Sasakian Statistical Manifolds II

Hitoshi Furuhata(✉)

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
furuhata@math.sci.hokudai.ac.jp

1 Introduction

This article is a digest of [2,3] with additional remarks on invariant submanifolds of Sasakian statistical manifolds.

We set $\Omega = \{1, \ldots, n+1\}$ as a sample space, and denote by $\mathcal{P}^+(\Omega)$ the set of positive probability densities, that is, $\mathcal{P}^+(\Omega) = \{p : \Omega \to \mathbb{R}_+ \mid \sum_{x\in\Omega} p(x) = 1\}$, where $\mathbb{R}_+$ is the set of positive real numbers. Let M be a smooth manifold as a parameter space, and $s : M \ni u \mapsto p(\cdot, u) \in \mathcal{P}^+(\Omega)$ an injection with the property that $p(x, \cdot) : M \to \mathbb{R}_+$ is smooth for each $x \in \Omega$. Consider a family of positive probability densities on Ω parametrized by M in this manner. We define a $(0, 2)$-tensor field on M by

$$g_u(X, Y) = \sum_{x\in\Omega} \{X \log p(x, \cdot)\}\{Y \log p(x, \cdot)\}p(x, u)$$

for tangent vectors $X, Y \in T_uM$. We say that an injection $s : M \to \mathcal{P}^+(\Omega)$ is a *statistical model* if g_u is nondegenerate for each $u \in M$, namely, if g is a Riemannian metric on M, which is called the *Fisher information metric* for s. Define $\varphi : M \to \mathbb{R}^{n+1}$ for a statistical model s by $\varphi(u) = {}^t[2\sqrt{p(1, u)}, \ldots, 2\sqrt{p(n+1, u)}]$. It is known that the metric on M induced by φ from the Euclidean metric on $\mathbb{R}^{n+1}$ coincides with the Fisher information metric g. Since the image $\varphi(M)$ lies on the n-dimensional hypersphere $S^n(2)$ of radius 2, the Fisher information metric is considered as the Riemannian metric induced from the standard metric of the hypersphere. For example, we set

$$M = \{u = {}^t[u^1, \ldots, u^n] \in \mathbb{R}^n \mid u^j > 0,\ \sum_{l=1}^{n} u^l < 1\},$$

$$s : M \ni u \mapsto p(x, u) = \begin{cases} u^k, & x = k \in \{1, \ldots, n\}, \\ 1 - \sum_{l=1}^{n} u^l, & x = n+1. \end{cases}$$

Then $\varphi(M) = S^n(2) \cap (\mathbb{R}_+)^{n+1}$ and the Fisher information metric is the restriction of the standard metric of $S^n(2)$. It shows that a hypersphere with the standard metric plays an important role in information geometry. It is an interesting question whether a *whole* hypersphere plays another part there.

In this article, we give a certain statistical structure on an odd-dimensional hypersphere, and explain its background.

F. Nielsen and F. Barbaresco (Eds.): GSI 2017, LNCS 10589, pp. 179–185, 2017.
https://doi.org/10.1007/978-3-319-68445-1_21

2 Sasakian Statistical Structures

Throughout this paper, M denotes a smooth manifold, and $\Gamma(E)$ denotes the set of sections of a vector bundle $E \to M$. All the objects are assumed to be smooth. For example, $\Gamma(TM^{(p,q)})$ means the set of all the C^∞ tensor fields on M of type (p, q).

At first, we will review the basic notion of Sasakian manifolds, which is a classical topic in differential geometry (See [5] for example). Let $g \in \Gamma(TM^{(0,2)})$ be a Riemannian metric, and denote by ∇^g the Levi-Civita connection of g. Take $\phi \in \Gamma(TM^{(1,1)})$ and $\xi \in \Gamma(TM)$.

A triple (g, ϕ, ξ) is called an *almost contact metric structure* on M if the following equations hold for any $X, Y \in \Gamma(TM)$:

$$\begin{aligned}&\phi\,\xi = 0, \quad g(\xi, \xi) = 1,\\ &\phi^2 X = -X + g(X, \xi)\xi,\\ &g(\phi X, Y) + g(X, \phi Y) = 0.\end{aligned}$$

An almost contact metric structure on M is called a *Sasakian structure* if

$$(\nabla^g_X \phi)Y = g(Y, \xi)X - g(Y, X)\xi \tag{1}$$

holds for any $X, Y \in \Gamma(TM)$. We call a manifold equipped with a Sasakian structure a *Sasakian manifold*.

It is known that on a Sasakian manifold the formula

$$\nabla^g_X \xi = \phi X \tag{2}$$

holds for $X \in \Gamma(TM)$. A typical example of a Sasakian manifold is a hypersphere of odd dimension as mentioned below.

We now review the basic notion of statistical manifolds to fix the notation (See [1] and references therein). Let ∇ be an affine connection of M, and $g \in \Gamma(TM^{(0,2)})$ a Riemannian metric. The pair (∇, g) is called a *statistical structure* on M if (i) $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ and (ii) $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$ hold for any $X, Y, Z \in \Gamma(TM)$. By definition, (∇^g, g) is a statistical structure on M.

We denote by R^∇ the curvature tensor field of ∇, and by ∇^* the dual connection of ∇ with respect to g, and set $S = S^{(\nabla,g)} \in \Gamma(TM^{(1,3)})$ as the mean of the curvature tensor fields of ∇ and of ∇^*, that is, for $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned}&R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,\\ &Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),\\ &S(X, Y)Z = \frac{1}{2}\{R^\nabla(X, Y)Z + R^{\nabla^*}(X, Y)Z\}.\end{aligned} \tag{3}$$

A statistical manifold (M, ∇, g) is called a *Hessian manifold* if $R^\nabla = 0$. If so, we have $R^{\nabla^*} = S = 0$ automatically.

For a statistical structure (∇, g) on M, we set $K = \nabla - \nabla^g$. Then the following hold:

$$\begin{aligned} K \in \Gamma(TM^{(1,2)}), \\ K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z) \end{aligned} \tag{4}$$

for any $X, Y, Z \in \Gamma(TM)$. Conversely, if K satisfies (4), the pair $(\nabla = \nabla^g + K, g)$ is a statistical structure on M.

The formula

$$S(X,Y)Z = R^g(X,Y)Z + [K_X, K_Y]Z \tag{5}$$

holds, where $R^g = R^{\nabla^g}$ is the curvature tensor field of the Levi-Civita connection of g.

For a statistical structure (∇, g), we often use the expression like $(\nabla = \nabla^g + K, g)$, and write $K_X Y$ by $K(X,Y)$.

Definition 1. A quadruplet $(\nabla = \nabla^g + K, g, \phi, \xi)$ is called a *Sasakian statistical structure* on M if (i) (g, ϕ, ξ) is a Sasakian structure and (ii) (∇, g) is a statistical structure on M, and (iii) $K \in \Gamma(TM^{(1,2)})$ for (∇, g) satisfies

$$K(X, \phi Y) + \phi K(X,Y) = 0 \quad \text{for} \quad X, Y \in \Gamma(TM). \tag{6}$$

These three conditions are paraphrased in the following three conditions ([3, Theorem 2.17]: (i') (g, ϕ, ξ) is an almost contact metric structure and (ii) (∇, g) is a statistical structure on M, and (iii') they satisfy

$$\nabla_X(\phi Y) - \phi \nabla^*_X Y = g(\xi, Y)X - g(X,Y)\xi, \tag{7}$$

$$\nabla_X \xi = \phi X + g(\nabla_X \xi, \xi)\xi. \tag{8}$$

We get the following formulas for a Sasakian statistical manifold:

$$K(X, \xi) = \lambda g(X, \xi)\xi, \quad g(K(X,Y), \xi) = \lambda g(X, \xi) g(Y, \xi), \tag{9}$$

where

$$\lambda = g(K(\xi, \xi), \xi). \tag{10}$$

Proposition 2. *For a Sasakian statistical manifold* $(M, \nabla, g, \phi, \xi)$*,*

$$S(X,Y)\xi = g(Y,\xi)X - g(X,\xi)Y \tag{11}$$

holds for $X, Y \in \Gamma(TM)$*.*

Proof. By (9), we have $[K_X, K_Y]\xi = 0$, from which (5) implies $S = R^g$. It is known that R^g is written as the right hand side of (11) (See [5]). □

A quadruplet $(\widetilde{M}, \widetilde{\nabla} = \nabla^{\widetilde{g}} + \widetilde{K}, \widetilde{g}, \widetilde{J})$ is called a *holomorphic statistical manifold* if $(\widetilde{g}, \widetilde{J})$ is a Kähler structure, $(\widetilde{\nabla}, \widetilde{g})$ is a statistical structure on $\widetilde{M}$, and

$$\widetilde{K}(X, \widetilde{J}Y) + \widetilde{J}\widetilde{K}(X,Y) = 0 \tag{12}$$

holds for $X, Y \in \Gamma(T\widetilde{M})$. The notion of Sasakian statistical manifold can be also expressed in the following: The *cone* over M defined below is a holomorphic statistical manifold. Let $(M, \nabla = \nabla^g + K, g, \phi, \xi)$ be a statistical manifold with an almost contact metric structure. Set $\widetilde{M}$ as $M \times \mathbb{R}_+$, and define a Riemannian metric $\widetilde{g} = r^2 g + (dr)^2$ on $\widetilde{M}$. Take a vector field $\Psi = r\dfrac{\partial}{\partial r} \in \Gamma(T\widetilde{M})$, and define $\widetilde{J} \in \Gamma(T\widetilde{M}^{(1,1)})$ by $\widetilde{J}\Psi = \xi$ and $\widetilde{J}X = \phi X - g(X, \xi)\Psi$ for any $X \in \Gamma(TM)$. Then, $(\widetilde{g}, \widetilde{J})$ is an almost Hermitian structure on $\widetilde{M}$, and furthermore, (g, ϕ, ξ) is a Sasakian structure on M if and only if $(\widetilde{g}, \widetilde{J})$ is a Kähler structure on $\widetilde{M}$. We construct connection $\widetilde{\nabla}$ on $\widetilde{M}$ by

$$\begin{cases} \widetilde{\nabla}_\Psi \Psi = -\lambda\xi + \Psi, \\ \widetilde{\nabla}_X \Psi = \widetilde{\nabla}_\Psi X = X - \lambda g(X, \xi)\Psi, \\ \widetilde{\nabla}_X Y = \nabla_X Y - g(X, Y)\Psi, \end{cases}$$

that is,

$$\widetilde{K}(\Psi, \Psi) = -\lambda\xi, \quad \widetilde{K}(X, \Psi) = -\lambda g(X, \xi)\Psi, \quad \widetilde{K}(X, Y) = K(X, Y)$$

for $X, Y \in \Gamma(TM)$, where λ is in (10). We then have that $(M, \nabla, g, \phi, \xi)$ is a Sasakian statistical manifold if and only if $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{J})$ is a holomorphic statistical manifold (A general statement is given as [2, Proposition 4.8 and Theorem 4.10]). It is derived from the fact that the formula (12) holds if and only if both (6) and (9) hold.

Example 3. Let S^{2n-1} be a unit hypersphere in the Euclidean space $\mathbb{R}^{2n}$. Let J be a standard almost complex structure on $\mathbb{R}^{2n}$ considered as $\mathbb{C}^n$, and set $\xi = -JN$, where N is a unit normal vector field of S^{2n-1}. Define $\phi \in \Gamma(T(S^{2n-1})^{(1,1)})$ by $\phi(X) = JX - \langle JX, N\rangle N$. Denote by g the standard metric of the hypersphere. Then such a (g, ϕ, ξ) is known as a standard Sasakian structure on S^{2n-1}. We set

$$K(X, Y) = g(X, \xi)g(Y, \xi)\xi \tag{13}$$

for any $X, Y \in \Gamma(TS^{2n-1})$. Since K satisfies (4) and (6), we have a Sasakian statistical structure $(\nabla = \nabla^g + K, g, \phi, \xi)$ on S^{2n-1}.

Proposition 4. *Let (M, g, ϕ, ξ) be a Sasakian manifold. Set ∇ as $\nabla^g + fK$ for $f \in C^\infty(M)$, where K is given in (13). Then (∇, g, ϕ, ξ) is a Sasakian statistical structure on M. Conversely, we define $\nabla_X Y = \nabla^g_X Y + L(X, Y)V$ for some unit vector field V and $L \in \Gamma(TM^{(0,2)})$. If (∇, g, ϕ, ξ) is a Sasakian statistical structure, then $L \otimes V$ is written as $L(X, Y)V = fg(X, \xi)g(Y, \xi)\xi$ for some $f \in C^\infty(M)$, as above.*

Proof. The first half is obtained by direct calculation. To get the second half, we have by (4),

$$\begin{aligned} 0 &= L(X, Y)V - L(Y, X)V = \{L(X, Y) - L(Y, X)\}V, \\ 0 &= g(L(X, Y)V, Z) - g(Y, L(X, Z)V) = g(L(X, Y)Z - L(X, Z)Y, V). \end{aligned} \tag{14}$$

Substituting V for Z in (14), we have

$$L(X,Y) = L(V,V)g(X,V)g(Y,V).$$

Accordingly, we get by (6),

$$0 = L(X,\phi Y)V + \phi\{L(X,Y)V\} = L(V,V)g(X,V)\{-g(Y,\phi V)V + g(Y,V)\phi V\},$$

which implies that $\phi V = 0$ if $L(V,V) \neq 0$, and hence $V = \pm\xi$. □

3 Invariant Submanifolds

Let $(\widetilde{M}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a Sasakian manifold, and M a submanifold of $\widetilde{M}$. We say that M is an *invariant submanifold* of $\widetilde{M}$ if (i) $\widetilde{\xi}_u \in T_uM$, (ii) $\widetilde{\phi}X \in T_uM$ for any $X \in T_uM$ and $u \in M$. Let $g \in \Gamma(TM^{(0,2)})$, $\phi \in \Gamma(TM^{(1,1)})$ and $\xi \in \Gamma(TM)$ be the restriction of $\widetilde{g}$, $\widetilde{\phi}$ and $\widetilde{\xi}$, respectively. Then it is shown that (g,ϕ,ξ) is a Sasakian structure on M.

A typical example of an invariant submanifold of a Sasakian manifold S^{2n-1} in Example 3 is an odd dimensional unit sphere. Furthermore, we have the following example. Let $\iota : Q \to \mathbb{C}P^{n-1}$ be a complex hyperquadric in the complex projective space, and $\widetilde{Q}$ the principal fiber bundle over Q induced by ι from the Hopf fibration $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$. We denote the induced homomorphism by $\widetilde{\iota} : \widetilde{Q} \to S^{2n-1}$. Then it is known that $\widetilde{\iota}(\widetilde{Q})$ is an invariant submanifold (See [4], [5]).

We briefly review the statistical submanifold theory to study invariant submanifolds of a Sasakian statistical manifold. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold, and M a submanifold of $\widetilde{M}$. Let g be the metric on M induced from $\widetilde{g}$, and consider the orthogonal decomposition with respect to $\widetilde{g}$: $T_u\widetilde{M} = T_uM \oplus T_uM^{\perp}$. According to this decomposition, we define an affine connection ∇ on M, $B \in \Gamma(TM^{\perp} \otimes TM^{(0,2)})$, $A \in \Gamma((TM^{\perp})^{(0,1)} \otimes TM^{(1,1)})$, and a connection $\nabla^{\perp}$ of the vector bundle $TM^{\perp}$ by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{15}$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. Then (∇, g) is a statistical structure on M. In the same fashion, we define an affine connection ∇^* on M, $B^* \in \Gamma(TM^{\perp} \otimes TM^{(0,2)})$, $A^* \in \Gamma((TM^{\perp})^{(0,1)} \otimes TM^{(1,1)})$, and a connection $(\nabla^{\perp})^*$ of $TM^{\perp}$ by using th dual connection $\widetilde{\nabla}^*$ instead of $\widetilde{\nabla}$ in (15).

We remark that $\widetilde{g}(B(X,Y),N) = g(A^*_N X, Y)$ for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, and remark that ∇^* coincides with the dual connection of ∇ with respect to g. See [1] for example.

Theorem 5. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a Sasakian statistical manifold, and M an invariant submanifold of $\widetilde{M}$ with $g, \phi, \xi, \nabla, B, A, \nabla^{\perp}, \nabla^*, B^*, A^*, (\nabla^{\perp})^*$ defined as above. Then the following hold:*

(i) *A quintuplet $(M, \nabla, g, \phi, \xi)$ is a Sasakian statistical manifold.*
(ii) $B(X,\xi) = B^*(X,\xi) = 0$ *for any $X \in \Gamma(TM)$.*

(iii) $B(X, \phi Y) = B(\phi X, Y) = \widetilde{\phi} B^*(X, Y)$ *for any* $X, Y \in \Gamma(TM)$*. In particular,* $\mathrm{tr}_g B = \mathrm{tr}_g B^* = 0$*.*
(iv) *If* B *is parallel with respect to the Van der Weaden-Bortolotti connection* $\widetilde{\nabla}'$ *for* $\widetilde{\nabla}$*, then* B *and* B^* *vanish. Namely, if* $(\widetilde{\nabla}'_X B)(Y, Z) = \nabla^{\perp}_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) = 0$ *for* $Z \in \Gamma(TM)$*, then* $B^*(X, Y) = 0$*.*
(v) $\widetilde{g}(\widetilde{S}(X, \widetilde{\phi} X)\widetilde{\phi} X - S(X, \phi X)\phi X, X) = 2\widetilde{g}(B^*(X, X), B(X, X))$ *for* $X \in \Gamma(TM)$*, where* $S = S^{(\nabla, g)}$ *and* $\widetilde{S} = S^{(\widetilde{\nabla}, \widetilde{g})}$ *as in (3).*

Corollary 6. *Let* $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ *be a Sasakian statistical manifold of constant* $\widetilde{\phi}$*-sectional curvature* c*, and* M *an invariant submanifold of* $\widetilde{M}$*. The induced Sasakian statistical structure on* M *has constant* ϕ*-sectional curvature* c *if and only if* $\widetilde{g}(B^*(X, X), B(X, X)) = 0$ *for any* $X \in \Gamma(TM)$ *orthogonal to* ξ*.*

If we take the Levi-Civita connection as $\widetilde{\nabla}$, the properties above reduce to the ones for an invariant submanifold of a Sasakian manifold. It is known that an invariant submanifold of a Sasakian manifold of constant $\widetilde{\phi}$-sectional curvature c is of constant ϕ-sectional curvature c if and only if it is totally geodesic. It is obtained by setting $B = B^*$ in Corollary 6. It is an interesting question whether there is an interesting invariant submanifold having nonvanishing B with the above property.

Outline of Proof of Theorem 5. The proof of (i) can be omitted.

By (i) and (8), we calculate that $\nabla_X \xi + B(X, \xi) = \widetilde{\nabla}_X \xi = \widetilde{\phi} X + \widetilde{g}(\widetilde{\nabla}_X \xi, \widetilde{\xi})\widetilde{\xi} = \phi X + g(\nabla_X \xi, \xi)\xi$. Comparing the normal components, we have (ii).

By (7), we have $\widetilde{g}(Y, \widetilde{\xi})X - \widetilde{g}(Y, X)\widetilde{\xi} = \widetilde{\nabla}_X(\widetilde{\phi} Y) - \widetilde{\phi}\widetilde{\nabla}^*_X Y = \nabla_X(\phi Y) + B(X, \phi Y) - \widetilde{\phi}(\nabla^*_X Y + B^*(X, Y)) = g(Y, \xi)X - g(Y, X)\xi + B(X, \phi Y) - \widetilde{\phi} B^*(X, Y)$. Comparing the normal components, we have (iii).

By (i) and (ii), we get that $0 = \nabla^{\perp}_X B(Y, \xi) - B(\nabla_X Y, \xi) - B(Y, \nabla_X \xi) = -B(Y, \phi X) = -\widetilde{\phi} B^*(X, Y)$, which implies (iv).

To get (v), we use the Gauss equation in the submanifold theory. The tangential component of $R^{\widetilde{\nabla}}(X, Y)Z$ is given as

$$R^{\nabla}(X, Y)Z - A_{B(Y,Z)}X + A_{B(X,Z)}Y,$$

for $X, Y, Z \in \Gamma(TM)$, which implies that

$$\begin{aligned} 2\widetilde{g}(\widetilde{S}(X, Y)Z, W) = {} & 2g(S(X, Y)Z, W) \\ & - \widetilde{g}(B^*(X, W), B(Y, Z)) + \widetilde{g}(B^*(Y, W), B(X, Z)) \\ & - \widetilde{g}(B(X, W), B^*(Y, Z)) + \widetilde{g}(B(Y, W), B^*(X, Z)). \end{aligned}$$

Therefore, we prove (v) from (iii). □

To get Corollary 6, we have only to review the definition. A Sasakian statistical structure (∇, g, ϕ, ξ) is said to be of constant ϕ-sectional curvature c if the sectional curvature defined by using S equals c for each ϕ-section, the plane spanned by X and ϕX for a unit vector X orthogonal to ξ: $g(S(X, \phi X)\phi X, X) = cg(X, X)^2$ for $X \in \Gamma(TM)$ such that $g(X, \xi) = 0$.

Acknowledgments. The author thanks the anonymous reviewers for their careful reading of the manuscript. This work was supported by JSPS KAKENHI Grant Number JP26400058.

References

1. Furuhata, H., Hasegawa, I.: Submanifold theory in holomorphic statistical manifolds. In: Dragomir, S., Shahid, M.H., Al-Solamy, F.R. (eds.) Geometry of Cauchy-Riemann Submanifolds, pp. 179–215. Springer, Singapore (2016). doi:10.1007/978-981-10-0916-7_7
2. Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K.: Kenmotsu statistical manifolds and warped product. J. Geom. To appear. doi:10.1007/s00022-017-0403-1
3. Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., Shahid, M.H.: Sasakian statistical manifolds. J. Geom. Phys. **117**, 179–186 (2017). doi:10.1016/j.geomphys.2017.03.010
4. Kenmotsu, K.: Invariant submanifolds in a Sasakian manifold. Tohoku Math. J. **21**, 495–500 (1969)
5. Yano, K., Kon, M.: Structures on Manifolds. World Scientific, Singapore (1984)