Appendix B
Background on Special Relativity and Quantum Field Theory

B.1 Basics of Special Relativity

We very briefly recall some basic concepts from special relativity. A very good introduction to the physics and mathematics of special relativity can be found in [95], covering much more than we need.

Special relativity is formulated on **Minkowski spacetime** $M = \mathbb{R}^4$ with a pseudo-Riemannian metric known as **Minkowski metric** η given by (we use units where the speed of light $c = 1$)

$$\eta_{\mu\nu} = \eta(\partial_\mu, \partial_\nu) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

This choice of signs $(+,-,-,-)$ is called the *West Coast metric*. Sometimes the *East Coast metric* with signature $(-,+,+,+)$ is used instead. The x^μ are the standard coordinates on $\mathbb{R}^4$, also written as

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

or $x^\mu = (t, \mathbf{x})$. Coordinate systems (charts) on Minkowski spacetime correspond to reference frames of moving observers. All inertial systems (unaccelerated orthonormal reference frames, also called Lorentz frames) with the same origin $(0, \mathbf{0})$ are related by **Lorentz transformations**

$$x'^\rho = \Lambda^\rho{}_\mu x^\mu.$$

M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
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Here the Einstein summation convention is understood and Λ is a matrix preserving the metric η

$$\eta(\Lambda v, \Lambda w) = \eta(v, w) \quad \forall v, w \in \mathbb{R}^4,$$

also written as

$$\Lambda^T \eta \Lambda = \eta$$

or

$$\eta_{\rho\sigma} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}.$$

If the origins are different, then the coordinate transformations between inertial systems are given by **Poincaré transformations**

$$x'^{\rho} = \Lambda^{\rho}{}_{\mu} x^{\mu} + a^{\rho},$$

where $a^{\rho} \in \mathbb{R}^4$ is a constant vector. Poincaré transformations are affine transformations.

Since the Minkowski distance between two points is independent of the chosen inertial frame, two points with distance zero *(lightlike)* in one frame have the same distance zero in any other Lorentz frame, meaning that the speed of light $c = 1$ is the same in any inertial frame.

The basis vectors of Lorentz frames transform as

$$e'_{\rho} = \left(\Lambda^{-1}\right)^{\mu}{}_{\rho} e_{\mu},$$

so that the vector $\Delta x^{\mu} e_{\mu}$ is invariant:

$$\begin{aligned} \Delta x'^{\rho} e'_{\rho} &= \Delta x^{\mu} \Lambda^{\rho}_{\mu} \left(\Lambda^{-1}\right)^{\nu}{}_{\rho} e_{\nu} \\ &= \Delta x^{\mu} e_{\mu}. \end{aligned}$$

The same vector a on spacetime can be expressed in the frame e_{μ} or the frame e'_{ρ}:

$$a^{\mu} e_{\mu} = a = a'^{\rho} e'_{\rho}.$$

This implies that

$$a'^{\rho} = \Lambda^{\rho}{}_{\mu} a^{\mu}.$$

Similarly, a 1-form ω can be expressed in the frame dx^{μ} or the frame dx'^{ρ}:

$$\omega_{\mu} dx^{\mu} = \omega = \omega'_{\rho} dx'^{\rho}.$$

This implies that

$$\omega'_\rho = \left(\Lambda^{-1}\right)^\mu{}_\rho \omega_\mu.$$

If a is a vector with components a^μ in the frame e_μ and we set

$$a_\nu = \eta_{\nu\mu} a^\mu,$$

then a_ν transforms as the components of a 1-form. Similarly, if we define the matrix $\eta^{\mu\nu}$ as the inverse of the matrix $\eta_{\mu\nu}$ (in the Minkowski case this is the same matrix) and if ω_μ are the components of a 1-form in the frame dx^μ, then

$$\omega^\nu = \eta^{\nu\mu} \omega_\mu$$

transforms as the components of a vector. This is the idea behind lowering and raising indices, which can be extended to arbitrary tensors.

Let $u \in \mathbb{R}^4$ be the velocity vector of a particle of mass $m > 0$. Going to the rest frame of the particle, the vector u has components $u = (1, \mathbf{0}) = (1, 0, 0, 0)$, which implies that $\eta(u, u) = 1$ independent of the chosen frame. In any frame we define the 4-momentum

$$mu = (E, \mathbf{p}),$$

where E is the energy and $\mathbf{p}$ the 3-momentum of the particle in that frame. Then $\eta(u, u) = 1$ implies

$$m^2 = E^2 - \mathbf{p}^2.$$

If we introduce again the speed of light c, then $\eta(u, u) = c^2$,

$$mu = \left(\frac{E}{c}, \mathbf{p}\right),$$

and

$$m^2 c^4 = E^2 - \mathbf{p}^2 c^2.$$

Relativistic theories of physics are theories formulated in Minkowski spacetime whose laws are invariant under Poincaré transformations. This means that the laws of physics are independent of where and when experiments are performed (invariance under space and time translations), how the experiments are oriented in space (invariance under rotations) and whether they are performed in different inertial systems moving with constant velocity (invariance under Lorentz boosts). For example, the principle of relativity claims that the laws of physics are the same here and in the Andromeda galaxy, they are the same now and in 1 million years and

they are the same on board two spacecrafts flying with arbitrary, constant velocities in different directions.

B.2 A Short Introduction to Quantum Field Theory

From classical gravity and electromagnetism we are used to thinking of matter as particles and interactions as carried by fields. However, according to quantum field theory, matter and interactions can both be described by particles *and* fields. Quantum field theory can be thought of as a unification of the concepts of classical fields and point particles and thus as a unification (in some sense) of interactions and matter (supersymmetric quantum field theories are a unification of both concepts in an even stronger sense). The remarkable consequences of this approach are that forces between matter particles can be reduced to couplings between different types of fields and that symmetry groups, such as gauge symmetries, can act through representations on both interaction and matter fields.

In the following sections we briefly want to discuss the basics of quantum field theory and the relation between particles and fields. Our intention is to give a short overview and interpretation, without any calculations or trying to be mathematically rigorous. We also assume a basic familiarity with quantum mechanics.

B.2.1 Quantum Field Theory and Quantum Mechanics

Quantum field theory (QFT) is a quantum theory, in some sense similar to quantum mechanics (QM):

- A quantum system has a **Hilbert space** V with a Hermitian scalar product $\langle\cdot|\cdot\rangle$. Elements of the vector space V are **state vectors (states)** $|v\rangle$ (we normalize these vectors to unit norm). We think of the state of the system as being time-dependent $|v, t\rangle$ (Schrödinger picture). However, we can equivalently think of the states as being time-independent and instead the operators as being time-dependent (Heisenberg picture, usually preferred in QFT).
- We cannot measure the state of a system directly, we can only measure the value of **observables**, described by Hermitian operators A on V. If $|v\rangle$ is an eigenvector of A with eigenvalue a,

$$A|v\rangle = a|v\rangle,$$

  and we measure the observable A if the system is in the state $|v\rangle$, then the value is the eigenvalue a. For an arbitrary state $|w\rangle$, the expectation value of the observable A is related to $\langle w|A|w\rangle$.

- We are also interested in **transition amplitudes** between states, given by scalar products $\langle w|v\rangle$. The amplitudes determine transition probabilities (the probability that the system in the state $|v\rangle$ is found after a measurement in the state $|w\rangle$) by taking the absolute value squared of this complex number.
- There is a Hermitian **Hamiltonian operator** H which determines the evolution of states between times t_0 and t (by convention $t_0 = 0$): we define the unitary operator

$$U(t,0) = e^{-\frac{i}{\hbar}Ht},$$

where $\hbar$ is the Planck constant (note that the exponential of a skew-Hermitian operator is unitary). Then time evolution of states is given by

$$|v,t\rangle = U(t,0)|v,0\rangle.$$

One of our aims is to determine the time evolution operator $U(t,0)$. Ideally we would like to diagonalize H, i.e. find an eigenbasis for H of states $|n\rangle$ of energy E_n,

$$H|n\rangle = E_n|n\rangle,$$

because such states have a very simple time evolution:

$$|n,t\rangle = e^{-iE_nt}|n,0\rangle,$$

where $e^{-iE_nt} \in \mathrm{U}(1)$ is just a complex number of absolute value 1. In general, in an interacting theory, this will be practically impossible.
- We can change from the Heisenberg picture to the Schrödinger picture and vice versa as follows: the Schrödinger-type operator is the Heisenberg-type operator taken at $t_0 = 0$:

$$A_S = A_H(0).$$

The time evolution of the Heisenberg-type operator is then given by the Hamiltonian:

$$A_H(t) = e^{\frac{i}{\hbar}Ht}A_S e^{-\frac{i}{\hbar}Ht}.$$

So far everything should be familiar from QM. We now discuss what is peculiar about QFT.

B.2.2 Free Quantum Field Theory on 0-Dimensional Space

Suppose that space is 0-dimensional and consists only of a single point. A real-valued field at this point is just a time dependent real number $\phi(t)$. The simplest type of quadratic Lagrangian for this field is

$$\mathscr{L} = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}m^2\phi^2.$$

This Lagrangian is known as the **harmonic oscillator**. The Euler–Lagrange equation for this Lagrangian is the ordinary differential equation

$$\ddot{\phi} + m^2\phi = 0.$$

The Hilbert space of the associated quantum theory can be described as follows: let

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathbb{C}.$$

The basis states corresponding to this direct sum decomposition are denoted by

$$|n\rangle, \quad n = 0, 1, 2, 3, \ldots$$

The vector space $\mathscr{H}$ is the Hilbert space of the harmonic oscillator.

- The states $|n\rangle$ are eigenvectors for the Hamiltonian H with energy E_n growing linearly with n. These states are interpreted as the *discrete set* of different vibrational modes of the field at the point.
- There is a Hermitian **number operator** N (an observable) whose eigenvectors are $|n\rangle$ with eigenvalue n:

$$N|n\rangle = n|n\rangle.$$

B.2.3 Free Quantum Field Theory on d-Dimensional Space

Free quantum field theories (and to a certain degree, weakly interacting, perturbative quantum field theories) have an interpretation in terms of **particles**.

Canonical Quantization

We consider the case of field theories on d-dimensional Euclidean space (for simplicity we assume $d = 3$). A real-valued field is now a real function $\phi(t, \mathbf{x})$

depending on time t and the space coordinate $\mathbf{x}$. The simplest type of quadratic Lagrangian for this field is the **Klein–Gordon Lagrangian**.

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{B.1}$$

The Euler–Lagrange equation for this Lagrangian is the linear wave equation

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0,$$

called the Klein–Gordon equation.

The Hilbert space of the associated quantum field theory can be described as follows: let V_1 be the Hilbert space of a single free **bosonic** particle. It is spanned by the basis states $|\mathbf{p}\rangle = |1_\mathbf{p}\rangle$, where $\mathbf{p} \in \mathbb{R}^3$ is the momentum of the particle, related to its energy by $m^2 = E^2 - \mathbf{p}^2$. In these states the particle is totally delocalized in position space. States where the particle is localized both in momentum and position space with a certain minimal width (given by Heisenberg's uncertainty principle) can be obtained as linear combinations of the states $|\mathbf{p}\rangle$, called **wave packets**.

A general construction in quantum theory implies that the Hilbert space of n indistinguishable particles of the same type is given by

$$V_n = \mathrm{Sym}^n V_1.$$

n-particle states are thus (linear combinations of) symmetrized tensor products of 1-particle states. We then form the (bosonic) **Fock space**

$$\mathscr{F} = \mathrm{Sym}^* V_1 = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V_1$$

that contains states with an arbitrary number of particles. It turns out that the Fock space $\mathscr{F}$ is a suitable Hilbert space for the quantum field theory described by the Klein–Gordon Lagrangian (B.1).

- A basis for the Fock space is given by states

  $$|n_{\mathbf{p}_1}, \ldots, n_{\mathbf{p}_r}\rangle, \quad n_{\mathbf{p}_i} = 0, 1, 2, 3, \ldots,$$

  where $\mathbf{p}_i \in \mathbb{R}^3$ are vectors and $n_{\mathbf{p}_i}$ is the number of particles of momentum $\mathbf{p}_i$. The total number of particles in this state is

  $$n = \sum_{i=0}^{r} n_{\mathbf{p}_i}.$$

- These states are eigenvectors of the Hamiltonian H with energy growing linearly with the numbers $n_{\mathbf{p}_i}$ and they again correspond to different vibrational modes of the field.
- The basis state $|0\rangle$ of the 1-dimensional space $V_0 \cong \mathbb{C}$, where all $n_{\mathbf{p}_i}$ are zero, is called the **vacuum**. The vacuum is the unique eigenstate of the Hamiltonian of eigenvalue 0.
- For every vector $\mathbf{p}$ there is a Hermitian **number operator** $N_{\mathbf{p}}$ with eigenvectors

$$N_{\mathbf{p}}|n_{\mathbf{p}}, n_{\mathbf{p}_1}, \ldots, n_{\mathbf{p}_r}\rangle = n_{\mathbf{p}}|n_{\mathbf{p}}, n_{\mathbf{p}_1}, \ldots, n_{\mathbf{p}_r}\rangle, \quad \text{where} \quad \mathbf{p}_i \neq \mathbf{p} \quad \forall i = 1, \ldots, r.$$

  The number operator is an observable which returns the number of particles of a given momentum in a given quantum state.
- The classical field $\phi(\mathbf{x})$ becomes in the QFT a field of Schrödinger-type operators $\hat{\phi}(\mathbf{x})$, depending on the space point $\mathbf{x}$ (more precisely, an operator-valued distribution), that all act on the same Hilbert space V. This field $\hat{\phi}$ of operators is called the **quantum field**. Together with the adjoint quantum field $\hat{\phi}^\dagger$ it creates and annihilates particles in the point $\mathbf{x}$, i.e. adds or removes these particles from the state in the Hilbert space. In the Heisenberg picture the field depends on the point $(t, \mathbf{x})$ in spacetime.

Similarly, the Fock space V for a free **fermionic** field can be generated using antisymmetrized tensor products:

$$\mathscr{F} = \Lambda^* V_1 = \bigoplus_{n=0}^{\infty} \Lambda^n V_1.$$

In this case the numbers $n_{\mathbf{p}}$ can only take the values 0 or 1.

These descriptions of the Fock spaces make it clear that the Hilbert space in quantum field theory is infinite-dimensional in two ways: the one-particle space V_1 is infinite-dimensional because the vector space $\mathbb{R}^3$ of momenta $\mathbf{p}$ has infinitely many elements (this is related to the fact that space is continuous and infinitely extended). In addition there is the infinite direct sum over the number of particles $N \in \mathbb{N}_0$ that we already encountered in the case of quantum mechanics of the harmonic oscillator (for the harmonic oscillator the vector space corresponding to V_1 is 1-dimensional).

B.2.4 Unitary Representation of the Poincaré Group

As one of the general axioms of QFT on 4-dimensional Minkowski spacetime we assume that the Hilbert space of the quantum theory carries a **unitary representation** of the universal covering group $\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$ of the Poincaré group. Note

that the non-compact simple Lie group $\mathrm{SL}(2,\mathbb{C})$ does not admit non-trivial finite-dimensional unitary representations according to Theorem 2.1.44.

B.2.5 Interacting Quantum Field Theories

A typical question in QFT is to calculate *scattering amplitudes*: Suppose we send in an (idealized) collider a total number of n particles with certain momenta $\mathbf{p}_i$ and want to determine the probability that we find after collision n' particles with certain momenta $\mathbf{p}'_j$. This process is governed by the laws of quantum theory: in general we can only calculate a *probability* for the process or transition to happen, we cannot predict the outcome completely, even if we know the initial state exactly. Since the numbers n and n' as well as the types of particles and their momenta can be different, certain particles get created and others annihilated in the scattering process.[1]

To describe interacting QFTs, like the ϕ^4-theory with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4,$$

we make the following assumptions, known as the **interaction picture**:

- The Hilbert space V of the interacting theory is the same Hilbert space as in the free theory (the Fock space). The Schrödinger-type field operators (operator-valued distributions) are also the same as in the free theory.
- The Hamiltonian H can be calculated from the Lagrangian $\mathscr{L}$ of the field theory, expressed through the fields, that we collectively denote by ϕ. The Hamiltonian of the interacting theory is of the form

  $$H = H_0 + H_I,$$

  where H_0 is the Hamiltonian of the free theory and H_I is the interaction part. Since the Hamiltonian H of the interacting theory is different from the Hamiltonian H_0 of the free theory, we expect the vacuum state $|\Omega\rangle$ of the interacting theory to be different from the vacuum state $|0\rangle$ of the free theory, even though both states (under our assumption) are elements of the same Fock space V.
- The time-dependence of the Heisenberg-type field operators will be different in the free and interacting theories, because the Hamiltonians are different. One considers two types of time-dependent quantum fields: the *Heisenberg picture field* is given the time evolution according to the full Hamiltonian H and the *interaction picture field* is given the time evolution according to the free Hamiltonian H_0.

[1] Bound states, like atoms and hadrons, are described in QFTs using other methods.

Scattering of particles in a collider can now be described as follows. We assume that for time $t \to -\infty$ in the distant past and for time $t' \to +\infty$ in the distant future the particles are far apart and can be considered as free. We can then think of the collections of particles that we send into and get out of the collider as states in the Hilbert space V:

$$|v, t\rangle = |n_{\mathbf{p}_1}, \ldots, n_{\mathbf{p}_r}, t\rangle,$$
$$|v', t'\rangle = |n_{\mathbf{p}'_1}, \ldots, n_{\mathbf{p}'_s}, t'\rangle.$$

We want to calculate the scalar product

$$\langle v', t'|U(t', t)|v, t\rangle.$$

If we know these scalar products, we can calculate the transition amplitudes between any two states, because the momentum states form a basis for the Hilbert space. Since the time evolution operator U is defined via the Hamiltonian H, it follows that the scalar product for different ingoing and outgoing states will be non-zero only if the action of H on particle states creates and annihilates certain particles.

More precisely, in a free field theory the Hamiltonian is quadratic in the fields ϕ and can be diagonalized. The eigenbasis is just given by the particle momentum states

$$|n_{\mathbf{p}_1}, \ldots, n_{\mathbf{p}_r}\rangle$$

which are fixed by the action of H up to multiplication by the eigenvalue (the total energy E of the collection of particles). It follows that in a free field theory particles do not get created or annihilated. The transition amplitude between different particle momentum states is zero and all scattering processes are trivial: we get the same particles with the same momenta out that we sent into the collider. The vibrational modes of the field described by these states are constant, independent of time.

In an interacting theory the Hamiltonian contains **anharmonic terms**, i.e. terms of order three or higher in the fields ϕ. Such Hamiltonians lead to non-trivial creation and annihilation of particles and thus to non-trivial scattering processes. Heuristically, the vibrational modes of the fields change with time and, since the fields are coupled and the corresponding equations of motion are **non-linear wave equations**, the vibrations of one field can start vibrations of another field.

The description so far assumed that the Hilbert space and the action of the Schrödinger-type field operators are the same for the interacting theory as for the free theory, only the vacuum state and the Hamiltonian have changed. This assumption is merely a first approximation and actually not consistent, according to *Haag's Theorem*: Schrödinger-type quantum fields for a free and an interacting theory cannot be the same; see, for example, [116, p. 391]. To define quantum field theories in a mathematically rigorous way is the aim of *constructive quantum field theory (CQFT)* and *algebraic quantum field theory (AQFT)*.

B.2.6 Path Integrals

Transition amplitudes can also be calculated using **path integrals**. Path integrals for interacting theories, in particular, gauge theories, are more convenient than canonical quantization. The path integral approach to quantum theory was originally developed by P.A.M. Dirac and R.P. Feynman.

To understand the idea of path integrals, recall that standard integrals are integrals of functions over finite-dimensional vector spaces (or finite-dimensional manifolds). Path integrals are integrals of functions (also called functionals) over *infinite-dimensional* vector spaces (or infinite-dimensional manifolds). These vector spaces arise naturally as the spaces of all fields of a certain type on spacetime, i.e. the space $\mathscr{C}^\infty(M, W)$, where M is spacetime and W is the vector space in which the field ϕ takes values (more generally, we could consider the space of all sections of a vector bundle over M). Path integrals over the infinite-dimensional vector space $\mathscr{C}^\infty(M, W)$ can be approximated by standard integrals over a finite-dimensional vector space if spacetime is replaced by a lattice with finite lattice spacing $a > 0$ (and finite extension).

The path integrals that appear in QFT are of the form

$$G(x_1, \ldots, x_n) = \frac{1}{\mathscr{N}} \int \mathscr{D}\phi \, \phi(x_1) \ldots \phi(x_n) \exp\left(\frac{i}{\hbar} \int_M \mathscr{L}(\phi) \, \mathrm{dvol}\right),$$

called **Green's functions** or **correlators**. Here $x_1, \ldots, x_n \in M$ are points in spacetime, ϕ is the field, $\mathscr{D}\phi$ is the **path integral measure** on the space $\mathscr{C}^\infty(M, W)$ and $\mathscr{L}$ is the Lagrangian of the field theory. The number $\mathscr{N}$ is a normalization constant. It is clear that (say for a complex scalar field ϕ, taking value in the complex numbers $W = \mathbb{C}$) for fixed points $x_1, \ldots, x_n$ the map

$$\begin{aligned} F\colon \mathscr{C}^\infty(M, \mathbb{C}) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \phi(x_1) \ldots \phi(x_n) \exp\left(\frac{i}{\hbar} \int_M \mathscr{L}(\phi) \, \mathrm{dvol}\right) \end{aligned}$$

is a function (functional) on the vector space $\mathscr{C}^\infty(M, \mathbb{C})$. The path integral

$$G(x_1, \ldots, x_n) = \frac{1}{\mathscr{N}} \int \mathscr{D}\phi \, F(\phi)$$

is the integral of this function over the infinite-dimensional space $\mathscr{C}^\infty(M, \mathbb{C})$.

Notice that the field ϕ here is the classical field, not the quantum field of operators. The approach to QFT using path integrals is independent of the approach using Hilbert spaces and quantum fields. However, it can be shown that if one knows all the Green's functions, then the Hilbert space together with the quantum fields can be reconstructed (this is known as the *Wightman Reconstruction Theorem*).

In general, it is very difficult to calculate or even define these path integrals precisely. Usually, this can only be done in the case of the free field with a quadratic Lagrangian $\mathscr{L}$. Scattering amplitudes can be calculated from Green's functions using the *LSZ reduction formula*, named after H. Lehmann, K. Symanzik and W. Zimmermann.

B.2.7 Series Expansions

The actual calculation of scattering amplitudes is a formidable task and often can only be done approximately, using power series expansions. In general, the Green's functions are functions of the coupling constant(s) g and the Planck constant $\hbar$:

$$G = G(x_1, \ldots, x_n, g, \hbar).$$

There are mainly two types of series expansions:

Perturbation Theory

Perturbation theory works if the coupling constant g is small so that the full Lagrangian $\mathscr{L}$ is a small perturbation

$$\mathscr{L} = \mathscr{L}_0 + g\mathscr{L}_{\text{int}}$$

of the free Lagrangian $\mathscr{L}_0$. The Green's functions for $\mathscr{L}_0$ are known and the Green's functions for $\mathscr{L}$ can be calculated in a series expansion in orders of g, by expanding the exponential

$$\exp\left(\frac{ig}{\hbar}\int_M \mathscr{L}_{\text{int}}(\phi)\,\text{dvol}\right)$$

in a power series in orders of g and then interchanging the path integral and the infinite sum (this step is mathematically *not* justified [50]). The terms in the power series expansion are described by **Feynman diagrams**. With the order of g increasing in each step by 1, the terms in the series expansion for a process with fixed external lines (in-coming and out-going particles) are called **leading order (LO)**, **next-to-leading order (NLO)**, **next-to-next-to-leading order (NNLO)**, and so on.

In perturbation theory, the full interacting Lagrangian is treated as a small perturbation of the free Lagrangian. Since the states of a free quantum field have an interpretation in terms of particles, it makes sense to think of perturbation theory as describing **(weakly) interacting particles**. Feynman diagrams, that depict these interactions, are the hallmark of perturbative quantum field theory.

Internal lines in Feynman diagrams represent intermediate **virtual particles** which are off-shell, i.e. do not satisfy the mass energy relation $m^2 = E^2 - \mathbf{p}^2$ (even though momentum and energy are conserved at each vertex). In contrast to the in- and out-state particles, virtual particles are therefore not "real" particles and cannot be detected (the photons that mediate interactions between electrons are different from the photons that we can see or detect with cameras).

A problem of perturbation theory is that the perturbation expansion in orders of the coupling constant g actually converges only for $g = 0$, i.e. the radius of convergence of the power series is zero (there is an argument due to Freeman Dyson that a QFT cannot be well-defined for negative values of the coupling constant g, hence the expansion around $g = 0$ must have radius of convergence equal to zero). This implies that the perturbation expansion only makes sense as an **asymptotic expansion**: up to a certain optimal order of g the series expansion approximates the Green's function better and better, but then, adding terms of higher order, the series expansion starts getting worse and eventually diverges.

Semi-Classical Approximation

Semi-classical approximation can be used if the Planck constant $\hbar$ is a (relatively) small number. The power series for the Green's functions is then an expansion in orders of $\hbar$. The lowest term of order zero is the **classical contribution** and terms in higher order of $\hbar$ are **quantum corrections**. In Feynman diagrams the classical contribution corresponds to **tree diagrams**, whereas quantum corrections correspond to **loop diagrams**. With respect to path integrals the semi-classical approximation is an expansion around the critical points of the Lagrangian, i.e. the classical solutions of the field equations (for $\hbar \to 0$ the path integral localizes at these classical solutions).

Non-Perturbative Quantum Field Theories

Note that we do not claim that the Green's functions are analytic in g or $\hbar$. Most smooth functions are, of course, not analytic, because analytic functions are determined everywhere in their domain of definition by their values in an arbitrarily

small neighbourhood of the center of expansion. The series expansions in QFT will therefore only be approximately accurate for small values of g and $\hbar$ and unusable if these parameters are large. In particular, if g is large, the QFT is called **non-perturbative**.

The term *non-perturbative* is essentially a synonym for *non-analytic*. Non-perturbative effects in QFT will typically become dominant if the coupling constant g is large. The particle interpretation breaks down (at least for the fundamental fields) for strongly interacting, non-perturbative QFTs.

B.2.8 Renormalization

The calculation of the contribution of Feynman diagrams with loops[2] involves certain integrals that can diverge and lead to infinite Green's functions. The idea of **renormalization** is to *absorb* the infinities that occur in the Green's functions into the parameters (in particular, the masses and coupling constants), which then become infinite themselves, while the Green's functions become finite. For this to work the parameters have to go in "the right way" to infinity, so that the Green's functions stay finite. More precisely, the parameters are no longer constants, but certain functions of a *cutoff*, and go to infinity when the cutoff is removed, whereas the Green's functions remain finite.

Alternatively, renormalization can be understood as adding to the original Lagrangian of the field theory a counterterm Lagrangian that cancels the divergences of the Green's functions. If we include terms (interactions) of the same form as the counterterms in the original Lagrangian, then adding counterterms is equivalent to a renormalization of parameters. A QFT is called **renormalizable** if finitely many counterterms are needed to cancel the divergences and **non-renormalizable** if infinitely many counterterms are needed. Non-renormalizable theories contain infinitely many different types of interactions and infinitely many parameters, but can still be useful (cf. [125, Sect. 21.2.2]).

The process of renormalization can be explained with a classical analogy, first observed by M. Abraham and H. Lorentz: the electric field of a charged point particle is of the form

$$\mathbf{E}(r) = \frac{\alpha \mathbf{r}}{r^3}$$

[2] Only loop diagrams require renormalization.

where $\alpha \neq 0$ is some constant and $\mathbf{r}$ is the radial vector. The energy density u of the electric field is proportional to $|\mathbf{E}|^2$, hence of the form

$$u(r) = \frac{\beta}{r^4}$$

with a constant $\beta \neq 0$. It follows that the total energy of the field is

$$\int_{\mathbb{R}^3} \frac{\beta}{r^4} \mathrm{dvol} = 4\pi\beta \int_0^\infty \frac{1}{r^2} dr.$$

This integral, when extended all the way to 0, is infinite. It follows that a charged point particle, like an electron, has an infinite energy in its electric field. If this energy is added to the bare rest mass of the electron via $E = mc^2$, corresponding to an *electromagnetic mass*, the total mass becomes infinite, which seems like a contradiction.

The idea is to set the bare (unobservable) rest mass m_B of the electron equal to $-\infty$, so that when we add the infinite energy due to the electric field the total (observable) mass m becomes finite. We define a cutoff $\epsilon > 0$ and set

$$I(\epsilon) = 4\pi\beta \int_\epsilon^\infty \frac{1}{r^2} dr,$$

which is finite for all $\epsilon > 0$. This is called **regularization** of the divergent integral. We also define

$$m_B(\epsilon) = m - \frac{I(\epsilon)}{c^2},$$

where m is the observed mass of the electron, known from experiments. This is called **renormalization** of the mass. The bare mass is thus a function of the cutoff ϵ and goes to $-\infty$ if we let $\epsilon \to 0$. However, the total mass is now

$$m_B(\epsilon) + \frac{I(\epsilon)}{c^2}$$

which is constant and equal to the finite mass m for all $\epsilon > 0$. We see that we have hidden the infinity from the divergent integral $I(0)$ in the renormalization of the mass m_B.

In general, the divergences encountered in QFTs can be traced to two aspects of space: space is continuous (leading to UV divergences) and space is infinitely extended (leading to IR divergences). Both aspects imply that QFTs, which describe time-dependent fields defined on space, have to deal with systems with infinitely many degrees of freedom, the crucial difference to QM. A QFT can be regularized by introducing cutoffs: a UV cutoff essentially means to reduce space to a lattice with finite lattice spacing $a > 0$ (corresponding to an upper cutoff on the norm $|\mathbf{p}|$

of the momentum) and an IR cutoff means to consider the theory in a finite volume $V < \infty$ of space (corresponding to a discrete set of momenta). Both regularizations together reduce the QFT to a system with finitely many degrees of freedom (in continuous time), essentially a version of QM.

B.2.9 Further Reading

Introductory accounts of quantum field theory are [51] and [86]. Extensive discussions from a physics point of view can be found, for example, in [124, 125, 132] and [143–145]. Mathematically rigorous discussions can be found in the classic references [17, 69, 134] and in [38, 45, 82]. Perturbation theory, semi-classical approximation and renormalization are very well and comprehensibly explained from a mathematical point of view in the lecture notes [50].