Chapter 2
Lie Groups and Lie Algebras: Representations and Structure Theory

At least locally, fields in physics can be described by maps on spacetime with values in vector spaces. Since symmetry groups in field theories act on fields, it is important to understand (linear) actions of Lie groups and Lie algebras on vector spaces, known as *representations*.

For example, we shall see that, in the Standard Model, three *Dirac spinors* for each *quark flavour* are combined and form a vector in a representation space $\mathbb{C}^3$ of the gauge group SU(3) of quantum chromodynamics. Similarly, two *left-handed Weyl spinors*, known as the *left-handed electron* and the *left-handed electron neutrino*, are combined to form a vector in a representation space $\mathbb{C}^2$ of the gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$ of the electroweak interaction.

It turns out that every Lie group and Lie algebra has a special representation, known as the *adjoint representation*. The adjoint representation can be used to define the *Killing form*, a canonical symmetric bilinear form on every Lie algebra. Both the adjoint representation and the Killing form are important tools for the classification of Lie algebras. The adjoint representation is also important in physics, because *gauge bosons* correspond to fields on spacetime that transform under the adjoint representation of the gauge group.

The purpose of this chapter is to describe representations of Lie groups and Lie algebras in general as well as the structure of semisimple and compact Lie algebras. We also discuss special scalar products on Lie algebras which will be used in Sect. 7.3.1 to construct Lagrangians for gauge boson fields. We only cover the basics of the representation and structure theory of Lie groups and Lie algebras. Much more details can be found in the references mentioned at the beginning of Chap. 1, which are also the references for this chapter.

M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
https://doi.org/10.1007/978-3-319-68439-0_2

2.1 Representations

2.1.1 Basic Definitions

We begin with the basic concept of representations of Lie groups and Lie algebras.

Definition 2.1.1 Let G be a Lie group and V a vector space over the real or complex numbers. Then a **representation** of G on V is a Lie group homomorphism

$$\rho: G \longrightarrow \mathrm{GL}(V)$$

to the Lie group $\mathrm{GL}(V)$ of linear isomorphisms of V. One sometimes writes $\mathrm{GL}(V) = \mathrm{Aut}(V)$, the Lie group of linear automorphisms of V. The Lie group $\mathrm{GL}(V)$ is by definition isomorphic to a general linear group of the form $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and n is the dimension of V.

If the representation is clear from the context, we sometimes write

$$\rho(g)v = g \cdot v = gv$$

for $g \in G, v \in V$. A representation ρ of a Lie group G is called **faithful** if ρ is injective.
For a Lie group representation ρ the identities

$$\rho(gh) = \rho(g) \circ \rho(h)$$

and

$$\rho\left(g^{-1}\right) = \rho(g)^{-1}$$

hold for all $g, h \in G$. Note that the definition of a representation ρ requires that the map ρ is a homomorphism in the algebraic sense and differentiable (in fact, by Theorem 1.8.14 it suffices to demand that the map ρ is continuous).

Example 2.1.2 By Theorem 1.2.7 any compact Lie group has a faithful representation on some finite-dimensional, complex vector space.

Definition 2.1.3 Let ρ_V, ρ_W be representations of a Lie group G on vector spaces V and W. Then a **morphism** of the representations is a G-**equivariant** linear map $f: V \to W$, so that

$$f(\rho_V(g)v) = \rho_W(g)f(v),$$

i.e.

$$f(gv) = gf(v) \quad \forall g \in G, v \in V.$$

Such a map f is also called an **intertwining map**. An **isomorphism** or **equivalence** of representations is a G-equivariant isomorphism.

Definition 2.1.4 Let $\rho: G \to \mathrm{GL}(V)$ be a representation of a Lie group G. Suppose that $H \subset G$ is an embedded Lie subgroup. Then the restriction

$$\rho|_H: H \longrightarrow \mathrm{GL}(V)$$

of the Lie group homomorphism ρ to H is a representation of H, called a **restricted representation**.

We define representations of Lie algebras in a similar way.

Definition 2.1.5 Let $\mathfrak{g}$ be a (real or complex) Lie algebra and V a vector space over the real or complex numbers. Then a **representation** of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\phi: \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V)$$

to the linear endomorphisms of V (linear maps $V \to V$). If the representation is clear from the context, we sometimes write

$$\phi(X)v = X \cdot v = Xv$$

for $X \in \mathfrak{g}, v \in V$. A representation ϕ of a Lie algebra $\mathfrak{g}$ is called **faithful** if ϕ is injective.

For a Lie algebra representation the following identity holds:

$$\phi([X, Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X) \quad \forall X, Y \in \mathfrak{g}.$$

Example 2.1.6 By Ado's Theorem 1.5.25 any Lie algebra has a faithful representation on some finite-dimensional vector space.

Remark 2.1.7 Note that if the Lie algebra is complex, then we require the representation $\phi: \mathfrak{g} \to \mathrm{End}(V)$ to be a complex linear map.

Definition 2.1.8 Let ϕ_V, ϕ_W be representations of a Lie algebra $\mathfrak{g}$ on vector spaces V and W. Then a **morphism** of the representations is a $\mathfrak{g}$-**equivariant** linear map $f: V \to W$, so that

$$f(\phi_V(X)v) = \phi_W(X)f(v),$$

i.e.

$$f(Xv) = Xf(v) \quad \forall X \in \mathfrak{g}, v \in V.$$

Such a map f is also called an **intertwining map**. An **isomorphism** or **equivalence** of representations is a $\mathfrak{g}$-equivariant isomorphism.

Definition 2.1.9 Let $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Then the restriction

$$\phi|_{\mathfrak{h}}\colon \mathfrak{h} \longrightarrow \mathrm{End}(V)$$

of the Lie algebra homomorphism ϕ to $\mathfrak{h}$ is a representation of $\mathfrak{h}$, called **restricted representation**.

Remark 2.1.10 Unless stated otherwise we only consider representations of Lie groups and Lie algebras on real and complex vector spaces and these vector spaces are finite-dimensional.

Remark 2.1.11 Both types of homomorphisms are called *representations*, because we represent elements in the Lie group or Lie algebra by linear maps on a vector space, i.e. (after a choice of basis for the vector space) by matrices.

Representations of Lie groups and their associated Lie algebras are related:

Proposition 2.1.12 (Induced Representations) *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a Lie group G on a vector space V. Then the differential $\rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$ is a representation of the Lie algebra $\mathfrak{g}$.*

Proof The proof follows from Theorem 1.5.18, because the differential of a Lie group homomorphism is a Lie algebra homomorphism. □

With Theorem 1.7.16 we get the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho_*} & \mathrm{End}(V) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & \mathrm{GL}(V) \end{array}$$

Note that the exponential map on the right is just the standard exponential map on endomorphisms (defined in the same way as for matrices, using composition instead of matrix multiplication). We can thus write the commutativity of the diagram as

$$\rho(\exp X) = e^{\rho_* X} \quad \forall X \in \mathfrak{g}.$$

This means: if we know how a Lie algebra element $X \in \mathfrak{g}$ acts in a representation on the vector space V, then we know how the group element $\exp X \in G$ acts on V.

Assuming Theorem 1.5.20 we get the following:

Corollary 2.1.13 (Integrability Theorem for Representations) *Let G be a connected and simply connected Lie group. Suppose $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ is a representation of the Lie algebra of G. Then there exists a unique representation $\rho\colon G \to \mathrm{GL}(V)$ such that $\rho_* = \phi$.*

The discussion in Example 1.5.21 shows that this may not hold if G is not simply connected. In particular, if

$$\phi\colon \mathfrak{so}(2) \cong \mathfrak{u}(1) \longrightarrow \mathrm{End}(V)$$

is a representation and X the generator of $\mathfrak{u}(1)$ with $\exp(2\pi iX) = 1$, then a necessary condition that ϕ comes from a representation

$$\rho\colon \mathrm{U}(1) \longrightarrow \mathrm{GL}(V)$$

is that

$$e^{2\pi i\phi(X)} = \mathrm{Id}_V.$$

Example 2.1.14 For any constant $k \in \mathbb{Z}$ there is a complex 1-dimensional representation

$$\begin{aligned} \rho_k\colon \mathrm{U}(1) &\longrightarrow \mathrm{U}(1) \subset \mathrm{GL}(\mathbb{C}) \\ z &\longmapsto z^k. \end{aligned}$$

We say that these representations have **winding number** k. In the Standard Model these representations appear in connection with the *weak hypercharge gauge group* $\mathrm{U}(1)_Y$.

Example 2.1.15 The Lie groups $\mathrm{GL}(n, \mathbb{R})$ (and $\mathrm{GL}(n, \mathbb{C})$) have canonical representations on $\mathbb{R}^n$ (and $\mathbb{C}^n$) by matrix multiplication on column vectors from the left. These representations induce representations for all linear groups, called **standard**, **defining** or **fundamental representations** (by a fundamental representation we will always mean the defining representation). There are similar, induced representations of the corresponding Lie algebras.

Definition 2.1.16 A representation of a Lie group G (or Lie algebra $\mathfrak{g}$) on a vector space V is called **irreducible** if there is **no proper invariant** subspace $W \subset V$, i.e. no vector subspace W, different from 0 or V, such that $G\cdot W \subset W$ (or $\mathfrak{g}\cdot W \subset W$). A representation is called **reducible** if it is not irreducible.

Example 2.1.17 The 0-dimensional and every 1-dimensional representation are irreducible, because in these cases there are no proper vector subspaces at all.

Definition 2.1.18 A **singlet representation** is a representation of a Lie group or Lie algebra on a 1-dimensional (real or complex) vector space. Similarly, a **doublet** or **triplet representation** is a representation on a 2- or 3-dimensional vector space. A representation of a Lie group or Lie algebra on an n-dimensional vector space is sometimes denoted by **n**, in particular, if the dimension uniquely determines the representation.

Example 2.1.19 (Trivial Representations) Let G be a Lie group and V a real or complex vector space. Then

$$\begin{aligned}\rho\colon G &\longrightarrow \mathrm{GL}(V)\\ g &\longmapsto \mathrm{Id}_V,\end{aligned}$$

where every group element gets mapped to the identity, is a representation, called a **trivial representation**. It is irreducible precisely if V is 1-dimensional. Similarly, if $\mathfrak{g}$ is a Lie algebra, then

$$\begin{aligned}\phi\colon \mathfrak{g} &\longrightarrow \mathrm{End}(V)\\ g &\longmapsto 0\end{aligned}$$

is a trivial representation. Again, it is irreducible precisely if V is 1-dimensional. We will later study a class of Lie algebras where *every* representation is either trivial or faithful, see Exercise 2.7.9.

It is a curious fact that the fundamental and trivial representations of SU(3) and SU(2), together with the winding number representations of U(1) in Example 2.1.14, suffice to describe *all* matter particles (and the Higgs field) in the Standard Model; see Sect. 8.5. The gauge bosons corresponding to these gauge groups are described by the *adjoint representation* that we discuss in Sect. 2.1.5.

Example 2.1.20 The fundamental representation of the Lie algebra $\mathfrak{su}(2)$ is an (irreducible) doublet representation on the vector space $\mathbb{C}^2$. Recall from Example 1.5.32 that there exists an isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. The fundamental representation of $\mathfrak{so}(3)$ thus also defines an (irreducible) triplet representation of $\mathfrak{su}(2)$ on $\mathbb{R}^3$ (and $\mathbb{C}^3$). It can be proved that $\mathfrak{su}(2)$ has a unique (up to equivalence) irreducible complex representation V_n of dimension $n+1$ for every natural number $n \geq 0$ (see, e.g. [24]).

Example 2.1.21 (The Heisenberg Lie Algebra and Quantum Mechanics) Recall from Example 1.5.38 that the Heisenberg Lie algebra $\mathfrak{nil}_3$ is a 3-dimensional real Lie algebra spanned by vectors p, q, z with Lie brackets

$$\begin{aligned}[q,p] &= z,\\ [q,z] &= 0,\\ [p,z] &= 0.\end{aligned}$$

Let $\hbar \in \mathbb{R}$ be some real number. A **central representation** of $\mathfrak{nil}_3$ is a representation

$$\mathfrak{nil}_3 \longrightarrow \mathrm{End}(V)$$

on a complex vector space V such that z gets mapped to $i\hbar \cdot \mathrm{Id}_V$. If we denote the images of q and p in $\mathrm{End}(V)$ by $\hat{q}$ and $\hat{p}$, then

$$[\hat{q}, \hat{p}] = i\hbar$$

and the other two commutation relations are satisfied trivially (on the right-hand side we do not write the identity map of V explicitly). This is the **canonical commutation relation** of quantum mechanics.

2.1.2 Linear Algebra Constructions of Representations

There are several well-known constructions that yield new vector spaces from given ones. If the given vector spaces carry a representation, then usually the new vector spaces carry induced representations. We first recall the following notion from complex linear algebra.

Definition 2.1.22 Let V be a complex vector space. Then we define the **complex conjugate** vector space $\bar{V}$ as follows:

1. As a set and abelian group $\bar{V} = V$.
2. Scalar multiplication is defined by

$$\begin{aligned} \mathbb{C} \times \bar{V} &\longrightarrow \bar{V} \\ (\lambda, v) &\longmapsto \bar{\lambda} v. \end{aligned}$$

If $f: V \to V$ is a complex linear map, then the same map (on the set $\bar{V} = V$) is denoted by $\bar{f}: \bar{V} \to \bar{V}$ and is still complex linear. The identity map $V \to \bar{V}$ is complex antilinear.

Definition 2.1.23 Let V and W be real or complex vector spaces with representations

$$\begin{aligned} \rho_V: G &\longrightarrow \mathrm{GL}(V) \\ \rho_W: G &\longrightarrow \mathrm{GL}(W) \end{aligned}$$

of a Lie group G. Then there exist the following representations of G, where $g \in G$ and $v \in V, w \in W$ are arbitrary:

1. The **direct sum** representation $\rho_{V \oplus W}$ on $V \oplus W$, defined by

$$g(v, w) = (gv, gw).$$

2. The **tensor product** representation $\rho_{V \otimes W}$ on $V \otimes W$, defined by

$$g(v \otimes w) = gv \otimes gw.$$

3. The **dual** representation ρ_{V^*} on V^*, defined by

$$(g\lambda)(v) = \lambda\left(g^{-1}v\right), \quad \forall \lambda \in V^*.$$

4. The **exterior power** representation $\rho_{\Lambda^k V}$ on $\Lambda^k V$, defined by

$$g(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = gv_1 \wedge gv_2 \wedge \ldots \wedge gv_k, \quad \forall v_1 \wedge v_2 \wedge \ldots \wedge v_k \in \Lambda^k V.$$

5. The **homomorphism space** representation $\rho_{\mathrm{Hom}(V,W)}$ on $\mathrm{Hom}(V,W)$, defined by

$$(gf)(v) = gf\left(g^{-1}v\right), \quad \forall f \in \mathrm{Hom}(V,W).$$

6. If V is a complex vector space, then the **complex conjugate** representation $\rho_{\bar{V}}$ on $\bar{V}$ is defined by

$$\rho_{\bar{V}}(g)v = \overline{\rho_V(g)}v.$$

Suppose in addition that

$$\tau_W : H \longrightarrow \mathrm{GL}(W)$$

is a representation of a Lie group H. Then there exists the following representation, where $h \in H$ is arbitrary:

7. The **(outer) tensor product** representation $\rho_V \otimes \tau_W$ on $V \otimes W$ of the Lie group $G \times H$, defined by

$$(g,h)(v \otimes w) = gv \otimes hw,$$

   for $g \in G, h \in H$.

It is easy to check that each of these maps is indeed a representation.

Remark 2.1.24 The direct sum representation $\rho_V \oplus \tau_W$ on $V \oplus W$ of the Lie group $G \times H$, defined by

$$(g,h)(v,w) = (gv, hw),$$

is less important, because it is can be reduced to the representations ρ_V and τ_W, each tensored with the trivial 1-dimensional representation.

Remark 2.1.25 If V is a complex representation space for a Lie group, we then get in total four complex representations which have the same dimension as V: V, V^*, $\bar{V}$ and $\bar{V}^*$.

The representations of the Lie group

$$G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

that appear in the Standard Model of elementary particles are direct sums of outer tensor product representations of the form

$$U \otimes V \otimes W,$$

where U, V, W are certain representations of the factors SU(3), SU(2), U(1) of G. See Sect. 8.5 for details.

Example 2.1.26 We describe these constructions using matrices. Consider the column vector spaces $V = \mathbb{K}^n$, $W = \mathbb{K}^m$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Representations ρ_V and ρ_W of a Lie group G take values in the matrix Lie groups $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{GL}(m, \mathbb{K})$. We can then identify the canonical representations of G on the vector spaces

$$V \oplus W, \quad V^*, \quad \mathrm{Hom}(V, W), \quad \Lambda^2 V^* \quad \text{and} \quad \bar{V} \text{ (if } V \text{ is complex)}$$

with the following representations:

1. $V \oplus W$ can be identified with $\mathbb{K}^{n+m}$. For a column vector $(x, y)^T \in \mathbb{K}^{n+m}$ the direct sum representation is given by
$$\rho_{V\oplus W}(g) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
2. V^* can be identified with a row vector space that we here denote by $(\mathbb{K}^n)^*$. For a row vector $s \in (\mathbb{K}^n)^*$ the dual representation is given by
$$\rho_{V^*}(g)s = s \cdot \rho_V(g)^{-1}.$$
3. $\mathrm{Hom}(V, W)$ can be identified with the vector space $\mathrm{Mat}(m \times n, \mathbb{K})$. For a matrix $A \in \mathrm{Mat}(m \times n, \mathbb{K})$ the representation on the homomorphism space is given by
$$\rho_{\mathrm{Hom}(V,W)}(g)A = \rho_W(g) \cdot A \cdot \rho_V(g)^{-1}.$$
4. $\Lambda^2 V^*$ is the space of skew-symmetric, bilinear maps
$$\lambda \colon V \times V \longrightarrow \mathbb{K}$$

and can be identified with $\mathfrak{so}(n, \mathbb{K})$, the space of skew-symmetric $n \times n$-matrices, by sending λ to the matrix A with coefficients $A_{ij} = \lambda(e_i, e_j)$. The representation on $\Lambda^2 V^*$ is then given by

$$\rho_{\Lambda^2 V^*}(g)A = \left(\rho_V(g)^{-1}\right)^T \cdot A \cdot \rho_V(g)^{-1}.$$

5. If $V = \mathbb{C}^n$, then $\bar{V} = \mathbb{C}^n$ as an abelian group and every complex scalar (and hence every complex matrix) acts as the complex conjugate. For a column vector $z \in \mathbb{C}^n$ the complex conjugate representation is given by

$$\rho_{\bar{V}}(g)z = \overline{\rho_V(g)} \cdot z.$$

There are analogous constructions for representations of Lie algebras:

Definition 2.1.27 Let V and W be real or complex vector spaces with representations

$$\phi_V \colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$$
$$\phi_W \colon \mathfrak{g} \longrightarrow \mathrm{End}(W)$$

of a Lie algebra $\mathfrak{g}$. Then there exist the following representations of $\mathfrak{g}$, where $X \in \mathfrak{g}$ and $v \in V, w \in W$ are arbitrary:

1. The **direct sum** representation $\phi_{V \oplus W}$ on $V \oplus W$, defined by

$$X(v, w) = (Xv, Xw).$$

2. The **tensor product** representation $\phi_{V \otimes W}$ on $V \otimes W$, defined by

$$X(v \otimes w) = (Xv) \otimes w + v \otimes (Xw).$$

3. The **dual** representation ϕ_{V^*} on V^*, defined by

$$(X\lambda)(v) = \lambda(-Xv), \quad \forall \lambda \in V^*.$$

4. The **exterior power** representation $\phi_{\Lambda^k V}$ on $\Lambda^k V$, defined by

$$X(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \ldots \wedge Xv_i \wedge \ldots \wedge v_k, \quad \forall v_1 \wedge v_2 \wedge \ldots \wedge v_k \in \Lambda^k V.$$

5. The **homomorphism space** representation $\phi_{\mathrm{Hom}(V,W)}$ on $\mathrm{Hom}(V, W)$, defined by

$$(Xf)(v) = Xf(v) + f(-Xv), \quad \forall f \in \mathrm{Hom}(V, W).$$

6. If V is a complex vector space and $\mathfrak{g}$ a real Lie algebra, then the **complex conjugate** representation $\phi_{\bar{V}}$ on $\bar{V}$ is defined by

$$\phi_{\bar{V}}(X)v = \overline{\phi_V(X)}v.$$

Suppose in addition that

$$\psi_W : \mathfrak{h} \longrightarrow \mathrm{End}(W)$$

is a representation of a Lie algebra $\mathfrak{h}$. Then there exists the following representation, where $Y \in \mathfrak{h}$ is arbitrary:

7. The **(outer) tensor product** representation $\phi_V \otimes \psi_W$ on $V \otimes W$ of the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$, defined by

$$(X, Y)(v \otimes w) = Xv \otimes w + v \otimes Yw,$$

for $X \in \mathfrak{g}, Y \in \mathfrak{h}$.

Remark 2.1.28 Perhaps the most interesting case in the proof that these maps define representations is the dual representation for both Lie groups and Lie algebras. To check that the formulas here define representations is the purpose of Exercise 2.7.1.

Both constructions are related:

Proposition 2.1.29 *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let ρ be any of the representations of G on $V \oplus W$, $V \otimes W$, V^*, $\Lambda^k V$,* Hom(V, W) *or $\bar{V}$ (or of $G \times H$ on $V \otimes W$) from Definition 2.1.23. Then the induced representation ρ_* of $\mathfrak{g}$ (or of $\mathfrak{g} \oplus \mathfrak{h}$) is the corresponding one from Definition 2.1.27.*

Proof The proof follows by differentiating the representation of G (or of $G \times H$). □

*2.1.3 *The Weyl Spinor Representations of* SL(2, ℂ)

We discuss an extended example that is relevant for some theories in physics, like the Standard Model or supersymmetry (see reference [146, Appendix A]). Let $G = \mathrm{SL}(2, \mathbb{C})$. As we will discuss in Sect. 6.8.2 in more detail, the group $\mathrm{SL}(2, \mathbb{C})$ is the *(orthochronous) Lorentz spin group*, i.e. the universal covering of the identity component of the Lorentz group of 4-dimensional spacetime.

We denote by $V = \mathbb{C}^2$ the fundamental $\mathrm{SL}(2, \mathbb{C})$-representation. Then we get the following four complex doublet representations, where $M \in \mathrm{SL}(2, \mathbb{C})$ and $\psi \in \mathbb{C}^2$:

1. The fundamental representation V:

$$\psi \longmapsto M\psi.$$

2. The dual representation V^*:

$$\psi^T \longmapsto \psi^T M^{-1}.$$

3. The complex conjugate representation $\bar{V}$:

$$\bar{\psi} \longmapsto \bar{M}\bar{\psi}.$$

4. The dual of the complex conjugate representation $\bar{V}^*$:

$$\bar{\psi}^T \longmapsto \bar{\psi}^T(\bar{M})^{-1}.$$

Here we denote the elements of the vector spaces V^*, $\bar{V}$ and $\bar{V}^*$ for clarity by ψ^T, $\bar{\psi}$ and $\bar{\psi}^T$.

Remark 2.1.30 In physics the components of the vectors in the spaces V, V^*, $\bar{V}$ and $\bar{V}^*$ are denoted by ψ_α, ψ^α, $\bar{\psi}_{\dot{\alpha}}$ and $\bar{\psi}^{\dot{\alpha}}$. We could denote these representations by $\mathbf{2}$, $\mathbf{2}^*$, $\bar{\mathbf{2}}$ and $\bar{\mathbf{2}}^*$.

Definition 2.1.31 In this situation the representation of $\mathrm{SL}(2,\mathbb{C})$ on V is called the **left-handed Weyl spinor** representation and the representation on $\bar{V}^*$ is called the **right-handed Weyl spinor** representation. Both representations are also called **chiral spinor representations**.
We want to show that the remaining two representations are isomorphic to the left- and right-handed Weyl spinor representations.

Definition 2.1.32 We define

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition 2.1.33 *We have the following equivalent description of* $\mathrm{SL}(2,\mathbb{C})$*:*

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ M \in \mathrm{Mat}(2\times 2,\mathbb{C}) \mid M^T\epsilon M = \epsilon \right\}.$$

Proof The proof is an easy calculation; see Exercise 2.7.2. □

Proposition 2.1.34 *The map*

$$\begin{aligned} f\colon V &\longrightarrow V^* \\ \psi &\longmapsto \psi^T\epsilon \end{aligned}$$

is an isomorphism of representations. Similarly the map

$$\begin{aligned} \bar{f}\colon \bar{V} &\longrightarrow \bar{V}^* \\ \bar{\psi} &\longmapsto \bar{\psi}^T\epsilon \end{aligned}$$

is an isomorphism of representations.

Proof We only have to show SL(2, ℂ)-equivariance of the maps. This follows by applying Proposition 2.1.33:

$$\begin{aligned} f(M\psi) &= (M\psi)^T \epsilon \\ &= \psi^T M^T \epsilon \\ &= (\psi^T \epsilon) M^{-1} \\ &= f(\psi) M^{-1} \end{aligned}$$

and

$$\begin{aligned} \bar{f}(\bar{M}\bar{\psi}) &= (\bar{M}\bar{\psi})^T \epsilon \\ &= \bar{\psi}^T \bar{M}^T \epsilon \\ &= (\bar{\psi}^T \epsilon) \bar{M}^{-1} \\ &= \bar{f}(\bar{\psi}) \bar{M}^{-1}. \end{aligned}$$

□

See Sect. 6.8 and Lemma 8.5.5 for more details about these isomorphisms.

2.1.4 Orthogonal and Unitary Representations

It is often useful to consider representations compatible with a scalar product on the vector space. Recall that a scalar product on a real vector space is called **Euclidean** if it is bilinear, symmetric and positive definite. A scalar product on a complex vector space is called **Hermitian** if it is sesquilinear (complex linear in the second argument and complex antilinear in the first argument), conjugate symmetric (exchanging the first and second argument changes the scalar product by complex conjugation) and positive definite.

Definition 2.1.35 A representation $\rho\colon G \to \mathrm{GL}(V)$ of a Lie group G on a Euclidean (or Hermitian) vector space $(V, \langle \cdot\,, \cdot \rangle)$ is called **orthogonal** (or **unitary**) if the scalar product is G-invariant, i.e.

$$\langle gv, gw \rangle = \langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle,$$

for all $g \in G$, $v, w \in V$. Equivalently, the map ρ has image in the orthogonal subgroup $\mathrm{O}(V)$ (or the unitary subgroup $\mathrm{U}(V)$) of the general linear group $\mathrm{GL}(V)$, determined by the scalar product $\langle \cdot\,, \cdot \rangle$.

In an orthogonal representation the group literally acts through rotations (and possibly reflections) on a Euclidean vector space. There is a similar notion for representations of Lie algebras.

Definition 2.1.36 A representation $\phi\colon \mathfrak{g} \rightarrow \mathrm{End}(V)$ of a real Lie algebra $\mathfrak{g}$ on a Euclidean (or Hermitian) vector space $(V, \langle \cdot\,,\cdot \rangle)$ is called **skew-symmetric** (or **skew-Hermitian**) if it satisfies

$$\langle Xv, w\rangle + \langle v, Xw\rangle = \langle \phi(X)v, w\rangle + \langle v, \phi(X)w\rangle = 0,$$

for all $X \in \mathfrak{g}$, $v, w \in V$. Equivalently, the map ϕ has image in the orthogonal Lie subalgebra $\mathfrak{o}(V)$ (or the unitary Lie subalgebra $\mathfrak{u}(V)$) of the general linear algebra $\mathfrak{gl}(V)$, determined by the scalar product $\langle \cdot\,,\cdot \rangle$.
We can similarly define invariance of a form on a vector space under representations of a Lie group or Lie algebra in the case where the form is not non-degenerate or not positive definite.

Invariant scalar products for Lie group and Lie algebra representations are related:

Proposition 2.1.37 (Scalar Products and Induced Representations) *Let $\rho\colon G \rightarrow \mathrm{GL}(V)$ be a representation of a Lie group G and $\langle \cdot\,,\cdot \rangle$ a G-invariant Euclidean (or Hermitian) scalar product on V, i.e. the representation ρ is orthogonal (or unitary). Then the induced representation $\rho_*\colon \mathfrak{g} \rightarrow \mathrm{End}(V)$ of the Lie algebra $\mathfrak{g}$ is skew-symmetric (or skew-Hermitian).*

Proof We have by Theorem 1.7.16

$$\rho(\exp tX) = \exp(t\rho_* X)$$

and hence by Corollary 1.7.30

$$\begin{aligned}\langle v, w\rangle &= \langle \rho(\exp tX)v, \rho(\exp tX)w\rangle \\ &= \langle \exp(t\rho_* X)v, \exp(t\rho_* X)w\rangle \\ &= \left\langle e^{t\rho_* X}v, e^{t\rho_* X}w\right\rangle \quad \forall t \in \mathbb{R}.\end{aligned}$$

Differentiating both sides by t in $t = 0$ and using the product rule we get:

$$0 = \langle (\rho_* X)v, w\rangle + \langle v, (\rho_* X)w\rangle.$$

This implies the claim. □
Let $\phi\colon \mathfrak{g} \rightarrow \mathrm{End}(V)$ be a unitary representation of a real Lie algebra $\mathfrak{g}$ on a complex vector space V. Then $\phi(X)$ is a skew-Hermitian endomorphism for all $X \in \mathfrak{g}$, hence $i\phi(X)$ is Hermitian. This implies that the endomorphism $i\phi(X)$ can be diagonalized with real eigenvalues (and $\phi(X)$ can be diagonalized with imaginary eigenvalues).

Definition 2.1.38 The eigenvalues of $-i\phi(X)$ are called **charges** of $X \in \mathfrak{g}$ in the unitary representation ϕ.
The minus sign in $-i\phi(X)$ is convention: we can write $\phi(X)$ as iA_X, where A_X is a Hermitian operator, and the charges are the eigenvalues of A_X.

If $\mathfrak{h} \subset \mathfrak{g}$ is an abelian subalgebra, then the operators $i\phi(X)$ for all $X \in \mathfrak{h}$ commute and can be diagonalized simultaneously. This idea is related to the notion of *weights* of a representation and used extensively in the classification of representations of Lie algebras and Lie groups (in a certain sense, that can be made precise, irreducible representations are thus determined by their charges).

Existence of Invariant Scalar Products

It is an important fact that representations of *compact* Lie groups always admit *an invariant scalar product*.

Theorem 2.1.39 (Existence of Invariant Scalar Products for Representations of Compact Lie Groups) *Let G be a compact Lie group and $\rho: G \to \mathrm{GL}(V)$ a representation on a real (or complex) vector space. Then we can find a G-invariant Euclidean (or Hermitian) scalar product on V, hence the given representation ρ becomes orthogonal (or unitary) for this scalar product.*

The proof uses the existence of an **integral** over differential forms σ of top degree n on oriented n-manifolds M:

$$\int_M \sigma \in \mathbb{R}, \quad \sigma \in \Omega^n(M).$$

If $\phi: M \to N$ is an orientation preserving diffeomorphism between oriented n-manifolds, then we have the transformation formula

$$\int_N \sigma = \int_M \phi^* \sigma \quad \forall \sigma \in \Omega^n(N).$$

We now prove Theorem 2.1.39.

Proof Suppose G has dimension n and let $X_1, \ldots, X_n$ be a basis of T_eG. We set $\tilde{X}_i$ for the corresponding *right-invariant* vector fields on G, defined by

$$\tilde{X}_i(p) = D_eR_p(X_i) \quad \forall p \in G.$$

This basis has a dual basis of right-invariant 1-forms $\omega^1, \ldots, \omega^n$. Then the wedge product

$$\sigma = \omega^1 \wedge \cdots \wedge \omega^n$$

is a nowhere vanishing, right-invariant differential form on G of top degree. We can assume that the orientation of G coincides with the orientation defined by σ, so that

$$\int_G \sigma > 0,$$

which is finite, because G is compact. Let $\langle\langle\cdot\,,\cdot\rangle\rangle$ denote an arbitrary Euclidean (or Hermitian) scalar product on V. We construct a new scalar product by *averaging* this scalar product over the action of the group G:

$$\langle v, w\rangle = \int_G \tau_{v,w}\sigma,$$

where $\tau_{v,w}$ is the smooth function

$$\begin{aligned} \tau_{v,w}\colon G &\longrightarrow \mathbb{R} \\ h &\longmapsto \langle\langle hv, hw\rangle\rangle \end{aligned}$$

(here the representation ρ is implicit and we use that G is compact, so that this integral is finite).

We claim that $\langle\cdot\,,\cdot\rangle$ is a G-invariant Euclidean (or Hermitian) scalar product on V: It is clear that $\langle\cdot\,,\cdot\rangle$ is bilinear and symmetric (or sesquilinear and conjugate symmetric in the complex case). For $v \neq 0$ the function $\tau_{v,v}$ is strictly positive on G. As a consequence the integral is

$$\langle v, v\rangle \geq 0 \quad \forall v \in V$$

with equality only if $v = 0$. Therefore $\langle\cdot\,,\cdot\rangle$ is a positive definite Euclidean (or Hermitian) scalar product on G.

We finally show G-invariance of the new scalar product: Let $g \in G$ be fixed. Then

$$R^*_{g^{-1}}\tau_{gv,gw} = \tau_{v,w} \quad \forall v, w \in V.$$

This follows from a short calculation:

$$\begin{aligned} (R^*_{g^{-1}}\tau_{gv,gw})(h) &= \tau_{gv,gw}(hg^{-1}) \\ &= \langle\langle hg^{-1}(gv), hg^{-1}(gw)\rangle\rangle \\ &= \tau_{v,w}(h), \end{aligned}$$

where we used that ρ (which is implicit) is a representation. This implies

$$R^*_{g^{-1}}(\tau_{gv,gw}\sigma) = \tau_{v,w}\sigma,$$

because σ is right-invariant. Since $R_{g^{-1}}$ is an orientation preserving diffeomorphism from G to G we get:

$$\begin{aligned}\langle gv, gw\rangle &= \int_G \tau_{gv,gw}\sigma\\ &= \int_G R^*_{g^{-1}}(\tau_{gv,gw}\sigma)\\ &= \int_G \tau_{v,w}\sigma\\ &= \langle v, w\rangle\end{aligned}$$

for all $g \in G$ and $v, w \in V$. □

Decomposition of Representations

The existence of an invariant scalar product for every representation of a compact Lie group has an important consequence.

Theorem 2.1.40 (Decomposition of Representations) *Let $\rho: G \to \mathrm{GL}(V)$ be a representation of a Lie group G on a finite-dimensional real (or complex) vector space V. Suppose that there exists a G-invariant Euclidean (or Hermitian) scalar product on V (this is always the case, by Theorem 2.1.39, if G is compact). Then V decomposes as a direct sum*

$$(V, \rho) = (V_1, \rho_1) \oplus \ldots \oplus (V_m, \rho_m)$$

of irreducible G-representations (V_i, ρ_i).

Proof The proof follows, because if $W \subset V$ is a subspace with $\rho(G)W \subset W$, then the orthogonal complement $W^\perp$ with respect to a G-invariant scalar product also satisfies $\rho(G)W^\perp \subset W^\perp$. We have

$$(V, \rho) = (W, \rho_W) \oplus (W^\perp, \rho_{W^\perp}).$$

We can thus continue splitting V until we arrive at irreducible representations (after finitely many steps, since V is finite-dimensional). □

Remark 2.1.41 One of the aims of representation theory for Lie groups G is to understand irreducible representations and to decompose any given representation (at least for compact G) into irreducible ones according to Theorem 2.1.40.

For instance, for $G = \mathrm{SU}(2)$, we can consider the tensor product representation $V_n \otimes V_m$, where V_n, V_m are the irreducible complex representations of dimension $n+1$ and $m+1$ mentioned in Example 2.1.20. The tensor product $V_n \otimes V_m$ is reducible under SU(2) and its decomposition into irreducible summands V_k is determined by the *Clebsch–Gordan formula*. This formula appears in quantum mechanics in the theory of the angular momentum of composite systems.

Remark 2.1.42 One of the basic topics in *Grand Unified Theories* is to study the restriction of representations of a compact Lie group G to embedded Lie subgroups $H \subset G$. If the representation ρ of G is irreducible, it may happen that the representation $\rho|_H$ of H is reducible and decomposes as a direct sum. The actual form of the decomposition of a representation ρ under restriction to a subgroup $H \subset G$ is called the **branching rule**.

For instance, there exist certain 5- and 10-dimensional irreducible representations of the Grand Unification group $G = \mathrm{SU}(5)$ that decompose under restriction to the subgroup $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (more precisely, to a certain $\mathbb{Z}_6$ quotient of this group; see Sect. 8.5.7) into the fermion representations of the Standard Model. Details of this calculation can be found in Sect. 9.5.4.

Remark 2.1.43 Suppose a Lie group G has a unitary representation on a complex vector space V and $e_1, \ldots, e_n$ is some orthonormal basis for V. If we decompose V into invariant, irreducible subspaces according to Theorem 2.1.40, then we can choose an associated orthonormal basis $f_1, \ldots, f_n$, adapted to the decomposition of V (spanning the G-invariant subspaces) and related to the original basis by a unitary matrix. In general, the basis $\{f_i\}$ will be different from $\{e_i\}$.

In the Standard Model where $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ this is related to the concept of *quark mixing*. The complex vector space V of fermions, which carries a representation of G, has dimension 45 (plus the same number of corresponding antiparticles) and is the direct sum of two G-invariant subspaces (sectors): a lepton sector of dimension 9 (where we do not include the hypothetical right-handed neutrinos) and a quark sector of dimension 36. Counting in this way, the Standard Model thus contains at the most elementary level 90 fermions (particles and antiparticles).

The quark sector has a natural basis of so-called *mass eigenstates*, given by the quarks of six different flavours u, d, c, s, t, b, each one appearing in three different colours and two chiralities (6 basis vectors for each flavour), yielding in total 36 quarks. However, the basis given by these flavours does not define a splitting into subspaces invariant under SU(2). The SU(2)-invariant subspaces are spanned by a basis of so-called *weak eigenstates* that can be obtained from the mass eigenstates by a certain unitary transformation. The matrix of this unitary transformation is known as the *Cabibbo–Kobayashi–Maskawa (CKM) matrix*, which has to be determined by experiments. The CKM matrix and quark mixing will be explained in more detail in Sect. 8.8.2.

Unitary Representations of Non-Compact Lie Groups

It is an important fact that certain *non-compact* Lie groups do not admit non-trivial *finite-dimensional* unitary representations according to the following theorem (a proof can be found in [12, Chap. 8.1B]):

Theorem 2.1.44 *A connected, simple, non-compact Lie group does not admit finite-dimensional unitary complex representations except for the trivial representation.*
See Definition 2.4.27 for the notion of *simple* Lie groups. For example, the Lie group $G = \mathrm{SL}(2, \mathbb{C})$ is simple and non-compact, hence every non-trivial unitary representation of G is infinite-dimensional. This has important consequences for quantum field theory, see Sect. B.2.4. Of course, $\mathrm{SL}(2, \mathbb{C})$ admits non-trivial finite-dimensional *non-unitary* representations, like the fundamental representation on $\mathbb{C}^2$.

2.1.5 The Adjoint Representation

We want to define a particularly important representation of a Lie group and its Lie algebra. The vector space carrying the representation has the same dimension as the Lie group or Lie algebra (we follow [142] in this subsection).

Recall that for an element g of a Lie group G we defined the inner automorphism (conjugation)

$$\begin{aligned} c_g = L_g \circ R_{g^{-1}} \colon G &\longrightarrow G \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

The differential $(c_g)_* \colon \mathfrak{g} \to \mathfrak{g}$ is an automorphism of the Lie algebra $\mathfrak{g}$, in particular a linear isomorphism.

Theorem 2.1.45 (Adjoint Representation of a Lie Group) *The map*

$$\begin{aligned} \mathrm{Ad} \colon G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}(g) = \mathrm{Ad}_g = (c_g)_* \end{aligned}$$

is a Lie group homomorphism, i.e. a representation of the Lie group G on the vector space $\mathfrak{g}$, called the **adjoint representation** *or* **adjoint action of the Lie group** *G. We sometimes write Ad_G instead of Ad.*

Proof Note that

$$c_{gh} = c_g \circ c_h \quad \forall g, h \in G.$$

Hence

$$\mathrm{Ad}_{gh} = (c_{gh})_* = (c_g)_* \circ (c_h)_* = \mathrm{Ad}_g \circ \mathrm{Ad}_h.$$

This shows that Ad is a homomorphism in the algebraic sense. We have to show that Ad is a smooth map. It suffices to show that for every $v \in \mathfrak{g}$ the map

$$\mathrm{Ad}(\cdot)v \colon G \longrightarrow \mathfrak{g}$$

is smooth, because if we choose a basis for the vector space $\mathfrak{g}$, it follows that Ad is a smooth matrix representation. The map $\mathrm{Ad}(\cdot)v$ is equal to the composition of smooth maps

$$G \longrightarrow TG \times TG \longrightarrow T(G \times G) \longrightarrow TG$$

given by

$$g \longmapsto ((g, 0), (e, v)) \longmapsto ((g, e), (0, v)) \longmapsto D_{(g,e)}c(0, v),$$

where we set

$$c \colon G \times G \longrightarrow G$$
$$(g, x) \longmapsto gxg^{-1}.$$

This implies the claim. □

The following identity (whose proof is left as an exercise) is sometimes useful.

Proposition 2.1.46 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and ρ a representation of G on a vector space V with induced representation ρ_* of $\mathfrak{g}$. Then*

$$\rho_*(\mathrm{Ad}_g X) \circ \rho(g) = \rho(g) \circ \rho_*(X) \quad \forall X \in \mathfrak{g}.$$

Example 2.1.47 The adjoint representation is very simple in the case of abelian Lie groups G: if G is abelian, then $c_g = \mathrm{Id}_G$ for all $g \in G$ and thus $\mathrm{Ad}_g = \mathrm{Id}_{\mathfrak{g}}$ for all $g \in G$, hence the adjoint representation is a trivial representation.

We consider a more general example: Let $G \subset \mathrm{GL}(n, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be a closed subgroup of a general linear group with Lie algebra $\mathfrak{g}$. Fix $Q \in G$.

Proposition 2.1.48 (Adjoint Representation of Linear Groups) *The adjoint action*

$$\mathrm{Ad}_Q \colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

is given by

$$\mathrm{Ad}_Q X = Q \cdot X \cdot Q^{-1},$$

where $\cdot$ denotes matrix multiplication and we identify elements $Q \in G$ and $X \in \mathfrak{g}$ with matrices in the canonical way.

Proof Define a curve $\gamma(t) = e^{tX}$ and take the derivative

$$\mathrm{Ad}_Q X = \left.\frac{d}{dt}\right|_{t=0} Q \cdot \gamma(t) \cdot Q^{-1} = Q \cdot X \cdot Q^{-1}.$$

□

In this situation, the Lie algebra $\mathfrak{g}$ on which the adjoint representation acts is naturally a vector space of *matrices*.

Example 2.1.49 We consider the adjoint representation of the Lie group SU(3). The Lie algebra $\mathfrak{su}(3)$ consists of the skew-Hermitian, tracefree matrices. As a real vector space, $\mathfrak{su}(3)$ has dimension 8 and is spanned by $i\lambda_a$, with $a = 1, \ldots, 8$, where λ_a are the Gell-Mann matrices from Example 1.5.33. We can define an explicit isomorphism

$$\begin{aligned} &\mathbb{R}^8 \longrightarrow \mathfrak{su}(3) \\ &G \longmapsto X = \sum_{a=1}^{8} iG_a\lambda_a = i \begin{pmatrix} G_3 + \frac{1}{\sqrt{3}}G_8 & G_1 - iG_2 & G_4 - iG_5 \\ G_1 + iG_2 & -G_3 + \frac{1}{\sqrt{3}}G_8 & G_6 - iG_7 \\ G_4 + iG_5 & G_6 + iG_7 & -\frac{2}{\sqrt{3}}G_8 \end{pmatrix}. \end{aligned}$$

On such a matrix X the group element $Q \in \mathrm{SU}(3)$ acts as

$$\mathrm{Ad}_Q X = Q \cdot X \cdot Q^{-1}.$$

Using the isomorphism $\mathbb{R}^8 \cong \mathfrak{su}(3)$ we could write this as an explicit representation on $\mathbb{R}^8$.

The following observation is sometimes useful.

Lemma 2.1.50 (Adjoint Representation of Direct Product) *Let $G = H \times K$ be a direct product of Lie groups. Then the adjoint representation of G on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ is the direct sum of the adjoint representations of H on $\mathfrak{h}$ and K on $\mathfrak{k}$:*

$$\mathrm{Ad}_{(h,k)}(X, Y) = (\mathrm{Ad}_h X, \mathrm{Ad}_k Y) \quad \forall (h,k) \in H \times K, (X, Y) \in \mathfrak{h} \oplus \mathfrak{k}.$$

Proof Let γ be a curve in H through e, tangent to $X \in \mathfrak{h}$. Then for $(h,k) \in H \times K$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (h,k)(\gamma(t), e)\left(h^{-1}, k^{-1}\right) &= \left.\frac{d}{dt}\right|_{t=0} \left(h\gamma(t)h^{-1}, e\right) \\ &= (\mathrm{Ad}_h X, 0). \end{aligned}$$

Similarly for a vector in $\mathfrak{k}$. □

Example 2.1.51 We consider the adjoint representation of the Standard Model Lie group

$$H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

We can write a group element $Q \in H$ as a block matrix

$$Q = \begin{pmatrix} Q_{\mathrm{SU}(3)} & & \\ & Q_{\mathrm{SU}(2)} & \\ & & Q_{\mathrm{U}(1)} \end{pmatrix},$$

with $Q_K \in K$ for $K = \mathrm{SU}(3), \mathrm{SU}(2), \mathrm{U}(1)$. We can similarly write the elements of the Lie algebra of H as a block matrix: with the notation from Examples 1.5.29, 1.5.32 and 1.5.33, the Lie algebra $\mathfrak{su}(3)$ is spanned by $i\lambda_a$, where λ_a are the Gell-Mann matrices, the Lie algebra $\mathfrak{su}(2)$ is spanned by $i\sigma_a$, where σ_a are the Pauli matrices, and the Lie algebra $\mathfrak{u}(1)$ is spanned by i. We can then define an isomorphism

$$\begin{aligned} \mathbb{R}^8 \oplus \mathbb{R}^3 \oplus \mathbb{R} &\longrightarrow \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \\ (G, W, B) &\longmapsto X = \left(\sum_{a=1}^{8} iG_a\lambda_a, \sum_{a=1}^{3} iW_a\sigma_a, iB \right) \\ &= i \begin{pmatrix} G_3 + \frac{1}{\sqrt{3}}G_8 & G_1 - iG_2 & G_4 - iG_5 & & & \\ G_1 + iG_2 & -G_3 + \frac{1}{\sqrt{3}}G_8 & G_6 - iG_7 & & & \\ G_4 + iG_5 & G_6 + iG_7 & -\frac{2}{\sqrt{3}}G_8 & & & \\ & & & W_3 & W_1 - iW_2 & \\ & & & W_1 + iW_2 & -W_3 & \\ & & & & & B \end{pmatrix}. \end{aligned}$$

According to Lemma 2.1.50 the adjoint action is given by multiplication of block matrices:

$$\mathrm{Ad}_Q X = Q \cdot X \cdot Q^{-1}.$$

The representation Ad_H describes the representation of the gauge boson fields in the Standard Model. The coefficients G_a, W_a and B (possibly with a different normalization) are known as the **gluon fields**, **weak gauge fields** and **hypercharge gauge field**; see Sect. 8.5.5 for more details.

Like any other representation of a Lie group, the adjoint representation of G induces a representation of the associated Lie algebra.

Theorem 2.1.52 (Adjoint Representation of a Lie Algebra) *The map*

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}),$$

given by

$$\mathrm{ad} = \mathrm{Ad}_*,$$

is a Lie algebra homomorphism, i.e. a representation of the Lie algebra $\mathfrak{g}$ *on the vector space* $\mathfrak{g}$*, called the* **adjoint representation of the Lie algebra** $\mathfrak{g}$*. We sometimes write* $\mathrm{ad}_{\mathfrak{g}}$ *instead of* ad*. We have the following commutative diagram according to Theorem 1.7.16:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

The map ad *satisfies the formula*

$$\mathrm{ad}(X)(Y) = \mathrm{ad}_X Y = [X, Y] \quad \forall X, Y \in \mathfrak{g}.$$

Proof We only have to prove the formula $\mathrm{ad}_X Y = [X, Y]$. For left-invariant vector fields X, Y on G, where X has flow ϕ_t, we have according to the commutative diagram

$$\begin{aligned} \mathrm{ad}_X Y &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp tX} Y_e \\ &= \frac{d}{dt}\bigg|_{t=0} (c_{\exp tX})_* Y_e \\ &= \frac{d}{dt}\bigg|_{t=0} (R_{\exp -tX})_*(L_{\exp tX})_* Y_e \\ &= \frac{d}{dt}\bigg|_{t=0} (R_{\exp -tX})_* Y_{\exp tX} \\ &= \frac{d}{dt}\bigg|_{t=0} (\phi_{-t})_* Y_{\phi_t(e)} \\ &= [X, Y]_e. \end{aligned}$$

Here we used Proposition 1.7.12 and Theorem A.1.46. □

We can write the formula given by the commutative diagram as

$$\mathrm{Ad}_{\exp X} = e^{\mathrm{ad}_X} \quad \forall X \in \mathfrak{g}.$$

A direct consequence of Example 2.1.47 is the following:

Corollary 2.1.53 *If G is an abelian Lie group, then the adjoint representation* ad *is trivial, hence the Lie algebra* $\mathfrak{g}$ *is abelian.*
It can be shown that the converse also holds (for connected Lie groups), cf. Exercise 2.7.7.

Remark 2.1.54 We can *define* for any Lie algebra $\mathfrak{g}$, even if it does not belong *a priori* to a Lie group, the map

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}),$$

by exactly the same formula

$$\mathrm{ad}_X Y = [X, Y] \quad \forall X, Y \in \mathfrak{g}.$$

Then this map is a representation of $\mathfrak{g}$ (by the Jacobi identity), again called the adjoint representation.

Remark 2.1.55 One should be careful not to confuse the fundamental and the adjoint representation for a linear group. In general, the dimensions are already different. For example, in the case of $\mathrm{SU}(n)$ the dimension of the fundamental representation is n, while the adjoint representation has dimension $n^2 - 1$. For a linear group the fundamental representation acts canonically on a vector space of column vectors, while the adjoint representation acts on a vector space of matrices.

Example 2.1.56 The homomorphism $\phi\colon S^3 \to \mathrm{SO}(3)$ from Example 1.3.8 is the adjoint representation of $S^3 = \mathrm{SU}(2)$.

2.2 Invariant Metrics on Lie Groups

Since a Lie group G is a manifold, we can study metrics (Riemannian or pseudo-Riemannian) on it. We are interested in particular in the following types of metrics.

Definition 2.2.1 Let s be a metric on a Lie group G.

1. The metric s is called
   - **left-invariant** if $L_g^* s = s$ for all $g \in G$
   - **right-invariant** if $R_g^* s = s$ for all $g \in G$.

   Equivalently, either all left translations or all right translations are isometries.
2. The metric s is called **bi-invariant** if it is both left- and right-invariant.

It is clear that every metric induces a scalar product on $\mathfrak{g} \cong T_e G$. On the other hand, given an arbitrary scalar product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$, it is easy to construct

- a *left-invariant metric* on G by

$$s(X, Y) = \langle L_{g^{-1}*}(X), L_{g^{-1}*}(Y) \rangle$$

- a *right-invariant metric* on G by

$$s(X, Y) = \langle R_{g^{-1}*}(X), R_{g^{-1}*}(Y)\rangle,$$

for all $g \in G$ and $X, Y \in T_gG$.

However, in general we only get a *bi-invariant metric* in this way if G is abelian (if G is not abelian, then $L_g \neq R_g$ for some $g \in G$). Bi-invariant metrics have the following characterization:

Theorem 2.2.2 (Bi-Invariant Metrics and Ad-Invariance) *Let s be a left-invariant metric on a Lie group G. Then s is bi-invariant if and only if the scalar product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{g}$ defined by the metric s is* Ad*-invariant, i.e.*

$$\langle \mathrm{Ad}_g v, \mathrm{Ad}_g w\rangle = \langle v, w\rangle$$

for all $g \in G$ and $v, w \in \mathfrak{g}$.

Proof Let X and Y be vectors in T_pG. Then we can calculate:

$$\begin{aligned}(R_g^* s)_p(X, Y) &= \langle L_{(pg)^{-1}*}R_{g*}(X), L_{(pg)^{-1}*}R_{g*}(Y)\rangle \\ &= \langle \mathrm{Ad}_{g^{-1}} \circ L_{p^{-1}*}(X), \mathrm{Ad}_{g^{-1}} \circ L_{p^{-1}*}(Y)\rangle\end{aligned}$$

and

$$s_p(X, Y) = \langle L_{p^{-1}*}(X), L_{p^{-1}*}(Y)\rangle,$$

where in both equations we used that s is left-invariant. This implies the claim, because $L_{p^{-1}*}$ is an isomorphism of vector spaces. □

Theorem 2.2.3 (Ad-Invariant Scalar Products for Compact Lie Groups) *Let G be a compact Lie group. Then there exists a Euclidean (positive definite) scalar product $\langle\cdot\,,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ which is* Ad*-invariant. The adjoint representation is orthogonal with respect to this scalar product.*

Proof This follows from Theorem 2.1.39, because Ad is a representation of the compact Lie group G on the vector space $\mathfrak{g}$. □

The existence of positive definite Ad-invariant scalar products on the Lie algebra of compact Lie groups is very important in gauge theory, in particular, for the construction of the gauge-invariant *Yang–Mills Lagrangian*; see Sect. 7.3.1. We will study such scalar products in more detail in Sect. 2.5 after we have discussed the general structure of compact Lie groups. The

(continued)

fact that these scalar products are positive definite is important from a phenomenological point of view, because only then do the kinetic terms in the Yang–Mills Lagrangian have the right sign (the gauge bosons have positive kinetic energy [148]).

Here is a corollary to Theorem 2.2.2 and Theorem 2.2.3:

Corollary 2.2.4 *Every compact Lie group admits a bi-invariant Riemannian metric.*

Remark 2.2.5 It can be shown that the geodesics of a bi-invariant metric on a Lie group G through the neutral element e are of the form $\gamma(t) = \exp(tX)$, with $X \in \mathfrak{g}$. The notions of exponential map for geodesics and Lie groups thus coincide for bi-invariant Riemannian metrics.

2.3 The Killing Form

We want to consider a special Ad-invariant inner product on every Lie algebra $\mathfrak{g}$, which in general is neither non-degenerate nor positive or negative definite. This is the celebrated Killing form.

Theorem 2.3.1 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The* **Killing form** *$B_{\mathfrak{g}}$ on $\mathfrak{g}$ is defined by*

$$\begin{aligned} B_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{K} \\ (X, Y) &\longmapsto \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y). \end{aligned}$$

This is a $\mathbb{K}$-bilinear, symmetric form on $\mathfrak{g}$.

Remark 2.3.2 Note that the Killing form for complex Lie algebras is also symmetric and complex bilinear and not Hermitian.

Proof For $Z \in \mathfrak{g}$ we have

$$\mathrm{ad}_X \circ \mathrm{ad}_Y(Z) = [X, [Y, Z]].$$

In particular, $B_{\mathfrak{g}}$ is indeed bilinear. To show that the Killing form is symmetric, recall the definition of the **trace** $\mathrm{tr}(f)$ of a linear endomorphism f of a vector space V: If $v_1, \ldots, v_n$ is a basis of V and we define the representing matrix of f by

$$f(v_j) = \sum_{i=1}^{n} f_{ij} v_i,$$

then

$$\mathrm{tr}(f) = \sum_{i=1}^{n} f_{ii}.$$

This number does not depend on the choice of basis for V: If $\phi: V \to V$ is an arbitrary isomorphism, then

$$\mathrm{tr}\left(\phi \circ f \circ \phi^{-1}\right) = \mathrm{tr}(f).$$

We also have

$$\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$$

for all endomorphisms $f, g: V \to V$. This shows, in particular, that $B_{\mathfrak{g}}$ is symmetric. □

Theorem 2.3.3 (Invariance of Killing Form Under Automorphisms) *Let $\sigma: \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism of $\mathfrak{g}$. Then the Killing form $B_{\mathfrak{g}}$ satisfies*

$$B_{\mathfrak{g}}(\sigma X, \sigma Y) = B_{\mathfrak{g}}(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, this holds in particular for the automorphism $\sigma = \mathrm{Ad}_g$ with $g \in G$ arbitrary.

Proof Note that

$$\mathrm{ad}_X Y = [X, Y].$$

Since σ is a Lie algebra automorphism we have

$$\mathrm{ad}_{\sigma X} Y = [\sigma X, Y] = \sigma([X, \sigma^{-1} Y] = \sigma \circ \mathrm{ad}_X(\sigma^{-1} Y).$$

Thus

$$\mathrm{ad}_{\sigma X} = \sigma \circ \mathrm{ad}_X \circ \sigma^{-1}.$$

We get for the Killing form:

$$\begin{aligned} B_{\mathfrak{g}}(\sigma X, \sigma Y) &= \mathrm{tr}(\mathrm{ad}_{\sigma X} \circ \mathrm{ad}_{\sigma Y}) \\ &= \mathrm{tr}\left(\sigma \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ \sigma^{-1}\right) \\ &= B_{\mathfrak{g}}(X, Y). \end{aligned}$$

□

Corollary 2.3.4 *The Killing form $B_{\mathfrak{g}}$ defines a bi-invariant symmetric form on any Lie group G.*

Remark 2.3.5 We will determine in Sect. 2.4 when the Killing form is non-degenerate or definite (in the case of a real Lie algebra).

Proposition 2.3.6 (ad Is Skew-Symmetric with Respect to the Killing Form) *Let $\mathfrak{g}$ be a Lie algebra with Killing form $B_{\mathfrak{g}}$. Then*

$$B_{\mathfrak{g}}(\mathrm{ad}_X Y, Z) + B_{\mathfrak{g}}(Y, \mathrm{ad}_X Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}.$$

Proof This follows from Theorem 2.3.3 and Proposition 2.1.37 if $\mathfrak{g}$ is the Lie algebra of a Lie group G. In the general case we use the formula

$$\mathrm{ad}_{\mathrm{ad}_X Y} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X \quad \forall X, Y \in \mathfrak{g},$$

which follows from the Jacobi identity. The definition of the Killing form implies

$$\begin{aligned} B_{\mathfrak{g}}(\mathrm{ad}_X Y, Z) + B_{\mathfrak{g}}(Y, \mathrm{ad}_X Z) &= \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z) - \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Z \circ \mathrm{ad}_X) \\ &= 0, \end{aligned}$$

because the trace is invariant under cyclic permutations. □

2.4 *Semisimple and Compact Lie Algebras

In this section we discuss some results concerning the general structure of Lie algebras and Lie groups (we follow [83] and [153]). There are two elements that play a key role in the theory of Lie algebras:

- The adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$, together with its invariant subspaces, known as *ideals*.
- The Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$.

Both notions are related: the definition of the Killing form $B_{\mathfrak{g}}$ involves the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ and the adjoint representation is skew-symmetric with respect to the Killing form.

The idea is to proceed in a similar way to Theorem 2.1.40 and try to decompose $\mathfrak{g}$ with the adjoint representation into irreducible, pairwise $B_{\mathfrak{g}}$-orthogonal pieces. This works out particularly well for a type of Lie algebra known as a *semisimple* Lie algebra. The next step is to classify the pieces where the adjoint representation is irreducible. These are called *simple* Lie algebras. We will discuss the classification for the simple Lie algebras coming from compact Lie groups, which appear in physics as gauge groups.

2.4.1 Simple and Semisimple Lie Algebras in General

Definition 2.4.1 Let $\mathfrak{g}$ be a Lie algebra. For subsets $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ we define $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{g}$ as the set of all finite sums of elements of the form $[X, Y]$ with $X \in \mathfrak{a}, Y \in \mathfrak{b}$.

Definition 2.4.2 Let $\mathfrak{g}$ be a Lie algebra.

1. An **ideal** in $\mathfrak{g}$ is a vector subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$. Equivalently,
$$\mathrm{ad}_{\mathfrak{g}}\mathfrak{a} \subset \mathfrak{a}.$$
2. The **center** of $\mathfrak{g}$ is defined as
$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \equiv 0\}.$$
3. The **commutator** of $\mathfrak{g}$ is defined as $[\mathfrak{g}, \mathfrak{g}]$.

The following is easy to check.

Lemma 2.4.3 *For any Lie algebra the commutator is an ideal and the center is an abelian ideal.*

Proposition 2.4.4 *The kernel of the adjoint representation of a Lie algebra* $\mathfrak{g}$ *is the center* $\mathfrak{z}(\mathfrak{g})$. *The adjoint representation is faithful if and only if* $\mathfrak{z}(\mathfrak{g}) = 0$.

Proof We have $\mathrm{ad}_X \equiv 0$ if and only if $[X, \mathfrak{g}] \equiv 0$. □

This implies Ado's Theorem 1.5.25 for Lie algebras with trivial center.

Definition 2.4.5 Let $\mathfrak{g}$ be a Lie algebra.

1. The Lie algebra $\mathfrak{g}$ is called **simple** if $\mathfrak{g}$ is non-abelian and $\mathfrak{g}$ has no non-trivial ideals (different from 0 and $\mathfrak{g}$).
2. The Lie algebra $\mathfrak{g}$ is called **semisimple** if $\mathfrak{g}$ has no non-zero abelian ideals.

Simple Lie algebras are sometimes defined equivalently as follows:

Lemma 2.4.6 *A Lie algebra* $\mathfrak{g}$ *is simple if and only if* $\mathfrak{g}$ *has dimension at least two and* $\mathfrak{g}$ *has no non-trivial ideals.*

Proof If $\mathfrak{g}$ is non-abelian, then it has dimension at least two. On the other hand, if $\mathfrak{g}$ is abelian and has dimension at least two, then $\mathfrak{g}$ has non-trivial (abelian) ideals. □

It is clear that every simple Lie algebra is semisimple.

Lemma 2.4.7 *If* $\mathfrak{g}$ *is simple, then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof The commutator $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, hence equal to $\mathfrak{g}$ or 0. The second possibility is excluded, because $\mathfrak{g}$ is not abelian. □

We can characterize simple Lie algebras as follows:

Proposition 2.4.8 (Criterion for Simplicity) *A Lie algebra* $\mathfrak{g}$ *is simple if and only if* $\mathfrak{g}$ *is non-abelian and the adjoint representation* $\mathrm{ad}_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *is irreducible.*

Proof The claim follows from the definition of an ideal. □

We can also characterize semisimple Lie algebras (we only prove one direction following [83]; the proof of the converse, which would take us too far afield, can be found in [77, 83]):

Theorem 2.4.9 (Cartan's Criterion for Semisimplicity) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if the Killing form* $B_{\mathfrak{g}}$ *is non-degenerate.*

Proof We only prove that the Killing form is degenerate if the Lie algebra is not semisimple. Let $\mathfrak{a}$ be a non-zero abelian ideal in $\mathfrak{g}$. We choose a complementary vector space $\mathfrak{s}$ with

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}.$$

Let $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$ be arbitrary elements. Then

$$[X, \mathfrak{a}] = 0,$$
$$[X, \mathfrak{s}] \subset \mathfrak{a},$$
$$[Y, \mathfrak{a}] \subset \mathfrak{a}.$$

Under the splitting $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$, the endomorphisms ad_X and ad_Y thus have the form

$$\mathrm{ad}_X = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$
$$\mathrm{ad}_Y = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

It follows that

$$\mathrm{ad}_X \circ \mathrm{ad}_Y = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

and

$$B_{\mathfrak{g}}(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = 0.$$

□

Remark 2.4.10 In general, the Killing form of a semisimple Lie algebra is indefinite, i.e. pseudo-Euclidean.

Assuming Cartan's Criterion we can prove the following.

Theorem 2.4.11 (Structure of Semisimple Lie Algebras) *If a Lie algebra* $\mathfrak{g}$ *is semisimple, then* $\mathfrak{g}$ *is the direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$$

of ideals $\mathfrak{g}_i$*, each of which is a simple Lie algebra, and which are pairwise orthogonal with respect to the Killing form.*

Proof We ultimately would like to apply Theorem 2.1.40 and decompose the adjoint representation on $\mathfrak{g}$ into irreducible summands, orthogonal with respect to the Killing form. There is one problem which requires some work: the Killing form $B = B_{\mathfrak{g}}$ is non-degenerate, but not (positive or negative) definite. Therefore it is not immediately clear that orthogonal complements of invariant subspaces lead to a direct sum decomposition.

Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and

$$\mathfrak{a}^{\perp} = \{X \in \mathfrak{g} \mid B(X, Y) = 0 \quad \forall Y \in \mathfrak{a}\}$$

the orthogonal complement with respect to the Killing form B. Then $\mathfrak{a}^{\perp}$ is also an ideal in $\mathfrak{g}$, because

$$B(\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}^{\perp}, \mathfrak{a}) = -B(\mathfrak{a}^{\perp}, \mathrm{ad}_{\mathfrak{g}}\mathfrak{a}) \subset B(\mathfrak{a}^{\perp}, \mathfrak{a}) = 0,$$

by Proposition 2.3.6. Furthermore, $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{a}^{\perp}$ is an abelian ideal in $\mathfrak{g}$: it is clear that the intersection of two ideals is an ideal and

$$B(\mathrm{ad}_{\mathfrak{b}}\mathfrak{b}, \mathfrak{g}) = -B(\mathfrak{b}, \mathrm{ad}_{\mathfrak{b}}\mathfrak{g}) \subset B(\mathfrak{b}, \mathfrak{b}) = 0.$$

This implies that $\mathfrak{b}$ is abelian, because B is non-degenerate. Since $\mathfrak{g}$ is semisimple, it follows that $\mathfrak{a} \cap \mathfrak{a}^{\perp} = 0$.

This implies

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$$

and the restriction of the Killing form to $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$ (which is just the Killing form on these Lie algebras) is non-degenerate. We can continue splitting the (finite-dimensional) Lie algebra $\mathfrak{g}$ in this fashion until we arrive at irreducible, non-abelian (simple) ideals. □

Remark 2.4.12 In addition to semisimple and abelian Lie algebras there are other classes of Lie algebras, like solvable and nilpotent Lie algebras, which we have not discussed in detail.

2.4.2 Compact Lie Algebras

We are particularly interested in *compact* Lie algebras, including compact simple and compact semisimple Lie algebras.

Definition 2.4.13 A real Lie algebra $\mathfrak{g}$ is called **compact** if it is the Lie algebra of some compact Lie group.

Remark 2.4.14 Even if $\mathfrak{g}$ is compact, there could exist non-compact Lie groups whose Lie algebra is also $\mathfrak{g}$. For example, the abelian Lie algebra $\mathfrak{u}(1)$ is the Lie algebra of the compact Lie group $\mathrm{U}(1) = S^1$ and of the non-compact Lie group $\mathbb{R}$.

Example 2.4.15 Note that the abelian Lie algebra $\mathbb{R}^n = \mathfrak{u}(1)\oplus\ldots\oplus\mathfrak{u}(1)$, for $n \geq 1$, is compact, but neither simple nor semisimple.

Theorem 2.4.16 (Killing Form of Compact Lie Algebras) *Suppose $\mathfrak{g}$ is a compact real Lie algebra. Then the Killing form $B_\mathfrak{g}$ is negative semidefinite: We have*

$$B_\mathfrak{g}(X,X) = 0 \quad \forall X \in \mathfrak{z}(\mathfrak{g}),$$
$$B_\mathfrak{g}(X,X) < 0 \quad \forall X \in \mathfrak{g} \setminus \mathfrak{z}(\mathfrak{g}).$$

Proof We follow the proof in [14]. Since $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, according to Theorem 2.2.3 there exists a positive definite scalar product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ which is Ad_G-invariant. Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathfrak{g}$ with respect to this scalar product. We get

$$\langle \mathrm{ad}_X \circ \mathrm{ad}_X Y, Y \rangle = -||\mathrm{ad}_X Y||^2 \quad \forall X, Y \in \mathfrak{g}$$

for the associated norm $|| \cdot ||$. This implies

$$\begin{aligned} B_\mathfrak{g}(X,X) &= \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X) \\ &= \sum_{i=1}^n \langle \mathrm{ad}_X \circ \mathrm{ad}_X e_i, e_i \rangle \\ &= -\sum_{i=1}^n ||\mathrm{ad}_X e_i||^2 \\ &\leq 0. \end{aligned}$$

Equality holds if and only if $\mathrm{ad}_X \equiv 0$ on $\mathfrak{g}$, i.e. $X \in \mathfrak{z}(\mathfrak{g})$. □

Remark 2.4.17 Note as an aside that the notion of a bilinear, symmetric form being (semi-)definite is only meaningful on real and not on complex vector spaces.

Corollary 2.4.18 *Let $\mathfrak{g}$ be a compact Lie algebra with trivial center, $\mathfrak{z}(\mathfrak{g}) = 0$. Then the Killing form $B_\mathfrak{g}$ is negative definite.*

Proof This follows from Theorem 2.4.16. □

Remark 2.4.19 The following converse to Corollary 2.4.18 can be proved (see [77]): if the Killing form of a real Lie algebra is negative definite, then it is compact with trivial center. In particular, every Lie subalgebra of a compact Lie algebra is compact.

Corollary 2.4.20 *Let* $\mathfrak{g}$ *be a compact Lie algebra. Then the Killing form* $B_{\mathfrak{g}}$ *is negative definite if and only if* $\mathfrak{g}$ *is semisimple.*

Proof One direction follows from Corollary 2.4.18, because semisimple Lie algebras have trivial center. The other direction follows from Theorem 2.4.9. □

Theorem 2.4.21 (Decomposition of Compact Lie Algebras) *Let* $\mathfrak{g}$ *be a compact Lie algebra with center* $\mathfrak{z}(\mathfrak{g})$. *Then there exists an ideal* $\mathfrak{h}$ *in* $\mathfrak{g}$ *such that*

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}.$$

The ideal $\mathfrak{h}$ *is a compact semisimple Lie algebra with negative definite Killing form.*

Proof Choose a positive definite scalar product $\langle \cdot\,,\cdot \rangle$ on $\mathfrak{g}$ which is Ad_G-invariant. Let $\mathfrak{h}$ be the orthogonal complement

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{g})^{\perp}$$

with respect to this scalar product. Then $\mathfrak{h}$ is an ideal, because

$$\langle \mathrm{ad}_{\mathfrak{g}}\mathfrak{h}, \mathfrak{z}(\mathfrak{g}) \rangle = -\langle \mathfrak{h}, \mathrm{ad}_{\mathfrak{g}}\mathfrak{z}(\mathfrak{g}) \rangle = 0.$$

It is clear that

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}.$$

By Theorem 2.4.16 the Killing form is negative definite on $\mathfrak{h}$, which is thus compact by Remark 2.4.19 and semisimple by Theorem 2.4.9. □

Corollary 2.4.22 (Structure of Compact Lie Algebras) *Let* $\mathfrak{g}$ *be a compact Lie algebra. Then* $\mathfrak{g}$ *is a direct sum of ideals*

$$\mathfrak{g} = \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1) \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s,$$

where the $\mathfrak{g}_i$ *are compact simple Lie algebras.*

Proof This follows from Theorem 2.4.11. The Lie algebras $\mathfrak{g}_i$ are compact by Remark 2.4.19. □

Using considerable effort it is possible to classify simple Lie algebras, one of the great achievements of 19th and 20th century mathematics. The result for compact simple Lie algebras is the following (see [83] for a proof):

Theorem 2.4.23 (Killing–Cartan Classification of Compact Simple Lie Algebras) *Every compact simple Lie algebra is isomorphic to precisely one of the following Lie algebras:*

1. $\mathfrak{su}(n+1)$ *for* $n \geq 1$.
2. $\mathfrak{so}(2n+1)$ *for* $n \geq 2$.
3. $\mathfrak{sp}(n)$ *for* $n \geq 3$.
4. $\mathfrak{so}(2n)$ *for* $n \geq 4$.
5. *An exceptional Lie algebra of type* $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$.

The families in the first four cases are also called $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ *in this order.*

Remark 2.4.24 The lower index n in the series $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ as well as in the exceptional cases $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ is the **rank** of the corresponding compact Lie group, i.e. the dimension of a **maximal torus subgroup** embedded in the Lie group.

Remark 2.4.25 The reason for the restrictions on n in the first four cases of the classical Lie algebras is to avoid counting Lie algebras twice, because we have the following isomorphisms (we only proved the first isomorphism in Sect. 1.5.5):

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \cong \mathfrak{sp}(1),$$
$$\mathfrak{sp}(2) \cong \mathfrak{so}(5),$$
$$\mathfrak{so}(6) \cong \mathfrak{su}(4).$$

There is also the abelian Lie algebra

$$\mathfrak{so}(2)$$

and the semisimple Lie algebra

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

cf. Exercise 1.9.21.

The basic building blocks of all compact Lie algebras are thus

- abelian Lie algebras
- the four families of classical compact non-abelian Lie algebras
- five exceptional compact Lie algebras.

In some sense, most compact Lie algebras are therefore classical or direct sums of classical Lie algebras.

It is sometimes convenient to know that we can choose for a compact semisimple Lie algebra a basis in such a way that the structure constants (see Definition 1.4.17) have a nice form. Let $\mathfrak{g}$ be a compact semisimple Lie algebra. According to Corollary 2.4.20 the Killing form $B_{\mathfrak{g}}$ is negative definite. Let $T_1, \ldots, T_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form:

$$B_{\mathfrak{g}}(T_a, T_b) = -\delta_{ab} \quad \forall a, b \in \{1, \ldots, n\}.$$

Proposition 2.4.26 *The structure constants f_{abc} for a $B_{\mathfrak{g}}$-orthonormal basis $\{T_a\}$ of a semisimple Lie algebra $\mathfrak{g}$ are totally antisymmetric:*

$$f_{abc} = -f_{bac} = f_{bca} = -f_{acb} \quad \forall a, b, c \in \{1, \ldots, n\}.$$

Proof This is Exercise 2.7.11. □

2.4.3 Compact Lie Groups

We briefly discuss the structure of compact Lie groups.

Definition 2.4.27 A connected Lie group G is called **simple** (or **semisimple**) if its Lie algebra is simple (or semisimple).

Corollary 2.4.28 *If G is simple, then Ad_G is an irreducible representation.*

Proof The claim follows from Proposition 2.4.8 because an Ad_G-invariant subspace in $\mathfrak{g}$ is also $\mathrm{ad}_{\mathfrak{g}}$-invariant. □

A proof of the following theorem can be found in [77].

Theorem 2.4.29 (Structure of Compact Lie Groups) *Let G be a compact connected Lie group. Then G is a finite quotient of a product of the form*

$$\tilde{G} \cong \mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \times G_1 \times \ldots \times G_s,$$

where the G_i are compact simple Lie groups.

Compact simple Lie groups and the abelian Lie group U(1) are therefore the building blocks of all compact connected Lie groups.

2.5 *Ad-Invariant Scalar Products on Compact Lie Groups

We know from Theorem 2.2.3 that compact Lie algebras admit scalar products that are invariant under the adjoint action. Such scalar products are important in gauge theory: they are necessary ingredients to construct the gauge-invariant Yang–Mills action and are related to the notion of *coupling constants*. We discuss, in particular, how to fix an Ad-invariant scalar product and how many different ones exist on a given compact Lie algebra.

We first consider Ad-invariant scalar products on compact *simple* Lie algebras. We need the following variant of a famous theorem of Schur.

Theorem 2.5.1 (Schur's Lemma for Scalar Products) *Let* $\rho\colon G \to \mathrm{GL}(V)$ *be an irreducible representation of a Lie group* G *on a real vector space* V *and* $\langle \cdot , \cdot \rangle_1$, $\langle \cdot , \cdot \rangle_2$ *two* G*-invariant symmetric bilinear forms on* V*, so that* $\langle \cdot , \cdot \rangle_2$ *is positive definite. Then there exists a real number* $a \in \mathbb{R}$ *such that*

$$\langle \cdot , \cdot \rangle_1 = a \langle \cdot , \cdot \rangle_2.$$

Remark 2.5.2 The assumption that the group representation is irreducible is important.

Proof We follow the proof in [153]. Let $L\colon V \to V$ be the unique linear map defined by (using non-degeneracy of the second scalar product)

$$\langle v, w \rangle_1 = \langle v, Lw \rangle_2 \quad \forall v, w \in V.$$

We have

$$\begin{aligned} \langle w, Lv \rangle_2 &= \langle w, v \rangle_1 \\ &= \langle v, w \rangle_1 \\ &= \langle v, Lw \rangle_2, \end{aligned}$$

hence L is self-adjoint with respect to the second scalar product. We can split V into the eigenspaces of L which are orthogonal with respect to the second scalar product. Since both bilinear forms are G-invariant we have

$$\begin{aligned} \langle gv, gLw \rangle_2 &= \langle v, Lw \rangle_2 \\ &= \langle v, w \rangle_1 \\ &= \langle gv, gw \rangle_1 \\ &= \langle gv, L(gw) \rangle_2. \end{aligned}$$

We conclude that $\rho(g) \circ L = L \circ \rho(g)$ for all $g \in G$ and thus the eigenspaces of L are G-invariant. Since the representation ρ is irreducible, V itself must be an eigenspace and hence $L = a \cdot \mathrm{Id}_V$. This implies the claim. □

Theorem 2.5.3 (Ad-Invariant Scalar Products on Compact Simple Lie Algebras) *Let* G *be a compact simple Lie group. Then there exists up to a positive factor a unique* Ad*-invariant positive definite scalar product on the Lie algebra* $\mathfrak{g}$.

The negative of the Killing form is an example of such an Ad*-invariant positive definite scalar product.*

Proof Existence follows from 2.2.3. Uniqueness follows from Corollary 2.4.28 and Theorem 2.5.1. The claim about the Killing form follows from Corollary 2.4.18. □

Let $T = \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ denote an n-dimensional torus and $\langle \cdot , \cdot \rangle_{\mathfrak{t}}$ a positive definite scalar product on its Lie algebra

$$\mathbb{R}^n = \mathfrak{t} = \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1).$$

Since the adjoint representation of an abelian Lie group is trivial, any inner product on an abelian Lie algebra is Ad-invariant. With respect to the standard Euclidean scalar product on $\mathbb{R}^n$, the scalar product $\langle \cdot , \cdot \rangle_{\mathfrak{t}}$ is determined by a positive definite symmetric matrix.

Theorem 2.5.4 (Ad**-Invariant Scalar Products on General Compact Lie Algebras)** *Let G be a compact connected Lie group of the form*

$$G = \mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \times G_1 \times \ldots \times G_s,$$

where the G_i are compact simple Lie groups. Let $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$ be an Ad_G-invariant positive definite scalar product on the Lie algebra $\mathfrak{g}$ of G. Then $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$ is the orthogonal direct sum of:

1. *a positive definite scalar product $\langle \cdot , \cdot \rangle_{\mathfrak{t}}$ on the center $\mathfrak{t} = \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1)$;*
2. *Ad_{G_i}-invariant positive definite scalar products $\langle \cdot , \cdot \rangle_{\mathfrak{g}_i}$ on the Lie algebras $\mathfrak{g}_i$.*

Conversely, the direct sum of any positive definite scalar product $\langle \cdot , \cdot \rangle_{\mathfrak{t}}$ on the abelian Lie algebra $\mathfrak{t}$ and any Ad_{G_i}-invariant positive definite scalar products $\langle \cdot , \cdot \rangle_{\mathfrak{g}_i}$ on the simple Lie algebras $\mathfrak{g}_i$ is an Ad_G-invariant positive definite scalar product on $\mathfrak{g}$.

Proof Let $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$ be an Ad_G-invariant positive definite scalar product on the Lie algebra $\mathfrak{g}$. We have to show that it decomposes as an orthogonal direct sum of scalar products on the summands. For any fixed $i = 1, \ldots, s$ we can write $G = G_i \times H$ with a compact Lie group H. Fix an arbitrary $Y \in \mathfrak{h}$ and let

$$\begin{aligned} f\colon \mathfrak{g}_i &\longrightarrow \mathbb{R} \\ X &\longmapsto \langle X, Y \rangle_{\mathfrak{g}}. \end{aligned}$$

Then f is a linear 1-form on $\mathfrak{g}_i$ and its kernel is a vector subspace of codimension zero or one. Let $g \in G_i$ and $X \in \mathfrak{g}_i$. Then by Lemma 2.1.50

$$\begin{aligned} f(\mathrm{Ad}_g X) &= \langle \mathrm{Ad}_g X, Y \rangle_{\mathfrak{g}} \\ &= \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle_{\mathfrak{g}} \\ &= \langle X, Y \rangle_{\mathfrak{g}} \\ &= f(X). \end{aligned}$$

This implies that the kernel of f is Ad_{G_i}-invariant. Since the adjoint representation of G_i is irreducible by Corollary 2.4.28 and since $\dim \mathfrak{g}_i > 1$, the kernel of f cannot have codimension 1. Therefore f must vanish identically.

This proves that the scalar product $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ decomposes as an orthogonal direct sum of scalar products $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}_i}$ on $\mathfrak{g}_i$ and $\langle \cdot\,,\cdot \rangle_{\mathfrak{h}}$ on $\mathfrak{h}$. The scalar product $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}}$ is Ad_G-invariant, hence $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}_i}$ is Ad_{G_i}-invariant and $\langle \cdot\,,\cdot \rangle_{\mathfrak{h}}$ is Ad_H-invariant. We continue to split the scalar product $\langle \cdot\,,\cdot \rangle_{\mathfrak{h}}$ on $\mathfrak{h}$ until the remaining Lie algebra is the center.

Conversely, if $\langle \cdot\,,\cdot \rangle_{\mathfrak{t}}$ is a scalar product on $\mathfrak{t} = \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1)$ and $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}_i}$ are Ad_{G_i}-invariant scalar products on $\mathfrak{g}_i$, then the orthogonal direct sum

$$\langle \cdot\,,\cdot \rangle_{\mathfrak{g}} = \langle \cdot\,,\cdot \rangle_{\mathfrak{t}} \oplus \langle \cdot\,,\cdot \rangle_{\mathfrak{g}_1} \oplus \ldots \oplus \langle \cdot\,,\cdot \rangle_{\mathfrak{g}_s}$$

is Ad_G-invariant by Lemma 2.1.50. □

In the situation of Theorem 2.5.4 the Ad_G-invariant scalar product $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ is determined by certain constants:

1. The scalar product $\langle \cdot\,,\cdot \rangle_{\mathfrak{t}}$ is determined by a positive definite symmetric matrix with respect to the standard Euclidean scalar product on $\mathbb{R}^n$.
2. The scalar products $\langle \cdot\,,\cdot \rangle_{\mathfrak{g}_i}$ are determined by positive constants relative to some fixed Ad_{G_i}-invariant positive definite scalar product on the simple Lie algebras $\mathfrak{g}_i$ (like the negative of the Killing form).

Definition 2.5.5 The constants that determine an Ad_G-invariant positive definite scalar product on the compact Lie algebra $\mathfrak{g}$ are called **coupling constants** in physics.

Example 2.5.6

1. In the Standard Model, where $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, there are three coupling constants, one for each factor.
2. In GUTs with a simple gauge group, like $G = \mathrm{SU}(5)$ or $G = \mathrm{Spin}(10)$, there is only a single coupling constant.

2.6 *Homotopy Groups of Lie Groups

In this section we collect some results (without proofs) on the homotopy groups of compact Lie groups. The following fact is elementary and can be found in textbooks on topology:

Proposition 2.6.1 (Fundamental Group of Topological Groups) *The fundamental group $\pi_1(G)$ of any connected topological group G is abelian.*

Regarding the order of the fundamental group of Lie groups it can be shown that (for a proof, see [24, Sect. V.7]):

Theorem 2.6.2 (Fundamental Group of Compact Semisimple Lie Groups) *Let G be a compact connected Lie group. Then $\pi_1(G)$ is finite if and only if G is semisimple. In particular, every compact simple Lie group has a finite fundamental group.*

The only-if direction follows from Theorem 2.4.29. As an example, it is possible to calculate the fundamental group of the classical Lie groups (see, for example, [129]).

Proposition 2.6.3 (Fundamental Groups of Classical Compact Groups) *The fundamental groups of the classical compact linear groups are:*

1. *Special orthogonal groups:*

$$\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z},$$
$$\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2 \ \forall n \geq 3.$$

2. *Unitary groups (for all $n \geq 1$):*

$$\pi_1(\mathrm{U}(n)) \cong \mathbb{Z}.$$

3. *Special unitary and symplectic groups (for all $n \geq 1$):*

$$\pi_1(\mathrm{SU}(n)) = 1,$$
$$\pi_1(\mathrm{Sp}(n)) = 1.$$

We have the following result on the second homotopy group (for a proof, see again [24, Sect. V.7]):

Theorem 2.6.4 (Second Homotopy Group of Compact Lie Groups) *Let G be a compact connected Lie group. Then $\pi_2(G) = 0$.*

The next theorem on the third homotopy group was proved by M.R. Bott using Morse theory [19]:

Theorem 2.6.5 (Third Homotopy Group of Compact Lie Groups) *Let G be a compact connected Lie group. Then $\pi_3(G)$ is free abelian, i.e. isomorphic to $\mathbb{Z}^r$ for some integer r. If G is compact, connected and simple, then $\pi_3(G) \cong \mathbb{Z}$.*

Combining Theorem 2.6.2 and Theorem 2.6.5 we get a topological criterion to decide whether a compact Lie group is simple:

Corollary 2.6.6 (Topological Criterion for Simplicity) *Let G be a compact connected Lie group. Then G is simple if and only if $\pi_1(G)$ is finite and $\pi_3(G) \cong \mathbb{Z}$.*

2.7 Exercises for Chap. 2

2.7.1 Verify that the dual representations on V^* defined in Definition 2.1.23 and Definition 2.1.27 are indeed representations of the Lie group G and the Lie algebra $\mathfrak{g}$.

2.7.2 Let

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Prove the following equivalent description of $\mathrm{SL}(2, \mathbb{C})$:

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ M \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \mid M^T \epsilon M = \epsilon \right\}.$$

2.7.3

1. Let $W \cong \mathbb{C}^2$ denote the fundamental representation of $\mathfrak{su}(2)$ and $\bar{W}$ the complex conjugate representation. Show that there exists a matrix $A \in \mathrm{GL}(2, \mathbb{C})$ such that
$$AMA^{-1} = \bar{M} \quad \forall M \in \mathfrak{su}(2).$$
Conclude that W and $\bar{W}$ are isomorphic as $\mathfrak{su}(2)$-representations.
2. Let $V_k \cong \mathbb{C}$ denote the representation of $\mathfrak{u}(1)$ with winding number $k \neq 0$. Prove that V_k and $\bar{V}_k$ are not isomorphic as $\mathfrak{u}(1)$-representations.
3. Let $V \cong \mathbb{C}^2$ denote the fundamental representation of the real Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and $\bar{V}$ the complex conjugate representation. Prove that V and $\bar{V}$ are not isomorphic as $\mathfrak{sl}(2, \mathbb{C})$-representations.
4. Does the complex conjugate representation make sense for complex representations of complex Lie algebras, like the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$?

Remark It can be shown that the fundamental representation of $\mathfrak{su}(n)$ for every $n \geq 3$ is not isomorphic to its complex conjugate. The only other compact simple Lie algebras which have complex representations not isomorphic to their conjugate are $\mathfrak{so}(4n+2)$ for every $n \geq 1$ (Weyl spinor representations) and E_6 (a 27-dimensional representation), see [104]. This is one of the reasons why Lie groups such as $\mathrm{SU}(5)$, $\mathrm{Spin}(10)$ or E_6 appear as gauge groups of Grand Unified Theories; see Sect. 8.5.3.

2.7.4 Determine the charges of the basis element $\tau_3 \in \mathfrak{su}(2)$ in:

1. the fundamental representation of $\mathfrak{su}(2)$ on $\mathbb{C}^2$;
2. the representation of $\mathfrak{su}(2)$ on $\mathbb{C}^3$ via the isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ and the complex fundamental representation of $\mathfrak{so}(3)$.

2.7.5

1. Consider the Lie group $\mathrm{SU}(2)$ with the fundamental representation on $\mathbb{C}^2$. Each of the basis vectors τ_1, τ_2, τ_3 of $\mathfrak{su}(2)$ from Example 1.5.32 generates

a one-parameter subgroup isomorphic to U(1). Determine the explicit branching rule for the fundamental representation on $\mathbb{C}^2$ under restriction to these circle subgroups, i.e. determine the corresponding decomposition of $\mathbb{C}^2$ into invariant complex subspaces together with the winding numbers of the induced representations.
2. Do the same exercise with the complex representation of SU(2) on $\mathbb{C}^3$ via the universal covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and the complex fundamental representation of SO(3).
3. Do the same exercise for the Lie group SU(3) with the fundamental representation on $\mathbb{C}^3$ and the circle subgroups generated by the basis vectors $v_1, \ldots, v_8$ of $\mathfrak{su}(3)$, where $v_a = \frac{i\lambda_a}{2}$ with the Gell-Mann matrices λ_a from Example 1.5.33 (cf. Exercise 1.9.26).

2.7.6 Consider the embedding

$$\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$$

from Exercise 1.9.10. Let $V = \mathbb{C}^{2n}$ be the complex fundamental representation of $\mathrm{SO}(2n)$. Determine the branching rule of the representation V under restriction to the subgroup $\mathrm{U}(n) \subset \mathrm{SO}(2n)$. It may be helpful to first consider the case $n = 1$.

2.7.7 Let G be a Lie group. The **center** of G is defined as

$$Z(G) = \{g \in G \mid gh = hg \quad \forall h \in G\}.$$

Suppose that G is connected.

1. Prove that the center $Z(G)$ is the kernel of the adjoint representation Ad_G. Conclude that $Z(G)$ is an embedded Lie subgroup in G with Lie algebra given by the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.
2. Prove that $\mathfrak{g}$ is abelian if and only if G is abelian.
3. Prove that Ad_G is trivial if and only if G is abelian. Conclude that the left-invariant and right-invariant vector fields on a connected Lie group G coincide if and only if G is abelian.

2.7.8 Consider the Lie algebra isomorphism of $\mathfrak{so}(3)$ with $(\mathbb{R}^3, \times)$ from Exercise 1.9.14.

1. Determine the symmetric bilinear form on $\mathbb{R}^3$ corresponding under this isomorphism to the Killing form $B_{\mathfrak{so}(3)}$.
2. Interpret the high school formula

$$z \cdot (x \times y) = -y \cdot (x \times z) \quad \forall x, y, z \in \mathbb{R}^3,$$

where $\cdot$ denotes the scalar product, in light of the first part of this exercise.

2.7.9

1. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $\phi\colon \mathfrak{g} \to \mathfrak{h}$ a Lie algebra homomorphism. Suppose that $\mathfrak{g}$ is simple. Show that ϕ is either injective or the trivial homomorphism. In particular, every representation of a simple Lie algebra is either faithful or trivial.
2. Show that every complex 1-dimensional representation of a semisimple Lie algebra is trivial.
3. Show that every homomorphism from a connected semisimple Lie group to $\mathrm{U}(1)$ is trivial. Find a non-trivial homomorphism from $\mathrm{U}(n)$ to $\mathrm{U}(1)$.

2.7.10 Let $\mathfrak{g}$ be a real Lie algebra. The **complexification** of $\mathfrak{g}$ is the complex Lie algebra

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$$

with the Lie bracket from $\mathfrak{g}$ extended $\mathbb{C}$-bilinearly. Show that if $\mathfrak{g}_{\mathbb{C}}$ is (semi-)simple, then $\mathfrak{g}$ is (semi-)simple.

Remark The following converses can be shown: If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}_{\mathbb{C}}$ is semisimple (this uses Theorem 2.4.9) and if $\mathfrak{g}$ is compact simple, then $\mathfrak{g}_{\mathbb{C}}$ is simple (see [83]).

2.7.11 Prove Proposition 2.4.26: the structure constants f_{abc} for a $B_{\mathfrak{g}}$-orthonormal basis $\{T_a\}$ of a semisimple Lie algebra $\mathfrak{g}$ are totally antisymmetric:

$$f_{abc} = -f_{bac} = f_{bca} = -f_{acb} \quad \forall a, b, c \in \{1, \ldots, n\}.$$

2.7.12 Let τ_1, τ_2, τ_3 be the basis of the Lie algebra $\mathfrak{su}(2)$ from Example 1.5.32. Fix an arbitrary, positive real number $g > 0$ and let

$$\beta_a = g\tau_a \in \mathfrak{su}(2) \quad (a = 1, 2, 3).$$

Define a unique positive definite scalar product $\langle \cdot, \cdot \rangle_g$ with associated norm $||\cdot||_g$ on $\mathfrak{su}(2)$ so that $\beta_1, \beta_2, \beta_3$ form an orthonormal basis. Determine the relation between $\det(X)$ and the norm $||X||_g$ for $X \in \mathfrak{su}(2)$. Show that the scalar product $\langle \cdot, \cdot \rangle_g$ is $\mathrm{Ad}_{\mathrm{SU}(2)}$-invariant.

2.7.13 Consider the Lie algebra $\mathfrak{su}(2)$.

1. Calculate the Killing form $B_{\mathfrak{su}(2)}$ directly from the definition and determine the constant g so that $-B_{\mathfrak{su}(2)} = \langle \cdot, \cdot \rangle_g$, where $\langle \cdot, \cdot \rangle_g$ is the scalar product from Exercise 2.7.12.
2. Fix an arbitrary, positive, real number $\lambda > 0$ and set

$$\begin{aligned} F_\lambda\colon \mathfrak{su}(2) \times \mathfrak{su}(2) &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto \lambda \mathrm{tr}(X \cdot Y), \end{aligned}$$

where tr denotes the trace and $\cdot$ the matrix product. Show that $-F_\lambda$ is a negative definite $\mathrm{Ad}_{\mathrm{SU}(2)}$-invariant scalar product on $\mathfrak{su}(2)$. Determine the constant λ so that $F_\lambda = B_{\mathfrak{su}(2)}$.

2.7.14 Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with Killing form $B_{\mathfrak{sl}(2,\mathbb{R})}$. Show that there exists a constant $\lambda \in \mathbb{R}$ such that

$$B_{\mathfrak{sl}(2,\mathbb{R})}(X,Y) = \lambda \mathrm{tr}(X \cdot Y) \quad \forall X, Y \in \mathfrak{sl}(2,\mathbb{R}),$$

where tr denotes the trace of the matrix and $\cdot$ the matrix product. Determine this constant λ. Is the Killing form $B_{\mathfrak{sl}(2,\mathbb{R})}$ definite? or non-degenerate?

2.7.15 Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

1. Show that the Killing form of the Lie algebra $\mathfrak{gl}(n,\mathbb{K})$ can be calculated as

$$B_{\mathfrak{gl}(n,\mathbb{K})}(X,Y) = 2n\mathrm{tr}(X \cdot Y) - 2\mathrm{tr}(X)\mathrm{tr}(Y).$$

   A suitable basis for $\mathfrak{gl}(n,\mathbb{K})$ to evaluate the trace on the left-hand side is given by the elementary matrices E_{ij} with a 1 at the intersection of the i-th row and j-th column and zeros elsewhere.
2. Let $\mathfrak{h}$ be an ideal in a Lie algebra $\mathfrak{g}$. Prove that for all $X, Y \in \mathfrak{h}$

$$B_{\mathfrak{h}}(X,Y) = B_{\mathfrak{g}}(X,Y).$$

3. Show that the Killing form of the Lie algebra $\mathfrak{sl}(n,\mathbb{K})$ is equal to

$$B_{\mathfrak{sl}(n,\mathbb{K})}(X,Y) = 2n\mathrm{tr}(X \cdot Y).$$

   Compare with Exercise 2.7.14.

2.7.16

1. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_\mathbb{C}$ its complexification as in Exercise 2.7.10. Under the canonical inclusion $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$ as the real part show that for all $X, Y \in \mathfrak{g}$

$$B_{\mathfrak{g}}(X,Y) = B_{\mathfrak{g}_\mathbb{C}}(X,Y).$$

2. Explain the difference between the results for the Killing form in Exercise 2.7.13 and Exercise 2.7.14, given the isomorphism of complex Lie algebras

$$\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{R})_\mathbb{C}$$

   from Exercise 1.9.18.
3. Show that every complex matrix A can be written uniquely as $A = B + iC$ with B, C skew-Hermitian. Conclude that

$$\mathfrak{u}(n)_\mathbb{C} \cong \mathfrak{gl}(n,\mathbb{C}),$$

$$\mathfrak{su}(n)_\mathbb{C} \cong \mathfrak{sl}(n,\mathbb{C}).$$

4. Show that the Killing forms of the Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ can be calculated as

$$B_{\mathfrak{u}(n)}(X, Y) = 2n\mathrm{tr}(X \cdot Y) - 2\mathrm{tr}(X)\mathrm{tr}(Y),$$
$$B_{\mathfrak{su}(n)}(X, Y) = 2n\mathrm{tr}(X \cdot Y).$$

Compare with Exercise 2.7.13.

2.7.17 Consider the basis of $\mathfrak{su}(3)$ given by the elements $i\lambda_a$, where λ_a are the Gell-Mann matrices from Example 1.5.33, with $a = 1, \ldots, 8$. Show that these basis vectors are orthogonal with respect to the Killing form $B_{\mathfrak{su}(3)}$ and determine $B_{\mathfrak{su}(3)}(i\lambda_a, i\lambda_a)$ for all a.

2.7.18

1. The **rank** of a compact Lie group G is the maximal dimension of an embedded torus subgroup $T \subset G$. Prove that the rank of a product $G \times H$ of compact Lie groups G and H is the sum of the ranks of G and H (you can assume without proof that a connected abelian Lie group is a torus).
2. Classify compact semisimple Lie algebras of rank $r = 1, 2, 3, 4$, assuming Theorem 2.4.23.