Dynamical Analysis of Fractional-Order Hyper-chaotic System

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Abstract. The purpose of this paper is to study the dynamical behavior of fractional order hyper-chaotic complex systems based on the bifurcation theorem. The variation of the system parameters and fractional order can induce the bifurcation by the simulation results.

Keywords: Fractional-order system · System order · System parameters

1 Introduction

The fractional calculus has been widely concerned in mathematics. However, At the beginning of its development, it has been paid much attention in the field of pure mathematics, Famous mathematician Leibniz and Euler give their initial understanding of fractional calculus [1]; In 1819, based on the Gamma function, Lacroix gave the first definition of fractional calculus, And a simple fractional calculus [2]. This has inspired mathematicians to study fractional calculus, However, due to the lack of a reasonable physical interpretation of fractional calculus, The lack of application background, in the field of application of slow development.

Since the first discovery of chaotic attractor in Lorenz numerical experiments, People put forward many chaotic systems, such as Chen system, *Lü* system, Duffing system, VanderPol system [3–6]. Early research focused on low dimensional chaotic systems. Because the hyperchaotic system has at least two positive Lyapunov exponents, And it contains more abundant and more complex dynamic behaviors than the low dimensional chaotic system, so it is more suitable for the design of digital cryptography and secure communication. It is a fractional calculus operator that can describe the dynamical behavior of chaotic system more accurately. Initial sensitivity and pseudo randomness are common properties of fractional order chaotic systems. In addition, it also has some properties of fractional order systems, such as the ability to reflect the historical information of the system, strong historical memory and so on. Therefore, the study of fractional order chaotic systems has extensive theoretical significance and practical value.

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2 Related Work

2.1 Fractional-Order Hyper-chaotic Complex System

In the literature [7], the author puts forward the integer orderhyper-chaotic system and makes a detailed analysis. Based on this, this paper presents a fractional order hyper-chaotic system with complex variables:

$$\begin{cases} D_*^{q_1} y_1 = a(y_1 - y_2) + y_4 \\ D_*^{q_2} y_2 = by_2 - y_1 y_3 + y_3 \\ D_*^{q_3} y_3 = \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - cy_3 \\ D_*^{q_4} y_4 = \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - dy_4 \end{cases} \quad (1)$$

The $y = (y_1, y_2, y_3, y_4)^{\mathrm{T}}$ is state variable, $y_1 = x_1 + ix_2, y_2 = x_3 + ix_4$ is complex state variable, $y_3 = x_5, y_4 = x_6$ is real state variable, $i = \sqrt{-1}$, state variables can be divided into imaginary and real parts, according to the linear property of the Caputo differential operator, the system (1) can be written as follows:

$$\begin{cases} D_*^{q_1} x_1 = a(x_2 - x_1) + x_6 \\ D_*^{q_1} x_2 = a(x_3 - x_2) + x_6 \\ D_*^{q_2} x_3 = bx_4 - x_2 x_5 + x_6 \\ D_*^{q_2} x_4 = bx_4 - x_1 x_5 + x_6 \\ D_*^{q_3} x_5 = x_1 x_3 + x_2 x_4 - cx_5 \\ D_*^{q_4} x_6 = x_1 x_3 + x_2 x_4 - dx_6 \end{cases} \quad (2)$$

2.2 Dynamic Behavior Analysis of the System

2.2.1 System Equilibrium Point

The equilibrium point of the system can be obtained by the following formula:

$$\begin{cases} D_*^{q_1} x_1 = a(x_3 - x_1) + x_6 = 0 \\ D_*^{q_1} x_2 = a(x_1 - x_2) = 0 \\ D_*^{q_2} x_3 = bx_3 - x_1 x_4 + x_6 = 0 \\ D_*^{q_2} x_4 = bx_4 - x_2 x_3 = 0 \\ D_*^{q_3} x_5 = x_1 x_3 + x_2 x_5 - cx_5 = 0 \\ D_*^{q_4} x_6 = x_1 x_2 + x_2 x_3 - dx_6 = 0 \end{cases} \quad (3)$$

According to the second equations of equation set (3), we have $x_1 = x_2$, by fourth equations, we have $x_2(b - x_5) = 0$, then $x_2 = 0$ or $x_5 = b$. if $x_5 = b$, then $a = b$, $x_6 = \frac{cx_5}{d} = \frac{cb}{d}$. Thus the equilibrium points of the system (3) are distributed in the circle which center for $(\frac{c}{2d}, 0)$ and radius of $r = \left(\sqrt{4bcd^2 + c^2}\right)/2d$, the equation of the circle can be written as:

$$(x_1 - \frac{c}{2d})^2 + x_2^2 = (\frac{\sqrt{4bcd^2 + c^2}}{2d})^2 \tag{4}$$

Let $x_1 - \frac{c}{2d} = r\cos\theta$, $x_2 = x_4 = r\sin\theta$, The balance point is as follows:

$$E_\theta = (r\cos\theta + \frac{c}{2b}, r\sin\theta, r\cos\theta - \frac{c}{2b}, r\sin\theta, b, \frac{bc}{d})$$

for $\theta \in [0, 2\pi]$.
If $x_2 = 0$, the equilibrium points of the system (3) are (0, 0, 0, 0, 0, 0), and there are two isolated unstable points:

$$(s_1, 0, \frac{das_1}{da + s_1}, 0, \frac{das_1^2}{c(da + s_1)}, \frac{as_1^2}{(da + s_1)}),$$
$$(s_2, 0, \frac{das_2}{da + s_2}, 0, \frac{das_2^2}{c(da + s_2)}, \frac{as_2^2}{(da + s_2)}),$$

For

$$s_1 = \frac{c}{2d} + \frac{1}{2}\sqrt{(\frac{c}{d})^2 + 4bc}, s_2 = \frac{c}{2d} - \frac{1}{2}\sqrt{(\frac{c}{d})^2 + 4bc}$$

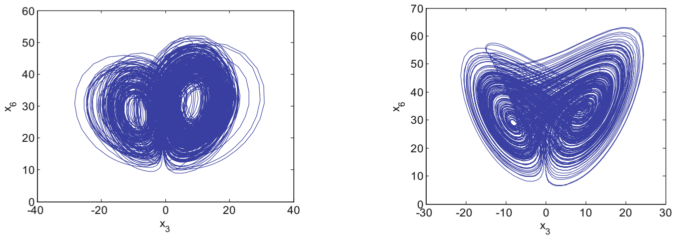
At the same time, the Jacobi matrix E_0 of the system (3) is

$$J_{E_0} = \begin{bmatrix} -a & 0 & a & 1 & 0 & 1 \\ 0 & -a & 0 & a & 0 & 0 \\ 0 & 0 & b & 0 & a & 1 \\ 0 & 0 & 0 & b & 0 & a \\ 0 & 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & -d \end{bmatrix} \tag{5}$$

The characteristic equation is $(\lambda + d)(\lambda + c)(\lambda - b)^2(\lambda + a)^2 = 0$, and it's characteristic value $\lambda_1 = -c, \lambda_2 = -d, \lambda_3 = \lambda_4 = b, \lambda_5 = \lambda_6 = -a$, the equilibrium point (0, 0, 0, 0, 0, 0) is stable, if b have negative eigenvalue, c, a, d is positive, otherwise the system (3) is unstable.

2.2.2 The Influence of System Order Variation on the System

We let $\boldsymbol{q_1} = \boldsymbol{q_2} = \boldsymbol{q_3} = \boldsymbol{q_4} = \alpha = 0.15$, and selection system parameter $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{23}, \boldsymbol{20}, \boldsymbol{4}, \boldsymbol{1})$, The fractional order system is an equal order system. When$q_1 = q_2 = q_3 = q_4 = \alpha = 0.80$, the fractional order hyperchaotic complex system has abalancepoint. if has $\alpha = 0.87$, the attractor of the system is shown in Fig. 1a. If we continued increase $\alpha = 0.90$, the attractor of the system is shown in Fig. 1b, It can be seen that the attractor with order 0.87 is completely different.



(a) (b)

Fig. 1.

Because the fractional order system is very complex, we only consider some special classes,

Selected $\boldsymbol{q_2} = \boldsymbol{q_3} = \boldsymbol{q_4} = 0.80$, when the fractional order $q_1 \in [0.81, 0.99]$, the fractional order q_1 of the system is shown in Fig. 2. When the order $q_1 = 0.86$, the fractional order hyperchaotic complex passes through the fork type bifurcation into the chaotic state. When selecting fractional order $q_1 = q_3 = q_4 = 0.92$, $q_2 \in [0.80, 0.95]$, the dynamic behavior of the system is shown in Fig. 3, As we can see in Fig. 3, the fractional order hyperchaotic complex system is chaotic in the range $q_2 \in [0.897, 1]$, when $q_2 \in (0.80, 0.997)$, there is a period doubling window, when $q_2 < 0.82$, the system is a fixed point.

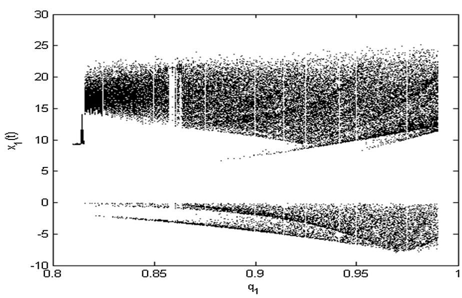


Fig. 2. Dynamical behavior of complex system when order $q_1 \in (0.80, 0.90)$

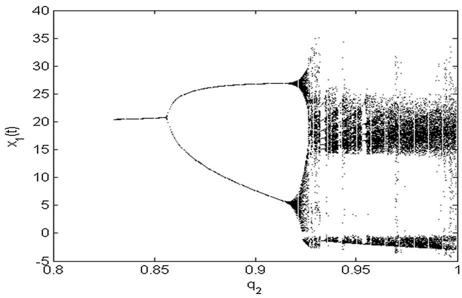


Fig. 3. Dynamical behavior of complex system when order $q_2 \in (0.80, 0.99)$

We do a lot of numerical simulation, for fractional order $q_3 = q_4$, $q_3 \in [0.80, 1]$, $q_4 \in [0.80, 1]$, fractional order hyperchaotic complex system has no obvious bifurcation behavior in this range.

2.2.3 The Influence of System Parameter Change on the System

First, we select the order of the system $\boldsymbol{q_1} = \boldsymbol{q_2} = \boldsymbol{q_3} = \boldsymbol{q_4} = 0.91$, the dynamic behavior of the system is further analyzed by means of bifurcation diagrams and phase diagrams.

Second, let the system parameters $(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{20}, \boldsymbol{5}, \boldsymbol{4})$, let parameters increase from 31.5 to 33, the bifurcation diagram of the system with parameters is shown in Fig. 4, it can be seen that the system starts from 31.5, and the system enters chaos through a series of period doubling bifurcations. When the system parameters starts from 33.5, then the tangent bifurcation occurs and system enters the chaotic state. after that, a series of period doubling bifurcations occurred,

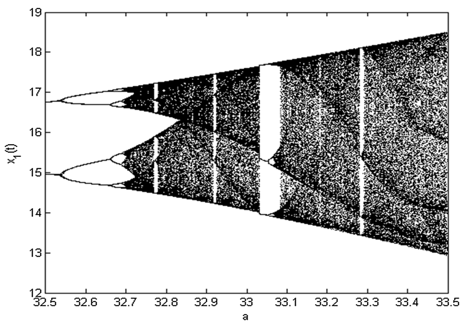


Fig. 4. Dynamical behavior of complex system when parameter $a \in (32.5, 33.5)$

Third, when the system parameters are selected by $(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{40}, \boldsymbol{5}, \boldsymbol{4})$, the system increases from 16 to 20, and a series of periodic doubling bifurcations occur. at the same time, the phase diagram of the system in three-dimensional phase space is obtained, the system can be seen in the case when the parameter is increased to 19.25, When the parameter b > 19.25, the system goes through a series of period doubling bifurcation into chaos, the phase diagram can be seen, the system has a cycle, cycle three and chaotic attractor.

Finally, the parameters of the system are changed with the change of parameters, the bifurcation diagram of the system with parameter d can be described, when the parameter $d = 0.3$ takes a tangent bifurcation, then the system enters a period doubling window, when the parameter $d = 1.12$ appears two order period doubling bifurcation, and then goes into chaos.

3 Conclusion

The existence and stability of the equilibrium points of the system are studied in this paper, and then the dynamic behavior of the system is discussed in terms of the order of the fractional order system and the bifurcation of the system parameters.

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