less than minus one for V^-. The desired conclusion follows from this along with fact that $(\psi, d\pi^\sigma(\mathbf{P}_0)\psi) \geq (\psi, d\pi^\sigma(i\partial_\tau), \psi)$ on V^+ and $(\psi, d\pi^\sigma(\mathbf{P}_0)\psi) \leq (\psi, d\pi^\sigma(i\partial_\tau)\psi)$ on V^- [8] where $(\,\cdot,\,\cdot\,)$ is the inner product in the Hilbert space completion of $C^\infty(\bar{M})$.

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Quantum isometry groups and Born reciprocity in 3d gravity

Prince K. Osei

Abstract Born reciprocity (or semidualisation) is an algebraic operation defined using quantum group (Lie bialgebra) methods. It is shown that this map provides a way of relating quantum groups that emerge in the application of the combinatorial quantisation programme to the Chern-Simons formulation of 3d gravity. It leads to the interpretation of the semiduality relation bewtween pairs of quantum groups arising from the same classical action as a physical equivalence of associated quantum theories after a suitable exchange of position and momentum degrees of freedom.

1 Introduction

Gravity in three dimensions [8] has become a large research subject area as it provides a fertile testing ground for ideas about quantum gravity in the more physical four dimensions. In particular, it provides a perfect setting for exploring the proposed connection between quantum gravity and noncommutative geometry [10]. Noncommutative spacetime in 3d quantum gravity has been studied in different approaches, see for example [4, 12, 14, 16, 19, 31] and [30] for a recent account. Any deformation of either the geometry into a noncommutative version or of the isometry group into a Hopf algebra are equivalent. Our focus here is to explore the role of Hopf algebras as tools that one can easily use in this framework.

The first aim of this talk is to give an account of the various physical and mathematical interpretations of Born reciprocity applied to Hopf algebras (Lie bialgebras) in the context of 3d gravity [17, 25, 26]. It turns out that it provides an equivalence relation between quantum groups that arise in the combinatorial quantisation of the Chern-Simons formulation of 3d gravity.

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2 Classical 3d gravity and geometric structures

The solutions to Einstein equations in 3d are locally isometric to certain model spacetimes which are completely determined by the signature of spacetime and the sign (or vanishing) of the cosmological constant Λ. The isometry groups of these model spacetimes are local isometries of 3d gravity. The structures of the corresponding isometry groups G_λ for the various values of Λ are summarised in Table 1. The family of Lie algebras that arise as isometry Lie algebras in 3d gravity is

Table 1: Local isometry groups in $3d$ gravity.

Λ	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$ISO(3) = SU(2) \ltimes \mathbb{R}^3$	$ISO(2,1) = SU(1,1) \ltimes \mathbb{R}^3$
$\Lambda > 0$	$SO(4) \cong \frac{(SU(2)\times SU(2))}{\mathbb{Z}_2}$	$SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$
$\Lambda < 0$	$SO(3,1) \cong \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$	$SO(2,2) \cong \frac{(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))}{\mathbb{Z}_2}$

denoted by $\mathfrak{g}_\lambda$. The Lie bracket of $\mathfrak{g}_\lambda$ is

$$[J_a, J_b] = \varepsilon_{abc} J^c, \quad [J_a, P_b] = \varepsilon_{abc} P^c, \quad [P_a, P_b] = \lambda \varepsilon_{abc} J^c, \tag{1}$$

where J_a, $a = 0,1,2$ are the generators of the Lie algebra $\mathfrak{g}$ of $so(3)$ or $so(2,1)$ and the metric $\eta_{ab} = \eta^{ab}$, is the Euclidean metric $\mathrm{diag}(1,1,1)$ or the Lorentzian metric $\mathrm{diag}(1,-1,-1)$ with $\varepsilon_{012} = \varepsilon^{012} = 1$. The Lie algebra $\mathfrak{g}_\lambda$ admits a two-parameter family of symmetric Ad-invariant bilinear non-degenerate forms [22, 23, 32]. The most general such inner product is given by the linear combination

$$(\cdot,\cdot)_\tau = \alpha(\cdot,\cdot)_s + \beta(\cdot,\cdot)_t, \tag{2}$$

in terms of two real parameters α, β. The form (2) is non-degenerate provided $\tau\bar{\tau} = \alpha^2 - \lambda\beta^2 \neq 0$, where $\tau = \alpha + \theta\beta \in R_\lambda$, $\theta^2 = \lambda$ and R_λ is a commutative ring obtained from $\mathbb{R}^2$ with the usual addition and λ-dependent multiplication law [23].

3 Chern-Simons formulation of 3d gravity and combinatorial quantisation

Our interest here is the Chern-Simons action with the most general inner product (2) [5, 23, 24]. Consider a three 3d manifold M^3 of the product topology $\mathbb{R} \times \Sigma$, where Σ is an oriented two dimensional manifold, possibly with handles and punctures (Physically, Σ represents 'space' and the puncture particles). The gauge field of Chern-Simons theory is locally a one-form A on the spacetime with values in the Lie algebra $\mathfrak{g}_\lambda$. In terms of the generators J_a and P_a, it is given by $A = \omega_a J^a + e_a P^a$, where $\omega = \omega^a J_a$ is the spin connection on the frame bundle and the one-form e_a as a dreibein. The curvature of this connection combines the Riemann curvature R, the torsion T and a cosmological term, see [23] for details.

The Chern-Simons action for the gauge field A is

$$I_\tau(A) = \int_M (A \wedge dA)_\tau + \frac{1}{3}(A \wedge [A,A])_\tau. \tag{3}$$

The equations of motion which follow from the general action is the flatness condition for the Cartan connection,

$$F = dA + A \wedge A = 0. \tag{4}$$

Integrating by parts and ignoring boundary terms, we identify the term proportional to α as the usual Einstein-Hilbert action for 3d gravity with a cosmological constant and the β term as a non-gravitational term dual to the gravitational one. One can make contact with the physical constants of 3d gravity via the identification $\alpha = \frac{1}{16\pi G}, \lambda = -c^2\Lambda$, where G is Newton's constant in 2+1 dimensions, c is the speed of light and taken to be imaginary in the case of Euclidean signature.

The flatness condition (4) implies that the phase space of 3d gravity on M^3 in the Chern-Simons framework can be characterised by the space of flat G_λ-connections on M^3, modulo gauge transformations. Indeed, the phase space is the moduli space of flat G_λ-connections on Σ equipped with the so called Atiyah-Bott symplectic structure defined in terms of (2) [32]. The application of the combinatorial quantisation programme to the Chern-Simons formulation of 3d gravity [1–3,7,20,21,28] has provided a systematic way of studying the role of quantum groups and noncommutative geometry in 3d quantum gravity. The starting and most crucial point of this construction is a description of the Poisson structure on an extended classical phase space in terms of a classical r-matrix (a solution of the classical Yang-Baxter equation(CYBE)) [9,15]. This description, originally formulated by Fock and Rosly [11], requires that the r-matrix contain the information of the inner product used in defining the Chern-Simons action in a compatible way. More precisely, an r-matrix is compatible with a Chern-Simons action if it satisfies the CYBE and if its symmetric part is equal to the Casimir associated to the Ad-invariant, non-degenerate symmetric bilinear form (2) used in defining the Chern-Simons action. These r-matrices are also known to be the classical limit of universal quantum R-matrices associated to certain quantum groups [29]. These quantum groups are deformations of the classical isometry groups and thus natural candidates for the quantum isometry groups of 3d quantum gravity. Classical r-matrices therefore provide a bridge between Chern-Simons theory and Hopf algebras allowing the Hopf algebras to emerge in a natural way.

The resulting quantum picture is a deformation of the model spacetimes into non-commutative spaces, a replacement of the local isometry groups with quantum isometry groups (QIGs). Table 2 gives the list of relevant QIGs. Again, we refer to the review [29] for a detailed discussion and further references on the precise

Table 2: Quantum isometry groups in 3d gravity.

Λ	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$D(U(\mathfrak{su}(2)))$	$D(U(\mathfrak{su}(1,1)))$
$\Lambda > 0$	$D(U_q(\mathfrak{su}(2)))$, q root of unity	$D(U_q(\mathfrak{su}(1,1)))$ $q \in \mathbb{R}$
$\Lambda < 0$	$D(U_q(\mathfrak{su}(2)))$, $q \in \mathbb{R}$	$D(U_q(\mathfrak{sl}(2,\mathbb{R})))$, $q \in U(1)$

description of the classical phase space, the construction of the physical Hilbert space and the role of the QIGs.

The compatibility requirement of Fock and Rosly's described above does not uniquely specify the classical r-matrices. Other solutions are possible and known. If one takes the relevant Casimir operator for the 'gravitational' bilinear form $\alpha(\cdot,\cdot)_s$ $K_t = \alpha^{-1}(J_a \otimes P^a + P_a \otimes J^a)$, it is known that [22, 28], the class of compatible r-matrices whose associated quantum groups are displayed in Table 2 (except for the case $\Lambda > 0$) is given by

$$r_D = 2\alpha^{-1}(P_a \otimes J^a + \varepsilon_{abc} n^a J_b \otimes J^c), \quad n_a n^a = -\lambda. \tag{5}$$

This equips the Lie algebra $\mathfrak{g}_\lambda$ with the structure of the classical double or double cross sum. However, in the Lorentzian case with vanishing cosmological constant for example, besides the double r-matrix which comes from the family (5), there is also the r-matrix [22, 23],

$$r_{B_0} = 2\alpha^{-1}\left(\frac{1}{2}(P_a \otimes J^a + J_a \otimes P^a) + \varepsilon_{abc} m^a (P^b \otimes J^c + J^b \otimes P^c)\right), \quad \mathbf{m}^2 = -1. \tag{6}$$

This endows the Lie algebra $\mathfrak{g}$ with a bicross sum structure. The associated quantum group for the double r-matrix in this case is the quantum double $D(U(\mathfrak{su}(1,1)))$ while that of (6) is the bicrossproduct or κ-Poincaré quantum group

$$\mathbb{C}[AN(2)] \blacktriangleright\!\!\triangleleft_s U(\mathfrak{sl}(2,\mathbb{R})).$$

Consequently for each classical r-matrix which satisfies Fock-Rosly compatibility requirement, the associated quantum group is an equally valid candidate and, therefore, the associated quantum groups are equally valid contenders for the role of quantum isometry groups. For a detailed and general construction of the most general r-matrix compatible with the generalised Casimir K_τ associated to the inner product (2), we refer the reader to a forthcoming paper [27].

The big picture here is to understand the set of all possible QIGs which arise in in 3d gravity via the combinatorial quantisation procedure and how they are mathematically and physically related. A key observation made in [25] is that the class of compatible r-matrices are related by twisting. The combinatorial quantisation procedure can therefore be viewed as defining an equivalence class of QIGs, with equivalence essentially given by twisting.

4 Born reciprocity in 3d gravity

A profound feature of the quantum groups in the family of quantum doubles and bicrossproduct quantum groups associated with 3d quantum gravity turns out to be related by a map called semidualisation or Born reciprocity. In physical terms, this map can be interpreted as the exchange of position and momentum degrees of freedom. This was discovered in [17] in the Euclidean setting and elaborated and extended to the Lorentzian setting in [25]. It was also studied at the infinitesimal Lie bialgebra level in [26]. A mathematically precise version of these ideas is contained in the book [15]. Majid's approach is in turn stimulated by Born's proposal of a reciprocity between momenta and positions [6].

Consider a Hopf algebra H which factorizes into two sub-Hopf algebras built on $H_1 \otimes H_2$ as a vector space and viewed as a double crossproduct Hopf algebra $H_1 \bowtie H_2$. This induced the actions $\triangleright : H_2 \otimes H_1 \to H_1$ and $\triangleleft : H_2 \otimes H_1 \to H_2$ of each Hopf algebra on the vector space of the other. We say that one has a matched pair of interacting Hopf algebras. On the other hand, given such a matched pair (H_1, H_2), one can reconstruct the associated double crossproduct $H_1 \bowtie H_2$ from these actions. There is a canonical covariant left action of $H_1 \bowtie H_2$ on H_2^* (the dual of H_2) as an algebra. The Drinfeld quantum double $D(H) = H \bowtie H^{*\mathrm{op}}$ can be seen as an example of a double crossproduct. The semidual of the associated matched pair data is obtained by dualising the data containing H_2 to give the bicrossproduct Hopf algebra $H_2^* \blacktriangleright\!\!\triangleleft H_1$ which covariantly acts on H_2 from the right as an algebra. Again, we refer to the book [15] for a comprehensive treatment. Observe that both the quantum double and the bicrossproduct quantum group can be obtained by semidualising two different sets of matched pair data resulting from factors of the same Hopf algebra.

Physically, for a quantum group built from factors (in our context momentum and rotations) acting on some space (position space), the semidual of the data can then thought of as a map which exchanges the position and momentum degrees of freedom. For example, in the semidual of the universal enveloping algebra of the Euclidean Lie algebra $U(\mathfrak{su}(2)) \triangleright\!\!< \mathbb{R}^3$, one keeps the angular momentum generators and replaces momenta (which generate translations in space) by position coordinates (which generate translations in momentum space) to obtain the bicrossproduct Hopf algebra $(\mathbb{R}^*)^3 >\!\!\triangleleft U(\mathfrak{su}(2))$. Another interpretation of semiduality at the Hopf algebra level is to interpret both the original and the semidual generators in the same way, but to think of semiduality as a map between different regimes.

Consider now Table 3 which provides a list of the local isometry groups arising in 3d gravity with some corresponding matched pairs of right crossproducts or double crossproducts. In Table 4 we list the corresponding semiduals of the universal enveloping algebras of the Lie algebras for the groups and factorisation given in Table 3. For example, in the Lorentzian case and with $\lambda > 0$, semidualisation of $U(\mathfrak{sl}(2,\mathbb{R}) \triangleright\!\!< U(\mathfrak{sl}(2,\mathbb{R}))$ gives the quantum double $D(U(\mathfrak{sl}(2,\mathbb{R}))) = \mathbb{C}(SL(2,\mathbb{R})) >\!\!\triangleleft U(\mathfrak{sl}(2,\mathbb{R}))$. The notation $\blacktriangleright\!\!\triangleleft_s$, $\blacktriangleright\!\!\triangleleft_l$ and $\blacktriangleright\!\!\triangleleft_t$ for the left-right bicrossproducts with spacelike, lightlike and timelike deformations respectively. We refer to [25] for further details.

Table 3: Local isometry groups in 3d gravity and their factorisations.

	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\lambda = 0$	$\tilde{E}_3 = SU(2) \triangleright\!< \mathbb{R}^3$	$\tilde{P}_3 = \begin{cases} SL(2,\mathbb{R}) \triangleright\!< \mathbb{R}^3 \\ SL(2,\mathbb{R}) \bowtie_l AN(2) \end{cases}$
$\lambda > 0$	$\widetilde{SO}(4) = SU(2) \triangleright\!< SU(2)$	$\widetilde{SO}(2,2) = \begin{cases} SL(2,\mathbb{R}) \triangleright\!< SL(2,\mathbb{R}) \\ SL(2,\mathbb{R}) \bowtie_s AN(2) \end{cases}$
$\lambda < 0$	$SL(2,\mathbb{C}) = SU(2) \bowtie AN(2)$	$SL(2,\mathbb{C}) = SL(2,\mathbb{R}) \bowtie_t AN(2)$

Table 4: Semiduals of local isometry groups in 3d gravity.

	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\lambda = 0$	$(\mathbb{R}^*)^3 > \!\!\triangleleft U(\mathfrak{su}(2))$	$(\mathbb{R}^*)^3 > \!\!\triangleleft U(\mathfrak{sl}(2,\mathbb{R}))$ $\mathbb{C}(AN(2)) \blacktriangleright\!\!\triangleleft_l U(\mathfrak{sl}(2,\mathbb{R}))$
$\lambda > 0$	$D(U(\mathfrak{su}(2)))$	$D(U(\mathfrak{sl}(2,\mathbb{R})))$ $\mathbb{C}(AN(2)) \blacktriangleright\!\!\triangleleft_s U(\mathfrak{sl}(2,\mathbb{R}))$
$\lambda < 0$	$\mathbb{C}(AN(2)) \blacktriangleright\!\!\triangleleft U(\mathfrak{su}(2))$	$\mathbb{C}(AN(2)) \blacktriangleright\!\!\triangleleft_t U(\mathfrak{sl}(2,\mathbb{R}))$

Semidualisation is defined for Lie bialgebras which are double cross sums $\mathfrak{g} \bowtie \mathfrak{m}$, where $\mathfrak{m}, \mathfrak{g}$ are Lie bialgebras with $\mathfrak{m}$ being a left $\mathfrak{g}$-module Lie algebra and $\mathfrak{g}$ a right $\mathfrak{m}$-module Lie coalgebra obeying certain compatibility conditions. In the semidual Lie bialgebra, the Lie bialgebra $\mathfrak{m}$ is replaced by its $\mathfrak{m}^*$ and leads to the bicross sum $\mathfrak{m}^* \blacktriangleright\!\!\triangleleft \mathfrak{g}$. At this infinitesimal Lie bialgebra level, the interpretation of semidualisation is essentially the exchange of degrees of freedom discussed above. The bicross sum $\mathfrak{m}^* \blacktriangleright\!\!\triangleleft \mathfrak{g}$ is coboundary and its classical r-matrix can easily be obtained. We refer to [15] for details.

The application to double cross sum decompositions of the local isometry Lie algebras arising in 3d gravity, semidualisation yields the main class of non-trivial r-matrices for the Euclidean and Poincaré group in three dimensions. In addition, the construction links the r-matrices with the Bianchi classification of three dimensional real Lie algebras [26].

5 Conclusion and prospects

It remains a substantial and significant challenge to flush out the detailed picture sketched here and to clarify the quantisation ambiguities in the application of the combinatorial quantisation procedure. A starting point would be to understand the structure of all possible QIGs that can arise in this construction and how they are related. To the best of our knowledge, the quantum groups that emerge are either quantum doubles or bicrossproducts and these semiduals of one another. Thus precise relation between any pair of quantum groups associated to the same action (and thus to the same set of physical parameters) is provided by Born reciprocity. One could therefore give the interpretation that semiduality relations between different quantum groups can generally be interpreted as a physical equivalence of associated quantum theories after a suitable exchange of position and momentum degrees of freedom. It turns out that in the context of 3d gravity, the bicrossproduct quantum group and the quantum double are indeed mathematically related by a Drinfel'd twist [18]. Thus, the combinatorial quantisation procedure does not define the QIGs uniquely but rather, it defines an equivalence class of quantum groups, with equivalence essentially given by Drinfel'd and module algebra twist. We refer to a forth coming paper [18] for further details.

Note also that in the application of semiduality to the local isometry groups, we start off with match pairs whose factorisation depends on the cosmological constant. However, after semidualisation the cosmological constant becomes a deformation parameter which is related to Planck mass. This leads to the interpretation of semiduality as the exchange of the cosmological length scale and the Planck mass in the context of 3d quantum gravity. In order to see the cosmological constant in the semidual regime, one has to q-deform the universal enveloping algebras of the isometry Lie algebras before applying the Born reciprocity map. We refer to [17,25] for details. At the infinithesimal Lie bialgebra level, it is conceivable that the semiduality could be used to classify of 6-dimensional Lie algebras.

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A new procedure for constructing basis vectors of irreducible representations of SU(3) under the SU(3)⊃SO(3) basis

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Abstract An effective algebraic angular momentum projection procedure for constructing basis vectors of an irreducible representation of the Lie group SU(3) under the non-canonical SU(3)⊃SO(3)⊃SO(2) basis from those of the canonical U(3)⊃ U(2)⊃ U(1) basis is outlined. The expansion coefficients are components of the null-space vectors of a projection matrix with, in general, four nonzero elements in each row, where the projection matrix is derived from known matrix elements of the U(3) generators in the canonical basis. The advantage of the new procedure lies in the fact that the Hill-Wheeler integral involved in Elliott's projection operator method used previously is avoided, thereby achieving faster numerical calculations with improved accuracy. However, the Gram-Schmidt orthonormalization is still needed in order to provide orthonormalized basis vectors.

1 Introduction

The non-canonical group chain SU(3)⊃ SO(3)⊃ SO(2) has been useful in nuclear shell-model calculations since the pioneering work of Elliott [1, 2]. In Elliott's model, the essential rotational features of the Bohr-Mottelson collective model can be well reproduced in a shell-model framework by introducing the quadrupole-quadrupole interaction within a three-dimensional harmonic oscillator mean-field [1, 2], where the quadrupole operators are generators of SU(3), while the angular momentum operators are generators of its subgroup SO(3). The SU(3)⊃SO(3) basis has been adopted in many studies due to its importance [4–6].

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SU(3) and other unitary groups are also useful in quantum interferometry [3]. In both nuclear shell-model calculations and quantum interferometry, the dimension of an irreducible representation (irrep) of SU(3) can be huge and may approach the classical asymptotic limits. In [1,2], the basis vectors of SU(3)⊃SO(3)⊃SO(2) are projected from a specific (extremal) SU(3)⊃SU(2)⊗U(1) state by using the angular momentum projection, which are often called the Elliott states [7]. Based on these studies, the practical algorithm for calculating various coupling coefficients of SU(3), including those of SU(3)⊃SO(3), has been formulated [8]. An optimized code for generating Clebsch-Gordan (CG) coefficients of SU(3)⊃SO(3) based on vector coherent state theory [9] has also been developed [10]. In the above mentioned codes, the original Elliott-Harvey angular momentum projection is adopted. The main complexities in practical calculations are two-fold. One of them lies in the fact that these calculations use a projection operator constructed by integration of the product of the rotational group element and its matrix element (Wigner's D-function) of a given angular momentum over the Euler angles. While the projection formalism can be straightforwardly implemented in computer codes, it needs to address challenges related to the accuracy and computing time for evaluating coupling coefficients of SU(3)⊃SO(3), because of the use of the Hill-Wheeler integral [11]. The other difficulty is related to the fact that the Elliott states are non-orthogonal, and to calculate overlaps of the Elliott states needed in many cases is also time consuming.

Very recently, we proposed a simple and effective angular momentum projection procedure [12] to construct the non-canonical O(5)⊃O(3) basis vectors of an irrep from the basis vectors of O(5)⊃O_1(3)⊗U(1). We observe that the canonical U(3)⊃U(2)⊃U(1) basis plays a similar role of U(5)⊃U(3)⊗U(2) used to construct the basis vectors of an irrep of O(5)⊃O(3). Thus, it should also be possible to construct SU(3)⊃SO(3)⊃SO(2) basis vectors of an irrep directly from those of the irrep under the U(3)⊃U(2)⊃U(1) basis with a similar simpler algebraic formalism.

2 Canonical and non-canonical bases of SU(3)

The generators of U(N) can be denoted by $\{E_{ij}\}$ $(1 \le i, j \le N)$ satisfying the following commutation and Hermitian conjugation relations:

$$[E_{ij},\, E_{lk}] = \delta_{jl} E_{ik} - \delta_{ik} E_{lj}, \quad (E_{ij})^{\dagger} = E_{ji}. \tag{1}$$

There is an obvious subgroup U(N-1) of U(N) generated by $\{E_{ij}\}$ $(1 \le i, j \le N-1)$. Thus, one gets the canonical chain of U(N) with U(N)⊃U(N-1)⊃ $\cdots$ ⊃U(2)⊃U(1), for which basis vectors of an irrep were constructed first by Gel'fand and Zetlin [13], and then discussed by many others in various ways [14]. The reduction U(N)↓U(N-1) for any N≥ 2 is multiplicity-free. By removing the first order algebraic invariant (Casimir operator), $C_1(U(N)) = \sum_{i=1}^{N} E_{ii}$, which is obviously commutative with all generators $\{E_{ij}\}$ $(1 \le i, j \le N)$ of U(N), the remaining N^2-1 generators generate SU(N). Let $[\nu_1, \nu_2, \ldots, \nu_N]$, where $\nu_1, \nu_2, \ldots, \nu_N$ are positive integers obeying $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_N$, be an irrep of U(N). It is well known that an irrep $[\nu_1+m, \nu_2+m, \ldots, \nu_N+m]$, where $m \geq -\nu_N$ is an integer, and $[\nu_1, \nu_2, \ldots, \nu_N]$ have the same dimension and the representation matrices of any element of U(N) for these two irreps differ only by an overall phase factor. Therefore, for the SU(3) case, an irrep can be denoted by $[n'_{13}, n'_{23}, n'_{33}] \equiv [n_{13} = n'_{13} - n'_{33}, n_{23} = n'_{23} - n'_{33}, 0]$, where $[n'_{13}, n'_{23}, n'_{33}]$ is used to label the corresponding irrep of U(3), where n_{i3} are zero or integers obeying the betweenness condition $n_{13} \geq n_{23} \geq 0$. Incidentally, for the (λ, μ) labels of an $SU(3)$ irrep used in the Elliott model, the relation is $(\lambda, \mu) = (n_{13} - n_{23}, n_{23}) = (n'_{13} - n'_{23}, n'_{23} - n'_{33})$. Therefore, the irrep denoted as $[n_{13},\ n_{23}]$ of SU(3) is also the irrep $[n_{13},\ n_{23}, 0]$ of U(3). The general (canonical) basis vectors of the irrep $[n_{13}, n_{23}, n_{33}]$ of U(3) under the basis U(3)⊃U(2)⊃U(1) may be denoted by [13],

$$\left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12},\ n_{22}] \\ n_{11} \end{array} \right\rangle, \tag{2}$$

with the betweenness conditions: $n_{13} \geq n_{12} \geq n_{23} \geq n_{22} \geq n_{33},\ n_{12} \geq n_{11} \geq n_{22}$.

The matrix representations of U(N) in the canonical basis are well-known [13]. It can be observed that the matrix elements of the generators $\{E_{ij}\}$ $(1 \leq i, j \leq 3)$ are all given in functions of two-number differences among the six quantum numbers n_{ij} $(1 \leq i \leq j \leq 3)$ satisfying the betweenness conditions of (2). One can check that there is an exact correspondence between the basis vectors of U(3)⊃U(2)⊃U(1):

$$\left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12},\ n_{22}] \\ n_{11} \end{array} \right\rangle = \left| \begin{array}{c} [n_{13} - n_{33},\ n_{23} - n_{33},\ 0] \\ [n_{12} - n_{33},\ n_{22} - n_{33}] \\ n_{11} - n_{33} \end{array} \right\rangle, \tag{3}$$

under which the matrix representations of U(3) are the same.

After a linear transformation, the generators of SU(3) can also be expressed in its non-canonical basis, i.e., in the SU(3)⊃SO(3) basis, with generators given by

$$\begin{aligned} L_0 = E_{11} - E_{22},\ L_+ = \sqrt{2}(E_{13} + E_{32}),\ \ L_- = (L_+)^\dagger = \sqrt{2}(E_{31} + E_{23}), \\ Q_2 = E_{12},\ Q_1 = \sqrt{\tfrac{1}{2}}(E_{32} - E_{13}),\ \ Q_0 = \sqrt{\tfrac{1}{6}}(E_{11} + E_{22} - 2E_{33}), \\ Q_{-1} = -(Q_1)^\dagger = \sqrt{\tfrac{1}{2}}(E_{31} - E_{23}),\ \ Q_{-2} = (Q_2)^\dagger = E_{21}, \end{aligned} \tag{4}$$

where $\{L_+, L_-, L_0\}$ are generators of the subgroup SO(3), which may be identified as the angular momentum operators satisfying the usual commutation relations:

$$[L_0,\ L_\pm] = \pm L_\pm,\ [L_+,\ L_-] = 2L_0, \tag{5}$$

and Q_u $(u = 2, 1, \ldots, -2)$ are quadrupole (moment) operators realized in the Elliott model for nuclei.

3 Basis vectors of an irrep of SU(3) under the SU(3)⊃SO(3) basis

The basis vector (2) is also an eigenstate of L_0 with eigenvalue $M = 2n_{11} - n_{12} - n_{22}$. For a given irrep $[n_{13}, n_{23}, 0]$ of U(3) [or SU(3)], all possible basis vectors of it under the U(3)⊃U(2)⊃U(1) basis shown in (2) restricted by the betweenness conditions form a complete set. Therefore, the SU(3)⊃SO(3)⊃SO(2) basis vectors can be expanded in terms of the basis vectors of U(3)⊃U(2)⊃U(1) with the restriction on the SO(2) quantum number $M = 2n_{11} - n_{12} - n_{22}$. In order to find all basis vectors of U(3)⊃U(2)⊃U(1) of the irrep $[n_{13}, n_{23}]$ of $SU(3)$ with fixed M, it suffices to consider all possible irreps $[n_{12}, n_{22}]$ of U(2) embedded in the canonical chain satisfying the betweenness conditions of (2) for this case. Practically, we only need to construct the highest weight state of SO(3) with $M = L$.

According to the restriction $M = 2n_{11} - n_{12} - n_{22}$ and the betweenness conditions of (2), we find that all possible basis vectors within the U(3) irrep $[n_{13}, n_{23}, 0]$ and with $M = k \geq 0$ are given as follows:

$$\left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k + 2q - t, t] \\ k + q \end{array} \right\rangle, \tag{6}$$

where

$$\begin{array}{c} 0 \leq k \leq n_{13}, \quad 0 \leq t \leq n_{23}, \\ \mathrm{Max}\left[t, \mathrm{IntM}[\tfrac{1}{2}(t - k + n_{23})]\right] \leq q \leq \mathrm{Int}[\tfrac{1}{2}(n_{13} - k + t)], \end{array} \tag{7}$$

in which Int$[x]$ is the integer part of x, and IntM$[x]$ is the largest integer closest to x defined by

$$\mathrm{IntM}[x] = \begin{cases} \mathrm{Int}[x] + 1 & \text{if } x - \mathrm{Int}[x] > 0, \\ \mathrm{Int}[x] & \text{if } x - \mathrm{Int}[x] = 0. \end{cases} \tag{8}$$

The basis vector of SU(3)⊃SO(3)⊃SO(2) for the SO(3) highest-weight state may be expanded in terms of (6) as

$$\left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \; L = M = k \end{array} \right\rangle = \sum_{t=0}^{n_{23}} \sum_{q=\mathrm{Max}\left[t, \mathrm{IntM}[\frac{1}{2}(t-k+n_{23})]\right]}^{\mathrm{Int}[\frac{1}{2}(n_{13}-k+t)]} c_{qt}^{(\zeta)}([n_{13}, n_{23}, 0], L) \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k + 2q - t, t] \\ k + q \end{array} \right\rangle, \tag{9}$$

where L is the angular momentum quantum number, ζ is the multiplicity label needed in the reduction $[n_{13}, n_{23}] \downarrow L$, and $\{c_{qt}^{(\zeta)} \equiv c_{qt}^{(\zeta)}([n_{13}, n_{23}, 0], L)\}$ are the expansion coefficients, which must satisfy

$$\sqrt{\frac{1}{2}} L_+ \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \; L = M = k \end{array} \right\rangle = (E_{13} + E_{32}) \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \; L = M = k \end{array} \right\rangle = 0. \tag{10}$$

By using the explicit matrix elements shown in [13], Eq. (10) can be written as

$$\sqrt{\tfrac{1}{2}} L_+ \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \; L = M = k \end{array} \right\rangle =$$

$$
\begin{aligned}
&=\sum_{t,\,q}\Bigg\{c^{(\zeta)}_{q,\,t}\left[\frac{(k+q-t+1)(n_{13}-k-2q+t)(k+2q-t-n_{23}+1)(k+2q-t+2)}{(k+2q-2t+1)(k+2q-2t+2)}\right]^{\frac{1}{2}}+\\
&+c^{(\zeta)}_{q+1,\,t}\left[\frac{(q-t+1)(n_{13}-k-2q+t-1)(k+2q-t-n_{23}+2)(k+2q-t+3)}{(k+2q-2t+2)(k+2q-2t+3)}\right]^{\frac{1}{2}}+\\
&+c^{(\zeta)}_{q+1,\,t+1}\left[\frac{(k+q-t+1)(n_{13}-t+1)(n_{23}-t)(t+1)}{(k+2q-2t+1)(k+2q-2t+2)}\right]^{\frac{1}{2}}-\\
&-c^{(\zeta)}_{q,\,t-1}\left[\frac{(q-t+1)(n_{13}-t+2)(n_{23}-t+1)t}{(k+2q-2t+2)(k+2q-2t+3)}\right]^{\frac{1}{2}}\Bigg\}\left|\begin{array}{c}[n_{13},n_{23},0]\\ [k+2q-t+1,t]\\ k+q+1\end{array}\right\rangle=0,
\end{aligned}
\tag{11}
$$

which, thus, leads to the following four-term relation to determine the expansion coefficients $\{c^{(\zeta)}_{q,t}\}$:

$$
\begin{aligned}
0=c^{(\zeta)}_{q,\,t}&\left[\frac{(k+q-t+1)(n_{13}-k-2q+t)(k+2q-t-n_{23}+1)(k+2q-t+2)}{k+2q-2t+1}\right]^{\frac{1}{2}}+\\
&+c^{(\zeta)}_{q+1,\,t}\left[\frac{(q-t+1)(n_{13}-k-2q+t-1)(k+2q-t-n_{23}+2)(k+2q-t+3)}{k+2q-2t+3}\right]^{\frac{1}{2}}+\\
+c^{(\zeta)}_{q+1,\,t+1}&\left[\frac{(k+q-t+1)(n_{13}-t+1)(n_{23}-t)(t+1)}{k+2q-2t+1}\right]^{\frac{1}{2}}-c^{(\zeta)}_{q,\,t-1}\left[\frac{(q-t+1)(n_{13}-t+2)(n_{23}-t+1)t}{k+2q-2t+3}\right]^{\frac{1}{2}}.
\end{aligned}
\tag{12}
$$

One can thus construct a matrix equation of (12) with

$$
\mathbf{P}([n_{13},n_{23}],k)\mathbf{c}^{(\zeta)}=\Lambda\mathbf{c}^{(\zeta)},\tag{13}
$$

where $\mathbf{c}^{(\zeta)}\equiv\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)$. Entries of the matrix $\mathbf{P}([n_{13},n_{23}],k)$ can easily be read from Eq. (12), which will be called the projection matrix. The components of the eigenvector $\mathbf{c}^{(\zeta)}$ corresponding to $\Lambda=0$ provide the expansion coefficients $\{c^{(\zeta)}_{q,t}\}$ of (9). Once the matrix $\mathbf{P}([n_{13},n_{23}],k)$ is constructed, it can be verified that the number of $\Lambda=0$ solutions of Eq. (13) for sufficiently large n_{13} equals exactly to the number of rows of $\mathbf{P}([n_{13},n_{23}],k)$ with all entries zero. However, some entries of $\mathbf{P}([n_{13},n_{23}],k)$ will be zero or become complex for some specific values of n_{13} and n_{23}. In such cases, a nonzero solution of $\{c^{(\zeta)}_{q,t}([n_{13},n_{23}],k)\}$ does not exist. Actually, the eigenvectors $\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)$ belong to the null space of $\mathbf{P}([n_{13},n_{23}],k)$. Namely, components of the ζ-th vector in the null space of $\mathbf{P}([n_{13},n_{23}],k)$ determined by (13) are the corresponding expansion coefficients $\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)$ of the basis vector of SU(3)⊃SO(3)⊃SO(2) with $L=k$ in terms of the basis vectors of the canonical chain of U(3) needed in (9). Since there are many ways currently available to find null-space vectors of a matrix, to find solutions of Eq. (13) with $\Lambda=0$ becomes practically easy. Furthermore, the SU(3)⊃SO(3)⊃SO(2) basis vectors (9) constructed from the expansion coefficients obtained according to (12) are also non-orthogonal with respect to the multiplicity label ζ. The Gram-Schmidt process may be adopted in order to construct orthonormalized basis vectors of SU(3)⊃SO(3)⊃SO(2). Thus, the multiplicity of $L=k$ for the given irrep $[n_{13},n_{23}]$ is given by the number of linearly independent null space vectors of $\mathbf{P}([n_{13},n_{23}],k)$.

4 Summary

In this paper, an effective angular momentum projection to construct basis vectors of an irrep of SU(3) under the SU(3)⊃SO(3)⊃SO(2) basis from those of the canonical U(3)⊃U(2)⊃ U(1) basis is outlined. We show that the expansion coefficients can be obtained as components of the null-space vectors of a projection matrix, for which, in general, there are only four nonzero elements in each row. There are currently available well-optimized algorithms for computing the null-space vectors of a matrix. Hence, to evaluate the expansion coefficients for the SU(3)⊃SO(3)⊃SO(2) basis in terms of the basis of the canonical chain becomes more practical than Elliott's projection operator method. Since the expansion coefficients are components of null-space vectors of the projection matrix, there is always arbitrariness in choosing these vectors [12]. Therefore, the null-space vectors are not orthogonal in general. The Gram-Schmidt orthonormalization is still needed in order to obtain orthonormalized basis vectors. It will be our next work to compile a code for calculating SO(3)-reduced matrix elements of the SU(3) generators and coupling coefficients of SU(3)⊃SO(3) according to the new projection method proposed in this paper. We can then check the runtime and compare it with other existing codes using Elliott's projection operator method, from which the efficiency of the new method can then be actually revealed.

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Massive Dirac field in 3D and induced equations for higher spin fields

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Abstract On the example of the free massive Dirac in flat three-dimensional space-time, we show how the linearised equations of motion for higher spin fields can be obtained from the induced action by coupling higher spin fields to conserved currents. The result is important because a classical analysis leads to many different formulations of free higher spin equations, and not all of them are expected to be a good starting point for introduction of interactions. Our result breaks the degeneracy. We express the results by using a metric-like description for higher spin fields, in which the equations of motion can be elegantly written in generic form that is valid for all spins.

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1 Introduction

Constructing interacting QFT's with fundamental massless higher spin (HS) fields (with spin $s > 2$) in $D \geq 4$ spacetime dimensions presents an interesting challenge. On the one hand, we know how to construct free QFT's with such fields, and we know that free QFT's of lower-spin fields contain conserved HS currents which simply beg to be coupled with corresponding HS fields. On the other hand, there are obstacles and stringent constraints, some of them in the form of no-go theorems, which, however, can be by-passed, e.g., by going to (A)dS spacetime or using higher-derivative couplings. Knowing that lower integer spin cases describe the known fundamental forces ($s = 1$ electroweak and strong, and $s = 2$ gravitation) the challenge could be of utmost importance.

The uncertainty about how to couple the HS field φ to matter fields ψ (fields with $s < 2$) already starts from the choice of the equation of motion (EOM) for *free* HS fields. The standard approach, which is a straightforward generalisation of the lower spin cases, is Fronsdal's formulation [1] in which free spin-s higher spin field $\varphi_{\mu_1\cdots\mu_s}(x)$ is the symmetric rank-s tensor satisfying the Fronsdal equation[1]

$$\mathscr{F} \equiv \Box\varphi - \partial\,\partial\cdot\varphi + \partial^2\varphi' = 0 \tag{1}$$

where prime on the HS field denotes a single trace (contraction of one pair of spacetime indices). The Fronsdal equation is invariant under the gauge transformations

$$\delta\varphi_{\mu_1\cdots\mu_s} = s\,\partial_{(\mu_1}\Lambda_{\mu_2\cdots\mu_s)} \tag{2}$$

if Λ is traceless. In Fronsdal's formulation the field φ propagates only free massless spin-s excitations.

Let us also mention the Maxwell-like formulation in which the EOM is

$$\mathscr{M} \equiv \Box\varphi - \partial\,\partial\cdot\varphi = 0 \tag{3}$$

where gauge symmetry is constrained by the condition $\partial\cdot\Lambda = 0$. In the Maxwell-like formulation, the field φ propagates excitations of spins $s, s-2, s-4, \ldots$.

Before starting to construct interacting theories with HS fields, one should take the following observations, related to the linearised theory, into account. (i) We have seen that in both formulations mentioned above, the gauge group is restricted, in the sense that the gauge parameter field is constrained in some way. This happens in all formulations of free HS fields with equations which are 2nd order in derivatives. As restricting the gauge group is not something we like to do, because it usually leads to problems after quantisation, it is more promising to formulate the free theory such that it obeys the unconstrained gauge symmetry (2). In fact, one can achieve this by introducing compensator fields in the equations of motion, but with the consequence of making the equations either higher-order in derivatives, or nonlocal [2]. Eqs. (1) and (3) can be recovered by gauge fixing. (ii) The spin 1 and 2 cases strongly suggest

[1] We shall focus here on higher spin fields with integer spin.

that the expressions may be complicated and in the form of infinite series, without using a covariant formulation (which in the spin-2 case is the Riemann geometry). We do not know the full covariant formulation, which reflects the fact that we do not have a satisfactory understanding of the HS gauge structure. However, there is a promising linearised covariant, metric-like, description developed in [3–5], which is a rather straightforward generalisation of the spin 1 and 2 cases. Using the metric-like description, the EOMs in Fronsdal's and Maxwell-like formulations, with a special choice for compensator fields α and D, can be written as [6]:

$$R'_{\mu_{s-2}\nu_s} = 0 \qquad \Longrightarrow \qquad \mathscr{F} = \partial^3\alpha \tag{4}$$

$$\partial \cdot R_{\mu_{s-1}\nu_s} = 0 \qquad \Longrightarrow \qquad \mathscr{M} = 2\,\partial^2 D, \tag{5}$$

where R is the rank-2s tensor field, of s-th order in derivatives, known as a linearised spin-s Riemann tensor. By taking the trace and gradient of the equations above, one arrives at alternative formulations of free HS theories, adding to degeneracy [6]. (iii) As we already noted, there are many different formulations of the free higher spin field EOM, presumably not all giving a good starting point for introducing interactions. For example, in the spin-2 case, Fronsdal equation (1) is the linearisation of the equation $R_{\mu\nu} = 0$, so it is a better starting point than (3). We need a method, or at least a hint, which we can use to make a correct choice. (iv) For spin $s \geq 2$, one can anticipate, based on the $s = 2$ example, that the full (nonlinear) gauge structure is non-abelian. However, in the lowest order we expect that the coupling to matter is accomplished by linearly coupling to the current in a way that does not introduce spurious excitations, which typically means that the current should be conserved, $\partial \cdot J = 0$. This forces a condition on the linearised HS field EOM, which is not satisfied by the metric-like EOM's (4) and (5).[2] In the spin-2 case, we know the way out: instead of using $R_{\mu\nu} = 0$ as a free EOM, one should use $G_{\mu\nu} = 0$, where $G_{\mu\nu}$ is the linearised Einstein tensor, which is divergence-free at the linearised level. The idea is to generalise this to HS fields.

In the rest of the paper we show how induced action may resolve these issues.

2 Induced action

The method we use is to couple HS fields, as background fields, to conserved HS currents in free QFT with "normal" matter (spin 0 and 1), and calculate the quantum effective action (i.e., the induced action) in the IR region. We assume that as is the case for spins 1 and 2, all relevant structures of the HS theories, should appear in the induced action. We performed calculations in $d = 3$ spacetime dimensions, but we believe that the result can be straightforwardly generalised to $d \geq 4$.

[2] Candidates of HS EOM which allow linearised coupling to the matter were constructed in [7]. However, the construction is rather involved and leaves a lot of degeneracy.

We shall demonstrate our method by using as a matter field two-component massive Dirac field $\psi(x)$ in the flat three dimensional spacetime, which we assume to be quantised. The free field Lagrangian is[3]

$$\mathscr{S}_0 = \int d^3x\, \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{6}$$

This theory possess conserved currents of any spin. They are given by (integer spins)

$$J^{(0)}_{\mu_1\cdots\mu_s} = i^{s-1}\bar{\psi}\gamma_\mu \sum_{j=0}^{\left[\frac{s-1}{2}\right]} \frac{(u_\mu - v_\mu)^{s-2j-1}\,(2w\,\eta_{\mu\mu} - 4\,u_\mu v_\mu)^j}{(2j+1)!\,(s-2j-1)!}\,\psi \tag{7}$$

where

$$u_\mu \equiv \overrightarrow{\partial}_\mu \quad , \quad v_\mu \equiv \overleftarrow{\partial}_\mu \quad , \quad w \equiv u^\mu v_\mu - m^2. \tag{8}$$

For spin-1, one gets $U(1)$ current, and for spin-2, the energy-momentum tensor.

The existence of the conserved currents suggests the possibility of coupling the Dirac field to the background HS field. Our goal is to calculate a corresponding effective (or induced) action and EOM for HS field. Now, we do not know the exact way how to consistently construct such a coupling, but if we are interested only in the linearised induced EOM (LIEOM), then it is given by[4]

$$\langle\langle J^{(s)}_{\mu_1\cdots\mu_s}(x)\rangle\rangle_{\text{lin}} = i\int d^dy\,\langle 0|\,T\left\{J^{(s)}_{\mu_1\cdots\mu_s}(x)J^{(s)}_{\nu_1\cdots\nu_s}(y)\right\}|0\rangle_{\text{d.f.}}\,\varphi_{\nu_1\cdots\nu_s}(y), \tag{9}$$

where "d.f" denotes divergence-free part of the 2-point correlator. We have calculated this 2-point correlator, using dimensional regularisation, up to $s = 10$, and the detailed results will be presented in [10].

3 Spin-3 case

The spin-3 current 2-point correlator was calculated in [8]. By using this result in (9), one obtains in the IR region

$$\begin{aligned}\langle\langle J_{\mu\mu\mu}(x)\rangle\rangle_{\text{lin}} = |m|&\left[\sum_{r=0}^{\infty} a_r\left(\frac{\Box}{m^2}\right)^r\right]\partial^\alpha F_{\alpha\mu\mu\mu}\\ &+\frac{1}{|m|}\left[\sum_{r=0}^{\infty} c_r\left(\frac{\Box}{m^2}\right)^r\right](\eta_{\mu\mu}\Box - \partial_\mu\partial_\mu)\partial^\alpha F_{\alpha\mu}\\ &+m|m|\left[\sum_{r=0}^{\infty} b_r\left(\frac{\Box}{m^2}\right)^r\right]G_{\mu\mu\mu} + \frac{m}{|m|}\left[\sum_{r=0}^{\infty} d_r\left(\frac{\Box}{m^2}\right)^r\right]C_{\mu\mu\mu},\end{aligned} \tag{10}$$

[3] This theory is not parity invariant, as parity changes the sign of the mass term.

[4] For even spins there are also "cosmological constant" contributions, which we ignore here.

where a_r, b_r, c_r, d_r are some numerical coefficients (which are all nonvanishing for $r=0$) and the symmetrisation in μ indices is understood. The tensor $F_{\alpha\mu\nu\rho}$ is obtained from the spin-3 Ricci tensors (tracings of spin-3 Riemann tensor)

$$F_{\alpha\mu\nu\rho} \equiv R'_{\alpha(\mu\nu\rho)} - \frac{1}{2}R''_{\alpha(\mu}\eta_{\nu\rho)} = \partial_{[\alpha}\Big(\mathscr{F}_{(\mu]\nu\rho)} - \frac{1}{2}\mathscr{F}_{(\mu]}\eta_{\nu\rho)}\Big), \tag{11}$$

and has the following properties:

$$\partial^\mu\partial^\alpha F_{\alpha\mu\nu\rho} = 0 \qquad , \qquad F^\alpha{}_{\alpha\nu\rho} = 0, \tag{12}$$

which is prompting us to call it the spin-3 field strength tensor. The tensor $G_{\mu\nu\rho}$ is the "dual" of the Riemann tensor, while $C_{\mu\nu\rho}$ is the generalised Cotton tensor. Both tensors are symmetric and divergence-free, and the Cotton tensor is also traceless.

The parity-even sector of (10), which consists of the first two terms, is 4th-order and higher in derivatives, while the parity-odd sector, is 3rd-order and higher. Note that the parity-odd sector now also contains two tensorially independent terms.

In the IR limit (lowest-derivative order) one gets

$$\langle\langle J_{\mu\nu\rho}(x)\rangle\rangle_{\text{linIR}} = a_0\,|m|\,\partial^\alpha F_{\alpha\mu\nu\rho} + b_0\,m|m|\,G_{\mu\nu\rho} \tag{13}$$

The form is similar to the spin-1 case, however, there are important differences some of which we have already noted above. The parity even part has the structure of the traced covariantised Maxwell-like formulation (5) (where replacing the Ricci tensor $R'_{\alpha\mu\nu\rho}$ by $F_{\alpha\mu\nu\rho}$ guarantees conservation). In the parity-odd sector, the leading term is not the Cotton tensor, which in the spin 1 and 2 cases follows from a Chern-Simons Lagrangian term, but the $G_{\mu\nu\rho}$ tensor which is *not conformal*. In the spin 1 and 3 cases, there is only one tensorially independent parity-odd term, which then trivially must be the Cotton tensor.

4 General spin

We have calculated the HS current 2-point correlators up to $s=10$, and full results will be presented in [10]. There are $[s/2]+1$ independent tensorial structures in the parity-even sector, and $[(s+1)/2]$ in the parity-odd sector. The IR limit, in symbolic notation, is given by

$$\begin{aligned}\tilde{J}^{(\text{IR})}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k) &= i\,a_s\,|m|^{2\lfloor\frac{s}{2}\rfloor-1}k^{2\lceil\frac{s}{2}\rceil}(\pi^2_{\mu\nu}-\pi_{\mu\mu}\pi_{\nu\nu})^{\lfloor\frac{s}{2}\rfloor}\pi_{\mu\nu}^{s\bmod 2}\\ &\quad + b_s\,m|m|^{2\lceil\frac{s}{2}\rceil-3}k^{2\lceil\frac{s-1}{2}\rceil}(\pi^2_{\mu\nu}-\pi_{\mu\mu}\pi_{\nu\nu})^{\lfloor\frac{s-1}{2}\rfloor}\pi_{\mu\nu}^{(s-1)\bmod 2}(k\cdot\varepsilon)_{\mu\nu}\end{aligned}$$

where a_s and b_s are some numerical coefficients. The first term is parity-even and the second is parity-odd. We used the standard projector $\pi_{\mu\nu}(k)\equiv\eta_{\mu\nu}-k_\mu k_\nu/k^2$.

Plugging this into (9), and expressing the result in the metric-like formulation, one gets for the IR limit of LIEOM

$$\langle\langle J_{\mu_1\cdots\mu_s}\rangle\rangle_{\text{linIR}} = a_s\,|m|^{s-2}\,\partial^\nu \varepsilon_{\nu\rho(\mu_1}\, G^\rho{}_{\mu_2\cdots\mu_s)} + b_s\, m|m|^{s-2}\, G_{\mu_1\cdots\mu_s} \quad (s \text{ odd}) \quad (14)$$

$$\langle\langle J_{\mu_1\cdots\mu_s}\rangle\rangle_{\text{linIR}} = a_s\,|m|^{s-1}\, G_{\mu_1\cdots\mu_s} + b_s\, m|m|^{s-3}\,\partial^\nu \varepsilon_{\nu\rho(\mu_1}\, G^\rho{}_{\mu_2\cdots\mu_s)} \quad (s \text{ even}) \quad (15)$$

where $G_{\mu_1\cdots\mu_s}$ is the symmetric divergence-free "dual" of the Riemann tensor

$$G_{\mu_1\cdots\mu_s} \equiv \varepsilon_{\mu_1\nu_1\rho_1}\cdots\varepsilon_{\mu_s\nu_s\rho_s}\, R^{\nu_1\rho_1\cdots\nu_s\rho_s} \qquad , \qquad \partial^{\mu_1}\, G_{\mu_1\cdots\mu_s} = 0. \qquad (16)$$

We see that LIEOM, in the metric-like formulation, has the same form for all spins, but that parity-odd and even sectors get interchanged when passing from odd to even spins. The parity-even part for odd spins is similar to the (multiply traced) Maxwell-like formulation (5), while for even spins, it is similar to the (multiply traced) Ricci-like formulation (4). The conserved tensor G for even spins plays the role of the Einstein tensor, while for odd spins it contributes to the parity-odd sector.

5 Conclusion and outlook

The idea we put forward is that induced action can be a useful tool for constructing higher-spin actions. We have shown that by assuming a standard minimal linear coupling of the background higher-spin field to the conserved current in a free QFT, the induced action gives us the form of the linearised equations of motion for higher spin fields, and thus resolves the degeneracy found in the literature. This suggests that by going to higher order (i.e., higher-point correlators), one may gain an important understanding about the nonlinear higher-spin structure, or find anomalies forbidding such couplings.

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Current algebra for a generalized two-sites Bose-Hubbard model

Gilberto N. Santos Filho

Abstract I present a current algebra for a generalized two-sites Bose-Hubbard model and use it to get the quantum dynamics of the currents. Different choices of the Hamiltonian parameters yield different dynamics. The current algebra is isomorphic to the $SO(3)$-algebra of the angular momentum. Using the wave functions I discuss the symmetries of the system. The Hamiltonian has one conserved quantity, the total number of atoms N, that is related to its global $U(1)$ gauge symmetry. The $\mathbb{Z}_2$ symmetry is associated with the parity of the wave function and is related to the parity of N. I generalize the Heisenberg equation of motion to write the second time derivative of any operator.

1 Introduction

Since the first experimental verification of the Bose-Einstein condensation (BEC) [1–3] occurred more than seven decades after its theoretical prediction [4, 5], a great deal of progress has been made in the theoretical and experimental study of this many-body physical phenomenon [6, 7]. Looking in this direction, a laser was used in an experiment to divide a BEC into two parts to study the interference phenomenon between two BECs [8, 9]. These two BECs can be coupled by Josephson's tunnelling [10, 11] with atoms transiting between the condensates in the same way Cooper's pairs go through a Josephson junction in a superconductor. This system is equivalent to a two-wells system with the particles tunnelling across a barrier between the wells. To study this system a model, known as the *canonical Josephson Hamiltonian*, was proposed by Leggett [7]. Since then many models have been used to study the BECs such as the quantum dynamics of the tunnelling of atoms between the two condensates, the entanglement, the quantum phase transitions and

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the quantum metrology [12–16]. The algebraic Bethe ansatz method has been used to solve and study some of these models [17–22]. I will consider here a generalized issue of the models [7, 20] by introducing on-well energies and leaving free choice for the interaction parameters that also permits the study of the tunnelling between two condensates with atoms of different species (different chemical elements) or atoms in different states in each condensate. The on-well energies are determined as the internal states of atoms in the condensates, by the kinetic energies and interaction of the atoms and/or the external potentials. I will study in this work the current algebra and the quantum dynamics of the currents for this model using a generalization of the Heisenberg equation of motion that makes it possible to write the second time derivative of the current operators. The generalized model is described by the Hamiltonian

$$\hat{H} = \sum_{i,j=1}^{2} K_{ij}\hat{N}_i\hat{N}_j - \sum_{i=1}^{2}(U_i - \mu_i)\hat{N}_i - \sum_{\substack{i,j=1 \\ i\neq j}}^{2} \Omega_{ij}\hat{a}_i^{\dagger}\hat{a}_j, \tag{1}$$

where $\hat{a}_i^{\dagger}$ $(\hat{a}_i)$ denote the single-particle creation (annihilation) operators and $\hat{N}_i = \hat{a}_i^{\dagger}\hat{a}_i$ are the corresponding boson number operators in each condensate. The boson operator total number of particles, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity, $[\hat{H}, \hat{N}] = 0$. The couplings K_{ij}, with $K_{ij} = K_{ji}$ $(i \neq j)$, provides the interaction strength between the bosons and they are proportional to the s-wave scattering length, Ω_{ij} are the amplitude of tunnelling, μ_i are the external potentials and $U_i = K_{ii} - \kappa_i$ are the on-well energies per particle, with κ_i the kinetic energies in each condensate.

2 Symmetries

The Hamiltonian (1) is invariant under the $\mathbb{Z}_2$ mirror transformation $\hat{a}_j \to -\hat{a}_j, \hat{a}_j^{\dagger} \to -\hat{a}_j^{\dagger}$, and under the global $U(1)$ gauge transformation $\hat{a}_j \to e^{i\alpha}\hat{a}_j$, where α is an arbitrary c-number and $\hat{a}_j^{\dagger} \to e^{-i\alpha}\hat{a}_j^{\dagger}, \;\; j = 1, 2$. For $\alpha = \pi$, we get again the $\mathbb{Z}_2$ symmetry. The global $U(1)$ gauge invariance is associated with the conservation of the total number of atoms $\hat{N} = \hat{N}_1 + \hat{N}_2$ and the $\mathbb{Z}_2$ symmetry is associated with the parity of the wave function by the relation $\hat{P}\,|\Psi\rangle = (-1)^N|\Psi\rangle$, where $\hat{P}$ is the parity operator, $[\hat{H}, \hat{P}] = 0$, and

$$|\Psi\rangle = \sum_{n=0}^{N} C_{n,N-n}\frac{(\hat{a}_1^{\dagger})^n}{\sqrt{n!}}\frac{(\hat{a}_2^{\dagger})^{N-n}}{\sqrt{(N-n)!}}|0,0\rangle. \tag{2}$$

There is also the permutation symmetry of the atoms of the two wells if we have $U_1 - \mu_1 = U_2 - \mu_2$ and $\Omega_{12} = \Omega_{21}$. When we have $U_1 - \mu_1 \neq U_2 - \mu_2$ or $\Omega_{12} \neq \Omega_{21}$ we break the symmetry. The wave function (2) is symmetric under this permutation

$$\hat{\mathscr{P}}\,|\Psi\rangle = \sum_{n=0}^{N} C_{N-n,n}\frac{(\hat{a}_1^\dagger)^{N-n}}{\sqrt{(N-n)!}}\frac{(\hat{a}_2^\dagger)^n}{\sqrt{n!}}|0,0\rangle = |\Psi\rangle, \tag{3}$$

where $\hat{\mathscr{P}}$ is the permutation operator and $[\hat{H},\hat{\mathscr{P}}] = 0$ if $\Delta U = \Delta\mu$ and $\Omega_{12} = \Omega_{21}$.

3 Current algebra

The quantum dynamics of any operator $\hat{O}$ in the Heisenberg picture is determined by the Heisenberg equation of motion

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H},\hat{O}]. \tag{4}$$

The boson operator total number of particles, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity, $[\hat{H},\hat{N}] = 0$, and it is commutable compatible operator (CCO) with the number operators of bosons in each well, $[\hat{N},\hat{N}_1] = [\hat{N},\hat{N}_2] = [\hat{N}_1,\hat{N}_2] = 0$. These, in addition, do not commute with the Hamiltonian and their time evolution is determined by the Josephson tunnelling current operator,

$$\hat{\mathscr{J}} = \frac{1}{2i}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1), \tag{5}$$

in coherent opposite phases because of the conservancy of $\hat{N}$, with

$$[\hat{H},\hat{N}_1] = +2i\Omega\,\hat{\mathscr{J}}, \quad [\hat{H},\hat{N}_2] = -2i\Omega\,\hat{\mathscr{J}}, \tag{6}$$

and

$$\hat{N}_1(t) = \hat{N}_1(t_0) - 2\frac{\Omega}{\hbar}\int_{t_0}^{t}\hat{\mathscr{J}}(\tau)\,d\tau, \quad \hat{N}_2(t) = \hat{N}_2(t_0) + 2\frac{\Omega}{\hbar}\int_{t_0}^{t}\hat{\mathscr{J}}(\tau)\,d\tau. \tag{7}$$

Hereafter and in the Eqs. (6) and (7) above, we will consider $\Omega_{12} = \Omega_{21} = \Omega$.

The tunnelling current $\hat{\mathscr{J}}$ together with the imbalance current operator

$$\hat{\mathscr{I}} = \frac{1}{2}(\hat{N}_1 - \hat{N}_2), \tag{8}$$

and the coherent correlation tunnelling current operator

$$\hat{\mathscr{T}} = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1), \tag{9}$$

generate the current algebra

$$[\hat{\mathscr{T}},\hat{\mathscr{J}}] = +i\hat{\mathscr{I}}, \quad [\hat{\mathscr{T}},\hat{\mathscr{I}}] = -i\hat{\mathscr{J}}, \quad [\hat{\mathscr{J}},\hat{\mathscr{I}}] = +i\hat{\mathscr{T}}. \tag{10}$$

With the identification $\hat{L}_x \equiv \hbar\hat{\mathscr{T}}$, $\hat{L}_y \equiv \hbar\hat{\mathscr{J}}$ and $\hat{L}_z \equiv \hbar\hat{\mathscr{I}}$ we can write it in the standard compact way of the angular momentum

$$[\hat{L}_k, \hat{L}_l] = i\hbar\varepsilon_{klm}\hat{L}_m, \tag{11}$$

where ε_{klm} is the antisymmetric Levi-Civita tensor with $k,l,m = x,y,z$ and $\varepsilon_{xyz} = +1$.

We have two Casimir operators for this current algebra. One of them is the total number of particles, $\hat{\mathscr{C}}_1 = \hat{N}$, related to the $U(1)$ symmetry and the other is related to the angular momentum algebra and the $O(3)$ symmetry, $\hat{\mathscr{C}}_2 = \hat{\mathscr{T}}^2 + \hat{\mathscr{I}}^2 + \hat{\mathscr{J}}^2$.

We can show that $\hat{\mathscr{C}}_2$ is just a function of $\hat{\mathscr{C}}_1$,

$$\hat{\mathscr{C}}_2 = \frac{\hat{\mathscr{C}}_1}{2}\left(\frac{\hat{\mathscr{C}}_1}{2} + 1\right). \tag{12}$$

The Casimir operators $\hat{\mathscr{C}}_1$ and $\hat{\mathscr{C}}_2$, the boson number of particles in each well $\hat{N}_1$, $\hat{N}_2$, and the imbalance current operator, $\hat{\mathscr{I}}$, are CCO and so they have the same set of eigenfunctions and can simultaneously have well defined values

$$\hat{\mathscr{C}}_2|n_1,n_2\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)|n_1,n_2\rangle \text{ and } \hat{\mathscr{I}}|n_1,n_2\rangle = \frac{1}{2}(n_1 - n_2)|n_1,n_2\rangle. \tag{13}$$

We also can use the realization of the $SU(2)$ algebra

$$\hat{\mathscr{L}}_\pm = \frac{1}{\hbar}(\hat{L}_x \pm i\hat{L}_y), \quad \hat{\mathscr{L}}_z = \frac{1}{\hbar}\hat{L}_z, \tag{14}$$

with the commutation relations

$$[\hat{\mathscr{L}}_z, \hat{\mathscr{L}}_\pm] = \pm\hat{\mathscr{L}}_\pm, \quad [\hat{\mathscr{L}}_+, \hat{\mathscr{L}}_-] = 2\hat{\mathscr{L}}_z, \tag{15}$$

that we can write as

$$[\hat{\mathscr{L}}_k, \hat{\mathscr{L}}_l] = \varepsilon_{kl-}\hat{\mathscr{L}}_+ + \varepsilon_{kl+}\hat{\mathscr{L}}_- + 2\varepsilon_{zkl}\hat{\mathscr{L}}_z, \tag{16}$$

with $k,l = z,+,-$ and $\varepsilon_{z+-} = +1$.

The $SU(2)$ algebra has three Casimr operators, $\hat{\mathscr{C}}_1$, $\hat{\mathscr{C}}_3$ and $\hat{\mathscr{C}}_4$, with

$$\hat{\mathscr{C}}_3 = \hat{\mathscr{L}}_+\hat{\mathscr{L}}_- + \hat{\mathscr{L}}_z^2 - \hat{\mathscr{L}}_z \quad \text{and} \quad \hat{\mathscr{C}}_4 = \hat{\mathscr{L}}_-\hat{\mathscr{L}}_+ + \hat{\mathscr{L}}_z^2 + \hat{\mathscr{L}}_z. \tag{17}$$

We can show that these Casimir operators are equal to $\hat{\mathscr{C}}_2$. In the deformed $SU(2)$ and $O(3)$ algebras they are different [23].

Using the commutation relations of the currents (10), it is easy to calculate the anticommutators

$$[\hat{\mathscr{T}}, \hat{\mathscr{I}}]_+ = 2\hat{\mathscr{I}}\hat{\mathscr{T}} - i\hat{\mathscr{J}}, \ [\hat{\mathscr{T}}, \hat{\mathscr{J}}]_+ = 2\hat{\mathscr{J}}\hat{\mathscr{T}} + i\hat{\mathscr{I}}, \ [\hat{\mathscr{J}}, \hat{\mathscr{I}}]_+ = 2\hat{\mathscr{I}}\hat{\mathscr{J}} + i\hat{\mathscr{T}}. \tag{18}$$

We will use these anticommutators together with the commutators (10) in calculating quantum dynamics of the currents.

4 Quantum dynamics of currents

We can rewrite the Hamiltonian (1) using those currents operators

$$\hat{H} = \alpha \hat{\mathscr{I}}^2 + \beta \hat{\mathscr{C}}_1 \hat{\mathscr{I}} - 2\Omega \hat{\mathscr{T}} + \frac{\hat{\mathscr{C}}_1}{2}\left(\frac{\hat{\mathscr{C}}_1}{2}\rho + \xi\right), \tag{19}$$

where

$$\alpha = K_{11} - 2K_{12} + K_{22}, \qquad \beta = K_{11} - K_{22},$$
$$\rho = K_{11} + 2K_{12} + K_{22}, \qquad \xi = 2(\mu_1 - U_1).$$

The quantum dynamics of the currents (5), (8) and (9) is determined by the current algebra, their commutation relations with the Hamiltonian and the parameters. We can use the Heisenberg equation of motion (4) to write the second time derivative of any operator $\hat{O}$ in the Heisenberg picture as

$$\frac{d^2\hat{O}}{dt^2} = \left(\frac{i}{\hbar}\right)^2 [\hat{H},[\hat{H},\hat{O}]] \quad \text{or} \quad \frac{d^2\hat{O}}{dt^2} = \frac{i}{\hbar}[\hat{H},\frac{d\hat{O}}{dt}]. \tag{20}$$

Using any of the Eqs. (20), we found the following equations for the quantum dynamics of the currents

$$\frac{d^2\hat{\mathscr{I}}}{dt^2} + 4\frac{\Omega^2}{\hbar^2}\hat{\mathscr{I}} = -4\frac{\Omega\alpha}{\hbar^2}\hat{\mathscr{I}}\hat{\mathscr{T}} + 2i\frac{\Omega\alpha}{\hbar^2}\hat{\mathscr{J}} - 2\frac{\Omega\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{T}}, \tag{21}$$

$$\begin{aligned}\frac{d^2\hat{\mathscr{J}}}{dt^2} + \frac{1}{\hbar^2}\left[\alpha^2 + \beta^2\hat{\mathscr{C}}_1^2 + 4\Omega^2\right]\hat{\mathscr{J}} = &-4\frac{\alpha^2}{\hbar^2}\hat{\mathscr{I}}^2\hat{\mathscr{J}} - 2i\frac{\alpha^2}{\hbar^2}\hat{\mathscr{I}}\hat{\mathscr{T}} - 2\frac{\alpha\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{I}}\hat{\mathscr{J}}\\ &- 4\frac{\alpha\Omega}{\hbar^2}\hat{\mathscr{J}}\hat{\mathscr{T}} - 2i\frac{\alpha\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{T}} - 2i\frac{\alpha\Omega}{\hbar^2}\hat{\mathscr{I}},\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{d^2\hat{\mathscr{T}}}{dt^2} + \frac{1}{\hbar^2}\left(\alpha^2 + \beta^2\hat{\mathscr{C}}_1^2\right)\hat{\mathscr{T}} = &-4\frac{\alpha^2}{\hbar^2}\hat{\mathscr{I}}\hat{\mathscr{I}}\hat{\mathscr{T}} + 4i\frac{\alpha^2}{\hbar^2}\hat{\mathscr{I}}\hat{\mathscr{J}} - 4\frac{\alpha\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{I}}\hat{\mathscr{T}}\\ &+ 2i\frac{\alpha\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{J}} - 4\frac{\Omega\alpha}{\hbar^2}(\hat{\mathscr{I}}^2 - \hat{\mathscr{J}}^2)\\ &- 2\frac{\Omega\beta}{\hbar^2}\hat{\mathscr{C}}_1\hat{\mathscr{I}}.\end{aligned} \tag{23}$$

5 Summary

I showed that a current algebra appears when we calculate the quantum dynamics of the tunnelling of the atoms between two condensates. I generalized the Heisenberg equation of motion to write the second time derivative of any operator. Then I calculated the quantum dynamics of these currents and showed that different dynamics appear when we consider different choices of the parameters of the Hamiltonian.

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The role of "escort fields" in the relation between massless and massive vector (tensor) mesons

Bert Schroer

Abstract The relation between massive and massless vector potentials is reviewed in light of recent progress concerning positivity-preserving renormalizable interactions and the ensuing weakening of localization. One obtains new insights beyond those which were extracted from the standard gauge theoretic setting.

1 Positivity preserving relation between massive potentials and their massless counterpart

Since a systematic presentation of an ongoing new development in QFT within a conference talk is hardly possible, the chosen alternative will be to present some of the relevant new ideas by starting from a simple concrete observation and leaving details to published papers [1] [9].

A *massless* point-local (pl) positivity-obeying vector potential does not exist[1] but one may encode the unitary $(m = 0, h = 1)$ Wigner representation into a unique string-local (sl) covariant field which transforms under the Lorentz group as a (1/2,1/2) vector potential. It acts in the same Wigner-Fock space

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[1] A fact which probably was already known to Wigner since there exists no (1/2,1/2) spinorial representation which corresponds to the unique $(m = 0, h = 1)$ Wigner unitary representation [2].

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$$A_\mu(x,e) = \int_0^\infty d\lambda F_{\mu\nu}(x+\lambda e)e^\nu,\ F_{\mu\nu} := \partial_\mu A_\nu^P - \partial_\nu A_\mu^P \tag{1}$$

$$\left\langle A_\mu(x,e)A_\nu(x',e')\right\rangle = \frac{1}{(2\pi)^{3/2}}\int e^{-ip(x-x')}M_{\mu\nu}(p;e,e')\frac{d^3p}{2p_0} \tag{2}$$

$$M_{\mu\nu}(p;e,e') = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p\cdot e_{i\varepsilon}p\cdot e'_{-i\varepsilon}} + \frac{p_\mu e_\nu}{p\cdot e_{i\varepsilon}} + \frac{p_\nu e'_\mu}{p\cdot e'_{-i\varepsilon}}$$

where the denominators $p\cdot e_{\pm i\varepsilon} = \lim_{\varepsilon\to\pm 0} p\cdot e + i\varepsilon$ indicate that the Fourier transform is the distribution valued boundary limit of the Heavyside function. Hence the field fluctuates in the spacelike string direction e, $e^2 = -1$ as it does in x, and as a consequence coalescent string directions require using Wick-ordering just as coinciding $x's$.

This line integral leads to a well-defined operator-valued distribution in the massive as well as in the massless case. Although the associated Wigner representations are inequivalent, the massive correlation functions pass smoothly to their massless counterparts.

In fact in the *massive* case, one may start from a Proca potential and use the resulting field strength for the construction of associated covariant sl potentials which acts in the same Hilbert space and are relatively Einstein-causal, i.e., both potentials are members of the same sl local equivalence ("Borchers") class [4]. This construction suggests that there may be an interesting role to play for another this time scalar sl field ϕ in this equivalence class,

$$\phi(x,e) = \int_0^\infty d\lambda A_\nu^P(x+\lambda e)e^\nu \tag{3}$$

$$A_\mu(x,e) = A_\mu^P(x) + \partial_\mu\phi(x,e). \tag{4}$$

Such massive covariant scalar sl fields appeared first as "theoretical toys" in [6]; later it became clear that they play an important role in a linear relation between pl and sl potentials (4). Both vector potentials are one-forms which the Poincaré Lemma associates to the $F_{\mu\nu}$ two-form.

The proof uses the representation of free Proca potential in terms of an u intertwiner functions [2]

$$A_\mu^P(x) = \frac{1}{(2\pi)^{3/2}}\int e^{ipx}\sum_{s_3} u_{\mu,s_3}(p)a^*(p,s_3) + h.c.$$

The ϕ and A_μ are linear combinations of the same $s=1$ creation/annihilation operators and their intertwiners $u_{s_3}(p,e)$ $u_{\mu,s_3}(p,e)$ can be read off from the definition of the two fields (3, 4). The last step is to verify that the intertwiner of A_μ agrees with that obtained from (1). For reasons which will become clear later, the scalar sl field ϕ will be referred to as an *escort* field.

There is a similar formula in gauge theory [8] in which ϕ is the scalar negative metric pl Stueckelberg field whose two-point function differs from that of a scalar pl free field by a minus sign. Instead of maintaining the degrees of freedom of the

$s = 1$ Wigner representation, it increases them by adding unphysical $s = 0$ degrees of freedom[2]. In the presence of interactions (e.g., “massive QED”) the BRST gauge formalism requires a further increase in terms of ghost degrees of freedom.

It will be shown that the use of sl potentials permits staying with only physical degrees of freedom; or to use a metaphor, the string-local quantum field theory (SLFT) is the result of applying Ockham’s razor to gauge theory. Considering the historical origin of the standard pl formulation from Lagrangian quantization, we are accustomed to consider sl fields as somewhat artificial objects which one may introduce after the renormalized perturbation calculations of interacting pl fields have been done, but which are not to be directly used in renormalized perturbation theory. But this viewpoint ignores the conceptual autonomy of QFT; there is nothing natural about that parallelism to classical field theory in the form of Lagrangian quantization and the sl fields are certainly not “Euler-Lagrange”.

All massive *free* fields exist as pl fields and are uniquely fixed in terms of their physical spin and their covariance[3], but there are no positivity-obeying renormalized *interacting* pl fields in case the interaction involves $s \geq 1$ fields; such pl interactions are only renormalizable in an indefinite metric setting. The use of positivity-preserving sl fields as well as that of pl gauge fields leads to a lowering of short distance dimensions but there is a huge physical difference. The sl fields achieve this by relieving the $d_{sd} = s+1$ pl fluctuations in x and allowing part of them to convert into directional fluctuations (whereas in gauge theory this results from brute force compensation of part of pl physical with unphysical degrees of freedom).

By repeated integrations along e, one can lower the d_{sd} to zero [9], but we will see that what one needs for renormalizable sl perturbation theory is $d_{sd} = 1$. Gauge theory achieves this by brute force compensation between part of the positive with negative metric degrees of freedom at the price of losing the physical causal localization (apart from local observables)

To obtain a wider conceptual orientation which does not refer to quantization, it is helpful to take notice of a powerful theorem of algebraic QFT (AQFT) [11]. Reconverting its physical content into the setting of covariant fields instead of Haag’s LQP nets of operator algebras, it says that in a theory with local observables and a mass gap, the superselected charge-carrying fields can always be assumed sl[4]. These physical sl fields are the mediators (“interpolating fields”) between the abstract causality principles of QFT and the observed world of particles.

The B-F theorem secures their existence in any theory with a mass gap. The Hilbert space of such a theory has the form of a Wigner-Fock particle space (the problem of asymptotic completeness does not exist in perturbation theory). As expected the theorem shows that all on-shell objects as particles and the S-matrix are e-independent; the main purpose of lessening the tightness of point-like localization

[2] This gauge theoretic version of (4) facilitates the understanding as to why in the passing to the unitary gauge [9] one encounters polynomially unbounded pl matter fields.

[3] Positivity obeying *massless* pl vector potentials and their tensorial counterparts do not exist.

[4] The semi-infinite spacelike strings are the cores of causally closed spacelike cones as points of pl fields are the cores of arbitrarily small causally closed double cones.

is better control of the vacuum polarization clouds which are responsible for the unbounded momentum space increase of pl $s \geq 1$ nonrenormalizable models.

The theorem does not reveal for which (positivity-maintaining) interactions one is forced to use sl fields. Here renormalized perturbation theory leads to a clear answer; pl fields *must* be used for interactions between $s < 1$ fields, whereas interactions involving $s \geq 1$ fields require using sl fields. Before commenting on interactions, it is helpful to present properties that are rather direct consequences of the above relations.

From the definition of the sl vector potential (1, 2), it follows that they have a massless limit whereas A_μ^P and ϕ (3, 4) have mass divergencies which mutually compensate on the right-hand side of (4). The sl two-point function (2) contains in addition to the Gupta-Bleuler (Feynman gauge) $-g_{\mu\nu}$ contribution rational in p terms whose presence is the price for maintaining positivity while improving the short distance behavior from $d_{sd}^P = 2$ to 1. The massless sl potential is the unique positivity maintaining covariant vector potential of Wigner's unitary helicity $h = 1$ representation.

Saying that the massive $s = 1$ sl field converges to the massless helicity $h = 1$ field does not mean that the operators converge but rather refers to the convergence of the correlation functions. The operator representations remain unitarily inequivalent but they can be reconstructed from the correlation function (the GNS reconstruction [7]); for free fields one only needs the 2-point-function [4].

This is particularly important in the presence of interactions since only massive $s \geq 1$ fields permit an operator formulation in a particle Wigner-Fock Hilbert space, whereas the operator formulation of the massless limit has to be reconstructed from the zero mass limits of the correlation functions. This is not a shortcoming of the perturbative approach but rather reflects the fact that important parts of our understanding of interactions involving massless $s \geq 1$ fields as infraparticles and confinement are still missing.

The Coulomb (radiation) potential is obtained from the sl potential by integrating over the spacelike directions e within the $t = 0$ hypersurface; the result is the unique rotation invariant vector potential. Note that both potentials live in the same Hilbert space and cannot be related by a gauge transformation. Gauge symmetry and gauge-related potentials only exist in a description with *additional unphysical (indefinite metric) degrees of freedom*. Replacing in the degrees of freedom conserving sl escort ϕ field in (4) by the scalar indefinite metric degrees of freedom-carrying Stueckelberg field one obtains instead of the physical sl potential the positivity-violating scalar Gupta-Bleuler potential (the Feynman gauge) [8]. In the presence of interactions, one has to introduce additional unphysical "ghost" degrees of freedom.

Massless $s \geq 1$ potentials share a curious topological property which has no massive counterpart; they violate Haag duality (HD) for multiply-connected localization regions. HD is violated if there exists operators which commute with all operators that are localized in the causal complement $\mathscr{O}'$ of a region $\mathscr{O}$, but which fail to be localized in $\mathscr{O}$ [7]. Such a violation is still consistent with Einstein causality (EC). The simplest illustration is provided by a magnetic flux through a toroidal region $\mathscr{T}$ (a thickend Wilson loop in terms of $A_\mu(x,e)$) [11].

The quantum field theoretical Aharonov-Bohm effect is closely related to this phenomenon; in that case the torus is closed at spacelike conformal infinity [9]. The use of the pl indefinite metric potential does not permit distinguishing between HD and EC because (in contrast to its field strength) it does not account for the correct causal localization. The somewhat eerie feeling concerning the A-B effect with respect to Einstein causality has its origin in that one's heuristic understanding is inclined to identify Haag duality with Einstein causality.

The deviation from physical localization increases in the presence of interactions and there seems to be agreement that the more than 5 decades old problems of a spacetime understanding of infrared and confinement phenomena (which depend on the correct long distance localization of fields) cannot be understood in gauge theory. Attempts to amend this in terms of using a different topology in state space [13] or by recovering physical fields as formally gauge invariant composites [14] seem to be no replacement for the positivity- and physical localization-preserving SLFT.

Another more tangible application of sl fields is related to the problem of massless conserved currents. The simplest case is the conserved current of a complex vector field B_ν. The pl definition $j_\mu = B_\nu^* \overleftrightarrow{\partial_\mu} B^\nu$ admits no massless limit but using the sl $B_\nu(x,e)$ one obtains a conserved sl current $j_\mu(x,e)$ with a well-defined global charge. In the massive case, there are two conserved currents. The pl and its sl counterpart lead to the same global charge. In the massless limit, only the sl current survives.

This zero mass situation has a much richer analog for the conserved energy-momentum tensor. In that case the problem with zero mass starts at $s=2$ since the $s=1$ tensor can still be expressed in terms of the massless field strength. The Weinberg-Witten No-Go theorem [15] excludes *pl* E-M tensors, however sl tensors which lead to the same global charges continue to exist. Without going into details [16], I mention the extension of (3) to arbitrary high spin s which leads to a linear relation in which the derivative of s escort fields of tensor degree from zero to $s-1$ appear [1].

Fields such as the escort fields, which maintain the cardinality of degrees of freedom (but whose introduction as separate entities is nevertheless necessary) are quite common in quantum mechanical many-body problems when, as the possible result of phase transitions, the basic degrees of freedom reorganize themselves and change the physical properties.

An illustration[5] is provided by the bosonic two-electron Cooper pairs in the microscopic BCS description of superconductivity which among other things account for the short range nature of London's screened (short range) vector potentials inside a superconductor.

In QFT there is a smooth passage from massless photons described in terms of a positivity obeying long range sl vector potential to a short range massive pl Proca counterpart, but in order to achieve this, one needs the intervention of the scalar sl escort ϕ. As will be seen in the next section, this is independent of what kind

[5] For a rigorous illustration of such composites, see Haag's presentation of the role of the Cooper pairs in the BCS model [12]..

of matter field one couples to the massive vector meson, be it complex spinor ψ, complex scalar φ, or scalar Hermitian H matter[6]. The escort fields play precisely that role which in the metaphoric presentation of the Higgs mechanism ("fattening of the photons by swallowing Goldstones") is incorrectly attributed to the H.

The correct understanding of the "fattening" is very important because it clears the head for its real raison d'être in models involving massive self-interacting vector mesons.

2 Lowest order perturbation theory using sl $s=1$ fields

The previous mainly kinematic observations would remain an academic exercise if the escort fields were not to play an essential dynamic role in the renormalization theory of interactions involving $s \geq 1$ fields. The simplest model which reveals this role is "massive QED". In this case the pl interaction density[7] L^P can be rewritten with the help of (4) and the use of the current conservation as

$$L^P = A^P_\mu j_\mu = L - \partial^\mu V_\mu(x,e) \; with \; L = A_\mu(x,e) j_\mu(x), \; V_\mu := \phi(x,e) j_\mu \tag{5}$$

$$d_e(L - \partial^\mu V_\mu(x,e)) = 0 = d_e L - \partial^\mu Q_\mu(x,e) \; with \; Q_\mu = d_e V_\mu. \tag{6}$$

We will refer to this requirement as the L, V_μ (or L, Q_μ) *pair condition*. It preserves the heuristic physical content of L^P in the adiabatic S-matrix limit, in lowest order,

$$S^{(1)} = \int L^P = \int L. \tag{7}$$

In this way the $d_{sd}^{int} = 5$ (3 from the j_μ and 2 from A^P) of L^P is lowered to $d_{sd}^{int} = 4$ of L at the expense of the occurrence of the $d_{sd}^{int}(\partial V) = 5$ divergence term. But in models with a mass gap, this term does not contribute to the first order adiabatic S-matrix limit (7).

Note that the pair condition has no analog for $s < 1$; linear relations as (4) between pl and their lower d_{sd} sl kinsmen only exist for $s \geq 1$, and attempts to use sl $L's$ for $s < 1$ would end in total delocalization.

At this point a short interlude may be helpful. Recently it had been claimed that a free $s = 1/2$ and $d_{sd} = 3/2$ Dirac fermion field can be transformed into an "Elko" field with $d_{sd} = 1$ (formula (100) in [18]). But a pl field *whose two-point function agrees with that of a 4-component free $d_{sd} = 1$ scalar field is really a four-component scalar field* (and not a camouflaged $s = 1/2$ field); this is the content of a well-known theorem (the Jost-Schroer theorem in [4] page 163). The existence of

[6] A significant difference is that the H coupling disappears in the Maxwell limit; this accounts for the fact that it was for a long time overlooked.

[7] Renormalized perturbation theory does not require the Euler-Lagrange quantization setting but only an interaction density L in terms of free fields [17].

an "Elko fermion" would contradict facts about QFT which were already known in the 60s; in particular it would clash with the spin-statistics theorem. It also would be in contradiction with Weinberg's complete classification of all pl fields for any spin.

The short distance dimension of a pl field can only be lowered by integrating it along a spacelike half-line as in (3); starting from a pl $d_{sd} = 3/2$ Dirac field, one obtains a sl $d_{sd} = 1/2$ Dirac field. But this cannot be used to convert the nonrenormalizable 4-Fermi coupling into a renormalizable interaction since the above pair condition can only be fulfilled for spins $s \geq 1$. For more details on the use of sl fields in renormalization theory, we refer to [19].

The fact that divergence operations as in $\partial^\mu TV_\mu L'$ cannot be taken inside the T-products means that the second order validity of the pair requirement

$$(d_e + d_{e'})TLL' - \partial^\mu TQ_\mu L' - \partial'^\mu TLQ'_\mu = 0 \tag{8}$$

$$hence\ (d_e + d_{e'})S^{(2)} = 0 \tag{9}$$

is a normalization condition (which has no counterpart in $s < 1$ renormalization theory). Whereas in massive spinor QED, this (and all higher perturbative extensions) is automatically fulfilled, its fulfillment in scalar QED induces the expected $A \cdot A\, |\varphi|^2$ term [8] [9]. In contrast to gauge theory where this contribution is induced by imposing BRST gauge invariance on S, in the present SLFT setting the e-independence of S is a consequence of the causal localization principles.

One expects that this perturbation theory of the S-matrix works for all orders. One also hopes that the extension of this adiabatic equivalence also works for the construction of correlation functions of sl fields. Formally speaking this corresponds to the independence from *internal* $e's$ in suitable sums of Feynman graphs after having integrated over inner $x's$. Second order calculations of sl fields have been started in [21].

This short presentation is not the place to present calculational details for which we refer to the above references and forthcoming work. However it may be of interest to mention those points for which the SLFT formulation leads to a very different physical interpretation of the Higgs model.

1. Instead of postulating a Mexican hat potential, the SLFT construction starts from a point-local AAH interaction between a massive vector potential and a Hermitian scalar field. The split into a string-local L and the divergence of a vector field V leads to an L which also depends on the escort field. The required e-independence of the second order S-matrix induces a four-order self-interaction which is of the same form as that derived from the postulated spontaneous symmetry breaking Mexican hat potential. For details see [8, 9].
2. Massive *self-interacting* vector mesons lead to a new physical phenomenon. The L, Q_μ pair condition can be satisfied but there is a hitch; the second order contains an induced renormalizability violating $d_{sd} = 5$ contribution which, if left uncompensated, would wreck the existence of the model. The only way to save this situation is to introduce a coupling to a lower spin H field so that the second

order H-contraction produces a $d_{sd} = 5$ compensating term. Such compensations one expected from different spin components of a supersymmetric coupling but the present situation is different in that the $d_{sd} = 5$ compensation is not a fringe benefit of a higher symmetry but rather the raison d'être for the H-particle.

3. The model of A-selfinteractions shows another lesser understood phenomenon in that the interaction requires the A self-coupling parameters to obey the symmetry of a Lie algebra without any symmetry requirement. Whereas in the BRST gauge formalism one can blame that on the quantization of a classical fibre bundle gauge theory, the SLFT formulation is solely built on causally localized quantum matter.
4. Any interaction involving higher spin $s \geq 2$ fields which passes the first order L, Q_μ requirement will have $Q'_\mu s$ with $d_{sd} \geq 2$ which inevitably lead to second order $d_{sd} > 5$ contribution and hence needs the compensation with lower spin contributions. It would be interesting to know from what s on the compensatory mechanism breaks down. Such spin s fields will be inert in that they only exist in the form of free fields but may (through their energy momentum tensor) interact with classical gravitational fields.

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String-localized infinite spin fields and inert matter

Bert Schroer

Abstract Positive energy ray representations of the Poincaré group are naturally subdivided into three classes according to their mass and spin content: m>0, m=0 finite helicity and m=0 infinite spin. Whereas the intrinsic noncompact localization accounts for its inertness, the possible existence of finite spin $s \geq 2$ inert matter is less clear.

1 Wigner's infinite spin representation and string-localization

Wigner's famous 1939 theory of unitary representations of the Poincaré group $\mathscr{P}$ was the first systematic and successful attempt to classify relativistic particles according to the *intrinsic* principles of relativistic quantum theory [1]. As we know nowadays, his massive and massless spin/helicity class of positive energy ray representations of $\mathscr{P}$ does not only cover all known particles, but their "covariantization" [2] leads also to a complete description of all covariant point-local free fields, except that point-local (pl) fields are not available for the infinite spin class. In view of its still insufficiently understood physical properties, we will refer to this matter as the "Wigner stuff" (WS).

The problem with WS is that the standard method of associating local fields through covariantization in terms of $u(p,s_3)$ intertwiners [2] (which convert the unitary Wigner representation into the covariant transformation law of a free field which acts in a Wigner-Fock Hilbert space) is not available. Although there have been constructions of covariant wave functions ever since Wigner's 1939 classification (for a recent covariantization, see [3]), the causal localization properties of WS require addressing the problem of localization in a more intrinsic way.

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The oldest result in this direction was a No-Go theorem by Yngvason [4] which excluded a description in terms of pl (Wightman) fields. The application of the concept of modular localization to Wigner's irreducible representation spaces supported the idea that the tightest localized subspaces are associated to arbitrary narrow spacelike cones [5], which in turn led to modular construction of string-local (sl) fields whose strings are cores of such cones [6]. The issue of WS localization was finally closed by a theorem which excludes compact localization [7].

The solution of the WS localization problem in [6] begged the question whether the improvement of short distance behavior by relaxing the tightness of localization from pl to sl could improve the renormalizability properties. It is well-kown that as a result of the increase of short distance dimensions of pl fields with spin $d_{sd} = s+1$ interactions involving $s \geq 1$ lead to nonrenormalizable pl interaction densities[1] $d_{sd}(L^P) > 4$.

The makeshift solution for preserving $s = 1$ renormalizability has been to yield on the positivity requirement of quantum theory. This leads to Gauge Theory (GT) which is an incomplete QFT since only the restricted set of gauge invariant operators possesses a physical content. The work on sl free fields in [6] suggested that there should also exist a physical way which maintains and achieves renormalizability by using the better short distance properties of sl fields.

The result has been described in several publications and may be summed up as follows, see [8–10]. The nonrenormalizable pl $L^P(x)$ corresponds to a pair $L(x,e), V_\mu(x,e)$ with

$$L^P = L - \partial^\mu V_\mu, \; d_e(L - \partial^\mu V_\mu) = 0 \tag{1}$$
$$S^{(1)} = \int L^P = \int L$$

where the second equation is the L, V_μ pair condition in the d_e differential calculus on the $d = 1+2$ dimensional de Sitter space of spacelike string directions e, $e^2 = -1$. This condition has the immediate consequence that the lowest order S-matrix inherits the best of the two worlds, the e-independence from L^P and the $d_{sd} \leq 4$ power counting bound (pcb) from L, since $\partial^\mu V_\mu$ divergence terms do not contribute to the adiabatic limit in models with a mass gap.

The origin of the pair property are linear relations between massive spin $s \geq 1$ pl potentials with $d_{sd} = s+1$ and their $d_{sd} = 1$ sl counterparts. For $s = 1$, this relations has the form

$$A_\mu(x,e) = A_\mu^P(x) + \partial_\mu \phi(x,e) \tag{2}$$

where the scalar sl "escort" field $\phi(x,e)$ is just as the two other two free fields, a linear combination of the three $s = 1$ Wigner creation/annihilation operators; the difference lies in their different intertwiner functions Replacing the A^P in $L^P = A_\mu^P j^\mu$ by $A - \partial V$, one obtains (1).

[1] The short distance dimension of a field is defined in terms of the short distance scaling behavior of its 2-point function.

For massive scalar QED $j^\mu(x) = \varphi^* \overleftrightarrow{\partial^\mu} \varphi$, the implementation of the second order L, V_μ pair condition *induces* the expected quadratic second order term $A \cdot A\,|\varphi|^2$. Different from counterterms, induced terms do not introduce new parameters. Induction of interaction terms is an epiphenomenon of $s \geq 1$ sl renormalization theory which requires the L, V pair condition; it has no counterpart in $s < 1$ pl interactions. Renormalization counterterms which come with new coupling parameters appear in both cases. Induction means in particular that there is no need to impose the $\partial_\mu \to \partial_\mu - iA_\mu$ rule from classical fibre bundles, since it is a consequence of the principles of local quantum physics (LQP) namely causal localization + (Hilbert space) positivity which are its defining properties.

Passing from interactions between a vector meson with complex matter to couplings with scalar Hermitian matter H in the form of a $A \cdot AH$ interaction, the higher order induction becomes much richer; in addition to the expected $A \cdot AHH$ and $AA\phi\phi$ terms, there are now induced cubic and quartic H, ϕ self-interactions which have been mistakenly attributed to spontaneous symmetry breaking [9] [10]. For self-interacting massive vector potentials, there is a new phenomenon. Whereas a L, V_μ pair exists, second order induction leads to a renormalizability violating the $d_{sd} = 5$ term. The only way the model can be saved is to introduce an additional coupling to a scalar Hermitian field H whose second order compensates this pcb-violating term. This, and not SSB, is the raison d'être for the H.

2 Inert higher spin fields

Perturbative interactions which are pl nonrenormalizable may turn out to be perfectly renormalizable in the SLFT setting. On the other hand, there are good reasons to believe that interactions which are not sl renormalizable do not correspond to a model of QFT. This view receives support from a powerful structural theorem of algebraic QFT [11] (AQFT). Converting its content into the standard setting of QFT, it states that in $s \geq 1$ models with a mass gap, each particle possesses *interpolating sl fields* (fields related to the particle by large time LSZ scattering theory).

Fields which cannot interact with other fields or themselves, and therefore only exist in the form of free fields, will be referred to as "inert". In contrast to massless finite spin s sl free fields which can be constructed as $m \to 0$ limits of massive $d_s = 1$ sl fields (for fermions $d_{sd} = 3/2$), it is not possible to represent the sl fields associated to WS in this way. This is the reason why their presence in an interaction turns out to be inconsistent with the pair property.

The result at the end of the previous section implies that $s = 1$ fields are fully reactive both with respect to lower spin fields and in the form of self-interactions. But without additional nontrivial calculations it is not clear whether $s = 2$ fields are reactive or inert. What is easy to see is that $d_{sd}(V_\mu) > 4$ and that $d_{sd}(V_\mu)$ increases with s, so that increasing spin requires an increasing number of second order compensating terms which are second order contributions from interactions of the highest spin with lower spin fields. According to this qualitative observation, the problem of second order compensatory preservation of renormalizability for interactions which fulfill the first order L, V_μ pair condition increases with s, so that one expects the existence of a maximal s beyond which fields will be inert.

Inert matter is highly desirable in connection with the problem of astrophysical darkness. Since free fields of any spin possess an energy-momentum tensor, they couple to classical gravity and contribute to gravitational backreaction. However inertness seems to lead to a "catch 22 situation" since a complete absence of reactivity gets into conflict with the standard model of cosmology according to which all matter originated in a (very reactive) big bang. But since dark matter surrounds us in abundance, the failure to register its reactivity in underground experiments would confront us with a similar problem.

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Toeplitz quantization of the quantum group $SU_q(2)$

Stephen Bruce Sontz

Abstract After a brief review of the general theory of Toeplitz quantization in a non-commutative setting, we present an example with the symbol space being the quantum group $SU_q(2)$. This includes creation and annihilation operators as well as their commutation relations. The general way for introducing Planck's constant into this theory is also presented in the example. This seems to be the first example of a quantization of a quantum group.

1 Toeplitz operators: the general theory

The idea is to start with a complex vector space $\mathscr{S}$, which has this extra structure: a multiplication and a conjugation. So, $\mathscr{S}$ is a $*$-algebra, called the *symbol space*. We also suppose that we have $\mathscr{P} \subset \mathscr{S}$, where $\mathscr{P}$ is closed under all these operations: sum, scalar product, and multiplication (but not necessarily conjugation). So, $\mathscr{P}$ is a sub-algebra, though not necessarily a sub-$*$-algebra. The particular symbol $\mathscr{P}$ was chosen, since it could represent a sub-algebra of non-commuting 'holomorphic polynomials'. If that is the case, then $\mathscr{H}$ can be interpreted as a generalized Segal-Bargmann space of 'holomorphic functions'.

In mathematics, the elements of $\mathscr{S}$ are typically functions with values in $\mathbb{C}$, and so the multiplication in $\mathscr{S}$ is commutative. But for us, the interesting case will be when the multiplication in $\mathscr{S}$ is not commutative, that is, $\mathscr{S}$ is a non-commutative algebra. As physicists we can think of $\mathscr{S}$ as representing the 'functions' defined on some (non-existent!) phase space. Next, we assume that there is a *(linear) projection map* $P : \mathscr{S} \to \mathscr{P}$, that is, P restricted to $\mathscr{P}$ is the identity function.

Now we take an element (or symbol) $f \in \mathscr{S}$ and use it to define a linear map $T_f : \mathscr{P} \to \mathscr{P}$ by

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$$T_f(\psi) := P(\psi f) \text{ for all } \psi \in \mathscr{P}.$$

We say that T_f is the *(right) Toeplitz operator* with *symbol* f. Left Toeplitz operators are similarly defined, in which case the symbol multiplies from the left. Their very similar theory is not discussed here.

2 Toeplitz quantization

Since we want to relate this to quantum mechanics, we must introduce a complex Hilbert space $\mathscr{H}$. As far as I am aware, a Hilbert space is absolutely necessary in order to develop any quantum theory in physics. Therefore $\mathscr{H}$ has an inner product, that is, for all $\phi_1, \phi_2 \in \mathscr{H}$ we have a complex number $\langle \phi_1, \phi_2 \rangle$, linear in ϕ_2 and conjugate linear in ϕ_1.

Question: How do we relate a Hilbert space to Toeplitz operators?

Answer: We assume that $\mathscr{P}$ is a dense subspace of a Hilbert space $\mathscr{H}$. Then the Toeplitz operator $T_f : \mathscr{P} \to \mathscr{P}$ is a *densely defined* linear operator acting *in* $\mathscr{H}$ (but not *on* $\mathscr{H}$). If T_f is self-adjoint, then it is a quantum mechanical observable. Let $\mathscr{L}(\mathscr{P}) := \{A : \mathscr{P} \to \mathscr{P} \mid A \text{ is linear}\}$ be the set of all linear maps of $\mathscr{P}$ to itself. The *linear* map $T : \mathscr{S} \to \mathscr{L}(\mathscr{P})$ given by $f \mapsto T_f$ is called the *Toeplitz quantization* of the symbol space $\mathscr{S}$. As seen later on, T is not necessarily $*$-*linear*.

This is called a quantization in part because it conforms to this maxim: *Quantization is operators in place of 'functions'*. This Toeplitz quantization is a type of *second quantization*, that is, it quantizes a theory that is already quantum in the sense of its already being non-commutative. Unlike many other quantization schemes, this Toeplitz quantization does not use a *measure* or a *generalized integral*.

At this point one can continue to develop this into a general theory of Toeplitz quantization. (See [3] and [4].) This includes creation and annihilation operators, their canonical commutation relations and Planck's constant $\hbar > 0$.

Instead, in this note I will present an example of the general theory. This is the latest example of several which I have developed. Due to the required brevity of this paper, proofs will be given in a paper [4] to be published later.

3 The quantum group $SU_q(2)$

One of the earliest and most basic *quantum groups* is $SU_q(2)$. (See [5].) We define this as the universal $*$-algebra over the complex field $\mathbb{C}$ generated by two elements, say a and c, that satisfy these relations where $0 \neq q \in \mathbb{R}$:

$$ac = qca \qquad ac^* = qc^*a \qquad cc^* = c^*c$$
$$a^*a + c^*c = 1 \qquad aa^* + q^2c^*c = 1.$$

In this example the symbol space $\mathscr{S}$ is $SU_q(2)$. This algebra is commutative if and only if $q = 1$.

We can think of a and c as being 'holomorphic' elements, while a^* and c^* are then 'anti-holomorphic' elements. Using this intuition, we now define the algebra of 'holomorphic polynomials' to be

$$\mathscr{P} := \text{algebra}\{a,c\} = \text{Free}\{a,c\}/\langle ac - qca\rangle, \tag{1}$$

the sub-algebra of $SU_q(2)$ generated by the elements a and c. It turns out that as an algebra, $\mathscr{P}$ is the *Manin quantum plane* (with parameter q).

For $k \in \mathbb{Z}$ and $l,m \in \mathbb{N}$ we define

$$\varepsilon_{klm} := \begin{cases} a^k\, c^l\, (c^*)^m = q^{km}\, (c^*)^m\, a^k\, c^l & \text{for } k \geq 0, \\ (a^*)^{-k}\, c^l\, (c^*)^m = (a^*)^{-k}\, (c^*)^m\, c^l & \text{for } k < 0. \end{cases} \tag{2}$$

Then it is known that the set $\{\varepsilon_{klm} \mid k \in \mathbb{Z},\, l,m \in \mathbb{N}\}$ is a vector space basis of $SU_q(2)$. We also define a sesquilinear form on $\mathscr{S} = SU_q(2)$. We first define this on pairs of basis vectors and then extend to the unique sesquilinear form on $SU_q(2)$. We use the convention that *sesquilinear* means anti-linear in the first entry and linear in the second. So for $k,r \in \mathbb{Z}$ and $l,m,s,t \in \mathbb{N}$ we define

$$\langle \varepsilon_{klm}, \varepsilon_{rst} \rangle_{\mathscr{S}} := w(l+t)\, \delta_{k,r}\, \delta_{l+t,m+s}. \tag{3}$$

Here $w : \mathbb{N} \to (0,\infty)$ is any strictly positive, real function whose values are called *weights*. Also, we are using the standard notation $\delta_{i,j}$ for the Kronecker delta. The motivation for this type of sesquilinear form can be found in my paper [2], though it is already implicit in Bargmann's paper [1].

We consider the restriction of the sesquilinear form to the sub-algebra $\mathscr{P}$. So for $k,l,r,s \in \mathbb{N}$, we have

$$\langle a^k c^l, a^r c^s \rangle_{\mathscr{S}} = \langle \varepsilon_{k,l,0}, \varepsilon_{r,s,0} \rangle_{\mathscr{S}} = w(l)\, \delta_{k,r}\, \delta_{l,s} = w(s)\, \delta_{k,r}\, \delta_{l,s}. \tag{4}$$

This shows that the inner product restricted to $\mathscr{P}$ is positive definite. So, for $k,l \geq 0$ we define

$$\varphi_{kl} := (w(l))^{-1/2} \varepsilon_{k,l,0} = (w(l))^{-1/2} a^k c^l. \tag{5}$$

Then $\Phi := \{\varphi_{k,l} \mid k,l \in \mathbb{N}\}$ is an orthonormal, vector space (Hamel) basis of $\mathscr{P}$. This turns $\mathscr{P}$ into a pre-Hilbert space whose completion with respect to this positive definite inner product is denoted by $\mathscr{H}$, a Hilbert space.

4 Toeplitz operators and quantization

Define the projection operator $P : SU_q(2) \to \mathscr{P}$ for all $f \in \mathscr{S} = SU_q(2)$ by

$$Pf := \sum_{i,j\geq 0} \langle \varphi_{i,j}, f\rangle_{\mathscr{S}}\ \varphi_{i,j}. \tag{6}$$

We note that only finitely many of the terms in the sum on the right side of this equation are non-zero. So, this infinite sum converges and gives a result in $\mathscr{P}$.

Moreover, the *Toeplitz operator* associated to the *symbol* $g \in SU_q(2)$ is $T_g = PM_g$ as in the general theory. Here M_g is the operator of multiplication on the right by the symbol g. Also, $T_g : \mathscr{P} \to \mathscr{P}$, that is, it linearly maps the Manin quantum plane to itself, and so is a densely defined linear operator in the Hilbert space $\mathscr{H}$. Finally, the map $SU_q(2) \to \mathscr{L}(\mathscr{P})$ given by $g \mapsto T_g$ is a Toeplitz quantization of the quantum group $SU_q(2)$. We do not claim that this is the only possible Toeplitz quantization of $SU_q(2)$. Far from it! Other choices could be made for the sub-algebra $\mathscr{P}$, for the projection P and for the sesquilinear form. I fully expect that all the quantum group deformations of the classical Lie groups have many Toeplitz quantizations.

5 Creation and annihilation operators

Creation operators are Toeplitz operators of the form T_g where the symbol g is a *holomorphic symbol*, that is, $g \in \mathscr{P}$.

The creation operators associated to $a, c \in \mathscr{P}$ are $T_a = PM_a = M_a$ and $T_c = PM_c = M_c$. (Recall P acts as the identity on $\mathscr{P}$.) Explicit calculations on the basis elements give for $i, j \geq 0$ that

$$T_a(\varphi_{i,j}) = q^{-j}\,\varphi_{i+1,j} \quad \text{and} \quad T_c(\varphi_{i,j}) = \left(\frac{w(j+1)}{w(j)}\right)^{1/2} \varphi_{i,j+1}. \tag{7}$$

Both T_a and T_c raise the total degree by 1, though T_a has bi-degree $(1,0)$ while for T_c the bi-degree is $(0,1)$.

These identities in turn immediately imply, as the reader can readily check, this q-commutation relation:

$$[T_c, T_a]_q \equiv T_c T_a - q\, T_a T_c = 0. \tag{8}$$

Annihilation operators are Toeplitz operators of the form T_g where the symbol g is an *anti-holomorphic symbol*, that is, $g^* \in \mathscr{P}$ or, equivalently, $g \in \mathscr{P}^*$. The annihilation operators for a^* and c^* are respectively $T_{a^*} = PM_{a^*}$ and $T_{c^*} = PM_{c^*}$. Explicit formulas on the basis elements for $i, j \geq 0$ can be readily calculated to be

$$T_{a^*}(\varphi_{i,j}) = q^j \left(1 - q^2 \frac{w(j+1)}{w(j)}\right) \varphi_{i-1,j},$$

$$T_{c^*}(\varphi_{i,j}) = \left(\frac{w(j)}{w(j-1)}\right)^{1/2} \varphi_{i,j-1},$$

where the right side of either identity is taken to be 0 if one of the sub-indices is -1.

6 Commutation relations, Planck's constant

First, here are some curious facts about the relation of the $*$-operation in $SU_q(2)$ and the adjoint operation of operators: $T_c^* = T_{c^*}$, but $T_a^* \neq T_{a^*}$. Hence this quantization is not $*$-linear. Those and the remaining results in this section are straightforward, though possibly tedious, calculations.

Here are some more commutation and q-commutation relations:

$$[T_a^*, T_a] = 0, \quad [T_c, T_a^*]_q = 0, \quad [T_a, T_c^*]_q = 0, \quad [T_c^*, T_c]_q = K, \quad \text{(CCR)}$$

where K is the bi-degree $(0,0)$ map defined by

$$K\,\varphi_{i,j} := \Big(\frac{w(j+1)}{w(j)} - q\frac{w(j)}{w(j-1)} \Big) \varphi_{i,j}. \tag{9}$$

The first three CCR's are *classical commutation relations*, that is, the result is zero. The last CCR is a *quantum commutation relation*, that is, the result is (in general) non-zero.

The full quantized version of $[T_c^*, T_c]_q = K$ is actually $[T_c^*, T_c]_q = \hbar^\alpha K$, where $\hbar > 0$ is Planck's constant. The power α is determined by dimensional analysis. This is the way that Planck's constant is introduced in general into this theory, namely, as the appropriate factor required in the quantum commutation relations in order that they be dimensionally correct. In this example α depends on the dimensions given to the element c. Note that for theoretical physics (where one takes $\hbar = 1$) this is not a relevant matter, but in mathematical physics one is interested in the *semi-classical limit* $\hbar \to 0$. I also wish to emphasize that this method for introducing Planck's constant into this theory differs sharply from that used in deformation quantization.

7 Concluding remarks

One principal goal of this research program is to introduce new ideas and methods from analysis and quantization into the study of non-commutative geometry and mathematical physics. In this regard, I have not only developed this theory of Toeplitz quantization for a general class of non-commutative algebras with auxiliary structures (as presented here), but also for some other exotic kinds of 'quantum spaces' that are not even algebras and might not necessarily have an inner product. However, a conjugation (also known as a $*$-operation) seems to be an essential structure. See [3] and references therein for details plus more examples of Toeplitz quantization in some other non-commutative settings.

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Dedication I wish to dedicate this paper to the memory of our dear, departed friend and colleague S. Twareque Ali. He will be missed by many, not just by me.

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The sinh-Gordon defect matrix generalized for n defects

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Abstract In this paper we obtain a general expression for the n-defect matrix for the sinh-Gordon model. This in turn generates general Bäcklund transformations (BT) for a system with n type-I defects through a gauge transformation.

1 Introduction

Integrable models are known to be characterized by an infinite number of conservation laws which are responsible for the stability of soliton solutions. In fact, these conservation laws may be regarded as hamiltonians generating time evolutions within a multi-time space. Each of these time evolutions are associated to a non-linear equation of motion and henceforth constitute an integrable hierarchy of equations with common conservation laws. Another peculiar feature of integrable models is the existence of Bäcklund transformations which relate two different field configurations of certain non-linear differential equation.

Bäcklund transformations (BT), among other applications, generate an infinite sequence of soliton solutions from a non-linear superposition principle (see [1]). These transformations have also been employed to describe integrable defects [2] in the sense that two solutions of an integrable model may be interpolated by a defect at certain spatial position. A BT connecting two-field configurations is the key ingredient to preserve the integrability of the system. Therefore, its systematic construction is important for the classification of integrable defects.

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The first type of Bäcklund transformation only involves the fields of the bulk theory, and is named type I. However, there exist integrable models for which such type of Bäcklund transformation are not allowed. This is the case of the Tzitzeica model where additional auxiliary fields are required [2–6]. These are called type II and consist of a new class of Bäcklund transformations. For the sine(sinh)-Gordon model, where a type I Bäcklund transformation exists, the type II Bäcklund transformation is shown to be constructed from the composition of two type I defects.

The novelty presented in this paper is to extend the composition of several consecutive Gauge-Bäcklund transformations for the sinh-Gordon model. This provides the generalization to the case of n defects by constructing the general defect matrix, as well as the corresponding general BT. This is a powerful method since the defect matrix appears to be universal and can be used as a generator of BT for all equations within a hierarchy [7]. Finally, we will present some solutions for such composite BT.

2 Gauge-Bäcklund transformation and defect matrices

The Lax pair for the sinh-Gordon model is given by

$$A_+(\phi_i) = \begin{pmatrix} \partial_+\phi_i & 1 \\ \lambda & -\partial_+\phi_i \end{pmatrix}, \qquad A_-(\phi_i) = \begin{pmatrix} 0 & \frac{e^{-2\phi_i}}{\lambda} \\ e^{2\phi_i} & 0 \end{pmatrix}, \tag{1}$$

where we denote ϕ_0 and ϕ_1 to be solutions for $x<0$ and $x>0$ regions, respectively. The defect is placed at $x=0$ and connects the two solutions by Bäcklund transformation. We assume the Lax pairs to be related by gauge transformation, i.e.,

$$K(\phi_0,\phi_1)A_\pm(\phi_1) = A_\pm(\phi_0)K(\phi_0,\phi_1) + \partial_\pm K(\phi_0,\phi_1) \tag{2}$$

where the defect matrix describing the transition from solutions ϕ_0 to ϕ_1 is given by

$$K_i \equiv K(\phi_{i-1},\phi_i) = \begin{pmatrix} 1 & -\frac{\sigma_i}{\lambda}e^{-(\phi_{i-1}+\phi_i)} \\ -\sigma_i e^{(\phi_{i-1}+\phi_i)} & 1 \end{pmatrix}, \tag{3}$$

and σ_i is the corresponding Bäcklund parameter. The gauge transformation (2) holds provided the following first order equations are satisfied:

$$\partial_+(\phi_0-\phi_1) = -2\sigma_1\sinh(\phi_0+\phi_1), \quad \text{and} \quad \partial_-(\phi_0+\phi_1) = -\frac{2}{\sigma_1}\sinh(\phi_0-\phi_1), \tag{4}$$

where $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. Equations (4) are type I Bäcklund transformations for the sinh-Gordon model.

Let us now consider the composition of two Bäcklund-gauge transformations $K^{(2)}(\phi_0,\phi_2) = K(\phi_1,\phi_2)K(\phi_0,\phi_1)$. From expression (3) we find

$$K^{(2)} = \begin{pmatrix} 1+\frac{\sigma_1\sigma_2}{\lambda}e^{p_1-p_2} & -\frac{1}{\lambda}\left(\sigma_1 e^{-p_1}+\sigma_2 e^{-p_2}\right) \\ -\sigma_1 e^{p_1}-\sigma_2 e^{p_2} & 1+\frac{\sigma_1\sigma_2}{\lambda}e^{-p_1+p_2} \end{pmatrix}. \tag{5}$$

Denoting $\eta = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1\sigma_2}$, $\sigma^2 = -\frac{1}{\sigma_1\sigma_2}$ and defining

$$\Lambda = -\phi_1 - \ln\left(2\sigma_2 e^{-\phi_0} + 2\sigma_1 e^{-\phi_2}\right) - \ln\frac{\sigma}{4}, \tag{6}$$

we obtain the type II Bäcklund transformations proposed in [5, 8], namely,

$$K^{(2)}(p,q,\Lambda) = \begin{pmatrix} 1-\frac{1}{\sigma^2\lambda}e^{q} & \frac{e^{\Lambda-p}}{2I\sigma}\left(e^{q}+e^{-q}+\eta\right) \\ -\frac{2}{\sigma}e^{p-\Lambda} & 1-\frac{1}{\lambda\sigma^2}e^{-q} \end{pmatrix}, \tag{7}$$

where $q=\phi_0-\phi_2, \quad p=\phi_0+\phi_2$.

Now we consider a system with n Type-I defects, each with a different parameter σ_i as showing in the following the diagram,

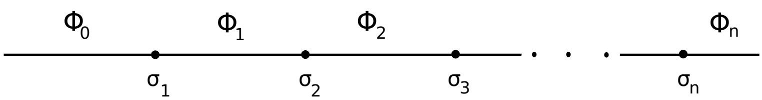

Fig. 1: Generalization for n type-I defects.

By defining $K^{(n)}$ in the following form,

$$K^{(n)} = K_n K_{n-1} \ldots K_2 K_1 = \begin{pmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ K_{21}^{(n)} & K_{22}^{(n)} \end{pmatrix}, \tag{8}$$

we find for even n:

$$\begin{aligned}
K_{11} &= 1+\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=1}^{n/2}\frac{1}{\lambda^r}\prod_{j=1}^{2r}\sigma_{i_j}e^{(-1)^{j+1}p_{i_j}}\right], \\
K_{12} &= -\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=0}^{(n-2)/2}\frac{1}{\lambda^{r+1}}\prod_{j=1}^{2r+1}\sigma_{i_j}e^{(-1)^{j}p_{i_j}}\right], \\
K_{21} &= -\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=0}^{(n-2)/2}\frac{1}{\lambda^{r}}\prod_{j=1}^{2r+1}\sigma_{i_j}e^{(-1)^{j+1}p_{i_j}}\right], \\
K_{22} &= 1+\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=1}^{n/2}\frac{1}{\lambda^r}\prod_{j=1}^{2r}\sigma_{i_j}e^{(-1)^{j}p_{i_j}}\right],
\end{aligned} \tag{9}$$

and for odd n:

$$
\begin{aligned}
K_{11} &= 1+\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=1}^{(n-1)/2}\frac{1}{\lambda^r}\prod_{j=1}^{2r}\sigma_{i_j}e^{(-1)^{j+1}p_{i_j}}\right],\\
K_{12} &= -\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=0}^{(n-1)/2}\frac{1}{\lambda^{r+1}}\prod_{j=1}^{2r+1}\sigma_{i_j}e^{(-1)^{j}p_{i_j}}\right],\\
K_{21} &= -\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=0}^{(n-1)/2}\frac{1}{\lambda^{r}}\prod_{j=1}^{2r+1}\sigma_{i_j}e^{(-1)^{j+1}p_{i_j}}\right],\\
K_{22} &= 1+\left[\prod_{a=1}^{n_\sigma}\left(\sum_{i_a=a}^{n-(n_\sigma-a)}\right)\right]\left[\sum_{r=1}^{(n-1)/2}\frac{1}{\lambda^r}\prod_{j=1}^{2r}\sigma_{i_j}e^{(-1)^{j}p_{i_j}}\right],
\end{aligned}
\tag{10}
$$

where n_σ is the number of parameters σ_{i_j} associated with each defect such that $i_1 < i_2 < i_3 < ... < i_n$, and $p_{i_j} = \phi_{i_j} + \phi_{i_j-1}$.

The next step is to derive a general expression for the BT corresponding to this $K^{(n)}$ defect matrix. In order to obtain Bäcklund transformations for n defects, we will assume $K^{(n)}$ to be the generator of the gauge transformation (2), leading to

$$
\begin{aligned}
\partial_+(\phi_0-\phi_n) &= -2\sum_{i=1}^{n}\sigma_i\sinh p_i\\
\partial_-(\phi_0-(-1)^n\phi_n) &= 2\sum_{i=1}^{n}\frac{(-1)^n}{\sigma_i}\sinh q_i\\
\partial_+ q_i &= -2\sigma_i\sinh p_i\\
\partial_- p_i &= -\frac{2}{\sigma_i}\sinh q_i
\end{aligned}
\tag{11}
$$

with $p_i = \phi_{i-1}+\phi_i,\ q_i = \phi_{i-1}-\phi_i$, and $i = 1,...,n$.

3 Bäcklund solutions

In this section we will consider some solutions of the sinh-Gordon model in the presence of two and three defects.

$\mathbf{n = 2}$

Consider now the fields ϕ_0 and ϕ_2 on each side of the defect with an intermediary field ϕ_1,

Vacuum → One Soliton → Vacuum Solution. First, we consider the following solution:

$$
\phi_0 = 0, \quad \phi_2 = 0, \quad \phi_1 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \quad \rho_1 = \exp\left(2k_1x_+ + \frac{2}{k_1}x_-\right), \tag{12}
$$

which satisfy the Bäcklund equations (11) with $n=2$ and the following conditions:

$$\sigma_1 = k_1, \quad \sigma_2 = -k_1. \tag{13}$$

Vacuum → One Soliton→ Two Soliton Solution. Another possible solution is

$$\phi_0 = 0, \quad \phi_1 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \quad \phi_2 = \ln\left(\frac{1+b_1\rho_1+b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}{1-b_1\rho_1-b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}\right),$$
$$\rho_j = \exp\left(2k_jx_+ + \frac{2}{k_j}x_-\right), \quad j=1,2, \tag{14}$$

where one must have $\alpha_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$ in the ϕ_2 component. Analogously, we get the following Bäcklund conditions:

$$\sigma_1 = k_1, \quad \sigma_2 = k_2, \quad b_1 = \frac{k_1+k_2}{k_1-k_2}. \tag{15}$$

$\mathbf{n=3}$

Finally putting a third defect at the same point as the others, we have the fields ϕ_0 and ϕ_3 on each side of the defects and two intermediary fields ϕ_1 and ϕ_2 at the defect points.

Vacuum → One Soliton→ Vacuum → One Soliton Solution. Now taking into account the solutions:

$$\phi_0 = 0, \quad \phi_2 = 0, \quad \phi_1 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \quad \phi_3 = \ln\left(\frac{1+\rho_2}{1-\rho_2}\right). \tag{16}$$

The Bäcklund conditions in order to satisfy Type-II BT are: $\sigma_1 = k_1$, $\sigma_2 = -k_1$, $\sigma_3 = k_2$.

Vacuum → One Soliton→ Two Soliton → Three Soliton Solution. Lastly, we assume:

$$\phi_0 = 0, \quad \phi_1 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \quad \phi_2 = \ln\left(\frac{1+b_1\rho_1+b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}{1-b_1\rho_1-b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}\right),$$
$$\phi_3 = \ln\left(\frac{1+R_1+R_2+R_3+\alpha_{12}R_1R_2+\alpha_{13}R_1R_3+\alpha_{23}R_2R_3+\alpha_{123}R_1R_2R_3}{1-R_1-R_2-R_3+\alpha_{12}R_1R_2+\alpha_{13}R_1R_3+\alpha_{23}R_2R_3-\alpha_{123}R_1R_2R_3}\right),$$
$$\rho_j = \exp\left(2k_jx_+ + \frac{2}{k_j}x_-\right), \quad R_j = a_j\rho_j, \quad j=1,2,3, \tag{17}$$

where one must have $\alpha_{123} = \alpha_{12}\alpha_{13}\alpha_{23}$, with

$$\alpha_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2, \alpha_{23} = \left(\frac{k_2-k_3}{k_2+k_3}\right)^2, \alpha_{13} = \left(\frac{k_1-k_3}{k_1+k_3}\right)^2,$$

in order to ensure that ϕ_2 and ϕ_3 satisfy the sinh-Gordon equation. In this case the Bäcklund conditions are $\sigma_1 = k_1, \sigma_2 = k_2, \sigma_3 = k_3, b_1 = \frac{k_1+k_2}{k_1-k_2}$, $a_1 = \left(\frac{k_1+k_3}{k_1-k_3}\right) b_1$ and

$a_2 = \left(\frac{k_2+k_3}{k_2-k_3}\right) b_2$, where b_2 is a free parameter. It is worth mentioning that the BT and their solutions for a four-defect system have been also computed, and the results have shown the expected behaviour.

4 Conclusion

In this paper we considered the sinh-Gordon model and provided general formulas for the defect matrix when n defects are considered. Our construction involves the product of n Type-I defect matrices. In addition, we have calculated their respective BT in a general way through gauge transformations and provided a few simple examples for $n = 2,3$.

It is important to point out that since the BT are constructed as gauge transformations, they preserve the zero curvature representation. The later describes a hierarchy of integrable equations based upon a universal Lax operator. These two facts induce the idea of the universality of the Bäcklund-Gauge transformation within the hierarchy. We have verified [9, 10] that the constructed defect matrix indeed gives the correct BT for the mKdV equation. It provides a systematic construction of BT for all higher grade evolution equations within the mKdV hierarchy. Several examples were verified for KdV hierarchies [7] as well.

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Higher-genus amplitudes and resurgence in SUSY double-well matrix model for 2D IIA superstrings

Fumihiko Sugino

Abstract We compute higher-genus one-point functions of non-SUSY operators in a SUSY double-well matrix model which has been discussed to correspond with 2D type IIA superstring theory on a nontrivial Ramond-Ramond background. We discuss the validity of resurgence theory in the model by comparing resurgent string perturbation series with instanton contributions.

1 Supersymmetric double-well matrix model

We start with a simple matrix model given by the action [1]

$$S = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2-\mu^2) + \bar{\psi}(\phi\psi+\psi\phi)\right]. \tag{1}$$

B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ matrices whose components are Grassmann numbers. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

from which one can see the nilpotency: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$. After integrating out B, we have a scalar potential of a double-well shape, $\frac{1}{2}(\phi^2-\mu^2)^2$. In the case of $\mu^2 > 2$, a large-N saddle point solution for the eigenvalue distribution of the matrix ϕ: $\rho(\lambda) \equiv \frac{1}{N}\mathrm{tr}\,\delta(\lambda-\phi)$ is given by

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$$\rho(\lambda) = \begin{cases} \frac{\nu_+}{\pi}\lambda\sqrt{(\lambda^2-a^2)(b^2-\lambda^2)} & (a<\lambda<b) \\ \frac{\nu_-}{\pi}|\lambda|\sqrt{(\lambda^2-a^2)(b^2-\lambda^2)} & (-b<\lambda<-a), \end{cases} \tag{4}$$

where $a=\sqrt{\mu^2-2}$ and $b=\sqrt{\mu^2+2}$. The filling fractions (ν_+,ν_-) satisfying $\nu_++\nu_-=1$ indicate that ν_+N (ν_-N) eigenvalues are around the right (left) minimum of the double-well. The large-N free energy and the expectation values $\left\langle \frac{1}{N}\mathrm{tr}B^n\right\rangle$ ($n=1,2,\ldots$) evaluated at the solution turn out to all vanish [1]. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parametrized by (ν_+,ν_-) at large N. On the other hand, in case of $\mu^2<2$, a non SUSY saddle point solution is obtained [2]. Transition between the SUSY phase ($\mu^2>2$) and the SUSY broken phase ($\mu^2<2$) is of the third order.

The partition function after B, ψ and $\bar\psi$ are integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H=\phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. Refs. [3, 4] discuss that the difference of the integration region only affects nonperturbative quantities in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion.

The Nicolai mapping changes the SUSY operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ ($n=1,2,\ldots$) to regular operators $\frac{1}{N}\mathrm{tr}H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix model (the $c=-2$ topological gravity) at least perturbatively in $1/N$. However, the non-SUSY operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}$ ($n=0,1,2,\ldots$) are mapped to $\pm\frac{1}{N}\mathrm{tr}H^{n+1/2}$ that are singular at the origin. They are not observables in the $c=-2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ in the original setting (1). Correlation functions among operators

$$\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\psi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\bar\psi^{2n+1} \qquad (n=0,1,2,\ldots) \tag{5}$$

at the solution (4) exhibit logarithmic singular behavior of powers of $\ln(\mu^2-2)$ [5].

In Ref. [5], it has been discussed that the matrix model corresponds to two-dimensional type IIA superstring theory [6, 7] on an nontrivial Ramond-Ramond background from the points of view of symmetry and spectrum. Furthermore, in Ref. [8], it has been confirmed at the tree level by performing the explicit computations of correlation functions in both sides. In Ref. [9], nonperturbative instanton computations have been carried out in the matrix model. The instantons induce dynamical SUSY breaking, and the breaking survives after taking the double scaling limit. This suggests that the target-space SUSY in the two-dimensional superstring theory is also spontaneously broken due to nonperturbative dynamics. The double scaling limit defined as

$$N\to\infty, \quad \mu^2\to 2 \quad \text{with } s\equiv N^{2/3}(\mu^2-2) \text{ fixed} \tag{6}$$

gives the correspondence of the matrix model to two-dimensional superstring theory beyond the tree level. In its weakly coupled region (s: large), instanton effects can be seen in the matrix model which are nonperturbative in $1/N$. Although such effects are typically of the order e^{-N} and vanish in the simple large-N limit, interestingly they remain after taking the double scaling limit (6). In Ref. [10], the exact expression of the nonperturbative free energy is given by a solution of the Painlevé II differential equation using the technique of random matrix theory [11].

This matrix model is simple but exhibits quite interesting features as correspondence with two-dimensional type IIA superstring theory on an nontrivial Ramond-Ramond background, nonperturbative spontaneous SUSY breaking, and so on. It should be worth being explored to get deep insights into issues of the Ramond-Ramond background and nonperturbative aspects of superstring theory.

2 Higher-genus amplitudes in the matrix model

In this section, we calculate the one-point function of ϕ_{2k+1} to all orders in the string perturbation theory. Since the operators are not protected by SUSY, nontrivial large-order behavior is expected here. As discussed in [5], the one-point function at the (ν_+,ν_-) filling fraction is simply related to that at the $(1,0)$ filling fraction by

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi_{2k+1}\right\rangle^{(\nu_+,\nu_-)} = (\nu_+-\nu_-)\left\langle \frac{1}{N}\mathrm{tr}\,\phi_{2k+1}\right\rangle^{(1,0)}. \tag{7}$$

So, it is sufficient to consider the sector of the $(1,0)$ filling fraction alone. The object is recast in the contour integral of the resolvent of ϕ^2 as

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi_{2k+1}\right\rangle^{(1,0)} = \oint_{[a,b]}\frac{dz}{2\pi i}z^{2k+1}\cdot 2z\left\langle \frac{1}{N}\mathrm{tr}\frac{1}{z^2-\phi^2}\right\rangle^{(1,0)} + \dots, \tag{8}$$

where the integration contour surrounds the support of the eigenvalue distribution $[a,b]$. "..." stands for nonuniversal analytic terms in s which we will ignore below. Notice that the resolvent is protected by SUSY because ϕ^2 is essentially equivalent with the auxiliary variable B. The resolvent can be explicitly computed at each order of the $1/N$ expansion using the result of the Gaussian matrix model [12]. After taking the double scaling limit (6), we end up with the following genus expansion [13]:

$$\begin{aligned} N^{\frac{2}{3}(k+2)}\left\langle \frac{1}{N}\mathrm{tr}\,\phi^{2k+1}\right\rangle^{(1,0)} &= \frac{1}{2\pi^{3/2}}\Gamma\left(k+\frac{3}{2}\right)\sum_{h=0}^{\left[\frac{k+2}{3}\right]}\left(-\frac{1}{12}\right)^h\frac{s^{k-3h+2}}{h!(k-3h+2)!}\ln s \\ &+\frac{(-1)^{k+1}}{2\pi^{3/2}}\Gamma\left(k+\frac{3}{2}\right)\sum_{h=\left[\frac{k+2}{3}\right]+1}^{\infty}\frac{(3h-k-3)!}{h!}\frac{s^{k+2-3h}}{12^h}. \end{aligned} \tag{9}$$

The infinite series in the second line is divergent and not Borel summable. In fact, as a result of the Borel resummation, we have

$$(\text{2nd line}) \simeq \frac{1}{4\pi}\frac{1}{3^{k+5/2}}\frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})}\int_0^\infty dz\left(1-\frac{z^2}{z_0^2}\right)^{k+5/2}e^{-z}$$
$$+(\text{less singular}) \tag{10}$$

with $z_0 \equiv \frac{4}{3}s^{3/2}$. The integrand has a branch cut singularity at $z = z_0$ which sits on the integration contour. The result of the integral changes depends on avoiding the singularity upwards or downwards. The difference gives the amount of the nonperturbative ambiguity:

$$i(-1)^{k+1}\frac{\Gamma(k+\frac{3}{2})}{2\pi\cdot 3^{k+5/2}}\frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})}\int_{z_0}^\infty dz\left(\frac{z^2}{z_0^2}-1\right)^{k+5/2}e^{-z}, \tag{11}$$

that is of the order $e^{-\frac{4}{3}s^{3/2}}$, coinciding with the leading instanton contribution [9,10].

3 Resurgence in the matrix model

Recently, resurgence theory has been discussed in quantum mechanical systems and matrix models, which tells us that ambiguity from large-order behavior of perturbation series should cancel with ambiguity from instanton contributions so that the total expression is well-defined (for example, see [14, 15]). It is interesting to compute instanton effects of the one-point function and check whether the resurgence program works in our case.

The one-point function can be expressed in terms of orthogonal polynomials as

$$\left\langle\frac{1}{N}\mathrm{tr}\phi^{2k+1}\right\rangle^{(1,0)} = \frac{1}{N}\sum_{n=0}^{N-1}\int_{-\infty}^{\mu^2}dx\,e^{-\frac{N}{2}x^2}(\mu^2-x)^{k+1/2}P_n(x)^2, \tag{12}$$

where $P_n(x)$ is a monic polynomial of x of the degree n and normalized by

$$\int_{-\infty}^{\mu^2}dx\,e^{-\frac{N}{2}x^2}P_n(x)P_m(x) = h_n\delta_{n,m}. \tag{13}$$

Note that $P_n(x)$ is not identical with the orthogonal polynomial of the standard Gaussian matrix model $P_n^{(H)}(x)$ given by the Hermite polynomial, because the integration region of (13) is different from the case of the standard Gaussian matrix model, i.e., whole real line. However, the difference only affects the nonperturbative contribution, and the perturbative expansion (9) can be reproduced by replacing $P_n(x)$ with $P_n^{(H)}(x)$ and evaluating the integral with the integration region being $x \in (-\infty, 2]$ [1].

[1] Relevant eigenvalues to perturbative contributions in the sector of the $(1,0)$ filling fraction, $a < \lambda$ $(< b)$ in (4), correspond to $(-2 <)$ $x < 2$ in (12) due to the Nicolai mapping $x = \mu^2 - \lambda^2$. In the double scaling limit, a significant contribution comes from the neighborhood of $x = 2$.

Let us consider the integral (12) with the region $[2, \mu^2]$, which is the outside of perturbative eigenvalue distribution and is expected to give nonperturbative contribution. We magnify the region around $x = 2$ by introducing a new variable ξ as $x = 2 + N^{-2/3}\xi$. As far as considering the lowest order of nonperturbative effects, i.e., one-instanton contribution, we may replace $P_n(x)$ in the integrand with $P_n\,(H)(x)$. Then, we have

$$N^{\frac{2}{3}(k+2)} \left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle^{(1,0)} \Bigg|_{1-\mathrm{inst.}} \sim \int_u^s d\xi\, \frac{e^{-\frac{4}{3}\xi^{3/2}}}{8\pi\xi} (s-\xi)^{k+1/2} \left[1 + \mathscr{O}(\xi^{-3/2})\right] \tag{14}$$

with u being the $\mathscr{O}(1)$ cutoff. The integrand is valid for ξ large as the symbol "$\sim$" means, and we compute the integral around the saddle point $\xi = \xi_* = \mathscr{O}(s)$. Note that expansion by the instanton number is good for s large. The saddle point is given as

$$\xi_* = s + \frac{2k+1}{4} s^{-1/2} + \mathscr{O}(s^{-2}). \tag{15}$$

In order for the Gaussian integral around ξ_* to be reliable in the integral region $[u, s]$, we first take k as being large and negative, since $s \gg -k \gg 1$. The steepest descent paths are $\xi - \xi_* = \pm ix$ ($x \in \mathbf{R}$) corresponding to $s \to s\,e^{\pm i\varepsilon}$. After performing the integral, we eventually rotate back k to positive. The final result we arrive at is

$$N^{\frac{2}{3}(k+2)} \left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle^{(1,0)} \Bigg|_{1-\mathrm{inst.}} \simeq (\pm i)(-1)^{k+1} \frac{1}{2^{k+4}\sqrt{\pi}} \frac{e^{-\frac{4}{3}s^{3/2}}}{s^{\frac{1}{2}k+\frac{7}{4}}} e^{-k-1/2} \left(k + \frac{1}{2}\right)^{k+1} \left[1 + \mathscr{O}(s^{-3/2})\right] \tag{16}$$

for $s \to s\,e^{\pm i\varepsilon}$.

Note that avoiding the singularity onwards or downwards in (10) corresponds to $s \to s\,e^{-i\varepsilon}$ or $s\,e^{+i\varepsilon}$, respectively. Hence, we can see that the ambiguity from the one-instanton contribution (16) cancels with the leading ambiguity from the Borel resummed perturbation series (11), when k is large and the use of the Stirling formula for $\Gamma(k + \frac{3}{2})$ is allowed.

4 Summary and discussion

We have computed the one-point functions of the non-SUSY operators $\frac{1}{N}\mathrm{tr}\ \phi_{2k+1}$ ($k = 0, 1, 2, \ldots$) in the SUSY double-well matrix model to all orders of genus expansion. The series is divergent and not Borel summable. We have explicitly checked that the leading ambiguity arising from the Borel resummation procedure cancels with that from the one-instanton contribution as the resurgence theory suggests as far as k is large. It is extremely interesting to investigate the instanton contribution for the case of k not large, and to examine whether the resurgence theory works or not.

In addition, the resurgence theory suggests that subleading ambiguities from the Borel resumed series should be canceled with ambiguities from contributions of higher instanton numbers. Checking these cases is also worth pursuing.

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3D Higher spin gravity and the fractional quantum Hall effect

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Abstract This article is based on the talk "Fractional Spin Gravity" presented at the 31*st International Colloquium on Group Theoretical Methods in Physics*, Rio de Janeiro 19-25 June 2016. There we emphasised a direct implication of the works by N. Boulanger, P. Sundell and the author on fractional spin extensions of 3*D* higher spin gravity [1,2]. This is that higher spin gravity may govern interactions of anyons in the fractional quantum Hall effect. More generally we suggest here also that fractional spin states of matter in $2+1$ interact with their background geometry by means of higher spin gravity.

1 Introduction: The fractional quantum Hall effect

The fractional quantum Hall effect (**FQHE**) occurs in a planar electric conductor when a perpendicular strong magnetic field yields the condensation of an electron gas. The vacuum of this system consists of a quantum condensate of electrons trapped in Landau levels which form a new state of matter [3]. The phases of the condensate are characterized by the value of the Landau level filling factor, ν, i.e., the number of Landau levels that are filled to maximal capacity. The Hall conductance, σ_H, is induced by Lorentz forces on charges in a perpendicular direction to the current. Remarkably enough, σ_H exhibits rational quantization. Its experimental values at FQHE conditions remain approximately constant, with respect to variations of the external magnetic field (B) in certain intervals, at a rational number of times the fundamental unit of conductivity (e^2/h); these are the Hall plateaux. Thus at FQHE conditions, the Hall conductance reads $\sigma_H = \nu e^2/h$, where e is the electron charge, h the Planck constant, and $\nu \in \mathbb{Q}$ is a rational number. The integer quantum Hall effect refers to the cases of the ν integer. It is well understood in

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terms of Landau level quantization without many-body electron interactions. It was predicted in [4] and then experimentally observed in [5] for which Von Klitzing was awarded a Nobel Prize. It was not expected however that the filling factors would be quantized at values $\nu < 1$, until it was experimentally observed [3]. A Nobel Prize went to Tsui and Stormer for this discovery, and to Laughlin for his explanation [6]. $\nu = p/q$ represents the ratio of the number of electronic charges (p) to the number ($q = B/\phi_0$) of units of magnetic flux quanta $\phi_0 := hc/e$. When the magnetic field varies an integer number of times the field quanta ϕ_0, a phase transition must occur, meaning that some charges should either abandon a complete Landau level, jumping to the next excited state, or, otherwise decay to a lower Landau level. This is why the occurrence of values $\nu < 1$ are counterintuitive, as it means that there might be less than one unit of electric charge per magnetic field quanta, i.e., that exited states (and holes) have fractional charge.

To explain this odd experimental fact, the Coulomb many-body electron-electron interaction should be taken into account [6]. Indeed, the Coulomb interaction of electrons lowers the gaps between Landau levels, breaking them into fractions. As a consequence, the excitations of the many-electron problem appear to have fractional charge and angular momentum. From variational principles Laughling [6] was able to guess an ansatz for the wave function of that problem, which was actually accurate. Laughlin imposed Wigner lattice symmetries on the many-electron configuration, as electrons repel each other, maximizing their distance inside a given Landau level. The anzats for the ground states and excited states are dubbed "Laughlin's" wave functions. He showed that they carry fractional charge; later on it was shown that they also carry fractional spin and statistics [7–9]. The excitated states behave as free particles, and they are known as "anyons". Anyons have been observed experimentally [10–14]. Anyons are frequently presented as bound states of fractional charges and magnetic fluxes, i.e., magnetic instantons in the plane attached to charges (see e.g., [8, 15, 16]). The braided statistics of anyons can be interpreted as result of the braiding of magnetic fluxes attached to rotating fractional charges.

The existence of anyons was predicted theoretically before the experiments. In fact, based on the analysis of the topology of the configuration space of identical particles in $2+1$ dimensions and given that the fundamental group of the rotation group in the spatial plane is $\mathbb{Z}$ (and not $\mathbb{Z}_2$ as for higher dimensions), Leinaas and Myrheim [17] argued in 1977 the existence of generalized braid statistics, generalizing bosonic and fermionic. Their article was considered an academic curiosity, the existence of statistics different from the bosonic and fermionic, but in spite of their mathematical possibility, it was too radical to be believed. Anyons appear also from group representation theory. Indeed, the Poincaré group, which is the isometry group of flat space-time, has also real-valued ($s \in \mathbb{R}$) spin representations in $2+1$ dimensions (for a review see [18]). Wigner's view on the existence of particles (fields) as carrying representations of the space-time isometries, together with Fradkin's belief that "all that is consistent is possible, and all that is possible happens" (in the 89's Dirac Medal ceremony) are validated by the fractional quantum Hall effect: anyons do exist.

The Hall setup can be imagined as a "toy universe" which passes through topological phase-transitions when the magnetic field (or the magnetic length $\ell_B = \sqrt{\hbar/eB}$ which is the typical scale at FQHE conditions) overpasses some critical values, but it stays stable for small variations. In particular, in FQHE phases the anyons make their appearance playing the role of fundamental particles. It is natural therefore to try to describe them by using equations of motion, in the same way that Dirac's equation describes spin $1/2$ massive/massless fields. Some examples of anyon wave equations were given by Jackiw and Nair in [19], and by Cortes and Plyushchay in [20]. The interested reader may consult reference [18] for a review of these and other anyon wave equations.

In [2] a topological (massless) first order non-linear action principle for real-valued spin fields coupled to gauge gravity interactions was proposed, and which can be reduced to a Chern-Simons model. In the latter reference, one of the remarkable results is that the gravitational interaction of fractional spin fields are indeed of the higher spin (**HS**) gravity type [21, 22]. It is therefore very suggestive that higher-spin gravity might describe the gravitational interactions of fractional spin fields having FQHE conditions. We shall argue below on theoretical grounds that gravitating anyons must couple to infinite-dimensional extensions of the Lorentz connection, which can be described using the tools of HS gravity [23–26]. Quoting Wigner's and Fradkin's ideas, we would expect that a systematic study of the interactions of anyons based on HS gravity should be consistent with their phenomenology in the FQHE.

2 Fractional higher spin gravity

In an arbitrary $2+1D$ curved background, the description of fractional spin fields requires the introduction of a Lorentz connection taking values in a discrete serie representation [27, 28], which are infinite dimensional, and so, the fractional spin fields must also have infinite components. The formulation of a Chern-Simons action principle, which make use of traces of product of Lorentz generators, might be inconsistent when (infinite) matrix representations are used since traces may either diverge or converge to wrong (Lorentz symmetry broken) values. These problems can be fixed using Vasiliev's HS gravity technology that depends on regularized (super) trace definitions (see e.g., [25]). In doing this we were able in reference [2] to write down an action principle for fractional spin gravity. As we will see, there is a technical need suggesting we look at HS gravity in order to construct a consistent framework of gauge gravity interactions of fractional spin fields. From another point of view, since fractional spin fields must transform under an infinite dimensional (discrete) representations of the Lorentz algebra, they will also admit the action of the universal enveloping algebra of the Lorentz algebra. The latter algebra defines also the gauge algebra of HS Chern-Simons gravity [21, 22]; therefore it becomes natural to promote the whole HS algebra, not only the Lorentz generators, to gauge fields mediating the interaction between fractional spin fields. By doing this the spin-2 gravity interactions of fractional spin fields are extended by infinite many fields with arbitrary (half-)integer spins. Thus, as pointed out, HS gravity provides natural interactions for fractional spin fields. Let us write down the equations of motion of the fractional spin gravity theory [1, 2] in order to observe this more explicitly. These are:

$$\mathrm{d}W + W\,W + \Psi\overline{\Psi} = 0\,, \qquad \mathrm{d}U + U\,U + \overline{\Psi}\Psi = 0\,, \tag{1}$$

$$\mathrm{d}\Psi + W\,\Psi + \Psi\,U = 0\,, \qquad \mathrm{d}\overline{\Psi} + \overline{\Psi}\,W + U\,\overline{\Psi} = 0\,, \tag{2}$$

which are more succinctly expressed as the vanishing of the gauge curvature,

$$\mathrm{d}\mathbb{A} + \mathbb{A}^2 = 0\,, \qquad \mathbb{A} = \begin{bmatrix} W & \Psi \\ \overline{\Psi} & U \end{bmatrix}\,, \tag{3}$$

where $\mathbb{A}$ contains the fusion of HS gravity connection (W), a fractional spin one forms (ψ), and a $U \in U(\infty) \otimes U(\infty)$ non-abelian internal field. As shown in [1,2], the ψ fields are valued in a non-polynomial class of functions of universal enveloping algebra generators. Thus the density $\Psi\overline{\Psi}$, when they are Taylor expanded, is source of the field strength ($\mathrm{d}W + W\,W$) of HS gravity for all spins. The same is valid for the non-abelian curvature $\mathrm{d}U + U\,U$. Thus non-trivial anyon distributions are the sources of higher-spin gravity and non-abelian interactions. When $\Psi = 0 = U$ the system (1) is equivalent to the Chern-Simons HS gravity [21, 22]. For definiteness and simplicity let us choose W-valued in a representation of the HS algebra $Aq^e_+(2;\mu)$ [22], up-to tensor-product extensions. W has an expansion of the type

$$W = \sum_{a=0,1,2;n=0,1,\ldots,\infty} dx^I \frac{1}{n!} W_I^{a_1 a_2 \ldots a_n} J_{a_1 a_2 \ldots a_n}\,, \tag{4}$$

up-to-supersymmetric extensions by fermionic (spinor) components, and idempotent generators. dx^I are line elements and $J_{a_1 a_2 \ldots a_n}$ are symmetric tensors belonging to the universal enveloping algebra of the Lorentz algebra generated by elements J_a. Hence $J_{a_1 a_2 \ldots a_n} = J_{(a_1} \cdots J_{a_n)}$ consists of symmetric products with spin n. The parameter μ in $Aq^e_+(2;\mu)$ determines the lowest spin of the anyons [1, 2],

$$s = \tfrac{1+\mu}{4}\,. \tag{5}$$

It was noticed in [1, 2] that for critical values, $\mu = -(2\ell+1)$, $\ell = 0, 1, 2, \ldots$, anyons become bosons or fermions since $s = -\frac{\ell}{2}$, and the $Aq^e_+(2;\nu)$ algebra is truncated to a matrix algebra $Mat_{\ell+1}(\mathbb{C})$ [22], while the $u(\infty)$ algebra is truncated to $u(\ell)$. With suitable reality conditions, W and U will take values respectively in the algebras $sl(\ell+1,\mathbb{R})$ and $u(\ell)$ (up-to-tensor products), while the boson/fermion (before anyon) Ψ and $\bar{\Psi}$ transform under the one-sided action of these algebras (cf. (1)-(2)) of spin s. Thus the model (1) contains $SL(\ell)$-type of HS gravities.

Though HS gravity interactions in a setup such as the Hall effect are expected to be weak, because of their topological nature, it does not mean that they are trivial. Indeed the braided statistics of anyons is yield by Chern-Simons interactions [8].

Thus, even if HS gravity fields do not propagate, non-trivial topological configurations might be reflected in bulk-edge effects, as the Hall conductance for instance. However, one needs further research to find out which effects of HS gravity can be measured. Our goal here has been to point out that HS gravity may be predictive in FQHE and similar experiments.

3 Conclusions

We have argued that the effects of higher spin gravity [1, 2] may be observed in the FQHE and similar experiments. Consider for instance the statistical phases of anyons in the Quantum Hall effect, given in terms of the filling factor by $\exp(i\pi\nu)$, and the statistical phases $\exp(-i2\pi s)$ (see (5)) in the models [1,2]. Comparing both statistical phases, we obtain that the filling factor of the FQHE

$$\nu = 2s = \tfrac{1+\mu}{4}, \tag{6}$$

is related to the μ-parameter of the fractional spin algebra (1)-(2). On the one hand, for critical values, $\mu = -(2\ell+1)$, the model [1, 2] is reduced to $SL(\ell)\otimes SL(\ell)$ (matrix) Chern-Simons HS gravity and the spin of the fractional spin fields become bosonic/fermionic $|s| = \frac{\ell}{2}$. Thus $SL(\ell)\otimes SL(\ell)$ HS gravities may be related to gravity interactions of the fundamental excitations/holes in the *integer quantum Hall effect* ($\nu = \ell$). More generally, for non-critical values of the HS algebra parameter μ, and the related non-integer filling factor (6), we would expect that their interactions will be described by the fractional spin gravity model (1)-(2) or extensions of it. In this way, we expect that fractional spin gravity [1, 2] will contribute to a better comprehension of the FQHE and related phenomena (e.g., [30–32]).

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The "odd" Gelfand-Zetlin basis for representations of general linear Lie superalgebras

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Abstract We introduce a new Gelfand-Zetlin (GZ) basis for covariant representations of $\mathfrak{gl}(n|n)$. The patterns in this basis are fixed according to a chain of subalgebras, all of which are Lie superalgebras themselves. The basic generators consist of odd elements only. This GZ basis is interesting because the limit when n goes to infinity becomes clear. This could be used in the description of systems with an infinite number of parabosons and parafermions.

1 Introduction and motivation

The generalization of bosons and fermions to so-called parabosons and parafermions was initiated by Green in 1953 [1]. In this process, the (anti-)commutation relations for the boson and fermion operators were replaced by certain triple relations [1, 2]. This allows for more freedom when it comes to representations: where the standard bosons and fermions (with certain conditions such as a unique vacuum vector) allow only one irreducible unitary representation (namely the Fock space), parabosons and parafermions allow several such representations each characterized by a number p, the order of statistics. For the case $p = 1$, the relations for parabosons and parafermions reduce to those for standard bosons and fermions.

The above generalization is especially interesting because of the underlying mathematical structure. A system consisting of k parafermions $f_j^{\pm}$ $(j = 1,\ldots,k)$ is known to correspond to the defining relations of the Lie algebra $\mathfrak{so}(2k+1)$ [3, 4]. Similarly, a system consisting of n parabosons $b_j^{\pm}$ $(j = 1,\ldots,n)$ corresponds to the

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defining relations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ [5]. When it comes to a combined system of k parafermions and n parabosons (referred to as the parastatistics operators), there is some choice for the mixed triple relations [2]. The most natural choice implies that such a combined system corresponds to the defining relations of the Lie superalgebra $\mathfrak{osp}(2k+1|2n)$ [6].

The construction of the corresponding parastatistics Fock space of order p, which corresponds to an infinite-dimensional unitary representation of $\mathfrak{osp}(2k+1|2n)$, is far from trivial. This construction, including the explicit action of the parastatistics operators in an appropriate basis, was completed only recently [7].

For people working in quantum field theory, the main interest is in such systems with an infinite degree of freedom, i.e., where $k,n \to \infty$. In order to consider this, recall that the main ingredient in the construction of the parastatistics Fock space of order p is the branching $\mathfrak{osp}(2k+1|2n) \supset \mathfrak{gl}(k|n)$, and the use of Gel'fand-Zetlin (GZ) patterns of covariant representations of $\mathfrak{gl}(k|n)$ to label the states of this Fock space. The GZ-basis for covariant representations of $\mathfrak{gl}(k|n)$ was constructed in [8], and proceeds according to the subalgebra chain

$$\mathfrak{gl}(k|n) \supset \mathfrak{gl}(k|n-1) \supset \cdots \supset \mathfrak{gl}(k|1) \supset \mathfrak{gl}(k) \supset \mathfrak{gl}(k-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1).$$

The labels of the GZ-basis vectors for $\mathfrak{gl}(k|n)$ follow similar rules as those of the classical GZ-basis for the Lie algebra $\mathfrak{gl}(n)$ [9]. For example, a basis vector for a covariant representation of $\mathfrak{gl}(4|3)$ is given by

$$\left| \begin{array}{llll:lll}
\mu_{17} & \mu_{27} & \mu_{37} & \mu_{47} & \mu_{57} & \mu_{67} & \mu_{77} \\
\mu_{16} & \mu_{26} & \mu_{36} & \mu_{46} & \mu_{56} & \mu_{66} & \\
\mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} & & \\
\hdashline
\mu_{14} & \mu_{24} & \mu_{34} & \mu_{44} & & & \\
\mu_{13} & \mu_{23} & \mu_{33} & & & & \\
\mu_{12} & \mu_{22} & & & & & \\
\mu_{11} & & & & & &
\end{array} \right).$$

In such a μ-triangle, all $\mu_{ij} \in \mathbb{Z}_+$, satisfy conditions such as

- betweenness conditions ($1 \le i \le j \le k-1$ or $k+1 \le i \le j \le k+n-1$)
$$\mu_{i,j+1} \ge \mu_{ij} \ge \mu_{i+1,j+1}$$
- θ-conditions or 0-1-conditions ($1 \le i \le k$, $k+1 \le s \le k+n$)
$$\mu_{is} - \mu_{i,s-1} \equiv \theta_{i,s-1} \in \{0,1\}.$$

For a complete description of the conditions, see [8]. Note also that the top row of the above μ-triangle corresponds to the highest weight of the covariant representation. A GZ-basis also includes the explicit action of a set of generators on the basis vectors: for the standard GZ-basis given above; this set consists of the Chevalley generators of $\mathfrak{gl}(k|n)$ [8, Theorem 7] (corresponding to the distinguished set of simple roots).

Although this GZ-basis is perfectly well suited for the finite rank case of $\mathfrak{gl}(k|n)$, the problem is that it cannot be extended to a class of irreducible representations (irreps) of the infinite rank Lie superalgebra $\mathfrak{gl}(\infty|\infty)$. In order to solve this, one needs to use a different GZ-basis according to a different chain of subalgebras:

$\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \mathfrak{gl}(n-1|n-2) \supset \cdots \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1)$.
This can then be “reversed” in order to give a GZ-basis for $\mathfrak{gl}(\infty|\infty)$:
$\mathfrak{gl}(1) = \mathfrak{gl}(1|0) \subset \mathfrak{gl}(1|1) \subset \mathfrak{gl}(2|1) \subset \mathfrak{gl}(2|2) \subset \mathfrak{gl}(3|2) \subset \mathfrak{gl}(3|3) \subset \cdots \subset \mathfrak{gl}(\infty|\infty)$.
Thus, first we need to construct a new GZ-basis (the “odd” GZ-basis) for $\mathfrak{gl}(n|n)$ according to the above subalgebra chain. A striking property is that the generators for which the action takes its simplest form is now different: they are the (positive and negative) root vectors corresponding to a non-distinguished simple root system of $\mathfrak{gl}(n|n)$ consisting of odd roots only (justifying the name “odd” GZ-basis).

All results of the current proceedings contribution have been given in [10]. Here we shortly review the problem and list some additional properties and remarks.

2 Overview of the main results

The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(k|n)$ is defined by [11]:

$$\mathfrak{gl}(k|n) = \{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\},$$

with $A \in M_{k\times k}$, $B \in M_{k\times n}$, $C \in M_{n\times k}$ and $D \in M_{n\times n}$. The even subalgebra $\mathfrak{gl}(k|n)_{\bar{0}}$ has $B=0$ and $C=0$; the odd subspace $\mathfrak{gl}(k|n)_{\bar{1}}$ has $A=0$ and $D=0$. It is convenient to use the ordered set $\{-k,\ldots,-2,-1;1,2,\ldots n\}$ as the index set for the rows and columns of the above matrices. The Weyl basis is given by elements E_{ij} $(i,j = -k,\ldots,-2,-1;1,2,\ldots,n)$, with Lie superalgebra bracket

$$[\![E_{ab},E_{cd}]\!] = \delta_{bc}E_{ad} - (-1)^{\deg(E_{ab})\deg(E_{cd})}\delta_{ad}E_{cb}.$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is span(E_{jj}) with $j = -k,\ldots,-2,-1;\ 1,2,\ldots,n$. The dual space $\mathfrak{h}^*$ (or weight space) is spanned by the forms ε_i $(i = -k,\ldots,-2,-1;$ $1,2,\ldots,n)$. For $\Lambda \in \mathfrak{h}^*$,

$$\Lambda = \textstyle\sum_{i=-k\ (i\neq 0)}^{n} m_{ir}\varepsilon_i,$$

the components are written as $(r = k+n)$

$$[m]^r = [m_{-k,r},\ldots,m_{-2,r},m_{-1,r};m_{1r},m_{2r},\ldots,m_{nr}].$$

The roots of $\mathfrak{gl}(k|n)$ are the elements $\varepsilon_i - \varepsilon_j$ $(i \neq j)$; the positive roots consist of $\varepsilon_i - \varepsilon_j$ $(i<j)$, and the positive odd roots of $\varepsilon_i - \varepsilon_j$ with $i<0$ and $j>0$. The distinguished set of simple roots [11] is

$$\varepsilon_{-k}-\varepsilon_{-k+1},\ \varepsilon_{-k+1}-\varepsilon_{-k+2},\ldots,\varepsilon_{-1}-\varepsilon_1,\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{n-1}-\varepsilon_n.$$

In general, an integral dominant weight Λ corresponds to a finite-dimensional irrep $V(\Lambda)$ and vice versa. Here, we are only dealing with covariant representations: these are labelled by a partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ such that λ is inside the (k,n)-hook: $\lambda_{k+1} \leq n$ [12]. The corresponding highest weight: $\Lambda^\lambda \equiv [m]^r$ is determined by [13]

$$m_{ir} = \lambda_{k+i+1}, \quad -k \leq i \leq -1,$$
$$m_{ir} = \max\{0, \lambda'_i - k\}, \quad 1 \leq i \leq n,$$

where λ' is the partition conjugate to λ. Conversely, if $[m]^r$ is integral dominant and $m_{-1,r} \geq \#\{i : m_{ir} > 0,\ 1 \leq i \leq n\}$, then this corresponds to the covariant module with

$$\lambda_i = m_{i-k-1,r}, \quad 1 \le i \le k,$$
$$\lambda_{k+i} = \#\{j : m_{jr} \le i,\ 1 \le j \le n\}, \quad 1 \le i \le n.$$

The main property of covariant representations is that their character is known to be a supersymmetric Schur function [12]. With $x_i = e^{\varepsilon_i}$ $(i \le -1)$ and $y_i = e^{\varepsilon_i}$ $(1 \le i)$,

$$\mathrm{char}\, V([\Lambda^\lambda]) = s_\lambda(x_{\bar{k}}, \ldots, x_{\bar{2}}, x_{\bar{1}} | y_1, y_2, \ldots, y_n).$$

(For convenience, we sometimes write $\bar{j}$ instead of $-j$, as in the indices of the x's).

Using properties of these supersymmetric Schur functions [14], one can "peel off" a variable y_n or a variable $x_{\bar{k}}$. This allows the decomposition of a covariant representation of $\mathfrak{gl}(n|n)$ according to the subalgebra chain $\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1)$. Labelling the highest weights of the respective covariant representations as follows:

$$\begin{aligned}
&\mathfrak{gl}(n|n) \ \leftrightarrow\ [m]^r = [m_{-n,r}, \ldots, m_{-2,r}, m_{-1,r}; m_{1r}, m_{2r}, \ldots, m_{nr}]\\
&\mathfrak{gl}(n|n-1) \ \leftrightarrow\ [m]^{r-1} = [m_{-n,r-1}, \ldots, m_{-1,r-1}; m_{1,r-1}, \ldots, m_{n-1,r-1}]\\
&\mathfrak{gl}(n-1|n-1) \ \leftrightarrow\ [m]^{r-2} = [m_{-n+1,r-2}, \ldots, m_{-1,r-2}; m_{1,r-2}, \ldots, m_{n-1,r-2}],
\end{aligned}$$

the decompositions $\mathfrak{gl}(n|n) \to \mathfrak{gl}(n|n-1)$ and $\mathfrak{gl}(n|n-1) \to \mathfrak{gl}(n-1|n-1)$ are given by, respectively,

$$V([m]^r) = \bigoplus_k V_k([m]^{r-1}), \qquad V([m]^{r-1}) = \bigoplus_k V_k([m]^{r-2})$$

according to the rules

$$\begin{aligned}
&(1)\ m_{ir} - m_{i,r-1} = \theta_{i,r-1} \in \{0,1\} \quad (-n \le i \le -1)\\
&(2)\ m_{ir} - m_{i,r-1} \text{ and } m_{i,r-1} - m_{i+1,r} \in \mathbb{Z}_+ \quad (1 \le i \le n-1)\\
&(3)\ m_{i,r-2} - m_{i,r-1} = \theta_{i,r-2} \in \{0,1\} \quad (1 \le i \le n-1)\\
&(4)\ m_{i,r-1} - m_{i+1,r-2} \text{ and } m_{i+1,r-2} - m_{i+1,r-1} \in \mathbb{Z}_+ \quad (-n \le i \le -2).
\end{aligned}$$

This process can now be repeated, and thus one obtains a new GZ-basis for covariant representations $V([m]^r)$ of $\mathfrak{gl}(n|n)$. The m-patterns of these vectors take the form

$$|m)^r = \left|\begin{array}{cccccc|cccccc}
m_{\bar{n}r} & m_{\overline{n-1},r} & \cdots & m_{\bar{2}r} & m_{\bar{1}r} & & m_{1r} & m_{2r} & \cdots & m_{n-2,r} & m_{n-1,r} & m_{nr}\\
m_{\bar{n},r-1} & m_{\overline{n-1},r-1} & \cdots & m_{\bar{2},r-1} & m_{\bar{1},r-1} & & m_{1,r-1} & m_{2,r-1} & \cdots & m_{n-2,r-1} & m_{n-1,r-1} & \\
 & m_{\overline{n-1},r-2} & \cdots & m_{\bar{2},r-2} & m_{\bar{1},r-2} & & m_{1,r-2} & m_{2,r-2} & \cdots & m_{n-2,r-2} & m_{n-1,r-2} & \\
 & m_{\overline{n-1},r-3} & \cdots & m_{\bar{2},r-3} & m_{\bar{1},r-3} & & m_{1,r-3} & m_{2,r-3} & \cdots & m_{n-2,r-3} & & \\
 & & \ddots & \vdots & \vdots & & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & \\
 & & & m_{\bar{2}4} & m_{\bar{1}4} & & m_{14} & m_{24} & & & & \\
 & & & m_{\bar{2}3} & m_{\bar{1}3} & & m_{13} & & & & & \\
 & & & & m_{\bar{1}2} & & m_{12} & & & & & \\
 & & & & m_{\bar{1}1} & & & & & & &
\end{array}\right),$$

where the inbetweenness conditions and θ-conditions to be satisfied for the integers m_{ij} follow from the above rules (1)-(4), and we have followed the same notational convention as before: $\bar{j}$ stands for $-j$. The set of all vectors $|m)^r$ satisfying these conditions constitute a basis in $V([m]^r)$ [10].

Recall that in the standard GZ-basis the action of the Lie superalgebra is determined by the (diagonal) action of the Cartan subalgebra elements E_{ii} and the explicit action of the Chevalley generators, i.e., the root vectors

$$E_{-n,-n+1},\ldots,E_{-2,-1},E_{-1,1},E_{1,2},\ldots,E_{n-1,n},$$

corresponding to the simple roots (in the distinguished basis) and those corresponding to the negatives of the simple roots. In this distinguished choice for the simple roots, there is only one odd simple root, depicted by a cross in the Dynkin diagram:

$$\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\cdots\!\!-\!\!\circ\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\circ\!\!-\!\!\cdots\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\,.$$

In the “odd” GZ-basis, the situation is different. For the GZ-patterns $|m)^r$, one can again give the (diagonal) action of the Cartan subalgebra elements E_{ii}. The set of positive root vectors for which an explicit action can be computed is now different and given by

$$E_{-1,1},E_{-2,1},E_{-2,2},E_{-3,2},E_{-3,3},\ldots,E_{-n,n-1},E_{-n,n},$$

consisting of odd roots only. (Similarly, there is the action of the corresponding set of negative root vectors.) Thus the root vectors $E_{\pm\alpha}$ correspond to the following choice of simple roots (with only odd roots):

$$\varepsilon_{-1}-\varepsilon_1,\varepsilon_1-\varepsilon_{-2},\varepsilon_{-2}-\varepsilon_2,\varepsilon_2-\varepsilon_{-3},\ldots,\varepsilon_{-n+1}-\varepsilon_{n-1},\varepsilon_{n-1}-\varepsilon_{-n},\varepsilon_{-n}-\varepsilon_n$$

with a Dynkin diagram of the form

$$\otimes\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\cdots\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\otimes\,.$$

The main result of [10] is the determination of the explicit action of the above generators on the new GZ basis vectors. We shall not repeat these formulae here, but just note that the action of the E_{ii} is a simple diagonal action on $|m)^r$, whereas the action of the remaining generators takes the form

$$E_{-i,i}|m)^r=\sum_{k=-i}^{-1}A_{ik}|m)^r_{+(k,2i-1)}+\sum_{k=1}^{i-1}A_{ik}|m)^r_{+(k,2i-1)}$$

$$E_{-i-1,i}|m)^r=\sum_{k=-i}^{-1}B_{ik}|m)^r_{-(k,2i)}+\sum_{k=1}^{i}B_{ik}|m)^r_{-(k,2i)}$$

$$E_{i,-i}|m)^r=\sum_{k=-i}^{-1}C_{ik}|m)^r_{-(k,2i-1)}+\sum_{k=1}^{i-1}C_{ik}|m)^r_{-(k,2i-1)}$$

$$E_{i,-i-1}|m)^r=\sum_{k=-i}^{-1}D_{ik}|m)^r_{+(k,2i)}+\sum_{k=1}^{i}D_{ik}|m)^r_{+(k,2i)}.$$

Herein, $|m)^r_{\pm(ij)}$ is the pattern obtained from $|m)^r$ by replacing the entry m_{ij} by $m_{ij} \pm 1$; the actual expressions for the matrix elements $A_{ik}, B_{ik}, C_{ik}, D_{ik}$ can be found in [10, Theorem 4]. Observe that the action of a generator makes changes in only one row of the pattern of $|m)^r$.

The major advantage is that the "odd" GZ basis for $\mathfrak{gl}(n|n)$ can easily be extended to the infinite rank Lie superalgebra $\mathfrak{gl}(\infty|\infty)$, defined as the set of matrices with index set $\{\ldots,-3,-2,-1;1,2,3,\ldots\} = \mathbb{Z}^* \equiv \mathbb{Z} \setminus \{0\}$ with only a finite number of nonzero elements, and with the appropriate bracket. A highest weight is an infinite sequence $[m] \equiv [\ldots, m_{-k}, \ldots, m_{-2}, m_{-1}; m_1, m_2, \ldots, m_k, \ldots]$, and provided these numbers satisfy certain conditions, the corresponding highest weight representation $V([m])$ is a covariant representation. The basis vectors of $V([m])$ consist of "infinite GZ-patterns": similar to those of $\mathfrak{gl}(n|n)$, but with the above sequence as top row and consisting of an infinite set of rows in a triangular pattern. These GZ-patterns should – apart from inbetweenness conditions and θ-conditions – also satisfy a *stability condition*. The set of infinite stable GZ-patterns $|m)$ form a basis of the irreducible representation $V([m])$, and the transformation of the basis under the action of the $\mathfrak{gl}(\infty|\infty)$ generators is easily obtained from the finite rank case [10].

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Bannai–Ito algebras and the $osp(1,2)$ superalgebra

Hendrik De Bie, Vincent X. Genest, Wouter van de Vijver, and Luc Vinet

Abstract The Bannai–Ito algebra $B(n)$ of rank $(n-2)$ is defined as the algebra generated by the Casimir operators arising in the n-fold tensor product of the $osp(1,2)$ superalgebra. The structure relations are presented and representations in bases determined by maximal Abelian subalgebras are discussed. Comments on realizations as symmetry algebras of physical models are offered.

1 Introduction

The Bannai–Ito (BI) algebra $B(3)$ of rank one is the associative algebra with three generators $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$ obeying the relations

$$\{\Gamma_{12}, \Gamma_{23}\} = \Gamma_{13} + \omega_{13}, \quad \{\Gamma_{12}, \Gamma_{13}\} = \Gamma_{23} + \omega_{23}, \quad \{\Gamma_{13}, \Gamma_{23}\} = \Gamma_{12} + \omega_{12}, \qquad (1)$$

where $\{A, B\} = AB + BA$ and $\omega_{12}, \omega_{13}, \omega_{23}$ are central. It has been introduced in [10] to encode the bispectrality of the BI polynomials. Indeed, the Dunkl shift operators of which the BI polynomials are eigenfunctions, the spectrum variable of the

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recurrence relation, and the anticommutator of these two operators satisfy (1), up to affine transformations. This algebra is also isomorphic to a degeneration of the double affine Hecke algebra of type $(C_1^\vee, C_1)$ [8] and has appeared in a variety of contexts. For a review, see [3].

Interestingly, the algebra (1) arises in the context of the representation theory of the Lie superalgebra $osp(1,2)$, more specifically in the recoupling schemes for the tensor product of three irreducible representations. In this framework, the BI polynomials are seen to be essentially the Racah coefficients of $osp(1,2)$, that is the elements of the matrices relating the bases associated to the coupling of the first two factors of the three-fold product to the basis corresponding to the situation where the last two factors are initially regrouped. This connection of $B(3)$ to $osp(1,2)$ extends to n-fold tensor products and leads to the BI algebra $B(n)$ of arbitrary rank.

This will be presented in the following. We shall also give indications of how representations of $B(n)$ can be constructed in bases associated to maximal Abelian subalgebras. We shall conclude by mentioning some applications of $B(n)$.

2 $osp(1,2)$ and the Bannai–Ito algebra

The $osp(1,2)$ superalgebra can be presented as follows. It is generated by two odd elements $J_\pm$ and one even element J_0 that obey

$$[J_0, J_\pm] = \pm J_\pm, \quad \{J_+, J_-\} = 2J_0,$$

with $[a,b] = ab - ba$. The $\mathbb{Z}_2$ grading can be accounted for by introducing the grade involution P and including the relations

$$[J_0, P] = 0, \qquad \{J_\pm, P\} = 0, \qquad P^2 = 1.$$

The Casimir operator in the universal enveloping algebra $\mathscr{U}(osp(1,2))$

$$\Gamma = \frac{1}{2}\left([J_-, J_+] - 1\right)P = J_0 P - J_+ J_- P - P/2, \tag{2}$$

is found to commute with all generators. There is an algebra morphism $\Delta : osp(1,2) \to osp(1,2) \otimes osp(1,2)$ called comultiplication that acts as follows on the generators:

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes P + 1 \otimes J_\pm, \qquad \Delta(P) = P \otimes P,$$

and is coassociative, i.e., $(\Delta \otimes 1)\Delta = \Delta(1 \otimes \Delta)$. The coproduct can be iterated to form higher tensor powers of $osp(1,2)$. For a positive integer n, define $\Delta^{(n)} : osp(1,2) \to osp(1,2)^{\otimes n}$ as $\Delta^{(n)} = (1^{\otimes(n-2)} \otimes \Delta) \circ \Delta^{(d-1)}$, with $\Delta^{(1)} = \mathrm{Id}$.

Let $[n] = \{1, \ldots, n\}$ and let $A = \{a_1, \ldots, a_k\}$ be an ordered k-subset of $[n]$. For $1 \le k \le n$, one has a realization of $osp(1,2)$ in $osp(1,2)^{\otimes n}$ for any A. This realization, denoted by $osp^A(1,2)$, has generators

$$J_{\pm}^{A} = \sum_{a_i \in A} J_{\pm}^{(a_i)} \prod_{j=a_i+1}^{a_k} P^{(j)}, \qquad J_0^{S} = \sum_{a_i \in A} J_0^{(a_i)}, \qquad P^{A} = \prod_{a_i \in A} P^{(a_i)},$$

where $J_{\pm}^{(i)}, J_0^{(i)}, P^{(i)}$ denote the generators of the i^{th} factor of $osp(1,2)$ in $osp(1,2)^{\otimes n}$. We can now define the following elements in $\mathscr{U}(osp(1,2)^{\otimes n})$:

$$\Gamma_A = J_0^A P^A - J_+^A J_-^A P^A - P^A/2. \tag{3}$$

Clearly $\Gamma_{\{i\}}$, $i = 1,\ldots,n$, are the Casimir elements corresponding to each of the factors in $osp(1,2)^{\otimes n}$; these are constant multiples, say λ_i, if one considers products of irreducible representations. $\Gamma^{[n]}$ is the total Casimir operator of $osp(1,2)^{\otimes n}$. It will be convenient to take $\Gamma_{\emptyset} = -1/2$. We define the Bannai–Ito algebra $B(n)$ as the algebra generated by the elements Γ_A with $A \subset [n]$.

Let us now determine the structure relations. Consider first the case $n = 3$. There are seven generators in this instance: $\Gamma_{\{i\}} \equiv \Gamma_i$, $i = 1,2,3$, $\Gamma_{\{i,j\}} \equiv \Gamma_{ij}$ for $\{i,j\} = \{1,2\},\{1,3\},\{2,3\}$, and $\Gamma_{[3]} \equiv \Gamma_{123}$. Here, Γ_1, Γ_2, Γ_3 and Γ_{123} are central. A direct calculation shows that

$$\{\Gamma_{ij}, \Gamma_{jk}\} = \Gamma_{ik} + 2\Gamma_j \Gamma_{ijk} + 2\Gamma_i \Gamma_k, \qquad i \neq j \neq k,$$

which coincides with the $B(3)$ defining relations (1) when the central ω_{ik} are identified as $\omega_{ik} = 2\Gamma_j \Gamma_{ijk} + 2\Gamma_i \Gamma_k$. This is the result obtained in [6]. The structure relations for the higher rank extension are obtained from this result using the following argument. Take any triple of pairwise disjoint subsets of $[n]$ called K, L, and M. There is an obvious isomorphism

$$osp^K(1,2) \otimes osp^L(1,2) \otimes osp^M(1,2) \cong osp(1,2) \otimes osp(1,2) \otimes osp(1,2),$$

which leads to an embedding of $B(3)$ into $B(n)$. Indeed, in view of this isomorphism, the Casimir elements Γ_K, Γ_L, Γ_M, $\Gamma_{K\cup L}$, $\Gamma_{K\cup M}$, $\Gamma_{L\cup M}$, and $\Gamma_{K\cup L\cup M}$ will generate $B(3)$ and we shall have for instance

$$\{\Gamma_{K\cup L}, \Gamma_{L\cup M}\} = \Gamma_{K\cup M} + 2\Gamma_L \Gamma_{K\cup L\cup M} + 2\Gamma_K \Gamma_M. \tag{4}$$

We wish to know $\{\Gamma_A, \Gamma_B\}$ for any two subsets A and B of $[n]$. To that end, take $K = A \setminus B$, $L = A \cap B$, $M = B \setminus A$ and make the corresponding Casimir operators in $\mathscr{U}(osp(1,2)^{\otimes n})$ using (3). The relation (4) becomes

$$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A\cup B)\setminus(A\cap B)} + 2\Gamma_{A\cap B}\Gamma_{A\cup B} + 2\Gamma_{A\setminus(A\cap B)}\Gamma_{B\setminus(A\cap B)}. \tag{5}$$

This provides the desired structure relations for $B(n)$, namely the relations obeyed by the Casimir elements Γ_A labeled by subsets A of $[n]$ [1].

3 Maximal Abelian subalgebras, representation bases, and connection coefficients

We wish to indicate here how representations of $B(n)$ can be obtained from the knowledge of representations of $B(3)$. To that end, we shall first introduce bases for representation spaces that are associated to maximal Abelian subalgebras of $B(n)$.

3.1 Maximal Abelian subalgebras

We readily see from (5) that $[\Gamma_A, \Gamma_B] = 0$ if $A \subset B$, $B \subset A$ or $A \cap B = \emptyset$; recall that $\Gamma_\emptyset = -1/2$. It follows that $\mathscr{Y}_n = \langle \Gamma_{[2]}, \Gamma_{[3]}, \ldots, \Gamma_{[n-1]} \rangle$ forms an Abelian subalgebra (AS) of $B(n)$ that is readily seen to be maximal. Note that $\Gamma_\emptyset$, $\Gamma_{[1]}$ and $\Gamma_{[n]}$ are not included in $\mathscr{Y}_n$ as they are central in $B(n)$. Other such maximal AS can be obtained by applying a permutation and taking $\pi\mathscr{Y}_n = \langle \Gamma_{\pi[2]}, \Gamma_{\pi[3]}, \ldots, \Gamma_{\pi[n-1]} \rangle$.

3.2 Bases for representation spaces

Bases for representations spaces can now be obtained by taking their elements to be joint eigenvectors of the operators (hereafter denoted by the same symbols) representing the generators of the various maximal AS. Given one such basis, one would wish to provide the action of the generators in the complement of the AS in order to construct the representation of $B(n)$. We shall indicate how this can be accomplished from knowledge of the connection coefficients between bases associated to different maximal AS. With this understood, we shall complete the picture with a characterization of the connection coefficients.

Suppose that a basis has been picked and that we want to give the action of a generator Γ on the elements of this basis. It is easy to see that every generator of $B(n)$ belongs to a maximal AS. There is a thus another basis, call it prime, in which Γ is diagonal. Now if the connection coefficients between the elements of the original bases and those of the prime basis are known, it follows from linear algebra that the action of Γ in the original basis can be written down. This applies to any generator. Hence if all bases associated to maximal AS can be connected, the action of all generators in a single basis can be obtained with the help of the connection coefficients.

3.3 Connection coefficients

Irreducible representations of $B(3)$ have been constructed, and as a result the connection coefficients (CCs) between the bases associated to the AS generated respectively by Γ_{12}, Γ_{13} and Γ_{23} are known; see [1, 4, 6, 10]. We shall simply set the notation and recall the main features. Consider an irreducible representation of $B(3)$ and let $\langle \phi_k \rangle$ be a set of basis vectors on which Γ_{12} acts diagonally, say $\Gamma_{12}\phi_k = \mu_k \phi_k$. The central elements are multiples of the identity: $\Gamma_i \phi_k = \lambda_i \phi_k$, $\Gamma_{123}\phi_k = \lambda_{123}\phi_k$ for $i = 1,2,3$. It has been found that Γ_{13} and Γ_{23} act in a tridiagonal fashion in the basis $\langle \phi_k \rangle$ and one has for instance $\Gamma_{23}\phi_k = a_{k,k-1}\phi_{k-1} + a_{k,k}\phi_k + a_{k,k+1}\phi_{k+1}$. The coefficients $a_{k,k}$, $a_{k,k\pm 1}$ have been explicitly determined from the properties of $B(3)$. Now if $\langle \psi_k \rangle$ denotes the basis in which Γ_{23} is diagonal, the CCs defined by

$$\psi_k = \sum_s B_{ks}(\lambda_1, \lambda_2, \lambda_3, \lambda_{123})\phi_s,$$

are nothing but the Racah coefficients of $osp(1,2)$. Knowing the action of Γ_{23} in both bases $\langle \phi_k \rangle$ and $\langle \psi_k \rangle$, one finds that B_{ks} satisfies a three-term recurrence relation which shows that these CCs can be expressed in terms of BI polynomials.

Let us now discuss the rank two case $B(4)$ to illustrate how one bootstraps from rank one to higher ranks. First consider the CCs between two bases associated to two maximal AS that differ by only one generator. An example is $(\Gamma_{12}, \Gamma_{123})$, $(\Gamma_{12}, \Gamma_{124})$. Γ_{123} and Γ_{124} will preserve the eigenspaces of the common generator Γ_{12}. Now note that Γ_{123} and Γ_{124} are also generators of a rank one BI algebra. Indeed, let $K = \{1,2\}$, $L = \{3\}$, $M = \{4\}$, $\Gamma_{K\cup L} = \Gamma_{123}$, $\Gamma_{K\cup M} = \Gamma_{124}$, $\Gamma_{L\cup M} = \Gamma_{34}$ provide an embedding of $B(3)$ into $B(4)$. These generators all commute with Γ_{12} and the basis vectors with fixed eigenvalues of Γ_{12} will support representations of $B(3)$. The representation theory of the rank one BI algebra tells us that the CCs will again be BI polynomials. Thus since $\langle \phi_{j_1,j_2} \rangle$ and $\langle \psi_{j_1,j_2} \rangle$ are the bases diagonalizing the maximal AS of our example with $\Gamma_{12}\phi_{j_1,j_2} = \mu^{12}_{j_1}\phi_{j_1,j_2}$ and $\Gamma_{12}\psi_{j_1,j_2} = \mu^{12}_{j_1,j_2}\psi_{j_1,j_2}$, we have

$$\psi_{j_1,j_2} = \sum_k W_{j_2 k}(\mu^{12}_{j_1}, \lambda_3, \lambda_4, \lambda_{1234})\phi_{j_1,k}.$$

This then allows us to obtain the actions of Γ_{124} and Γ_{34} on the basis vectors ϕ_{j_1,j_2}. To find the action of other generators, one must consider the relations of the basis $\langle \phi_{j_1,j_2} \rangle$ with other subalgebra-type bases. It can be seen that there is always a path between any given basis to all the others that is made out of intermediary segments where the corresponding AS only differ by one element. The CCs between any two bases are then obtained by iterating for each of those segments the procedure just described when only one generator is different. The resulting CCs will hence be given by a product of BI polynomials. Furthermore, since all generators are part of maximal AS and are diagonal in the corresponding bases, knowing the CCs allows us to obtain the action of all generators in a chosen basis. These considerations extend from $B(4)$ to $B(n)$ and it follows that the representations of $B(n)$ in a fixed subalgebra-type basis can be fully characterized.

4 Conclusion

We conclude by mentioning that the Bannai–Ito algebra $B(n)$ has arisen in various systems. These models are obtained from particular realizations of $osp(1,2)$:

- The Dunkl Laplacian when Dunkl operators are used [7];
- The Dirac–Dunkl equation when Clifford algebras are introduced [1,4];
- The superintegrable model with reflections governed by the Hamiltonian

$$H = \sum_{1\leq i<j\leq n} J_{ij}^2 + \sum_{k=1}^{n} \frac{\mu_k(\mu_k - r_k)}{x_k^2}$$

  with $r_k f(x_1,\ldots,x_k,\ldots,x_n) = f(x_1,\ldots,-x_k,\ldots,x_n)$ and $J_{ij} = i(x_j\partial_{x_i} - x_i\partial_{x_j})$ when gauge-transformed parabosonic operators are called upon [2,5]. In this connection, see [9,11,12]

All these models have the Bannai–Ito algebra as a symmetry algebra. We may suspect that $B(n)$ and its representations will keep appearing in different guises.

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Exact state revival in a spin chain with next-to-nearest neighbour interactions

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Abstract An extension with next-to-nearest neighbour interactions of the simplest XX spin chain with perfect state transfer (PST) is presented. The conditions for PST and entanglement generation (balanced fractional revival) can be obtained exactly and are discussed.

1 Introduction

Certain spin chains have been known to model advantageously devices that effect perfectly the transfer of quantum states between locations [1, 2, 3]. Calling upon their dynamics to realize the transport has the merit of minimizing the need for external interventions and of protecting coherence. Analytic models have been found for which the ocurrence of this perfect state transfer (PST) is demonstrated from an exact analysis. As a rule, the couplings between spins must be non-uniform. The simplest such spin chain is of the XX type with parabolic couplings only between its nearest neighbours [4]. It is referred to as the Krawtchouk model in view of the family of orthogonal polynomials that emerge in its description. This model is proving quite useful not only as a paradigm example but also as a test bed for experimentalists. As it turns out, these spin models have a translation in terms of arrays of optical waveguides in view of the mathematical equivalence of the single excitation

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dynamics of spin chains with the coupled mode theory of optical lattices. Recent experimental implementations [5, 6] have in fact been carried in this framework using the Kratwchouk model. Restricting to nearest-neighbour (NN) interactions is obviously an approximation in this context and it becomes relevant to examine, exactly if possible, the situation beyond this restriction. It is with this perspective that we present in Sect. 2 an analytic extension of the NN Krawtchouk model that includes next-to-nearest neighbour (NNN) couplings. The conditions for PST along that chain can again be found exactly and will be given in Sect. 3.

There is another phenomenon of importance for quantum information that can be realized in spin chains, namely entanglement generation. This is obtained by end-to-end balanced fractional revival whereby a wavepacket initially at one end is reproduced simultaneously (with half the intensity) at both ends. The NN Krawtchouk model does not exhibit this effect but it proves possible when NNN interactions are included. This will be covered in Sect. 4. A summary and remarks on experiments that this analysis suggests will form the concluding section.

2 The model

We shall condier a spin chain with the following Hamiltonian of type XX on $\left(\mathbb{C}^2\right)^{\otimes(N+1)}$ where each of the $(N+1)$ spins interacts with its nearest and next-to-nearest neighbours on the left and on the right:

$$H = \frac{1}{2}\sum_{\ell=0}^{N-1}\left[J^{(1)}_{\ell+1}\left(\sigma^x_\ell\sigma^x_{\ell+1}+\sigma^y_\ell\sigma^y_{\ell+1}\right)+J^{(2)}_{\ell+2}\left(\sigma^x_\ell\sigma^x_{\ell+2}+\sigma^y_\ell\sigma^y_{\ell+2}\right)\right]+\frac{1}{2}\sum_{\ell=0}^{N}B_\ell\left(\sigma^z_\ell+1\right). \tag{1}$$

As usual, σ^x_ℓ, σ^y_ℓ, σ^z_ℓ stand for the Pauli matrices with the index ℓ indicating on which of the $\mathbb{C}^2$ factors they act. The nearest-neighbours couplings are taken to be the same as those of the Krawtchouk model: $J^{(1)}_n = \beta J_n$ with $J_n = \frac{1}{2}\sqrt{n(N-n+1)}$ and β a parameter. The next-to-nearest neighbour couplings are given by $J^{(2)}_n = \alpha J_{n-1}J_n$ with α another parameter and the local magnetic fields are $B_n = \alpha\left(J_n^2+J_{n+1}^2\right)$. Note that when $\alpha = 0$, the NN Krawtchouk model with no magnetic fields is recovered. Owing to rotational symmetry about the z-axis, H preserves the number of spins that are up over the chain, i.e., the number of eigenstates of σ^z_ℓ with eigenvalue $+1$. In the following, we shall only need to consider chain states that have a single spin up. A natural basis for that subspace is given by the vectors $|n\rangle = (0,0,\cdots,0,1,0,\cdots,0)^{\mathsf{T}}$, $n = 0,\ldots,N$, with only 1 in the n$^{\text{th}}$ position corresponding to the only spin up at the n$^{\text{th}}$ site. The action of H on those states is given by $H|n\rangle = J^{(2)}_{n+2}|n+2\rangle + J^{(1)}_{n+1}|n+1\rangle + B_n|n\rangle + J^{(1)}_n|n-1\rangle + J^{(2)}_n|n-2\rangle$. Now consider the operator J that acts as follows on the vectors $|n\rangle : J|n\rangle = J_{n+1}|n+1\rangle + J_n|n-1\rangle$. It follows that $J^2|n\rangle = J_{n+1}J_{n+2}|n+2\rangle + \left(J^2_{n+1}+J^2_n\right)|n\rangle + J_nJ_{n-1}|n-2\rangle$. We thus observe that

$$H|n\rangle = \left(\alpha J^2+\beta J\right)|n\rangle. \tag{2}$$

Let $|x_s\rangle$ be the eigenstates of J with eigenvalues x_s: $J|x_s\rangle = x_s|x_s\rangle$. In view of Eq. (2), these will be eigenstates of H with eigenvalues $E_s = \alpha x_s^2 + \beta x_s$. As it turns out, the eigenvalues and eigenvectors of J can be obtained from angular momentum theory. Let L_z and $L_\pm$ be the $\mathfrak{su}(2)$ generators represented in the standard fashion by

$$L_z|\ell,m\rangle = m|\ell,m\rangle \quad , \quad L\pm|\ell,m\rangle = \sqrt{(\ell \mp m)(\ell \pm m+1)}|\ell,m\pm 1\rangle \tag{3}$$

on the usual angular momentum states $|\ell,m\rangle$, $-\ell \leq m \leq \ell$.

Identify $|\ell,m\rangle = |\frac{N}{2}, n-\frac{N}{2}\rangle \equiv |n\rangle$, $n = 0, 1, \cdots, N$. Then $L_x|n\rangle = \frac{1}{2}(L_+ + L_-)|n\rangle = J_{n+1}|n+1\rangle + J_n|n-1\rangle$ and the action of L_x is seen to be that of J. Since $L_x = e^{-i\frac{\pi}{2}L_y} L_z e^{i\frac{\pi}{2}L_y}$, the spectrum of $L_x = J$ is the same as the spectrum of L_z, thus $x_s = s - \frac{N}{2}$. Now consider the expansion of the eigenstates $|x_s\rangle$ on the vectors of the occupational basis

$$|x_s\rangle = e^{-i\frac{\pi}{2}L_y}|s\rangle = \sum_{n=0}^{N} \langle n|\, e^{-i\frac{\pi}{2}L_y}|s\rangle\, |n\rangle = \sum_{n=0}^{N} \sqrt{\omega_s}\chi_n(x_s)|n\rangle\,. \tag{4}$$

At this point, either from the 3-term recurrence relation $J_{n+1}\chi_{n+1}(x) + J_n\chi_{n-1}(x) = x\chi_n(x)$ that follows from $J|x_s\rangle = x_s|x_s\rangle$ or the knowledge of the Wigner $\mathscr{D}$ functions, we find that the expansion coefficients are given by the normalized Krawtchouk polyomials which are defined as follows:

$$\chi_n(x) = (-1)^n \sqrt{\binom{N}{n}}\, {}_2F_1\left(\begin{matrix} -n, -s \\ -N \end{matrix} \middle| 2\right) \tag{5}$$

with the hypergeometric series given by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{z^k}{k!} \tag{6}$$

and $(a)_k = a(a+1)\cdots(a+k-1)$. These polyomials are orthogonal with respect to the binomial distribution: $\omega_s = \frac{N!}{s!(N-s)!}\left(\frac{1}{2}\right)^N$. Since $\langle n|\, e^{-i\frac{\pi}{2}L_y}|s\rangle = \sqrt{\omega_s}\chi_n(x_s)$ are elements of an orthogonal matrix, we also have the inverse expansion

$$|n\rangle = \sum_{s=0}^{N} \sqrt{\omega_s}\chi_n(x_s)|x_s\rangle\,. \tag{7}$$

For later purposes, observe that when $n = N$:

$$\chi_N(x_s) = (-1)^N \sum_{k=0}^{N} (-s)_k \frac{2^k}{k!} = (-1)^N \sum_{k=0}^{N} \frac{s!}{(s-k)!} \frac{(-2)^k}{k!} = (-1)^{N+s}. \tag{8}$$

3 Perfect state transfer

Let us examine the conditions for PST, that is the transfer with probability one, after time T, of a spin up from one end of the chain to the other. This will happen if

$$e^{-iTH}|0\rangle = e^{i\phi}|N\rangle\,, \tag{9}$$

where ϕ is some phase. In order to analyze this condition, use the expansion in Eq. (7) in terms of the eigenstates of H with eigenvalues $E_s = \alpha x_s^2 + \beta x_s$ to find that Eq. (9) amounts to $e^{-i\phi}e^{-iTE_s} = \chi_N(x_s) = (-1)^{N+s}$ in view of Eq. (8). This last equation can be rewritten as follows in terms of the exponents:

$$TE_s = -\phi + \pi\,(N+s+2L_s)\ , \quad s = 0,1,\ldots,N, \tag{10}$$

where L_s are arbitrary integers that may depend on s. Let us consider first the NN model, with $\alpha = 0$, and verify that PST occurs. In this case $E_s = \beta\left(s-\frac{N}{2}\right)$, and one has

$$T\beta\left(s-\tfrac{N}{2}\right) = -\phi + \pi\,(N+s+2L_s)\,. \tag{11}$$

This shows that the integer numbers L_s must depend linearly on s and take the form $L_s = \ell s + m$ with ℓ and m integers. With ϕ appropriately chosen to take care of the constant terms, Eq. (11) reveals that PST will be achieved at times T given by $T = \frac{\pi}{\beta}(2\ell+1)$, $\ell = 0,1,\ldots$ with the minimal time for PST in the NN model being $T = \frac{\pi}{\beta}$.

Can PST be maintained in the presence of NNN interactions? The answer is in the affirmative provided certains conditions are verified by the parameters α and β. When $\alpha \neq 0$, the eigenvalues E_s of H are given by $E_s = \alpha\left(s-\frac{N}{2}\right)^2 + \beta\left(s-\frac{N}{2}\right)$ and condition (10) reads

$$T\left[\alpha\left(s-\tfrac{N}{2}\right)^2 + \beta\left(s-\tfrac{N}{2}\right)\right] = -\phi + \pi N + \pi s + 2\pi L_s. \tag{12}$$

Although more involved, the analysis of this equation proceeds in a way analogous to that of Eq. (11). The reader will find the details in [7]. The upshot is the following. As distinct from the NN model, PST does not always occur. It will happen in the model with NNN interactions if $\frac{\alpha}{\beta}$ is rational, in other words if $\frac{\alpha}{\beta} = \frac{p}{q}$, with p and q co-prime integers. The minimal PST time is $T = \frac{\pi}{\beta}q$. Moreover if p is odd, q and N must be either both odd or both even.

4 Fractional revival

We discuss next the possibility of observing fractional revival (FR) at the two ends of the chain. This FR phenomenon will occur after time τ if

$$e^{-iH\tau}|0\rangle = \mu|0\rangle + \nu|N\rangle \tag{13}$$

with $|\mu|^2 + |\nu|^2 = 1$. Note that PST is a special case of FR with $\mu = 0$ ($|\nu| = 1$). Furthermore, it is readily recognized that when $|\mu| = |\nu| = \frac{1}{\sqrt{2}}$, the state obtained at time τ is maximally entangled as a balanced coherent sum of $|0\rangle = |\uparrow\downarrow\downarrow\cdots\downarrow\rangle$ and $|N\rangle = |\downarrow\downarrow\cdots\downarrow\uparrow\rangle$. Now upon using expansion (7), condition (13) is translated into $e^{-i\tau E_s} = e^{i\phi}\left(\mu' + \nu'(-1)^{N+s}\right)$, $\mu = e^{i\phi}\mu'$, $\nu = e^{i\phi}\nu'$ and μ' is chosen real without loss of generality. Taking the modulus on both sides, we see that $Re(\mu'\nu') = 0$. Given that μ' is real, ν' must thus be imaginary. We shall write $\mu' = \cos\theta$, $\nu' = i\sin\theta$ which makes the FR condition become

$$e^{-iE_s\tau} = e^{i\phi}\left(\cos\theta + i(-1)^{N+s}\sin\theta\right). \tag{14}$$

In this parametrization, up to integer multiples of π, $\theta = \frac{\pi}{2}$ corresponds to PST. The conditions for FR at two sites in NN spin chains of type XX have been thoroughly analyzed in [8]. Let us first examine here if FR can be found in the NN Krawtchouk model. For $E_s = \beta\left(s - \frac{N}{2}\right)$, (14) splits into the following two Eqs. according to the parity of s:

$$\beta\tau\left(2s + j - \tfrac{N}{2}\right) = -\phi - (-1)^{N+j}\theta + 2\pi L_s^{(j)}\,, \tag{15}$$

where $L_s^{(j)}$, $j = 0,1$, are two independent sequences of integers that must be of the form $L_s^{(j)} = \gamma_j s + \delta_j$, with γ_j and δ_j integers. It follows from (15) that $\gamma_0 = \gamma_1 = 1,2,\ldots$ and that $\tau = \pi\frac{\gamma_0}{\beta}$. Moreover, apart from a relation determining the phase ϕ in terms of the parameters, one finds that $\theta = (-1)^N\left[\frac{\gamma_0}{2} + (\delta_0 - \delta_1)\right]\pi$. Therefore, up to sign and integer multiples of π, θ can only take the values 0 and $\frac{\pi}{2}$. This means that only PST and perfect return are possible. We thus reach the conclusion that FR at two sites cannot happen in the NN Krawtchouk model. Let us now turn to the NNN extension. In this case, the FR condition (14) yields relations analogous to (15) with the l.h.s replaced by $\left[\alpha\left(2s + j - \frac{N}{2}\right)^2 + \beta\left(2s + j - \frac{N}{2}\right)\right]\tau$ and the sequences of integers $L_s^{(j)}$ having instead a quadratic form: $L_s^{(j)} = \xi_j s^2 + \eta_j s + \zeta_j$, $j = 0,1$, where for each j, independently, ξ_j and η_j can be simultaneously integer or half-integer while ζ_j is integer. Once again, we refer the reader to [7] for the detailed analysis of what these equations entail. The findings are as follows. FR can happen in NNN spin chains that have $\frac{\alpha}{\beta} = \frac{p}{q}$ with p and q co-prime integers and p odd; again, q and N must have the same parity. When these conditions are met $\theta \simeq \frac{\pi}{4}$, entanglement generation or balanced FR will be realized and its first occurrence will be observed at time $\tau = q\frac{\pi}{2\beta}$.

The picture with respect to FR is thus as follows. While it does not occur in the NN Krawtchouk spin chain, the presence of additional NNN interactions allows this phenomenon to take place under the circumstances that we have spelled out. However, the only form of FR at sites 0 and N that can be realized is of the balanced type which corresponds to the generation of maximally entangled state.

5 Conclusion

Summing up, we have provided an analytic model with NNN interactions that extends the simplest XX spin chain with PST, namely the NN Krawtchouk model. This extended model involves two parameters α and β. The NN model is recovered when $\alpha = 0$. When $\alpha \neq 0$, for PST to occur, we must have $\frac{\alpha}{\beta} = \frac{p}{q}$ where p and q are co-prime integers. If FR is to happen, it can only be of the balanced type and p must be odd and in that case N must be of the same parity as q.

It would now be quite interesting to obtain an experimental validation of these results. Discussions are underway regarding the design of an optical array in which entanglement generation would be observed as per the predictions and specifications of the analysis that we have described here.

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A triality between weak mutually unbiased bases, zeros of their analytic representations, and finite geometries

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Abstract Quantum systems with variables in $\mathbb{Z}(d)$ are considered, and three different structures are studied. We show that there is a correspondence (triality) between (1) weak mutually unbiased bases; (2) their analytic representation in the complex plane based on Theta functions, and their zeros; (3) finite geometries in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space

1 Introduction

There has been much work on mutually unbiased bases in systems $\Sigma(d)$ with variables in $\mathbb{Z}(d)$. Two bases $|B_1;n\rangle$ and $|B_2;m\rangle$ (where $n,m \in \mathbb{Z}(d)$) are mutually unbiased [1] if

$$|\langle B_1;n|B_2;m\rangle| = \frac{1}{\sqrt{d}}. \tag{1}$$

In the case that $d = p$ (where p is a prime number), $\mathbb{Z}(p)$ is a field, and it is known that there are $d+1$ such bases. Related is the result that there are $d+1$ mutually unbiased bases, in systems with variables in the Galois field $GF(p^e)$. It is a very difficult problem to find the number of mutually unbiased bases in the general case that $\mathbb{Z}(d)$ is a ring.

From this we might conjecture that the concept of mutually unbiased bases is tailored for fields, and another revised concept is needed for rings. Refs [2, 3] have introduced the concept of weak mutually unbiased bases, and have shown that it fits naturally to the concept of rings. For simplicity we discuss the case where $d = p_1 p_2$, where p_1, p_2 are prime numbers. Then the $\mathbb{Z}(d)$ factorizes as $\mathbb{Z}(p_1) \times \mathbb{Z}(p_2)$, and

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consequently the quantum system $\Sigma(d)$ factorizes in terms of two 'factor systems' $\Sigma(p_1)$ and $\Sigma(p_2)$ (there are bijective maps that relate the states in $\Sigma(d)$ to tensor products of states in the $\Sigma(p_1)$ and $\Sigma(p_2)$). The weak mutually unbiased bases (WMUB) are tensor products of mutually unbiased bases in the factor systems $\Sigma(p_1)$ and $\Sigma(p_2)$, which both have prime dimension. An alternative, equivalent definition of weak mutually unbiased bases is

$$|\langle B_1;n|B_2;m\rangle| = \frac{1}{\sqrt{f}};\ \text{or}\ 0;\quad f|d. \tag{2}$$

It has been shown in ref [3] that there is a duality (correspondence) between mutually unbiased bases, and the finite geometry in the $\mathbb{Z}(d)\times\mathbb{Z}(d)$ phase space. This geometry is very interesting, because it is an example of a non-near-linear geometry. Near-linear geometries [4, 5] are based on the axiom that two lines have at most one point in common. The geometries studied here violate this axiom and this is intimately related to the fact that $\mathbb{Z}(d)$ is a ring.

In ref [6], we studied another aspect of these systems. We considered an analytic representation based on Theta functions, and its zeros. We then showed that there is a triality (correspondence) between this analytic representation and its zeros, the finite geometry in the $\mathbb{Z}(d)\times\mathbb{Z}(d)$ phase space, and the weak mutually unbiased bases.

Here we review briefly this work, with a minimum of mathematical formulas. The emphasis is on the overall physical picture and how this can evolve in future work, rather than the mathematical proofs of the various statements (which have been given in ref [6]). The proofs use a factorization of the WMUB, their analytic representation, and the lines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$, which is based on the Chinese remainder theorem.

2 Lines and sublines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$

$\mathbb{Z}(d)\times\mathbb{Z}(d)$ is a non-near-linear geometry. Straight lines in it do not obey the axiom that two lines have at most one point in common. Two lines can have a 'subline' in common. This is related to the fact that the additive group $\mathbb{Z}(d)$ with $d = p_1p_2$, has non-trivial subgroups (the $\mathbb{Z}(p_1)$ and $\mathbb{Z}(p_2)$). In this case two lines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$ might have in common a 'subline' in $\mathbb{Z}(p_1)\times\mathbb{Z}(p_1)$ or in $\mathbb{Z}(p_2)\times\mathbb{Z}(p_2)$. The concept of 'subline' is intimately related to the fact that d has non-trivial divisors, and consequently $\mathbb{Z}(d)$ is a ring but it is not a field. There is a partial order that relates the finite geometry in $\mathbb{Z}(d)\times\mathbb{Z}(d)$ and its sub-geometries (based on non-trivial subgroups of $\mathbb{Z}(d)$), which has been studied in ref [10].

A line through the origin in $\mathbb{Z}(d)\times\mathbb{Z}(d)$ is defined as

$$\mathscr{L}(\nu,\mu) = \{(\nu\alpha,\mu\alpha)\mid \alpha\in\mathbb{Z}(d)\}.$$

The number of points in $\mathscr{L}(\nu,\mu)$ is $d/\mathfrak{G}(\nu,\mu,d)$ (where $\mathfrak{G}(\nu,\mu,d)$ is the greatest common divisor of these integers). If $\mathfrak{G}(\nu,\mu,d) = 1$, the corresponding line is maximal line with d points. If $\mathfrak{G}(\nu,\mu,d) > 1$, the corresponding line is a subline, and the number of points is a divisor of d. In the case of prime d, there are no sublines. There are $\psi(d) = (p_1+1)(p_2+1)$ (Dedekind psi) maximal lines through the origin.

As an example, we consider the case with $d = 15$. Then $\mathscr{L}(2,3)$ is a maximal line with 15 points, and the

$$\mathscr{L}(6,9) = \{(6,9),(12,3),(3,12),(9,6),(0,0)\} \subset \mathscr{L}(2,3)$$
$$\mathscr{L}(10,0) = \{(10,0),(5,0),(0,0)\} \subset \mathscr{L}(2,3) \tag{3}$$

are sublines. Fig.1 shows the lines $\mathscr{L}(1,13)$ (circles), and $\mathscr{L}(1,7)$ (crosses), in the case $d = 15$. The two lines have in common the three points $(0,0)$, $(5,5)$, $(10,10)$.

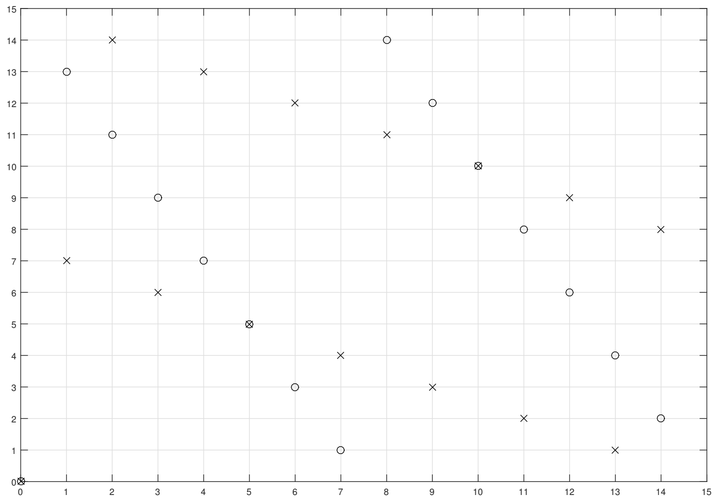

Fig. 1: The lines $\mathscr{L}(1,13)$ (circles), and $\mathscr{L}(1,7)$ (crosses), in the case $d = 15$. The two lines have in common the three points $(0,0)$, $(5,5)$, $(10,10)$.

3 Quantum systems with variables in $\mathbb{Z}(d)$ and analytic representations with Theta functions

We consider quantum systems with variables in $\mathbb{Z}(d)$ [7–9]. We also consider the position basis $|X;m\rangle$, where $m \in \mathbb{Z}(d)$ (odd d), and the momentum basis $|P;m\rangle$:

$$|P;n\rangle = F|X;n\rangle; \quad F = d^{-1/2}\sum_{m,n}\omega_d(mn)|X;m\rangle\langle X;n|$$
$$\omega_d(m) = \exp\left(i\frac{2\pi m}{d}\right). \tag{4}$$

We define an analytic representation, in which an arbitrary state

$$|g\rangle = \sum_m g_m|X;m\rangle; \qquad \sum_m |g_m|^2 = 1 \tag{5}$$

is represented by

$$G(z) = \pi^{-1/4}\sum_{m=0}^{d-1} g_m^* \,\Theta_3\left[\frac{\pi m}{d} - z\frac{\pi}{d};\frac{i}{d}\right]. \tag{6}$$

This is defined on a cell $\mathfrak{S} = [0,d]\times[0,d]$, because

$$G(z+d) = G(z)$$
$$G(z+id) = G(z)\exp\left(-\pi d - 2i\pi z\right). \tag{7}$$

The scalar product is given by

$$\langle g_2|g_1^*\rangle = \frac{\sqrt{2\pi}}{d^{5/2}}\int_{\mathfrak{S}} dz_R dz_I \exp\left(\frac{-2\pi}{d}z_I^2\right) G_1(z)G_2(z^*),$$

$G(z)$ has exactly d zeros ζ_r in each cell, with sum

$$\sum_{r=1}^{d}\zeta_r = \frac{d^2}{2}(1+i).$$

Therefore in each cell $d-1$ zeros are independent.

4 Duality between weak mutually unbiased bases and lines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$

The weak mutually unbiased bases are tensor products of mutually unbiased bases in the factor systems of prime dimension. An alternative, equivalent definition is given in Eq.(2).

There is a duality between weak mutually unbiased bases and lines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$, as follows. A pair of maximal lines through the origin, belongs to one of the following three categories:

- They have only the origin as the common point. There are $d\psi(d)/2$ such pairs of maximal lines.

- They have p_2 points in common. There are $p_1\psi(d)/2$ such pairs of maximal lines.
- They have p_1 points in common. There are $p_2\psi(d)/2$ such pairs of maximal lines.

This corresponds to a pair of weak mutually unbiased bases, with absolute value of the overlap equal to

- $d^{-1/2}$. There are $d\psi(d)/2$ such pairs of bases,
- $p_1^{-1/2}$. There are $p_1\psi(d)/2$ such pairs of bases,
- $p_2^{-1/2}$. There are $p_2\psi(d)/2$ such pairs of bases.

5 Triality between weak mutually unbiased bases, zeros of their analytic representations, and finite geometries

The duality of the previous section is extended to include the lines of zeros that represent the vectors in WMUB. The vectors in a WMUB are represented with analytic functions, whose zeros have the following properties:

- The d zeros of any vector in a WMUB are on a straight line.
- The d vectors in a WMUB have zeros on parallel straight lines. The slope labels the WMUB.
- The d vectors in a WMUB have a total of d^2 zeros. **The set of these zeros is the same for all WMUB**.

In Fig.2 we show the lines of zeros of the WMUB corresponding to the lines in Fig.1. There are d parallel lines of zeros for each WMUB, and only a representative is shown. Comparison of the figures 1, 2 shows the correspondence between the lines in $\mathbb{Z}(d)\times\mathbb{Z}(d)$ and the lines of zeros representing the vectors of WMUB. The precise mathematical formalism is given in ref [6].

6 Discussion

In the general context of finite quantum systems with variables in $\mathbb{Z}(d)$, we considered three different structures. The first is weak mutually unbiased bases. The second is their analytic representation in Eq.(6) and in particular the zeros of analytic functions that represent the vectors in WMUB. The third is the $\mathbb{Z}(d)\times\mathbb{Z}(d)$ as non-near-linear finite geometry. We have shown that there is a triality between these three structures.

This can be interpreted as an indication that the WMUB are more suitable concept than the MUB, for the case that d is a non-prime number and $\mathbb{Z}(d)$ is a ring (which is not a field).

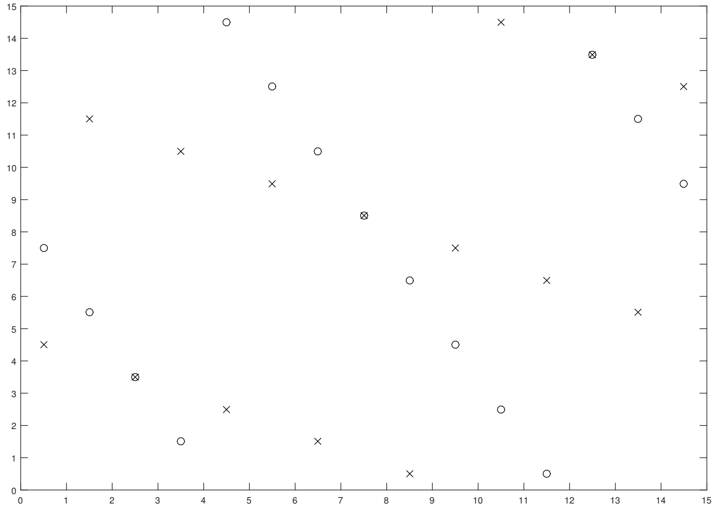

Fig. 2: The lines of zeros of the two WMUB corresponding to the lines in Fig.1. There are d parallel lines of zeros corresponding to the d vectors in a given WMUB, and only a representative one is shown. It is seen that this figure is a 'shifted version' of Fig.1.

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The U(2) Fourier group for rectangular pixellated images

Alejandro R. Urzúa and Kurt Bernardo Wolf

Abstract We study the model of optics in which images are two-dimensional pixellated arrays of values on a screen, and in particular the unitary transformations that have direct correspondence with those in the paraxial geometric and wave optical models. This correspondence is established for the U(2) Fourier group that consists of rotations, gyrations, and two-dimensional Fourier transformations.

1 Introduction

The finite model of optics regards images as matrices $\mathbf{f} = \|f(q_x, q_y)\|$ with generally complex elements $f(q_x, q_y)$, whose columns and rows are seen as integer coordinates $q_x|_{-j_x}^{j_x}$, $q_y|_{-j_y}^{j_y}$ that count the $N_x \times N_y = (2j_x + 1) \times (2j_y + 1)$ pixels in a generally rectangular screen. We shall consider N_x, N_y to be odd integers so as to have a pixel at the center of the screen and to simplify our computations by having j_x, j_y integers as SO(3) irreducible representation labels. Unitary transformations of these images will be represented by $N_xN_y \times N_xN_y$ matrices that will be elements of the *Fourier* group $\mathrm{U}(2)_\mathrm{F}$ to be defined below.

In the paraxial geometric optical model with $2D$ screens (which is isomorphic to $2D$ classical mechanics) where positions $\mathbf{q} = (q_x, q_y)$ and momenta $\mathbf{p} = (p_x, p_y)$ are continuous and form a $4D$ phase space, the Fourier group $\mathrm{U}(2)_\mathrm{F}$ [6] consists of *rotations* between the x–y components, *gyrations* in the (q_x, p_y) and (q_y, p_x) planes jointly, and *fractional Fourier* transformations that rotate the (q_x, p_x) and (q_y, p_y) planes independently. In the wave optical model (isomorphic to $2D$ quantum me-

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chanics), the images on a 2*D* screen are given by functions $f(q_x, q_y)$ where $\mathrm{U}(2)_\mathrm{F}$ is given by integral linear canonical transforms [3–5]. We are particularly interested in gyrations which, in the wave model, relate cartesian Hermite-Gauss with Laguerre-Gauss modes of angular momentum.

In previous works on the discrete model and its transformations [1, 9, 10] only *square* $N \times N$ screens were considered, where *discrete* angular momentum states have a relatively clear analogy with the wave model counterparts. Their definition on *rectangular* screens $N_x \neq N_y$ is not immediately obvious; the thrust in this contribution is to extend it to the generic case.

2 The model of discrete optics

Since this work has appeared in extended version in Ref. [8], we shall dispense with its lengthy recount of the one-dimensional quantum harmonic oscillator dynamical algebra osc_1, generated by position and momentum operators $\overline{Q}$, $\overline{P}$, $\overline{1}$ and the Hamiltonian $\overline{H} = \frac{1}{2}(\overline{P}^2 + \overline{Q}^2)$.[1] To build the discrete model of optics, with pixellated screens, we *pre-contract* this Lie algebra osc_1 to su(2), which is well known to group theorists.

To start with, we assign to the three generators of su(2) the following roles:

$$\text{position } Q \equiv J_1, \quad \text{momentum } P \equiv -J_2, \quad \text{mode } H - j1 \equiv J_3, \tag{1}$$

in the representation j, where the Casimir operator has eigenvalue $j(j+1)$, and whose dimension is $2j+1$ –the number of pixels in our one-dimensional screen, where we sense the image 'wavefunctions' $f(q)$, on the discrete eigenvalues $q|_{-j}^{j}$ of Q. The commutators $[J_i, J_j] = \mathrm{i}J_k$, with i, j, k cyclic, thus become

$$[H, Q] = -\mathrm{i}P, \quad [H, P] = \mathrm{i}Q, \quad [Q, P] = \mathrm{i}H. \tag{2}$$

The first two expressions are the geometric and dynamical Hamilton equations respectively, and the last determines the model to be that of discrete, finite optics. Furthermore, as is known, when the density and number of pixels $j \to \infty$, su(2) can be contracted to osc_1 [1], so the discrete model limits smoothly to the continuous model.

In two dimensions quite naturally we build $\mathrm{su}(2)_x \oplus \mathrm{su}(2)_y$ with two mutually commuting sets of generators, Q_i, P_i, H_i, $i \in \{x, y\}$. In the continuous model of osc_2 one further builds the real symplectic Lie algebra sp(4,R) with the ten symmetric quadratic products $\overline{Q}_i\overline{Q}_j$, $\overline{P}_i\overline{P}_j$ and $\frac{1}{2}\{\overline{Q}_i, \overline{P}_j\}_+$, which generates all linear transformations in the classical and quantum models of osc_2 that preserve this algebra. Now, this sp(4,R) algebra contains a maximal compact subalgebra u(2) that generates the *Fourier* group $\mathrm{U}(2)_\mathrm{F}$, whose center is the fractional isotropic Fourier transform (FT).

[1] We use the overbars to indicate that the symbols refer to the *continuous* model of paraxial wave optics, as they do for quantum mechanics.

We define and name the following four quadratic operators,

$$\text{symmetric FT } \overline{L}_0 := \tfrac{1}{4}(\overline{P}_x^2+\overline{P}_y^2+\overline{Q}_x^2+\overline{Q}_y^2-2\overline{1}) = \tfrac{1}{2}(\overline{H}_x+\overline{H}_y), \tag{3}$$

$$\text{antisymmetric FT } \overline{L}_1 := \tfrac{1}{4}(\overline{P}_x^2-\overline{P}_y^2+\overline{Q}_x^2-\overline{Q}_y^2) = \tfrac{1}{2}(\overline{H}_x-\overline{H}_y), \tag{4}$$

$$\text{gyration } \overline{L}_2 := \tfrac{1}{2}(\overline{P}_x\overline{P}_y+\overline{Q}_x\overline{Q}_y), \tag{5}$$

$$\text{rotation } \overline{L}_3 := \tfrac{1}{2}(\overline{Q}_x\overline{P}_y-\overline{Q}_y\overline{P}_x) =: \tfrac{1}{2}\overline{M}, \tag{6}$$

where $\overline{M} = 2\overline{L}_3$ is the 'physical' angular momentum operator, and whose commutation relations are those of u(2),

$$[\overline{L}_0, \overline{L}_k] = 0, \quad [\overline{L}_i, \overline{L}_j] = \mathrm{i}\overline{L}_k. \tag{7}$$

This structure will be *imported* [2] to the discrete model, below.

3 Discrete bases for discrete optics

Consider now a one-dimensional pixellated 'screen' of $N = 2j+1$ pixels, and the space of all 'images' $f(q)$, $q \in \{-j, \dots, j-1, j\}$. Evidently we have a 'pixel basis' with Kronecker deltas, $\delta_{q^\star}(q) = \delta(q-q^\star)$, which consists of 'black' zeros and 1 on the pixel $q^\star$, determined by the eigenvalues of $Q = J_1$ in (1), in the representation j of su(2). Similarly, we can build the *mode* eigenbasis $\Psi_n^{(j)}(q)$ of H using that of $J_3 = H - j1$. Since $Q = J_1$ and J_3 are related through a $\frac{1}{2}\pi$ rotation generated by J_2, their images on the screen will be related by a Wigner 'little-d' function, so

$$\Psi_n^{(j)}(q) := d^j_{n-j,q}(\tfrac{1}{2}\pi) = \frac{(-1)^n}{2^j}\sqrt{\binom{2j}{n}\binom{2j}{j+q}}K_n(j+q;\tfrac{1}{2},2j), \tag{8}$$

where $K_n(s;\frac{1}{2},2j) = {}_2F_1(-n,-s;-2j;2) = K_s(n;\frac{1}{2},2j)$ is the symmetric Kravchuk polynomial, $n|_0^{2j}$, $q|_{-j}^{j}$, $s|_0^{2j}$, and which forms an su(2) *multiplet* of $2j+1$ functions.

In generally two-dimensional, rectangular pixellated screens of $N_x \times N_y$ pixels, we build the *cartesian mode* real eigenbasis with an evident notation

$$\Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y) := \Psi_{n_x}^{(j_x)}(q_x)\Psi_{n_y}^{(j_y)}(q_y). \tag{9}$$

All images $f(q_x,q_y)$ on the screen can be expanded as

$$f(q_x,q_y) = \sum_{n_x,n_y} f_{n_x,n_y}\Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y), \quad f_{n_x,n_y} = \sum_{q_x,q_y} f(q_x,q_y)\Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y), \tag{10}$$

because the basis is orthonormal and complete. The two fractional Fourier-*Krachuk* transforms of the mode eigenbasis (9) are generated by $L_0 := \frac{1}{2}(H_x+H_y)$ and $L_1 := \frac{1}{2}(H_x-H_y)$, in direct correspondence with (3) and (4),

$$\begin{aligned} \mathscr{K}_{\rm iso}(\phi) : \Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y) &= e^{-{\rm i}\frac{1}{2}\phi(n_x+n_y)}\,\Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y), \\ \mathscr{K}_{\rm aniso}(\alpha) : \Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y) &= e^{-{\rm i}\frac{1}{2}\alpha(n_x-n_y)}\,\Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y). \end{aligned} \tag{11}$$

where $\mathscr{K}_{\rm iso}(\phi) := \exp(-2{\rm i}\phi L_0)$ and $\mathscr{K}_{\rm aniso}(\alpha) := \exp(-2{\rm i}\alpha L_1)$. These transformations are *domestic* to the finite model.

4 Importation of $U(2)_F$ transformations

Now we shall *import* the transformations generated by $\overline{L}_2$ and $\overline{L}_3$ in (5) and (6), as gyration and rotation, using the same 'little-d' Wigner coefficients and phases that describe rotations around the 2- and 3-axis for any U(2) group realization. Out of the n_x, n_y mode numbers we posit that $n := n_x + n_y$ yields their *total mode* (energy), while $\mu := \frac{1}{2}(n_x - n_y)$ is their *angular momentum* within multiplets of spin $\lambda(n)$.

Gyrations and rotations of the cartesian basis (9) are then defined respectively as

$$\mathscr{G}(\gamma) : \Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y) = e^{-{\rm i}\pi(n_x-n_y)/4} \sum_{n'_x+n'_y=n} e^{{\rm i}\pi(n'_x-n'_y)/4} d^{\lambda(n)}_{\mu,\mu'}(2\gamma)\,\Psi^{(j_x,j_y)}_{n'_x,n'_y}(q_x,q_y), \tag{12}$$

$$\mathscr{R}(\theta) : \Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y) = \sum_{n'_x+n'_y=n} d^{\lambda(n)}_{\mu,\mu'}(2\theta)\,\Psi^{(j_x,j_y)}_{n'_x,n'_y}(q_x,q_y), \tag{13}$$

where $\mathscr{G}(\gamma) = \exp(-2{\rm i}\gamma L_2)$ and $\mathscr{R}(\theta) = \exp(-2{\rm i}\theta L_3)$. The relations between $\mu, \lambda(n)$ and n_x, n_y, in a rectangle $N_x \geq N_y$, are given in Ref. [8] and take three forms depending on whether $0 \leq n \leq 2j_y$, $2j_y < n < 2j_x$, or $2j_x \leq 2(j_x+j_y)$, belonging to the lower, middle and top sections of the cartesian multiplet shown in Fig. 1 (left). Importantly, we note that the domestic Fourier-Kravchuk transforms and the imported gyrations and rotations compose as all U(2) transformations do, and that they are *unitary* in the vector space of images $f(q_x,q_y) \in {\rm R}^{N_xN_y}$ under the usual sesquilinear inner product.

Importantly, the gyration $\mathscr{G}(\frac{1}{4}\pi)$ transforms —in the *continuous* model— from Hermite-Gauss to Laguerre-Gauss beams. We use the same transformation to *define* the modes of 'rectangular angular momentum,' or Laguerre-*Kravchuk* modes:

$$\Lambda^{(j_x,j_y)}_{n,m}(q_x,q_y) := \mathscr{G}(\tfrac{1}{4}\pi) : \Psi^{(j_x,j_y)}_{n_x,n_y}(q_x,q_y) \tag{14}$$

$$= e^{-{\rm i}\pi(n_x-n_y)/4} \sum_{n'_x+n'_y=n} e^{{\rm i}\pi(n'_x-n'_y)/4} d^{\lambda(n)}_{\mu,\mu'}(\tfrac{1}{4}\pi)\,\Psi^{(j_x,j_y)}_{n'_x,n'_y}(q_x,q_y). \tag{15}$$

These are shown in Fig. 1 (right); they are complex: $\Lambda^{(j_x,j_y)}_{n,m}(q_x,q_y) = \Lambda^{(j_x,j_y)}_{n,-m}(q_x,q_y)^*$, as the Laguerre-Gauss modes are, and are orthogonal and complete in the ordinary sesquilinear ${\rm R}^{N_xN_y}$ inner product. Since $\mathscr{R}(\theta) = \mathscr{K}_{\rm aniso}(-\frac{1}{4}\pi)\mathscr{G}(\theta)\mathscr{K}_{\rm aniso}(\frac{1}{4}\pi)$, under rotations these modes are only multiplied by a phase $e^{-{\rm i}m\theta}$.

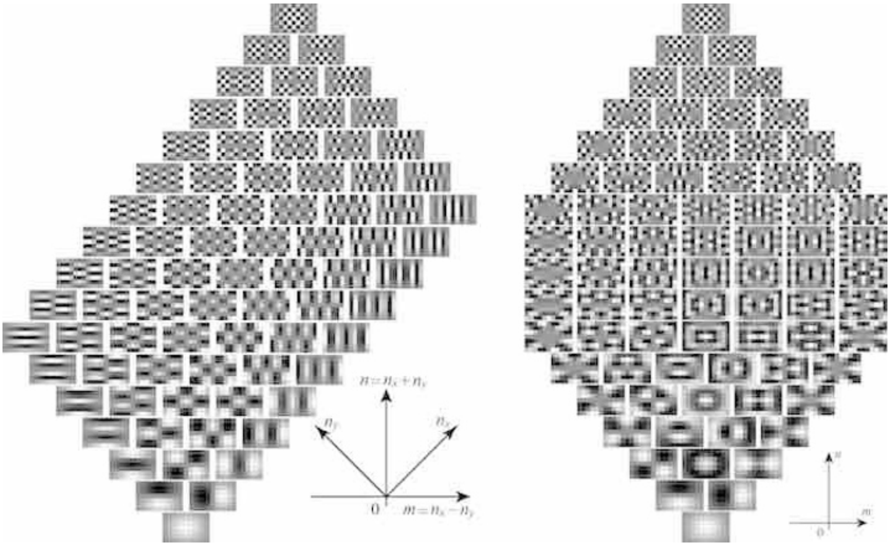

Fig. 1: *Left*: The 77 cartesian modes in (9) for $N_x = 11$ and $N_y = 7$, $(j_x, j_y) = (5,3)$, referred to the diagonal axes $n_x|_0^{10}$, $n_y|_0^6$, and to *total mode* $n = n_x + n_y$ and 'angular momentum' $m = n_x - n_y$. *Right*: The 77 'Laguerre-Kravchuk modes' in (15), with $n|_0^{18}$ and 'rectangular angular momentum' $m|_{-\lambda(n)}^{\lambda(n)}$, $\lambda(n) \leq 3$. Note the three regions (lower and upper triangles, and the intermediate rhomboid of states) correspond in both multiplets. Each 'screen' has 11×7 pixels.

5 Rotation of rectangular pixellated images

Heretofore we had examined rotations only for square pixellated screens $N_x = N_y$, which could also be unitarily transformed to circular, polar-pixellated screens. In the present work we generalize the cartesian pixellation to rectangular shape with a plausible definition of angular momentum states. Rotation in rectangular screens of generic images $f(q_x, q_y)$ can be effected either expanding it in the $\Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y)$ or in the $\Lambda_{n,m}^{(j_x,j_y)}(q_x,q_y)$ bases. The latter contains complex elements, so we have preferred the former because it is real as in (10), and rotate through,

$$\mathscr{R}(\theta) : f(q_x,q_y) = \sum_{n_x,n_y} f_{n_x,n_y}^{(\theta)} \Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y), \tag{16}$$

$$\begin{aligned} f_{n_x,n_y}^{(\theta)} &= \sum_{q_x,q_y} f(q_x,q_y) \mathscr{R}(\theta) : \Psi_{n_x,n_y}^{(j_x,j_y)}(q_x,q_y) \\ &= \sum_{q_x,q_y} \sum_{n'_x,n'_y} f(q_x,q_y)\, d_{\mu,\mu'}^{\lambda(n)}(2\theta)\, \Psi_{n'_x,n'_y}^{(j_x,j_y)}(q_x,q_y). \end{aligned} \tag{17}$$

Fig. 2: Successive rotations of a 0−to−1 image "$\mathscr{B}$" on a 61×37 pixellated screen $(j_x, j_y) = (30, 18)$, by angles 0, $\frac{1}{6}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$, and π. We chose to cut the bottom and upper graylevels of the intermediate rotations to give better visual information. The cut levels are: all values below −0.30 are black; all values above 1.30 are white.

In Fig. 2 we show a sample of succesive rotations of an image, on which we can give a few observations.

Rotations by $\frac{1}{2}\pi$ will exchange the long and short dimensions of the rectangle, so we expect —and see— that Gibbs-like oscillations take place which in a few pixels overshoot the maximum initial values of the original image. The same occurs for any other rotation, except for multiples of π, where the initial image is reproduced. The Gibbs phenomenon is unavoidable when unitarity is preserved, as we know from discrete and finite Fourier analysis. Rotation algorithms provided by interpolation algorithms loose information and cannot be inverted or concatenated as unitary transformations do.

Questions further afield are the possibility of generalizing image preservation through a change of the pixel coordinates. A map to polar pixellation of *annular* screens was postulated, as they can be built with the same number of pixels; this map was computed and tested [7], but the essential quality of image recognizability fades rapidly with $|N_x - N_y|$. On the other hand, screens with some kind of *elliptic* equal-area pixellation have not been found. Yet we can underline that our present work has achieved to find the correct analogue of the action of the geometric and wave 4-parameter Fourier group $\mathrm{U(2)_F}$ on generic images on rectangular pixellated screens.

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Part IV
Short Articles

Finding a dictionary between tensor models and crystallization theory

Grace Itunuoluwa Akinwande

Abstract Crystallization theory has been useful in representing piecewise-linear (PL) manifolds in any dimension as D-regular edge-colored bipartite graphs. PL manifolds are of increasing interest to theoretical physicists as a tool to explore discretized versions of geometry in quantum gravity. We provide here a dictionary and compare quantities which have appeared in two contexts: in the so-called crystallization theory of simplicial manifolds (mathematics) and in colored tensor models which are simplicial models in view of providing a quantum version of general relativity (physics).

1 Crystallization Theory and the Regular Genus

The representation of PL manifolds has been difficult beyond dimension 4. But crystallization theory [2] has been very useful in representing compact PL manifolds of any dimension making use of a class of edge-colored graphs.

A $(D+1)$-colored graph, G, is a graph that has an edge-coloration of $D+1$ colors, with the color set $\Delta_D = \{0,1,\dots,D\}$, such that each pair of adjacent edges does not have the same color.

A graph encoded manifold also called gem is a $(D+1)$-colored graph, G, that represents a D-dimensional manifold M^D, that is, the pseudocomplex associated to G is homeomorphic to M^D. A crystallization is a contracted gem.

A simplicial manifold is obtained by gluing the facets of D-simplices according to certain rules (see [2]). On the other hand, Pezzana's existence theorem states that every closed connected manifold admits a crystallization [5,6].

A regular embedding of a colored graph on a surface is such that each region in the embedding is bounded either by a cycle or by an open path of the $(D+1)$-

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colored graph, G, with edges alternatively colored by ε_j and ε_{j+1}, $j \in \mathbb{Z}_{D+1}$, where $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_D)$. These embeddings are $\frac{1}{2}D!$ up-to-uniqueness. The regular genus of G is given as $\rho(G) = \min_\varepsilon\{\rho_\varepsilon(G)\}$, where $\rho_\varepsilon(G)$ is the genus of the embedding ε. Furthermore, the regular genus of a manifold M is defined as

$$\mathscr{G}(M) = \min\{\rho(G) \mid G \text{ represents } M\}.$$

2 From tensor models to simplicial manifolds

Tensor models generalize matrix models [3]. To make contact with PL manifolds and crystallization theory, we will focus on the colored version of tensor models [4]. The partition function of a colored tensor model of rank D is given by

$$\mathscr{Z}_N(\lambda, \bar{\lambda}) = \int d\bar{\psi} d\psi e^{-(\sum_{i=0}^{D} \sum_{\vec{n}} \bar{\psi}^i_{\vec{n}_i} \psi^i_{\vec{n}_i} + \lambda \sum_{\vec{n}} \prod_{i=0}^{D} \psi^i_{\vec{n}_i} + \bar{\lambda} \sum_{\vec{n}} \prod_{i=0}^{D} \bar{\psi}^i_{\vec{n}_i})},$$

where $\psi^i_{\vec{n}_i}$ is a rank D complex tensor. At the perturbative level, they naturally give rise to Feynman graphs which are bipartite $(D+1)$-colored graphs. The d-bubbles of the graph are maximally connected subgraphs consisting of edges with exactly d colors, $d \in \Delta_D$. The sets that are indexed by these bubbles give rise to a simplicial manifold (see Figure 1).

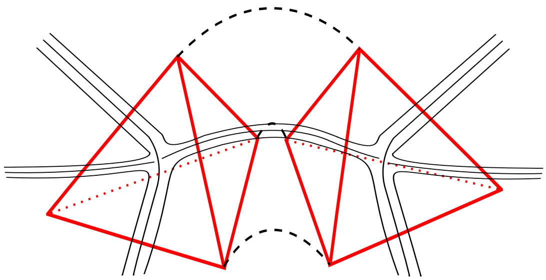

Fig. 1 A Feynman graph in a rank 3 tensor model which is a gluing of two tetrahedra.

Another important ingredient in colored tensor models is the notion of jackets. A jacket is a ribbon graph determined by a $(D+1)$-permutation cycle, namely τ, such that each of its faces is given by $(\tau^k(0), \tau^{k+1}(0))$, $k \in \Delta_D$. It should be noted that the notion of jacket coincides with the notion of regular embedding in crystallization theory.

The Gurau degree, $\omega(G)$, of a $(D+1)$-colored graph, G, is defined as

$$\omega(G) = \sum_{\mathscr{J}} g_{\mathscr{J}},$$

where the sum is performed over all jackets and $g_{\mathscr{J}}$ is the genus of a jacket $\mathscr{J}$. Thus, $\rho(G) \leq \omega(G)$, and the two quantities coincide in dimension 2.

3 Conclusion

There are many quantities and notions in crystallization theory which match with notions in colored tensor models. In [1], we discuss and show the following table:

Table 1: Comparison of notions and quantities in both theories (SM: Simplicial manifold).

Crystallization theory	Colored tensor models
D-residue	D-bubble
SM: attaching simplices to vertices	SM: indexing D-bubbles by all bubbles
h-dipole cancellation	h-dipole contraction
h-dipole insertion	h-dipole creation
Regular embedding, F_ε	Jacket, $\mathscr{J}$
Heegaard surface, Σ	Jacket, $\mathscr{J}$

The Gurau degree is related to the number of faces, and hence to a discrete version of the integral of the curvature. One could possibly make this statement more precise. Finally, if we define $\omega(M) = \min\{\omega(G) \mid G \text{ represents } M\}$ for a manifold M, one could ask if this quantity $\omega(M)$ coincides, or at least can be compared with the notion of scalar curvature.

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Quantum cosmology of scalar-tensor theories and self-adjointness

Carla R. Almeida

Abstract Self-adjointness of the Hamiltonian operator obtained from quantizing the classical Brans-Dicke action with a non-minimally coupled scalar field in mini-superspace is addressed. The matter field plays the role of time, introduced by Schultz's formalism by means of a perfect fluid. We determine the conditions for self-adjointness of the Hamiltonian operator.

1 Introduction

We canonically quantise the classical Brans-Dicke theory with a non-minimally coupled scalar-field having a matter component. Considering a FLRW metric and a change of coordinates, the Wheeler-DeWitt equation gives us the following Hamiltonian operator:

$$\hat{H} = -\frac{b^{3\alpha}}{\varphi^{\frac{3\alpha}{2}}}\left\{\frac{\varphi^{\frac{1}{2}}}{b}\left[\partial_b^2 + \frac{p}{b}\partial_b\right] - \varpi\frac{\varphi^{\frac{3}{2}}}{b^3}\left[\varphi\partial_\varphi^2 + q\partial_\varphi\right]\right\}, \tag{1}$$

where b is related with the scalar factor, α is the matter constant of the EoS, ϖ is a constant, and p,q are the ordering factors. This operator is symmetric only if we consider the measure

$$\langle\psi,\phi\rangle = \int \bar{\psi}\phi b^{p-3\alpha+1}\varphi^{q+\frac{(3\alpha+5)}{2}}dbd\varphi. \tag{2}$$

Notice that it depends on the ordering factors.

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2 Self-adjointness

To verify the self-adjointness of an operator or the existence of self-adjoint operators, we have to determine the number of linearly independent solutions of the eigenvalue equation $\hat{H}^*\Psi = \pm i\Psi$. For our case, the von Neumann theorem assures us that the Hamiltonian operator is (essentially) self-adjoint or has self-adjoint extensions for every kind of matter chosen. However, the eigenvalue equation is only separable for $\alpha = 1/3$ (radiative matter) and $\alpha = 1$ (stiff matter). For $\alpha = 1/3$, the eigenvalue equation becomes a system of differential equations:

$$\left(-\partial_b^2 - \frac{p}{b}\partial_b + \frac{k^2}{b^2}\right)X(b) = 0; \quad \varpi\left(\varphi^2\partial_\varphi^2 + q\varphi\partial_\varphi\right)Y(\varphi) = k^2 Y(\varphi), \tag{3}$$

where k is the constant of separation. We proceed with the von Neumann method showing that there are square-integrable solutions, that is, the Hamiltonian has self-adjoint extensions, only if $\varpi < 0$, $q = 1$ and $p < -1$ or $p > 3$. Therefore, we have:

(i) For $\varpi < 0$, $q = 1$ and $p < -1$ or $p > 3$, the operator has self-adjoint extensions.
(ii)For every other case, the operator is already self-adjoint.

For $\varpi < 0$ the Hamiltonian operator $\hat{H}$ is positive and bounded from below; then we can chose a unique extension, called Friedrich's extension, which always preserves the ground state. A similar analysis and similar results can be obtained for the case of stiff matter $\alpha = 1$.

2.0.1 Conclusion

In this work [1], we quantized the Hamiltonian of the classical Brans-Dicke theory and determined the conditions for its self-adjointness. As expected, we were able to reproduce the results obtained in [2] for a conformal matter field ($\alpha = 1/3$), showing the equivalence of the Einstein and Jordan frames.

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Luminosity of ultrahigh energy cosmic rays as a probe of black strings

Rita C. Anjos, Carlos H. Coimbra-Araújo

Abstract Ultrahigh energy cosmic rays (UHECRs) can originate from extragalactic sources as Active Galactic Nuclei. We propose a mechanism to calculate bounds on the upper limits of the AGN luminosity fraction that can be converted into UHECRs. This result comes from the mechanism powered by central black holes to produce the AGN luminosity and observation of UHECRs and gamma-rays from experiments to reconstruct proton and iron luminosities of a given AGN source.

1 Introduction

This contribution provides a better understanding of the effects of luminosities from shower, AGN mechanisms and braneworld AGN corrections, and provides a background model for considering UHECR luminosity. We investigate the bolometric luminosity that comes from the accretion mechanism. We argue that the jet contribution comes as a quantity that is proportional to the bolometric luminosity, i.e., a fraction of such luminosity, assuming that any geometrically thick or hot inner region of an accretion disk produces powerful jets [1–3]. The method connects an upper limit on the integral GeV-TeV gamma-ray flux and upper limit on the UHECR luminosity through the cascading process that takes place during propagation of the cosmic rays in background radiation fields. The simulated spectra were normalized with an upper limit on the energy spectrum measured by the Pierre Auger Observatory [4, 5]

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2 Luminosity of black strings and cosmic rays

The luminosity L_{acc} of AGNs due to black hoke (BH) accretion is given by equation

$$L_{acc}^{Theory} = \frac{GM\dot{M}}{6R_{-}}, \tag{1}$$

where $\dot{M}$ denotes the accretion rate and depends on some specific model of accretion, $R_{-} = R_{brane}$, R_{+kerr} and $R_{+kerbrane}$ are the horizons of BHs which are, respectively, static with extra dimension corrections, rotating, without and with, extra dimension corrections. The luminosity of AGNs are produced essentially by the accretion mechanism of supermassive black holes [3]. We can consider that the jet contribution comes as a proportional quantity to the bolometric luminosity (L_{acc}^{Theory}), namely, a fraction (η_{CR}), assuming that any geometrically thick or hot inner region of an accretion flow can advance magnetic field fluctuations to produce powerful jets. We can write the fraction of the total luminosity going into UHECR as

$$L_{CR}^{Theory} = \eta_{CR} L_{acc}^{Theory}. \tag{2}$$

The luminosity decreases for some cases when extra dimension effects are taken into consideration [1].

3 Conclusion

We calculated the luminosity due to accretion of nine sources. The luminosities calculated based on theory were compared to an upper limit on the UHECR luminosity. The comparison resulted in being able to determine upper limits on the energy conversion from accretion to UHECR. The theoretical estimations of this conversion efficiency represents important information about the energy balance in BH [1].

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About nonlinear coherent states in graphene

Erik Díaz-Bautista, David J. Fernández C.

Abstract We analyze the nonlinear coherent states for electrons in graphene interacting with a homogeneus magnetic field which is orthogonal to the layer surface. We also evaluate the corresponding Heisenberg uncertainty relation.

1 Introduction

Graphene is a single layer of carbon atoms arranged in a hexagonal honeycomb lattice whose conduction and valence bands meet at the Dirac points, six locations in momentum space on the edge of the first Brillouin zone [1]. Around the Dirac points, low energy electrons interacting with a magnetic field $\mathbf{B}$, which is orthogonal to the layer surface, are ruled by the Dirac-Weyl equation [2,3]:

$$v_F\, \boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}/c)\, \Psi(x,y) = E\, \Psi(x,y), \tag{1}$$

where $v_F \sim 0.003\,c$ is the Fermi velocity, $\boldsymbol{\sigma}$ is the vector of Pauli matrices, $\Psi(\mathbf{r}) = (\psi^+(\mathbf{r}),\ \psi^-(\mathbf{r}))^{\mathrm{T}}$, E is the energy, $-e$ is the electron charge and $\mathbf{A} = B(x)\hat{e}_y$. For a constant magnetic field $\mathbf{B} = B_0\hat{e}_z$, $B_0 > 0$, its solutions are given by the following eigenvalues with their related normalized eigenvectors:

$$E_n = \pm\hbar v_F \sqrt{n\omega}, \quad \Psi_n(x,y) = \left(\psi_{n-1}^+(x),\ \psi_n^-(x)\right)^{\mathrm{T}} e^{iky}/\sqrt{2}, \quad n = 0,1,2,\ldots, \tag{2}$$

where $\Psi_0(x,y) \equiv \left(0,\ \psi_0^-(x)\right)^{\mathrm{T}} e^{iky}$ and $\psi_n^{\pm}(x) = \psi_n^{\pm}(z(x))$ are the standard harmonic oscillator eigenfunctions with $z = \sqrt{\omega/2}\,(x + 2k/\omega)$, $\omega = 2eB_0/c\hbar$.

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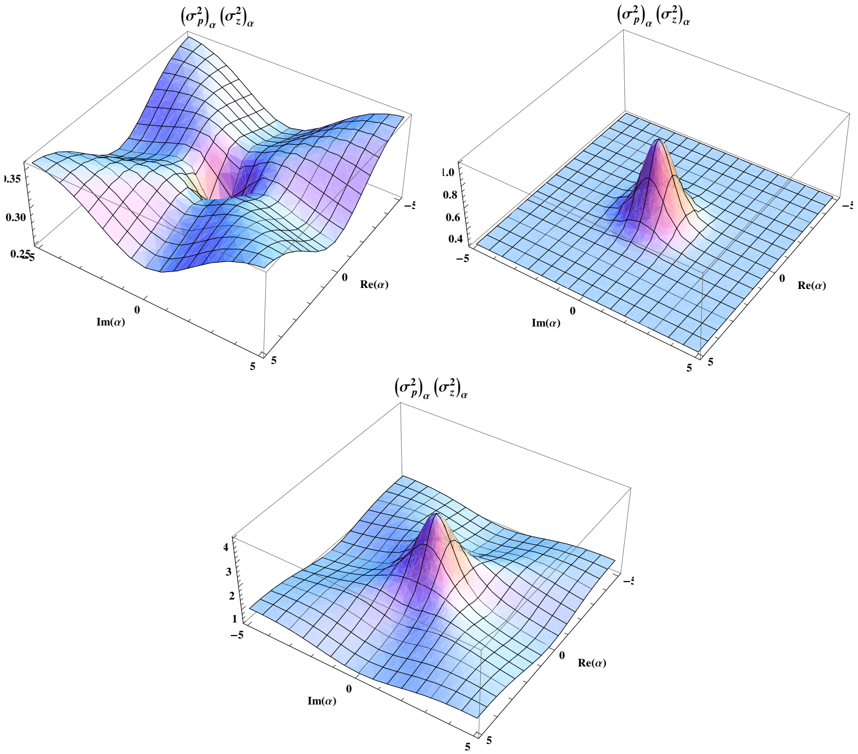

Fig. 1: Heisenberg uncertainty relation $(\sigma_z)^2_\alpha(\sigma_p)^2_\alpha$ as a function of α for some choices of $g(\hat{N})$.

2 Annihilation operator

The annihilation operator is not unique. We will choose $\hat{A}^- = \mathrm{diag}\,(f(\hat{N})\hat{\vartheta}^-,\ g(\hat{N}+\hat{1})\hat{\vartheta}^-)$, since $\hat{A}^-\Psi_n = \sqrt{n}\ g(n)\Psi_{n-1}$, where $f(\hat{N}) = \sqrt{\hat{N}+\hat{2}}\,g(\hat{N}+\hat{2})/\sqrt{\hat{N}+\hat{1}}$, $\hat{\vartheta}^\pm = (z \mp \partial_z)/\sqrt{2}$ and $\hat{N} = \hat{\vartheta}^+\hat{\vartheta}^-$. The coherent states (CS) Ψ_α are built as eigenstates of $\hat{A}^-$ with eigenvalue $\alpha \in \mathbb{C}$. Explicit forms of Ψ_α appear for simple g-choices:

$$\Psi_\alpha = \begin{cases} \dfrac{1}{\sqrt{2\,{}_0F_1(1;|\alpha|^2)-1}}\left[\Psi_0 + \displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}\alpha^n}{n!}\Psi_n\right], & \text{if } g(\hat{N}) = \sqrt{\hat{N}}, \quad (3) \\ e^{-|\alpha|^2/2}\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\Psi_{n+1}, & \text{if } g(\hat{N}+\hat{1}) = \frac{\sqrt{\hat{N}}}{\sqrt{\hat{N}+\hat{1}}}, \quad (4) \\ \dfrac{1}{\sqrt{{}_0F_2(1,2;|\alpha|^2)}}\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^n}{n!\sqrt{(n+1)!}}\Psi_{n+2}, & \text{if } g(\hat{N}+\hat{2}) = \frac{\hat{N}\sqrt{\hat{N}+\hat{1}}}{\sqrt{\hat{N}+\hat{2}}}. \quad (5) \end{cases}$$

3 Conclusions

For graphene in a constant magnetic field, we have identified annihilation and creation operators $\hat{A}^{\pm}$ and built its CS as eigenstates of $\hat{A}^{-}$. Due to the non-uniqueness of $\hat{A}^{-}$, we can find different sets of CS and the Heisenberg uncertainty relation for each one is calculated. It achieves a minimum, equal to 1/4, for the CS of Eq. (3), and it reaches a maximum for those of Eqs. (4) and (5), depending on which is the lowest energy of the excited states involved in their linear combination (see Fig. 1).

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Painlevé IV solutions from systems with a harmonic oscillator gapped spectrum

MI Estrada-Delgado and David J. Fernández C.

Abstract Supersymmetry transformations of order k are applied to the harmonic oscillator for generating potentials V_k^j whose spectra have a gap of thickness $k+1$ with respect to the initial spectrum. The system's extremal states are identified and, since the conditions ensuring that the Hamiltonian has third order ladder operators and thus it is connected with the PIV equation are satisfied, solutions to this equation can be found. An alternative supersymmetry transformation is applied to the harmonic oscillator by adding the levels needed to reproduce the spectrum of V_k^j, up to a constant energy displacement. The three new extremal states are as well identified and we get the corresponding solutions to the PIV equation. Finally, the PIV solutions found through both transformations are analysed.

1 Supersymmetric quantum mechanics

In the k-th order intertwining technique $k+1$ Hamiltonians H_j and $2k$ first order operators $A_l^{\pm}$ given by

$$H_j = -\frac{1}{2}\frac{d^2}{dx^2} + V_j(x), \;\; A_l^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + \alpha_l(x,\varepsilon_l)\right), \;\; j \in \mathbb{N}_0 \leq k, \;\; l \in \mathbb{N} \leq k, \quad (1)$$

are intertwined in the way $H_i A_i^+ = A_i^+ H_{i-1}$ with $i \in \mathbb{N} \leq k$.

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With the substitution of $\alpha_1(x,\varepsilon_l) = u_l'/u_l$, it turns out that we require k solutions u_l of the initial stationary Schrödinger equation associated to ε_l. Moreover, the SUSY partners generated from the harmonic oscillator through connected seed solutions become associated with the PIV equation, allowing us to generate solutions to such equation in a simple way [1].

2 Equivalent SUSY transformations

Choosing as factorization energy $\varepsilon_1 = -\frac{5}{2}$ $(k=1)$, the potential will be non-singular if $\nu \in (-1,1)$, as proved in [1]. In particular, for $\nu = 0$ we denote the potential as V_1^{-3}. On the other hand, a second-order transformation which employs the first two excited states leads to the SUSY partner potential V_2^1. Plots of these potentials and the associated spectra are shown in Figure 1a and 1b.

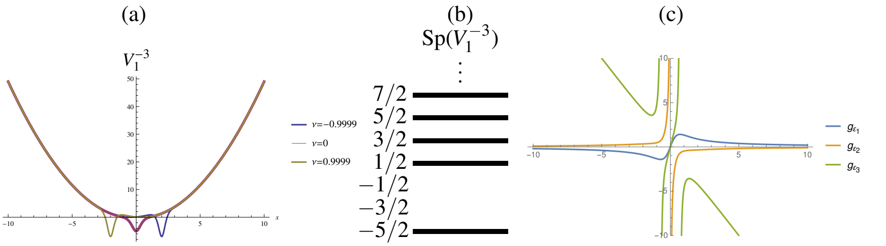

Fig. 1: (a) Plot of three potentials of the family of SUSY partners of the oscillator generated by adding a new level at $-\frac{5}{2}$ for different ν's, in particular, the one for $\nu = 0$ is called V_1^{-3}; (b) Spectrum associated to these SUSY partner potentials, which is the same for V_2^1 but displaced down by 3; (c) Solutions to the *PIV* equation associated to V_2^1 and V_1^{-3}.

The eigenfunctions for both potentials turn out to be the same and, after identifying and sorting the extremal states, the PIV solutions are calculated (see Fig. 1c).

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Interior solution for a translating cylinder of matter

Pedro Henrique Meert Ferreira and Maria de Fátima Alves da Silva

Abstract Solutions to Einstein's Field Equations with cylindrical symmetry have drawn much attention because they are significantly different from what the Newtonian picture predicts. Solutions of this kind are widely known and many cases have been studied. In particular, Van Stockum spacetime describes the interior of a cylinder with rotating dust; this solution is known to satisfy junction conditions with Lewis spacetime, which is the solution to the exterior part, i.e., gravitational field produced in vacuum by a rotating cylinder. We notice that the Van Stockum line element has a symmetry between its axial and angular coordinates; such a metric allows us to study a cylinder with a fluid moving along the axis of symmetry. We are currently studying this solution, and here we present some consequences such as the equation of state that the fluid should satisfy as well as energy conditions.

1 Outline and discussion of the problem

From a work of 1937, W. J. van Stockum showed that solutions for axisymmetric distributions of matter exist; his own solution is for dust rotating around an axis of symmetry [1], i.e., a cylinder of matter. We notice that van Stockum's original solution is symmetric under the change $z \leftrightarrow \varphi$ (within a multiplicative factor to correct dimensions); we associate this solution to an infinitely long cylinder of matter translating along the z axis. We are motivated to do so in order to verify whether there is a frame dragging effect, which is known to occur in solutions associated to rotating sources. Once we have a source for such a spacetime, we can calculate quantities associated to the exterior geometry, namely, Lewis spacetime with translation. Af-

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ter performing the interchange of coordinates, we are faced with the following line element, written in geometric units $8\pi G = c = 1$:

$$ds^2 = -dt^2 + e^{-\alpha^2 r^2}\left(\lambda d\varphi^2 + dr^2\right) + \frac{r^2\left(1-\alpha^2 r^2\right)}{\lambda} dz^2 + \frac{\alpha r^2}{\sqrt{\lambda}}\left(dtdz + dzdt\right), \quad (1)$$

where we adopted the conventional cylindrical coordinates $\{0,1,2,3\} \mapsto \{t,r,z,\varphi\}$. The parameter α is a constant related to angular momentum density in the original solution in which particles are rotating; by analogy we presume that this parameter is related to linear momentum density. λ is a real positive constant factor which asserts that the line element has the correct dimensions, thus $\lambda \sim [length]^2$.

We use the anisotropic energy-momentum tensor

$$T_{\mu\nu} = (\rho + p_z)\, u_\mu u_\nu + p_z g_{\mu\nu} + \left(p_\varphi - p_z\right) P_\mu P_\nu + (p_r - p_z)\, S_\mu S_\nu, \quad (2)$$

where p_i stand for the pressure in the ith component. u_μ is the covariant four velocity which satisfies the normalization condition $u_\mu u^\mu = -1$, and P^μ, S^μ are orthonormal space-like four vectors. Using (1) and (2) we solve Einstein's equation, $G_{\mu\nu} = kT_{\mu\nu}$, for the energy density, ρ, pressure components, p_i, and the velocity along de z axis which we call v.

By solving Einstein's equations for the most general case, one obtains

$$\rho = \frac{4F\alpha^2 e^{\alpha^2 r^2}}{G^2}\left\{1 - \frac{\left(\alpha^2 r^4 - \frac{H^2}{G^2}\right)\left(\frac{G^2}{F} - 1\right)}{\frac{H^2}{G^2} + \frac{r^2}{\lambda}\left(1-\alpha^2 r^2\right)}\right\}, \; p_z = \frac{4\alpha^2 e^{\alpha^2 r^2}\left(\alpha^2 r^4 - \frac{H^2}{G^2}\right)}{\frac{H^2}{G^2} + \frac{r^2}{\lambda}\left(1-\alpha^2 r^2\right)},$$

where we defined the following functions: $F(v) = 1 - \frac{\alpha r^2}{\sqrt{\lambda}} v + \frac{r^2(1-\alpha^2 r^2)}{\lambda} v^2$, $G(v) = -1 + \frac{\alpha r^2}{\sqrt{\lambda}} v$ and $H(v) = \frac{\alpha r^2}{\sqrt{\lambda}} + \frac{r^2(1-\alpha^2 r^2)}{\lambda} v$. The only solution for v that does not require α, λ, or both parameters to be complex or imaginary is the solution of a co-moving reference frame, i.e., $v = 0$. Setting this value for v, we find from Einstein's equation that $\lambda = 1$ and from this result one can see that $p_z = 0$ and $\rho \propto e^{\alpha^2 r^2}$, which means that we have a cylinder of dust with rigid translation along the axis of symmetry. It is interesting to notice that although we claim that α is related to linear momentum density –we are not able to prove it, the only possibility is $v = 0$. We expect this solution to be related to the static spacetime by a boost, as it happens to be in a newtonian theory, in which the potential is the same independent of any translation or rotational motion of the source. Also, the so-called frame dragging is not present in this simple case. We expect to find it in the case of differential translation, as suggested by [2], which we are currently investigating.

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A classical calculation of the W-boson magnetic moment

Alexandre Hefren de Vasconcelos Júnior

Abstract We aim to calculate the *electromagnetic* correction for the magnetic moment of an electric-charged massive elementary particle of spin-1, the intermediate vector W-boson of the weak interaction. It is a classical (or semi-classical) calculation, which is not based on diagrams. Our purpose is not to substitute the standard calculation, but to highlight the notion of extended fields. Is it possible to simulate quantum effects by using a notion of extended field? Interestingly, there is a concept in the algebraic approach to QFT called modular localization [1] that makes use of string instead of pointlike localization.

1 Introduction

The giromagnetic adimensional factor of elementary particles provides us with the most successful result in the history of science. That is because the agreement between theoretical calculations and experimental data is astonishingly precise [2].

The theory behind this calculations is quantum field theory (QFT); the first result was derived by Schwinger [3] in 1948. It was about the leptonic magnetic moment of the electron calculated using Feynman diagrams. The result is extremely famous in terms of the anomaly $a_e = \alpha/2\pi$. The pertubative approach based on quantization of classical interactions and gauge theory is very fruitful. Nevertheless, the Standard Model (SM) is not settled as a mathematically complete quantum field theory. Following the idea presented in [4], we give special status to the Compton wavelength and work the bosonic case.

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2 The calculations

From the electroweak sector of the SM, the lagrangian for the W-boson is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}W^{\star}_{\mu\nu}W^{\mu\nu} + m^2 W^{\star}W + ieqF_{\mu\nu}W^{\star\mu}W^{\nu}. \tag{1}$$

Noether's second theorem gives us the off-shell current which, in the Lorenz gauge, reads

$$\begin{aligned}\partial_\nu F^{\nu\mu} = \Box A^\mu = J^\mu = -2eqIm\left[W_\nu(\partial^\mu W^{\star\nu}) + W^{\star\nu}(\partial^\nu W^\mu)\right] + 2e^2q^2 W^{\star\nu}W_\nu A^\mu \\ -e^2q^2\left[W^{\star\mu}W_\nu + W^\mu W^\star_\nu\right]A^\nu - ieq\partial_\nu(W^\nu W^{\star\mu} - W^{\star\nu}W^\mu). \end{aligned}\tag{2}$$

For the magnetic sector, $\mu = i$, the Ampère-Maxwell is given by

$$\begin{aligned}&-\nabla^2 A^i + 2e^2q^2(\mathbf{W}^\star\cdot\mathbf{W})A^i - e^2q^2\left(W^{i\star}\mathbf{W}\cdot\mathbf{A} + W^i\mathbf{W}^\star\cdot\mathbf{A}\right)\\ &\quad = +2eqIm\left[\mathbf{W}\cdot\partial^i\mathbf{W}^\star - \mathbf{W}^\star\cdot\partial W^i\right] - ieq\partial_j\left(W^jW^{i\star} - W^{j\star}W^i\right).\end{aligned}\tag{3}$$

Thinking about corrections for the magnetic moment in $\mathscr{O}(e^2q^2)$, we define the (factored) total current $\widetilde{J}$ by $J^\mu \equiv eq\widetilde{J}^\mu$. In the Lorenz gauge with static approximation, the potential associated with $\widetilde{J}$ is

$$\mathbf{A} = \frac{eq}{4\pi}\int\frac{\widetilde{\mathbf{J}}(\mathbf{x}')d^3\mathbf{x}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{4}$$

For simplicity, we consider the non-relativistic limit and ignore the back-reaction effect, so the subsidiary condition is $W^0 = 0$. The convective term of the current, related to the massive boson, is therefore absent. Let us define the pure electromagnetic spin current

$$\widetilde{\widetilde{J}}^{i} = -i\partial_j\left(W^jW^{i\star} - W^{j\star}W^i\right) = -i\partial_j\left[W_+W_+^\star\left(\varepsilon^\star_{+_i}\varepsilon_{+_j} - \varepsilon_{+_i}\varepsilon^\star_{+_j}\right)\right], \tag{5}$$

and the associated particle density $\frac{\rho_{part}}{m} = 2\mathbf{W}^\star\cdot\mathbf{W}$. Choosing a particular polarization for the W boson, $\varepsilon^+_\mu = \frac{1}{\sqrt{2}}(0,1,i,0)$, the spin current takes the form

$$\widetilde{\widetilde{J}}^{i} = -\frac{i}{2m}\partial_j\left[\rho_{part}\left(\varepsilon^\star_{+_i}\varepsilon_{+_j} - \varepsilon_{+_i}\varepsilon^\star_{+_j}\right)\right]. \tag{6}$$

Going back to the general expression of the current,

$$\begin{aligned}\widetilde{\widetilde{J}}^{i}(\mathbf{r}) = \widetilde{J}^{i} - \left(\frac{2e^2q^2}{4\pi}\right)\int\frac{\widetilde{J}^{i}(\mathbf{r}')W^{\nu\star}W_\nu\, d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\\ + \left(\frac{e^2q^2}{4\pi}\right)\int\frac{\widetilde{J}^{j}(\mathbf{r}')\left[W^{i\star}W_j + W^iW^\star_j\right]d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned}\tag{7}$$

Now, we can express the total current in terms of the spin current by

$$\widetilde{\widetilde{J}}^{i} = \widetilde{J}^{i}\left(1 - \frac{e^2q^2}{16\pi^2}\right), \tag{8}$$

where we used the main heuristic consideration of taking $|\mathbf{r}-\mathbf{r}'| \equiv \lambda$ as a fixed distance related to the Compton wavelength associated with the W-boson. Therefore, $m\lambda = 2\pi$ appears immediately. Also, there was a delta "function" associated to the particle's density inside the integrals.

For the magnetic moment, the famous classical expression is

$$\mu = \frac{1}{2}\int \mathbf{r}\times\left(eq\widetilde{\mathbf{J}}\right) d^3r', \tag{9}$$

which can be expressed as

$$\mu = \frac{eq}{2m}\left(1+\frac{\alpha}{4\pi}\right)^{-1} 2\int\left(\left(\varepsilon_+^{\star}\mathbf{S}\varepsilon_+\right)\frac{\rho_{part}}{2}\right)d^3r' = \frac{eq}{2m}2\left(1-\frac{\alpha}{4\pi}\right)^{-1}\mathbf{s}, \tag{10}$$

with the spin vector related to the euclidean rotation generator $\mathbf{s}=\int\frac{\rho_{part}}{2}\left(\varepsilon_+^{\star}\mathbf{S}\varepsilon_+\right)d^3r'$. Finally, for $\alpha \ll 1$, one gets

$$\mu = \frac{eq}{2m}2\left(1+\frac{\alpha}{4\pi}\right)\mathbf{s}. \tag{11}$$

The experimental result from 2001 [5] gives us the value of the magnetic moment of the W-boson, $\mu_W\left(\frac{2M}{eq}\right) = 2.22^{+0.20}_{-0.19}$.

The electromagnetic corrections are small compared to the contributions of the others interactions for the magnetic moment of the W-boson and this goes to the boson being very massive. Nevertheless, the electromagnetic corrections scale as expected, i.e., as α/π.

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Quantum angle from $\mathrm{E}(2)$ coherent states quantization of motion on the circle

Diego Noguera

Abstract Covariant integral quantisation using coherent states for the Euclidean Group $\mathrm{E}(2)$ is applied to construct quantum observables for the motion on the circle. An important issue of our approach is a self-adjoint angle operator with excellent localisation properties.

1 Introduction

A crucial property of coherent states (CS) in view of integral quantisation is the resolution of the identity. Our construction of CS is based on unitary irreducible representations of the Euclidean group $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, and it is strongly influenced by the seminal paper by De Bièvre [1] and Chapter 9 of the book [2]. An interesting and original outcome of our approach is a self-adjoint angle operator showing satisfying localisation on the circle. Details are given in [3].

2 $\mathrm{E}(2)$ coherent states

The existence of our coherent states is encapsulated in the following result (proof is given in [3]).

Theorem 1. *Let* $\boldsymbol{\kappa} = (\kappa\cos\gamma, \kappa\sin\gamma) \in \mathbb{R}^2$, $\boldsymbol{\lambda} = (\lambda\cos\zeta, \lambda\sin\zeta) \in \mathbb{R}^2$, *and* $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$. *The vectors* $\eta_{p,q}(\alpha) = e^{i[\kappa p\cos(q-\alpha+\gamma)+\lambda\cos(q-\alpha+\zeta)]}\eta(\alpha - q)$ *form a family of CS for* $E(2)$*, which resolves the identity on* $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ *if* η *is admissible in the following sense:*

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$$I_{L^2(\mathbb{S}^1,\mathrm{d}\alpha)} = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\mathrm{d}q}{c_\eta} |\, \eta_{p,q} \rangle\langle \eta_{p,q} \,|, \; 0 < c_\eta := \frac{2\pi}{|\boldsymbol{\kappa}|} \int_{\mathbb{S}^1} \frac{|\eta(q)|^2 \mathrm{d}q}{|\sin(\arg\boldsymbol{\kappa} - q)|} < \infty, \quad (1)$$

where supp η *is an interval excluding the roots of* $|\sin(\arg\boldsymbol{\kappa} - q)|$.

Consequently, the covariant integral quantization [2] of a function f is defined by

$$A_f = \frac{1}{c_\eta} \int_{\mathbb{R}\times\mathbb{S}^1} \mathrm{d}p\,\mathrm{d}q\, f(p,q) |\, \eta_{p,q} \rangle\langle \eta_{p,q} \,| \,. \quad (2)$$

3 Quantum angle operator

For the 2π periodic angle function $a(\alpha)$ defined by $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, the **angle operator** computed from (2) is the multiplication operator

$$A_{a} \psi(\alpha) = \left(E_{\eta;\gamma} * a\right)(\alpha)\psi(\alpha) \quad \text{where } E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta} \frac{|\eta(\alpha)|^2}{|\sin(\gamma - \alpha)|} \,. \quad (3)$$

With the conditions on supp η given in Th. 1, the convolution $E_{\eta;\gamma} * a$ becomes

$$\left(E_{\eta;\gamma} * a\right)(\alpha) = \int_{\alpha-\gamma}^{\alpha+\pi-\gamma} \mathrm{d}q\, E_{\eta;\gamma}(\alpha - q) a(q) \,, \quad (4)$$

which corresponds to the spectrum of A_{a}. As an example, for $\gamma = \pi/2$ and $a = \kappa c_\eta / 2\pi$, the expresssion (4) yields to

$$\left(E_{\eta,\frac{\pi}{2}} * a\right)(\alpha) = \alpha - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}x \frac{x|\eta(x)|^2}{a\cos(x)} - \begin{cases} \int_{\frac{\pi}{2}}^{\alpha} \mathrm{d}x \frac{2\pi|\eta(x)|^2}{a\cos(x)} & 0 \leqslant \alpha < \frac{\pi}{2}, \\ 0 & \frac{\pi}{2} \leqslant \alpha < \frac{3\pi}{2}, \\ \int_{-\frac{\pi}{2}}^{\alpha-2\pi} \mathrm{d}x \frac{2\pi|\eta(x)|^2}{a\cos(x)} & \frac{3\pi}{2} \leqslant \alpha < 2\pi. \end{cases} \quad (5)$$

A further analysis of A_{a}, the quantisation of many other relevant obsevables, and the study of their semi-classical analysis are found in [3].

4 Conclusions

The coherent states given in Th. 1 allow us to map the angle function a to the bounded self-adjoint multiplication operator A_{a} on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$. Its spectrum is given by the periodic function (4), where the additional term to α can be made arbitrarily small almost everywhere through suitable choices of the function $\eta(\alpha)$.

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Free-energy formalism for inhomogeneous nonlinear Fokker-Planck equations

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Abstract We extend the free-energy formalism recently introduced for homogeneous Fokker–Planck equations to a wide class of inhomogeneous nonlinear Fokker–Planck equations, providing sufficient conditions for the equation coefficients to obtain a free-energy that does not increase with time. Some properties of the stationary solutions of these Fokker–Planck equations are discussed.

Consider a Fokker-Planck equation (FPE) in $(1+1)$ dimensions, i.e., a continuity equation for the probability density $\rho(x,t)$

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J[x,\rho(x,t)]}{\partial x}, \tag{1}$$

with a probability-current density given by

$$J[x,\rho(x,t)] := A(x)\Psi[\rho(x,t)] - D(x)\Omega[\rho(x,t)]\frac{\partial \rho(x,t)}{\partial x}. \tag{2}$$

We assume the following boundary conditions $\forall t \geq 0$:

$$\lim_{x\to\pm\infty} \rho(x,t) = \lim_{x\to\pm\infty} \frac{\partial \rho(x,t)}{\partial x} = \lim_{x\to\pm\infty} J[x,\rho] = 0. \tag{3}$$

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Moreover, we assume that $\forall x \in R$, $0 < D(x) < \infty$ and $\Omega[\rho] > 0$ almost everywhere. We search therefore for a trace-form free-energy-like functional

$$F(t) := \int_{-\infty}^{\infty} f[x, \rho(x,t)] \mathrm{d}x \tag{4}$$

$$f[x, \rho(x,t)] := \varphi(x)\rho(x,t) - \Theta s[\rho(x,t)], \tag{5}$$

where f is a free-energy density, φ is an effective potential, s is an entropy density such that $s[0] = s[1] = 0$, and $\Theta > 0$ is a parameter that plays the role of a temperature. Evaluating the time derivative of F, and imposing Eq. (1), we obtain

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -\int_{-\infty}^{\infty} \Theta D(x)\Psi(x) \left[-\frac{A(x)}{D(x)} + \frac{\Omega[\rho]}{\Psi[\rho]}\frac{\partial \rho(x,t)}{\partial x}\right] \times \left[\frac{1}{\Theta}\frac{\partial \varphi(x)}{\partial x} - \left.\frac{\mathrm{d}^2 s[y]}{\mathrm{d}y^2}\right|_{y=\rho(x,t)} \frac{\partial \rho(x,t)}{\partial x}\right] \mathrm{d}x.$$

We assume, without loss of generality, that $\Psi[\rho]$ is positive. The integrand is non-negative, i.e., the free energy is non-increasing along the entire time evolution, if

$$\frac{1}{\Theta}\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} = -\frac{A(x)}{D(x)}, \quad \frac{\mathrm{d}^2 s[\rho]}{\mathrm{d}\rho^2} = -\frac{\Omega[\rho]}{\Psi[\rho]}. \tag{6}$$

The relations above have been obtained for the first time in Refs. [1, 2] for the homogenous case $D(x) \equiv D = constant$.

One may wonder whether the structure of J presented in Eq. (2) and adopted in Eq. (1), might be substituted by a more general structure like

$$J[x, \rho(x,t)] = \tilde{\Psi}[x, \rho(x,t)] - \tilde{\Omega}[x, \rho(x,t)]\frac{\partial \rho(x,t)}{\partial x}. \tag{7}$$

It turns out that the structure of our free-energy functional as defined in Eqs. (4) and (5) is not compatible with the structure of the above probability-current density unless $\tilde{\Psi}[x, \rho(x,t)] = A(x)\Psi[\rho(x,t)]$ and $\tilde{\Omega}[x, \rho(x,t)] = D(x)\Omega[\rho(x,t)]$. If we instead do not specify the structure of $f[x, \rho]$, we can still write down a set of equations such that $\frac{\mathrm{d}F(t)}{\mathrm{d}t} \leq 0$ which, in turn, constrain the pair $\tilde{\Psi}, \tilde{\Omega}$ as follows:

$$\left.\begin{aligned} \frac{\partial^2 f[x,\rho]}{\partial x \partial \rho} &= \tilde{\Psi}[x, \rho] \\ \frac{\partial^2 f[x,\rho]}{\partial \rho^2} &= \tilde{\Omega}[x, \rho] \end{aligned}\right\} \Rightarrow \frac{\partial \tilde{\Psi}[x, \rho]}{\partial \rho} = \frac{\partial \tilde{\Omega}[x, \rho]}{\partial x}, \tag{8}$$

where the implied condition is not satisfied *a priori*.

Let us now come back to the factorized probability-current density in Eq. (2). It can be shown [3] that if a stationary distribution of Eq. (1), $\rho_{st}(x)$, exists, then it is unique, coinciding with the limit distribution $\rho_{st}(x) = \lim_{t\to\infty} \rho(x,t)$, and it can be written in the form

$$\rho_{st}(x) = \exp_s[-\Theta^{-1}\varphi(x) + c], \tag{9}$$

where, given $g(\rho) := \frac{\mathrm{d}s(\rho)}{\mathrm{d}\rho}$, $\exp_s(x) := g^{-1}(-x)$ is a deformed exponential associated with the (generalized) entropy density s and c is a normalization constant. Hence, even for a fixed entropic form, one can obtain a wide class of stationary distributions by an appropriate choice of the argument of the deformed exponential, and in particular of $D(x)$.

Observe that (A, Ψ, D, Ω) and $(\gamma A, \gamma^{-1}\Psi, \delta D, \delta^{-1}\Omega)$, with $\gamma, \delta \neq 0$, lead to the same FPE. Therefore the quantities $\Theta^{-1}\partial_x\varphi$ and s are given up to a common multiplicative constant, unless additional information is available. In particular, the parameter Θ can be only fixed on the basis of the specific properties of the model under consideration. Supposing φ fixed in this way, and evaluating the second moment of the stationary distribution $\int_{-\infty}^{\infty} x^2 \rho_{st}(x)\mathrm{d}x$, the following identity can be written

$$\Theta = \frac{-\int_{-\infty}^{\infty} x^3 \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \left(\frac{\mathrm{d}^2 s[z]}{\mathrm{d}z^2}\Big|_{z=\rho_{st}(x)} \right)^{-1} \mathrm{d}x}{3\int_{-\infty}^{\infty} x^2 \rho_{st}(x)\mathrm{d}x}. \tag{10}$$

Summarizing, we have extended the free-energy formalism introduced in [1, 2], to a wide class of inhomogeneous nonlinear Fokker–Planck equations. The connection with q-statistics (and its associated nonadditive entropy S_q) can be straightforwardly obtained as a particular case. A more complete analysis of the formalism, addressing properties of the free-energy functional, entropy production in the process of relaxation towards the equilibrium, derivation of the stationary solution, accompanied by a proof of the existence of a unique limit distribution, is being published elsewhere [3].

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Relativistic deformation of Helmholtz wavefields

Cristina Salto-Alegre, Amalia Torre, and Kurt Bernardo Wolf

We investigate the aberration of images on a plane screen produced by a three-dimensional Helmholtz wavefield when the sphere of plane wave directions is subject to relativistic deformation. This aberration was originally studied in the geometric optics model in Refs. [1, 2] and [3, Sect. 5.7], and for the wave model in $[1]_{(\mathrm{II})}$, albeit with simplifications. Here we apply the relativistic deformation to the monochromatic wave model provided by solutions of the Helmholtz equation, $(\partial_x^2+\partial_y^2+\partial_z^2+k^2)\,f(x,y,z)=0$, which can be written equivalently in *evolution* form as a 2×2 matrix operator equation,

$$\mathbf{H}\,\mathbf{f}=\partial_z\mathbf{f}, \qquad \mathbf{H}:=\begin{pmatrix}0 & 1\\ -\Delta_k & 0\end{pmatrix}, \quad \mathbf{f}:=\begin{pmatrix}f(\mathbf{r},z)\\ f_z(\mathbf{r},z)\end{pmatrix}, \tag{1}$$

where $\mathbf{r}=(x,y)$, k is a wavenumber, and $\Delta_k:=\partial_x^2+\partial_y^2+k^2$. The wavefield is $f(\mathbf{r},z)$ and $f_z(\mathbf{r},z)=\partial_z f(\mathbf{r},z)$ is its normal derivative at a $z=$ constant plane.

The operator $\mathbf{H}$ generates translations of (1)in the z-direction from the standard screen: $\mathbf{f}(\mathbf{r},z)=\exp(z\mathbf{H})\,\mathbf{f}(\mathbf{r})$, where $\mathbf{f}(\mathbf{r}):=\mathbf{f}(\mathbf{r},z)|_{z=0}$. Solutions to the Helmholtz equation are a superposition of plane waves whose directions lie on the sphere. We can imagine being in a space vehicle; at relativistic speeds, the sphere of stars will be deformed so that they will apparently crowd towards the direction of motion, the angles to it will map as $\tan\frac{1}{2}\theta\mapsto\tan\frac{1}{2}\theta'=e^{-\chi}\tan\frac{1}{2}\theta$. One can deform the Euclidean symmetry algebra of translations and rotations that accompanies (1) into a Lorentz algebra $[1]_{(\mathrm{II})}$ to translate the boost map as generated by

$$\mathbf{B}_{\mathbf{r}}=\frac{1}{k}\begin{pmatrix}(\hat{D}+1)\partial_{\mathbf{r}}+k^2\mathbf{r} & 0\\ 0 & (\hat{D}+2)\partial_{\mathbf{r}}+k^2\mathbf{r}\end{pmatrix}, \quad \mathbf{B}_z=\frac{1}{k}\begin{pmatrix}0 & 0\\ k^2-(\hat{D}+2)\Delta_k & 0\end{pmatrix}, \tag{2}$$

where $\hat{D}:=\mathbf{r}\cdot\partial_{\mathbf{r}}$ and $\hat{C}:=k^2-(\hat{D}+2)\Delta_k$. The relativistic deformation of functions on the screen is $\mathbf{f}(\mathbf{r})\mapsto\mathbf{f}_\gamma(\mathbf{r})=\exp(\mathrm{i}\gamma\boldsymbol{\omega}\cdot B)\,\mathbf{f}(\mathbf{r})$, with the 3-vector $\boldsymbol{\omega}$ of directions

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on the sphere and $B = (\mathbf{B_r}, \mathbf{B}_z)$; for finite boosts γ in the z direction, the operator is

$$\frac{k}{2\pi^{3/2}} \begin{pmatrix} \left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} (\frac{\gamma}{k})^{2n} (\hat{D}\hat{C})^n\right) \mathrm{sinc}\,(k|\mathrm{r}-\mathrm{r}_0|) \\ \frac{\mathrm{i}\gamma}{k}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} (\frac{\gamma}{k})^{2n} (\hat{C}\hat{D})^n\right) \hat{C}\,\mathrm{sinc}\,(k|\mathrm{r}-\mathrm{r}_0|) \end{pmatrix}. \tag{3}$$

The function of minimal width in a Helmholtz field is a spherical Bessel function $j_0(r)$, which is expressible as a trigonometric function: $\psi(\mathbf{r}-\mathbf{r}_o) = (k/2\pi^{3/2}) \times \mathrm{sinc}\,(k|\mathrm{r}-\mathrm{r}_0|)$, with zero normal derivative $\psi_z(\mathbf{r}-\mathbf{r}_o) = 0$. In Fig. ?? we show the series (3) succesively computed up to the terms $n = 0, 1, \ldots, 5$. Previous work [1]$_{\text{(II)}}$ presented results for $n \leq 2$ on a Gaussian —which is not properly a Helmholtz field.

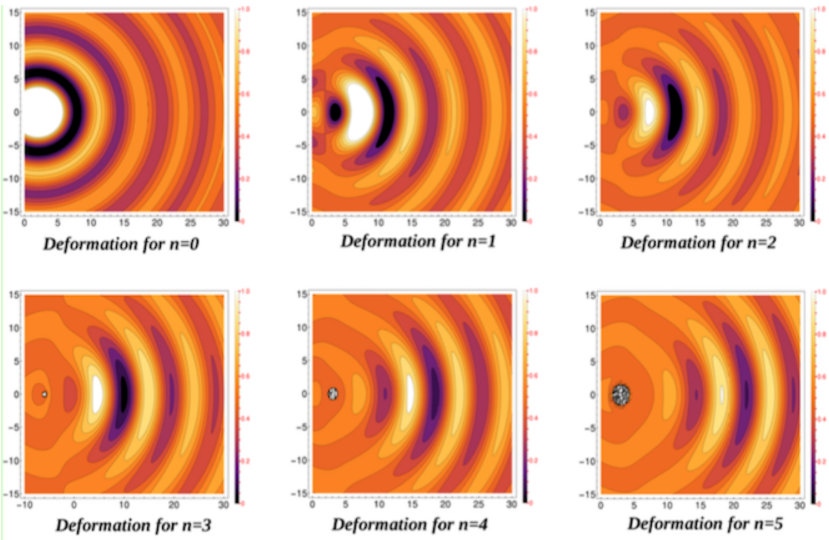

Fig. 1: The relativistic coma aberration of the Helmholtz wavefield $\psi(\mathbf{r}-\mathbf{r}_o)$ as given by the n first terms in the series (3), for $n = 0, 1, \ldots, 5$, $\gamma = -2$, and centered on $\mathbf{r}_o = (3,0)$ —marked in the second row; color range is $(0,1)$. Note that the apex maximum is displaced to the right while there are increasing $\sim n$ oscillations that form the coma with the characteristic $\sim 60°$ opening angle.

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Part V
Memorials: S. T. Ali and L. Boyle

Syad Twareque Ali

Jean-Pierre Antoine and Jean-Pierre Gazeau

Nitya kaaler utshab taba
Bishyer-i-dipaalika
Aami shudhu tar-i-mateer pradeep
Jaalao tahaar shikhaa [1]

– Tagore

Fig. 1: Excursion on the Ganges river during the Bose School and Conference on Current Topics in Physics, held in Dakha, Bangladesh, December 15–21, 2007: Twareque Ali with two participants.

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[1] Thine is an eternal celebration ... – A cosmic Festival of Lights! ... Therein I am a mere flicker of a wicker lamp ... O kindle its flame, (my Master!); from the preface of AAG.

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The career of Twareque exhibits several contrasting features. First, he is one of the few people we know who had FOUR successive nationalities. Born a citizen of the British Empire, he became a Pakistani in 1947, a Bangladeshi in 1971, and finally a Canadian.

Actually Twareque travelled a lot professionally. After a MSc in Dhaka, Bangladesh, in 1966, and a PhD in Rochester, USA, in 1973, he occupied positions successively at ICTP, Trieste (Italy), U. of Toronto, U. of Prince Edward Island, TU Clausthal (Germany), finally Concordia U., Montréal. In addition, he has been invited professor at uncountably many foreign universities throughout the world and invited speaker at as many conferences.

During his whole life, his scientific worldline was fully coherent. Indeed his entire career can be summarized in those keywords which were his favorite topics: phase space, positive operator valued measures, reproducing kernels, quantization, coherent states, orthogonal polynomials ... Throughout these years, Twareque has relentlessly preached mathematical rigor in the field of coherent states, too often left to the rather sloppy treatment of quantum optics fans (in Wikipedia, for instance !).

Let us be more specific. In the 1970s and 1980s, he devoted much time to measurement problems in (fuzzy) phase space and stochastic, Galilean and Eisteinian, quantum mechanics, mostly with E. Prugovečki, in Toronto (which led to a memorable and lasting dispute with Gerald Kaiser), G. Emch in Rochester and H.-D. Doebner in Clausthal. His 1985 monumental review paper [1] is a pedagogical landmark for the domain. Then he gradually focused on coherent states for the Galileo group, the Poincaré group, and other semi-direct product groups. These were the topics on which we started our collaboration at the end of the eighties and continued up to the end. Along the way, eleven joint papers were written (plus four with JPA alone and ten with JPG), three books edited (Białowieża proceedings), two more written as co-authors [2, 3] and a special issue of J. Phys.A [4] was published. A notable result of the joint enterprise was the extension of square integrability of group representations to homogeneous spaces [5] and the introduction of continuous frames in Hilbert spaces — the key to many applications, including wavelets [6]. Then he started to be interested in quantization, mostly Berezin or coherent states quantization, and in the mathematics of signal processing. A particularly nice achievement was his review paper with M. Engliš [7]. In the last years, he studied noncommutative quantum mechanics, quaternionic Hilbert spaces and complex orthogonal polynomials.

An important part of Twareque's life is the organization of meetings. Two series are notable. First, the Białowieża Workshops on Geometric Methods in Physics, famous for the wild forest, bisons, vodka, Russian mathematicians of the highest caliber and, of course, their charismatic chief organizer, Anatol Odzijewicz. From 1992 on, the XIth meeting, Twareque was instrumental in transforming a small local workshop into a full-fledged international event, still going strong — the last edition was Nr.XXXV, last July.

Another remarkable achievement is the series of workshops in Havana, Cuba, organized jointly by Concordia U. and the University of Havana. Here again, Twareque was one of the “chevilles ouvrières” of the meeting, who succeeded in attracting a number of recurrent distinguished participants to these beautiful surroundings and almost singlehandedly took care of the proceedings. More recently Twareque has also become a faithful participant in the school and workshops on mathematical physics (COPROMAPH) organized in Cotonou, Bénin, by Norbert Hounkonnou.

To conclude, being more explicit about numerous (around 150) Twareque’s contributions listed by “order of importance” would be illusory because there are so many different, even contradictory, classification criteria. We would like just to stress the fact that Twareque had a very deep and subtle knowledge and understanding of Quantum Physics, in its foundations as well as in its working mathematical tools. And, above all, he never showed arrogance for that. It was always extremely pleasant to work with him because of his intelligence and extreme tolerance for the ideas advanced by others. Open-mindedness was one of his great qualities. His clean, polished mathematical style is a pleasure to read, even if his papers are sometimes dense and compact. His nonscientific writings have the same quality, even poetry — we still remember a memorable poem composed by him in Białowieża! His talks, of which we attended quite a number, were always extremely clear and pedagogical. His ideas were often thought provoking. Altogether we feel fortunate to have been able to work with him for so long, and so do surely all his other collaborators.

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Laurence Boyle - In Memoriam

Maia Angelova

Fig. 1: LL Boyle born 15th May 1942 in Oxford and died 6th October 2015 in Kent

Lewis Laurence Boyle, always known as Laurence, was born in Oxford on 15 May 1942. He was brought up the eldest of a large family, seven children, brothers and sisters, of Catherine and Donald Boyle on the outskirts of Oxford, near the village of Iffley. Laurence's early years were in many ways typical of a child brought up during the years of austerity, and with the national housing shortage,

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the Boyle family lived with their maternal grandparents until Laurence was thirteen. This was a stimulating environment for the obviously bright little boy, where education was highly regarded. With the shortage of toys, even paper being rationed until 1953, Laurence collected what was available, especially tickets, both train and bus, timetables and stamps. Meccano and Dinky toys were treats.

Laurence attended Notre Dame Prep School in Oxford, where he rapidly proved his abilities, particularly in Mathematics, and he went on to secondary school when he was only ten years of age. Rather small, a year younger than his classmates and with poor eyesight, Laurence was not a born sportsman, but the Salesian College in Cowley provided a good range of interesting activities and clubs, and it was at school there that he developed what were to be three life-long interests, geology, chess and languages, especially Spanish.

Laurence was very adept with timetables, and from an early age was organising the family holidays by train. Aged 12 he was trusted by his parents to arrange the travel for by then the parents and four children to Southport, including finding the most economical method, (mid-week tickets), purchasing the tickets with money from his mother, and even taking the luggage in advance to Littlemore station by pram. The trips, including several to Ireland, went very smoothly, with no parental input other than the arrangement of accommodation.

Aged 16, Laurence went on his own to Spain to spend a month with the relations of a Spanish priest who was a friend of his grandparents. Again, all the travel arrangements were done by Laurence himself, (and Spain was certainly not a holiday destination at that time!) The travel went well, but the family he stayed with spoke only Basque, so Laurence returned home with a basic knowledge of that language rather than improved Spanish. The ease with which Laurence had learned basic Basque gave him the confidence to learn many languages, and he eventually spoke about twelve, being able to give scientific lectures in a number of them.

Laurence was actively involved with the local geology group since age of 14. He was one of the founding members of the Oxford Geology Group.

When he was sixteen, Laurence sat the Oxford University entrance exam ”for practice” and was awarded a place at Christ Church, Oxford. At that time, the university regulations prohibited a student starting before he or she was eighteen, so Laurence stayed on at school, eventually attaining eight A levels, before reading Chemistry as his first degree.

He graduated from Oxford University with the degrees:BA (Hons) (1963) in Chemistry with supplementary Mineralogy and BSc (1965) in Chemistry and Mathematics and an MA (1966) (Figures 2 and 3). He obtained a DPhil in Theoretical Chemistry in 1966 under the supervision firstly of Professor A.D. Buckingham, FRS who left to take up a post in Princeton, and secondly under Professor C.A. Coulson, FRS in the Mathematical Institute at Oxford.

Laurence was appointed as an Assistant Lecturer in Chemistry at the University of Kent at Canterbury in October 1966 at the age of 24, being one of the second cohort of academic staff after the University first opened a year earlier. He did not undertake a period of post-doctoral work, and his tenured post at Kent was his first and only ever job! He retired in 2009 [1].

At the University of Kent, Laurence was involved in teaching, research and administration. He taught a very wide range of courses, including Spectroscopy, Electron Paramagnetic Resonance, Quantum Chemistry, Advanced Ligand Field Theory, Bonding, Crystal Structures, Theoretical Organic Chemistry, Geochemistry for Chemists and for Environmental Physical Scientists, and Rocks and Minerals. As part of the Environmental Physical Sciences degree course, his geology teaching involved organising and running single-handedly annual week-long field trips during the summer vacation to Scotland, as well as day trips to Boulogne. He gave many 'outreach' lectures on symmetry in chemistry for visiting sixth-form students.

He was a College Tutor and Natural Sciences Faculty Senior Tutor for eight years, and a Convenor for the very successful Chemical Physics degree. This course, which ran for many years and attracted some very bright students, was essentially unique among UK undergraduate degree programmes [1]. Laurence's role involved co-ordinating all the teaching arrangements and organising the examinations, including setting the papers and dealing with the external examiners. He was Chemistry Library Purchasing Officer in which role he was able to exploit his love of books. In particular, he assembled a very impressive array of back-runs of many major scientific journals, which he procured on the second-hand market as library collections elsewhere were being disbanded. Sadly, because of storage space requirements and associated costs, this legacy of Laurence has now largely disappeared [1].

Fig. 2: Lewis Laurence Boyle BSc, MA, DPhil

Fig. 3: LL Boyle in Christ Church, Oxford with his tutor, Dr Paul W Kent, on the right and friends.

By 1994 he had supervised 13 PhD students and 4 MSc (by research) students. He had received 13 post-doctoral workers from abroad: six Spaniards, one Pole, three Bulgarians, one Indian, one Iranian and one Ukrainian. He had lectured in 20 different countries in, to use his own words, "the most appropriate of five languages". He had a constant stream of professors, mainly from Eastern Europe, who visited and worked with him during his time at Kent [1]. He published with his PhD students, his post-doctoral students and his collaborators (see examples [2-4].)

Laurence specialized in Applications of Group Theory to Chemistry, Spectroscopy and Crystallography. He published about 80 papers, the latest one in 2014, in scientific journals and proceedings of international conferences. He was most proud of his paper with Kerie Green (Figure 4, right-hand side) on the complete derivation of the representation groups and projective representations of the classical point groups [2]. He was considered a leading expert on symmetries of vibrational spectroscopy and crystallographic space groups (for example [3]).

Laurence was a very keen editor of academic work. His last editorial work was of AJ Ceuleman's Group Theory Applied to Chemistry published by Springer in 2013. In the Acknowledgements of the book AJ Ceulemans wrote: "I am very grateful to L. Laurence Boyle for the critical reading of the entire manuscript, taking out remaining mistakes and inconsistencies."

I met Laurence in the summer of 1987 in Varna, where we attended the 16th International Colloquium on Group-Theoretical Methods in Physics (ICGTMP), organised by Chavdar Palev and Vladimir Dobrev. I had just completed my PhD from the University of Sofia under the supervision of Dr. Josef Kotzev. I worked with Laurence for a year in 1988 as a Visiting Research Fellow at the University of Kent, funded by The British Council. The project was on co-representations of magnetic space groups. I had two shorter visits to Kent in 1990 and 1991, after which I moved to Oxford University and Somerville College and continued to work with Laurence until the end of 1996. We wrote several papers together (see for example [4]).

Fig. 4: Left: Ms Miriam Lewis Laurence's long term partner; right: Dr. Kerie Green, a former PhD student of L L Boyle with whom he published [2] and who is now a Senior lecturer in Mathematics at the University of South Wales.

Laurence was appointed to the Standing Committee in 1979 in anticipation of his organizing the Xth International Colloquium on Group-Theoretical Methods in Physics (ICGTMP) in Canterbury in 1981 with Arthur Cracknell (Dundee). The chairman at the time was Aloysio Janner (Nijmegen) and Laurence was appointed Honorary Secretary, a post which he held for 29 years. When I and Wojtek Zakrzewski (Durham) organised the 28th ICGTMP in Newcastle upon Tyne in the summer of 2010, Laurence helped with many useful suggestions.

Laurence organised a number of large international conferences, undertook a Royal Society Study Visit to South Korea, and was a member of numerous external committees. He organised the 3rd Wigner Symposium (Oxford, 1993) with Allan Solomon (Open University) and Maia Angelova (then Somerville College, Oxford), The 24th Quantum Theory Conference (Guernsey,1993) and two smaller conferences in a stately home near Canterbury in fulfilment of a contract with the European Commission. He was actively involved in the establishment of the series of symposia on Quantum Theory and Symmetries and was a member of its International Board (Figure 5).

Fig. 5: Laurence Boyle with Reidun Twarock and Vladimir Dobrev in Prague from the Quantum Theory and Symmetry QTS7 conference in 2011.

Fig. 6: Laurence Boyle with his sister Julia in Downing Street.

He loved books, geology, antique furniture and the challenges of travelling. He enjoyed dining out. He spoke several languages fluently. Laurence was a valued member of our community (Figures 6), was an excellent researcher and teacher, and a very good organiser of academic events. He had many close friends (Figure 4) and loved his family (Figure 7). He will remain in our hearts.

Fig. 7: XXVIII ICGTMP Goup28 Newcastle upon Tyne July 2010, organised by M Angelova and W Zakrzewski

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I would like to thank Dr Katy Kidd, Laurence's sister for some of the pictures and stories about Laurence's early years, Ms Miriam Lewis, Laurence's long term partner for stories about Laurence's social life in Kent, Dr ... student at chemistry at Christ Church Oxford with Laurence and close friend and colleague, Professor John Todd, friend and colleague of Laurence at the University of Kent or facts and stories about Laurence's academic life at University of Kent at Canterbury, Professor AJ Ceulemans for stories about Laurence's passion about symmetry and group theory in chemistry, Professors V Dobrev and G Pogosyan for pictures and facts from Laurence's involvement in the ICGTMP Standing Committee work.

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Colloquium group photo at the campus of UFRJ, nearby CBPF where parallel sessions were hold. Image by Alvaro Farias (2016).

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