Chapter 7
The Classical Lagrangians of Gauge Theories

If we consider, from an abstract point of view, a field theory involving several types of fields on spacetime (scalar fields, gauge fields, spinors, etc.), then the **Lagrangian** of the field theory is the formula that contains the dynamics and all interactions between these fields. In classical field theory, the equations of motion, i.e. the field equations, that govern the evolution of the fields over time, are derived from the Lagrangian. In quantum field theory, the Lagrangian (through the **action**, the integral of the Lagrangian over spacetime) enters the formula for path integrals that are used to calculate correlators and scattering amplitudes for elementary particles.

Given that the structure of the common Lagrangians is quite simple, it is truly remarkable that the enormous complexity and intricacy of quantum field theories are already contained in the Lagrangians. The Lagrangians can be considered the fundamental cornerstones of field theories.

Lagrangians can be categorized depending on which types of fields and interactions they involve: there are Lagrangians for free fields, Lagrangians for a single interacting field and Lagrangians for several interacting fields. As a general rule, Lagrangians which are **harmonic**, i.e. quadratic in the fields, correspond to free theories, while Lagrangians which contain **anharmonic** terms of order three or higher in the fields lead in the quantum field theory to the creation and annihilation of particles and thus to interactions. **Interactions between fields** (in particular, in the case of weakly interacting, perturbative quantum field theories) are depicted using **Feynman diagrams**. Interacting quantum field theories are usually very complicated and in many cases (including the Standard Model) not fully understood.

There are *a priori* countless Lagrangians that one could consider for a given set of fields. The Lagrangians that are important in physics are mainly restricted by three principles:

1. Existence of symmetries.
2. The quantum field theory should be renormalizable.
3. The quantum field theory should be free of gauge anomalies.

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We will briefly discuss how these principles restrict the possible Lagrangians and then study the Lagrangians that appear in the Standard Model of elementary particles. These Lagrangians are called:

- the Yang–Mills Lagrangian
- the Klein–Gordon and Higgs Lagrangian
- the Dirac Lagrangian
- Yukawa coupling (itself not a complete dynamic Lagrangian)

The Lagrangians in the Standard Model are all Lorentz invariant and gauge invariant. Lorentz invariance here means invariance under local Lorentz transformations of the spacetime manifold, acting on each tangent space. This implies that for fixed values of the fields the Lagrangian is a scalar function on spacetime. Lorentz invariance for a field theory involving spinors always means invariance under the orthochronous Lorentz spin group.

There are numerous books and articles on field theory and the Standard Model. In the present and the following chapter on the Standard Model we mainly rely on the following references:

- The book [16] by David Bleecker is one of the best mathematical treatments of symmetry breaking. Our discussion of symmetry breaking and the Higgs mechanism in Chap. 8 draws heavily from it.
- The article [9] by John C. Baez and John Huerta is an excellent mathematical exposition of the representations of the Standard Model and Grand Unified Theories. Our notation for the representations of the Standard Model in Chap. 8 mainly follows this reference.
- The book [100] by Ulrich Mosel is a concise summary of the Standard Model with a very good exposition of the Lagrangians and symmetry breaking. The explicit Lagrangians for the Standard Model that we derive in Chap. 8 mainly follow the notation in Mosel's book.
- The book [137] by Mark Thomson is an excellent modern and readable treatment of particle physics with many interesting details and explanations concerning experimental and theoretical aspects.
- The topic of the book [22] by Gustavo Castelo Branco, Luís Lavoura and João Paulo Silva is CP violation, but it also has a very clear description of the Standard Model and its Lagrangians.
- The book [62] by Carlo Giunti and Chung W. Kim focuses mainly on neutrino physics, but also contains in the first chapters a concise and modern description of the Standard Model, including details about the Lagrangians and quark mixing.
- Our main references for results from quantum field theory are the book [125] by Matthew Schwartz and the books [143–145] by Steven Weinberg.
- The website [105] of the Particle Data Group contains many up-to-date experimental values for elementary particles as well as some succinct theoretical discussions.

- A great source for the history and development of the Standard Model is the book [79] by Lillian Hoddeson, Laurie Brown, Michael Riordan and Max Dresden (editors).[1] Historical remarks can also be found on the official website [117] for the Nobel Prize in Physics. In particular, [118–120] and [121] contain very readable background material for the Nobel Prizes in Physics 2004, 2008, 2013 and 2015. A short history of gauge theory in physics and mathematics can be found in the book review [92] for Bleecker's book.

Further references are the books [33, 71] and [113] (on the complete Standard Model), [112] (on the electroweak theory), [42] (on QCD), [124, 132] (on QFT in general) and [39, 41, 101, 102, 114, 122, 123] (on the mathematics of the Standard Model) as well as the lecture notes [141].

7.1 Restrictions on the Set of Lagrangians

The Lagrangians that occur in physics are restricted from the infinite set of possible Lagrangians by certain principles that we want to discuss in this section.

7.1.1 Existence of Symmetries

The Lagrangian (or the action) of a field theory should be invariant under certain transformations of the fields, i.e. under certain symmetry groups. Particular examples are:

- Lorentz symmetry
- gauge symmetry
- conformal symmetry
- supersymmetry

We have to distinguish two meanings of symmetries in field theories. Here we think of the primary meaning: the Lagrangian for the fields and thus the *laws of physics*, not the field configurations or their initial values themselves, are invariant under symmetry transformations.

The secondary meaning of symmetry (invariance of the *actual field configuration*) is also sometimes of significance in physics. For example, the action of general relativity for the spacetime metric is invariant under the full orientation preserving diffeomorphism group of the spacetime manifold. A specific metric, however, is invariant only under a much smaller symmetry group, the *isometry group* of this metric (which could just consist of a single element, the identity map).

[1] I thank Anthony Britto for pointing out this reference.

Similarly the actions of supergravity theories are invariant under all local supersymmetries, but a specific supersymmetric configuration is invariant under a much smaller group of supersymmetries (a generic field configuration is not supersymmetric at all).

A third example is spontaneously broken gauge theories, which we consider in detail in Chap. 8. In this case the Lagrangian is invariant under gauge transformations with values in a Lie group G, but due to the existence of the Higgs condensate, the vacuum configuration is invariant only under gauge transformations with values in a subgroup $H \subset G$.

The existence of **gauge symmetries** is particularly important: it can be shown that a quantum field theory involving massless spin 1 bosons can be consistent (i.e. unitary, see Sect. 7.1.3) only if it is gauge invariant [125, 143]. This is the reason why we demand Lagrangians involving vector fields (or 1-forms) to be invariant under gauge transformations.

7.1.2 The Quantum Field Theory Should Be Renormalizable

The quantum field theory associated to the Lagrangian should be renormalizable to yield in the end (after renormalization of the parameters, such as coupling constants and masses, cf. Sect. B.2.8) finite results that can be compared with experiments and used to adjust the free parameters of the theory. A simple calculation of the *mass dimension* of summands in the Lagrangian determines which terms have a chance to yield renormalizable theories.

For example, let

$$\mathscr{L} = \mathscr{L}(\phi_1, \ldots, \phi_n)$$

be a renormalizable Lagrangian, where $\phi_1, \ldots, \phi_n$ denote certain fields on spacetime (not necessarily scalars). Suppose that $\mathscr{L}$ is Lorentz invariant and, say, gauge invariant (for instance, $\mathscr{L}$ could be the Yang–Mills Lagrangian or the Klein–Gordon Lagrangian). Then for all natural numbers k the k-th power $\mathscr{L}^k$ will also be Lorentz invariant and gauge invariant. However, in almost all cases, for $k \geq 2$, the Lagrangian $\mathscr{L}^k$ will be non-renormalizable, because it has the wrong mass dimension.

Demanding that the quantum field theory is renormalizable thus greatly restricts the possible terms that can appear in Lagrangians. Calculating mass dimensions (power counting), it can be shown that in 4-dimensional spacetime the only renormalizable and gauge invariant Lagrangians are sums of the Lagrangians that we discuss in this chapter[2] (where in the case of the Higgs Lagrangian for a scalar

[2]An exception, that we do not discuss in this book, is the *topological theta term* $\langle F_M^A, *F_M^A \rangle_{\mathrm{Ad}(P)}$, that appears in some modifications of QCD and in supersymmetric gauge theories.

field ϕ the potential has to be a polynomial in ϕ of degree less than or equal to 4). See [143, Sect. 12.3] for details.

This is very satisfying, because it means, from the point of view of quantum field theory in 4-dimensional spacetime, that there will be no additional types of interactions. It also turns out that all of the allowed Lagrangians actually appear in the Standard Model (in a certain specific form, i.e. with a specific gauge group G, specific charged fermions, etc.). The restriction of renormalizability does not hold for *effective* Lagrangians, i.e. Lagrangians that are only used for calculations at low energies.

7.1.3 The Quantum Field Theory Should Be Free of Gauge Anomalies

Symmetries of the classical field theory, like gauge symmetries, do not necessarily hold in the quantum field theory. The reason is that the measure involved in the definition of path integrals may not be invariant under the symmetry. If this happens, the symmetry is called **anomalous**.

In quantum theory, we demand that the Hilbert space of the system does not contain both vectors of positive norm and negative norm (states of negative norm are called **ghost states**). This property is sometimes called **unitarity** (a vector space with a positive definite Hermitian scalar product is also known as a unitary vector space).[3] If unitarity does not hold, i.e. there exist both states of positive and negative norm in the Hilbert space, then the scalar product does not have a probability interpretation (see Exercise 7.9.1), violating a fundamental axiom of quantum theory.

It is possible to show that in 4-dimensional Minkowski spacetime, anomalies of gauge symmetries imply that the quantum theory violates unitarity (this is related to the fact that the Lorentz metric is indefinite and that the scalar product on the Hilbert space of the quantum field theory must be Poincaré invariant; see [125, Chap. 8] for details). It follows that the quantum theory has to be free of gauge anomalies. In practice, this restricts the possible representations and charges of the fermions: the contributions of the fermions in the theory to the gauge anomaly depend on both the gauge groups and fermion representations and have to cancel each other. The Standard Model is anomaly free, see Sect. 8.5.8. For more details on anomalies, see [125, Chap. 30].

[3]There is another concept of unitarity (unitarity of the S-matrix, i.e. of time evolution) that we do not consider here.

One therefore has to be careful: even if a gauge theory is well-defined on the classical level, this may not be true for the associated quantum theory. In particular, vanishing of gauge anomalies has to be checked for every theory beyond the Standard Model, like Grand Unified Theories or supersymmetric extensions.

7.1.4 The Lagrangian of the Standard Model

Our aim in this chapter is to understand each term in the following Lagrangian, which is essentially the Lagrangian of the Standard Model and could be called the **Yang–Mills–Dirac–Higgs–Yukawa Lagrangian**:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_D[\Psi, A] + \mathscr{L}_H[\Phi, A] + \mathscr{L}_Y[\Psi_L, \Phi, \Psi_R] + \mathscr{L}_{YM}[A] \\ &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) + \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - 2g_Y \mathrm{Re}\left(\overline{\Psi}_L \Phi \Psi_R\right) - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.\end{aligned}$$

7.2 The Hodge Star and the Codifferential

Throughout this chapter, (M, g) is an n-dimensional oriented pseudo-Riemannian manifold. In physics, M is spacetime and g usually has Lorentzian signature. In mathematics, M is an arbitrary manifold and g is often taken to be Riemannian.

We first want to understand the *Yang–Mills Lagrangian* $\mathscr{L}_{YM}[A]$ for a connection A on a principal bundle $P \to M$ and derive the associated equation of motion, called the *Yang–Mills equation*. This equation is most easily stated using the *codifferential*, whose definition involves the *Hodge star operator*. The metric g on the manifold M enters the Yang–Mills equation precisely through the Hodge star. In this section we discuss as a mathematical preparation the Hodge star operator, the codifferential and some related concepts. We follow the exposition in [14].

7.2.1 Scalar Products on Forms and the Hodge Star Operator

The metric g together with the orientation of the manifold M define a **canonical volume form** dvol_g on M: If $e_1, \ldots, e_n$ is an oriented, orthonormal basis of T_pM, then dvol_g is characterized by

$$\mathrm{dvol}_g(e_1, \ldots, e_n) = +1.$$

Lemma 7.2.1 *If (U, ϕ) is an oriented chart for M with local coordinates x^μ, then*

$$\mathrm{dvol}_g = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n,$$

where

$$|g| = |\det(g_{\mu\nu})|$$

is the absolute value of the determinant of the matrix with entries

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu).$$

Proof This is Exercise 7.9.2. □

We denote by $g^{\mu\nu}$ the entries of the matrix inverse to the matrix with entries $g_{\mu\nu}$. We can raise indices of tensors in the standard way using $g^{\mu\nu}$. For example,

$$T^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma},$$

where the Einstein summation convention is understood.

The semi-Riemannian metric g on M defines bundle metrics on the vector bundles of k-forms $\Lambda^k T^*M$ for all k. This yields scalar products between sections of these bundles that we can write explicitly as follows:

Definition 7.2.2 For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we define the **scalar product of forms**

$$\langle \cdot, \cdot \rangle \colon \Omega^k(M, \mathbb{K}) \times \Omega^k(M, \mathbb{K}) \longrightarrow \mathscr{C}^\infty(M, \mathbb{K})$$

as follows: for real-valued k-forms $\omega, \eta \in \Omega^k(M, \mathbb{R})$ on M we set

$$\begin{aligned}
\langle \omega, \eta \rangle &= \sum_{\mu_1 < \ldots < \mu_k} \omega_{\mu_1 \ldots \mu_k} \eta^{\mu_1 \ldots \mu_k} \\
&= \frac{1}{k!} \sum_{\mu_1 \ldots \mu_k} \omega_{\mu_1 \ldots \mu_k} \eta^{\mu_1 \ldots \mu_k} \\
&= \frac{1}{k!} \omega_{\mu_1 \ldots \mu_k} \eta^{\mu_1 \ldots \mu_k},
\end{aligned}$$

where

$$\omega_{\mu_1 \ldots \mu_k} = \omega(\partial_{\mu_1}, \ldots, \partial_{\mu_k})$$

in a local chart (U, ϕ) of M and the second and third sum extend over all k-tuples $\mu_1 \ldots \mu_k$.

For complex-valued k-forms $\omega, \eta \in \Omega^k(M, \mathbb{C}) \cong \Omega^k(M, \mathbb{R}) \otimes \mathbb{C}$ on M we set

$$\langle \omega, \eta \rangle = \sum_{\mu_1 < \ldots < \mu_k} \bar{\omega}_{\mu_1 \ldots \mu_k} \eta^{\mu_1 \ldots \mu_k}.$$

These scalar products are well-defined, independent of the choice of local chart. The associated **norm** is given in both cases by

$$|\omega|^2 = \langle \omega, \omega \rangle.$$

Remark 7.2.3 On a pseudo-Riemannian manifold the norm is in general not positive definite. In particular, $|\omega|^2 = 0$ does not imply $\omega = 0$. For this reason we usually try to avoid the notation $|\omega|^2$.

Definition 7.2.4 The **Hodge star operator**

$$*\colon \Omega^k(M, \mathbb{K}) \longrightarrow \Omega^{n-k}(M, \mathbb{K})$$

is the linear map defined for real-valued forms by

$$\langle \omega, \eta \rangle \mathrm{dvol}_g = \omega \wedge *\eta \quad \forall \omega, \eta \in \Omega^k(M, \mathbb{R})$$

and for complex-valued forms by

$$\langle \omega, \eta \rangle \mathrm{dvol}_g = \bar{\omega} \wedge *\eta \quad \forall \omega, \eta \in \Omega^k(M, \mathbb{C}).$$

Choosing a local frame, it can be shown that this uniquely defines $*$.

Remark 7.2.5 This definition of the Hodge star operator for pseudo-Riemannian manifolds does not necessarily coincide with the definition sometimes found in the literature. Baum [14], for instance, uses the definition

$$*' = (-1)^t * .$$

We continue to use our definition.

Suppose $e_1, \ldots, e_n$ is an oriented, orthonormal basis of tangent vectors with

$$g(e_i, e_i) = g_{ii} = g^{ii} = \pm 1.$$

Let $\alpha^1, \ldots, \alpha^n$ be the dual basis of 1-forms with $\alpha^i(e_j) = \delta^i_j$. Then

$$\mathrm{dvol}_g = \alpha^1 \wedge \ldots \wedge \alpha^n$$

and we have:

Lemma 7.2.6 *The Hodge star operator is given by*

$$*(\alpha^{m_1} \wedge \ldots \wedge \alpha^{m_k}) = g^{m_1 m_1} \cdots g^{m_k m_k} \epsilon_{m_1 \ldots m_k m_{k+1} \ldots m_n} \alpha^{m_{k+1}} \wedge \ldots \wedge \alpha^{m_n}.$$

In this formula there is on the right-hand side ***no*** *summation over indices, $\{m_{k+1}, \ldots, m_n\}$ is a complementary set to $\{m_1, \ldots, m_k\}$ and ϵ is totally antisymmetric with*

$$\epsilon_{123\ldots n} = 1.$$

In particular,

$$*\mathrm{dvol}_g = (-1)^t \cdot 1,$$
$$*1 = \mathrm{dvol}_g.$$

Definition 7.2.7 Let $\Omega_0^k(M, \mathbb{K})$ denote the differential forms with compact support on M. Then we define the **L^2-scalar product of forms**

$$\langle \cdot , \cdot \rangle_{L^2} \colon \Omega_0^k(M, \mathbb{K}) \times \Omega_0^k(M, \mathbb{K}) \longrightarrow \mathbb{K}$$

by

$$\langle \omega, \eta \rangle_{L^2} = \int_M \langle \omega, \eta \rangle \mathrm{dvol}_g.$$

We have to restrict the L^2-scalar product to forms with compact support, because otherwise the integral may not be finite.

We can generalize these constructions to twisted differential forms. Suppose that $E \to M$ is a $\mathbb{K}$-vector bundle with bundle metric $\langle \cdot , \cdot \rangle_E$. Together with the semi-Riemannian metric g we then get induced bundle metrics on the vector bundle $\Lambda^k T^*M \otimes E$ of twisted k-forms for all k. More explicitly we can write:

Definition 7.2.8 We define the **scalar product of twisted forms**

$$\langle \cdot , \cdot \rangle_E \colon \Omega^k(M, E) \times \Omega^k(M, E) \longrightarrow \mathscr{C}^\infty(M)$$

as follows: choose a local frame $e_1, \ldots, e_r$ for E over $U \subset M$ and expand k-forms F, G twisted with E as

$$F = \sum_{i=1}^r F_i \otimes e_i,$$

$$G = \sum_{j=1}^r G_j \otimes e_j,$$

with $F_i, G_j \in \Omega^k(U, \mathbb{K})$. Then we set

$$\langle F, G\rangle_E = \sum_{i,j=1}^{r} \langle F_i, G_j\rangle \langle e_i, e_j\rangle_E.$$

This scalar product is independent of the choice of local frame $\{e_i\}$.

We can also define a **Hodge star operator on twisted forms**

$$*\colon \Omega^k(M, E) \longrightarrow \Omega^{n-k}(M, E)$$

by

$$*F = \sum_{i=1}^{r} (*F_i) \otimes e_i$$

and an L^2**-scalar product of twisted forms**

$$\langle \cdot\,, \cdot\rangle_{E,L^2}\colon \Omega_0^k(M, E) \times \Omega_0^k(M, E) \longrightarrow \mathbb{K}$$

by

$$\langle \omega, \eta\rangle_{E,L^2} = \int_M \langle \omega, \eta\rangle_E \mathrm{dvol}_g.$$

7.2.2 The Codifferential

Let (M, g) be an oriented semi-Riemannian manifold of dimension n and signature (s, t). We have the usual exterior differential

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

on forms.

Definition 7.2.9 We define the **codifferential**

$$d^*\colon \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$$

by

$$d^* = (-1)^{t+nk+1} * d * .$$

The codifferential has the following interesting property:

Theorem 7.2.10 (Codifferential on Forms Is Formal Adjoint of Differential) *Let M be a manifold without boundary. Then the codifferential d^* is the formal adjoint of the differential d with respect to the L^2-scalar product on forms with compact support, i.e.*

$$\langle d\omega, \eta\rangle_{L^2} = \langle \omega, d^*\eta\rangle_{L^2}$$

for all $\omega \in \Omega_0^k(M)$, $\eta \in \Omega_0^{k+1}(M)$.

Proof We calculate the difference

$$\langle d\omega, \eta\rangle - \langle \omega, d^*\eta\rangle$$

with respect to the (pointwise) scalar product of forms. According to Exercise 7.9.3

$$**\colon \Omega^{n-k}(M) \longrightarrow \Omega^{n-k}(M)$$

is given by

$$** = (-1)^{t+(n-k)k}.$$

We have

$$\begin{aligned}\big(\langle d\omega, \eta\rangle - \langle \omega, d^*\eta\rangle\big)\,\mathrm{dvol}_g &= (d\omega) \wedge *\eta - \omega \wedge *\left(d^*\eta\right) \\ &= (d\omega) \wedge *\eta + (-1)^k \omega \wedge (d * \eta) \\ &= d(\omega \wedge *\eta).\end{aligned}$$

This implies the claim by Stokes' Theorem A.2.24. □

We want to generalize the definition of the codifferential to twisted forms. Let $E \to M$ be a $\mathbb{K}$-vector bundle with a scalar product $\langle\cdot,\cdot\rangle_E$ and a compatible covariant derivative ∇. In Sect. 5.12 we defined the associated exterior covariant derivative (or covariant differential)

$$d_\nabla\colon \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E).$$

Definition 7.2.11 We define the **covariant codifferential**

$$d_\nabla^*\colon \Omega^{k+1}(M,E) \longrightarrow \Omega^k(M,E)$$

by

$$d_\nabla^* = (-1)^{t+nk+1} * d_\nabla * .$$

We then get the following analogue of Theorem 7.2.10.

Theorem 7.2.12 (Covariant Codifferential on Twisted Forms Is Formal Adjoint of Covariant Differential) *Let M be a manifold without boundary. Then the covariant codifferential d_∇^* is the formal adjoint of the exterior covariant differential d_∇ with respect to the L^2-scalar product on forms with compact support, i.e.*

$$\langle d_\nabla \omega, \eta \rangle_{E,L^2} = \left\langle \omega, d_\nabla^* \eta \right\rangle_{E,L^2}$$

for all $\omega \in \Omega_0^k(M,E)$, $\eta \in \Omega_0^{k+1}(M,E)$.

Proof We follow the proof in [14]. Since d_∇ and d_∇^* are linear, it suffices to prove the statement for forms ω, η of the form

$$\omega = \sigma \otimes e, \quad \sigma \in \Omega_0^k(M), \quad e \in \Gamma(E),$$
$$\eta = \mu \otimes f, \quad \mu \in \Omega_0^{k+1}(M), \quad f \in \Gamma(E).$$

Then

$$d_\nabla \omega = (d\sigma) \otimes e + (-1)^k \sigma \wedge \nabla e$$

and

$$\begin{aligned} d_\nabla^* \eta &= (-1)^{t+nk+1} * d_\nabla * (\mu \otimes f) \\ &= (-1)^{t+nk+1} * \big((d * \mu) \otimes f + (-1)^{n-k-1} (*\mu) \wedge \nabla f \big) \\ &= (d^* \mu) \otimes f + (-1)^{t+nk+n-k} * ((*\mu) \wedge \nabla f) \,. \end{aligned}$$

In particular, with Exercise 7.9.3,

$$*d_\nabla^* \eta = -(-1)^k (d * \mu) \otimes f - (-1)^{n-1} (*\mu) \wedge \nabla f.$$

We introduce a scalar product

$$\langle \cdot\,, \cdot \rangle_E \colon \Omega^1(M,E) \otimes \Gamma(E) \longrightarrow \Omega^1(M)$$

by setting

$$\langle \omega \otimes a, b\rangle_E = \omega\langle a, b\rangle_E \quad \forall \omega \in \Omega^1(M), a, b \in \Gamma(E)$$

and extending linearly.

For the difference of the pointwise scalar products we then get

$$\begin{aligned}\left(\langle d_\nabla\omega, \eta\rangle_E - \left\langle \omega, d_\nabla^*\eta\right\rangle_E\right) \mathrm{dvol}_g &= \langle (d\sigma)\otimes e, \mu\otimes f\rangle_E + (-1)^k\langle \sigma\wedge\nabla e, \mu\otimes f\rangle_E \\ &\quad - \langle \sigma\otimes e, (d^*\mu)\otimes f\rangle_E \\ &\quad + (-1)^{n-1}(\sigma\wedge *\mu)\wedge\langle \nabla f, e\rangle_E \\ &= d(\sigma\wedge *\mu)\langle e, f\rangle_E \\ &\quad + (-1)^{n-1}(\sigma\wedge *\mu)\wedge(\langle\nabla e, f\rangle_E + \langle \nabla f, e\rangle_E) \\ &= d\left((\sigma\wedge *\mu)\langle e, f\rangle_E\right).\end{aligned}$$

In the final step we used that ∇ is compatible with the scalar product on E. The claim now follows by Stokes' Theorem A.2.24. □

7.3 The Yang–Mills Lagrangian

In this section we define the Yang–Mills Lagrangian and derive the associated Yang–Mills equation. We fix the following data:

- an n-dimensional oriented pseudo-Riemannian manifold (M, g)
- a principal G-bundle $P \to M$ with compact structure group G of dimension r
- an Ad-invariant positive definite scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, determined by certain coupling constants, as in Sect. 2.5
- a $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$-orthonormal vector space basis $T_1, \ldots, T_r$ for $\mathfrak{g}$.

The Ad-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ determines a bundle metric on the associated real vector bundle $\mathrm{Ad}(P) = P\times_{\mathrm{Ad}}\mathfrak{g}$ that we denote by $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)}$.

7.3.1 The Yang–Mills Lagrangian

Let A be a connection 1-form on the principal bundle P with curvature 2-form $F^A \in \Omega^2(P, \mathfrak{g})$. According to Corollary 5.13.5 the curvature defines a twisted 2-form

$$F_M^A \in \Omega^2(M, \mathrm{Ad}(P)).$$

Definition 7.3.1 The **Yang–Mills Lagrangian** is defined by

$$\mathscr{L}_{YM}[A] = -\frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}\,.$$

For a fixed connection A, the Yang–Mills Lagrangian is a global smooth function

$$\mathscr{L}_{YM}[A]\colon M \longrightarrow \mathbb{R}.$$

Theorem 7.3.2 *The Yang–Mills Lagrangian is gauge invariant, i.e.*

$$\mathscr{L}_{YM}[f^*A] = \mathscr{L}_{YM}[A]$$

for all bundle automorphisms $f \in \mathscr{G}(P)$ and all connections A on P.

Proof Theorem 5.4.4 implies that the curvature form $F^A \in \Omega^2(P, \mathfrak{g})$ transforms as

$$F^{f^*A} = \mathrm{Ad}_{\sigma_f^{-1}} \circ F^A.$$

Let $f\cdot$ denote the action of f on the adjoint bundle, given by Theorem 5.3.8. Then

$$F_M^{f^*A} = f^{-1} \cdot F_M^A.$$

Since the scalar product $\langle \cdot\,, \cdot\rangle_{\mathfrak{g}}$ is Ad-invariant, it follows that $\langle \cdot\,, \cdot\rangle_{\mathrm{Ad}(P)}$ is invariant under the action of f^{-1}. This implies the claim. □

We want to find a formula for the Yang–Mills Lagrangian in local coordinates and in a local gauge. Let $s\colon U \to P$ be a local gauge. Then the local field strength is given by

$$F_s^A = s^*F^A \in \Omega^2(U, \mathfrak{g}).$$

The scalar product on the Lie algebra $\mathfrak{g}$ defines a scalar product

$$\langle \cdot\,, \cdot\rangle_{\mathfrak{g}}\colon \Omega^2(U, \mathfrak{g}) \times \Omega^2(U, \mathfrak{g}) \longrightarrow \mathscr{C}^\infty(U, \mathfrak{g}).$$

As before we set in a chart with coordinates x^μ

$$F_{\mu\nu}^A = F_s^A(\partial_\mu, \partial_\nu).$$

We can expand

$$F_s^A = F_s^{Aa} \otimes T_a$$

and

$$F^A_{\mu\nu} = F^{Aa}_{\mu\nu} T_a,$$

where $F^{Aa}_s \in \Omega^2(U)$ are real-valued differential forms, $F^{Aa}_{\mu\nu} \in \mathscr{C}^\infty(U)$ are real-valued smooth functions on U and we sum over the indices a.

We can then write the Yang–Mills Lagrangian locally as

$$\begin{aligned}\mathscr{L}_{YM}[A] &= -\frac{1}{2}\left\langle F^A_s, F^A_s\right\rangle_{\mathfrak{g}} \\ &= -\frac{1}{4}\left\langle F^A_{\mu\nu}, F^{A\mu\nu}\right\rangle_{\mathfrak{g}} \\ &= -\frac{1}{4} F^{Aa}_{\mu\nu} F^{A\mu\nu}_a,\end{aligned}$$

where we sum over all μ, ν. The local field strength is given by

$$F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

and

$$F^{Aa}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bca} A^b_\mu A^c_\nu$$

with the **structure constants** defined by

$$[T_a, T_b] = \sum_{c=1}^{r} f_{abc} T_c.$$

Lemma 7.3.3 *The structure constants of the Lie algebra* $\mathfrak{g}$ *with respect to a* $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$*-orthonormal basis* $\{T_a\}$ *satisfy*

$$f_{abc} + f_{bac} = 0$$

and

$$f_{bca} + f_{bac} = 0$$

for all indices a, b, c. *In particular,*

$$f_{bca} = f_{abc}.$$

Proof The first claim is clear, because the Lie bracket is antisymmetric. The second claim follows, because the T_a are an orthonormal basis and the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$

on $\mathfrak{g}$ is Ad-invariant: we have

$$\langle [T_b, T_c], T_a \rangle_{\mathfrak{g}} + \langle T_c, [T_b, T_a] \rangle_{\mathfrak{g}} = 0,$$

which implies the claim. □

We can therefore also write the structure equation for the curvature as

$$F^{Aa}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu.$$

This implies the following explicit formula for the Yang–Mills Lagrangian:

$$\begin{aligned} \mathscr{L}_{YM}[A] &= -\frac{1}{4} F^{Aa}_{\mu\nu} F^{A\mu\nu}_a \\ &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) \\ &\quad - \frac{1}{2} f_{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} \\ &\quad - \frac{1}{4} f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}. \end{aligned} \tag{7.1}$$

The term in the second line is quadratic in the gauge field. It describes free (non-interacting) gauge bosons and is the only term if the group G is abelian. The terms in the third and fourth line are cubic and quartic in the gauge field and describe a direct interaction between the gauge bosons in non-abelian gauge theories. In the case of QCD these terms are called 3-gluon vertex and 4-gluon vertex. Figure 7.1 shows the Feynman diagrams for these vertices.

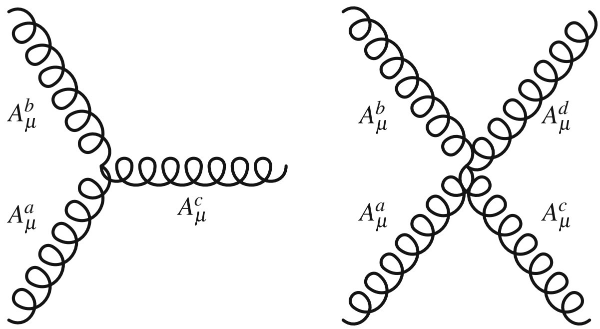

Fig. 7.1 Interaction vertices for non-abelian gauge bosons

Remark 7.3.4 In physics, the quantum field theory for a gauge field A determined by the Yang–Mills Lagrangian, without any additional matter fields, is known as **pure Yang–Mills theory** or **gluodynamics**. For non-abelian Lie groups, the quantum version of pure Yang–Mills theory predicts particles, known as **glueballs**, which only consist of gauge bosons (gluons in QCD). The Clay Millennium Prize Problem [37] on the mass gap is to prove that the masses of glueballs in a quantum pure Yang–Mills theory on $\mathbb{R}^4$ with compact simple gauge group G are bounded from below by a positive (non-zero) number.

Remark 7.3.5 The term "gauge invariance" was invented by Hermann Weyl in 1929 for the U(1) gauge theory of electromagnetism. Gauge theory for non-abelian structure groups G was first developed by Chen Ning Yang (Nobel Prize in Physics 1957) and Robert L. Mills (for $G = \mathrm{SU}(2)$) in the 1950s.

7.3.2 The Yang–Mills Equation

We assume now that

- the semi-Riemannian manifold (M, g) is closed, i.e. compact and without boundary.

Definition 7.3.6 Let $\mathscr{A}(P)$ denote the space of all connection 1-forms A on the principal bundle P. This is by the discussion in Sect. 5.13 a (usually infinite-dimensional) affine space over the vector space

$$\Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \xrightarrow{\cong} \Omega^1(M, \mathrm{Ad}(P)),$$

with isomorphism given by the map Λ. For $\alpha \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ we set

$$\alpha_M = \Lambda(\alpha) \in \Omega^1(M, \mathrm{Ad}(P)).$$

Definition 7.3.7 The **Yang–Mills action** for a principal G-bundle $P \to M$ is the smooth map

$$S_{YM}\colon \mathscr{A}(P) \longrightarrow \mathbb{R},$$

defined by

$$\begin{aligned} S_{YM}[A] &= -\frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P),L^2} \\ &= -\frac{1}{2}\int_M \langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}\,\mathrm{dvol}_g. \end{aligned}$$

The integral is well-defined, because M is compact.

Definition 7.3.8 We call a connection A on the principal bundle P a **critical point** of the Yang–Mills action if

$$\left.\frac{d}{dt}\right|_{t=0} S_{YM}[A + t\alpha] = 0$$

for all variations

$$\alpha \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \cong \Omega^1(M, \mathrm{Ad}(P)).$$

For a connection A on P we denote by d_A the associated covariant differential and by d_A^* the covariant codifferential. We want to prove:

Theorem 7.3.9 *A connection A on a principal bundle $P \to M$ is a critical point of the Yang–Mills action if and only if A satisfies the* ***Yang–Mills equation***

$$d_A^* F_M^A = 0,$$

i.e.

$$d_A * F_M^A = 0.$$

Proof We follow the proof in [14]. According to the structure equation in Theorem 5.5.4 we can calculate

$$\begin{aligned} F^{A+t\alpha} &= d(A + t\alpha) + \frac{1}{2}[A + t\alpha, A + t\alpha] \\ &= F^A + t(d\alpha + [A, \alpha]) + \frac{1}{2}t^2[\alpha, \alpha]. \end{aligned}$$

This implies

$$F_M^{A+t\alpha} = F_M^A + t\,(d_A\alpha_M) + \frac{1}{2}t^2[\alpha_M, \alpha_M].$$

We get with Theorem 7.2.12

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \left\langle F_M^{A+t\alpha}, F_M^{A+t\alpha} \right\rangle_{\mathrm{Ad}(P),L^2} &= 2\left\langle d_A\alpha_M, F_M^A \right\rangle_{\mathrm{Ad}(P),L^2} \\ &= 2\left\langle \alpha_M, d_A^* F_M^A \right\rangle_{\mathrm{Ad}(P),L^2}. \end{aligned}$$

Since the scalar product on the Lie algebra $\mathfrak{g}$ is non-degenerate, the L^2-scalar product on $\Omega^1(M, \mathrm{Ad}(P))$ is non-degenerate. It follows that A is a critical point of the Yang–Mills Lagrangian if and only if $d_A^* F_M^A = 0$. □

In a local gauge $s\colon U \to P$ the Yang–Mills equation can be written as

$$d * F_s^A + \left[A_s, *F_s^A\right] = 0.$$

Remark 7.3.10 Recall that *any* connection A on the principal bundle P has to satisfy the **Bianchi identity**, which can be written according to Theorem 5.14.2 as

$$d_A F_M^A = 0.$$

Atiyah and Bott [6] have noted that the curvature F_M^A of a connection A that satisfies in addition to the Bianchi identity the Yang–Mills equation $d_A * F_M^A = 0$ can thus be considered as a **harmonic form** (in a non-linear sense if G is non-abelian) in $\Omega^2(M, \mathrm{Ad}(P))$ (compare with Exercise 7.9.5). The Yang–Mills equation is a second-order partial differential equation for the connection A.

Remark 7.3.11 Note that the Yang–Mills equation depends through the Hodge star operator on the pseudo-Riemannian metric g on M. If the equation holds for one metric, it does not necessarily hold for another metric.

Example 7.3.12 (Maxwell's Equations) In the case when $G = \mathrm{U}(1)$, the local curvature forms F_s are independent of the choice of local gauge s and define a global 2-form $F_M \in \Omega^2(M, \mathfrak{u}(1))$, see Corollary 5.6.4. The Bianchi identity and Yang–Mills equation are then given by

$$dF_M = 0,$$
$$d * F_M = 0.$$

These are **Maxwell's equations** for a source-free electromagnetic field (on a general n-dimensional oriented pseudo-Riemannian manifold). On Minkowski spacetime of dimension 4 we can use the construction in Sect. 5.7 to write Maxwell's equations in terms of the electric and magnetic field.

Maxwell's equations generalize to any abelian Lie group G. Note that in this case both the Bianchi identity and the Yang–Mills equation are linear. For a non-abelian structure group these equations are non-linear (and therefore much harder to solve).

We could study the Yang–Mills equation on any of the examples of principal bundles that we defined in Chap. 4, in particular, on the Hopf fibrations over projective spaces or on the canonical principal bundles over homogeneous spaces, once (pseudo-)Riemannian metrics on the base manifolds have been defined.

Definition 7.3.13 We call a connection A on a principal bundle a **Yang–Mills connection** if it satisfies the Yang–Mills equation.
Since the Yang–Mills equations do not depend on the choice of local gauge, the gauge group $\mathscr{G}(P)$ of the principal bundle $P \to M$ acts on the space of Yang–Mills connections. We can therefore set:

Definition 7.3.14 The **Yang–Mills moduli space** of a principal bundle $P \to M$ over a pseudo-Riemannian manifold (M, g) is the space of Yang–Mills connections A modulo the gauge group $\mathscr{G}(P)$.
The moduli space is usually the quotient of an infinite-dimensional space by the action of an infinite-dimensional group. It is therefore non-trivial to define, for example, a smooth structure on the moduli space.

Example 7.3.15 (Instantons) Let $P \to M$ be a principal G-bundle over an oriented *Riemannian* 4*-manifold* (M, g). In this case the Hodge star operator satisfies $** = 1$ on 2-forms on M. We consider connections A on P with curvature $F_M^A \in \Omega^2(M, \mathrm{Ad}(P))$ such that either

$$*F_M^A = F_M^A$$

or

$$*F_M^A = -F_M^A.$$

Connections that satisfy these identities are called **self-dual** and **anti-self-dual instantons**, respectively (see Exercise 7.9.3 for the notion of self-duality).

Since any connection A satisfies the Bianchi identity, instantons automatically satisfy the Yang–Mills equation. The instanton equations are examples of **BPS equations**, i.e. special first order equations (here for the gauge field A) whose solutions are (often) automatically solutions of the second order field equations (here the Yang–Mills equations). BPS equations appear in many other parts of physics, for example, in the theory of magnetic monopoles (Bogomolny equations) or in supergravity (Killing spinor equations).

The instanton equations are preserved under the action of the gauge group $\mathscr{G}(P)$ and we can define **instanton moduli spaces**. These moduli spaces, especially for structure groups $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, are the cornerstone of **Donaldson theory**, which revolutionized the understanding of smooth 4-manifolds in the 1980s.

7.3.3 Massive Gauge Bosons

The Yang–Mills Lagrangian

$$\begin{aligned}\mathscr{L}_{YM} &= -\frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)} \\ &= -\frac{1}{4}F_{\mu\nu}^{Aa}F_a^{A\mu\nu}\end{aligned}$$

describes *massless* gauge bosons. Arguments from physics show that gauge bosons of mass m are described by adding (in a local gauge) a term of the form

$$\frac{1}{2}m^2 A_a^\nu A_\nu^a \tag{7.2}$$

to the Yang–Mills Lagrangian. We could try to write this Lagrangian in an invariant form as above, such as

$$\frac{1}{2}m^2\left\langle A_M, A_M\right\rangle_{\mathrm{Ad}(P)},$$

however the gauge field A does not define an element $A_M \in \Omega^1(M, \mathrm{Ad}(P))$ (only the difference of two gauge fields is such a twisted form). This indicates that the Lagrangian in Eq. (7.2) is not well-defined, independent of local gauge. It is also easy to see directly that local gauge transformations $g\colon U \to G$, which are not constant, in general do not leave the Lagrangian in Eq. (7.2) invariant.

Remark 7.3.16 One of the main features of the Higgs mechanism, discussed in Chap. 8, is that it allows us to introduce a non-zero mass for gauge bosons with a *gauge invariant* Lagrangian. Introducing a mass for gauge bosons is necessary to describe the weak interaction as a gauge theory, because experiments show that the W- and Z-gauge bosons of the weak interaction have a non-zero mass.

7.4 Mathematical and Physical Conventions for Gauge Theories

In mathematics and physics slightly different conventions are used for scalar products, coupling constants and covariant derivatives. We want to compare these conventions in this section. We fix the following data:

- a compact Lie group G which is either simple or $\mathrm{U}(1)$ (the conventions below can be generalized to any compact Lie group)
- an Ad_G-invariant positive definite scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ (if $\mathfrak{g}$ is simple we can take the negative of the Killing form and for $\mathfrak{u}(1) \cong \mathbb{R}$ we can take any positive definite scalar product)

- a $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$-orthonormal basis $S_1,\dots,S_r$ of the Lie algebra $\mathfrak{g}$
- a real coupling constant $g > 0$.

1. In **mathematics** we choose the scalar product

$$\langle\cdot,\cdot\rangle'_{\mathfrak{g}} = \frac{1}{g^2}\langle\cdot,\cdot\rangle_{\mathfrak{g}}$$

with orthonormal basis

$$T_a = gS_a, \quad a = 1,\dots,r.$$

We expand the gauge field $A \in \Omega^1(P,\mathfrak{g})$ and curvature $F \in \Omega^2(P,\mathfrak{g})$ as

$$A = \sum_{a=1}^{r} A^a \otimes T_a,$$

$$F = \sum_{a=1}^{r} F^a \otimes T_a.$$

The covariant derivative (after a choice of local gauge) is

$$\nabla^A_\mu = \partial_\mu + A_\mu.$$

The local curvature is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Yang–Mills Lagrangian is

$$\mathscr{L}_{YM} = -\frac{1}{4}\left\langle F^{\mu\nu}, F_{\mu\nu}\right\rangle'_{\mathfrak{g}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a.$$

2. In **physics** we choose the Hermitian scalar product $\langle\cdot,\cdot\rangle_{i\mathfrak{g}}$ on $i\mathfrak{g}$ associated to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and the orthonormal basis

$$\frac{1}{i}S_a, \quad a = 1,\dots,r$$

of $i\mathfrak{g}$. We expand the gauge field $B \in \Omega^1(P, i\mathfrak{g})$ and curvature $G \in \Omega^2(P, i\mathfrak{g})$ as

$$B = \frac{1}{i}\sum_{a=1}^{r} B^a \otimes S_a,$$

$$G = \frac{1}{i}\sum_{a=1}^{r} G^a \otimes S_a.$$

There are two different sign conventions for the covariant derivative:

$$\nabla_\mu^B = \partial_\mu \pm igB_\mu.$$

The local curvature is

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \pm ig[B_\mu, B_\nu].$$

The Yang–Mills Lagrangian is

$$\mathscr{L}_{YM} = -\frac{1}{4}\left\langle G^{\mu\nu}, G_{\mu\nu}\right\rangle_{i\mathfrak{g}} = -\frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}.$$

3. The correspondence between the mathematical and physical conventions is given by setting

$$A = \pm igB,$$
$$F = \pm igG.$$

If the representation of the Lie group G on a vector space V is unitary, then the field A_μ will act as a skew-Hermitian operator and B_μ will act as a Hermitian operator. We have $\nabla_\mu^A = \nabla_\mu^B$ and

$$\begin{aligned} A^a &= \pm B^a, \\ F^a &= \pm G^a \\ &= \mp G_a \\ &= F_a. \end{aligned}$$

Most of the time we shall use the mathematical convention and indicate when we use the physical convention.

Remark 7.4.1 Note one interesting point that can be seen most clearly in the physical convention: The coupling constant g appearing in the covariant derivative (describing the coupling of the gauge field to other fields, as we will see below) is the same as the coupling constant appearing in front of the term $[B_\mu, B_\nu]$ in the curvature $G_{\mu\nu}$, describing the coupling between the gauge bosons in non-abelian gauge theories.

7.5 The Klein–Gordon and Higgs Lagrangians

So far we have considered pure gauge theories that involve only a gauge field (connection) A. In physics, however, we are also interested in matter fields that couple to the gauge field. We first consider the case of scalar fields, like the Higgs field. We again fix an oriented pseudo-Riemannian manifold (M, g).

7.5.1 The Pure Scalar Field

Definition 7.5.1 A **complex scalar field** is a smooth map

$$\phi : M \longrightarrow \mathbb{C}.$$

A **multiplet of complex scalar fields** is a smooth map

$$\phi : M \longrightarrow \mathbb{C}^r$$

for some $r > 1$.
We consider the standard Hermitian scalar product

$$\langle v, w \rangle = v^\dagger w$$

on $\mathbb{C}^r$. If ϕ is a multiplet of scalar field with values in $\mathbb{C}^r$, then the differential $d\phi$ is an element

$$d\phi \in \Omega^1 (M, \mathbb{C}^r) .$$

There is an induced Hermitian scalar product on the vector space-valued 1-forms $\Omega^1 (M, \mathbb{C}^r)$.

Definition 7.5.2 The **free Klein–Gordon Lagrangian** for a multiplet of complex scalar fields $\phi : M \to \mathbb{C}^r$ of mass m is defined by

$$\mathscr{L}_{KG}[\phi] = \langle d\phi, d\phi \rangle - m^2 \langle \phi, \phi \rangle .$$

For a given field ϕ the free Klein–Gordon Lagrangian defines a smooth map

$$\mathscr{L}_{KG}[\phi] : M \longrightarrow \mathbb{R}.$$

The expression $\langle d\phi, d\phi \rangle$ is called the **kinetic term** and the expression $-m^2 \langle \phi, \phi \rangle$ is called the **Klein–Gordon mass term**.

In local coordinates on M the kinetic term is given by

$$\langle d\phi, d\phi\rangle = \left\langle \partial^\mu \phi, \partial_\mu \phi \right\rangle.$$

It is also useful to consider a more general situation.

Definition 7.5.3 Let $V: \mathbb{R} \to \mathbb{R}$ be a smooth function, called a **potential**. Then the **Higgs Lagrangian** for a multiplet of complex scalar fields ϕ with potential V is defined by

$$\mathscr{L}_H[\phi] = \langle d\phi, d\phi\rangle - V(\phi),$$

where $V(\phi)$ denotes $V(\langle\phi,\phi\rangle)$ (of course it suffices to define the potential on $\mathbb{R}_{\geq 0}$). The Higgs field in the Standard Model, which we study in Chap. 8, is a multiplet of complex scalars described by a similar Lagrangian.
The potential V, if it contains terms of order higher than two in the field ϕ, describes a **direct interaction** between particles of the field ϕ. In the Standard Model, for instance, the potential V of the Higgs field is a quadratic polynomial in $\phi^\dagger\phi$, hence of order four in ϕ.

7.5.2 The Scalar Field Coupled to a Gauge Field

We now consider the case of a scalar field ϕ coupled to a gauge field A. We fix the following data:

- an n-dimensional oriented pseudo-Riemannian manifold (M, g)
- a principal G-bundle $P \to M$ with compact structure group G of dimension r
- a complex representation $\rho: G \to \mathrm{GL}(W)$ with associated complex vector bundle $E = P \times_\rho W \to M$
- a G-invariant Hermitian scalar product $\langle\cdot\,,\cdot\rangle_W$ on W with associated bundle metric $\langle\cdot\,,\cdot\rangle_E$ on the vector bundle E.

We then define:

Definition 7.5.4 If the dimension of W is one, then a smooth section of E is called a **complex scalar field** and if the dimension of W is greater than one, then a smooth section of E is called a **multiplet of complex scalar fields** (or simply a scalar field) and the vector space W is called a **multiplet space**.
With the exterior covariant derivative

$$d_A: \Gamma(E) \longrightarrow \Omega^1(M, E)$$

and the scalar product $\langle\cdot\,,\cdot\rangle_E$ on $\Omega^1(M,E)$ we set:

Definition 7.5.5 The **Klein–Gordon Lagrangian** for a multiplet of complex scalar fields $\Phi \in \Gamma(E)$ of mass m coupled to a gauge field A is defined by

$$\mathscr{L}_{KG}[\Phi, A] = \langle d_A\Phi, d_A\Phi\rangle_E - m^2\langle\Phi, \Phi\rangle_E.$$

For given fields Φ and A the Klein–Gordon Lagrangian is a smooth function

$$\mathscr{L}_{KG}[\Phi, A]\colon M \longrightarrow \mathbb{R}.$$

The associated action $S_{KG}[\Phi, A]$ is the integral over the Klein–Gordon Lagrangian (on a closed manifold M).
In local coordinates on M we can write the kinetic term as

$$\langle d_A\Phi, d_A\Phi\rangle_E = \left\langle \nabla^{A\mu}\Phi, \nabla^A_\mu\Phi \right\rangle_E.$$

It is sometimes useful to have an even more explicit local formula for the Klein–Gordon Lagrangian: Choosing a local gauge $s\colon U \to P$, we can write

$$\Phi|_U = [s, \phi],$$

where $\phi\colon U \to W$ is a smooth function. The covariant derivative is given by

$$\nabla^A_\mu\Phi = [s, \nabla^A_\mu\phi], \quad \nabla^A_\mu\phi = \partial_\mu\phi + A_\mu\phi.$$

The term $A_\mu\phi$ is called the **minimal coupling** (we suppress in the notation the induced representation ρ_* of the Lie algebra $\mathfrak{g}$ on W). We identify W with $\mathbb{C}^r$ and the scalar product on W with the standard Hermitian product

$$\langle v, w\rangle = v^\dagger w$$

on $\mathbb{C}^r$. Since the representation of G on W is unitary and the gauge field A_μ has values in $\mathfrak{g}$, this implies that A_μ acts through skew-Hermitian matrices on $\mathbb{C}^r$:

$$A_\mu^\dagger = -A_\mu.$$

In a local gauge s for the principal bundle, the Klein–Gordon Lagrangian can then be written as

$$\begin{aligned}\mathscr{L}_{KG}[\Phi, A] = &(\partial^\mu\phi)^\dagger(\partial_\mu\phi) - m^2\phi^\dagger\phi \\ &+ (\partial^\mu\phi)^\dagger(A_\mu\phi) - (\phi^\dagger A_\mu)(\partial^\mu\phi) \\ &- \phi^\dagger A^\mu A_\mu\phi.\end{aligned} \tag{7.3}$$

The two terms in the first line, which are quadratic in the field ϕ with values in $W \cong \mathbb{C}^r$, are the Klein–Gordon Lagrangian for a free multiplet of complex scalar fields of mass m, consisting of the kinetic term and the mass term.

The terms in the second and third line are cubic and quartic in the fields ϕ and A_μ. These **interaction terms** describe an interaction (or coupling) between the gauge field and the multiplet of scalar fields and thus an indirect interaction between particles of the scalar field, mediated by the gauge bosons (see the Feynman diagrams after Remark 5.9.5 for a depiction of the interaction between a scalar field and a gauge field).

We see here (and later in the case of the Dirac Lagrangian for fermions) that in gauge theories where G does not act diagonally on the multiplet vector space $W = \mathbb{C}^s$, the action of the gauge group leads to two related kinds of mixing:

- The representation of the gauge group G on W, defining the associated bundle E, mixes different components of the multiplet, i.e. different components are gauge equivalent. In other words, the identification of a section of E with a map to V and the splitting into components depends on the choice of gauge.
- Via the induced representation of the Lie algebra $\mathfrak{g}$ on W, the gauge field A pairs different components of the multiplet in the interaction vertices.

This has important consequences for the Standard Model, where different particles like the up and down quark or the electron and electron neutrino form $\mathrm{SU}(2) \times \mathrm{U}(1)$-doublets.

Definition 7.5.6 Sections Φ of an associated vector bundle $E = P \times_\rho V$ with

$$\rho_* \colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

non-trivial are called **charged scalars**. It follows that charged scalars have a non-trivial coupling to the gauge field A.

Theorem 7.5.7 *The Klein–Gordon Lagrangian of a multiplet of complex scalar fields, coupled to a gauge field, is gauge invariant:*

$$\mathscr{L}_{KG}\left[f^{-1}\Phi, f^*A\right] = \mathscr{L}_{KG}[\Phi, A]$$

for all bundle automorphisms $f \in \mathscr{G}(P)$.

We need the following lemma.

Lemma 7.5.8 *Let $f \in \mathscr{G}(P)$ be a bundle automorphism. Then*

$$d_{f^*A}\left(f^{-1}\Phi\right) = f^{-1}d_A\Phi.$$

Proof This follows from a calculation in local coordinates for $d_A\Phi(X) = \nabla_X^A\Phi$ with a vector field X. For a more invariant argument, note that by the definition of covariant derivatives using parallel transport in Sect. 5.9 it suffices to show that

$$D\left(f^{-1}\Phi, \gamma, x, f^*A\right) = D(\Phi, \gamma, x, A).$$

This follows from Exercise 5.15.9. □

We can now prove Theorem 7.5.7.

Proof The kinetic term $\langle d_A\Phi, d_A\Phi\rangle_E$ and the mass term $-m^2\langle\Phi, \Phi\rangle_E$ are both separately invariant under gauge transformations, because the scalar product $\langle\cdot\,,\cdot\rangle_W$ on the vector space W is G-invariant, hence $\langle\cdot\,,\cdot\rangle_E$ is invariant under the action of f^{-1}. □

In the Klein–Gordon Lagrangian for a scalar field the gauge field A is non-dynamic, i.e. does not appear with derivatives, and is just a fixed background field. The total Lagrangian that describes the dynamics of the scalar field, the gauge field and their interactions is the **Yang–Mills–Klein–Gordon Lagrangian**

$$\mathscr{L}_{KG}[\Phi, A] + \mathscr{L}_{YM}[A] = \langle d_A\Phi, d_A\Phi\rangle_E - m^2\langle\Phi, \Phi\rangle_E - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.$$

We can also consider the case of a scalar field with a potential coupled to a gauge field.

Definition 7.5.9 The **Higgs Lagrangian** for a multiplet of complex scalar fields coupled to a gauge field is defined by

$$\mathscr{L}_H[\Phi, A] = \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi),$$

where $V(\Phi)$ is a gauge invariant potential. We only consider the case where

$$V(\Phi) = V(\langle\Phi, \Phi\rangle_E),$$

with a function $V\colon \mathbb{R} \to \mathbb{R}$.

This Lagrangian describes an interaction between particles of the scalar field and particles of the gauge field and in addition a direct interaction between the particles of the scalar field (if the potential V contains terms of order three or higher in Φ).

A similar argument to the one in Theorem 7.5.7 shows:

Theorem 7.5.10 *The Higgs Lagrangian of a multiplet of complex scalar fields with potential V and coupled to a gauge field is gauge invariant:*

$$\mathscr{L}_H\left[f^{-1}\Phi, f^*A\right] = \mathscr{L}_H[\Phi, A]$$

for all bundle automorphisms $f \in \mathscr{G}(P)$.

The sum of the Higgs and Yang–Mills Lagrangians is called the **Yang–Mills–Higgs Lagrangian**

$$\mathscr{L}_H[\Phi, A] + \mathscr{L}_{YM}[A] = \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.$$

Remark 7.5.11 It is sometimes useful to consider **real scalar fields** Φ, which are sections in vector bundles E associated to real orthogonal representations of the Lie group G. The Klein–Gordon Lagrangian for a real scalar field of mass m coupled to a gauge field is

$$\mathscr{L}_{KG}[\Phi, A] = \frac{1}{2}\langle d_A\Phi, d_A\Phi\rangle_E - \frac{1}{2}m^2\langle \Phi, \Phi\rangle_E.$$

There is an analogous generalization to real scalar fields with a potential V.

7.6 The Dirac Lagrangian

Fermions are described classically by spinor fields on spacetime. In this section we define a Lagrangian for fermions. We fix the following data:

- an n-dimensional oriented and time-oriented pseudo-Riemannian spin manifold (M, g) of signature (s, t)
- a spin structure $\mathrm{Spin}^+(M)$ together with complex spinor bundle $S \rightarrow M$
- a Dirac form $\langle \cdot\,, \cdot \rangle$ (not necessarily positive definite) on the Dirac spinor space $\Delta = \Delta_n$ with associated Dirac bundle metric $\langle \cdot\,, \cdot \rangle_S$. We abbreviate $\langle \Psi, \Phi\rangle_S$ by $\overline{\Psi}\Phi$.

We can then define the Dirac Lagrangian for a free spinor field.

Definition 7.6.1 The **Dirac Lagrangian** for a free spinor field $\Psi \in \Gamma(S)$ of mass m is defined by

$$\begin{aligned}\mathscr{L}_D[\Psi] &= \mathrm{Re}\langle\Psi, D\Psi\rangle_S - m\langle\Psi,\Psi\rangle_S\\ &= \mathrm{Re}\left(\overline{\Psi}D\Psi\right) - m\overline{\Psi}\Psi,\end{aligned}$$

where $D\colon \Gamma(S) \to \Gamma(S)$ denotes the Dirac operator. The expression $\mathrm{Re}\left(\overline{\Psi}D\Psi\right)$ is called the **kinetic term** and $-m\overline{\Psi}\Psi$ is called the **Dirac mass term**.
Taking the real part in the kinetic term is necessary, because the Lagrangian has to be real. If the Dirac form $\langle\cdot,\cdot\rangle_S$ has $\delta = -1$, then the calculation in Exercise 7.9.12 implies that

$$(\langle\Psi, D\Psi\rangle_S - \langle D\Psi,\Psi\rangle_S)\mathrm{dvol}_g = d\alpha$$

for some $(n-1)$-form α on M depending on the spinor $\Psi \in \Gamma(S)$. As a consequence the kinetic term of the Dirac Lagrangian satisfies

$$\begin{aligned}\mathrm{Re}\left(\langle\Psi, D\Psi\rangle_S\right)\mathrm{dvol}_g &= \frac{1}{2}\left(\langle\Psi, D\Psi\rangle_S + \langle\Psi, D\Psi\rangle_S^*\right)\mathrm{dvol}_g\\ &= \frac{1}{2}\left(\langle\Psi, D\Psi\rangle_S + \langle D\Psi,\Psi\rangle_S\right)\mathrm{dvol}_g\\ &= \langle\Psi, D\Psi\rangle_S\mathrm{dvol}_g - \frac{1}{2}d\alpha.\end{aligned}$$

This implies by Stokes' Theorem A.2.24 that the action defined by $\langle\Psi, D\Psi\rangle_S$ and its real part are the same if the manifold M has no boundary and Ψ has compact support.

7.6.1 The Fermion Field Coupled to a Gauge Field

Similar to a scalar field, a spinor can be coupled to a gauge field. This construction is very important, because it defines the interaction between matter particles (fermions) and gauge bosons in gauge theories (for example, the interaction between electrons and photons in QED or the interaction between quarks and gluons in QCD). We fix in addition to the data above the following data:

- a principal G-bundle $P \to M$ with compact structure group G of dimension r
- a complex representation $\rho\colon G \to \mathrm{GL}(V)$ with associated complex vector bundle $E = P \times_\rho V \to M$

- a G-invariant Hermitian scalar product $\langle \cdot\,, \cdot \rangle_V$ on V with associated bundle metric $\langle \cdot\,, \cdot \rangle_E$ on the vector bundle E. Together with the Dirac form on the spinor bundle S we get a Hermitian scalar product $\langle \cdot\,, \cdot \rangle_{S\otimes E}$ on the twisted spinor bundle $S \otimes E$. We again abbreviate $\langle \Psi, \Phi \rangle_{S\otimes E}$ by $\overline{\Psi}\Phi$.

Choosing a local gauge $s\colon U \to P$ and an orthonormal basis $v_1, \ldots, v_s$ for V, the twisted spinors Ψ, Φ correspond to multiplets

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_s \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_s \end{pmatrix},$$

where Ψ_i and Φ_j are sections of the spinor bundle S over U. The scalar product on $S \otimes E$ can then be written as

$$\overline{\Psi}\Phi = \sum_{j=1}^{s} \overline{\Psi}_j \Phi_j.$$

Definition 7.6.2 The **Dirac Lagrangian** for a twisted spinor field $\Psi \in \Gamma(S \otimes E)$ of mass m coupled to a gauge field A on the principal bundle P is defined by

$$\begin{aligned} \mathscr{L}_D[\Psi, A] &= \mathrm{Re}\langle \Psi, D_A \Psi \rangle_{S\otimes E} - m\langle \Psi, \Psi \rangle_{S\otimes E} \\ &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) - m\overline{\Psi}\Psi, \end{aligned}$$

where $D_A\colon \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ denotes the twisted Dirac operator. The associated action $S_D[\Psi, A]$ is the integral over the Dirac Lagrangian (on a closed manifold M).

Choosing in addition to the local gauge for P and the orthonormal basis for V a local vielbein e for the tangent bundle TM with associated local trivialization ϵ of $\mathrm{Spin}^+(M)$, we can write the Dirac Lagrangian as

$$\begin{aligned} \mathscr{L}_D[\Psi, A] =& \mathrm{Re} \sum_{j=1}^{s} i\overline{\psi}_j \Gamma^p \left(\partial_p - \frac{1}{4}\omega_{pqr}\Gamma^{qr}\right)\psi_j - \sum_{j=1}^{s} m\overline{\psi}_j \psi_j \\ &+ \mathrm{Re} \sum_{j=1}^{s} i\overline{\psi}_j \Gamma^p (A_p \psi)_j, \end{aligned} \tag{7.4}$$

(continued)

Definition 7.6.2 (continued)
where ψ is a map with values in $\Delta \otimes V$, ψ_j are maps with value in Δ, and Γ^p are physical gamma matrices. Here the two terms in the first line are the Dirac Lagrangian for a free multiplet of fermions, consisting of the **kinetic term**

$$\mathrm{Re} \sum_{j=1}^{s} i\overline{\psi}_j \Gamma^p \partial_p \psi_j,$$

a coupling between the spinor field and the metric g via ω_{pqr}, and the Dirac mass term. The term in the second line, which is cubic in the fields, is the **interaction term** that describes an interaction between the fermions and the gauge field and thus an indirect interaction between the fermions (see the Feynman diagram in Fig. 7.2 for the interaction between a fermion ψ and a gauge field A_p).

The gauge field A_p with values in the Lie algebra $\mathfrak{g}$ acts on the V part of ψ through the induced representation (suppressed in the notation). Since the gauge field A acts by skew-Hermitian matrices, the interaction term is automatically real and we can drop the symbol Re.

Definition 7.6.3 Sections Ψ of a twisted spinor bundle $S \otimes E$, where E is associated to a representation ρ of the gauge group G on a vector space V with

$$\rho_* \colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

non-trivial, are called **charged fermions**. It follows that charged fermions have a non-trivial coupling to the gauge field A.

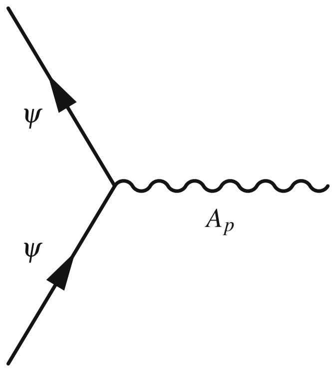

Fig. 7.2 Interaction vertex for fermion and gauge field

Theorem 7.6.4 *The Dirac Lagrangian for a twisted spinor field is gauge invariant:*

$$\mathscr{L}_D\left[f^{-1}\Psi, f^*A\right] = \mathscr{L}_D[\Psi, A]$$

for all bundle automorphisms $f \in \mathscr{G}(P)$.

Proof This is Exercise 7.9.11. □

Example 7.6.5 For the **strong interaction (QCD)** we have $G = \mathrm{SU}(3)$, $V \cong \mathbb{C}^3$ and there are six multiplets Ψ_f, called quarks, for the flavours $f = u, d, c, s, t, b$. The three components of every multiplet are called colours. The interaction term involving the gauge field A_μ with values in $\mathfrak{su}(3)$ (corresponding to the eight gluons) mixes different colours of a quark of a given flavour, but does not mix different flavours (different flavours of quarks are only mixed by the weak interaction). The Lagrangian for QCD can thus be written as

$$\mathscr{L}_D[\Psi, A] = \sum_f \left(\mathrm{Re}\left(\overline{\Psi}_f D_A \Psi_f\right) - m_f \overline{\Psi}_f \Psi_f\right),$$

where the sum runs over the six different flavours f.

Remark 7.6.6 Considering the mass term $-m\overline{\Psi}\Psi$ in the Dirac Lagrangian, it is clear that all components of the multiplet Ψ have the same mass m. We could try to generalize this and introduce different mass terms for different components of the multiplet. However, if the components with different masses are related by the action of a group element $g \in G$, then gauge invariance of the Lagrangian will be lost.

This could be a problem for the Standard Model, because we want to combine particles with very different masses, like the electron and electron neutrino, into $\mathrm{SU}(2) \times \mathrm{U}(1)$-doublets and at the same time keep the Lagrangian gauge invariant. It turns out that the situation in the Standard Model is even more difficult, because left-handed and right-handed fermions transform in different representations of $\mathrm{SU}(2) \times \mathrm{U}(1)$, so that a gauge invariant Dirac mass term is not defined, even if all components of the multiplet had the same mass. See Sect. 7.6.2 for more details. As we will discuss in Chap. 8, these problems can be solved by introducing a **Higgs field**.

We can again make both the spinor multiplet Ψ and the connection 1-form A dynamic by considering the **Yang–Mills–Dirac Lagrangian**

$$\mathscr{L}_D[\Psi, A] + \mathscr{L}_{YM}[A] = \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) - m\overline{\Psi}\Psi - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.$$

7.6.2 Lagrangians for Chiral Fermions

In this subsection we consider the case of an oriented and time-oriented Lorentzian spin manifold M of *even* dimension n with metric of signature $(1, n-1)$ or $(n-1, 1)$ (the most interesting case for the Standard Model is Minkowski spacetime of dimension $n = 4$). The Dirac bundle metric $\langle \cdot\,, \cdot \rangle_S$ on the spinor bundle $S \to M$ has a special property: both choices of the matrix A in Proposition 6.7.13 consist of a product of an odd number of gamma matrices. Hence if we decompose spinors $\Psi, \Phi \in \Gamma(S)$ into left-handed (positive) and right-handed (negative) components we get

$$\begin{aligned} \overline{\Psi}\Phi &= \langle \Psi, \Phi \rangle_S \\ &= \langle \Psi_L, \Phi_R \rangle_S + \langle \Psi_R, \Phi_L \rangle_S \\ &= \overline{\Psi}_L \Phi_R + \overline{\Psi}_R \Phi_L. \end{aligned} \tag{7.5}$$

In particular, we observe the following:

Proposition 7.6.7 (Dirac Bundle Metrics for Spinors on Lorentz Manifolds) *On even-dimensional oriented and time-oriented Lorentzian spin manifolds, for both choices of the matrix A in Proposition 6.7.13, the Dirac bundle metric is null on the subbundles S_L and S_R and pairs left-handed with right-handed spinors. In particular, the decomposition $S = S_L \oplus S_R$ of the spinor bundle into left-handed and right-handed Weyl spinors is not orthogonal with respect to the Dirac bundle metric.*

Remark 7.6.8 This is different from the situation on even-dimensional Riemannian spin manifolds, where we can take $A = I$ (the identity matrix) so that left-handed and right-handed Weyl spinors are orthogonal.

Formula (7.5) also holds if the spinors are sections of a twisted spinor bundle

$$S \otimes E = (S_L \otimes E) \oplus (S_R \otimes E).$$

We get:

Proposition 7.6.9 *On even-dimensional oriented and time-oriented Lorentzian manifolds, for both choices of the matrix A in Proposition 6.7.13, the (gauge invariant) Dirac Lagrangian for twisted spinors can be written as*

$$\begin{aligned} \mathscr{L}_D[\Psi, A] &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) - m\overline{\Psi}\Psi \\ &= \mathrm{Re}\left(\overline{\Psi}_L D_A \Psi_L + \overline{\Psi}_R D_A \Psi_R\right) - 2m\mathrm{Re}\left(\overline{\Psi}_L \Psi_R\right). \end{aligned}$$

In the second line all three Hermitian scalar products are taken in $S \otimes E$.

We want to generalize this discussion to the case of a twisted chiral spinor bundle. We consider a twisted chiral spinor bundle over a Lorentzian spin manifold of even dimension:

$$(S \otimes E)_+ = (S_L \otimes E_L) \oplus (S_R \otimes E_R).$$

Here E_L and E_R are complex vector bundles associated to representations

$$\rho_L : G \longrightarrow \mathrm{GL}(V_L),$$
$$\rho_R : G \longrightarrow \mathrm{GL}(V_R).$$

We fix G-invariant Hermitian scalar products on V_L and V_R which define Hermitian bundle metrics $\langle \cdot\,,\cdot \rangle_{E_L}$ and $\langle \cdot\,,\cdot \rangle_{E_R}$.

We can then define a *massless* Dirac Lagrangian as before:

$$\begin{aligned}\mathscr{L}_D[\Psi, A] &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) \\ &= \mathrm{Re}\left(\overline{\Psi}_L D_A \Psi_L + \overline{\Psi}_R D_A \Psi_R\right).\end{aligned}$$

In the second line the first scalar product is taken in $S \otimes E_L$ and the second scalar product in $S \otimes E_R$ (the Dirac operator only acts on the S-component and does not change the E-component). It is not difficult to check that this Lagrangian is gauge invariant.

However, if we now also want to define a Dirac mass term as before, we run into a problem that can ultimately be traced back to Proposition 7.6.7: the natural mass term

$$-m\overline{\Psi}\Psi = -2m\mathrm{Re}\left(\overline{\Psi}_L \Psi_R\right)$$

is so far *not* defined: it pairs a spinor with an E_L-component and a spinor with an E_R-component, but the Hermitian bundle metrics are only defined if both spinors have the same type of E-component.

We could try to introduce a Dirac mass term in this situation as follows:

Definition 7.6.10 Let V_R and V_L be unitary representations of a Lie group G. Then a **mass pairing** is a G-invariant form

$$\kappa : V_L \times V_R \longrightarrow \mathbb{C}$$

which is complex antilinear in the first argument and complex linear in the second. A mass pairing κ defines a form on the level of bundles

$$\kappa : E_L \times E_R \longrightarrow \mathbb{C}$$

that can then be used to define a gauge invariant Dirac mass term for chiral twisted spinors. However, the following theorem shows that in many cases a mass pairing vanishes identically:

Theorem 7.6.11 (Triviality of Mass Pairings) *Suppose that V_L and V_R are irreducible, unitary, non-isomorphic representations of G. Then every mass pairing κ is identically zero.*

Proof We can identify $\overline{V}_L^*$, the dual of the complex conjugate of V_L, with

$$\overline{V}_L^* = \{\alpha : V_L \to \mathbb{C} \mid \alpha \text{ is } \mathbb{C}\text{-antilinear}\}.$$

The induced G-representation on this space is defined by

$$(g \cdot \alpha)(v_L) = \alpha\left(g^{-1} \cdot v_L\right)$$

for $g \in G$, $v_L \in V_L$. The map

$$\begin{aligned} V_L &\longrightarrow \overline{V}_L^* \\ v_L &\longmapsto \langle \cdot\,, v_L \rangle_{V_L}, \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle_{V_L}$ is the G-invariant Hermitian form on V_L, defines a *complex linear* G-equivariant isomorphism.

Suppose a mass pairing $\kappa \neq 0$ exists. Then

$$\begin{aligned} V_R &\longrightarrow \overline{V}_L^* \\ v_R &\longmapsto \kappa(\cdot, v_R) \end{aligned}$$

is a complex linear G-equivariant map. Combining both maps we get a complex linear G-equivariant map

$$V_R \longrightarrow V_L$$

which is non-zero, because $\kappa \neq 0$. By Schur's Lemma this map has to be an isomorphism of the representations V_R and V_L (because the kernel and image of the map are G-invariant), contradicting our assumption. □

It is known from experiments that a realistic theory of particle physics has to involve twisted chiral fermions with a non-zero mass, because the weak interaction is not invariant under parity inversion (see Sect. 8.5). Together with Remark 7.3.16 and Remark 7.6.6 it follows that there are three situations in which it is not clear how to define mass terms and at the same time keep the Lagrangian gauge invariant:

- non-zero masses for gauge bosons
- different masses for fermions in the same gauge multiplet
- non-zero masses for twisted chiral fermions.

We shall see in Chap. 8 that the introduction of the **Higgs field** allows a very elegant solution of these problems: using the Higgs field we can define a fully gauge invariant Lagrangian that contains certain interaction terms between the gauge bosons and the Higgs field and the fermions and the Higgs field. In a specific type of gauge, called a *unitary gauge*, these interaction terms take the form of mass terms for the gauge bosons and fermions.

7.7 Yukawa Couplings

In this section we discuss Yukawa couplings which are used in the Standard Model to define a mass for twisted chiral fermions. Yukawa couplings are certain trilinear forms involving two twisted chiral spinors and one scalar field. The idea is that the G-representation on the scalar field precisely cancels the difference between the representations on the twisted chiral spinors so that the whole trilinear expression is gauge invariant. We consider the case of an oriented and time-oriented Lorentzian spin manifold (M, g) of dimension n with signature $(1, n-1)$ or $(n-1, 1)$ together with a principal G-bundle $P \to M$.

Definition 7.7.1 Suppose that V_L, V_R, W are unitary representation spaces of the compact Lie group G. Then we define a **Yukawa form** as a map

$$\tau\colon V_L \times W \times V_R \longrightarrow \mathbb{C}$$

which is invariant under the action of G, complex antilinear in V_L, real linear in W and complex linear in V_R.
Suppose τ is a Yukawa form. We then define:

Definition 7.7.2 For a real constant g_Y the G-invariant scalar

$$\begin{aligned}(\Delta_L \otimes V_L) \times W \times (\Delta_R \otimes V_R) &\longrightarrow \mathbb{R}\\ (\lambda_L \otimes v_L, \phi, \lambda_R \otimes v_R) &\longmapsto -2g_Y \mathrm{Re}\left(\overline{\lambda}_L \lambda_R \tau(v_L, \phi, v_R)\right)\end{aligned}$$

is called a **Yukawa coupling** (the constant g_Y is also called a Yukawa coupling and sometimes already appears in the definition of the Yukawa form τ). It defines a gauge invariant Lagrangian for which we use the shorthand notation

$$\begin{aligned}\mathscr{L}_Y[\Psi_L, \Phi, \Psi_R] &= -2g_Y \mathrm{Re}\left(\overline{\Psi}_L \Phi \Psi_R\right) \\ &= -g_Y\left(\overline{\Psi}_L \Phi \Psi_R\right) - g_Y\left(\overline{\Psi}_L \Phi \Psi_R\right)^*,\end{aligned}$$

where the Yukawa form τ is implicit,

$$\begin{aligned}\Psi_L &\in \Gamma(S_L \otimes E_L), \\ \Phi &\in \Gamma(F), \\ \Psi_R &\in \Gamma(S_R \otimes E_R),\end{aligned}$$

and E_L, F, E_R are the complex vector bundles associated to the principal bundle P via the G-representations V_L, W, V_R.

We will discuss in Chap. 8 how Yukawa coupling between two twisted chiral fermions and the Higgs field leads to masses for the fermions. The Lagrangian of the **Standard Model** is then essentially the sum of all the Lagrangians that we discussed in this chapter, i.e. the following **Yang–Mills–Dirac–Higgs–Yukawa Lagrangian**:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_D[\Psi, A] + \mathscr{L}_H[\Phi, A] + \mathscr{L}_Y[\Psi_L, \Phi, \Psi_R] + \mathscr{L}_{YM}[A] \\ &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) + \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - 2g_Y \mathrm{Re}\left(\overline{\Psi}_L \Phi \Psi_R\right) - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.\end{aligned}$$

Remark 7.7.3 In the discussions in this chapter, the pseudo-Riemannian metric g on the manifold M has been considered as a fixed background. Classically we can add a Lagrangian for the metric (like the Einstein–Hilbert Lagrangian) to make g dynamic. However, this approach does not yield a well-defined quantum field theory. Since we are mainly interested in the Standard Model, which is defined on flat Minkowski spacetime of dimension 4, we will not discuss aspects of quantum gravity.

7.8 Dirac and Majorana Mass Terms

So far we have considered Dirac mass terms for spinor fields. For a spinor $\Psi \in \Gamma(S)$ such a mass term is given by

$$-m\langle\Psi, \Psi\rangle_S = -m\overline{\Psi}\Psi,$$

where $\langle \cdot , \cdot \rangle$ is a Dirac form on the spinor space. We want to discuss a second type of mass term that is important in neutrino physics.

Definition 7.8.1 Let $(\cdot , \cdot)$ denote a Majorana form on the spinor space Δ as in Sect. 6.7.1. Then

$$-m\mathrm{Re}(\Psi, \Psi)_S$$

is called a **Majorana mass term**.

It is clear that both Dirac and Majorana mass terms are invariant under the action of the spin group. We want to compare these forms in the case of Minkowski spacetime of dimension 4. Recall from Sect. 6.8 that the Dirac form is defined by the matrix

$$A = \Gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

and the Majorana form is defined by the matrix

$$C = i\Gamma_0\Gamma_2 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}.$$

If we decompose a Dirac spinor Ψ into left-handed and right-handed Weyl spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

then the Dirac mass term is given by

$$\begin{aligned} -m\langle \psi, \psi \rangle &= -m\psi^\dagger A\psi \\ &= -m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right) \end{aligned}$$

and the Majorana mass term is given by

$$\begin{aligned} -m\mathrm{Re}(\psi, \psi) &= m\mathrm{Re}\left(\tilde{\psi}\psi\right) \\ &= -m\mathrm{Re}\left(\psi^T C\psi\right) \\ &= m\mathrm{Re}\left(i\psi_L^T\sigma_2\psi_L - i\psi_R^T\sigma_2\psi_R\right). \end{aligned}$$

Here we used the notation $\tilde{\psi} = \psi^T C$ for the Majorana conjugate from Definition 6.7.6. The important consequence is that the Dirac mass term is zero for spinors which have only one Weyl component ψ_L or ψ_R, while the Majorana mass term may be non-zero in this case.

We briefly want to discuss the extension of these Lorentz invariant mass terms to Lorentz and gauge invariant mass terms for charged fermions, i.e. sections of twisted spinor bundles $S\otimes E$. In the case of the Dirac mass term we saw in Sect. 7.6.2 that such an extension is always possible if both left-handed and right-handed Weyl spinor bundles are twisted with the same associated vector bundle E, using the Hermitian scalar product on $S\otimes E$, coming from the Dirac form on S and a Hermitian scalar product on E.

In the case of the Majorana mass term there is now a problem, because the *complex bilinear* Majorana form on S usually does not combine with the *Hermitian* scalar product on E. If E is the associated bundle $P\times_\rho V$, then we need a G-invariant complex bilinear form on the vector space V. However, even in simple situations such an invariant bilinear form does not exist:

Lemma 7.8.2 *Let*

$$\begin{aligned}\rho_k\colon \mathrm{U}(1) &\longrightarrow \mathrm{U}(1)\\ \alpha &\longmapsto \alpha^k\end{aligned}$$

be the complex representation of $\mathrm{U}(1)$ *on* $\mathbb{C}$ *of winding number k. Suppose that B is a* $\mathrm{U}(1)$*-invariant complex bilinear form on* $\mathbb{C}$*. Then* $B\equiv 0$.

Proof We have

$$\begin{aligned}B(z,z) &= B\left(\alpha^k z, \alpha^k z\right)\\ &= \alpha^{2k}B(z,z) \quad \forall \alpha\in \mathrm{U}(1), z\in\mathbb{C}.\end{aligned}$$

It follows that $B(z,z)=0$ for all $z\in\mathbb{C}$, hence $B\equiv 0$. □

This indicates that there is no straightforward extension of the Majorana mass term to charged fermions.

7.9 Exercises for Chap. 7

7.9.1 (From [125]) Let H be a Hilbert space with a bilinear form

$$\langle\cdot,\cdot\rangle\colon H\times H\longrightarrow H,$$

satisfying

$$\langle\psi,\phi\rangle = \langle\phi,\psi\rangle^*,$$

where * denotes complex conjugation. We say that the bilinear form has a **probability interpretation** if the following holds: for all vectors $\phi,\psi\in H$ with

$$|\langle\phi,\phi\rangle|^2 = |\langle\psi,\psi\rangle|^2 = 1$$

the following inequality holds:

$$|\langle \phi, \psi \rangle|^2 \leq 1.$$

1. Suppose that the bilinear form is positive definite. Prove that the bilinear form has a probability interpretation.
2. Suppose that there exist vectors ϕ, ψ in H such that

$$\langle \phi, \phi \rangle > 0,$$
$$\langle \psi, \psi \rangle < 0.$$

   Prove that the bilinear form does not have a probability interpretation.

7.9.2 Let (M, g) be an oriented pseudo-Riemannian manifold and (U, ϕ) an oriented chart for M with local coordinates x^μ. Prove that the volume form dvol_g is given by

$$\mathrm{dvol}_g = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n,$$

where

$$|g| = |\det(g_{\mu\nu})|$$

is the absolute value of the determinant of the matrix with entries

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu).$$

7.9.3 Let (M, g) be an n-dimensional oriented pseudo-Riemannian manifold of signature (s, t) and $*$ the Hodge star operator.

1. Prove that

$$**\colon \Omega^k(M) \longrightarrow \Omega^k(M)$$

   is given by

$$** = (-1)^{t+k(n-k)}.$$

2. Determine the even dimensions $n = 2k$ where $** = 1$ on $\Omega^k(M)$ if (M, g) is Riemannian or Lorentzian. In these dimensions we can define self-dual and anti-self-dual k-forms ω, satisfying $*\omega = \omega$ and $*\omega = -\omega$, respectively.

7.9.4 Let (M, g) be an n-dimensional oriented pseudo-Riemannian manifold of signature (s, t) and $*$ the Hodge star operator.

1. Let ∇ denote the Levi-Civita connection of g and suppose that $\alpha \in \Omega^1(M)$ is a 1-form. Prove that if α is parallel ($\nabla\alpha = 0$), then α is closed ($d\alpha = 0$).
2. Let $\eta \in \Omega^1(M)$ be a 1-form, $p \in M$ a point and $e_1, \ldots, e_n$ a local oriented g-orthonormal frame of the tangent bundle in an open neighbourhood of p with $(\nabla e_i)(p) = 0$ for all i. Let $\eta_i = \eta(e_i)$ and $\eta^i = g^{ii}\eta_i$ (no summation). Prove that at the point p

$$(*d*\eta)(p) = (-1)^t \left(\sum_{i=1}^{n} L_{e_i}\eta^i \right)(p).$$

7.9.5 Let (M, g) be a closed (compact without boundary) n-dimensional oriented pseudo-Riemannian manifold of signature (s, t). The **Laplace operator** on k-forms is defined by

$$\Delta = dd^* + d^*d \colon \Omega^k(M) \longrightarrow \Omega^k(M)$$

where d^* is the codifferential from Definition 7.2.9. A form ω is called **harmonic** if $\Delta\omega = 0$. Suppose that (M, g) is Riemannian.

1. Prove that

$$\omega \text{ is harmonic} \Leftrightarrow d\omega = 0 \text{ and } d^*\omega = 0.$$

2. Prove that

$$\omega \text{ is harmonic} \Leftrightarrow *\omega \text{ is harmonic.}$$

7.9.6 Let (M^4, g) be a pseudo-Riemannian 4-manifold with a principal bundle $P \rightarrow M$. Prove that the Yang–Mills action $S_{YM}[A]$ is invariant under a *conformal change* of the metric g:

$$g' = e^{2\lambda} g,$$

where $\lambda \in \mathscr{C}^\infty(M)$ is an arbitrary smooth function on M.

7.9.7

1. Prove that the connection A from Sect. 5.2.2 on the Hopf bundle $S^3 \rightarrow S^2$ with structure group $\mathrm{U}(1)$ satisfies the Yang–Mills equation (i.e. Maxwell's equations) if S^2 has the standard round Riemannian metric.
2. Prove that the Yang–Mills moduli space for the Hopf bundle $S^3 \rightarrow S^2$ over the round sphere S^2 consists of a single point.

7.9.8 Let $M = \mathbb{R}^{1,3}$ be Minkowski spacetime with the flat Minkowski metric η. Let $P \rightarrow M$ be a trivial principal G-bundle with a global gauge $s\colon M \rightarrow P$. For a connection A decompose the curvature $F = F^A$ as in Sect. 5.7 into generalized electric and magnetic fields E and B with values in the Lie algebra $\mathfrak{g}$.

1. Express the Bianchi identity and the Yang–Mills equation in terms of E, B and A.
2. Express the instanton equations $*F = F$ and $*F = -F$ in terms of E and B.

7.9.9

1. On the Hopf bundle $S^7 \to S^4$ with structure group SU(2) define in analogy to the construction in Sect. 5.2.2 an explicit connection 1-form $A \in \Omega^1\left(S^7, \mathfrak{su}(2)\right)$ using quaternions.
2. Prove that A is an anti-self-dual instanton for the standard round Riemannian metric on S^4.

7.9.10 Let (M, g) be a closed oriented pseudo-Riemannian manifold, $P \to M$ a principal G-bundle with compact structure group G and $E \to M$ an associated vector bundle with Hermitian bundle metric $\langle \cdot\,, \cdot \rangle_E$. We fix an Ad-invariant positive definite scalar product on the Lie algebra $\mathfrak{g}$ and consider the Yang–Mills–Higgs Lagrangian

$$\begin{aligned}\mathscr{L}_{YMH}[\Phi, A] &= \mathscr{L}_H[\Phi, A] + \mathscr{L}_{YM}[A] \\ &= \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.\end{aligned}$$

We are looking for critical points of the associated action S_{YMH} under variations of Φ and A.

1. Prove that variation of the field Φ leads to the field equation
$$d_A^* d_A \Phi = V'(\Phi)\Phi, \tag{7.6}$$
where V' is the derivative of $V\colon \mathbb{R} \to \mathbb{R}$ and $V'(\Phi) = V'(\langle \Phi, \Phi\rangle_E)$.
2. Show that elements $\alpha_M \in \Omega^1(M, \mathrm{Ad}(P))$, $\Phi \in \Gamma(E)$ define a canonical twisted 1-form $\alpha_M \cdot \Phi \in \Omega^1(M, E)$.
3. Prove that there exists a unique twisted 1-form
$$J_H(A, \Phi) \in \Omega^1(M, \mathrm{Ad}(P))$$
such that
$$\langle \alpha_M, J_H(A, \Phi)\rangle_{\mathrm{Ad}(P)} = 2\mathrm{Re}\left(\langle d_A\Phi, \alpha_M \cdot \Phi\rangle_E\right)$$
for all $\alpha_M \in \Omega^1(M, \mathrm{Ad}(P))$.
4. Show that variation of the connection A leads to the field equation
$$d_A^* F_M^A = J_H(A, \Phi). \tag{7.7}$$

Equations (7.6) and (7.7) are called **Yang–Mills–Higgs equations**.

7.9.11 Prove the statement in Theorem 7.6.4 concerning the gauge invariance of the Dirac Lagrangian.

7.9.12

1. Under the assumptions of Theorem 6.11.5, define a 1-form $\eta \in \Omega^1(M, \mathbb{C})$ by

$$\eta(X) = \langle X \cdot \Phi, \Psi \rangle_{S\otimes E} \quad \forall X \in \mathfrak{X}(M)$$

   and prove that

$$\langle D_A\Phi, \Psi\rangle_{S\otimes E} - \langle \Phi, D_A\Psi\rangle_{S\otimes E} = (-1)^t * d * \eta$$

   (Exercise 7.9.4 could be helpful).
2. Prove Theorem 6.11.5.
3. Discuss what can be said in the case $\delta = +1$ and the implications for the Dirac Lagrangian.

7.9.13 Let (M, g) be an n-dimensional closed oriented and time-oriented pseudo-Riemannian spin manifold, $S \to M$ a spinor bundle with Dirac bundle metric $\langle \cdot\,, \cdot \rangle_S$ with $\delta = -1$, $P \to M$ a principal G-bundle with compact structure group G and $E \to M$ an associated vector bundle with Hermitian bundle metric $\langle \cdot\,, \cdot \rangle_E$. We fix an Ad-invariant positive definite scalar product on the Lie algebra $\mathfrak{g}$ and consider the Yang–Mills–Dirac Lagrangian

$$\begin{aligned}\mathscr{L}_{YMD}[\Psi, A] &= \mathscr{L}_D[\Psi, A] + \mathscr{L}_{YM}[A] \\ &= \mathrm{Re}\left(\overline{\Psi} D_A \Psi\right) - m\overline{\Psi}\Psi - \frac{1}{2}\left\langle F_M^A, F_M^A\right\rangle_{\mathrm{Ad}(P)}.\end{aligned}$$

We are looking for critical points of the associated action S_{YMD} under variations of Ψ and A.

1. Prove that variation of the spinor Ψ leads to the **Dirac equation**

$$D_A\Psi = m\Psi.$$

2. Show that $\alpha_M \in \Omega^1(M, \mathrm{Ad}(P))$ and $\Psi \in \Gamma(S \otimes E)$ define via Clifford multiplication a canonical section $\alpha_M \cdot \Psi \in \Gamma(S \otimes E)$.
3. Prove that there exists a unique twisted 1-form

$$J_D(\Psi) \in \Omega^1(M, \mathrm{Ad}(P))$$

   such that

$$\langle \alpha_M, J_D(\Psi)\rangle_{\mathrm{Ad}(P)} = \mathrm{Re}\,(\langle \Psi, \alpha_M \cdot \Psi\rangle_{S\otimes E})$$

   for all $\alpha_M \in \Omega^1(M, \mathrm{Ad}(P))$.
4. Show that variation of the connection A leads to the field equation

$$d_A^* F_M^A = J_D(\Psi).$$