Chapter 3
Group Actions

There are different ways in which Lie groups can act as transformation or symmetry groups on geometric objects. One possibility, that we discussed in Chap. 2, is the representation of Lie groups on vector spaces. A second possibility, studied in this chapter, is Lie group actions on manifolds. Both concepts are related: A representation is a linear action of the group where the manifold is a vector space. Conversely, an action on a manifold can be thought of as a *non-linear representation* of the group. More precisely, a linear representation of a group corresponds to a homomorphism into the general linear group of a vector space. A group action then corresponds to a homomorphism of the group into the *diffeomorphism group* of a manifold.

Even though we are most interested in Lie group actions on manifolds, it is useful to consider more general types of actions: actions of groups on sets and actions of topological groups on topological spaces. We will also introduce several standard notions related to group actions, like *orbits* and *isotropy groups*. In the smooth case, if a Lie group G acts on a manifold M, then there is an induced *infinitesimal action* of the Lie algebra $\mathfrak{g}$, defining so-called *fundamental vector fields* on M. This map can be understood as the induced Lie algebra homomorphism from the Lie algebra of G to the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$.

In the case of smooth actions of a Lie group G on a manifold M, the interesting question arises under which conditions the *quotient space* M/G again admits the structure of a smooth manifold. The main (and rather difficult) result that we prove in this context is *Godement's Theorem*, which gives a necessary and sufficient condition that quotient spaces under general equivalence relations are smooth manifolds. The smooth structure on the quotient space is defined using so-called *slices* for the equivalence classes.

It turns out that the quotient space of a Lie group action admits the structure of a smooth manifold in particular in the following cases:

- A *compact* Lie group G acting smoothly and *freely* on a manifold M.
- A closed subgroup H of a Lie group G acting on G by right (or left) translations.

M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
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Both cases can be used to construct new and interesting smooth manifolds. In the second case, if the closed subgroup H acts on the *right* on G, then there is an additional *left* action of G on the quotient manifold G/H. This action is *transitive* and G/H is an example of a *homogeneous space*. We will study homogeneous spaces in detail in all of the three cases of group actions on sets, topological spaces and manifolds and prove that any homogeneous space is of the form G/H.

We finally apply the theory of group actions to construct the exceptional compact simple Lie group G_2, which plays an important part in *M-theory*, a conjectured theory of quantum gravity in 11 dimensions, and derive some of its properties.

General references for this chapter are [14, 24] and [142].

3.1 Transformation Groups

In this section we define group actions and study their basic properties. Since many statements in this section are quite elementary, we designate some of the proofs as exercises.

Before we begin with the formal definitions, let us consider some basic examples to get a bird's eye view of group actions. The simplest example is perhaps the canonical left action of the general linear group $\mathrm{GL}(V)$ on a vector space V, given by the map

$$\begin{aligned} \Phi\colon \mathrm{GL}(V) \times V &\longrightarrow V \\ (f, v) &\longmapsto \Phi(f, v) = f(v). \end{aligned} \tag{3.1}$$

A representation of a group G on V then corresponds to a group homomorphism

$$\phi\colon G \longrightarrow \mathrm{GL}(V),$$

defining a linear action of G on V.

We would like to extend this idea to other types of actions. Suppose that

- M is a set and $\mathrm{S}(M)$ the symmetric group of all bijections $M \to M$; or
- M is a topological space and $\mathrm{Homeo}(M)$ the homeomorphism group of M; or
- M is a manifold and $\mathrm{Diff}(M)$ the diffeomorphism group of M.

Replacing V by M and $\mathrm{GL}(V)$ by $\mathrm{S}(M)$ ($\mathrm{Homeo}(M)$, $\mathrm{Diff}(M)$) in Eq. (3.1) we get canonical actions of these automorphism groups on M. *Actions* of a group G on M are then given by homomorphisms ϕ of G into these groups and thus

(continued)

correspond to *non-linear representations* of G on M (which in the case for Homeo(M) and Diff(M) should in some sense be continuous and smooth).

In each of these cases, the images of the group G under the homomorphisms ϕ define subgroups of GL(V), S(M), Homeo(M) and Diff(M) that are usually easier to handle than the full automorphism groups themselves (which in the case of the diffeomorphism group, for example, are infinite-dimensional if $\dim M \geq 1$).

An explicit example of a Lie group action on a manifold is the famous *Hopf action* of $S^1 = \mathrm{U}(1)$ on S^3 defined by the map

$$\begin{aligned} \Phi: S^3 \times S^1 &\longrightarrow S^3 \\ (v, w, \lambda) &\longmapsto (v, w) \cdot \lambda = (v\lambda, w\lambda), \end{aligned}$$

where S^3 is the unit sphere in $\mathbb{C}^2$ and S^1 the unit circle in $\mathbb{C}$ (this is an example of a *right action*). It is clear that the map is well-defined, i.e. it preserves the 3-sphere, and it is smooth. The map also has the following properties:

1. $(v, w) \cdot (\lambda \cdot \mu) = ((v, w) \cdot \lambda) \cdot \mu$
2. $(v, w) \cdot 1 = (v, w)$

for all $(v, w) \in S^3$ and $\lambda, \mu \in S^1$. We shall see that these are the defining properties of group actions, ensuring that we obtain a homomorphism into the diffeomorphism group. In the case of the Hopf action we can think of it as a homomorphism

$$\phi: S^1 \longrightarrow \mathrm{Diff}(S^3).$$

The Hopf action will also be an important example in subsequent chapters.

We shall later study properties of this and other actions. For example, we can fix a point $(v_0, w_0) \in S^3$ and consider its *orbit* under the action:

$$\begin{aligned} S^1 &\longrightarrow S^3 \\ \lambda &\longmapsto (v_0, w_0) \cdot \lambda. \end{aligned}$$

In this case, the orbit map is injective for all $(v_0, w_0) \in S^3$ and the Hopf action is therefore called *free*.

3.2 Definition and First Properties of Group Actions

We now come to the formal definition of group actions.

Definition 3.2.1 A **left action** of a group G on a set M is a map

$$\begin{aligned} \Phi: G \times M &\longrightarrow M \\ (g, p) &\longmapsto \Phi(g, p) = g \cdot p = gp \end{aligned}$$

satisfying the following properties:

1. $(g \cdot h) \cdot p = g \cdot (h \cdot p)$ for all $p \in M$ and $g, h \in G$.
2. $e \cdot p = p$ for all $p \in M$.

The group G is called a **transformation group** of M.
We can think of a group action as moving a point $p \in M$ around in M as we vary the group element $g \in G$. This is very similar to the concept of a representation of a group on a vector space, where a vector is moved around as we vary the group element.

If G is a topological group, M a topological space and Φ continuous, then Φ is called a **continuous left action**. Similarly, if G is a Lie group, M a smooth manifold and Φ is smooth, then Φ is called a **smooth left action**. Here $G \times M$ carries the canonical product structure as a topological space or smooth manifold.

Similarly **right actions** of a group G on a set M are defined as a map

$$\begin{aligned} \Phi: M \times G &\longrightarrow M \\ (p, g) &\longmapsto \Phi(p, g) = p \cdot g = pg \end{aligned}$$

satisfying the following properties:

1. $p \cdot (g \cdot h) = (p \cdot g) \cdot h$ for all $p \in M$ and $g, h \in G$.
2. $p \cdot e = p$ for all $p \in M$.

There is, of course, also the notion of a continuous or smooth right action (most of the following statements hold for both left and right actions). We can turn every left action into a right action (and vice versa):

Proposition 3.2.2 *Let*

$$\begin{aligned} \Phi: G \times M &\longrightarrow M \\ (g, p) &\longmapsto g \cdot p \end{aligned}$$

be a left action of a group G on a set M. Then

$$\begin{aligned} M \times G &\longrightarrow M \\ (p, g) &\longmapsto p * g = g^{-1} \cdot p \end{aligned}$$

defines a right action of G on M.

Proof This is Exercise 3.12.1. □

A group action Φ is a map with two entries: a group element $g \in G$ and a point $p \in M$. It is useful to consider the maps that we obtain if we fix one of the entries and let only the other one vary.

Definition 3.2.3 Let $\Phi: G \times M \to M$ be a left action. For $g \in G$ we define the **left translation** by

$$\begin{aligned} l_g: M &\longrightarrow M \\ p &\longmapsto g \cdot p. \end{aligned}$$

Similarly, for a right action $\Phi: M \times G \to M$ and $g \in G$ we define the **right translation** by

$$\begin{aligned} r_g: M &\longrightarrow M \\ p &\longmapsto p \cdot g. \end{aligned}$$

For $p \in M$ the **orbit map** is given by

$$\begin{aligned} \phi_p: G &\longrightarrow M \\ g &\longmapsto g \cdot p \end{aligned}$$

for a left action and

$$\begin{aligned} \phi_p: G &\longrightarrow M \\ g &\longmapsto p \cdot g \end{aligned}$$

for a right action.

It is clear that for a continuous (smooth) left action the left translations l_g for all $g \in G$ and the orbit maps ϕ_p for all $p \in M$ are continuous (smooth) maps. The reason is that in the smooth case the map l_g is given by the composition of smooth maps

$$\begin{aligned} M &\longrightarrow G \times M \longrightarrow M \\ p &\longmapsto (g, p) \longmapsto g \cdot p \end{aligned}$$

and ϕ_p is given by the composition

$$\begin{aligned} G &\longrightarrow G \times M \longrightarrow M \\ g &\longmapsto (g, p) \longmapsto g \cdot p. \end{aligned}$$

The continuous case and the case of right actions follow similarly.

We could define left translations as above for any map $\Phi: G \times M \to M$ even if Φ does not satisfy *a priori* the axioms of a left action. It is easy to see that group actions are then characterized by the fact that all left translations l_g for $g \in G$ are bijections of M and

$$\begin{aligned} \phi: G &\longrightarrow \mathrm{S}(M) \\ g &\longmapsto l_g \end{aligned}$$

is a group homomorphism. In the case of a continuous (smooth) left action, the left translations define a group homomorphism

$$\phi: G \longrightarrow \mathrm{Homeo}(M)$$

and

$$\phi: G \longrightarrow \mathrm{Diff}(M),$$

respectively, into the group of homeomorphisms (diffeomorphisms) of M. Note that, as we said before, a continuous (smooth) group action is more than just a group homomorphism into the homeomorphism (diffeomorphism) group, because the group homomorphism has to be in addition continuous (smooth) in the argument $g \in G$ (one could make this precise by defining a topology or smooth structure on the homeomorphism and diffeomorphism groups, which in general are infinite-dimensional).

Here are some additional concepts for group actions (we define them in the general case for group actions on sets, but they apply verbatim for continuous and smooth group actions).

Definition 3.2.4 Let Φ be a left action of a group G on a set M.

1. The **orbit** of G through a point $p \in M$ is

$$\mathcal{O}_p = G \cdot p = \{g \cdot p \mid g \in G\}.$$

   The orbit is the image of the orbit map (see Fig. 3.1).
2. The **fixed point set** of a group element $g \in G$ is the set

$$M^g = \{p \in M \mid g \cdot p = p\}.$$

3. The **isotropy group** or **stabilizer** of a point $p \in M$ is

$$G_p = \{g \in G \mid g \cdot p = p\}.$$

   In physics, isotropy groups are also called **little groups**. It is an easy exercise to show that the isotropy group G_p is indeed a subgroup of G for all $p \in M$.

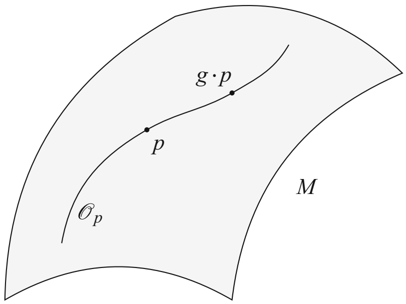


Fig. 3.1 Orbit of a group action

There are analogous definitions for right actions.

Remark 3.2.5 We shall see later in Corollary 3.8.10 that for a smooth action of a Lie group G on a manifold M the orbit $\mathcal{O}_p$ through every point $p \in M$ is an (immersed or embedded) submanifold of M.

Lemma 3.2.6 (Two Orbits Are Either Disjoint or Identical) *Let Φ be an action of a group G on a set M and $p \in M$ an arbitrary point. If $q \in \mathcal{O}_p$, then $\mathcal{O}_q = \mathcal{O}_p$. Hence the orbits of two points in M are either disjoint or identical.*
This means that orbits which intersect in one point are already identical.

Proof Suppose Φ is a left action. Then q is of the form $q = g \cdot p$ for some $g \in G$. We get

$$\mathcal{O}_q = G \cdot q = (G \cdot g) \cdot p = G \cdot p,$$

because the right translation $R_g : G \to G$ is a bijection. □

Remark 3.2.7 We can also phrase this differently: The relation

$$p \sim q \Leftrightarrow \exists g \in G : q = g \cdot p$$

for $p, q \in M$ defines an *equivalence relation* on M and the orbits of G are precisely the equivalence classes. M is therefore the disjoint union of the orbits of G.

Definition 3.2.8 Let Φ be an action of a group G on a set M. Then the following subset of the powerset of M

$$M/G = \{\mathcal{O}_p \subset M \mid p \in M\}$$

is called **the space of orbits** or the **quotient space** of the action.
Note that the *subsets* $\mathcal{O}_p$ of M become *elements* (*points*) in M/G. If we think of the subset $\mathcal{O}_p$ as a point in M/G, we also denote it by $[p]$ or $\bar{p}$. The map

$$\begin{aligned} \pi : M &\longrightarrow M/G \\ p &\longmapsto [p] \end{aligned}$$

is called the **canonical projection**. If $x \in M/G$, then a point $p \in M$ with $[p] = x$ is called a **representative** of x.

Concerning isotropy groups we can say the following.

Proposition 3.2.9 (Isotropy Groups Are (Closed) Subgroups) *Let Φ be an action of a group G on M and let $p \in M$ be any point. If the group action is continuous on a Hausdorff space M or smooth on a manifold M, then the stabilizer G_p is a closed subgroup of G. In particular, in the smooth case the stabilizer G_p is an embedded Lie subgroup of G by Cartan's Theorem 1.1.44.*

Proof This is an exercise. □

Suppose $\phi\colon G \to \mathrm{S}(M)$ is the group homomorphism induced from a group action. Then

$$\ker\phi = \bigcap_{p\in M} G_p.$$

In particular, for continuous actions on Hausdorff spaces or smooth actions on manifolds, the normal subgroup $\ker\phi$ is closed in G.

We want to compare the isotropy groups of points on the same G-orbit. Suppose p, q are points in M on the same G-orbit. It is easy to check that there exists an element $g \in G$ such that

$$c_g(G_p) = g \cdot G_p \cdot g^{-1} = G_q.$$

In particular, the isotropy groups G_p and G_q are isomorphic.

In the case of a smooth action of a Lie group G we call the Lie algebra $\mathfrak{g}_p$ of the stabilizer G_p of a point $p \in M$ the **isotropy subalgebra**. The following description of the isotropy subalgebra is useful in applications.

Proposition 3.2.10 (The Isotropy Subalgebra and the Orbit Map) *Let Φ be a smooth action of a Lie group G on a manifold M. Fix a point $p \in M$ and let ϕ_p denote the orbit map*

$$\phi_p\colon G \longrightarrow M$$

as before. Then the kernel of the differential

$$D_e\phi_p\colon \mathfrak{g} \longrightarrow T_pM$$

is equal to the isotropy subalgebra $\mathfrak{g}_p$.

Proof We assume that the action is on the left, the case of right actions follows similarly. If $X \in \mathfrak{g}_p$, then $\exp(tX) \in G_p$ for all $t \in \mathbb{R}$ and therefore

$$\phi_p(\exp(tX)) = \exp(tX) \cdot p = p \quad \forall t \in \mathbb{R}.$$

This implies that X is in the kernel of the differential $D_e\phi_p$. Conversely, suppose that X is in the kernel of $D_e\phi_p$. Then

$$\left.\frac{d}{d\tau}\right|_{\tau=0} (\exp(\tau X)\cdot p) = 0.$$

This implies

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=s} (\exp(tX)\cdot p) &= \left.\frac{d}{d\tau}\right|_{\tau=0} (\exp(sX)\cdot \exp(\tau X)\cdot p) \\
&= D_p l_{\exp(sX)} \left(\left.\frac{d}{d\tau}\right|_{\tau=0} \exp(\tau X)\cdot p \right) \\
&= 0 \quad \forall s \in \mathbb{R}.
\end{aligned}$$

Therefore, the curve $\exp(tX)\cdot p$ is constant and equal to $\exp(0)\cdot p = p$. This implies that $\exp(tX) \in G_p$ for all $t \in \mathbb{R}$ and thus $X \in \mathfrak{g}_p$ by Corollary 1.8.11. □

Definition 3.2.11 Let Φ be an action of a group G on a set M. We distinguish three cases, depending on whether the orbit map is surjective, injective or bijective for every $p \in M$.

1. The action is called **transitive** if the orbit map is surjective for every $p \in M$. In other words, M consists of only one orbit, $M = \mathcal{O}_p$ for every $p \in M$. We then call M a **homogeneous space** for G.
2. The action is called **free** if the orbit map is injective for every $p \in M$.
3. The action is called **simply transitive** if it is both transitive and free, i.e. if the orbit map for every $p \in M$ is a bijection from G onto M.

We leave it as an exercise to show the following properties of G-actions on M:

1. The orbit map is surjective for one $p \in M$ if and only if it is surjective for all $p \in M$.
2. The action is transitive if and only if M/G consists of precisely one point.
3. The orbit map of a point $p \in M$ is injective if and only if the isotropy group of p is trivial, $G_p = \{e\}$.
4. The action is free if and only if $g\cdot p \neq p$ for all $p \in M$, $g \neq e \in G$. Hence the action is free if and only if all points in M have trivial isotropy group or, equivalently, all group elements $g \neq e$ have empty fixed point set.

As a consequence of Proposition 3.2.10 we see:

Corollary 3.2.12 (Orbit Maps of Smooth Free Actions) *If Φ is a smooth free action of a Lie group G on a manifold M, then the orbit maps $\phi_p\colon G \rightarrow M$ are injective immersions for every point $p \in M$. If G is compact, then the orbit maps are embeddings.*

In the case of a free action of a compact Lie group G each orbit is therefore an embedded submanifold diffeomorphic to G.

Definition 3.2.13 An action Φ is called **faithful** or **effective** if the induced homomorphism $\phi: G \to \mathrm{S}(M)$ is injective.
It is not difficult to see that if one point in M has trivial isotropy group, then the action is faithful. We can always make a group action faithful by passing to the induced action of the quotient group $G/\ker\phi$.

It is sometimes important to compare actions of a group G on two sets M and N. In particular, we would like to have a notion of isomorphism of group actions.

Definition 3.2.14 Let $\Phi: G \times M \to M$ and $\Psi: G \times N \to N$ be left actions of a group G on sets M and N. Then a G**-equivariant** map $f: M \to N$ is a map such that

$$f(g \cdot p) = g \cdot f(p) \quad \forall p \in M, g \in G.$$

If G is a topological (Lie) group and the actions continuous (smooth), we demand in addition that f is continuous (smooth). A G-equivariant bijection (homeomorphism, diffeomorphism) is called an **isomorphism** of G-actions. There are analogous definitions in the case of right actions.

3.3 Examples of Group Actions

We discuss some common examples of group actions, in particular, smooth actions of Lie groups on manifolds.

It is quite easy to define group actions of *discrete abelian groups* on manifolds: Any diffeomorphism $f: M \to M$ defines a smooth group action

$$\mathbb{Z} \times M \longrightarrow M$$
$$(k, p) \longmapsto k \cdot p = f^k(p).$$

If f happens to be periodic, $f^n = \mathrm{Id}_M$ for some integer n, then this defines a smooth group action of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$:

$$\mathbb{Z}_n \times M \longrightarrow M$$
$$([k], p) \longmapsto [k] \cdot p = f^k(p).$$

If $f_1, \ldots, f_m$ are pairwise commuting diffeomorphisms of M, then

$$\mathbb{Z}^m \times M \longrightarrow M$$
$$(k_1, \ldots, k_m, p) \longmapsto (k_1, \ldots, k_m) \cdot p = f_1^{k_1} \circ \ldots \circ f_m^{k_m}(p)$$

is a smooth group action.

An example of a simply transitive action is to take $M = G$ for an arbitrary group G and let G act on itself by left (and right) translations:

$$\begin{aligned} \Phi: G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h. \end{aligned}$$

Another type of action that is easy to define is given by *group representations*. Let $\rho: G \to \mathrm{GL}(V)$ be a representation of a Lie group G on a real (or complex) vector space V. Then

$$\begin{aligned} \Phi: G \times V &\longrightarrow V \\ (g, v) &\longmapsto g \cdot v = \rho(g)v \end{aligned}$$

is a smooth left action on the manifold V (which is diffeomorphic to a Euclidean space). Such an action is called **linear**. We can also define a smooth right action by

$$\begin{aligned} \Phi: V \times G &\longrightarrow V \\ (v, g) &\longmapsto v \cdot g = \rho(g)^{-1}v. \end{aligned}$$

Note that it is important to take the inverse of $\rho(g)^{-1}$, otherwise the first property of a right action is in general not satisfied (see Exercise 3.12.1).

In both cases, the orbit of $0 \in V$ consists only of one point,

$$G \cdot 0 = \{0\}$$

and thus the isotropy group of 0 is all of G,

$$G_0 = G.$$

For a non-zero vector $v \neq 0$ the isotropy group in general will be a proper subgroup of G,

$$G_v \subsetneq G.$$

This is the basic mathematical idea behind **symmetry breaking** (from the full group G to the subgroup G_v), one of the centrepieces of the Standard Model that we discuss in Chap. 8.

For a linear representation, the homomorphism induced by the action has image in $\mathrm{GL}(V)$

$$\phi = \rho: G \longrightarrow \mathrm{GL}(V).$$

The action is faithful if and only if the representation is faithful.

Suppose in addition that $V \cong \mathbb{R}^n$ and the representation ρ is orthogonal. Then ρ has image in $\mathrm{O}(n)$ and the action maps the unit sphere S^{n-1} in $\mathbb{R}^n$ around the origin to itself. We therefore get a smooth left action

$$\begin{aligned} G \times S^{n-1} &\longrightarrow S^{n-1} \\ (g, v) &\longmapsto \rho(g)v. \end{aligned}$$

Similarly, if $V \cong \mathbb{C}^n$ and the representation ρ is unitary, so that it has image in $\mathrm{U}(n)$, then the action preserves the unit sphere S^{2n-1} in $\mathbb{C}^n$. We get a smooth left action

$$\begin{aligned} G \times S^{2n-1} &\longrightarrow S^{2n-1} \\ (g, v) &\longmapsto \rho(g)v. \end{aligned}$$

Finally, assume that $V \cong \mathbb{H}^n$ and the representation ρ is **quaternionic unitary**, by which we mean that ρ has image in $\mathrm{Sp}(n)$. Then the action preserves the standard symplectic scalar product (see Definition 1.2.9) and induces a smooth left action on the unit sphere S^{4n-1} in $\mathbb{H}^n$:

$$\begin{aligned} G \times S^{4n-1} &\longrightarrow S^{4n-1} \\ (g, v) &\longmapsto \rho(g)v. \end{aligned}$$

In each case we can similarly define right actions, using the inverses $\rho(g)^{-1}$. These actions on spheres are again called **linear**.

An important special case of this construction is the following:

Definition 3.3.1 Consider the groups $\mathbb{R}^*$, $\mathbb{C}^*$ and $\mathbb{H}^*$ of non-zero real, complex and quaternionic numbers. We define for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the following linear right actions by scalar multiplication:

$$\mathbb{K}^{n+1} \setminus \{0\} \times \mathbb{K}^* \longrightarrow \mathbb{K}^{n+1} \setminus \{0\}.$$

These actions are free and induce the following free linear right actions of the groups of real, complex and quaternionic numbers of unit norm on unit spheres:

$$\begin{aligned} S^n \times S^0 &\longrightarrow S^n \\ S^{2n+1} \times S^1 &\longrightarrow S^{2n+1} \\ S^{4n+3} \times S^3 &\longrightarrow S^{4n+3} \\ (x, \lambda) &\longmapsto x\lambda. \end{aligned}$$

These actions are called **Hopf actions**. The most famous example is the action of S^1 on S^3 that we already considered at the beginning of Sect. 3.1.

Note that $S^0 \cong \mathbb{Z}_2, S^1 \cong \mathrm{U}(1)$ and $S^3 \cong \mathrm{SU}(2)$. We shall see later in Example 3.7.34 that the *quotient spaces* under these free actions are smooth manifolds

$$\begin{aligned}\mathbb{RP}^n &= S^n/\mathbb{Z}_2\\ \mathbb{CP}^n &= S^{2n+1}/\mathrm{U}(1)\\ \mathbb{HP}^n &= S^{4n+3}/\mathrm{SU}(2)\end{aligned}$$

of dimension n, $2n$, $4n$, called **real**, **complex** and **quaternionic projective space**.

We consider another example of linear actions on spheres.

Theorem 3.3.2 (Linear Transitive Actions of Classical Groups) *The defining (fundamental) representations of* $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ *and* $\mathrm{Sp}(n)$ *define the following linear transitive actions on spheres with associated isotropy groups of the vector* e_1*:*

1. $\mathrm{O}(n)$*-action on* S^{n-1} *with isotropy group*

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{O}(n-1) \end{pmatrix} \cong \mathrm{O}(n-1).$$

Similarly, there is an

2. $\mathrm{SO}(n)$*-action on* S^{n-1} *with isotropy group isomorphic to* $\mathrm{SO}(n-1)$.
3. $\mathrm{U}(n)$*-action on* S^{2n-1} *with isotropy group isomorphic to* $\mathrm{U}(n-1)$.
4. $\mathrm{SU}(n)$*-action on* S^{2n-1} *with isotropy group isomorphic to* $\mathrm{SU}(n-1)$.
5. $\mathrm{Sp}(n)$*-action on* S^{4n-1} *with isotropy group isomorphic to* $\mathrm{Sp}(n-1)$.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *the defining (fundamental) representations of* $\mathrm{GL}(n, \mathbb{K})$ *and* $\mathrm{SL}(n, \mathbb{K})$ *define the following linear transitive actions with associated isotropy groups of the vector* e_1*:*

6. $\mathrm{GL}(n, \mathbb{K})$*-action on* $\mathbb{K}^n \setminus \{0\}$ *with isotropy group*

$$\begin{pmatrix} 1 & \mathbb{K}^{n-1} \\ 0 & \mathrm{GL}(n-1, \mathbb{K}) \end{pmatrix}.$$

7. $\mathrm{GL}(n, \mathbb{R})_+$*-action on* $\mathbb{R}^n \setminus \{0\}$ *with isotropy group*

$$\begin{pmatrix} 1 & \mathbb{R}^{n-1} \\ 0 & \mathrm{GL}(n-1, \mathbb{R})_+ \end{pmatrix}.$$

8. $\mathrm{SL}(n, \mathbb{K})$*-action on* $\mathbb{K}^n \setminus \{0\}$ *with isotropy group*

$$\begin{pmatrix} 1 & \mathbb{K}^{n-1} \\ 0 & \mathrm{SL}(n-1, \mathbb{K}) \end{pmatrix}.$$

Proof This is an exercise. The case of $\mathrm{SL}(n, \mathbb{H})$ uses the following lemma. □

Lemma 3.3.3 *For* $A \in \mathrm{Mat}(m \times m, \mathbb{H})$ *and* $v \in \mathbb{H}^m$ *the following equation holds:*

$$\det \begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} = \det(A).$$

Proof This is an exercise. □

We saw above that from a representation of a group on a vector space, we sometimes get group actions on other manifolds, in particular on spheres. We now show that from smooth actions on manifolds we also get representations on certain vector spaces.

Let $\Phi: G \times M \to M$ be a smooth (left) action of a Lie group G on a manifold M. Let $p \in M$ be a point and G_p its isotropy subgroup. By Proposition 3.2.9 the isotropy group G_p is an embedded Lie subgroup of G. The differential of the left translation l_g is a map

$$l_{g*} = D_p l_g : T_p M \longrightarrow T_p M,$$

for all $g \in G_p$. This is an isomorphism with inverse $l_{g^{-1}*}$.

Theorem 3.3.4 (Isotropy Representation) *The map*

$$\begin{aligned} \rho_p : G_p &\longrightarrow \mathrm{GL}(T_p M) \\ g &\longmapsto l_{g*} \end{aligned}$$

is a representation of the isotropy group G_p *on* T_pM*, called the* **isotropy representation**.

Proof We follow [142]. For $g, h \in G$ we calculate

$$\begin{aligned} \rho_p(gh) &= D_p l_{gh} \\ &= D_p(l_g \circ l_h) \\ &= \rho_p(g) \circ \rho_p(h), \end{aligned}$$

where we used the chain rule. Hence ρ_p is a group homomorphism. We want to show that ρ_p is smooth. Let $v \in T_pM$ be arbitrary and fixed. Then the map $\rho_p(\cdot)v$ is the composition of smooth maps

$$G_p \longrightarrow TG_p \times TM \longrightarrow T(G \times M) \longrightarrow TM$$

given by

$$g \longmapsto ((g, 0), (p, v)) \longmapsto ((g, p), (0, v)) \longmapsto D_{(g,p)}\Phi(0, v).$$

It follows that ρ_p is a smooth homomorphism, hence a representation. □

We get an analogous isotropy representation for right actions using the differential of right translations. Here is an almost trivial example of this construction.

Example 3.3.5 Let G be a Lie group, ρ a G-representation on a vector space V and $\Phi: G \times V \to V$ the induced linear action. Then the isotropy group of $0 \in V$ is all of G,

$$G_0 = G,$$

and the isotropy representation on $T_0V \cong V$ can be identified with ρ itself

$$\rho_0 = \rho,$$

because the action is linear.

Here is a more interesting example:

Example 3.3.6 Every Lie group G acts on itself on the left by conjugation:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto c_g(h) = ghg^{-1}. \end{aligned}$$

The isotropy group of $e \in G$ is the full group G,

$$G_e = G,$$

and the isotropy representation on $T_eG \cong \mathfrak{g}$ is the adjoint representation

$$\rho_e = \mathrm{Ad}_G.$$

The adjoint representation can thus be seen as a special case of the general construction of isotropy representations.

3.4 Fundamental Vector Fields

Suppose a Lie group G acts smoothly on a manifold M. We want to discuss a construction that defines for every vector in the Lie algebra $\mathfrak{g}$ a certain vector field on M. These vector fields correspond to an **infinitesimal action** of $\mathfrak{g}$ on M (the construction only works for smooth Lie group actions on manifolds).

We can think of this from an abstract point of view as follows: if $f: G \to H$ is a Lie group homomorphism, then we saw in Sect. 1.5.3 that there is an induced Lie algebra homomorphism

$$f_*: \mathfrak{g} \longrightarrow \mathfrak{h}.$$

Suppose now that the Lie group G acts smoothly on a manifold M. We know that this action corresponds to a homomorphism

$$\phi: G \longrightarrow \mathrm{Diff}(M),$$

where ϕ is in a certain sense smooth. We can ask whether there is again an induced homomorphism on the level of Lie algebras.

We first have to determine the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$: note that if Y is a vector field on M, then its *flow* generates a 1-parameter family of diffeomorphisms of M. If we think of the flow of Y as an exponential map applied to Y, it is clear that the Lie algebra of $\mathrm{Diff}(M)$ consists of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M with the standard commutator (this is plausible even if we do not formally define $\mathrm{Diff}(M)$ as an infinite-dimensional Lie group). Given a Lie group action of G on M we therefore look for an induced Lie algebra homomorphism

$$\phi_*: \mathfrak{g} \longrightarrow \mathfrak{X}(M).$$

For example, in the case of the Hopf action

$$\begin{aligned} \Phi: S^3 \times \mathrm{U}(1) &\longrightarrow S^3 \\ (v, w, \lambda) &\longmapsto (v\lambda, w\lambda) \end{aligned}$$

it follows from the definition below that the induced homomorphism

$$\phi_*: \mathfrak{u}(1) \cong i\mathbb{R} \longrightarrow \mathfrak{X}(S^3)$$

is given by

$$\phi_*(ix)_{(v,w)} = (ivx, iwx)$$

with $x \in \mathbb{R}$. Here $\phi_*(ix)$ is indeed a tangent vector field on S^3.

Definition 3.4.1 Let G be a Lie group and M a manifold. Suppose that $M \times G \to M$ is a right action. For $X \in \mathfrak{g}$ we define the associated **fundamental vector field** $\tilde{X}$ on M by

$$\tilde{X}_p = \left.\frac{d}{dt}\right|_{t=0} (p \cdot \exp(tX)).$$

If we denote by ϕ_p the orbit map for the right action,

$$\begin{aligned} \phi_p\colon G &\longrightarrow M \\ g &\longmapsto p \cdot g, \end{aligned}$$

then

$$\tilde{X}_p = (D_e\phi_p)(X_e).$$

Similarly, suppose that $G \times M \to M$ is a left action. Then we define the fundamental vector field by

$$\tilde{X}_p = \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX) \cdot p)$$

for $p \in M$. If we denote by ϕ'_p the following orbit map for the left action,

$$\begin{aligned} \phi'_p\colon G &\longrightarrow M \\ g &\longmapsto g^{-1} \cdot p, \end{aligned}$$

then

$$\tilde{X}_p = (D_e\phi'_p)(X_e).$$

The minus sign in the definition of the fundamental vector field for left actions has a reason that will become clear in Proposition 3.4.4.

The formula for the fundamental vector fields has the following interpretation: recall that vectors X in the Lie algebra define one-parameter subgroups, given by $\exp(tX)$ with $t \in \mathbb{R}$. The action of such a subgroup on a point $p \in M$ defines a curve in M and the fundamental vector field in p is given as the velocity vector of this curve at $t = 0$ (up to the sign in the case of left actions).

Example 3.4.2 Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a Lie group G on a vector space V. The representation defines a left action

$$\Phi\colon G \times V \longrightarrow V.$$

Let $\rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$ be the induced representation of the Lie algebra. For $X \in \mathfrak{g}$, the fundamental vector field $\tilde{X}$ is then given by

$$\begin{aligned}\tilde{X}_v &= \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX)\cdot v)\\ &= -\rho_*(X)(v) \quad \forall v \in V.\end{aligned}$$

Here are some properties of fundamental vector fields.

Proposition 3.4.3 (Fundamental Vector Fields of Free Actions) *Let G be a Lie group acting on a smooth manifold M. If the action is free, then the map*

$$\begin{aligned}\phi_*\colon \mathfrak{g} &\longrightarrow \mathfrak{X}(M)\\ X &\longmapsto \tilde{X}\end{aligned}$$

is injective.

Proof This follows from Proposition 3.2.10. □

Proposition 3.4.4 (Fundamental Vector Fields Define Lie Algebra Homomorphism) *Let G be a Lie group acting on a manifold M on the right or left. The map*

$$\begin{aligned}\phi_*\colon \mathfrak{g} &\longrightarrow \mathfrak{X}(M)\\ X &\longmapsto \tilde{X}\end{aligned}$$

that associates to a Lie algebra element the corresponding fundamental vector field on M is a Lie algebra homomorphism, i.e. it is an $\mathbb{R}$-linear map such that

$$\widetilde{[X,Y]} = [\tilde{X},\tilde{Y}] \quad \forall X, Y \in \mathfrak{g}.$$

In particular, the set of all fundamental vector fields is a Lie subalgebra of the Lie algebra of all vector fields on M.

Proof We prove the claim if G acts on the left on M. The proof for right actions follows similarly. Fix a point $p \in M$ and let ϕ'_p denote the following orbit map

$$\begin{aligned}\phi'_p\colon G &\longrightarrow M\\ g &\longmapsto g^{-1}\cdot p.\end{aligned}$$

The second definition of $\tilde{X}$,

$$\tilde{X}_p = (D_e\phi'_p)(X_e)$$

shows that the map

$$\begin{aligned}\phi_*\colon \mathfrak{g} &\longrightarrow \mathfrak{X}(M)\\ X &\longmapsto \tilde{X}\end{aligned}$$

is linear.

We want to show that the left-invariant vector field $X \in \mathfrak{g}$ and $\tilde{X} \in \mathfrak{X}(M)$ are ϕ'_p-related. For this we have to show that

$$\tilde{X}_{\phi'_p(a)} = (D_a\phi'_p)(X_a)$$

for all $a \in G$. We have, since X is a left-invariant vector field on G,

$$\begin{aligned}(D_a\phi'_p)(X_a) &= (D_a\phi'_p)(D_eL_a)\left(\frac{d}{dt}\bigg|_{t=0}\exp(tX)\right)\\ &= \frac{d}{dt}\bigg|_{t=0}\left(\exp(-tX)\left(a^{-1}\cdot p\right)\right)\\ &= \tilde{X}_{a^{-1}p}\\ &= \tilde{X}_{\phi'_p(a)}.\end{aligned}$$

The claim now follows from Proposition A.1.49. □

Remark 3.4.5 The reason why we defined in Definition 3.4.1 the fundamental vector field for left actions with a minus sign in $\exp(-tX)$ is so that

$$\widetilde{[X,Y]} = [\tilde{X},\tilde{Y}]$$

holds for all $X, Y \in \mathfrak{g}$. If we defined the fundamental vector field for left actions with $\exp(tX)$ instead (this is sometimes done in the literature), then we would get a minus sign here:

$$\widetilde{[X,Y]} = -[\tilde{X},\tilde{Y}] \quad \forall X, Y \in \mathfrak{g},$$

because on the left-hand side we have to change the sign once and on the right-hand side twice.

It is sometimes useful to know how fundamental vector fields behave under right or left translations on the manifold. It will turn out that even though fundamental vector fields are defined using the group action, they are in general not invariant under the action.

Proposition 3.4.6 (Action of Right and Left Translations on Fundamental Vector Fields) *Suppose a Lie group G acts on a manifold M. Let $X \in \mathfrak{g}$ and $g \in G$.*

1. If G acts on the right on M, then

$$r_{g*}(\tilde{X}) = \tilde{Y},$$

where

$$Y = \mathrm{Ad}_{g^{-1}}X \in \mathfrak{g}.$$

2. *If G acts on the left on M, then*

$$l_{g*}(\tilde{X}) = \tilde{Z},$$

where

$$Z = \mathrm{Ad}_g X \in \mathfrak{g}.$$

Proof We prove the statement for right actions, the statement for left actions follows similarly. At a point $p \in M$ we calculate

$$\begin{aligned}(r_{g*}(\tilde{X}))_p &= (D_{pg^{-1}}r_g)(\tilde{X}_{pg^{-1}})\\ &= (D_{pg^{-1}}r_g)\left(\frac{d}{dt}\bigg|_{t=0} pg^{-1}\cdot\exp(tX)\right)\\ &= (D_e\phi_p)\left(\frac{d}{dt}\bigg|_{t=0}\alpha_{g^{-1}}(\exp tX)\right),\end{aligned}$$

with the orbit map

$$\begin{aligned}\phi_p\colon G &\longrightarrow M\\ g &\longmapsto p\cdot g.\end{aligned}$$

On the other hand

$$\begin{aligned}Y &= \mathrm{Ad}_{g^{-1}}X\\ &= \frac{d}{dt}\bigg|_{t=0}\alpha_{g^{-1}}(\exp tX).\end{aligned}$$

This implies the claim by the second definition of the fundamental vector field. □

Corollary 3.4.7 (Translations of Fundamental Vector Fields Are Fundamental) *For a right (left) action of a Lie group G on a manifold M the right (left) translations of fundamental vector fields are again fundamental vector fields. If the Lie group G is abelian, then the fundamental vector fields are invariant under all right (left) translations.*

3.5 The Maurer–Cartan Form and the Differential of a Smooth Group Action

3.5.1 Vector Space-Valued Forms

Recall from Definition A.2.3 that a k-form on a real vector space V is defined as an alternating multilinear map

$$\lambda\colon \underbrace{V\times\cdots\times V}_{k} \longrightarrow \mathbb{R}.$$

The vector space of all k-forms on V is denoted by $\Lambda^k V^*$.

Suppose W is another real vector space. Then we define a **k-form on V with values in W** as an alternating, multilinear map

$$\lambda\colon \underbrace{V\times\cdots\times V}_{k} \longrightarrow W.$$

The vector space of all k-forms on V with values in W can be identified with the tensor product $\Lambda^k V^* \otimes W$.

Similarly we defined k-forms on a smooth manifold as alternating $\mathscr{C}^\infty(M)$-multilinear maps

$$\lambda\colon \underbrace{\mathfrak{X}(M)\times\cdots\times \mathfrak{X}(M)}_{k} \longrightarrow \mathscr{C}^\infty(M)$$

and we defined $\Omega^k(M)$ as the set of all k-forms on M; see Definition A.2.12.

We now define

$$\mathscr{C}^\infty(M,W)$$

as the set of all smooth maps from M into the vector space W (the vector space W has a canonical structure of a manifold, so that smooth maps into W are defined). A **k-form on M with values in W** is then an alternating $\mathscr{C}^\infty(M)$-multilinear map

$$\lambda\colon \underbrace{\mathfrak{X}(M)\times\cdots\times \mathfrak{X}(M)}_{k} \longrightarrow \mathscr{C}^\infty(M,W).$$

The set of all k-forms on M with values in W can be identified with $\Omega^k(M,W) = \Omega^k(M)\otimes_{\mathbb{R}} W$. One also calls the forms in $\Omega^k(M,W)$ **twisted with** W.

Remark 3.5.1 Note that there is no canonical wedge product of forms on a vector space or a manifold with values in a vector space W, because there is in general no canonical product $W\times W\to W$ (an exception is forms with values in $W=\mathbb{C}$, where there is indeed a canonical wedge product).

3.5.2 *The Maurer–Cartan Form*

The following notion of a vector space-valued form on a Lie group is useful for studying group actions and principal bundles. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Definition 3.5.2 The **Maurer–Cartan form** $\mu_G \in \Omega^1(G, \mathfrak{g})$ is the 1-form on G with values in $\mathfrak{g}$ defined by

$$(\mu_G)_g(v) = (D_g L_{g^{-1}})(v) \in T_e G \cong \mathfrak{g}$$

for all $g \in G$ and $v \in T_g G$. The Maurer–Cartan form is also called the **canonical form** or **structure form**.

The Maurer–Cartan form thus associates to a tangent vector v at the point $g \in G$ the unique left-invariant vector field X on G whose value at g is $X_g = v$ (equivalently, the generating vector of this vector field at $e \in G$).

Proposition 3.5.3 (Invariance of Maurer–Cartan Form Under Translations) *The Maurer–Cartan form has the following invariance properties under left and right translations:*

$$L_g^* \mu_G = \mu_G,$$
$$R_g^* \mu_G = \mathrm{Ad}_{g^{-1}} \circ \mu_G,$$

for all $g \in G$.

Proof We calculate for all $h \in G$ and $v \in T_h G$:

$$\begin{aligned}(R_g^* \mu_G)_h(v) &= (\mu_G)_{hg}(D_h R_g)(v) \\ &= (D_{hg} L_{g^{-1}h^{-1}})(D_h R_g)(v) \\ &= (D_e \alpha_{g^{-1}})(D_h L_{h^{-1}})(v) \\ &= \mathrm{Ad}_{g^{-1}}(\mu_G)_h(v).\end{aligned}$$

The statement for L_g follows similarly. □

3.5.3 *The Differential of a Smooth Group Action*

Recall that a smooth (right) action of a Lie group is a map $\Phi: M \times G \to M$ satisfying certain axioms. It is sometimes useful to determine the differential of this map in a given point $(x, g) \in M \times G$. The formula for this differential involves the Maurer–Cartan form.

Proposition 3.5.4 (The Differential of a Smooth Group Action) *Let G be a Lie group acting smoothly on the right on a manifold M,*

$$\Phi : M \times G \longrightarrow M.$$

Then under the canonical identification

$$T_{(x,g)} M \times G \cong T_x M \oplus T_g G,$$

the differential of the map Φ is given by

$$\begin{aligned} D_{(x,g)}\Phi : T_x M \oplus T_g G &\longrightarrow T_{xg} M \\ (X, Y) &\longmapsto (D_x r_g)(X) + \widetilde{\mu_G(Y)}_{xg}, \end{aligned}$$

where r_g denotes right translation and μ_G denotes the Maurer–Cartan form.

Proof Let $\phi_x : G \to M$ denote the orbit map

$$\phi_x(g) = xg.$$

Let $x(t)$ be a curve in M tangent to X and $g(t)$ a curve in G tangent to Y. Then

$$\begin{aligned} D_{(x,g)}\Phi(X, Y) &= D_{(x,g)}\Phi(X, 0) + D_{(x,g)}\Phi(0, Y) \\ &= D_{(x,g)}\Phi(\dot{x}(0), 0) + D_{(x,g)}\Phi(0, \dot{g}(0)) \\ &= (D_x r_g)(X) + (D_g \phi_x)(Y). \end{aligned}$$

Let $y \in \mathfrak{g}$ denote the left-invariant vector field corresponding to Y. Then $y = \mu_G(Y)$. In the proof of Proposition 3.4.4 we saw that

$$(D_g \phi_x)(Y) = \tilde{y}_{\phi_x(g)}$$

(we proved the statement for left actions, but the corresponding statement also holds for right actions). This proves the claim. □

3.6 Left or Right Actions?

In general there is no difference whether we assume that a group action is a left or right action. However, when we discuss homogeneous spaces in Sect. 3.8, there will be two different actions at the same time, which have to be compatible. We therefore make the following conventions:

- If we are interested in quotient spaces M/G, we take the G-action on M to be a *right action*. In particular, if $H \subset G$ is a subgroup and we want to consider G/H, then H acts on G on the right. When we consider principal bundles in Chap. 4, we will take the G-action on the principal bundle to be a right action as well. For example, the Hopf actions introduced in Definition 3.3.1 are right actions whose quotient spaces are the projective spaces.
- If we are interested in homogeneous spaces, i.e. spaces M with a transitive group action, we will take the G-action on M to be a *left action*. For example, the linear transitive actions on spheres introduced in Theorem 3.3.2 are left actions.

Usually we are not interested in the quotient space of a transitive group action, because it consists only of a single point, so that both cases do not overlap. Occasionally one encounters situations in the literature where we have a right G-action on M with quotient space M/G and a non-transitive left K-action on M/G. Then it makes sense to consider the quotient space $K\backslash M/G$ under the left K-action (we will not consider such quotients in the following).

3.7 *Quotient Spaces

An important objective in the study of group actions is to understand the quotient space of a given action. In this section we are specifically interested in the following question: Suppose that G is a Lie group acting smoothly on a manifold M. Under which circumstances does the quotient set M/G have the structure of a smooth manifold?

This question has many applications, because it is possible to construct new and interesting manifolds as quotients of this form (like projective spaces and lens spaces, to name only two examples). For instance, in the case of the Hopf action

$$\begin{aligned} \Phi\colon S^3 \times \mathrm{U}(1) &\longrightarrow S^3 \\ (v, w, \lambda) &\longmapsto (v\lambda, w\lambda), \end{aligned}$$

which is a *free* action, it can be shown that the quotient space $S^3/\mathrm{U}(1)$ is a smooth manifold diffeomorphic to $\mathbb{CP}^1 \cong S^2$.

It is useful to study the question of quotients in greater generality: we first consider quotients of manifolds (and topological spaces) under arbitrary equivalence relations and later the case of the equivalence relation defined by group actions.

We follow [130] for smooth manifolds and the excellent exposition in [139] in the general case. An additional reference is [89].

3.7.1 Quotient Spaces Under Equivalence Relations on Topological Spaces

Suppose X is a set and $\sim$ an equivalence relation on X. We can describe $\sim$ equivalently by a subset $R \subset X \times X$ so that

$$x \sim y \Leftrightarrow (x, y) \in R.$$

The equivalence class of an element $x \in X$ is the subset

$$[x] = \{y \in X \mid y \sim x\}.$$

As subsets of X, equivalence classes of two elements $x, x' \in X$ are either disjoint or identical. We denote by X/R the space of equivalence classes, called the **quotient space**

$$X/R = \{[x] \mid x \in X\}.$$

We have the canonical projection

$$\begin{aligned} \pi: X &\longrightarrow X/R \\ x &\longmapsto [x]. \end{aligned}$$

We now specialize to the case when X is a topological space. Then we define on X/R the usual **quotient topology** by setting $U \subset X/R$ open if and only if $\pi^{-1}(U) \subset X$ is open. It is easy to check that this indeed defines a topology on X/R. The canonical projection $\pi: X \to X/R$ is continuous. The following is well-known:

Lemma 3.7.1 *A map $f: X/R \to Y$ from a quotient space to another topological space is continuous if and only if $f \circ \pi$ is continuous:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle \pi}\downarrow & \overset{f\circ\pi}{\searrow} & \\ X/R & \xrightarrow[f]{} & Y \end{array}$$

We are first interested in the following question: under which conditions is the quotient space X/R Hausdorff? The answer is given by the following lemma.

Lemma 3.7.2 (Hausdorff Property of Quotient Spaces Under Equivalence Relations) *Let X be a topological space.*

1. *If X/R is Hausdorff, then $R \subset X \times X$ is closed.*
2. *If $\pi: X \to X/R$ is open and $R \subset X \times X$ is closed, then X/R is Hausdorff.*

Remark 3.7.3 Note that we do not need to assume that X is Hausdorff.

Proof We use in the proof the following standard fact from point set topology: a topological space Y is Hausdorff if and only if the diagonal

$$\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$$

is a closed subset in $Y \times Y$. In the following, we denote by Δ the diagonal in the space $X/R \times X/R$.

1. The map

$$\pi \times \pi: X \times X \longrightarrow X/R \times X/R$$

   is continuous. Since X/R is Hausdorff, the diagonal Δ is closed, hence the preimage $(\pi \times \pi)^{-1}(\Delta)$ is closed. We have

$$(x, y) \in (\pi \times \pi)^{-1}(\Delta) \Leftrightarrow (x, y) \in R.$$

   Hence $R = (\pi \times \pi)^{-1}(\Delta)$ is closed in $X \times X$.
2. The map $\pi \times \pi$ is open and $(X \times X) \setminus R$ is open, hence its image in $X/R \times X/R$ is open. We have

$$\begin{aligned}([x], [y]) \in (\pi \times \pi)((X \times X) \setminus R) &\Leftrightarrow [x] \neq [y] \\ &\Leftrightarrow ([x], [y]) \in (X/R \times X/R) \setminus \Delta.\end{aligned}$$

   It follows that Δ is closed and X/R is Hausdorff.

□

3.7.2 *Quotient Spaces Under Equivalence Relations on Manifolds*

We now consider the case of an equivalence relation R on a smooth manifold M and we would like to determine when the quotient space M/R is a smooth manifold. It is useful to demand that the smooth structure has the additional property that $\pi: M \to M/R$ is a submersion. Consider the following lemma.

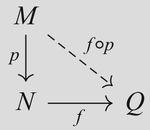
Lemma 3.7.4 (Surjective Submersions Admit Local Sections) *Let $p: M \to N$ be a surjective submersion between smooth manifolds. Then p admits* **smooth local**

sections, *i.e. for each* $x \in N$ *there exists an open neighbourhood* $U \subset N$ *of* x *and a smooth map* $s: U \to M$ *such that* $p \circ s = \mathrm{Id}_U$.

Proof This follows from the normal form theorem for submersions (see Theorem A.1.28), because locally submersions are projections. □

The following lemma is very useful in applications.

Lemma 3.7.5 (Smoothness of Maps Out of the Target Space of a Surjective Submersion) *Let* $p: M \to N$ *be a surjective submersion. Then a map* $f: N \to Q$ *is smooth if and only if* $f \circ p: M \to Q$ *is smooth. Moreover,* f *is a submersion if and only if* $f \circ p$ *is a submersion and* f *is surjective if and only if* $f \circ p$ *is surjective.*



Proof If f is smooth, then $f \circ p$ is smooth. Conversely, assume that $f \circ p$ is smooth. Let $x \in N$ and $U \subset N$ an open neighbourhood of x with a smooth section $s: U \to M$ for p. On U we have $p \circ s = \mathrm{Id}_U$, hence

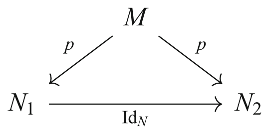
$$(f \circ p) \circ s = f.$$

Thus f is smooth on U and therefore on all of N.

The claim about submersions and surjectivity is clear, because p and its differential are surjective. □

Corollary 3.7.6 *Let* M *be a manifold and* $p: M \to N$ *a surjective map to a set* N. *Then* N *admits at most one structure of a smooth manifold so that* p *is a submersion.*

Proof Suppose N_1 and N_2 are structures of smooth manifolds on N so that p is a submersion in both cases. By Lemma 3.7.5 the identity map $\mathrm{Id}_N: N_1 \to N_2$ is a diffeomorphism.



□

Corollary 3.7.7 (Uniqueness of Smooth Manifold Structure on Quotient Spaces) *Let* M *be a smooth manifold and* R *an equivalence relation on* M. *Then* *there exists at most one smooth structure on M/R so that $\pi: M \to M/R$ is a submersion.*

Remark 3.7.8 Lemma 3.7.5 and Corollary 3.7.7 are the reasons why the smooth structure on M/R should have the property that $\pi: M \to M/R$ is a submersion.
We assume from now on that M is a smooth manifold and R an equivalence relation on M. We first derive a necessary condition for M/R to be a smooth manifold such that π is a submersion.

Lemma 3.7.9 *Let M/R have the structure of a smooth manifold so that $\pi: M \to M/R$ is a surjective submersion. Then R is a closed embedded submanifold of $M \times M$ and the restrictions of the projections*

$$\mathrm{pr}_i|_R: R \longrightarrow M$$

are surjective submersions, for $i = 1, 2$.

Proof The graph of the projection

$$\Gamma = \{(x, \pi(x)) \in M \times M/R \mid x \in M\}$$

is a closed embedded submanifold of $M \times M/R$ and

$$F = \mathrm{Id}_M \times \pi: M \times M \longrightarrow M \times M/R$$

is a submersion. Therefore $F^{-1}(\Gamma)$ is a closed embedded submanifold of $M \times M$. We have

$$\begin{aligned}(x, y) \in F^{-1}(\Gamma) &\Leftrightarrow (x, \pi(y)) \in \Gamma \\ &\Leftrightarrow \pi(x) = \pi(y) \\ &\Leftrightarrow (x, y) \in R.\end{aligned}$$

This shows that R is a closed embedded submanifold of $M \times M$.
The map $F|_R: R \to \Gamma$ is a surjective submersion. The projection $\mathrm{pr}_1|_\Gamma: \Gamma \to M$ is also a surjective submersion, because

$$\mathrm{pr}_1|_\Gamma \circ (\mathrm{Id}_M, \pi) = \mathrm{Id}_M: M \longrightarrow M.$$

It follows that

$$\mathrm{pr}_1|_\Gamma \circ F|_R: R \longrightarrow M$$

is a smooth surjective submersion. This map is equal to $\mathrm{pr}_1|_R$. The claim for $\mathrm{pr}_2|_R$ follows by symmetry of the equivalence relation. □

It is a non-trivial fact that the converse also holds.

Theorem 3.7.10 (Godement's Theorem on the Manifold Structure of Quotient Spaces) *Let R be an equivalence relation on a manifold M. Suppose that R is a closed embedded submanifold of $M \times M$ and $\mathrm{pr}_1|_R : R \to M$ a surjective submersion. Then M/R has a unique structure of a smooth manifold such that the canonical projection $\pi : M \to M/R$ is a surjective submersion.*

The proof of Godement's Theorem, which is not easy and quite technical, is deferred to Sect. 3.11. We first want to derive some consequences of it.

3.7.3 Quotient Spaces Under Continuous Group Actions

We begin more generally by considering the case of a topological group G acting continuously on the right on a topological space X. The map defining the action is

$$\Phi : X \times G \longrightarrow X.$$

We would like to determine under which conditions the quotient space X/G is Hausdorff (we do not need to assume that X itself is Hausdorff).

Lemma 3.7.11 *The canonical projection $\pi : X \to X/G$ is open.*

Proof Let U be an open subset of x. We have to show that $\pi^{-1}(\pi(U))$ is open in X. However,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} U \cdot g$$

and each of the sets $U \cdot g$ is open, because right translations are homeomorphisms. □

Corollary 3.7.12 (Hausdorff Property of Quotient Spaces Under Continuous Group Actions) *The quotient space X/G is Hausdorff if and only if the map*

$$\Psi : X \times G \longrightarrow X \times X$$
$$(x, g) \longmapsto (x, xg)$$

has closed image.

Proof According to Lemma 3.7.2 and since $\pi: X \to X/G$ is open, the space X/G is Hausdorff if and only if the equivalence relation $R \subset X \times X$ is closed. We have

$$(x, y) \in R \Leftrightarrow \exists g \in G: y = xg.$$

This shows that R is equal to the image of the map Ψ. □

Let G be a topological group and $H \subset G$ a subgroup with the subspace topology. Then H acts continuously on the right on G via right translations

$$\begin{aligned} \Phi: G \times H &\longrightarrow G \\ (g, h) &\longmapsto gh. \end{aligned}$$

We get a topological quotient space G/H.

Corollary 3.7.13 (Hausdorff Property of G/H) *Let G be a topological group and $H \subset G$ a subgroup. Then G/H is Hausdorff if and only if H is a closed set in the topology of G.*

Proof According to Corollary 3.7.12 we have to show that the image of

$$\begin{aligned} \Psi: G \times H &\longrightarrow G \times G \\ (g, h) &\longmapsto (g, gh) \end{aligned}$$

is closed if and only if H is closed. Consider the map

$$\begin{aligned} T: G \times G &\longrightarrow G \times G \\ (g, g') &\longmapsto (g, gg'). \end{aligned}$$

This map is a homeomorphism and we have $\Psi = T|_{G \times H}$. Hence the image of Ψ is closed in $G \times G$ if and only if $G \times H$ is closed in $G \times G$. This happens if and only if H is closed in G. □

As an aside we note the following result, which is useful in applications such as Example 3.8.11 (we follow [34] and [142] in the proof).

Proposition 3.7.14 (Connectedness of G and G/H) *Let G be a topological group and $H \subset G$ a closed subgroup. Suppose that H is connected. Then G/H is connected if and only if G is connected.*

Proof If G is connected, then G/H is connected, because the canonical projection $\pi: G \to G/H$ is surjective and continuous.

Conversely, suppose that G/H is connected and

$$G = U \cup V,$$

where U, V are open non-empty subsets of G. We have to show that U and V cannot be disjoint.

By Lemma 3.7.11 the sets $\pi(U)$ and $\pi(V)$ are open and non-empty in G/H with

$$G/H = \pi(U) \cup \pi(V).$$

Since G/H is connected there exists an element

$$[g] \in \pi(U) \cap \pi(V).$$

Because of $G = U \cup V$ we get

$$\mathcal{O}_g = gH = (gH \cap U) \cup (gH \cap V).$$

By construction $gH \cap U$, $gH \cap V$ are open and non-empty in gH. Since gH is connected, the claim follows. □

3.7.4 Proper Group Actions

We consider some topological notions that are useful in applications to group actions.

Definition 3.7.15 A topological space X is called **locally compact** if every point in X has a compact neighbourhood.

Lemma 3.7.16 *Let X be a locally compact Hausdorff space. Then a subset $A \subset X$ is closed if and only if the intersection of A with any compact subset of X is compact.*

Proof If A is closed, then the intersection with any compact subset of X is compact. Conversely, assume that $A \cap K$ is compact for every compact subset $K \subset X$. Let $x \in X \setminus A$. Since X is locally compact, there exists an open neighbourhood $U \subset X$ of x contained in a compact subset $K \subset X$. By assumption, $C = A \cap K$ is compact, hence closed in X, since X is Hausdorff. Then $U \setminus C = U \cap (X \setminus C)$ is an open neighbourhood of x contained in $X \setminus A$. This implies the claim. □

Definition 3.7.17 A continuous map $f: X \to Y$ between topological spaces is called **proper** if the preimage $f^{-1}(K)$ of every compact subset $K \subset Y$ is compact in X.

Lemma 3.7.18 *Let $f: X \to Y$ be a continuous proper map between topological spaces, where Y is locally compact Hausdorff. Then f is a closed map.*

Proof Let $A \subset X$ be a closed set. By Lemma 3.7.16 we have to show that $f(A) \cap K$ is compact for every compact subset $K \subset Y$. However,

$$f(A) \cap K = f\left(A \cap f^{-1}(K)\right).$$

Since f is proper, the set $f^{-1}(K)$ is compact and thus $A \cap f^{-1}(K)$ and $f\left(A \cap f^{-1}(K)\right)$ are compact. This implies the claim. □

Lemma 3.7.19 *Let $f: X \to Y$ be a closed continuous map between topological spaces such that $f^{-1}(y)$ is compact for all $y \in Y$. Then f is proper.*

Proof The proof is left as an exercise. □

We consider the following type of group actions.

Definition 3.7.20 A continuous action of a topological group G on a topological space X is called **proper** if the map

$$\begin{aligned} \Psi: X \times G &\longrightarrow X \times X \\ (x, g) &\longmapsto (x, xg) \end{aligned}$$

is proper.

Corollary 3.7.21 (Map Ψ Is Closed If Action Is Proper) *Let $X \times G \to X$ be a continuous, proper action of a topological group G on a topological space X, where X is locally compact Hausdorff. Then the map*

$$\begin{aligned} \Psi: X \times G &\longrightarrow X \times X \\ (x, g) &\longmapsto (x, xg) \end{aligned}$$

is closed. In particular, X/G is Hausdorff.

Proof This follows from Lemma 3.7.18 and Corollary 3.7.12. □

Here is a general situation in which group actions are proper.

Proposition 3.7.22 (Actions of Compact Topological Groups Are Proper) *Let $X \times G \to X$ be a continuous action of a topological group G on a Hausdorff space X. Suppose that G is compact. Then the action is proper.*

Proof Let $K \subset X \times X$ be a compact subset. Then

$$L = \mathrm{pr}_1(K)$$

is a compact subset of X. If $\Psi(x, g) = (x, xg) \in K$, then $x \in L$, hence

$$\Psi^{-1}(K) = \Psi^{-1}(K) \cap (L \times G).$$

However, $\Psi^{-1}(K)$ is closed in $X \times G$ and $L \times G$ is compact, hence $\Psi^{-1}(K)$ is compact. □

Corollary 3.7.23 *Let $X \times G \to X$ be a continuous action of a topological group G on a locally compact Hausdorff space X. Suppose that G is compact. Then X/G is Hausdorff.*

3.7.5 Quotient Spaces Under Smooth Group Actions

We have now arrived at the central topic in this section: to determine under which conditions the quotient of a smooth action of a Lie group on a smooth manifold is again a smooth manifold.

Definition 3.7.24 We call a smooth right action of a Lie group G on a manifold M **principal** if the action is free and the map

$$\Psi: M \times G \longrightarrow M \times M$$
$$(p, g) \longmapsto (p, pg)$$

is closed.

Theorem 3.7.25 (Manifold Structure on Quotient Spaces Under Principal Actions of Lie Groups) *Suppose that Φ is a principal right action of the Lie group G on the manifold M. Then M/G has a unique structure of a smooth manifold such that $\pi: M \to M/G$ is a submersion.*

Proof Since the action of G on M is free, the map Ψ is injective. We want to show that Ψ is an immersion: by Proposition 3.5.4 the differential of Ψ is given by

$$D_{(x,g)}(X, Y) = \Big(X, (D_x r_g)(X) + \widetilde{\mu_G(Y)}_{xg}\Big).$$

If $D_{(x,g)}(X, Y) = (0, 0)$, then $X = 0$ and $\widetilde{\mu_G(Y)}_{xg} = 0$. From Proposition 3.4.3 we get $\mu_G(Y) = 0$, hence $Y = 0$. This proves that the differential of Ψ is injective.

Since Ψ is a closed injective map, it is a homeomorphism onto its image R and thus an embedding. Hence R is a closed embedded submanifold of $M \times M$. According to Theorem 3.7.10 it remains to show that $\mathrm{pr}_1|_R: R \to M$ is a submersion. However,

$$\mathrm{pr}_1|_R \circ \Psi: M \times G \longrightarrow M$$

is just $\mathrm{pr}_1: M \times G \to M$ and thus a submersion. This implies the claim. □

Corollary 3.7.26 (The Differential of the Projection $\pi: M \to M/G$) *Suppose that Φ is a principal right action of the Lie group G on the manifold M. Then the dimension of the quotient manifold M/G is given by*

$$\dim(M/G) = \dim M - \dim G.$$

In particular, the kernel of the differential

$$D_p\pi: T_pM \longrightarrow T_{[p]}M/G$$

at a point $p \in M$ is equal to the tangent space $T_p\mathscr{O}_p$ of the G-orbit through p.

Proof The claim about the dimension of M/G follows from the proof of Theorem 3.7.10. The second claim then follows from Corollary 3.2.12. □

Remark 3.7.27 For a free action of a Lie group G on a manifold M the preimage $\Psi^{-1}(p,q)$ of every point $(p,q) \in M \times M$ is either empty or consists of a single element (and is thus a compact set). Lemma 3.7.19 implies that principal Lie group actions on manifolds are proper. Together with Corollary 3.7.21 we conclude that principal actions are equivalent to free proper Lie group actions on manifolds.
We formulate this as follows:

Corollary 3.7.28 (Free Proper Actions of Lie Groups Are Equivalent to Principal Actions) *Suppose that $M \times G \to M$ is a smooth free action of a Lie group G on a manifold M. Then the action is principal if and only if it is proper.*
Proposition 3.7.22 then implies:

Corollary 3.7.29 (Free Actions of Compact Lie Groups Are Principal) *Suppose that $M \times G \to M$ is a smooth free action of a compact Lie group G on a manifold M. Then the action is principal.*
We get the following corollary, which is very useful in applications.

Corollary 3.7.30 (Quotients Under Free Actions of Compact Lie Groups) *Let G be a compact Lie group acting smoothly and freely on a manifold M. Then M/G has a unique structure of a smooth manifold such that $\pi\colon M \to M/G$ is a submersion.*

Proof This follows from Theorem 3.7.25 and Corollary 3.7.29. □

Remark 3.7.31 (Fundamental Groups) If in the situation of Corollary 3.7.30 the manifolds M and G are connected, it follows from Exercise 3.12.6 that π_* maps the fundamental group of M surjectively onto the fundamental group of M/G. In particular, if M is simply connected, then M/G is simply connected.

Example 3.7.32 (Quotients Under Free Actions of Finite Groups) Finite groups with the discrete topology are compact. Hence if a finite group G acts freely and smoothly on a manifold M, then the quotient M/G is a smooth manifold such that the canonical projection is a submersion.

Example 3.7.33 (Lens Spaces) Let $p > 0$ be an integer and $\alpha = e^{2\pi i/p} \in S^1$ the corresponding root of unity. Let $q \neq 0$ be an integer coprime to p. We consider the following smooth action of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ on the unit sphere $S^3 \subset \mathbb{C}^2$:

$$\begin{aligned} S^3 \times \mathbb{Z}_p &\longrightarrow S^3 \\ (z_1, z_2, [k]) &\longmapsto (z_1, z_2) \cdot [k] = \left(z_1\alpha^k, z_2\alpha^{kq}\right). \end{aligned}$$

This action is free: if $z_1 \neq 0$ and $z_1\alpha^k = z_1$, then $[k] = 0$. If $z_2 \neq 0$ and $z_2\alpha^{kq} = z_2$, then again $[k] = 0$, because q is coprime to p. According to Example 3.7.32 the quotient

$$L(p,q) = S^3/\mathbb{Z}_p$$

under this action is a smooth 3-dimensional manifold. These manifolds are called **lens spaces**.

Example 3.7.34 (Projective Spaces Are Smooth Manifolds) The projective spaces

$$\begin{aligned} \mathbb{RP}^n &= S^n/\mathbb{Z}_2, \\ \mathbb{CP}^n &= S^{2n+1}/\mathrm{U}(1), \\ \mathbb{HP}^n &= S^{4n+3}/\mathrm{SU}(2) \end{aligned}$$

are quotients of manifolds under smooth free actions of compact Lie groups and therefore smooth manifolds such that the canonical projections are submersions.

Let G be a Lie group and $H \subset G$ a closed subgroup. According to Cartan's Theorem 1.1.44 the subgroup H is an embedded Lie subgroup of G. There is a smooth right action of H on G by right translations

$$\begin{aligned} \Phi\colon G \times H &\longrightarrow G \\ (g,h) &\longmapsto gh. \end{aligned}$$

Corollary 3.7.35 (Manifold Structure on G/H) *Let G be a Lie group and $H \subset G$ a closed subgroup. Then the right action of H on G is principal and G/H has a unique structure of a smooth manifold such that $\pi\colon G \to G/H$ is a submersion.*

Proof It is clear that the orbit maps

$$\begin{aligned} \phi_g\colon H &\longrightarrow G \\ h &\longmapsto gh \end{aligned}$$

are injective, hence the action is free. According to Theorem 3.7.25 it remains to show that the map

$$\begin{aligned} \Psi = \mathrm{Id}_G \times \Phi\colon G \times H &\longrightarrow G \times G \\ (g,h) &\longmapsto (g,gh) \end{aligned}$$

is closed. As in the proof of Corollary 3.7.13 we consider the map

$$T: G \times G \longrightarrow G \times G$$
$$(g, g') \longmapsto (g, gg').$$

This map is a diffeomorphism with $\Psi = T|_{G\times H}$. If $A \subset G \times H$ is closed, then A is closed in $G \times G$, since H is closed in G. This implies that $\Psi(A) = T(A)$ is closed in $G \times G$. □

Corollary 3.7.36 *Let G be a Lie group and $H \subset G$ a closed subgroup. Then the dimension of the quotient manifold G/H is given by*

$$\dim(G/H) = \dim G - \dim H.$$

In particular, the kernel of the differential

$$D_e\pi: T_eG \longrightarrow T_{[e]}G/H$$

is equal to the Lie algebra $\mathfrak{h}$ of H.

Proof This follows from Corollary 3.7.26. □

3.8 *Homogeneous Spaces

Recall that a set M together with a transitive action of a group G is called a *homogeneous space*. An example is the transitive action of $\mathrm{SO}(n)$ on the sphere S^{n-1} with isotropy group isomorphic to $\mathrm{SO}(n-1)$, cf. Theorem 3.3.2. In this section we study the structure of homogeneous spaces for actions of groups, topological groups and Lie groups. We are most interested in the case of Lie group actions, but the other two cases are useful as a warm-up. We will show that every homogeneous space is, up to isomorphism of group actions, of the form G/H, where H is a suitable subgroup of G.

3.8.1 Groups and Homogeneous Spaces

Let G be any group and $H \subset G$ a subgroup. Then H acts on the right on G. We get a quotient space G/H of orbits, also called **left cosets**.

Definition 3.8.1 We define a map

$$\Phi: G \times G/H \longrightarrow G/H$$
$$(g, [a]) \longmapsto g \cdot [a] = [ga].$$

Note that G acts on the *left* on the set of *left cosets* G/H.
We want to show that Φ is indeed a group action.

Proposition 3.8.2 (Φ Is a Transitive Left Action of G on G/H)

1. *The map Φ is a well-defined, transitive group action of G on the set G/H.*
2. *The isotropy group of $[e] \in G/H$ is equal to H. Therefore the isotropy group of any point in G/H is isomorphic to H.*

Proof This is an easy exercise. □

We now consider an arbitrary transitive group action of G on a set M. We want to show that up to an equivariant bijection this group action is of the form above.

Proposition 3.8.3 (Structure of Transitive Group Actions on Sets) *Let $G \times M \to M$ be a transitive left action of a group G on a set M. Fix an arbitrary point $p \in M$ and let G_p denote the isotropy group of p. Then $G_p \subset G$ is a subgroup and*

$$\begin{aligned} f\colon G/G_p &\longrightarrow M \\ [a] &\longmapsto a \cdot p \end{aligned}$$

is a well-defined G-equivariant bijection.

Proof Another easy exercise. □

This implies that every homogeneous G-space is of the form G/H for some subgroup $H \subset G$ (not only as a set, but as the space of an action). We will show in the following subsections that this result essentially still holds in the continuous and smooth category.

3.8.2 Topological Groups and Homogeneous Spaces

Let G be a topological group and $H \subset G$ a subgroup with the subspace topology. Consider the quotient space G/H with the subspace topology. According to Proposition 3.8.2 we get a transitive group action

$$\begin{aligned} \Phi\colon G \times G/H &\longrightarrow G/H \\ (g, [a]) &\longmapsto g \cdot [a] = [ga]. \end{aligned}$$

Proposition 3.8.4 (Φ Is a Continuous Action for Topological Groups) *Suppose G is a topological group and $H \subset G$ a subgroup. Then the transitive group action*

$$\Phi\colon G \times G/H \longrightarrow G/H$$

is continuous.

Proof Multiplication in G followed by projection onto G/H is continuous,

$$G \times G \longrightarrow G \longrightarrow G/H.$$

This implies, by the definition of the quotient topology on G/H, that the group action

$$\Phi\colon G \times G/H \longrightarrow G/H$$

is continuous. □

According to Corollary 3.7.13 the space G/H is Hausdorff if and only if H is a closed subset in G. We now consider the case of an arbitrary transitive continuous group action.

Proposition 3.8.5 (Structure of Transitive Continuous Group Actions on Topological Spaces) *Let $G \times M \to M$ be a transitive continuous left action of a topological group G on a Hausdorff space M. Fix an arbitrary point $p \in M$ and let G_p denote the isotropy group of p. Then $G_p \subset G$ is a closed subgroup and*

$$\begin{aligned} f\colon G/G_p &\longrightarrow M \\ [a] &\longmapsto a \cdot p \end{aligned}$$

is a well-defined continuous G-equivariant bijection between Hausdorff spaces. If G is compact, then f is a homeomorphism.

Proof The isotropy group G_p is closed in G by Proposition 3.2.9 (here we need that M is Hausdorff). It is clear by Proposition 3.8.3 that f is a well-defined equivariant bijection. It is also clear from the definition of the quotient topology that f is continuous. The final statement follows, because a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

Remark 3.8.6 In general, if G is non-compact, the map f is *not* a homeomorphism.

3.8.3 Lie Groups and Homogeneous Spaces

We now come to the case that we are most interested in: G is a Lie group and $H \subset G$ a closed subgroup. By Corollary 3.7.35 the quotient G/H has a unique structure of a smooth manifold such that $\pi\colon G \to G/H$ is a submersion. According to Proposition 3.8.2 we get a transitive group action

$$\begin{aligned} \Phi\colon G \times G/H &\longrightarrow G/H \\ (g,[a]) &\longmapsto g \cdot [a] = [ga]. \end{aligned}$$

Proposition 3.8.7 (Φ Is a Smooth Action for Lie Groups) *Suppose G is a Lie group and $H \subset G$ a closed subgroup. Then the transitive group action*

$$\Phi : G \times G/H \longrightarrow G/H$$

is smooth.

Proof Multiplication in G followed by projection onto G/H is smooth,

$$G \times G \longrightarrow G \longrightarrow G/H.$$

By Lemma 3.7.5 the map

$$\Phi : G \times G/H \longrightarrow G/H$$

is smooth. □

We can now determine the structure of smooth manifolds that are homogeneous under the action of a Lie group.

Theorem 3.8.8 (Structure of Transitive Smooth Group Actions on Manifolds) *Let $G \times M \to M$ be a transitive smooth left action of a Lie group G on a manifold M. Fix an arbitrary point $p \in M$ and let G_p denote the isotropy group of p. Then $G_p \subset G$ is a closed subgroup and*

$$\begin{aligned} f : G/G_p &\longrightarrow M \\ [a] &\longmapsto a \cdot p \end{aligned}$$

is a well-defined G-equivariant diffeomorphism between manifolds.

Proof It follows from Proposition 3.8.5 that f is well-defined, continuous, bijective and G-equivariant. By Corollary 3.7.35 the quotient space G/G_p is a smooth manifold. It remains to show that f is smooth and a diffeomorphism.

The map f is smooth by Lemma 3.7.5, because the orbit map

$$\begin{aligned} \phi_p : G &\longrightarrow M \\ a &\longmapsto a \cdot p \end{aligned}$$

is smooth. To show that f is a diffeomorphism it suffices to show that the differential of f is an isomorphism at every point of G/G_p. By G-equivariance of f we have

$$f([ga]) = g \cdot f([a]).$$

Since left translations are diffeomorphisms of G/G_p and M, the differential of f is an isomorphism at every point of G/G_p if and only if it is an isomorphism at $[e]$.

We first show that the differential of f is injective at $[e]$: let $U \subset G/G_p$ be an open neighbourhood of $[e]$ and $s: U \to G$ a local section with $\pi \circ s = \mathrm{Id}_U$, where $\pi: G \to G/G_p$ is the canonical projection. Without loss of generality $s([e]) = e$. Then $f = \phi_p \circ s$ and

$$D_{[e]}f = D_e\phi_p \circ D_{[e]}s.$$

We also have

$$\mathrm{Id}_{T_{[e]}G/G_p} = D_e\pi \circ D_{[e]}s.$$

This shows that $D_{[e]}s$ is injective and its image is a complementary subspace to the kernel of $D_e\pi$, which is the Lie algebra $\mathfrak{g}_p$ of G_p according to Corollary 3.7.36. The kernel of $D_e\phi_p$ is also equal to $\mathfrak{g}_p$ according to Proposition 3.2.10. This implies that the differential $D_{[e]}f$ is injective.

To show that $D_{[e]}f$ is surjective it suffices to show by G-equivariance that $D_{[a]}f$ is surjective at some point $[a] \in G/G_p$. This follows from the next lemma. □

Lemma 3.8.9 *Let $f: X \to Y$ be a surjective smooth map between manifolds. Then there exists a point $x \in X$ such that $D_x f$ is surjective.*

Proof According to Sard's Theorem A.1.27 there exists a regular value $y \in Y$ of f. Since f is surjective, there exists an $x \in X$ with $f(x) = y$. Then x is a regular point f, i.e. the differential $D_x f$ is surjective. □
Along the way we have shown the following more general result.

Corollary 3.8.10 (The Orbit Map Induces an Injective Immersion of G/G_p into M) *Let $G \times M \to M$ be a smooth left action of a Lie group G on a manifold M, not necessarily transitive. Fix a point $p \in M$. Then*

$$\begin{aligned} f: G/G_p &\longrightarrow M \\ [a] &\longmapsto a \cdot p \end{aligned}$$

is an injective immersion of the manifold G/G_p into M whose image is the orbit $\mathcal{O}_p$ of p. In particular, if the Lie group G is compact, then the orbit $\mathcal{O}_p$ is an embedded submanifold of M, diffeomorphic to G/G_p.

Example 3.8.11 In Theorem 3.3.2 we saw that the standard representation of $\mathrm{O}(n)$ on $\mathbb{R}^n$ induces a transitive action of $\mathrm{O}(n)$ on the unit sphere S^{n-1} with isotropy group of a point $e_1 \in S^{n-1}$ isomorphic to the subgroup $\mathrm{O}(n-1)$. Theorem 3.8.8 then implies that the orbit map descends to a diffeomorphism

$$\mathrm{O}(n)/\mathrm{O}(n-1) \xrightarrow{\cong} S^{n-1}.$$

In a similar way we get diffeomorphisms

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \xrightarrow{\cong} S^{n-1},$$
$$\mathrm{U}(n)/\mathrm{U}(n-1) \xrightarrow{\cong} S^{2n-1},$$
$$\mathrm{SU}(n)/\mathrm{SU}(n-1) \xrightarrow{\cong} S^{2n-1},$$
$$\mathrm{Sp}(n)/\mathrm{Sp}(n-1) \xrightarrow{\cong} S^{4n-1}.$$

We also get diffeomorphisms

$$\mathrm{GL}(n,\mathbb{K})/\left(\mathrm{GL}(n-1,\mathbb{K}) \times \mathbb{K}^{n-1}\right) \xrightarrow{\cong} \mathbb{K}^n \setminus \{0\},$$
$$\mathrm{GL}(n,\mathbb{R})_+/\left(\mathrm{GL}(n-1,\mathbb{R})_+ \times \mathbb{R}^{n-1}\right) \xrightarrow{\cong} \mathbb{R}^n \setminus \{0\},$$
$$\mathrm{SL}(n,\mathbb{K})/\left(\mathrm{SL}(n-1,\mathbb{K}) \times \mathbb{K}^{n-1}\right) \xrightarrow{\cong} \mathbb{K}^n \setminus \{0\}.$$

Note that the group structure on $\mathrm{GL}(n-1,\mathbb{K}) \times \mathbb{K}^{n-1}$ and $\mathrm{SL}(n-1,\mathbb{K}) \times \mathbb{K}^{n-1}$ is *not* the direct product structure.
We can now prove Theorem 1.2.22 on the connected components of the classical linear groups (the idea for this proof is from [34] and [142]).

Proof Let G be a Lie group and $H \subset G$ a closed connected subgroup. According to Proposition 3.7.14 the quotient manifold G/H is connected if and only if G is connected. We apply this inductively to the homogeneous spaces in Example 3.8.11. We do the case of $\mathrm{SO}(n)$ explicitly, the other cases are left as an exercise. It is clear that $\mathrm{SO}(1) = \{1\}$ is connected. Since S^{n-1} is connected for all $n \geq 2$, the diffeomorphism

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \xrightarrow{\cong} S^{n-1}$$

shows that $\mathrm{SO}(n)$ is connected for all $n \geq 2$. □
The following fact is sometimes useful:

Corollary 3.8.12 (Smooth Structure on Sets with a Transitive Lie Group Action) *Suppose that M is a set and $G \times M \to M$ a transitive left action of a Lie group G on M with closed isotropy group G_p, for some $p \in M$. Then*

$$f\colon G/G_p \longrightarrow M$$
$$[a] \longmapsto a \cdot p$$

is a bijection. The set M can be given a unique structure of a smooth manifold, so that f becomes a diffeomorphism. If G is compact, then M is compact.

We conclude that in this situation we get the manifold structure on M *for free*, without the (sometimes difficult) task of defining a topology and an atlas of smoothly compatible charts for M.

3.9 *Stiefel and Grassmann Manifolds

We discuss two examples of compact homogeneous spaces where the manifold structure is defined by Corollary 3.8.12.

Example 3.9.1 (Stiefel Manifolds) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider positive integers $k \leq n$. The **Stiefel manifold** $V_k(\mathbb{K}^n)$ is defined as the set of ordered k-tuples of orthonormal vectors in $\mathbb{K}^n$ with respect to the standard Euclidean (Hermitian, symplectic) scalar product on $\mathbb{R}^n$ ($\mathbb{C}^n$, $\mathbb{H}^n$) from Definition 1.2.9:

$$V_k(\mathbb{K}^n) = \{(v_1, \ldots, v_k) \mid v_i \in \mathbb{K}^n, \langle v_i, v_j \rangle_{\mathbb{K}^n} = \delta_{ij}\}.$$

We consider the case $\mathbb{K} = \mathbb{R}$ in detail. The group $\mathrm{O}(n)$ acts on the set $V_k(\mathbb{R}^n)$ via

$$A \cdot (v_1, \ldots, v_k) = (Av_1, \ldots, Av_k).$$

Since we can complete the vectors $v_1, \ldots, v_k$ to an orthonormal basis of $\mathbb{R}^n$ and $\mathrm{O}(n)$ acts transitively on orthonormal bases, it follows that the action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$ is also transitive. The isotropy group of the point

$$p = (e_1, \ldots, e_k) \in V_k(\mathbb{R}^n)$$

is equal to

$$\mathrm{O}(n)_p = \left\{ \begin{pmatrix} E_k & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{O}(n-k) \right\} \cong \mathrm{O}(n-k).$$

This holds, because if $C \in \mathrm{O}(n)$ satisfies $C \cdot p = p$, then C is of the form

$$C = \begin{pmatrix} E_k & B \\ 0 & A \end{pmatrix}$$

and $CC^T = E$ implies $AA^T = E$ and $BA^T = 0$, hence $A \in \mathrm{O}(n-k)$ and $B = 0$. It follows that the real Stiefel manifold admits the structure of a compact manifold given by

$$V_k(\mathbb{R}^n) = \mathrm{O}(n)/\mathrm{O}(n-k).$$

In particular, $V_k(\mathbb{R}^n)$ has dimension

$$\begin{aligned}\dim V_k(\mathbb{R}^n) &= \dim \mathrm{O}(n) - \dim \mathrm{O}(n-k)\\ &= \dim \mathfrak{o}(n) - \dim \mathfrak{o}(n-k)\\ &= \frac{1}{2}n(n-1) - \frac{1}{2}(n-k)(n-k-1)\\ &= nk - \frac{1}{2}k(k+1).\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}V_k(\mathbb{C}^n) &= \mathrm{U}(n)/\mathrm{U}(n-k),\\ V_k(\mathbb{H}^n) &= \mathrm{Sp}(n)/\mathrm{Sp}(n-k).\end{aligned}$$

It follows that the complex and quaternionic Stiefel manifolds are connected for all $k \leq n$. For real Stiefel manifolds and $k < n$ this follows from Exercise 3.12.12.

Example 3.9.2 (Grassmann Manifolds) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider non-negative integers $k \leq n$. The **Grassmann manifold** or **Grassmannian** $Gr_k(\mathbb{K}^n)$ is defined as the set of k-dimensional vector subspaces in $\mathbb{K}^n$:

$$Gr_k(\mathbb{K}^n) = \{U \subset \mathbb{K}^n \mid U \text{ is a } k\text{-dimensional vector subspace}\}.$$

We consider the case $\mathbb{K} = \mathbb{R}$. The group $\mathrm{O}(n)$ acts on the set $Gr_k(\mathbb{R}^n)$ via

$$A \cdot U = \{Au \in \mathbb{R}^n \mid u \in U\}.$$

This action is transitive, since we can choose a basis for U and the action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$ is transitive. The isotropy group of

$$p = \mathrm{span}(e_1, \ldots, e_k) \in Gr_k(\mathbb{R}^n)$$

is equal to

$$\mathrm{O}(n)_p = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathrm{O}(k), B \in \mathrm{O}(n-k) \right\} \cong \mathrm{O}(k) \times \mathrm{O}(n-k).$$

It follows that the real Grassmannian $Gr_k(\mathbb{R}^n)$ admits the structure of a compact manifold given by

$$Gr_k(\mathbb{R}^n) = \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k)).$$

Note that there is a diffeomorphism

$$Gr_{n-k}(\mathbb{R}^n) \cong Gr_k(\mathbb{R}^n).$$

The dimension of $Gr_k(\mathbb{R}^n)$ is equal to

$$\begin{aligned}\dim Gr_k(\mathbb{R}^n) &= \dim V_k(\mathbb{R}^n) - \dim \mathrm{O}(k)\\ &= nk - \frac{1}{2}k(k+1) - \frac{1}{2}k(k-1)\\ &= k(n-k).\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}Gr_k(\mathbb{C}^n) &= \mathrm{U}(n)/(\mathrm{U}(k) \times \mathrm{U}(n-k)),\\ Gr_k(\mathbb{H}^n) &= \mathrm{Sp}(n)/(\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)).\end{aligned}$$

There are diffeomorphisms

$$Gr_1(\mathbb{K}^{n+1}) \cong \mathbb{KP}^n$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

3.10 *The Exceptional Lie Group G_2

In this section we discuss the compact simple exceptional Lie group G_2. In particular, we want to show that G_2 has dimension 14. This is a nice application of homogeneous spaces and Stiefel manifolds. We follow the paper [26] by Robert Bryant.

Besides being mathematically interesting, the Lie group G_2 plays an important role in M-theory, a conjectured supersymmetric theory of quantum gravity in 11 dimensions, which is related to the superstring theories in dimension 10. If M-theory is a realistic theory of nature, with 4-dimensional spacetime, 7 of the 11 dimensions have to be very small (*compactified*). The vacuum or background of the theory is thus of the form $\mathbb{R}^4 \times K$, where $\mathbb{R}^4$ is Minkowski spacetime and K is a compact Riemannian 7-manifold. Moreover, for the background to be a solution of the supergravity equations of motion, preserving one supersymmetry in dimension 4 (the most interesting case from a phenomenological point of view), the Riemannian metric on the 7-dimensional compact manifold K has to have *holonomy group* equal to G_2 (assuming that the *flux* is set to zero). The first compact examples of Riemannian manifolds with holonomy equal to G_2 were constructed by Dominic Joyce.

A Riemannian metric has holonomy group G_2 precisely if the 7-manifold admits a certain type of 3-form that is parallel with respect to the Levi-Civita connection. We will introduce the linear model of the 3-form on a vector space of dimension seven and define G_2 as its isotropy group.

3.10.1 Definition of the 3-Form ϕ and the Lie Group G_2

We need some preparations: Let $V = \mathbb{R}^7$ with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ and standard orthonormal basis $\{e_j\}$. Let $\{\omega^i\}$ denote the dual basis of V^*, defined by

$$\omega^i(e_j) = \delta^i_j.$$

We use a shorthand notation for wedge products of the ω^i. For example,

$$\omega^{123} = \omega^1\omega^{23} = \omega^1 \wedge \omega^2 \wedge \omega^3.$$

Definition 3.10.1 We define a 3-form $\phi \in \Lambda^3 V^*$ by:

$$\phi = \omega^{123} + \omega^1(\omega^{45} + \omega^{67}) + \omega^2(\omega^{46} - \omega^{57}) - \omega^3(\omega^{47} + \omega^{56}).$$

Remark 3.10.2 The peculiar form of ϕ will be justified in Exercise 3.12.15. Other choices, however, are possible and lead to equivalent descriptions of G_2.
The group $GL(7, \mathbb{R})$ acts on the column vector space V on the left via the standard representation. There is an induced representation on $\Lambda^k V^*$ defined by (cf. Definition 2.1.23):

$$(g\alpha)(v_1, \ldots, v_k) = \alpha\left(g^{-1}v_1, \ldots, g^{-1}v_k\right) \quad g \in GL(7, \mathbb{R}), v_i \in V.$$

We think of this representation as a left action of $GL(7, \mathbb{R})$ on $\Lambda^k V^*$.

Definition 3.10.3 We define $G_2 \subset GL(7, \mathbb{R})$ as the isotropy group of the 3-form ϕ:

$$G_2 = GL(7, \mathbb{R})_\phi = \{g \in GL(7, \mathbb{R}) \mid g\phi = \phi\}.$$

This is a closed embedded Lie subgroup of $GL(7, \mathbb{R})$.

3.10.2 G_2 as a Compact Subgroup of $SO(7)$

Definition 3.10.4 For $x \in V$ we denote by $x \lrcorner \phi$ (contraction of ϕ with x) the 2-form on V defined by

$$(x \lrcorner \phi)(y, z) = \phi(x, y, z) \quad \forall y, z \in V.$$

The following map is very useful in the study of the Lie group G_2.

Definition 3.10.5 We set

$$\begin{aligned} b\colon V \times V &\longrightarrow \Lambda^7 V^* \\ (x, y) &\longmapsto b(x, y) = \frac{1}{6}(x \lrcorner \phi) \wedge (y \lrcorner \phi) \wedge \phi. \end{aligned}$$

Here are some properties of the map b.

Proposition 3.10.6 *The map b is symmetric and bilinear. It is G_2-equivariant, i.e. we have*

$$b(gx, gy) = g(b(x, y)) \quad \forall g \in \mathrm{G}_2\, x, y \in V.$$

A calculation shows that

$$b(x, y) = \langle x, y\rangle \cdot \mathrm{vol},$$

where $\mathrm{vol} = \omega^{1234567}$ *is the standard volume form of V.*

Proof It is clear that b is symmetric and bilinear. For $g \in \mathrm{G}_2$ and $x, y, z \in V$ we calculate

$$\begin{aligned} ((gx)\lrcorner\phi)(y, z) &= \phi(gx, y, z) \\ &= \phi\left(gx, gg^{-1}y, gg^{-1}z\right) \\ &= \left(g^{-1}\phi\right)\left(x, g^{-1}y, g^{-1}z\right) \\ &= \phi\left(x, g^{-1}y, g^{-1}z\right) \\ &= (x\lrcorner\phi)\left(g^{-1}y, g^{-1}z\right) \\ &= (g(x\lrcorner\phi))(y, z). \end{aligned}$$

Therefore

$$(gx)\lrcorner\phi = g(x\lrcorner\phi)$$

and

$$\begin{aligned} b(gx, gy) &= \frac{1}{6}((gx)\lrcorner\phi) \wedge ((gy)\lrcorner\phi) \wedge \phi \\ &= \frac{1}{6}(g(x\lrcorner\phi)) \wedge (g(y\lrcorner\phi)) \wedge g\phi \\ &= g(b(x, y)). \end{aligned}$$

The final property can be proved by a (tedious) direct calculation using the explicit form of ϕ. Because of symmetry and bilinearity of b it suffices to show that

$$b(e_i, e_j) = \delta_{ij} \cdot \mathrm{vol} \quad \forall i \leq j \in \{1, \ldots, 7\}.$$

We have

$$\begin{aligned}
e_1 \lrcorner \phi &= \omega^{23} + \omega^{45} + \omega^{67}, \\
e_2 \lrcorner \phi &= -\omega^{13} + \omega^{46} - \omega^{57}, \\
e_3 \lrcorner \phi &= \omega^{12} - \omega^{47} - \omega^{56}, \\
e_4 \lrcorner \phi &= -\omega^{15} - \omega^{26} + \omega^{37}, \\
e_5 \lrcorner \phi &= \omega^{14} + \omega^{27} + \omega^{36}, \\
e_6 \lrcorner \phi &= -\omega^{17} + \omega^{24} - \omega^{35}, \\
e_7 \lrcorner \phi &= \omega^{16} - \omega^{25} - \omega^{34}.
\end{aligned}$$

We then calculate all 28 wedge products of the form

$$(e_i \lrcorner \phi) \wedge (e_j \lrcorner \phi) \wedge \phi$$

with $i \leq j$. For example,

$$\begin{aligned}
(e_1 \lrcorner \phi) \wedge (e_1 \lrcorner \phi) \wedge \phi &= 6 \cdot \mathrm{vol}, \\
(e_1 \lrcorner \phi) \wedge (e_2 \lrcorner \phi) \wedge \phi &= 0.
\end{aligned}$$

The claim then follows from these calculations. □

Corollary 3.10.7 (G_2 **Is a Compact Subgroup of** SO(7)) *The following identity holds*

$$\langle gx, gy \rangle = (\det g)^{-1} \cdot \langle x, y \rangle \quad \forall g \in G_2, x, y \in V,$$

and

$$\det g = 1 \quad \forall g \in G_2.$$

In particular, G_2 *preserves the standard scalar product and orientation on* V *and is thus a compact embedded Lie subgroup of* SO(7).

Proof For any $g \in \mathrm{GL}(7, \mathbb{R})$ we have

$$\begin{aligned}(g\mathrm{vol})(e_1, \ldots, e_7) &= \mathrm{vol}\left(g^{-1}e_1, \ldots, g^{-1}e_7\right)\\ &= \det\left(g^{-1}I\right)\\ &= (\det g)^{-1},\end{aligned}$$

hence

$$g\mathrm{vol} = (\det g)^{-1} \cdot \mathrm{vol}.$$

By Proposition 3.10.6 this implies for all $g \in \mathrm{G}_2$

$$\begin{aligned}\langle gx, gy\rangle \mathrm{vol} &= b(gx, gy)\\ &= g(b(x, y))\\ &= \langle x, y\rangle g\mathrm{vol}\\ &= (\det g)^{-1}\langle x, y\rangle \mathrm{vol}.\end{aligned}$$

Therefore

$$\langle gx, gy\rangle = (\det g)^{-1} \cdot \langle x, y\rangle \quad \forall g \in \mathrm{G}_2, x, y \in V.$$

Consider the matrix $g^T g$. We have

$$(g^T g)_{ij} = \langle ge_i, ge_j\rangle = (\det g)^{-1}\delta_{ij},$$

hence

$$g^T g = (\det g)^{-1} I_7.$$

Calculating the determinant on both sides we get

$$\begin{aligned}(\det g)^2 &= \det\left(g^T g\right)\\ &= (\det g)^{-7},\end{aligned}$$

hence

$$(\det g)^9 = 1$$

and

$$\det g = 1 \quad \forall g \in \mathrm{G}_2.$$

We get

$$\langle gx, gy \rangle = \langle x, y \rangle \quad \forall g \in G_2$$

and with $\det g = 1$ it follows that G_2 is a subgroup of SO(7). Since G_2 is a closed subgroup and SO(7) is compact, it follows that G_2 is compact. □

3.10.3 *An* SU(2)*-Subgroup of* G_2

Definition 3.10.8 Let $P\colon V \times V \to V$ be the map defined by

$$\langle P(x, y), z \rangle = \phi(x, y, z) \quad \forall x, y, z \in V.$$

Proposition 3.10.9 *The map P is antisymmetric, bilinear and* G_2*-equivariant. We have $P(e_1, e_2) = e_3$.*

Proof The first two properties are clear. The third property follows because the standard scalar product on V is G_2-invariant and ϕ is G_2-invariant. The final claim follows immediately from the definition of ϕ. □

Consider the action

$$\begin{aligned} G_2 \times V_2\left(\mathbb{R}^7\right) &\longrightarrow V_2\left(\mathbb{R}^7\right) \\ (g, v_1, v_2) &\longmapsto g \cdot (v_1, v_2) = (gv_1, gv_2). \end{aligned}$$

This is the restriction of the standard action of O(7) on the Stiefel manifold $V_2\left(\mathbb{R}^7\right)$.

Definition 3.10.10 Let $H \subset G_2$ denote the isotropy group of the point $p = (e_1, e_2) \in V_2\left(\mathbb{R}^7\right)$ under this action.

Since P is G_2-equivariant and $P(e_1, e_2) = e_3$ we have $He_3 = e_3$. Therefore H is the subgroup of G_2 defined by

$$He_i = e_i \quad \forall i = 1, 2, 3$$

and the action of H restricts to an action on the orthogonal complement

$$W = \mathrm{span}(e_4, e_5, e_6, e_7).$$

Lemma 3.10.11 *The Lie group H is isomorphic to the subgroup of* SO(4)*, acting on W and fixing the 2-forms*

$$\begin{aligned} \beta_1 &= \omega^{45} + \omega^{67}, \\ \beta_2 &= \omega^{46} - \omega^{57}, \\ \beta_3 &= \omega^{47} + \omega^{56}. \end{aligned}$$

Proof This follows, because $H \subset \mathrm{G}_2$ and G_2 fixes the 3-form ϕ. □

Proposition 3.10.12 (H Is Isomorphic to $\mathrm{SU}(2)$) *The Lie group H is isomorphic to the subgroup of* $\mathrm{SO}(4)$*, acting on W and fixing the complex structure*

$$Je_4 = e_5,$$
$$Je_6 = e_7$$

and the complex volume form

$$\rho = (\omega^4 + i\omega^5) \wedge (\omega^6 + i\omega^7).$$

Hence H is isomorphic to $\mathrm{SU}(2)$.

Proof Since

$$\rho = \beta_2 + i\beta_3,$$

an element $g \in \mathrm{SO}(4)$ fixes ρ if and only if it fixes both β_2 and β_3. For any vector $v \in W$ we have

$$Jv = (v \lrcorner \beta_1)^*,$$

where $*$ denotes the vector dual to the 1-form with respect to the standard scalar product on W. It follows that $g \in \mathrm{SO}(4)$ fixes J if and only if it fixes β_1. □

3.10.4 The Dimension of G_2

Corollary 3.10.13 (Upper Bound on the Dimension of G_2) *The action of* G_2 *on the Stiefel manifold $V_2\left(\mathbb{R}^7\right)$ induces an injective immersion of* $\mathrm{G}_2/\mathrm{SU}(2)$ *into $V_2(\mathbb{R}^7)$. In particular,*

$$\dim \mathrm{G}_2 \leq 14$$

with equality if and only if the action of G_2 *on $V_2\left(\mathbb{R}^7\right)$ is transitive.*

Proof The first claim follows from Corollary 3.8.10. We have

$$\dim V_2\left(\mathbb{R}^7\right) = 7 \cdot 2 - \frac{1}{2} 2 \cdot 3 = 11,$$

according to the calculation in Example 3.9.1. Since $\dim \mathrm{SU}(2) = 3$ and the map

$$f \colon \mathrm{G}_2/\mathrm{SU}(2) \longrightarrow V_2\left(\mathbb{R}^7\right)$$

has injective differential, the second claim follows. The third claim follows since in the case of equality the map f is a submersion, hence has open image, and the image is closed, since G_2 is compact (it can be shown that $V_2(\mathbb{R}^7)$ is connected, cf. Exercise 3.12.12). □

Lemma 3.10.14 (Lower Bound on the Dimension of G_2) *The action of* $GL(7, \mathbb{R})$ *on* $\Lambda^3 V^*$ *induces an injective immersion*

$$h: GL(7, \mathbb{R})/G_2 \longrightarrow \Lambda^3 V^*$$
$$[g] \longmapsto g \cdot \phi.$$

Hence $\dim G_2 \geq 14$*, with equality if and only if the map h has open image.*

Proof The first claim again follows from Corollary 3.8.10. The second claim follows from

$$\dim GL(7, \mathbb{R}) = 7 \cdot 7 = 49,$$
$$\dim \Lambda^3 V^* = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

□

Collecting our results, we get the following theorem:

Theorem 3.10.15 (G_2 Has Dimension 14) *The Lie group* G_2 *has dimension* 14. *It acts transitively on the Stiefel manifold* $V_2(\mathbb{R}^7)$*. In particular, the standard representation of* G_2 *on* $V = \mathbb{R}^7$ *is irreducible. Moreover, the* $GL(7, \mathbb{R})$*-orbit of* ϕ *in* $\Lambda^3 V^*$ *is open.*

Remark 3.10.16 (G_2 Is a Simple Lie Group) A calculation of the homotopy groups of G_2, using the fibration

$$\begin{array}{ccc} SU(2) & \longrightarrow & G_2 \\ & & \downarrow \\ & & V_2(\mathbb{R}^7) \end{array}$$

shows that

$$\pi_0(G_2) = 0, \quad \pi_1(G_2) = 0, \quad \pi_3(G_2) = \mathbb{Z}.$$

Hence G_2 is connected, simply connected and simple, cf. Corollary 2.6.6. The details of this calculation can be found in [26].

3.11 *Godement's Theorem on the Manifold Structure of Quotient Spaces

In this section we want to prove Godement's Theorem 3.7.10. We continue to follow [130] and [139]. Let R be an equivalence relation on a manifold M. Suppose that R is a closed embedded submanifold of $M \times M$ and $\mathrm{pr}_1|_R: R \to M$ a surjective submersion. By symmetry of equivalence relations it follows that $\mathrm{pr}_2|_R: R \to M$ is also a surjective submersion. We endow M/R with the quotient topology.

3.11.1 Preliminary Facts

We want to prove two preliminary facts: we first show that the quotient M/R is Hausdorff.

Lemma 3.11.1 (The Quotient Space Is Hausdorff) *The canonical projection $\pi: M \to M/R$ is open and M/R is Hausdorff.*

Proof Suppose $U \subset M$ is open. We claim that

$$\pi^{-1}(\pi(U)) = \mathrm{pr}_1((M \times U) \cap R).$$

This holds because $x \in \pi^{-1}(\pi(U))$ if and only if there exists a $y \in U$ such that $(x, y) \in R$. Since $\mathrm{pr}_1|_R$ is a submersion and $(M \times U) \cap R$ is open in R, the set $\pi^{-1}(\pi(U))$ is an open subset of M, hence $\pi(U)$ is an open subset of M/R by the definition of the quotient topology. This proves that π is an open map. The claim about the Hausdorff property follows from Lemma 3.7.2, because R is by assumption a closed subset of $M \times M$. □

We denote the equivalence class of a point $x \in M$ under the equivalence relation R by $[x]$. We want to show that equivalence classes are embedded submanifolds of M.

Lemma 3.11.2 (Equivalence Classes Are Embedded Submanifolds of M) *Every equivalence of R is a closed embedded submanifold of M of dimension* $\dim R - \dim M$.

Proof We can write

$$[x] = \mathrm{pr}_1\left((\mathrm{pr}_2|_R)^{-1}(\{x\})\right),$$

because

$$(\mathrm{pr}_2|_R)^{-1}(\{x\}) = \{(y, x) \in M \times M \mid y \sim x\}.$$

Since $\mathrm{pr}_2|_R: R \to M$ is a submersion, the subset $K = (\mathrm{pr}_2|_R)^{-1}(\{x\})$ is an embedded submanifold of R of dimension $\dim R - \dim M$. However, K is contained in $M \times \{x\}$

on which pr_1 is a diffeomorphism onto M. Therefore $[x] = \mathrm{pr}_1(K)$ is an embedded submanifold of M of dimension $\dim R - \dim M$. □

3.11.2 The Slice Theorem

Our task is to show that the quotient space M/R has the structure of a smooth manifold. To define charts for M/R we construct so-called *slices* for the equivalence relation on open neighbourhoods for any point of M. In a second step we will then construct slices for *saturated* open neighbourhoods, which are the main tools needed to define the manifold structure on M/R.

Definition 3.11.3 Let $U \subset M$ be an open neighbourhood. Then a **slice** for the intersection of the equivalence classes of R with U is a closed embedded submanifold $S \subset U$ together with a surjective submersion $q: U \to S$ such that for every $x \in U$ the set $[x] \cap U$ intersects S precisely in the single point $q(x)$.

Theorem 3.11.4 (Slice Theorem) *Every point in M has an open neighbourhood $U \subset M$ with a slice (S, q) for the intersection of the equivalence classes of R with U.*

To prove the theorem fix $a \in M$ and let S' be any submanifold of M through a of dimension $\dim M - \dim[a]$ and transverse to the submanifold $[a]$. This means that

$$T_a S \oplus T_a[a] = T_a M.$$

We will show that we can find an open neighbourhood U of a in M such that $S = S' \cap U$ is a slice.

Lemma 3.11.5 *Consider*

$$Z = (\mathrm{pr}_2|_R)^{-1}\left(S'\right).$$

Then Z is a submanifold of R through (a, a) of dimension $\dim Z = \dim M$ *and* $\mathrm{pr}_1|_Z: Z \to M$ *is a local diffeomorphism around* (a, a).

Proof Since $\mathrm{pr}_2|_R$ is a submersion, it is clear that Z is a submanifold of R with

$$\dim R - \dim Z = \dim M - \dim S' = \dim[a] = \dim R - \dim M.$$

Hence $\dim Z = \dim M$. We have

$$Z = (M \times S') \cap R.$$

Since $a \in S'$ and $a \sim a$, it follows that $(a, a) \in Z$.

It remains to show that the differential of $\mathrm{pr}_1|_Z$ in (a, a) is an isomorphism onto $T_a M$. We consider the following submanifolds of Z through (a, a):

$$[a] \times \{a\} \text{ and the diagonal } \Delta_{S'} \subset S' \times S'.$$

The tangent spaces to these submanifolds are given by

$$T_a[a] \oplus 0 \text{ and } \Delta_{T_a S'}.$$

These tangent spaces have zero intersection and their dimensions are $\dim[a]$ and $\dim S' = \dim M - \dim[a] = \dim Z - \dim[a]$. Hence

$$T_{(a,a)}Z = (T_a[a] \oplus 0) \oplus \Delta_{T_a S'}.$$

The image of $T_{(a,a)}Z$ under the differential of $\mathrm{pr}_1|_Z$ is

$$T_a[a] + T_a S' = T_a M,$$

hence the differential of $\mathrm{pr}_1|_Z$ is surjective and thus an isomorphism. □

Note that

$$\mathrm{pr}_2|_Z : Z \longrightarrow S'$$

is a submersion. By Lemma 3.11.5 we can choose open neighbourhoods O and U' of $a \in M$ such that

$$\mathrm{pr}_1|_{Z \cap (O \times O)} : Z \cap (O \times O) \longrightarrow U'$$

is a diffeomorphism. Let s denote the inverse of this diffeomorphism and

$$q = \mathrm{pr}_2|_Z \circ s.$$

Then q is a submersion of U' onto an open subset of $S' \cap O$.

Our aim is to shrink U' to U so that $S = S' \cap U$ is a slice together with the restriction of q. Note that

$$s(x) = (x, q(x)) \in Z \cap (O \times O) \quad \forall x \in U'.$$

In particular, $U' \subset O$.

Lemma 3.11.6 *Let $x \in S' \cap U'$. Then $s(x) = (x, x)$ and $q(x) = x$. In particular, if $y \in U'$ and $q(y) \in U'$, then $q(q(y)) = q(y)$.*

Proof We have $\Delta_{S'} \subset R$, hence $\Delta_{S'} \subset Z$. Thus

$$(x, x) \in \Delta_{S'} \cap (U' \times U') \subset Z \cap (O \times O).$$

Moreover,

$$\mathrm{pr}_1(x, x) = x = \mathrm{pr}_1 \circ s(x),$$

since s is the inverse of $\mathrm{pr}_1|_{Z\cap(O\times O)}$. Since $\mathrm{pr}_1|_{Z\cap(O\times O)}$ is injective, this implies $s(x) = (x, x)$ and thus $q(x) = x$.

Finally, if $y \in U'$ and $q(y) \in U'$, then $x = q(y) \in S' \cap U'$ and the claim follows. □

Lemma 3.11.7 *Let*

$$U = U' \cap q^{-1}(U' \cap O),$$
$$S = S' \cap U.$$

Then U and S together with the restriction of q to U satisfy the requirements of Theorem 3.11.4.

Proof Clearly U is an open neighbourhood of a in M, because $a \in U'$ and $a \in S'$, hence $q(a) = a \in U' \cap O$ by Lemma 3.11.6. We also have $S \subset U$. Suppose $x \in U$. Then $x \in U'$ and $q(x) \in U' \cap O$. Thus $q(q(x)) = q(x) \in U' \cap O$ and therefore $q(x) \in U$ by definition of U. But also $q(x) \in S'$ by definition of q, hence $q(x) \in S$. Therefore the restriction of q to U defines a submersion

$$q: U \longrightarrow S.$$

If $x \in S$, then $x \in S' \cap U'$ and $q(x) = x$ by Lemma 3.11.6. This implies that q is surjective.

Finally, suppose that $x \in U$ and $y \in [x] \cap S$. Then

$$(x, y) \in ((M \times S) \cap R) \cap (O \times O) \subset Z \cap (O \times O),$$

because $U' \subset O$. Thus

$$(x, y) = s(x) = (x, q(x)),$$

hence $y = q(x)$. This proves the final requirement for the slice (S, q). □

Definition 3.11.8 If $V \subset M$ is a subset, then we denote the restriction of R to V by R_V. As a subset of $M \times M$ we have $R_V = (V \times V) \cap R$. We denote by $\pi_V: V \to V/R_V$ the canonical projection.

Corollary 3.11.9 (Slice for Open Subset Defines Local Manifold Structure on Quotient) *Every point in M has an open neighbourhood $U \subset M$ such that U/R_U has the structure of a smooth manifold and $\pi_U: U \to U/R_U$ is a surjective submersion.*

Proof Let $U \subset M$ be an open subset with a slice (S, q). Then the map $q\colon U \to S$ induces a bijection

$$\bar{q}\colon U/R_U \longrightarrow S.$$

We give U/R_U the structure of a smooth manifold such that $\bar{q}$ is a diffeomorphism. Then $\pi_U = \bar{q}^{-1} \circ q$ is a surjective submersion. □

3.11.3 Slices for Saturated Neighbourhoods and Proof of Godement's Theorem

Definition 3.11.10 A subset $V \subset M$ is called **saturated** if

$$V = \pi^{-1}(\pi(V)).$$

Equivalently, V is a union of equivalence classes. If U is an arbitrary subset of M, then $V = \pi^{-1}(\pi(U))$ is saturated.
We want to show that every point of M is contained in a *saturated* open neighbourhood with a slice. This is the main fact that we need to prove that M/R has the structure of a smooth manifold.

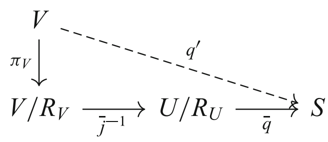
Corollary 3.11.11 (Slices for Saturated Open Subsets) *Let $U \subset M$ be an open subset with a slice (S, q) and V the saturated open subset $V = \pi^{-1}(\pi(U))$. Then there exists a surjective submersion $q'\colon V \to S$ so that (S, q') is a slice for V.*

Proof It is clear that $U \subset V$. Let $j\colon U \hookrightarrow V$ be the inclusion. We claim that there is a well-defined map

$$\bar{j}\colon U/R_U \longrightarrow V/R_V$$

and that this map is a bijection. The map is well-defined, because if $x, y \in U$ are equivalent, then they are equivalent in V. The map is also injective. Finally, the map is surjective, because if $x \in V$, then there exists a $y \in U$ with $(x, y) \in R$.

Using the bijection $\bar{q}\colon U/R_U \to S$ from the proof of Corollary 3.11.9, we get a well-defined map $q'\colon V \to S$:



The map q' has the following property: for $x \in V$, there exists a $y \in U$ such that $[x] = [y]$, i.e.

$$\bar{j}^{-1}([x]) = [y].$$

Then

$$\begin{aligned} q'(x) &= \bar{q} \circ \bar{j}^{-1}([x]) \\ &= q(y). \end{aligned}$$

This implies, since $S \subset U$,

$$\begin{aligned} [x] \cap S &= [x] \cap U \cap S \\ &= [y] \cap U \cap S \\ &= \{q(y)\} \\ &= \{q'(x)\}. \end{aligned}$$

Hence $[x]$ intersects S precisely in the point $q'(x)$.

Since $U \subset V$, the map q' is surjective. It remains to show that q' is a submersion. We claim that there is a commutative diagram

$$\begin{array}{ccc} (M \times U) \cap R & \xrightarrow{\mathrm{pr}_2} & U \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle q} \\ V & \xrightarrow[q']{} & S \end{array}$$

where the arrows on the left, right and top are submersions. The arrow on the right is a submersion, because (S, q) is a slice and the arrows on the top and on the left are submersions, because $\mathrm{pr}_1|_R, \mathrm{pr}_2|_R \colon R \to M$ are submersions. To show that the diagram is commutative, let $(x, y) \in (M \times U) \cap R$. Then $x \sim y$ and $x \in V$. The statement then is

$$q'(x) = q(y),$$

which we showed above. Lemma 3.7.5 then proves that q' is a submersion. □

Corollary 3.11.12 (Slice for Open Saturated Subset Defines Local Manifold Structure on Quotient) *Let $V \subset M$ be an open subset with a slice (S, q'). Then V/R_V has the structure of a smooth manifold so that $\pi_V \colon V \to V/R_V$ is a surjective submersion.*

We can now finish the proof of Godement's Theorem 3.7.10.

Proof We have shown that there exists a covering of M by open saturated sets V_i so that the open sets $W_i = V_i/R_{V_i} \subset M/R$ have the structure of a smooth manifold with

$$\pi_i : V_i \longrightarrow W_i$$

being surjective submersions. Suppose $V_i \cap V_j \neq \emptyset$. By Lemma 3.11.13 below we have to show that the manifold structures on $W_i \cap W_j$ induced from W_i and W_j are the same, i.e. the identity map between the open subsets $W_i \cap W_j \subset W_i$ and $W_i \cap W_j \subset W_j$ is a diffeomorphism. Since V_i and V_j are saturated, we have

$$\pi(V_i \cap V_j) = \pi(V_i) \cap \pi(V_j) = W_i \cap W_j.$$

The manifold structure induced from V_i and V_j on $V_i \cap V_j$ are the same. Since π is for each of these structures a submersion from $V_i \cap V_j$ onto $W_i \cap W_j$, it follows from Corollary 3.7.7 that the induced manifold structures on $W_i \cap W_j$ are the same. It is then also clear that

$$\pi : M \longrightarrow M/R$$

is a surjective submersion. □

We used (a slight generalization of) the following lemma, whose proof is clear:

Lemma 3.11.13 *Let X be a topological space, $W_1, W_2 \subset X$ open and*

$$\phi_1 : W_1 \longrightarrow U_1$$
$$\phi_2 : W_2 \longrightarrow U_2$$

homeomorphisms onto open subsets U_1, U_2 of $\mathbb{R}^n$. Define the unique smooth structure on W_i such that ϕ_i becomes a diffeomorphism, for $i = 1, 2$. Then the change of charts

$$\phi_2 \circ \phi_1^{-1} : \phi_1(W_1 \cap W_2) \longrightarrow \phi_2(W_1 \cap W_2)$$

is a diffeomorphism if and only if

$$\mathrm{Id} : W_1 \supset W_1 \cap W_2 \longrightarrow W_1 \cap W_2 \subset W_2$$

is a diffeomorphism.

3.12 Exercises for Chap. 3

3.12.1 Prove Proposition 3.2.2. Find an example of a left action

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, p) &\longmapsto g \cdot p \end{aligned}$$

so that

$$\begin{aligned} M \times G &\longrightarrow M \\ (p, g) &\longmapsto p * g = g \cdot p \end{aligned}$$

does *not* define a right action of G on M.

3.12.2 Let M be a Hausdorff space with a continuous left action of a topological group G. For a subset $K \subset G$ consider the **fixed point set**

$$M^K = \{p \in M \mid K \cdot p = p\}.$$

Prove the following:

1. If $K = \{k\}$ contains only one element, then M^K is a closed subset of M.
2. M^K is a closed subset of M for arbitrary subsets $K \subset G$.

3.12.3 The Lie group $G = \mathrm{SU}(2) \times \mathrm{U}(1)$ acts on $\mathbb{C}^2$ via

$$\left(A, e^{i\alpha}\right) \cdot v = e^{i\alpha} A v,$$

where Av denotes multiplication of the matrix $A \in \mathrm{SU}(2)$ with the column vector $v \in \mathbb{C}^2$. Let

$$p = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \in \mathbb{C}^2,$$

where $v_0 \in \mathbb{R}$, $v_0 \neq 0$.

1. Determine the isotropy subalgebra $\mathfrak{g}_p$ and the isotropy subgroup G_p. Which standard Lie group is G_p isomorphic to?
2. Determine the orbit $\mathcal{O}_p$ of p under the action of G. Which standard manifold is $\mathcal{O}_p$ diffeomorphic to?

In the electroweak gauge theory the *Higgs field* takes values in $\mathbb{C}^2$. The vector p is known as a *vacuum vector*. The isotropy group G_p is called the *unbroken subgroup*.

3.12.4 We consider S^3 with the Hopf action:

$$\begin{aligned} S^3 \times S^1 &\longrightarrow S^1 \\ \left(z, e^{i\alpha}\right) &\longmapsto ze^{i\alpha}. \end{aligned}$$

We identify

$$\begin{aligned} \mathbb{R}^4 &\longrightarrow \mathbb{C}^2 \\ (x_1, y_1, x_2, y_2) &\longmapsto (x_1 + iy_1, x_2 + iy_2). \end{aligned}$$

Let s denote the stereographic projection of S^3 through the point $(0, 1) \in S^3$:

$$\begin{aligned} s\colon S^3 \setminus \{(0, 1)\} &\longrightarrow \mathbb{R}^3 \\ (x_1, y_2, x_2, y_2) &\longmapsto \frac{1}{1 - x_2}(x_1, y_1, y_2). \end{aligned}$$

Let $\gamma_i\colon S^1 \to S^3$, for $i = 1, 2, 3$, denote the orbit maps of the points

$$p_1 = (1, 0), \quad p_2 = \frac{1}{\sqrt{2}}(1, 1), \quad p_3 = (0, 1)$$

on S^3 under the Hopf action. Consider the images

$$\begin{aligned} \sigma_i = s \circ \gamma_i\colon S^1 &\longrightarrow \mathbb{R}^3, \quad i = 1, 2 \\ \sigma_3 = s \circ \gamma_3\colon \mathbb{R} \cong S^1 \setminus \{1\} &\longrightarrow \mathbb{R}^3 \end{aligned}$$

of these curves under the stereographic projection. Determine and sketch σ_1, σ_2, σ_3 (for σ_2 it may be helpful to rotate the coordinate system, so that σ_2 lies in a coordinate plane.) Show that σ_1 and σ_2 are circles and σ_3 is a line. The circle σ_1 spans a flat disk in $\mathbb{R}^3$. Show that σ_2 intersects this disk transversely in one point. This means that σ_1, σ_2 and hence γ_1, γ_2 are *linked*.

Remark It is possible to show that all orbits of the Hopf action on S^3 are linked pairwise.

3.12.5 The aim of this exercise is to verify two propositions on fundamental vector fields in a special case with a direct calculation. The standard representation of the Lie group $\mathrm{SU}(2)$ on $\mathbb{C}^2$ induces a left-action

$$\mathrm{SU}(2) \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2.$$

We fix the vectors

$$\tau_a = -\frac{i\sigma_a}{2} \in \mathfrak{su}(2), \quad a = 1, 2, 3.$$

1. Determine the fundamental vector fields $\tilde{\tau}_a$ on $\mathbb{C}^2$ and show by direct calculation that

$$[\tilde{\tau}_a, \tilde{\tau}_b] = \widetilde{[\tau_a, \tau_b]} \quad \forall a, b \in \{1, 2, 3\},$$

   without using Proposition 3.4.4.
2. Let

$$A = \begin{pmatrix} r & -\bar{r} \\ r & \bar{r} \end{pmatrix} \in \mathrm{SU}(2), \quad r = \frac{1}{2} - \frac{1}{2}i.$$

   Calculate directly $l_{A*}(\tilde{\tau}_1)$ and compare with $\tilde{Z}$, where $Z = \mathrm{Ad}_A \tau_1$, without using Proposition 3.4.6.

3.12.6 (From [23]) Let G be a compact Lie group acting smoothly and freely on a manifold M. Let $\pi: M \to M/G$ be the canonical projection.

1. Prove that for every smooth curve $\gamma: I \to M/G$, defined on an interval I, there exists a smooth lift $\bar{\gamma}: I \to M$ with $\pi \circ \bar{\gamma} = \gamma$.
2. Suppose that M is connected and at least one of the orbits of G on M is connected (e.g. G is connected). Prove that π_* maps the fundamental group of M surjectively onto the fundamental group of M/G. In particular, if M is simply connected, then M/G is simply connected.

3.12.7 Let G and H be topological groups and M and N topological spaces. Suppose that G acts continuously on the right on M and H acts continuously on the right on N. Let $\phi: G \to H$ be a group homomorphism. Suppose that $f: M \to N$ is **ϕ-equivariant**, i.e.

$$f(p \cdot g) = f(p) \cdot \phi(g) \quad \forall p \in M, g \in G.$$

Prove the following:

1. If f is continuous, then f induces a continuous map $f_\phi: M/G \to N/H$.
2. If ϕ is an isomorphism and f a homeomorphism, then f_ϕ is a homeomorphism.

3.12.8 Use Exercise 3.12.7 to prove the following facts about lens spaces:

1. There exists a homeomorphism $L(p, q) \to L(p, -q)$.
2. If $qr \equiv 1 \bmod p$, then there exists a homeomorphism $L(p, q) \to L(p, r)$.

Remark According to a theorem of Reidemeister there exists a homeomorphism between lens spaces $L(p, q_1)$ and $L(p, q_2)$ only in these two cases, their combination, or in the trivial case $q_1 = q_2$.

3.12.9

1. Show that $\mathbb{CP}^1$ can be covered by two charts diffeomorphic to $\mathbb{C}$ and that $\mathbb{CP}^1$ is diffeomorphic to S^2.
2. Prove that $\mathbb{HP}^1$ is diffeomorphic to S^4.

3.12.10 Consider complex projective space $\mathbb{CP}^n = S^{2n+1}/S^1$. Show that there is a transitive left action of $\mathrm{SU}(n+1)$ on $\mathbb{CP}^n$ with isotropy group isomorphic to $\mathrm{U}(n)$. Deduce that there is a diffeomorphism

$$\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{U}(n).$$

3.12.11 Prove that there is a diffeomorphism $\mathbb{RP}^3 \cong \mathrm{SO}(3)$.

3.12.12 Show that for $k < n$ the real and complex Stiefel manifolds can be written as homogeneous spaces

$$V_k(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(n-k),$$
$$V_k(\mathbb{C}^n) = \mathrm{SU}(n)/\mathrm{SU}(n-k).$$

Deduce that for $k < n$ the real Stiefel manifolds $V_k(\mathbb{R}^n)$ are connected.

3.12.13 Consider the half-plane

$$H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

1. Show that the map

$$\mathrm{SL}(2,\mathbb{R}) \times H \longrightarrow H$$
$$(A, z) \longmapsto \frac{az+b}{cz+d},$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

is well-defined and defines a left-action of $\mathrm{SL}(2,\mathbb{R})$ on H.
2. Prove that this action is transitive and that the action defines a diffeomorphism between H and $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

3.12.14 (From [57]) According to Exercise 1.9.10 the group $\mathrm{SO}(2n)$ has a subgroup isomorphic to $\mathrm{U}(n)$. We would like to identify the homogeneous space $\mathrm{SO}(2n)/\mathrm{U}(n)$.

1. Let

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{Mat}(2n \times 2n, \mathbb{R}).$$

Show that the subgroup

$$H = \{A \in \mathrm{SO}(2n) \mid AJ_0 = J_0 A\}$$

of $\mathrm{SO}(2n)$ is isomorphic to $\mathrm{U}(n)$ (compare with Exercise 1.9.10).

2. Consider the set

$$\mathscr{J}^+\left(\mathbb{R}^{2n}\right) = \left\{J \in \mathrm{SO}(2n) \mid J^2 = -I_{2n}\right\}.$$

This is the set of *almost complex structures* on $\mathbb{R}^{2n}$, compatible with the scalar product and the orientation. The group $\mathrm{SO}(2n)$ acts on $\mathscr{J}^+\left(\mathbb{R}^{2n}\right)$ by conjugation

$$\begin{aligned} \mathrm{SO}(2n) \times \mathscr{J}^+\left(\mathbb{R}^{2n}\right) &\longrightarrow \mathscr{J}^+\left(\mathbb{R}^{2n}\right) \\ (A, J) &\longmapsto AJA^{-1}. \end{aligned}$$

Prove that this action is transitive.

3. Conclude that $\mathrm{SO}(2n)/\mathrm{U}(n) \cong \mathscr{J}^+\left(\mathbb{R}^{2n}\right)$.

Remark It can be shown that $\mathrm{SO}(4)/\mathrm{U}(2) \cong S^2$ and $\mathrm{SO}(6)/\mathrm{U}(3) \cong \mathbb{CP}^3$.

3.12.15 Let $V = \mathbb{R}^7$ with standard scalar product $\langle \cdot\,, \cdot \rangle$ and let $P\colon V \times V \to V$ denote the antisymmetric, bilinear G_2-equivariant map from Definition 3.10.8.

1. Let $x, y \in V$ be arbitrary vectors. Show that there exists an element $g \in \mathrm{G}_2$ such that (at the same time)

$$\begin{aligned} gx &= x_1 e_1, \\ gy &= y_1 e_1 + y_2 e_2, \end{aligned}$$

with real coefficients x_1, y_1, y_2.

2. Use the first part of this exercise to prove the identity

$$\langle P(x, y), P(x, y)\rangle = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2 \quad \forall x, y \in V.$$

3. Let

$$\mathbb{O} = \mathbb{R}e_0 \oplus V \cong \mathbb{R}^8$$

and define an $\mathbb{R}$-bilinear multiplication $\cdot$ on $\mathbb{O}$ by

$$\begin{aligned} e_0 \cdot e_0 &= e_0, \\ e_0 \cdot x = x &= x \cdot e_0, \\ x \cdot y &= -\langle x, y\rangle e_0 + P(x, y), \end{aligned}$$

for all $x, y \in V$. Let $(\cdot\,,\cdot)$ denote the scalar product on $\mathbb{O}$ so that $e_0, e_1, e_2, \ldots, e_7$ are orthonormal, with associated norm $||\cdot||$. Prove that

$$||z \cdot w||^2 = ||z||^2||w||^2 \quad \forall z, w \in \mathbb{O}.$$

Hence $\mathbb{O}$ is a real normed division algebra of dimension 8, known as the **octonions**.

4. Prove that

$$(gx) \cdot (gy) = g(x \cdot y) \quad \forall g \in \mathrm{G}_2, x, y \in V.$$

5. For

$$z = x_0 e_0 + x \in \mathbb{O}$$

with $x_0 \in \mathbb{R}$ and $x \in V$ define the conjugate

$$\bar{z} = x_0 e_0 - x.$$

Show that

$$z \cdot \bar{z} = \bar{z} \cdot z = ||z||^2 e_0.$$

This implies that every non-zero octonion has a multiplicative inverse.

6. Calculate $(e_1 \cdot e_2) \cdot e_4$ and $e_1 \cdot (e_2 \cdot e_4)$ and show that the octonions are not associative.

3.12.16 (From [27]) We continue with the notation from Exercise 3.12.15.

1. Use the first part of Exercise 3.12.15 to prove the identity

$$P(x, P(x, y)) = -\langle x, x\rangle y + \langle x, y\rangle x \quad \forall x, y \in V.$$

2. Let $x \in V$ be an arbitrary vector of norm 1 and V_x the orthogonal complement of $\mathbb{R}x$ in V. Then V_x is a real 6-dimensional vector subspace of V. Prove that multiplication of octonions defines a linear map

$$\begin{aligned} J_x \colon V_x &\longrightarrow V_x \\ v &\longmapsto x \cdot v \end{aligned}$$

with $J_x^2 = -\mathrm{Id}$, i.e. a complex structure on V_x.

3. Let S^6 be the unit sphere in V. Show that the restriction of the action of SO(7) on S^6 to the subgroup G_2 is transitive with isotropy group isomorphic to SU(3). Conclude that S^6 can be realized as a homogeneous space

$$S^6 \cong \mathrm{G}_2/\mathrm{SU}(3).$$

Remark Since the rank of the Lie group G_2 is 2, it does not contain Lie subgroups isomorphic to $\mathrm{SU}(n)$ for $n \geq 4$.

3.12.17 (From [73]) We continue with the notation from Exercise 3.12.15. Our aim is to show that G_2 contains a certain Lie subgroup isomorphic to SO(4).

1. Consider on $\mathrm{Im}\mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^7$ the representation of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ given by

$$(q_1, q_2) \cdot (a, b) = (\bar{q}_1 a q_1, q_1 b \bar{q}_2).$$

   Show that this representation defines an embedding of

$$\mathrm{SO}(4) \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$$

   into SO(7) (compare with Exercises 1.9.20 and 1.9.21).
2. Identify V with $\mathrm{Im}\mathbb{H} \oplus \mathbb{H}$ via the embeddings

$$i \mapsto e_1,\ j \mapsto e_2,\ k \mapsto e_3 \quad \text{on } \mathrm{Im}\mathbb{H}$$

   and

$$1 \mapsto e_4,\ i \mapsto e_5,\ j \mapsto e_6,\ k \mapsto e_7 \quad \text{on } \mathbb{H}.$$

   Prove that the embedding $\mathrm{SO}(4) \hookrightarrow \mathrm{SO}(7) = \mathrm{SO}(V)$ above has image in G_2 (for example, by showing that the Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ maps to the Lie algebra $\mathfrak{g}_2$ of G_2).

3.12.18 (From [73]) We continue with the notation from Exercises 3.12.15 and 3.12.17. A 3-dimensional oriented real vector subspace $U \subset V$ is called **associative** if the restriction $\phi|_U$ is positive, i.e. a volume form, where ϕ denotes the 3-form from the definition of the Lie group G_2. Let $G(\phi)$ denote the set of all associative subspaces of V.

1. Show that the action of G_2 on V induces an action of G_2 on $G(\phi)$.
2. Let $U \subset V$ be an associative subspace and $x, y \in U$ orthonormal. Prove that the vectors $x, y, x \cdot y$ span U. Show that the action of G_2 on $G(\phi)$ is transitive.
3. Show that the isotropy group H of $U_0 = \mathrm{span}(e_1, e_2, e_3) \in G(\phi)$ contains the subgroup $\mathrm{SO}(4) \subset \mathrm{G}_2$ from Exercise 3.12.17.
4. Let $h \in H$. Show that there exists an element $k \in \mathrm{SO}(4)$ such that

$$g = kh = (\mathrm{Id}, g_2) \in \mathrm{SO}(\mathrm{Im}\mathbb{H}) \times \mathrm{SO}(\mathbb{H})$$

with $g_2(1) = 1$. Show that for $q \in \mathrm{Im}\mathbb{H}$ we can write with multiplication of octonions

$$(0, q) = (q, 0) \cdot (0, 1) \in \mathrm{Im}\mathbb{H} \oplus \mathbb{H} = V.$$

Conclude that $g_2 = \mathrm{Id}$, hence $H = \mathrm{SO}(4)$ and

$$G(\phi) \cong \mathrm{G}_2/\mathrm{SO}(4).$$