# Chapter 4
# Fibre Bundles

What is gauge theory? It is not an overstatement to say that gauge theory is ultimately the theory of *principal bundles* and *associated vector bundles*. Besides full gauge theories, it also proves beneficial in certain situations to study the theory only involving principal bundles, sometimes called *Yang–Mills theory*. In physics, an example of a full gauge theory would be quantum chromodynamics (QCD), the theory of quarks, gluons and their interactions, while pure Yang–Mills theory would be a theory only of gluons, also called gluodynamics. Even such a simplified theory is very interesting – the Clay Millennium Prize Problem [37] on the mass gap, for instance, is a problem concerning the spectrum of *glueballs* in pure quantum Yang–Mills theory.

With the background knowledge of Lie groups, Lie algebras, representations and group actions, we will now study fibre bundles in general and more specifically principal bundles, vector bundles and associated bundles, which together form the core or the "stage" of gauge theories.

Fibre bundles can be thought of as twisted, non-trivial products between a base manifold and a fibre manifold. Principal and vector bundles are fibre bundles whose fibres are, respectively, Lie groups and vector spaces, so that the bundle admits a special type of *bundle atlas*, preserving some of the additional structure of the fibres.

The fundamental geometric object in a gauge theory is a principal bundle over spacetime with *structure group* given by the gauge group. The fibres of a principal bundle are sometimes thought of as an internal space at every spacetime point, not belonging to spacetime itself. The gauge group acts at every spacetime point on the internal space in a simply transitive way. *Connections* on principal bundles, that we discuss in Chap. 5, correspond to *gauge fields*, whose particle excitations in the associated quantum field theory are the *gauge bosons* that transmit interactions. Matter fields in the Standard Model, like quarks and leptons, or scalar fields, like the Higgs field, correspond to sections of vector bundles *associated* to the principal bundle (and twisted by *spinor bun-*

 
M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
https://doi.org/10.1007/978-3-319-68439-0_4

*dles* in the case of fermions). The ultimate reason for the interaction between matter fields and gauge fields is that both are related to the same principal bundle.

Fibre bundles are indispensable in gauge theory and physics in the situation where spacetime, the *base manifold*, has a non-trivial topology. This happens, for example, in string theory where spacetime is typically assumed to be a product $\mathbb{R}^4 \times K$ of Minkowski spacetime with a compact Riemannian manifold $K$. It also happens if we compactify (Euclidean) spacetime $\mathbb{R}^4$ to the 4-sphere $S^4$. In these situations, fields on spacetime often cannot be described simply by a map to a fixed vector space, but rather as *sections* of a non-trivial vector bundle.

Even in the case where the fibre bundles are trivial, for example, in the case of principal bundles and vector bundles over contractible manifolds like $\mathbb{R}^n$, there is still a small, but important difference between a trivial fibre bundle and the choice of an actual trivialization. We will see that this is similar to the difference in special relativity between Minkowski spacetime and the choice of an inertial system.

Fibre bundles are not only important in physics, but for a variety of reasons also in differential geometry and differential topology: many non-trivial manifolds can be constructed as (total spaces of) fibre bundles and numerous structures on manifolds, such as vector fields, differential forms and metrics, are defined using bundles. Mathematically, we are especially interested in the construction of *non-trivial* fibre bundles (trivial bundles are just globally products). We discuss the following methods that (potentially) yield non-trivial bundles:

- Mapping tori (Example 4.1.5) and the clutching construction (Sect. 4.6) yield fibre bundles over the circle $S^1$ and higher-dimensional spheres $S^n$.
- Principal group actions define principal bundles (Sect. 4.2.2; specific examples are the famous *Hopf fibrations* and principal bundles over homogeneous spaces).
- Actions of the structure group $G$ of a principal bundle $P \to M$ on another manifold $F$ (the general fibre) yield associated fibre bundles (Sect. 4.7) over $M$. In particular, all vector bundles can be obtained in this way.
- The tangent bundle $TM$ and frame bundle $\mathrm{Fr}(M)$ of smooth manifolds $M$ are specific examples of vector and principal bundles.
- In general, every fibre bundle can be constructed using a cocycle of transition functions (Exercise 4.8.9).

This chapter, like the previous one, contains many definitions and concepts. I hope that there are sufficiently many examples to illustrate the definitions and balance the exposition. References for this chapter for fibre bundles in general are [14, 84, 133] and [136] as well as [5, 25, 39, 74] and [78] for vector bundles in particular.

# 4.1 General Fibre Bundles

## *4.1.1 Definition of Fibre Bundles*

Before we begin with the definition of fibre bundles, we consider two very general notions: suppose $\pi: E \to M$ is a surjective differentiable map between smooth manifolds (occasionally we will consider the following notions even in the case of a surjective map $\pi: E \to M$ between sets).

**Definition 4.1.1**

1. Let $x \in M$ be an arbitrary point. The (non-empty) subset

$$E_x = \pi^{-1}(x) = \pi^{-1}(\{x\}) \subset E$$

is called the **fibre** of $\pi$ over $x$.

2. For a subset $U \subset M$ we set

$$E_U = \pi^{-1}(U) \subset E.$$

We can think of $E_U$ as the part of $E$ “above” $U$. It is clear that $E_U$ is the union of all fibres $E_x$, where $x \in U$.

3. A differentiable map $s: M \to E$ such that

$$\pi \circ s = \mathrm{Id}_M$$

is called a **(global) section** of $\pi$. A differentiable map $s: U \to E$, defined on some open subset $U \subset M$, satisfying

$$\pi \circ s = \mathrm{Id}_U$$

is called a **local section**.

Note that a differentiable map $s: U \to E$ is a (local) section of $\pi: E \to U$ if and only if $s(x) \in E_x$ for all $x \in U$.

For a general surjective map, the fibres $E_x$ and $E_y$ over points $x \neq y \in M$ can be very complicated and different, in particular, they may not be embedded submanifolds of $E$ and even when they are, they may not be diffeomorphic. The simplest example where these properties *do* hold is a product $E = M \times F$ with $\pi$ given by the projection onto the first factor.

**Fibre bundles** are an important generalization of products $E = M \times F$ and can be understood as *twisted* products. The fibres of a fibre bundle are still embedded submanifolds and are all diffeomorphic. However, the fibration in general is only **locally trivial**, i.e. locally a product, and not globally. We shall see later in

Corollary 4.2.9 and Corollary 4.5.12 that if the topology of $M$ is trivial (i.e. $M$ is contractible), then certain types of fibre bundles over $M$ (like principal and vector bundles) are always globally trivial. If $M$ has a non-trivial topology (for example, if $M$ is a sphere $S^n$), this may not be the case.

Consider, for instance, the Hopf action of $S^1 = \mathrm{U}(1)$ on $S^3$, introduced in Definition 3.3.1. This is a free action, i.e. the orbit of every point in $S^3$ is an embedded $S^1$ and the quotient space $S^3/\mathrm{U}(1)$ of this action is the smooth manifold $\mathbb{CP}^1 \cong S^2$.

However, it is clear (e.g. by considering fundamental groups) that $S^3$ cannot be diffeomorphic to $S^2 \times S^1$. We will see in Example 4.2.14 that $S^3$ really is the total space of a *non-trivial* $S^1$-bundle over $S^2$. We denote this bundle by

$$S^1 \longrightarrow S^3 \stackrel{\pi}{\longrightarrow} S^2$$

or

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

This is the celebrated *Hopf fibration*. The total space $S^3$ is simply connected even though the fibres $S^1$ are not. This is possible, because the fibre bundle is globally non-trivial.

General fibre bundles are defined as follows.

**Definition 4.1.2** Let $E, F, M$ be manifolds and $\pi : E \to M$ a surjective differentiable map. Then $(E, \pi, M; F)$ is called a **fibre bundle** (or **locally trivial fibration** or **locally trivial bundle**) if the following holds: For every $x \in M$ there exists an open neighbourhood $U \subset M$ around $x$ such that $\pi$ restricted to $E_U$ can be **trivialized**, i.e. there exists a diffeomorphism

$$\phi_U : E_U \longrightarrow U \times F$$

such that

$$\mathrm{pr}_1 \circ \phi_U = \pi,$$

hence the following diagram commutes:

$$\begin{array}{ccc} E_U & \xrightarrow{\phi_U} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U & \end{array}$$

We also write

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow{\scriptstyle\pi} \\ & & M \end{array}$$

or

$$F \longrightarrow E \xrightarrow{\pi} M$$

to denote a fibre bundle. We call

- $E$ the **total space**
- $M$ the **base manifold**
- $F$ the **general fibre**
- $\pi$ the **projection**
- $(U, \phi_U)$ a **local trivialization** or **bundle chart**.

See Fig. 4.1.

*Remark 4.1.3* The classic references [133] and [81] use the term *fibre bundle* in a more restrictive sense; see Remark 4.1.15.

It is easy to see, using a local trivialization $(U, \phi_U)$, that the fibre

$$E_x = \pi^{-1}(x)$$

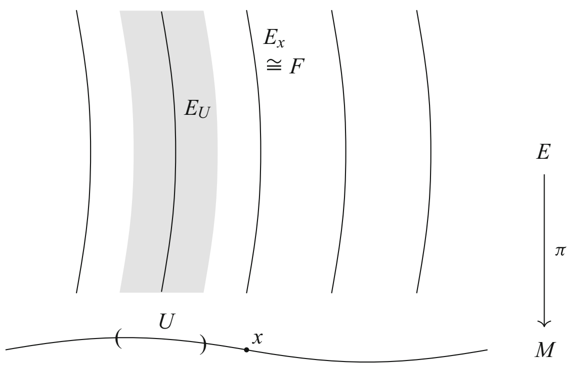


**Fig. 4.1** Fibre bundle

is an embedded submanifold of the total space $E$ for every $x \in M$ and the map $\phi_{Ux}$ defined by

$$\phi_{Ux} = \mathrm{pr}_2 \circ \phi_U|_{E_x} : E_x \longrightarrow F$$

is a diffeomorphism between the fibre over $x \in U$ and the general fibre.

Note that in a local trivialization the map

$$\phi_U : E_U \longrightarrow U \times F$$

is a diffeomorphism and

$$\mathrm{pr}_1 : U \times F \longrightarrow U$$

is a submersion (its differential is everywhere surjective). This implies that the projection $\pi : E \to M$ of a fibre bundle is always a submersion. The Regular Value Theorem A.1.32 then shows again that the fibres $E_x = \pi^{-1}(x)$ are embedded submanifolds of $E$.

*Example 4.1.4 (Trivial Bundle)* Let $M$ and $F$ be arbitrary smooth manifolds and $E = M \times F$. Then $\pi = \mathrm{pr}_1$ defines a fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & M \times F \\ & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & M \end{array}$$

This bundle is called **trivial**.

*Example 4.1.5 (Mapping Torus)* We discuss an example where the idea of a fibre bundle as a "twisted product" becomes very apparent. Let $F$ be a manifold and $\phi : F \to F$ a diffeomorphism. We construct a fibre bundle $E_\phi$ as follows: Take

$$F \times [0, 1]$$

modulo the equivalence relation defined by

$$(x, 0) \sim (\phi(x), 1).$$

The quotient $E_\phi = (F \times [0, 1]) / \sim$ is a fibre bundle over the circle $S^1$ with general fibre $F$:

$$\begin{array}{ccc} F & \longrightarrow & E_\phi \\ & & \downarrow{\scriptstyle \pi} \\ & & S^1 \end{array}$$

The bundle $E_\phi$ is called the **mapping torus** with general fibre $F$ and **monodromy** $\phi$. See Remark 4.6.4 for more details. We can think of the bundle $E_\phi$ as being obtained by gluing the two ends of $F \times [0, 1]$ together using the diffeomorphism $\phi$.

If $\phi$ is the identity, then the mapping torus is a trivial bundle, but if $\phi$ is not the identity, the mapping torus may be non-trivial. For example, for the fibre $F = S^1$ we can do the construction with $\phi$ the identity of $S^1$, in which case $E_\phi$ is diffeomorphic to the torus $T^2$, and with $\phi$ the reflection $z \mapsto \bar{z}$ on $S^1 \subset \mathbb{C}$, in which case $E_\phi$ is diffeomorphic to the Klein bottle. Since the Klein bottle is not diffeomorphic to $T^2$, the second example is a non-trivial $S^1$-bundle over $S^1$.

The *clutching construction* that we discuss in Sect. 4.6 is a generalization of the mapping torus construction which yields fibre bundles

$$\begin{array}{ccc} F & \longrightarrow & E_f \\ & & \big\downarrow \pi \\ & & S^n \end{array}$$

over spheres of arbitrary dimension.

### 4.1.2 Bundle Maps

**Definition 4.1.6** Let $F \to E \xrightarrow{\pi} M$ and $F' \to E' \xrightarrow{\pi'} M$ be fibre bundles over the manifold $M$. A **bundle map** or **bundle morphism** of these bundles is a smooth map $H: E \to E'$ such that

$$\pi' \circ H = \pi,$$

i.e. such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\quad H \quad} & E' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & M & \end{array}$$

A **bundle isomorphism** is a bundle map which is a diffeomorphism. If such an isomorphism exists, we write $E \cong E'$.

*Remark 4.1.7* Note that a morphism $H: E \to E'$ maps a point in the fibre of $E$ over $x \in M$ to a point in the fibre of $E'$ over the same point $x$. *A bundle map therefore covers the identity of $M$.* We could consider more general bundle maps between bundles over different manifolds $M$ and $N$ that cover a given smooth map $f: M \to N$. It is clear that a bundle isomorphism induces a diffeomorphism between the fibres of $E$ and $E'$ over any $x \in M$.

**Definition 4.1.8** Fibre bundles isomorphic to a trivial bundle as in Example 4.1.4 are also called **trivial**.
It is more difficult to construct non-trivial fibre bundles. The mapping tori defined in Example 4.1.5 are for many choices of $(F, \phi)$ non-trivial bundles. We will discuss other examples of (potentially) non-trivial bundles in Sect. 4.2.2 and Sect. 4.6.

*Remark 4.1.9* Let $F \to E \xrightarrow{\pi} M$ be a fibre bundle. The existence of a local trivialization over $U \subset M$ then means that the restricted bundle

$$\pi|_{E_U} : E_U \longrightarrow U$$

is isomorphic to the trivial bundle

$$\mathrm{pr}_1 : U \times F \longrightarrow U.$$

This in hindsight justifies why fibre bundles are called locally trivial.
Isomorphic bundles have diffeomorphic general fibres. The converse is not true in general: There may exist non-isomorphic bundles whose general fibres are diffeomorphic. In particular, as we shall see later in detail, there exist bundles not (globally) isomorphic to a trivial bundle.

We can characterize trivial bundles as follows:

**Proposition 4.1.10 (Trivial Bundles and Projections onto the General Fibre)** *Let $F \to E \xrightarrow{\pi} M$ be a fibre bundle. Then the bundle is isomorphic to a trivial bundle if and only if there exists a smooth map $\tau: E \to F$ such that the restrictions*

$$\tau|_{E_x} : E_x \longrightarrow F$$

*are diffeomorphisms for all $x \in M$.*

*Proof* If the bundle is trivial

$$\begin{array}{ccc} F & \longrightarrow & E = M \times F \\ & & \downarrow \mathrm{pr}_1 \\ & & M \end{array}$$

we can set $\tau = \mathrm{pr}_2$.

Conversely, assume that a map $\tau: E \to F$ exists which restricts to a diffeomorphism on each fibre. Consider the map

$$\begin{aligned} H: E &\longrightarrow M \times F \\ p &\longmapsto (\pi(p), \tau(p)). \end{aligned}$$

Then $H$ is a smooth with

$$\mathrm{pr}_1 \circ H = \pi.$$

The map $H$ is bijective, because it maps $E_x$ bijectively onto $F$.

We have to show that $H$ is a diffeomorphism. We claim that the differential of $H$ is an isomorphism for every point $p \in E$. Since the dimensions of $E$ and $M \times F$ agree (this follows from the existence of local trivializations for $E$), it suffices to show that the differential is surjective for every $p \in E$. The details are left as an exercise. □

### 4.1.3 Bundle Atlases

**Definition 4.1.11** A **bundle atlas** for a fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array}$$

is an open covering $\{U_i\}_{i \in I}$ of $M$ together with bundle charts

$$\phi_i: E_{U_i} \longrightarrow U_i \times F.$$

**Definition 4.1.12** Let $\{(U_i, \phi_i)\}_{i \in I}$ be a bundle atlas for a fibre bundle $F \to E \xrightarrow{\pi} M$. If $U_i \cap U_j \neq \emptyset$, we define the **transition functions** by

$$\phi_j \circ \phi_i^{-1}|_{(U_i \cap U_j) \times F}: (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F.$$

The transition functions are diffeomorphisms. These maps have a special structure, because they preserve fibres: For every $x \in U_i \cap U_j$ we get a diffeomorphism

$$\phi_{jx} \circ \phi_{ix}^{-1}: F \longrightarrow F.$$

The maps

$$\begin{aligned} \phi_{ji}: U_i \cap U_j &\longrightarrow \mathrm{Diff}(F) \\ x &\longmapsto \phi_{jx} \circ \phi_{ix}^{-1} \end{aligned}$$

into the group of diffeomorphisms of $F$ are also called transition functions. See Fig. 4.2.

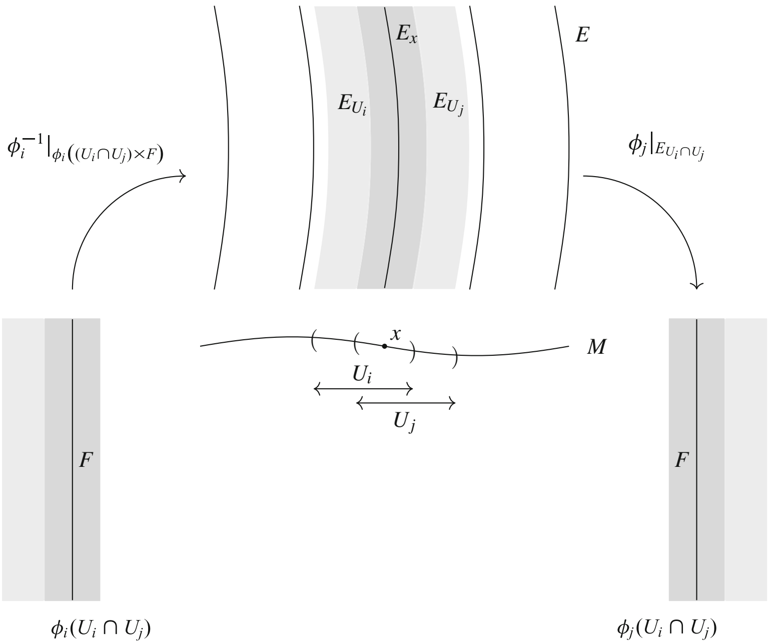


**Fig. 4.2** Transition functions

**Lemma 4.1.13 (Cocycle Conditions)** *The transition functions* $\{\phi_{ji}\}_{i,j \in I}$ *satisfy the following equations:*

$$\begin{aligned} \phi_{ii}(x) &= \mathrm{Id}_F \quad \textit{for } x \in U_i, \\ \phi_{ij}(x) \circ \phi_{ji}(x) &= \mathrm{Id}_F \quad \textit{for } x \in U_i \cap U_j, \\ \phi_{ik}(x) \circ \phi_{kj}(x) \circ \phi_{ji}(x) &= \mathrm{Id}_F \quad \textit{for } x \in U_i \cap U_j \cap U_k. \end{aligned}$$

*The third equation is called the* **cocycle condition***.*

*Proof* Follows immediately from the definitions. □

*Remark 4.1.14* Exercise 4.8.9 shows that a bundle can be (re-)constructed from its transition functions using a suitable quotient space. The three properties of Lemma 4.1.13 ensure the existence of a certain equivalence relation, used in the construction of this quotient space.

A bundle atlas is very similar to an atlas of charts for a manifold. One difference is that in the case of charts for a manifold we demand that the images of the charts are open sets in a Euclidean space $\mathbb{R}^n$. In the case of charts for a bundle the images are of the form $U \times F$. In both cases the transition functions are smooth. In the case of a bundle atlas, the transition functions have an additional special structure, because they preserve fibres.

**Table 4.1** Comparison between notions for manifolds and fibre bundles

| Manifold | Fibre bundle |
|---|---|
| Coordinate chart | Bundle chart |
| Coordinate transformation | Transition functions |
| Atlas | Bundle atlas |
| Trivial manifold with only one chart: $\mathbb{R}^n$ | Trivial bundle with only one bundle chart: $M \times F$ |
| Non-trivial manifold needs at least two charts (like $S^n$) | Non-trivial bundle needs at least two bundle charts (like a non-trivial bundle over $S^n$) |

We can compare the definitions of general manifolds and general fibre bundles as in Table 4.1.

*Remark 4.1.15* Some references, such as [133] and [81], use the term *fibre bundle* more restrictively. If the topological definition in these books is transferred to a smooth setting, the definition amounts to assuming that the transition functions of a bundle atlas are smooth maps to a Lie group $G$, acting smoothly as a transformation group on the fibre $F$, instead of maps to the full diffeomorphism group Diff$(F)$ of the fibre:

$$\begin{aligned} \phi_{ji}\colon U_i \cap U_j &\longrightarrow G \\ x &\longmapsto \phi_{jx} \circ \phi_{ix}^{-1}. \end{aligned}$$

Equivalently, a fibre bundle is with this definition always an associated bundle in the sense of Remark 4.7.8.

## 4.1.4 *Pullback Bundle

We want to show that we can *pull back* a bundle via a map between the base manifolds. Suppose

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \big\downarrow \pi_M \\ & & M \end{array}$$

is a fibre bundle.

**Lemma 4.1.16 (Restriction of Bundle over Submanifold in the Base)** *If $W \subset M$ is an embedded submanifold, then the restriction*

$$\begin{array}{ccc} F & \longrightarrow & E_W \\ & & \big\downarrow \pi_M \\ & & W \end{array}$$

*is a fibre bundle.*

*Proof* Let $\{(U_i, \phi_i)\}_{i\in I}$ be a bundle atlas for the fibre bundle $F \to E \to M$ with bundle charts

$$\phi_i : E_{U_i} \longrightarrow U_i \times F.$$

Then the sets $V_i = U_i \cap W$ form an open covering of $W$ and

$$\psi_i = \phi_i|_{E_{V_i}} : E_{V_i} \longrightarrow V_i \times F$$

are bundle charts for the restriction of $E$ over $W$. □

Suppose $f : N \to M$ is a differentiable map from some manifold $N$ to $M$. We set

$$f^*E = \{(x, e) \in N \times E \mid f(x) = \pi_M(e)\}$$

and

$$\begin{aligned} \pi_N : f^*E &\longrightarrow N \\ (x, e) &\longmapsto x. \end{aligned}$$

**Theorem 4.1.17 (Pullback Bundles)** *The map $\pi_N$ with*

$$\begin{array}{ccc} F & \longrightarrow & f^*E \\ & & \big\downarrow \pi_N \\ & & N \end{array}$$

*is a fibre bundle over $N$, called the* **pullback** *of $E$ under $f$.*

*Proof* We have an obvious fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & N \times E \\ & & \big\downarrow \pi_{N\times M} = \mathrm{Id}_N \times \pi_M \\ & & N \times M \end{array}$$

The graph

$$\Gamma_f = \{(x, f(x)) \in N \times M \mid x \in N\}$$

is an embedded submanifold of $N \times M$. Therefore the restriction

$$\begin{array}{ccc} F & \longrightarrow & \pi_{N\times M}^{-1}(\Gamma_f) \subset N \times E \\ & & \downarrow \pi_{N\times M} \\ & & \Gamma_f \end{array}$$

is a fibre bundle by Lemma 4.1.16. Note that

$$(x, e) \in \pi_{N\times M}^{-1}(\Gamma_f) \Leftrightarrow \pi(e) = f(x).$$

Hence as a set

$$\pi_{N\times M}^{-1}(\Gamma_f) = f^*E,$$

which defines a smooth structure on $f^*E$, and we have a fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & f^*E \\ & & \downarrow \pi_{N\times M} \\ & & \Gamma_f \end{array}$$

There exists a diffeomorphism

$$\begin{aligned} \tau\colon \Gamma_f &\longrightarrow N \\ (x, f(x)) &\longmapsto x. \end{aligned}$$

We can define a bundle over $N$ using the projection $\tau \circ \pi_{N\times M}$:

$$\begin{array}{ccccc} F & \longrightarrow & f^*E & & \\ & & \pi_{N\times M}\downarrow & \searrow & \\ & & \Gamma_f & \xrightarrow{\tau} & N \end{array}$$

But

$$\tau \circ \pi_{N\times M}(x, e) = \tau(x, \pi(e)) = x = \pi_N(x, e),$$

hence

$$\tau \circ \pi_{N\times M} = \pi_N.$$

This shows that

$$\begin{array}{ccc} F & \longrightarrow & f^*E \\ & & \big\downarrow \pi_N \\ & & N \end{array}$$

is a fibre bundle. □

*Remark 4.1.18* Note that the pullback bundle $f^*E$ has the same general fibre $F$ as the bundle $E$. The fibre of $f^*E$ over a point $x \in N$ is canonically diffeomorphic to the fibre of $E$ over $f(x) \in M$ via the map

$$\begin{aligned} \left(f^*E\right)_x &\longrightarrow E_{f(x)} \\ (x, e) &\longmapsto e. \end{aligned}$$

*Remark 4.1.19* It is not difficult to show that the pull-back of a trivial bundle is always trivial. The pull-back of a non-trivial bundle may be non-trivial or trivial, depending on the situation; see Exercise 4.8.2.

## *4.1.5 Sections of Bundles*

We want to study sections of fibre bundles. This is particularly simple in the case of trivial bundles.

**Definition 4.1.20** Let

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array}$$

be a fibre bundle. We denote the set of smooth global sections $s: M \to E$ by $\Gamma(E)$ and the set of smooth local sections $s: U \to E$, for $U \subset M$ open, by $\Gamma(U, E)$.
It is easy to see that for a trivial bundle $E$ there is a 1-to-1 correspondence between sections of $E$ and maps from the base manifold $M$ to the general fibre $F$. This implies:

**Corollary 4.1.21 (Existence of (Local) Sections)**

1. *Every trivial fibre bundle has smooth global sections (for example, under the above correspondence, we could take constant maps from the base $M$ to the fibre $F$).*
2. *Every fibre bundle has smooth local sections, since every fibre bundle is locally trivial.*

Note that non-trivial fibre bundles can, but do not need to have smooth *global* sections (for example, vector bundles, to be discussed later, always have global sections, but principal bundles in general do not). In particular, for a non-trivial bundle, a map from the base manifold to the general fibre usually does not define a section.

## 4.2 Principal Fibre Bundles

Principal fibre bundles are a combination of the concepts of fibre bundles and group actions: they are fibre bundles which also have a Lie group action so that both structures are compatible in a certain sense. Principal bundles together with so-called *connections* play an important role in gauge theory. Generally speaking, principal bundles are the primary place where Lie groups appear in gauge theories (Lie groups also appear as global symmetry groups, like the *flavour* or *chiral symmetry* in QCD; see Sect. 9.1).

### *4.2.1 Definition of Principal Bundles*

We consider again the Hopf action of $S^1$ on $S^3$, introduced in Definition 3.3.1, with quotient space equal to $\mathbb{CP}^1 \cong S^2$. If we accept for the moment that $S^3$ is the total space of an $S^1$-bundle over $S^2$,

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \big\downarrow \pi \\ & & S^2 \end{array}$$

(we will prove this in Example 4.2.14), then we can say the following: there is an action of the Lie group $S^1$ on the total space $S^3$ of the bundle which preserves the fibres and is simply transitive on them. In addition we will show that there is a special type of bundle atlas for the Hopf fibration which is compatible with this $S^1$-action.

This leads us to the following definition:

**Definition 4.2.1** Let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

be a fibre bundle with general fibre a Lie group $G$ and a smooth action $P \times G \rightarrow P$ on the right. Then $P$ is called a **principal $G$-bundle** if:

1. The action of $G$ **preserves the fibres of $\pi$ and is simply transitive on them**, i.e. the action restricts to

$$P_x \times G \longrightarrow P_x$$

and the orbit map

$$G \longrightarrow P_x$$
$$g \longmapsto p \cdot g$$

is a bijection, for all $x \in M, p \in P_x$.
2. There exists a **bundle atlas of $G$-equivariant bundle charts** $\phi_i : P_{U_i} \rightarrow U_i \times G$, satisfying

$$\phi_i(p \cdot g) = \phi_i(p) \cdot g \quad \forall p \in P_{U_i}, g \in G,$$

where on the right-hand side $G$ acts on $(x, a) \in U_i \times G$ via

$$(x, a) \cdot g = (x, ag).$$

We also call such an atlas a **principal bundle atlas** for $P$ and the charts in a principal bundle atlas **principal bundle charts**.

The group $G$ is called the **structure group** of the principal bundle $P$.

There are two features that distinguish a principal bundle $P \rightarrow M$ from a standard fibre bundle whose general fibre is a Lie group $G$:

1. there exists a right $G$-action on $P$, simply transitive on each fibre $P_x$, for $x \in M$;
2. the bundle $P$ has a principal bundle atlas.

If $P \rightarrow M$ is a principal $G$-bundle and $g \in G$, then we denote as before by $r_g$ the right translation

$$r_g : P \longrightarrow P$$
$$p \longmapsto p \cdot g.$$

The fibre $P_x$ is a submanifold of the total space $P$ for every $x \in M$ and the orbit map

$$\begin{aligned} G &\longrightarrow P_x \\ g &\longmapsto p \cdot g \end{aligned}$$

is an embedding for all $p \in P_x$, according to Corollary 3.8.10, because the stabilizer $G_p = \{e\}$ is trivial.

*Example 4.2.2* The trivial bundle

$$\begin{array}{ccc} G & \longrightarrow & M \times G \\ & & \big\downarrow \mathrm{pr}_1 \\ & & M \end{array}$$

has the canonical structure of a principal $G$-bundle with $G$-action

$$\begin{aligned} (M \times G) \times G &\longrightarrow M \times G \\ (x, h, g) &\longmapsto (x, hg) \end{aligned}$$

and the principal bundle atlas consisting of only one bundle chart

$$\mathrm{Id}: M \times G \longrightarrow M \times G.$$

*Example 4.2.3* If $G \to P \xrightarrow{\pi} M$ is a principal bundle and $f: N \to M$ a smooth map, then the pullback $f^*P$ has the canonical structure of a principal $G$-bundle over $N$ (this is Exercise 4.8.4).

Transition functions for a principal bundle atlas have a special form:

**Proposition 4.2.4 (Transition Functions of Principal Bundles)** *Let $P \to M$ be a principal $G$-bundle and $\{(U_i, \phi_i)\}_{i \in I}$ a principal bundle atlas for $P$. Then the transition functions take values in the subgroup $G$ of* Diff$(G)$,

$$\begin{aligned} \phi_{ji}: U_i \cap U_j &\longrightarrow G \subset \mathrm{Diff}(G) \\ x &\longmapsto \phi_{jx} \circ \phi_{ix}^{-1} \end{aligned}$$

*where an element $g \in G$ acts as a diffeomorphism on $G$ through left multiplication,*

$$g(h) = g \cdot h.$$

*Proof* For $x \in U_i \cap U_j$ we have a diffeomorphism

$$\phi_{jx} \circ \phi_{ix}^{-1}: G \longrightarrow G.$$

We set

$$g = \phi_{jx} \circ \phi_{ix}^{-1}(e).$$

Then by equivariance of the bundle charts

$$\phi_{jx} \circ \phi_{ix}^{-1}(h) = g \cdot h.$$

This implies the claim. □

The following criterion sometimes simplifies the task of showing that a group action on a manifold $P$ defines a principal bundle (we follow [14, Theorem 2.4]).

**Theorem 4.2.5 (Principal Bundles Defined via Local Sections)** *Let $G$ be a Lie group and $\pi: P \to M$ a smooth surjective map between manifolds with a smooth action $P \times G \to P$ on the right. Then $P$ is a principal $G$-bundle if and only if the following holds:*

1. *The action of $G$ preserves the fibres of $\pi$ and is simply transitive on them.*
2. *There exists an open covering $\{U_i\}_{i\in I}$ of $M$ together with local sections $s_i: U_i \to P$ of the map $\pi$.*

*Remark 4.2.6* Recall that we defined in Sect. 4.1.1 the notion of a section for any smooth surjective map, not only for fibre bundles.

*Proof* Suppose that $\pi: P \to M$ is a principal bundle. Choose a principal bundle atlas $\{(U_i, \phi_i)\}$ for $P$ with

$$\phi_i: P_{U_i} \longrightarrow U_i \times G.$$

Then the following maps are local sections

$$\begin{aligned} s_i: U_i &\longrightarrow P \\ x &\longmapsto \phi_i^{-1}(x, e), \end{aligned}$$

where $e \in G$ is the neutral element.

Conversely, suppose that an open covering $\{U_i\}_{i\in I}$ with sections $s_i: U_i \to P$ is given. According to the following lemma these sections define charts in a principal bundle atlas for $P$. □

**Lemma 4.2.7** *Let $G$ be a Lie group and $\pi: P \to M$ a smooth surjective map between manifolds with a smooth action $P \times G \to P$ on the right. Suppose that the* *action of G preserves the fibres of* $\pi$ *and is simply transitive on them. Let* $s\colon U \to P$ *be a local section for* $\pi$*. Then*

$$\begin{aligned} t\colon U \times G &\longrightarrow P_U \\ (x, g) &\longmapsto s(x) \cdot g \end{aligned}$$

*is a G-equivariant diffeomorphism.*

*Proof* Let $s\colon U \to P$ be a local section of the surjective map $\pi\colon P \to M$. We have to show that

$$\begin{aligned} t\colon U \times G &\longrightarrow P_U \\ (x, g) &\longmapsto s(x) \cdot g \end{aligned}$$

is a $G$-equivariant diffeomorphism. It is clear that $t$ is smooth, because the local section $s$ is smooth and the $G$-action on $P$ is smooth. The map $t$ is also $G$-equivariant by the definition of group actions and it is bijective: the reason is that the map

$$\begin{aligned} t(x, \cdot)\colon G &\longrightarrow P_x \\ g &\longmapsto s(x) \cdot g \end{aligned}$$

is bijective for every fixed $x \in U$, since the $G$-action on $P$ is simply transitive on the fibres. The set $P_U = \pi^{-1}(U)$ is an open subset of $P$. Since $t$ is smooth and surjective, Sard's Theorem A.1.27 implies that

$$\dim P = \dim P_U \leq \dim M + \dim G.$$

It remains to show that the differential of $t$ is injective in each point $(x, g) \in U \times G$. Then $t$ is a diffeomorphism.

The differential

$$D_{(x,g)}t\colon T_xM \times T_gG \longrightarrow T_{s(x)\cdot g}P$$

is given according to Proposition 3.5.4 by

$$D_{(x,g)}t(X, Y) = D_x(r_g \circ s)(X) + \widetilde{\mu_G(Y)}_{s(x)\cdot g}.$$

We set

$$s' = r_g \circ s.$$

The map

$$\begin{aligned} T_g G &\longrightarrow T_{s'(x)} P_x \\ Y &\longmapsto \widetilde{\mu_G(Y)}_{s'(x)} \end{aligned}$$

is an isomorphism, because the action of $G$ is simply transitive on the fibre $P_x$, cf. Corollary 3.2.12. We consider the map

$$\begin{aligned} T_x M &\longrightarrow T_{s'(x)} P \\ X &\longmapsto D_x s'(X). \end{aligned}$$

Note that $s'$ is also a local section of $P$ over $U$, since

$$\pi \circ s' = \mathrm{Id}_U.$$

The chain rule shows that

$$D_{s'(x)}\pi \circ D_x s' = \mathrm{Id}_{T_x M}.$$

This implies that $D_x s'$ is injective and the image of $D_x s'$ intersected with $T_{s'(x)} P_x \subset \ker D_{s'(x)}\pi$ is zero. We conclude that $D_{(x,g)} t$ is injective. □

A proof of the following theorem can be found in [81, Chap. 4, Corollary 10.3].

**Theorem 4.2.8 (Principal Bundles and Homotopy Equivalences)** *Let $f\colon M \to N$ be a smooth homotopy equivalence between manifolds and $G$ a Lie group. Then the pullback $f^*$ is a bijection between isomorphism classes of principal $G$-bundles over $N$ and principal $G$-bundles over $M$.*

In particular we get:

**Corollary 4.2.9 (Principal Bundles over Contractible Manifolds Are Trivial)** *If $M$ is a contractible manifold and $G$ a Lie group, then every principal $G$-bundle over $M$ is trivial. This holds, in particular, if $M = \mathbb{R}^n$ for some $n$.*

### 4.2.2 *Principal Bundles Defined by Principal Group Actions

Recall from Definition 3.7.24 that a smooth right action of a Lie group $G$ on a manifold $P$ is called principal if the action is free and the map

$$\begin{aligned} \Psi\colon P \times G &\longrightarrow P \times P \\ (p, g) &\longmapsto (p, pg) \end{aligned}$$

is closed. We want to show as an application of Theorem 4.2.5 that principal Lie group actions define principal bundles.

**Theorem 4.2.10 (Principal Lie Group Actions Define Principal Bundles)** *Let $\Phi$ be a principal right action of a Lie group $G$ on a manifold $P$. Then $P/G$ is a smooth manifold and*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & P/G \end{array}$$

*is a principal bundle with structure group $G$.*

*Proof* According to Theorem 3.7.25 the topological space $P/G$ has the unique structure of a smooth manifold so that $\pi : P \to P/G$ is a submersion. In particular, by Lemma 3.7.4, the projection $\pi$ admits local sections

$$s_i : U_i \longrightarrow P.$$

The claim then follows from Theorem 4.2.5. □

**Corollary 4.2.11 (Free Actions by Compact Lie Groups Define Principal Bundles)** *Let $G$ be a compact Lie group acting freely on a smooth manifold $P$. Then $P/G$ is a smooth manifold and*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & P/G \end{array}$$

*is a principal $G$-bundle.*

*Proof* This follows from Corollary 3.7.29. □

We can also prove the following converse to Theorem 4.2.10.

**Theorem 4.2.12 (Principal Bundles Define Principal Actions)** *Let*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

*be a principal $G$-bundle. Then the right action of $G$ on $P$ is principal.*

*Proof* The $G$-action on $P$ is free by the definition of principal bundles. If $G$ is compact, then the claim follows from Corollary 3.7.29. In the general case, consider the map

$$\begin{aligned} \Psi : P \times G &\longrightarrow P \times P \\ (p, g) &\longmapsto (p, p \cdot g). \end{aligned}$$

We have to show that $\Psi$ is closed.

Let $A \subset P \times G$ be a closed subset and $((p_i, q_i))_{i \in \mathbb{N}} \in \Psi(A)$ a sequence converging to $(p, q) \in P \times P$. There exist uniquely determined $g_i \in G$ such that $q_i = p_i \cdot g_i$, where $(p_i, g_i) \in A$ and

$$\Psi(p_i, g_i) = (p_i, q_i).$$

We want to show that the sequence $(g_i)_{i \in \mathbb{N}}$ converges in $G$.

Let $\pi(p) = x$ and $U \subset M$ be an open neighbourhood of $x$ with a principal bundle chart

$$\phi: P_U \longrightarrow U \times G.$$

There exists an integer $N$ such that for all $i \geq N$ the $p_i$ are contained in $P_U$. Then we can write

$$\begin{aligned}
\phi(p_i) &= (x_i, h_i), \\
\phi(q_i) &= (x_i, h_i g_i), \\
\phi(p) &= (x, h),
\end{aligned}$$

with certain $x_i \in U$ and $h_i, h \in G$. Since $q_i \to q$ and $x_i \to x$, it follows that

$$\phi(q) = \left(x, h'\right)$$

for some $h' \in G$. Since $h_i \to h$ and $h_i g_i \to h'$, it follows that the sequence

$$g_i = h_i^{-1}(h_i g_i)$$

converges in $G$ to

$$g = h^{-1} h'.$$

The set $A$ is closed, hence $(p, g) \in A$. We have $q = p \cdot g$ and we conclude that $(p, q)$ is in $\Psi(A)$. □

**Corollary 4.2.13** *Principal bundles with structure group G correspond precisely to principal G-actions.*

*Example 4.2.14 (Hopf Fibration)* Let

$$S^{2n+1} = \left\{ (w_0, \ldots, w_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum_{i=0}^{n} |w_i|^2 = 1 \right\}$$

be a sphere of odd dimension. Consider the Lie group $S^1 = \mathrm{U}(1) \subset \mathbb{C}$ (unit circle). It acts on the sphere $S^{2n+1}$ via

$$\begin{aligned} S^{2n+1} \times S^1 &\longrightarrow S^{2n+1} \\ (w, \lambda) &\longmapsto w\lambda. \end{aligned}$$

This is the Hopf action from Definition 3.3.1. The quotient $S^{2n+1}/\mathrm{U}(1)$ of this action can be identified with the complex projective space $\mathbb{CP}^n$. Corollary 4.2.11 implies that

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \pi \\ & & \mathbb{CP}^n \end{array}$$

is a principal $S^1$-bundle, called the **Hopf fibration** or **Hopf bundle**.

To give an alternative proof of this statement, we can also apply Theorem 4.2.5 directly (we follow [14, Example 2.7]). We have to find an open covering of $\mathbb{CP}^n$ together with local sections (the first condition in the theorem is clearly satisfied, because the action of $S^1$ on $S^{2n+1}$ is free). We set

$$\pi(w_0, \ldots, w_n) = [w_0 : \ldots : w_n] \in \mathbb{CP}^n$$

and define for $i = 0, \ldots, n$

$$U_i = \{[w] = [w_0 : \ldots : w_n] \in \mathbb{CP}^n \mid w_i \neq 0\}.$$

The subset $U_i$ is open in $\mathbb{CP}^n$, since $\pi$ is an open map by Lemma 3.7.11. We also set

$$v_i([w]) = \left( \frac{w_0}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, 1, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_n}{w_i} \right) \in \mathbb{C}^{n+1} \setminus \{0\}$$

and

$$\begin{aligned} s_i \colon U_i &\longrightarrow S^{2n+1} \\ [w] &\longmapsto \frac{v_i([w])}{||v_i([w])||}. \end{aligned}$$

These are well-defined local sections for the canonical projection $\pi$:

$$\pi \circ s_i = \mathrm{Id}_{U_i},$$

since $s_i([w])$ is a complex multiple of $w$. Therefore we see again that $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$ is a principal fibre bundle.

It is clear (considering fundamental groups, for example) that $S^{2n+1}$ is not diffeomorphic to $\mathbb{CP}^n \times S^1$. The Hopf fibration is thus an example of a non-trivial (principal) fibre bundle.

Similar arguments for the standard action of the Lie group $S^3 \subset \mathbb{H}$ on $S^{4n+3} \subset \mathbb{H}^{n+1}$ lead to a Hopf fibration

$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \pi \\ & & \mathbb{HP}^n \end{array}$$

over the quaternionic projective space $\mathbb{HP}^n$ (there is also a principal $\mathbb{Z}_2$-bundle $S^n \to \mathbb{RP}^n$ over real projective space). Special cases of this construction are the Hopf fibrations (see Exercise 3.12.9)

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

and

$$\begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \downarrow \pi \\ & & S^4 \end{array}$$

We consider another class of examples of principal bundles. Let $G$ be a Lie group and $H \subset G$ a closed subgroup, acting smoothly on $G$ by right translations:

$$\begin{aligned} \Phi\colon G \times H &\longrightarrow G \\ (g,h) &\longmapsto gh. \end{aligned}$$

According to Corollary 3.7.35 there is a (unique) smooth structure on the quotient space $G/H$, so that $\pi\colon G \to G/H$ is a submersion.

**Theorem 4.2.15 (The Canonical Principal Bundles over Homogeneous Spaces)** *If $G$ is a Lie group and $H \subset G$ a closed subgroup, then*

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \pi \\ & & G/H \end{array}$$

*is a principal bundle with structure group $H$.*

*Proof* This follows from Theorem 4.2.10. We can also verify the conditions of Theorem 4.2.5 directly. The first condition is clearly satisfied, because the action of $H$ on $G$ is free. By Lemma 3.7.4 there exist smooth local sections

$$s_i \colon U_i \longrightarrow G$$

for the canonical projection $\pi \colon G \to G/H$, where the open subsets $U_i \subset G/H$ cover $G/H$. This proves the claim. □

*Example 4.2.16 (Principal Bundles over Homogeneous Spheres)* From Example 3.8.11 we get the following principal bundles over spheres:

$$\begin{array}{ccc} \mathrm{O}(n-1) & \longrightarrow & \mathrm{O}(n) \\ & & \big\downarrow \pi \\ & & S^{n-1} \end{array}$$

$$\begin{array}{ccc} \mathrm{SO}(n-1) & \longrightarrow & \mathrm{SO}(n) \\ & & \big\downarrow \pi \\ & & S^{n-1} \end{array}$$

$$\begin{array}{ccc} \mathrm{U}(n-1) & \longrightarrow & \mathrm{U}(n) \\ & & \big\downarrow \pi \\ & & S^{2n-1} \end{array}$$

$$\begin{array}{ccc} \mathrm{SU}(n-1) & \longrightarrow & \mathrm{SU}(n) \\ & & \big\downarrow \pi \\ & & S^{2n-1} \end{array}$$

$$\begin{array}{ccc} \mathrm{Sp}(n-1) & \longrightarrow & \mathrm{Sp}(n) \\ & & \big\downarrow \pi \\ & & S^{4n-1} \end{array}$$

In particular, we get the following principal sphere bundles over spheres:

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathrm{SO}(3) \\ & & \big\downarrow \pi \\ & & S^2 \end{array}$$

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathrm{U}(2) \\ & & \big\downarrow \pi \\ & & S^3 \end{array}$$

$$\begin{array}{ccc} S^3 & \longrightarrow & \mathrm{SU}(3) \\ & & \big\downarrow \pi \\ & & S^5 \end{array}$$

$$\begin{array}{ccc} S^3 & \longrightarrow & \mathrm{Sp}(2) \\ & & \big\downarrow \pi \\ & & S^7 \end{array}$$

From the examples in Sect. 3.9 we also get principal bundles over the Stiefel and Grassmann manifolds, such as

$$\begin{array}{ccc} \mathrm{O}(n-k) & \longrightarrow & \mathrm{O}(n) \\ & & \big\downarrow \pi \\ & & V_k(\mathbb{R}^n) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{O}(k) \times \mathrm{O}(n-k) & \longrightarrow & \mathrm{O}(n) \\ & & \big\downarrow \pi \\ & & Gr_k(\mathbb{R}^n) \end{array}$$

and similarly for the complex and quaternionic Stiefel and Grassmann manifolds. According to the results in Sect. 3.10.4 there is a principal bundle

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathrm{G}_2 \\ & & \big\downarrow \\ & & V_2(\mathbb{R}^7) \end{array}$$

and according to Exercise 3.12.16 there is a principal bundle

$$\begin{array}{ccc} \mathrm{SU}(3) & \longrightarrow & \mathrm{G}_2 \\ & & \big\downarrow \\ & & S^6 \end{array}$$

### 4.2.3 Bundle Morphisms, Reductions of the Structure Group and Gauges

We define homomorphisms of principal bundles as follows:

**Definition 4.2.17** Suppose $G \to P \xrightarrow{\pi} M$ and $G' \to P' \xrightarrow{\pi'} M$ are principal bundles over the same base manifold $M$ and $f: G \to G'$ is a Lie group homomorphism. Then a **bundle morphism** between $P$ and $P'$ is an $f$**-equivariant** smooth bundle map $H: P \to P'$, i.e.

$$\pi' \circ H = \pi$$

and

$$H(p \cdot g) = H(p) \cdot f(g) \quad \forall p \in P, g \in G.$$

Given the principal $G'$-bundle $P'$ and the homomorphism $f: G \to G'$, the principal $G$-bundle $P$ together with the bundle morphism $H: P \to P'$ is also known as an $f$**-reduction** of $P'$.

If $f: G \to G'$ is an embedding, then $H$ is called a $G$**-reduction** of $P'$ and the image of $H$ is called a **principal** $G$**-subbundle** of $P'$. If $G = G', f = \mathrm{Id}_G$ and $H$ is a $G$-equivariant bundle isomorphism, then $H$ is called a **bundle isomorphism**.

A principal $G$-bundle isomorphic to the trivial bundle in Example 4.2.2 is also called trivial.

As before in the case of general bundles we could consider morphisms between principal bundles over different base manifolds $M$ and $N$ that cover a smooth map from $M$ to $N$.

The following notion is especially relevant in gauge theory.

**Definition 4.2.18** Let $\pi: P \to M$ be a principal bundle. A **global gauge** for the principal bundle is a global section $s: M \to P$. Similarly, a **local gauge** is a local section $s: U \to P$ defined on an open subset $U \subset M$.

Any local gauge defines a local trivialization of a principal bundle:

**Theorem 4.2.19 (Gauges Correspond to Trivializations)** *Let*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

*be a principal G-bundle and* $s: U \to P$ *a local gauge. Then*

$$\begin{aligned} t: U \times G &\longrightarrow P_U \\ (x, g) &\longmapsto s(x) \cdot g \end{aligned}$$

*is a G-equivariant diffeomorphism. In particular, if* $s: M \to G$ *is a global gauge, then the principal bundle is trivial, with trivialization given by the inverse of t:*

$$t^{-1}: P \longrightarrow M \times G.$$

*Proof* This follows from Lemma 4.2.7. □

*Remark 4.2.20* Note that for this construction to work we need the $G$-action on $P$. The result would not hold if we just had a fibre bundle with fibre $G$.

*Remark 4.2.21* Theorem 4.2.19 has the following interpretation, see Table 4.2: A local gauge defines a local trivialization of a principal $G$-bundle, i.e. an identification $\pi^{-1}(U) \cong U \times G$. *A choice of local gauge thus corresponds to the choice of a local coordinate system for a principal bundle in the fibre direction.* This can be compared, in special relativity, to the choice of an inertial system for Minkowski spacetime $M$, which defines an identification $M \cong \mathbb{R}^4$.

Of course, different choices of gauges are possible, leading to different trivializations of the principal bundle, just as different choices of inertial systems lead to different identifications of spacetime with $\mathbb{R}^4$. The idea of gauge theory is that physics *should be independent of the choice of gauge*. This can be compared to the theory of relativity which says that physics is independent of the choice of inertial system.

Note that, if we consider principal bundles over Minkowski spacetimes $\mathbb{R}^4$, it does not matter for this discussion that principal bundles over Euclidean spaces are always trivial by Corollary 4.2.9. What matters is the independence of the actual choice of *trivialization*, i.e. the choice of (global) gauge. Even on a trivial principal bundle there are non-trivial gauge transformations. This is very similar to special relativity, where spacetime is trivial, i.e. isometric to $\mathbb{R}^4$ with a Minkowski metric, but what matters is the independence of the actual trivialization, i.e. the choice of inertial system. Transformations between inertial systems are called *Lorentz transformations*, transformations between (local) gauges are called *gauge transformations*.

**Table 4.2** Comparison between notions for special relativity and gauge theory

| | Manifold | Trivialization | Transformations and invariance |
|---|---|---|---|
| Special relativity | Spacetime $M$ | $M \cong \mathbb{R}^4$ via inertial system | Lorentz |
| Gauge theory | Principal bundle $P \to M$ | $P \cong M \times G$ via choice of gauge | Gauge |

## 4.3 *Formal Bundle Atlases

We briefly return to the case of general fibre bundles. We are sometimes in the following situation: We have a manifold $M$, a set $E$ and a surjective map $\pi\colon E \to M$. However, we do not *a priori* have a topology or the structure of a smooth manifold on $E$. Under which circumstances can we define such structures, so that $\pi\colon E \to M$ becomes a smooth fibre bundle?

*Example 4.3.1* Let $M$ be a smooth manifold of dimension $n$. The tangent space $T_pM$ is an $n$-dimensional vector space for all $p \in M$. Let $TM$ be the disjoint union

$$TM = \dot{\bigcup}_{p\in M} T_pM$$

with the obvious projection $\pi\colon TM \to M$. How do we define the structure of a smooth manifold on the set $TM$, such that $TM$ becomes a fibre bundle over $M$, with fibres given by $T_pM$? We can also define for each tangent space $T_pM$ the dual vector space $T_p^*M$ or the exterior algebra $\Lambda^k T_p^*M$. How do we construct smooth fibre bundles that have these vector spaces as fibres?

The following notion is useful in this context (we follow [14, Sect. 2.1]).

**Definition 4.3.2** Let $M$ and $F$ be manifolds, $E$ a set and $\pi\colon E \to M$ a surjective map.

1. Suppose $U \subset M$ is open and

$$\phi_U\colon E_U \longrightarrow U \times F$$

   is a bijection with

$$\mathrm{pr}_1 \circ \phi_U = \pi|_{E_U}.$$

   Then we call $(U, \phi_U)$ a **formal bundle chart** for $E$.
2. A family $\{(U_i, \phi_i)\}_{i\in I}$ of formal bundle charts, where $\{U_i\}_{i\in I}$ is an open covering of $M$, is called a **formal bundle atlas** for $E$.
3. We call the charts in a formal bundle atlas $\{(U_i, \phi_i)\}_{i\in I}$ **smoothly compatible** if all transition functions

$$\phi_j \circ \phi_i^{-1}|_{(U_i\cap U_j)\times F}\colon (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F,$$

   for $U_i \cap U_j \neq \emptyset$, are smooth maps (i.e. diffeomorphisms).

We then have:

**Theorem 4.3.3 (Formal Bundle Atlases Define Fibre Bundles)** *Let $M$ and $F$ be manifolds, $E$ a set and $\pi\colon E \to M$ a surjective map. Suppose that $\{(U_i, \phi_i)\}_{i\in I}$ is a formal bundle atlas for $E$ of smoothly compatible charts. Then there exists a unique* *topology and a unique structure of a smooth manifold on E such that*

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

*is a smooth fibre bundle with smooth bundle atlas* $\{(U_i, \phi_i)\}_{i\in I}$.

The proof consists of several steps. We first define a topology on $E$: consider the bijections

$$\phi_i : E_{U_i} \longrightarrow U_i \times F.$$

We define a subset $O \subset E$ to be open if and only if

$$\phi_i(O \cap E_{U_i})$$

is open in $U_i \times F$ for all $i \in I$.

**Lemma 4.3.4 (The Topology on $E$ Defined by a Formal Bundle Atlas)** *This defines a topology on E which is Hausdorff and has a countable base. It is the unique topology on E such that all formal bundle charts* $\phi_i : E_{U_i} \to U_i \times F$ *are homeomorphisms.*

*Proof* We first show that this defines a topology on $E$: it is clear that $\emptyset$ and $E$ are open. It is also easy to see that arbitrary unions and finite intersections of open sets are open.

By definition the maps $\phi_i$ are open. Suppose that $O \subset E_{U_i}$ and $\phi_i(O)$ is open. Then for all $j \in I$

$$\begin{aligned} \phi_j(O \cap E_{U_j}) &= \left(\phi_j \circ \phi_i^{-1}\right)\left(\phi_i(O \cap E_{U_j} \cap E_{U_i})\right) \\ &= \left(\phi_j \circ \phi_i^{-1}\right)\left(\phi_i(O) \cap (U_j \cap U_i) \times F\right). \end{aligned}$$

It follows that $O$ is open in $E$ and that $\phi_i : E_{U_i} \to U_i \times F$ is a homeomorphism.

Since $M$ and $F$ are Hausdorff, it is not difficult to show that the topology on $E$ is Hausdorff, by considering for arbitrary points $p, q \in E$ first the case $\pi(p) \neq \pi(q)$ with $\pi(p) \in U_i$, $\pi(q) \in U_j$ and then the case $\pi(p) = \pi(q) \in U_i$.

To show that the topology on $E$ has a countable base we choose a countable base $\{V_j\}_{j\in J}$ for the topology of $M$ and a countable base $\{W_k\}_{k\in K}$ for the topology of $F$. Without loss of generality we can assume that the family $\{U_i\}_{i\in I}$ is countable, without changing the topology of $E$. Let $O \subset E$ be an arbitrary open set and $p \in O$ a point. Then $p \in O \cap E_{U_i}$ for some $i$ and there exist $j \in J$ and $k \in K$ such that

$$p \in \phi_i^{-1}(V_j \times W_k) \subset O \cap E_{U_i}.$$

This shows that the countable family

$$\{\phi_i^{-1}(V_j \times W_k)\}_{i\in I, j\in J, k\in K}$$

of open sets of $E$ forms a base.

The uniqueness statement for the topology of $E$ is clear. □

We can now finish the proof of Theorem 4.3.3.

*Proof* To define a smooth structure on $E$, we first define the smooth structure on $E_{U_i}$ such that the homeomorphism

$$\phi_i : E_{U_i} \longrightarrow U_i \times F$$

is a diffeomorphism. Then this defines a smooth structure on $E$, because the transition functions

$$\phi_j \circ \phi_i^{-1}|_{(U_i\cap U_j)\times F} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

are diffeomorphisms. This is the unique smooth structure on $E$ so that $\pi : E \to M$ is a smooth fibre bundle with general fibre $F$ and $\{(U_i, \phi_i)\}_{i\in I}$ is a smooth bundle atlas. □

## 4.4 *Frame Bundles

We want to apply Theorem 4.3.3 to define so-called *frame bundles*. Let $M$ be a smooth, $n$-dimensional manifold. For a point $p \in M$ we define the set of all bases of $T_pM$

$$\mathrm{Fr}_{\mathrm{GL}}(M)_p = \{(v_1, \ldots, v_n) \text{ basis of } T_pM\}$$

and define the disjoint union

$$\mathrm{Fr}_{\mathrm{GL}}(M) = \dot{\bigcup}_{p\in M} \mathrm{Fr}_{\mathrm{GL}}(M)_p.$$

There is a natural projection $\pi : \mathrm{Fr}_{\mathrm{GL}}(M) \to M$ and an action

$$\mathrm{Fr}_{\mathrm{GL}}(M) \times \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{Fr}_{\mathrm{GL}}(M),$$

given by

$$(v_1, \ldots, v_n) \cdot A = \left(\sum_{i=1}^n v_i A_{i1}, \ldots, \sum_{i=1}^n v_i A_{in}\right), \quad \forall (v_1, \ldots, v_n) \in \mathrm{Fr}_{\mathrm{GL}}(M)_p, A \in \mathrm{GL}(n, \mathbb{R}).$$

**Theorem 4.4.1 (Frame Bundles)** *The projection* $\pi$ *and the action of* $\mathrm{GL}(n,\mathbb{R})$ *define the structure of a principal* $\mathrm{GL}(n,\mathbb{R})$*-bundle*

$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{R}) & \longrightarrow & \mathrm{Fr}_{\mathrm{GL}}(M) \\ & & \downarrow \pi \\ & & M \end{array}$$

*This bundle is called the* **frame bundle** *of the manifold M.*

*Proof* We defined $\mathrm{Fr}_{\mathrm{GL}}(M)$ so far only as a set. It is clear that the action of $\mathrm{GL}(n,\mathbb{R})$ preserves the fibres of $\pi$ and is simply transitive on them. Let $(U_i, \psi_i)$ be a local manifold chart for $M$,

$$\psi_i\colon U_i \longrightarrow \mathbb{R}^n.$$

Then

$$\begin{aligned} s_i\colon U_i &\longrightarrow \mathrm{Fr}_{\mathrm{GL}}(M)_{U_i} \\ p &\longmapsto (\partial_{x_1}, \ldots, \partial_{x_n})\,(p) \end{aligned}$$

is a local section for $\pi$. We have

$$s_i(p) = \left((D_p\psi_i)^{-1}e_1, \ldots, (D_p\psi_i)^{-1}e_n\right).$$

We define the inverse of a formal bundle chart by

$$\begin{aligned} \phi_i^{-1}\colon U_i \times \mathrm{GL}(n,\mathbb{R}) &\longrightarrow \mathrm{Fr}_{\mathrm{GL}}(M)_{U_i} \\ (p, A) &\longmapsto s_i(p)\cdot A. \end{aligned}$$

The transition functions are

$$\phi_j \circ \phi_i^{-1}\colon (U_i \cap U_j) \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow (U_i \cap U_j) \times \mathrm{GL}(n,\mathbb{R})$$

with

$$\phi_j \circ \phi_i^{-1}(p, A) = \left(p, D_{\psi_i(p)}\left(\psi_j \circ \psi_i^{-1}\right)\cdot A\right).$$

These maps are smooth, because the transition functions $\psi_j \circ \psi_i^{-1}$ are smooth. This shows that the maps $\phi_i$ are smoothly compatible formal bundle charts and by Theorem 4.3.3 there exists a manifold structure on $\mathrm{Fr}_{\mathrm{GL}}(M)$ such that $\pi$ becomes a fibre bundle with general fibre $\mathrm{GL}(n,\mathbb{R})$.

The $\mathrm{GL}(n, \mathbb{R})$-action is smooth (by considering the action in the bundle charts) and the (inverse) bundle charts $\phi_i^{-1}$ are $\mathrm{GL}(n, \mathbb{R})$-equivariant:

$$\phi_i^{-1}((p,A) \cdot B) = s_i(p)(A \cdot B) = \phi_i^{-1}(p,A) \cdot B \quad \forall B \in \mathrm{GL}(n, \mathbb{R}).$$

Therefore $\pi \colon \mathrm{Fr_{GL}}(M) \to M$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle over $M$. □

*Remark 4.4.2 (Orthogonal Frame Bundles)* If $(M, g)$ is an $n$-dimensional Riemannian manifold, we can define a principal $\mathrm{O}(n)$-bundle

$$\begin{array}{ccc} \mathrm{O}(n) & \longrightarrow & \mathrm{Fr_O}(M) \\ & & \downarrow \pi \\ & & M \end{array}$$

whose fibre over $p \in M$ consists of the set of orthonormal bases in $T_pM$. If $M$ is in addition oriented, then there is also a principal $\mathrm{SO}(n)$-bundle

$$\begin{array}{ccc} \mathrm{SO}(n) & \longrightarrow & \mathrm{Fr_{SO}}(M) \\ & & \downarrow \pi \\ & & M \end{array}$$

defined using oriented orthonormal bases. There are similar constructions of orthonormal frame bundles for pseudo-Riemannian manifolds.

*Remark 4.4.3* A frame, i.e. a basis of a tangent space to a manifold, is in physics often called a **vielbein**, in particular in the case of an orthonormal frame to a Lorentz manifold (the word "vielbein" is German and means "many-leg". It is a generalization of the word **tetrad** in the 4-dimensional case.)

**Definition 4.4.4** Let $G$ be a Lie group. A principal $G$-subbundle of the frame bundle $\mathrm{Fr_{GL}}(M)$ of a smooth manifold $M$, i.e. a $G$-reduction of the frame bundle, is called a ***G*-structure** on $M$.
In particular, a Riemannian metric on $M^n$ defines an $\mathrm{O}(n)$-structure and, together with an orientation, an $\mathrm{SO}(n)$-structure on $M$.

## 4.5 Vector Bundles

We consider another class of fibre bundles, called *vector bundles*, that are ubiquitous in differential geometry and gauge theory. The prototype of a vector bundle is the tangent bundle $TM$ of a smooth manifold $M$. Moreover, in physics, matter fields in gauge theories are described classically by sections of vector bundles. In addition to [14] we follow in this section [25, 74] and [78].

### 4.5.1 Definitions and Basic Concepts

Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$.

**Definition 4.5.1** A fibre bundle

$$\begin{array}{ccc} V & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

is called a **(real or complex) vector bundle of rank** $m$ if:

1. The general fibre $V$ and every fibre $E_x$, for $x \in M$, are $m$-dimensional vector spaces over $\mathbb{K}$.
2. There exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ for $E$ such that the induced maps

$$\phi_{ix}: E_x \longrightarrow V$$

are vector space isomorphisms for all $x \in U_i$. We call such an atlas a **vector bundle atlas** for $E$ and the charts in a vector bundle atlas **vector bundle charts**. See Fig. 4.3.

A vector bundle of rank 1 is called a **line bundle**.

There are two features that distinguish a vector bundle $E \to M$ from a standard fibre bundle whose general fibre is a vector space $V$:

1. the vector space structure on each fibre $E_x$, for $x \in M$;
2. the bundle $E$ has a vector bundle atlas.

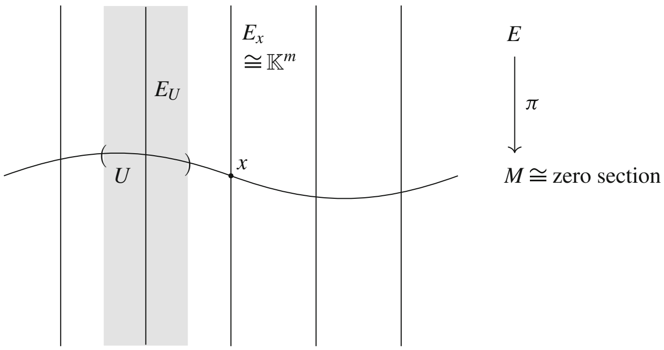


**Fig. 4.3** Vector bundle

The vector space structure on each fibre implies that we can add any two sections of a vector bundle $E$ and multiply sections with a scalar or a smooth function on the base manifold $M$ with values in $\mathbb{K}$.

*Example 4.5.2* The simplest example of a vector bundle is the trivial bundle $M \times \mathbb{K}^m$, often denoted by $\underline{\mathbb{K}}^m$. It has the canonical vector space structure on each fibre $\{p\} \times \mathbb{K}^m$, for $p \in M$, and the vector bundle atlas consisting of only one bundle chart

$$\mathrm{Id}\colon M \times \mathbb{K}^m \longrightarrow M \times \mathbb{K}^m.$$

Here is a more interesting example:

*Example 4.5.3 (The Tangent Bundle of a Smooth Manifold)* We want to show that the tangent bundle of a smooth manifold is canonically a smooth real vector bundle. Let $M$ be a smooth manifold of dimension $n$. We define the set

$$TM = \dot{\bigcup}_{p \in M} T_pM$$

with the canonical projection $\pi\colon TM \to M$. We claim that $TM$ has the structure of a smooth real vector bundle of rank $n$ over $M$: First, the general fibre $\mathbb{R}^n$ and each fibre $T_pM$ are $n$-dimensional real vector spaces. If

$$\psi_i\colon U_i \longrightarrow \phi_i(U_i) \subset \mathbb{R}^n$$

is a local manifold chart for $M$, then

$$\begin{aligned} \Psi_i\colon TM_{U_i} &\longrightarrow U_i \times \mathbb{R}^n \\ (p, v) &\longmapsto (p, D_p\psi_i(v)) \end{aligned}$$

is a formal bundle chart for $TM$. These formal bundle charts are smoothly compatible, because

$$\begin{aligned} \Psi_j \circ \Psi_i^{-1}\colon (U_i \cap U_j) \times \mathbb{R}^n &\longrightarrow (U_i \cap U_j) \times \mathbb{R}^n \\ (p, w) &\longmapsto \left(p, D_p\left(\psi_j \circ \psi_i^{-1}\right) w\right) \end{aligned}$$

is a smooth map. By Theorem 4.3.3, $\pi\colon TM \to M$ has the structure of a smooth fibre bundle with general fibre diffeomorphic to $\mathbb{R}^n$. Since the bundle charts $(U_i, \Psi_i)$ are linear isomorphisms on each fibre, the bundle $TM$ is a vector bundle of rank $n$.

*Remark 4.5.4* Note that sections of $TM$ are the same as vector fields on $M$:

$$\Gamma(TM) = \mathfrak{X}(M).$$

Transition functions for a vector bundle atlas have a special form:

**Proposition 4.5.5 (Transition Functions of Vector Bundles)** *Let $E \to M$ be a $\mathbb{K}$-vector bundle of rank m and $\{(U_i, \phi_i)\}_{i\in I}$ a vector bundle atlas for E. Then the transition functions take values in the subgroup* $\mathrm{GL}(m, \mathbb{K})$ *of* $\mathrm{Diff}(\mathbb{K}^m)$*,*

$$\phi_{ji}: U_i \cap U_j \longrightarrow \mathrm{GL}(m, \mathbb{K}) \subset \mathrm{Diff}(\mathbb{K}^m)$$
$$x \longmapsto \phi_{jx} \circ \phi_{ix}^{-1}.$$

The following definition applies only to real vector bundles.

**Definition 4.5.6** A real vector bundle $E \to M$ of rank $m$ is called **orientable** if it admits a vector bundle atlas $\{(U_i, \phi_i)\}_{i\in I}$ such that all transition functions map to

$$\phi_{ji}: U_i \cap U_j \longrightarrow \mathrm{GL}_+(m, \mathbb{R}),$$

where $\mathrm{GL}_+(m, \mathbb{R})$ denotes the subgroup of invertible matrices with positive determinant.
Clearly, if $E \to M$ is a *complex* vector bundle of rank $m$, then forgetting the complex structure it defines an underlying *real* rank $2m$ vector bundle $E_\mathbb{R} \to M$. The bundle $E_\mathbb{R}$ is always orientable, because the identification $\mathbb{C}^m = \mathbb{R}^{2m}$ as real vector spaces induces an embedding $\mathrm{GL}(m, \mathbb{C}) \subset \mathrm{GL}_+(2m, \mathbb{R})$ by Exercise 1.9.10.

There is a notion of a homomorphism between vector bundles over the same manifold.

**Definition 4.5.7** Let $V \to E \stackrel{\pi_E}{\to} M$ and $W \to F \stackrel{\pi_F}{\to} M$ be vector bundles over $M$ over the same field $\mathbb{K}$.

1. A smooth bundle map $L: E \to F$, satisfying $\pi_F \circ L = \pi_E$, is called a **vector bundle homomorphism** if the restriction to a fibre

$$L|_{E_x}: E_x \longrightarrow F_x$$

is a linear map for all $x \in M$. A vector bundle homomorphism which is injective (surjective) on each fibre is called a **vector bundle monomorphism (epimorphism)**. If $L: E \to E$ is a homomorphism, then $L$ is also called a **vector bundle endomorphism**.
2. A **vector bundle isomorphism** is a vector bundle homomorphism which is a diffeomorphism of the total spaces and an isomorphism on each fibre. A vector bundle is called **trivial** if it is isomorphic to the trivial bundle. If the tangent bundle $TM$ of a manifold $M$ is trivial, then $M$ is called **parallelizable**.

*Remark 4.5.8* According to Exercise 4.8.5, a vector bundle homomorphism which is an isomorphism on each fibre is a vector bundle isomorphism.
As before, we could consider vector bundle homomorphisms between vector bundles over different base manifolds $M$ and $N$ that cover a smooth map from $M$

to $N$. It is not difficult to prove with Remark 4.5.8 that a vector bundle $E \to M$ of rank $m$ is trivial if and only if it has $m$ global sections

$$v_1, \ldots, v_m : M \longrightarrow E,$$

such that $v_1(x), \ldots, v_m(x)$ form a basis of the fibre $E_x$, for all $x \in M$.

*Remark 4.5.9 (Sections of Vector Bundles)* Note that (contrary to principal fibre bundles) vector bundles always admit *global* sections: the section that is equal to zero everywhere on $M$ is a trivial example (the fibres of a vector bundle *are* vector spaces, so there is a canonical element, namely 0. The fibres of a principal bundle are *only diffeomorphic* to a Lie group, so the neutral element $e$ is not a canonical element in a fibre.) However, in the case of a vector bundle it is not clear that there are sections without zeros, and even if this is the case, it is not clear that there are $m$ sections which form a basis in each fibre.

*Example 4.5.10 (Parallelizable Spheres)* We want to show that the spheres $S^0$, $S^1$, $S^3$ and $S^7$ are parallelizable. This is trivial for $S^0$, which consists only of two points. We consider $S^1$ as the unit sphere in $\mathbb{C}$. For $x \in S^1$, the vector $ix \in \mathbb{C}$ is orthogonal to $x$ with respect to the standard Euclidean scalar product:

$$\mathrm{Re}\,\langle x, ix\rangle = \mathrm{Re}\, i||x||^2 = 0,$$

where $\langle z, w\rangle = \bar{z}w$ is the standard Hermitian scalar product on $\mathbb{C}$. This implies that

$$\begin{aligned} S^1 \times \mathbb{R} &\longrightarrow TS^1 \subset T\mathbb{C}|_{S^1} \\ (x, t) &\longmapsto (x, tix) \end{aligned}$$

is a trivialization of $TS^1$.

Similarly, we can consider $S^3$ as the unit sphere in $\mathbb{H}$. Then

$$\begin{aligned} S^3 \times \mathbb{R}^3 &\longrightarrow TS^3 \subset T\mathbb{H}|_{S^3} \\ (x, t_1, t_2, t_3) &\longmapsto (x, t_1 ix + t_2 jx + t_3 kx) \end{aligned}$$

is a trivialization of $TS^3$.

Finally, we consider $S^7$ as the unit sphere in the octonions $\mathbb{O}$. The octonions $\mathbb{O} \cong \mathbb{R}^8$ are spanned by $e_0, e_1, \ldots, e_7$, where

$$e_0^2 = e_0, \quad e_i^2 = -e_0 \quad \forall i = 1, \ldots, 7.$$

The map

$$\begin{aligned} S^7 \times \mathbb{R}^7 &\longrightarrow TS^7 \subset T\mathbb{O}|_{S^7} \\ (x, t_1, t_2, \ldots, t_7) &\longmapsto (x, t_1 e_1 x + t_2 e_2 x + \ldots + t_7 e_7 x) \end{aligned}$$

is a trivialization of $TS^7$.

It is a deep theorem due to J.F. Adams [2] that $S^0$, $S^1$, $S^3$ and $S^7$ are the only spheres which are parallelizable. This is related to the fact that division algebras exist only in dimension 1, 2, 4 and 8. A proof using K-theory can be found in [74]. See also Exercise 6.13.8.
A proof of the following theorem can be found in [5, 74] and [81].

**Theorem 4.5.11 (Vector Bundles and Homotopy Equivalences)** *Let $f: M \to N$ be a smooth homotopy equivalence between manifolds. Then the pullback $f^*$ is a bijection between isomorphism classes of vector bundles over $N$ and vector bundles over $M$ of the same rank and over the same field $\mathbb{K}$.*
In particular, we get:

**Corollary 4.5.12 (Vector Bundles over Contractible Manifolds Are Trivial)** *If $M$ is a contractible manifold, then every vector bundle over $M$ is trivial. This holds, in particular, if $M = \mathbb{R}^n$ for some $n$.*

## 4.5.2 Linear Algebra Constructions for Vector Bundles

A useful fact is that we can construct new vector bundles from given ones by applying linear algebra constructions fibrewise: suppose $E, F$ are vector bundles over $M$ over the same field $\mathbb{K}$. Then there exist canonically defined vector bundles

$$E \oplus F, \quad E \otimes F, \quad E^*, \quad \Lambda^k E, \quad \mathrm{Hom}(E, F)$$

over $M$. If $\mathbb{K} = \mathbb{C}$ there also exists a complex conjugate vector bundle $\bar{E}$. The fibres of these vector bundles are given by

$$(E \oplus F)_x = E_x \oplus F_x,$$

and similarly in the other cases. This follows from Theorem 4.3.3, because local vector bundle charts for $E$ and $F$ can be combined to yield smoothly compatible formal vector bundle charts for the set $E \oplus F$, defining the structure of a smooth vector bundle on $E \oplus F \to M$. Similarly in the other cases.

Purely linear algebraic constructions, such as the direct sum and tensor product of vector spaces, extend to smooth vector bundles and yield new vector bundles with canonically defined smooth bundle structures.

*Example 4.5.13* Consider the tangent bundle $TM \to M$. Then there exist canonically associated vector bundles $T^*M$ and $\Lambda^k T^*M$ over $M$. Sections of $\Lambda^k T^*M$ are $k$-forms on $M$:

$$\Gamma(\Lambda^k T^*M) = \Omega^k(M).$$

More generally, for a vector bundle $E \to M$, sections of the bundle $\Lambda^k T^*M \otimes E$ are $k$-forms on $M$ with values in $E$, i.e. elements of $\Omega^k(M, E)$: If $\omega \in \Omega^k(M, E)$, then at a point $x \in M$

$$\omega_x : T_xM \times \ldots \times T_xM \longrightarrow E_x$$

is multilinear and alternating. This generalizes the notion of vector space-valued forms in Sect. 3.5.1 to forms which have values in a vector bundle. One sometimes calls $\Lambda^k T^*M \otimes E$ the **bundle of $k$-forms over $M$ twisted with $E$**.
We want to define the concept of vector subbundle (following [25]):

**Definition 4.5.14** Let $\pi : E \to M$ be a $\mathbb{K}$-vector bundle of rank $m$. A subset $F \subset E$ is called a **vector subbundle** of rank $k$ if each point $p \in M$ has an open neighbourhood $U$ together with a vector bundle chart $(U, \phi)$ of $E$ such that

$$\phi(E_U \cap F) = U \times \mathbb{K}^k \subset U \times \mathbb{K}^m,$$

where $\mathbb{K}^k$ is the vector subspace $\mathbb{K}^k \times \{0\} \subset \mathbb{K}^m$. It follows that $F$ is an embedded submanifold of $E$ and $\pi|_F : F \to M$ has the canonical structure of a $\mathbb{K}$-vector bundle of rank $k$ over $M$.

*Example 4.5.15 (Normal Bundle of Spheres)* For $n \neq 0, 1, 3, 7$ the sphere $S^n$ does not have a trivial tangent bundle according to Adams' Theorem mentioned in Example 4.5.10. However, the **normal bundle** $\nu(S^n)$ of $S^n$ in $\mathbb{R}^{n+1}$ is trivial for any $n \geq 0$: The normal bundle is defined as

$$\nu(S^n) = \{(x, u) \in S^n \times \mathbb{R}^{n+1} \mid u \perp T_xS^n\},$$

with projection onto the first factor. It is clear that the normal bundle is a real line bundle. The following map is a trivialization of $\nu(S^n)$:

$$\begin{aligned} S^n \times \mathbb{R} &\longrightarrow \nu(S^n) \\ (x, t) &\longmapsto (x, tx). \end{aligned}$$

Note that

$$TS^n \oplus \nu(S^n) = T\mathbb{R}^{n+1}|_{S^n}.$$

We conclude that the sum of a non-trivial vector bundle (the tangent bundle to the sphere) and a trivial vector bundle (the normal bundle) can be trivial. One says that the tangent bundle of the sphere $S^n$ is **stably trivial**: It becomes trivial after taking the direct sum with a trivial bundle (here a trivial *line* bundle). Both $TS^n$ and $\nu(S^n)$ are vector subbundles of the trivial bundle $T\mathbb{R}^{n+1}|_{S^n}$. Note that this also means that a trivial vector bundle can have non-trivial subbundles.

**Definition 4.5.16** Let $E \to M$ be a $\mathbb{K}$-vector bundle over $M$. A **(Euclidean** or **Hermitian) bundle metric** is a metric on each fibre $E_x$ that varies smoothly with $x \in M$. More precisely, it is a section

$$\langle \cdot , \cdot \rangle \in \Gamma(E^* \otimes E^*) \quad (\mathbb{K} = \mathbb{R})$$

or

$$\langle \cdot , \cdot \rangle \in \Gamma(\bar{E}^* \otimes E^*) \quad (\mathbb{K} = \mathbb{C})$$

which defines in each point $x \in M$ a non-degenerate symmetric ($\mathbb{K} = \mathbb{R}$) or Hermitian ($\mathbb{K} = \mathbb{C}$) form

$$\langle \cdot , \cdot \rangle_x : E_x \times E_x \longrightarrow \mathbb{K}.$$

**Proposition 4.5.17 (Existence of Bundle Metrics)** *Every $\mathbb{K}$-vector bundle over a manifold $M$ admits a positive definite bundle metric.*

*Proof* This follows by a partition of unity argument, because a convex combination of positive definite metrics on a vector space is still a positive definite metric. □

For associated vector bundles we will give a more explicit construction of bundle metrics in Proposition 4.7.12.

*Example 4.5.18* The tangent bundle $TM$ of any submanifold $M^m \subset \mathbb{R}^n$ has a bundle metric induced from the standard Euclidean scalar product on $\mathbb{R}^n$. In particular, $TS^{n-1}$ has a canonical bundle metric.

**Proposition 4.5.19 (Orthogonal Complement of a Vector Subbundle)** *Let $E \to M$ be a vector bundle with a positive definite bundle metric and $F \subset E$ a vector subbundle. Then the orthogonal complement $F^\perp$ is a vector subbundle of $E$ and $F \oplus F^\perp$ is isomorphic to $E$.*

*Proof* This is Exercise 4.8.16. □

The following is clear.

**Proposition 4.5.20 (Transition Functions of Vector Bundles with a Metric)** *Let $E \to M$ be a $\mathbb{K}$-vector bundle of rank $m$ with a positive definite bundle metric. Choosing local trivializations given by orthonormal bases it follows that there exists a vector bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ with transition functions of the form*

$$\phi_{ji} : U_i \cap U_j \longrightarrow \mathrm{O}(m) \quad (\mathbb{K} = \mathbb{R})$$

*or*

$$\phi_{ji} : U_i \cap U_j \longrightarrow \mathrm{U}(m) \quad (\mathbb{K} = \mathbb{C}).$$

*If the bundle is real and orientable, then we can find a vector bundle atlas such that*

$$\phi_{ji}\colon U_i \cap U_j \longrightarrow \mathrm{SO}(m).$$

If $E \to M$ is a real (complex) vector bundle of rank $m$ with a positive definite bundle metric, then the set consisting of all vectors of length 1 in each fibre forms the **unit sphere bundle** $S(E) \to M$, which is a smooth $S^{m-1}$-bundle ($S^{2m-1}$-bundle) over $M$.

## 4.6 *The Clutching Construction

We want to describe a construction that yields (all) vector bundles over spheres $S^n$. The idea of this so-called **clutching construction** is to glue together trivial vector bundles over the northern and southern hemisphere of $S^n$ along the equator (we follow [74]).

Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, where $n \geq 1$. We define the north and south pole

$$N_+ = (0, \ldots, 0, +1) \in S^n,$$
$$N_- = (0, \ldots, 0, -1) \in S^n$$

and the open sets

$$U_+ = S^n \setminus \{N_+\},$$
$$U_- = S^n \setminus \{N_-\}.$$

Both $U_+$ and $U_-$ are diffeomorphic to $\mathbb{R}^n$ via the stereographic projection; cf. Example A.1.8. Let $f$ be any smooth map

$$f\colon S^{n-1} \longrightarrow \mathrm{GL}(k, \mathbb{K}),$$

where we think of $S^{n-1} \subset S^n$ as the equator of $S^n$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Such a map is called a **clutching function**. We write

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$
$$z = x_{n+1} \in \mathbb{R}.$$

Let $p$ denote the following retraction of the intersection $U_+ \cap U_-$ onto the equator:

$$p\colon U_+ \cap U_- \longrightarrow S^{n-1}$$
$$(x, z) \longmapsto \frac{x}{||x||}.$$

Here

$$||x||^2 = x_1^2 + \ldots + x_n^2$$

is the Euclidean norm. Note that this map is well-defined, because $x \neq 0$ on $U_+ \cap U_-$. We use the retraction to extend the clutching function to a smooth map on $U_+ \cap U_-$:

$$\bar{f} = f \circ p \colon U_+ \cap U_- \longrightarrow \mathrm{GL}(k, \mathbb{K}).$$

**Definition 4.6.1** Let $E_f = \tilde{E}/\sim$ be the quotient set of the disjoint union

$$\tilde{E} = \left(U_- \times \mathbb{K}^k\right) \dot{\bigcup} \left(U_+ \times \mathbb{K}^k\right)$$

by identifying

$$(x, z, v) \in (U_- \cap U_+) \times \mathbb{K}^k \subset U_- \times \mathbb{K}^k$$

with

$$(x, z, \bar{f}(x, z) \cdot v) \in (U_+ \cap U_-) \times \mathbb{K}^k \subset U_+ \times \mathbb{K}^k.$$

**Theorem 4.6.2 (Vector Bundle over a Sphere Defined by a Clutching Function)** *Via the projection*

$$\pi \colon E_f \longrightarrow S^n$$
$$[x, z, v] \longmapsto (x, z)$$

*the set $E_f$ has a canonical structure of a $\mathbb{K}$-vector bundle of rank $k$ over the sphere $S^n$:*

$$\begin{array}{ccc} \mathbb{K}^k & \longrightarrow & E_f \\ & & \downarrow \pi \\ & & S^n \end{array}$$

*Proof* (See also Exercise 4.8.9.) Note that the map $\pi$ is well-defined on the quotient set $E_f$ and surjective onto $S^n$. Let $\sigma$ denote the quotient map

$$\sigma \colon \left(U_- \times \mathbb{K}^k\right) \dot{\bigcup} \left(U_+ \times \mathbb{K}^k\right) \longrightarrow E_f.$$

The map $\sigma$ decomposes into injective maps $\sigma_\pm$ on $U_\pm \times \mathbb{K}^k$. For $(x, z, w) \in U_\pm \times \mathbb{K}^k$ define

$$[x, z, w]_\pm = \sigma_\pm(x, z, w).$$

Then

$$[x, z, v]_- = \left[x, z, \bar{f}(x, z) \cdot v\right]_+ .$$

The maps

$$\begin{aligned} \phi_\pm : E_{f\,U_\pm} &\longrightarrow U_\pm \times \mathbb{K}^k \\ [x, z, v] &\longmapsto \sigma_\pm^{-1}([x, z, v]_\pm) \end{aligned}$$

are well-defined formal bundle charts. We want to show that these formal bundle charts are smoothly compatible: We calculate

$$\begin{aligned} \phi_+ \circ \phi_-^{-1}(x, z, v) &= \sigma_+^{-1} \circ \sigma_-(x, z, v) \\ &= \sigma_+^{-1}[x, z, v]_- \\ &= \sigma_+^{-1}\left[x, z, \bar{f}(x, z) \cdot v\right]_+ \\ &= (x, z, \bar{f}(x, z) \cdot v), \end{aligned}$$

which is a smooth map. It follows from Theorem 4.3.3 that $\pi: E_f \to S^n$ has the structure of a fibre bundle. Since the bundle charts $\phi_+, \phi_-$ are linear isomorphisms on each fibre, it follows that $\pi: E_f \to S^n$ is a vector bundle with general fibre $\mathbb{K}^k$.

□

The following can be shown, see [5] or [74]:

**Theorem 4.6.3 (Vector Bundles over Spheres and Homotopy Classes of Clutching Functions)**

1. *Every complex vector bundle over $S^n$ of rank $k$ is isomorphic to a bundle $E_f$ for a certain clutching function $f: S^{n-1} \to \mathrm{U}(k)$, unique up to homotopy.*
2. *Similarly, every orientable real vector bundle over $S^n$ of rank $k$ is isomorphic to a bundle $E_f$ for a certain clutching function $f: S^{n-1} \to \mathrm{SO}(k)$, unique up to homotopy.*

Every vector bundle over a sphere can be constructed using a clutching function, because by Corollary 4.5.12 every vector bundle over $S^n$ is trivial over $U_+$ and $U_-$. Given an arbitrary $\mathbb{K}$-vector bundle $E \to S^n$ of rank $k$ we obtain an associated clutching function as follows:

- Let $E_\pm$ denote the restrictions of $E$ to $U_\pm$. Choose vector bundle trivializations

$$h_\pm : E_\pm \longrightarrow U_\pm \times \mathbb{K}^k.$$

- Consider

$$h_+ \circ h_-^{-1}\colon S^{n-1} \times \mathbb{K}^k \longrightarrow S^{n-1} \times \mathbb{K}^k.$$

This map is a linear isomorphism on each fibre and defines the clutching function

$$f\colon S^{n-1} \longrightarrow \mathrm{GL}(k, \mathbb{K}).$$

The clutching functions in Theorem 4.6.3 can be taken to have image in $\mathrm{U}(k)$ or $\mathrm{SO}(k)$, because there are deformation retractions

$$\mathrm{GL}(k, \mathbb{C}) \longrightarrow \mathrm{U}(k),$$
$$\mathrm{GL}_+(k, \mathbb{R}) \longrightarrow \mathrm{SO}(k).$$

Complex and real orientable vector bundles over $S^n$ are therefore essentially classified by the homotopy groups $\pi_{n-1}(\mathrm{U}(k))$ and $\pi_{n-1}(\mathrm{SO}(k))$.

*Remark 4.6.4 (Clutching Construction for Arbitrary Fibres)* Let $F$ be a smooth manifold and

$$f\colon S^{n-1} \longrightarrow \mathrm{Diff}(F)$$

a "smooth" map, again called a clutching function (the precise formulation in the general case is not completely trivial, because we did not define a smooth structure on the diffeomorphism group $\mathrm{Diff}(F)$). A similar construction to the one above yields a fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & E_f \\ & & \downarrow \pi \\ & & S^n \end{array}$$

over $S^n$ with general fibre $F$. The mapping torus construction in Example 4.1.5 can be seen as a special case of the clutching construction for $n = 1$: If $\phi\colon F \to F$ is the monodromy of the mapping torus, we choose

$$f\colon S^0 \longrightarrow \mathrm{Diff}(F)$$
$$-1 \longmapsto \mathrm{Id}_F$$
$$+1 \longmapsto \phi.$$

*Example 4.6.5 (Exotic 7-Spheres)* Another very nice application appears in Milnor's classic paper [94], where certain *exotic 7-spheres* (homeomorphic but not diffeomorphic to $S^7$) are defined as $S^3$-bundles over $S^4$, using the clutching construction with clutching function

$$f_{hj}: S^3 \longrightarrow \mathrm{SO}(4) \subset \mathrm{Diff}_+(S^3)$$

given by

$$f_{hj}(u) \cdot v = u^h v u^j.$$

Here $u, v \in S^3 = \mathrm{Sp}(1)$ and $h, j \in \mathbb{Z}$. For certain values of the integers $h$ and $j$, the $S^3$-bundle over $S^4$ determined by the clutching function $f_{hj}$ is an exotic 7-sphere.

Milnor's paper started the field known as *differential topology* and led to an extensive investigation of exotic spheres of arbitrary dimension. The study of the smooth topology of general 4-manifolds, using Donaldson theory and later Seiberg–Witten theory, also belongs to the field of differential topology.

## 4.7 Associated Vector Bundles

In Chap. 2 we studied the theory of Lie group representations. We now want to combine this theory with the theory of principal bundles from the present chapter.

We said before that principal bundles are the place where Lie groups appear in gauge theories. *Associated vector bundles*, which we discuss in this section, are precisely the place where *representations on vector spaces* are built into gauge theories. We can summarize this in the following diagram:

$$\begin{array}{ccc}
\text{Lie groups (gauge groups)} & \longrightarrow & \text{Representations on vector spaces} \\
\downarrow & & \downarrow \\
\text{Principal bundles} & \longrightarrow & \text{Associated vector bundles (matter fields)} \\
\downarrow & & \downarrow \\
\text{Connections (gauge fields)} & \longrightarrow & \text{Covariant derivatives (interaction/coupling)}
\end{array}$$

The third row will be explained in Chap. 5.

For example, in the Standard Model, one generation of fermions is described by associated complex vector bundles of rank 8 for left-handed fermions and rank 7 for right-handed fermions, associated to representations of the gauge group $SU(3) \times SU(2) \times U(1)$. Taking particles and antiparticles together we get two associated complex vector bundles of rank 15 (right-handed and left-handed) which are related by complex conjugation. The complete fermionic content of the Standard Model is described by the direct sum of three copies of these vector bundles (a complex vector bundle of rank 90), corresponding to the three generations. These constructions will be described in detail in Sect. 8.5.1.

### 4.7.1 Basic Concepts

As an introduction, consider again the Hopf fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \big\downarrow \pi \\ & & S^2 \end{array}$$

We are interested in the complex representations of $S^1$ on $\mathbb{C}$ with winding number $k \in \mathbb{Z}$:

$$\begin{aligned} \rho_k\colon S^1 &\longrightarrow \mathrm{U}(1) \\ z &\longmapsto z^k \end{aligned}$$

Our aim is to define an associated bundle

$$\gamma^k = S^3 \times_{\rho_k} \mathbb{C}.$$

This will be a complex line bundle over $S^2$

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \gamma^k \\ & & \big\downarrow \pi \\ & & S^2 \end{array}$$

whose transition functions are given by the transition functions of the Hopf fibration *composed* with the group homomorphism $\rho_k$.

The general definition is the following: Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. We then associate to each principal $G$-bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow{\scriptstyle \pi_P} \\ & & M \end{array}$$

and each representation

$$\rho: G \longrightarrow \mathrm{GL}(V)$$

of the structure group $G$ on a $\mathbb{K}$-vector space $V$ of dimension $k$ a vector bundle $E \to M$ with fibres isomorphic to $V$.

**Lemma 4.7.1** *Let $P$ be a principal $G$-bundle and $\rho$ a representation of the Lie group $G$ on a $\mathbb{K}$-vector space $V$. Then the map*

$$\begin{aligned} (P \times V) \times G &\longrightarrow P \times V \\ (p, v, g) &\longmapsto (p, v) \cdot g = \big(p \cdot g, \rho(g)^{-1} v\big) \end{aligned}$$

*defines a free principal right action of the Lie group $G$ on the product manifold $P \times V$. In particular, the quotient space $E = (P \times V)/G$ is a smooth manifold such that the projection $P \times V \to E$ is a submersion.*

*Proof* It is clear that

$$(P \times V) \times G \longrightarrow P \times V$$

is a right action, which is free since the action of $G$ on $P$ is free. If $G$ is compact, then the claim follows from Corollary 3.7.29. In the general case, the action is principal by an argument very similar to the one in the proof of Theorem 4.2.12. □

**Theorem 4.7.2 (Associated Vector Bundle Constructed as a Quotient)** *Let $P$ be a principal $G$-bundle and $\rho$ a representation of the Lie group $G$ on a $\mathbb{K}$-vector space $V$. Then the quotient space $E = (P \times V)/G$ has the structure of a $\mathbb{K}$-vector bundle over $M$, with projection*

$$\begin{aligned} \pi_E: E &\longrightarrow M \\ [p, v] &\longmapsto \pi_P(p) \end{aligned}$$

*and fibres*

$$E_x = (P_x \times V)/G$$

*isomorphic to* $V$. *The vector space structure on the fibre* $E_x$ *over* $x \in M$ *is defined by*

$$\lambda[p, v] + \mu[p, w] = [p, \lambda v + \mu w], \quad \forall p \in P, v, w \in V, \lambda, \mu \in \mathbb{K},$$

*where* $\pi_P(p) = x$.

*Proof* It is clear that $\pi_E$ is well-defined and that $V$ is isomorphic to the fibres $E_x$ via $v \mapsto [p_x, v]$ with a fixed $p_x \in P_x$. We need to find a vector bundle atlas for $E$. Let $(U, \phi_U)$ be a bundle chart for the principal bundle $P$:

$$\begin{aligned} \phi_U: P_U &\longrightarrow U \times G \\ p &\longmapsto (\pi_P(p), \beta_U(p)). \end{aligned}$$

We set

$$\begin{aligned} \psi_U: E_U &\longrightarrow U \times V \\ [p, v] &\longmapsto (\pi_P(p), \rho(\beta_U(p))v). \end{aligned}$$

Since $P \times V \to E$ is a submersion, the map $\psi_U$ is smooth. It is a diffeomorphism with smooth inverse

$$\begin{aligned} \psi_U^{-1}: U \times V &\longrightarrow E_U \\ (x, v) &\longmapsto \left[\phi_U^{-1}(x, e), v\right]. \end{aligned}$$

Its restriction to each fibre $E_x$ is a linear isomorphism to the vector space $V$. Thus $\psi_U$ defines a chart in a vector bundle atlas for $E$. □

**Definition 4.7.3** The vector bundle

$$E = P \times_\rho V = (P \times V)/G$$

is called the **vector bundle associated** to the principal bundle $P$ and the representation $\rho$ on $V$:

$$\begin{array}{ccc} V & \longrightarrow & P \times_\rho V \\ & & \downarrow \pi_E \\ & & M \end{array}$$

The group $G$ (or its image $\rho(G) \subset \mathrm{GL}(V)$) is known as the **structure group** of $E$.

*Remark 4.7.4* Note that in the definition of the vector space structure on the fibres $E_x$,

$$\lambda[p, v] + \mu[p, w] = [p, \lambda v + \mu w],$$

we have to choose both representatives with the *same* point $p$ in the fibre of $P$ over $x \in M$.

*Example 4.7.5* For every principal $G$-bundle $P \to M$ and every vector space $V$, the vector bundle associated to the trivial homomorphism

$$\begin{aligned} \rho: G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto \mathrm{Id}_V \end{aligned}$$

is a trivial vector bundle. See Exercise 4.8.20.
It is useful in applications to have a suitable description of local sections of an associated vector bundle.

**Proposition 4.7.6 (Local Sections of Associated Vector Bundles)** *Let $P$ be a principal bundle and $E = P \times_\rho V$ an associated vector bundle. Let $s: U \to P$ be a local gauge. Then there is a 1-to-1 relation between smooth sections $\tau: U \to E$ and smooth maps $f: U \to V$, given by*

$$\tau(x) = [s(x), f(x)] \quad \forall x \in U.$$

*In particular, the local gauge defines a preferred isomorphism between $V$ and every fibre $E_x$ over $x \in U$.*

*Proof* If $f: U \to V$ is a smooth map, then

$$\begin{aligned} U &\longrightarrow P \times V \\ x &\longmapsto (s(x), f(x)) \end{aligned}$$

is smooth and hence $\tau: U \to E$ is smooth. The map $\tau$ is a section, because

$$\pi_E \circ \tau(x) = \pi_P \circ s(x) = x.$$

Conversely, let $\tau: U \to E$ be a smooth section. Since $E_x = (P_x \times V)/G$ and the action of $G$ on $P_x$ is simply transitive, there is a unique $f(x) \in V$ such that

$$\tau(x) = [s(x), f(x)].$$

We have to show that $f: U \to V$ is smooth: Define a bundle chart $\phi_U$ of the principal bundle using the section $s$:

$$\begin{aligned}\phi_U^{-1}: U \times G &\longrightarrow P_U\\ (x, g) &\longmapsto s(x) \cdot g.\end{aligned}$$

Then with the notation in the proof of Theorem 4.7.2 we have $\beta_U(s(x)) = e$ and

$$\begin{aligned}\psi_U \circ \tau(x) &= \psi_U([s(x), f(x)])\\ &= (x, \rho(\beta_U(s(x)))f(x))\\ &= (x, f(x)).\end{aligned}$$

Since $\psi_U$ and $\tau$ are smooth, it follows that $f$ is smooth. □

Matter fields in physics are described by smooth sections of vector bundles $E$ associated to principal bundles $P$ via representations of the gauge group $G$ on a vector space $V$ (in the case of fermions the associated bundle $E$ is twisted in addition with a *spinor bundle* $S$, i.e. the bundle is $S \otimes E$). It follows that, given a local gauge of the gauge bundle $P$, the section in $E$ corresponds to a unique local map from spacetime into the vector space $V$.

In particular, since principal bundles on $\mathbb{R}^n$ are trivial by Corollary 4.2.9, we can describe matter fields on a spacetime diffeomorphic to $\mathbb{R}^n$ by unique maps from $\mathbb{R}^n$ into a vector space, *once a global gauge for the principal bundle has been chosen.* A (local) trivialization of the gauge bundle thus determines a unique (local) trivialization of all associated vector bundles.

**Definition 4.7.7** Let $E = P \times_\rho V$ be an associated vector bundle. If the representation

$$\rho_*: \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

is non-trivial, then the sections of $E$ are called **charged**.
This term will be explained in more detail in Sect. 5.9.

*Remark 4.7.8 (Associated Fibre Bundles with Arbitrary Fibres)* Given a principal bundle $P \to M$ with structure group $G$, a manifold $F$ and a smooth left action

$$\begin{aligned}\Psi: G \times F &\longrightarrow F\\ (g, v) &\longmapsto g \cdot v\end{aligned}$$

a similar construction using the quotient

$$P \times_{\Psi} F = (P \times F)/G$$

under the $G$-action

$$\begin{aligned} (P \times F) \times G &\longrightarrow P \times F \\ (p, v, g) &\longmapsto \big(p \cdot g, g^{-1} \cdot v\big) \end{aligned}$$

yields an associated fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & P \times_{\Psi} F \\ & & \big\downarrow \pi \\ & & M \end{array}$$

with structure group given by the image of $G$ in the diffeomorphism group $\mathrm{Diff}(F)$, determined by the action $\Psi$.

*Example 4.7.9 (Flat Bundles)* Here is an example of the construction in Remark 4.7.8. Let $M$ and $F$ be manifolds and

$$\psi: \pi_1(M) \longrightarrow \mathrm{Diff}(F)$$

a group homomorphism. This defines an action of the (discrete) group $\pi_1(M)$ on $F$. The universal covering

$$\pi_M: \tilde{M} \longrightarrow M$$

can be considered as a principal bundle with discrete structure group $\pi_1(M)$. The associated fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & \tilde{M} \times_{\psi} F \\ & & \big\downarrow \\ & & M \end{array}$$

is called the **flat bundle** with **holonomy** $\psi$. In the case of $M = S^1$ this yields again the mapping torus from Example 4.1.5. More generally, for $M = T^n$, a collection of $n$ pairwise commuting diffeomorphisms

$$f_i: F \longrightarrow F, \quad i = 1, \ldots, n$$

defines a flat bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & T^n \end{array}$$

## 4.7.2 Adapted Bundle Atlases for Associated Vector Bundles

We discuss a specific type of bundle atlas for associated vector bundles. Let $P \to M$ be a principal $G$-bundle and $E = P \times_\rho V$ an associated vector bundle, where $\rho: G \to \mathrm{GL}(V)$ is a representation. We choose a principal bundle atlas $\{(U_i, \phi_i)\}_{i\in I}$ for $P$, determined by local gauges $s_i: U_i \to P$.

**Definition 4.7.10** The principal bundle atlas for $P$ defines an **adapted bundle atlas** for $E$ with local trivializations

$$\psi_i: E_{U_i} \longrightarrow U_i \times V$$

whose inverses are given by

$$\psi_i^{-1}(x, v) = [s_i(x), v].$$

**Proposition 4.7.11 (Adapted Bundle Atlases and the Structure Group)** *Suppose the transition functions of the principal bundle charts for P are given by*

$$\phi_{ji} = \phi_{j\cdot} \circ \phi_{i\cdot}^{-1}: U_i \cap U_j \longrightarrow G.$$

*Then the transition functions for the adapted bundle atlas for E are*

$$\begin{aligned} \psi_{ji} = \psi_{j\cdot} \circ \psi_{i\cdot}^{-1}: U_i \cap U_j &\longrightarrow \mathrm{GL}(V) \\ x &\longmapsto \psi_{ji}(x) = \rho(\phi_{ji}(x)). \end{aligned}$$

*The transition functions of E thus have image in the subgroup* $\rho(G) \subset \mathrm{GL}(V)$, *where G is the structure group of P.*

*Proof* We have

$$s_i(x) = s_j(x) \cdot \phi_{ji}(x).$$

This implies

$$\begin{aligned} \psi_{ix}^{-1}(v) &= [s_j(x) \cdot \phi_{ji}(x), v] \\ &= \psi_{jx}^{-1} \circ \rho(\phi_{ji}(x))(v). \end{aligned}$$

Therefore

$$\psi_{ji}(x) = \psi_{jx} \circ \psi_{ix}^{-1} = \rho(\phi_{ji}(x)).$$

□

### 4.7.3 Bundle Metrics on Associated Vector Bundles

It is often important to consider bundle metrics on an associated vector bundle. We can construct such metrics as follows: let $\pi_P: P \to M$ be a principal bundle with structure group $G$, $\rho: G \to \mathrm{GL}(V)$ a representation and $E \to M$ the associated vector bundle $E = P \times_\rho V$.

**Proposition 4.7.12 (Bundle Metrics on Associated Vector Bundles from *G*-Invariant Scalar Products)** *Suppose that $\langle \cdot , \cdot \rangle_V$ is a $G$-invariant scalar product on $V$. Then the bundle metric $\langle \cdot , \cdot \rangle_E$ on the associated vector bundle $E$ given by*

$$\langle [p, v], [p, w] \rangle_{E_x} = \langle v, w \rangle_V,$$

*for arbitrary $p \in P_x$, is well-defined.*

*Proof* This is an easy calculation choosing two different representatives for the vectors in the fibre $E_x$. □

### 4.7.4 Examples

*Example 4.7.13 (From Vector Bundles to Principal Bundles and Back)* We claim that *every* vector bundle has the structure of an associated vector bundle for some principal bundle. We first consider the tangent bundle $TM$: Let $M$ be an $n$-dimensional smooth manifold and consider the frame bundle

$$\begin{array}{ccc} \mathrm{GL}(n, \mathbb{R}) & \longrightarrow & \mathrm{Fr}_{\mathrm{GL}}(M) \\ & & \downarrow \\ & & M \end{array}$$

Let

$$\rho_{\mathrm{GL}}: \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

be the standard representation, given by matrix multiplication from the left on column vectors. Then there exists an isomorphism of vector bundles

$$TM \cong \mathrm{Fr}_{\mathrm{GL}}(M) \times_{\rho_{\mathrm{GL}}} \mathbb{R}^n.$$

An isomorphism is given by

$$H\colon \mathrm{Fr}_{\mathrm{GL}}(M) \times_{\rho_{\mathrm{GL}}} \mathbb{R}^n \longrightarrow TM$$
$$[(v_1, \dots, v_n), (x_1, \dots, x_n)] \longmapsto \sum_{i=1}^{n} v_i x_i.$$

It is easy to check that the map $H$ is well-defined and a bundle isomorphism.

Choosing a Riemannian metric $g$ on $M$ we can define the orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}(M)$. Using the standard representation $\rho_{\mathrm{O}}$ of $\mathrm{O}(n)$ on $\mathbb{R}^n$ we get another vector bundle isomorphism

$$TM \cong \mathrm{Fr}_{\mathrm{O}}(M) \times_{\rho_{\mathrm{O}}} \mathbb{R}^n.$$

Similarly, every real vector bundle $E$ of rank $n$ is associated to a principal $\mathrm{GL}(n,\mathbb{R})$-bundle (and a principal $\mathrm{O}(n)$-bundle), defined using frames in the fibres of $E$. If $E$ is orientable, it is associated to a principal $\mathrm{SO}(n)$-bundle. Similar statements hold for complex vector bundles with principal $\mathrm{GL}(n,\mathbb{C})$- and $\mathrm{U}(n)$-bundles.
We get:

**Proposition 4.7.14** *Let $E \to M$ be a real or complex vector bundle. Then $E$ is associated to some principal* $\mathrm{O}(n)$*- or* $\mathrm{U}(n)$*-bundle $P \to M$.*
In particular, the vector bundles over spheres that we defined in Sect. 4.6 using the clutching construction are associated vector bundles. Note that the structure as an associated vector bundle is not unique: as we saw above in the case of the frame bundle, the same vector bundle can be associated to principal bundles with different Lie groups.

We can use our constructions of principal bundles over spheres, projective spaces, and Stiefel and Grassmann manifolds to define associated vector bundles over those manifolds.

*Example 4.7.15* Recall the principal bundle

$$\begin{array}{ccc} \mathrm{SO}(n-1) & \longrightarrow & \mathrm{SO}(n) \\ & & \big\downarrow \pi \\ & & S^{n-1} \end{array}$$

from Example 4.2.16. Then any representation of $\mathrm{SO}(n-1)$ on a real or complex vector space defines an associated vector bundle over the sphere $S^{n-1}$. A similar construction works for any of the other principal bundles over spheres given in Example 4.2.16. Alternatively, these bundles can also be realized (up to isomorphism) by the clutching construction.

The construction also applies to the principal bundles over Stiefel and Grassmann manifolds, like

$$\begin{array}{ccc} \mathrm{O}(n-k) & \longrightarrow & \mathrm{O}(n) \\ & & \downarrow \pi \\ & & V_k(\mathbb{R}^n) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{O}(k) \times \mathrm{O}(n-k) & \longrightarrow & \mathrm{O}(n) \\ & & \downarrow \pi \\ & & Gr_k(\mathbb{R}^n) \end{array}$$

These examples can be generalized: start with any smooth homogeneous space $G/H$ and consider the canonical principal bundle

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \pi \\ & & G/H \end{array}$$

according to Theorem 4.2.15. Then representations of $H$ define associated vector bundles over $G/H$, known as **homogeneous vector bundles**.

*Example 4.7.16* Let

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array}$$

be the Hopf fibration. We want to study complex line bundles associated to this principal $S^1$-bundle. For $k \in \mathbb{Z}$ consider the homomorphism

$$\begin{aligned} \rho_k\colon S^1 &\longrightarrow \mathrm{U}(1) \\ z &\longmapsto z^k \end{aligned}$$

of winding number $k$. Then the associated bundle

$$\gamma^k = S^{2n+1} \times_{\rho_k} \mathbb{C}$$

is a complex line bundle. The bundle $\gamma^0$ is trivial and $\gamma^k$ is isomorphic to

$$\gamma^k \cong \underbrace{\gamma^1 \otimes \ldots \otimes \gamma^1}_{k \text{ factors}} \quad (k > 0)$$

and

$$\gamma^k \cong \underbrace{\gamma^{1*} \otimes \ldots \otimes \gamma^{1*}}_{|k| \text{ factors}} \quad (k < 0).$$

See Exercise 4.8.21.

Similarly, using representations of $\mathrm{SU}(2) \cong S^3$ we can define vector bundles associated to the quaternionic Hopf fibration

$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \\ & & \mathbb{HP}^n \end{array}$$

*Example 4.7.17 (Adjoint Bundle)* An important general example of an associated vector bundle is the following: let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

be a principal bundle with structure group $G$. Consider the adjoint representation

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g}).$$

Then the associated vector bundle

$$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$$

is called the **adjoint bundle**. Its general fibre is isomorphic to the vector space underlying the Lie algebra $\mathfrak{g}$:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{Ad}(P) \\ & & \downarrow \\ & & M \end{array}$$

## 4.8 Exercises for Chap. 4

**4.8.1** The *Möbius strip* can be defined as the submanifold

$$M = \left\{ \left(e^{i\phi}, re^{i\phi/2}\right) \,\middle|\, \phi \in [0, 2\pi], r \in [-1, 1] \right\} \subset S^1 \times \mathbb{C}.$$

The projection $\pi: M \to S^1$ is defined as $\pi = \mathrm{pr}_1|_M$.

1. Show that $\pi: M \to S^1$ is a fibre bundle with general fibre $[-1, 1]$ (here we consider a small generalization of the notion of a fibre bundle to manifolds with boundary).
2. Prove that the boundary $\partial M$ is connected and that the bundle $\pi$ is not trivial.
3. Prove that the image of any smooth section $s: S^1 \to M$ intersects the zero section $z: S^1 \to M, z(\alpha) = (\alpha, 0)$.

*Hint*: Note that the map $S^1 \to S^1, e^{i\phi} \mapsto e^{i\phi/2}$ is not well-defined.

**4.8.2** Let $\pi: M \to S^1$ denote the Möbius strip from Exercise 4.8.1 and consider the map $f_n: S^1 \to S^1, f_n(z) = z^n$ for $n \in \mathbb{Z}$.

1. Show that the pull-back bundle $f_n^* M$ is isomorphic to the bundle $M_n \to S^1$ defined by

$$M_n = \left\{ \left(e^{i\psi}, re^{in\psi/2}\right) \,\middle|\, \psi \in [0, 2\pi], r \in [-1, 1] \right\} \subset S^1 \times \mathbb{C}$$

   (with projection onto the first factor).
2. Determine those $n \in \mathbb{Z}$ for which $f_n^* M$ is trivial and those for which it is non-trivial.

**4.8.3 (Fibre Sum)** Suppose that $F \to E \to M$ and $F' \to E' \to M'$ are two fibre bundles over $n$-dimensional manifolds $M$ and $M'$. Let $D$ and $D'$ be embedded open $n$-discs in $M$ and $M'$ together with trivializations $F \times D$ and $F' \times D'$ of the fibrations over $D$ and $D'$. We assume that $F$ and $F'$ are diffeomorphic and choose a diffeomorphism

$$\phi: F \longrightarrow F'.$$

We write $D$ and $D'$ minus the centre 0 as $S^{n-1} \times (0, 1)$ and fix a diffeomorphism $r$ from $(0, 1)$ to $(0, 1)$ which reverses orientation. Let $\tau: S^{n-1} \to S^{n-1}$ be the diffeomorphism which reverses the sign of one of the coordinates on $S^{n-1} \subset \mathbb{R}^n$. Consider the diffeomorphism

$$\begin{aligned} \psi: F \times (D \setminus 0) &\longrightarrow F' \times (D' \setminus 0) \\ (x, v, t) &\longmapsto (\phi(x), \tau(v), r(t)). \end{aligned}$$

The **fibre sum** $E\#_\psi E'$ is defined by gluing together the manifolds $M \setminus F$ and $M' \setminus F'$ along the diffeomorphism $\psi$. Prove that $E\#_\psi E'$ is a smooth fibre bundle over the connected sum $M\#M'$ with general fibre $F$.

**4.8.4** Let $G \to P \xrightarrow{\pi} M$ be a principal bundle and $f: N \to M$ a smooth map between manifolds. Prove that the pullback $f^*P$ has the canonical structure of a principal $G$-bundle over $N$.

**4.8.5**

1. Let $F \to E \xrightarrow{\pi} M$ and $F' \to E' \xrightarrow{\pi} M$ be fibre bundles and $H: E \to E'$ a bundle morphism. Suppose that $H$ maps every fibre of $E$ diffeomorphically onto a fibre of $E'$. Show that $H$ is a diffeomorphism and hence a bundle isomorphism.
2. Let $G \to P \xrightarrow{\pi} M$ and $G' \to P' \xrightarrow{\pi} M$ be principal bundles and $f: G \to G'$ a Lie group isomorphism. Show that every $f$-equivariant bundle morphism $H: P \to P'$ is a diffeomorphism.

**4.8.6 (From [14])** We consider the Hopf bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

The total space $S^3$ of this bundle admits two different $S^1$-actions: The standard action

$$\begin{aligned} S^3 \times S^1 &\longrightarrow S^3, \\ (w, \lambda) &\longmapsto w\lambda \end{aligned}$$

and the *reversed action*

$$\begin{aligned} S^3 \times S^1 &\longrightarrow S^3, \\ (w, \lambda) &\longmapsto w\lambda^{-1}. \end{aligned}$$

Both actions endow the same fibre bundle $S^1 \to S^3 \xrightarrow{\pi} S^2$ with the structure of a principal bundle. Prove that these principal bundles are not isomorphic *as principal bundles*.

**4.8.7** Recall the definition of lens spaces from Example 3.7.33. Show that the lens space $L(p, 1)$ is the total space of a principal fibre bundle over $S^2$ with structure group $S^1$.

**4.8.8** Show that there is a canonical free $\mathrm{O}(k)$-action on the Stiefel manifold $V_k(\mathbb{R}^n)$ and that this defines a principal $\mathrm{O}(k)$-bundle

$$\begin{array}{ccc} \mathrm{O}(k) & \longrightarrow & V_k(\mathbb{R}^n) \\ & & \downarrow \\ & & Gr_k(\mathbb{R}^n) \end{array}$$

**4.8.9** We want to discuss another way to construct fibre bundles. Let $M$, $F$ be smooth manifolds and $\{U_i\}_{i\in I}$ an open covering of $M$ together with diffeomorphisms

$$\phi_{ji}\colon (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

whenever $U_i \cap U_j \neq \emptyset$, satisfying

$$\mathrm{pr}_1 \circ \phi_{ji} = \mathrm{pr}_1.$$

We also write $\phi_{ji}(x) = \phi_{ji}(x, -)$ for $x \in U_i \cap U_j$. Let $\tilde{E}$ be the disjoint union

$$\tilde{E} = \dot{\bigcup}_{i\in I} U_i \times F.$$

1. Show that

$$(x, v) \sim (x', v') \Leftrightarrow \exists i, j \in I : x = x' \in U_i \cap U_j \text{ and } v' = \phi_{ji}(x)v$$

   defines an equivalence relation on $\tilde{E}$ if and only if the $\phi_{ji}$ satisfy the three conditions of Lemma 4.1.13.
2. Show that if the $\phi_{ji}$ satisfy the conditions of Lemma 4.1.13, then the quotient set

$$E = \tilde{E}/\sim$$

   has the canonical structure of a smooth fibre bundle over $M$ with general fibre $F$ and transition functions $\phi_{ji}$.

**4.8.10** Prove that the principal bundle

$$\begin{array}{ccc} \mathrm{SO}(n-1) & \longrightarrow & \mathrm{SO}(n) \\ & & \downarrow \pi \\ & & S^{n-1} \end{array}$$

from Example 4.2.16 is isomorphic to the frame bundle $\mathrm{Fr}_{\mathrm{SO}}(S^{n-1})$.

**4.8.11** Prove that a subset $F \subset E$ is a subbundle of the vector bundle $E$ if and only if $F$ is the image of a vector bundle monomorphism to $E$.

**4.8.12** Prove that

$$E = \{(U, v) \in Gr_k(\mathbb{K}^n) \times \mathbb{K}^n \mid v \in U\},$$

with projection onto the first factor, defines a $\mathbb{K}$-vector bundle over the Grassmann manifold $Gr_k(\mathbb{K}^n)$ of rank $k$. This bundle is called the **tautological vector bundle**. Particular examples, for $k = 1$, are the **tautological line bundles** over $\mathbb{RP}^{n-1}$ and $\mathbb{CP}^{n-1}$.

**4.8.13** We denote by $L \to S^1$ the infinite Möbius strip, defined by

$$L = \left\{\left(e^{i\phi}, re^{i\phi/2}\right) \,\middle|\, \phi \in [0, 2\pi], r \in \mathbb{R}\right\} \subset S^1 \times \mathbb{C}.$$

It follows from Exercise 4.8.1 that this is a non-trivial, real line bundle over the circle. Prove that the real vector bundle $L \oplus L \to S^1$ is trivial.

**4.8.14** Let $L \to S^1$ be the infinite Möbius strip.

1. Show that under the diffeomorphism $S^1 \cong \mathbb{RP}^1$ the infinite Möbius strip is isomorphic to the tautological line bundle over $\mathbb{RP}^1$.
2. Prove that the tautological line bundle over $\mathbb{RP}^n$ is non-trivial for all $n \geq 1$.

**4.8.15** Let $E \to M$ be a real vector bundle of rank $m$. Show that $E$ is orientable if and only if $\Lambda^m E$ is trivial.

**4.8.16** Let $E \to M$ be a $\mathbb{K}$-vector bundle of rank $m$ with a positive definite (Euclidean or Hermitian) bundle metric. Suppose that $F \subset E$ is a vector subbundle. Prove that the orthogonal complement $F^\perp$ is a vector subbundle of $E$ and that $F \oplus F^\perp$ is isomorphic to $E$.

**4.8.17** Determine the clutching function of the tangent bundle $TS^2 \to S^2$ geometrically as follows:

1. Draw two disks in the plane and label them $N$ and $S$. Draw on the boundary circle of disk $N$ four points $a, b, c, d$ counter-clockwise with 90° between consecutive points. Draw on the boundary circle of disk $S$ corresponding points $a, b, c, d$, such that the disks under identification of the boundary circles yield a sphere $S^2$.
2. Draw in the center of disk $N$ an orthonormal basis and label the vectors 1 and 2. Parallel transport this basis to the points $a, b, c, d$. Take these bases and draw the matching bases on the $S$ side in the points $a, b, c, d$. Call these bases $I$.
3. Take the basis at the point $a$ on disk $S$ and parallel transport it to the center of disk $S$. Then parallel transport this basis from the center to the points $b, c, d$. Call these bases $II$.

4. Determine how bases $I$ twist against bases $II$ and thus determine the clutching function, i.e. the degree of the map

$$f: S^1 \longrightarrow \mathrm{SO}(2) \cong S^1.$$

To fix the sign of the degree, you probably need at least one more point at 45° between $a$ and $b$, for example.

What do you get if you do something similar for $TS^3 \to S^3$ by realizing $S^3$ as two solid cubes identified along their six faces?

**4.8.18** Determine the clutching function of the tautological complex line bundle $E \to \mathbb{CP}^1 \cong S^2$. The total space of the line bundle is

$$E = \left\{([z], wz) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid z \neq 0, w \in \mathbb{C}\right\}$$

and $\mathbb{CP}^1$ is covered by

$$U_+ = \{[z:1] \in \mathbb{CP}^1 \mid z \in \mathbb{C}\},$$
$$U_- = \{[1:z] \in \mathbb{CP}^1 \mid z \in \mathbb{C}\}.$$

**4.8.19** Determine the clutching functions in the sense of Remark 4.6.4 for the Hopf fibrations

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \big\downarrow \pi \\ & & S^2 \end{array}$$

and

$$\begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \big\downarrow \pi \\ & & S^4 \end{array}$$

**4.8.20** Let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

be a principal $G$-bundle and $\rho: G \to \mathrm{GL}(V)$, $\rho_i: G \to \mathrm{GL}(V_i)$, for $i = 1, 2$, representations. Let

$$E = P \times_\rho V, \quad E_i = P \times_{\rho_i} V_i$$

be the associated vector bundles. Show that the dual bundle $E^*$, the direct sum $E_1 \oplus E_2$ and the tensor product $E_1 \otimes E_2$ are isomorphic to vector bundles associated to $P$. Determine the corresponding representations of $G$ and the vector bundle isomorphisms. Show that the vector bundle associated to the trivial representation is trivial.

**4.8.21 (From [14])** Let

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \gamma^k \\ & & \downarrow \pi \\ & & \mathbb{CP}^n \end{array}$$

be the complex line bundle defined in Example 4.7.16.

1. Prove that $\gamma^0$ is trivial and $\gamma^1$ is isomorphic to the tautological line bundle.
2. Prove that $\gamma^{-k} \cong \gamma^{k*}$ for all $k \in \mathbb{Z}$ and

$$\gamma^k \cong \underbrace{\gamma^1 \otimes \ldots \otimes \gamma^1}_{k \text{ factors}} \quad (k > 0).$$

**4.8.22** Let $E \to M$ be a complex vector bundle of rank $n \geq 2$. Show that $E$ is associated to a principal $\mathrm{SU}(n)$-bundle over $M$ if and only if $\Lambda^n E$ is a trivial complex line bundle.

**4.8.23** Let $E = P \times_\rho V$ be an associated vector bundle and $\alpha$ a section of the adjoint bundle $\mathrm{Ad}(P)$. Prove that $\alpha$ defines a canonical endomorphism of the vector bundle $E$.

**4.8.24** Let $M = G/H$ be a smooth homogeneous space and consider the canonical principal $H$-bundle

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \pi \\ & & M \end{array}$$

Suppose that $\rho: H \to \mathrm{GL}(V)$ is a representation with associated homogeneous vector bundle

$$\begin{array}{ccc} V & \longrightarrow & E = G \times_\rho V \\ & & \downarrow \pi \\ & & M \end{array}$$

1. Prove that there exists a canonical smooth left action of the Lie group $G$ on the total space $E$. Show that this action maps fibres of $E$ by linear isomorphisms onto fibres of $E$ and that any given fibre of $E$ can be mapped by a group element onto any other fibre.
2. Identify the space $\Gamma(E)$ of sections of the vector bundle $E$ over the manifold $M$ with a suitable vector subspace $\mathrm{Map}_H(G, V)$ of the vector space $\mathrm{Map}(G, V)$.

*Remark* The representation of $G$ on $\Gamma(E)$, induced by this construction from the representation of the closed subgroup $H$ on $V$, is denoted by $\mathrm{Ind}_H^G(V)$.

**4.8.25 (From [30])** Let $M = G/H$ be a smooth homogeneous space and consider the canonical principal $H$-bundle

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \big\downarrow \pi \\ & & M \end{array}$$

Prove that the tangent bundle $TM$ is isomorphic to the homogeneous vector bundle over $M$, defined by the representation $\rho$ of $H$ on the vector space $\mathfrak{g}/\mathfrak{h}$, given by

$$\rho(h)[v] = [\mathrm{Ad}_h v] \quad \forall h \in H, [v] \in \mathfrak{g}/\mathfrak{h},$$

where Ad denotes the adjoint representation of $G$.