Appendix A
Background on Differentiable Manifolds

From a mathematical point of view, gauge theories are described by a *spacetime M* together with certain *fibre bundles* (principal bundles, associated vector bundles, spinor bundles) over M. Spacetime and fibre bundles are assumed to have the structure of *differentiable manifolds*. Differentiable manifolds in turn are certain *topological spaces* that essentially have the property of being locally Euclidean, i.e. locally look like an open set in some $\mathbb{R}^n$, and that have a differentiable structure, so that we can define differentiable maps (and their derivatives), vector fields, differential forms, etc. on them.

We briefly sketch the definitions of these concepts. More details can be found in any textbook on differentiable manifolds or differential geometry, like [84] and [142].

A.1 Manifolds

A.1.1 Topological Manifolds

Topological manifolds are topological spaces with certain additional structures. They are a first step towards *differentiable manifolds*, which are the main spaces that we will consider in this book.

Definition A.1.1 An **n-dimensional topological manifold**, also called a **topological n-manifold**, is a topological space M such that:

1. M is locally Euclidean, i.e. locally homeomorphic to $\mathbb{R}^n$. This means that around every point $p \in M$ there exists an open neighbourhood $U \subset M$ that is homeomorphic to some open set $V \subset \mathbb{R}^n$ (both open sets with the subspace topology).

M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
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2. M is Hausdorff.
3. M has a countable basis for its topology.

The local homeomorphisms $\phi: M \supset U \to V \subset \mathbb{R}^n$ (and sometimes the subsets U) are called **charts** or **local coordinate systems** for M. Axiom (a) says that we can cover the whole manifold M by charts. Note that the dimension n is assumed to be the same over the whole manifold. Axiom (c) is of a technical nature and usually can be neglected for our purposes. We often denote an n-manifold by M^n.

Example A.1.2 The simplest topological n-manifold is $M = \mathbb{R}^n$ itself. We can cover M by one chart $\phi: \mathbb{R}^n \to \mathbb{R}^n$, given by the identity.

Example A.1.3 Another example of a topological n-manifold is the **n-sphere** $M = S^n$ for $n \geq 0$. We define

$$S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}.$$

Here $||x||$ denotes the Euclidean norm. We endow S^n with the subspace topology of $\mathbb{R}^{n+1}$. It follows that S^n is Hausdorff, compact and has a countable basis for its topology.

We thus only have to cover S^n by charts that define local homeomorphisms to $\mathbb{R}^n$. A very useful choice are two charts given by **stereographic projection**. We think of $\mathbb{R}^n$ as the hyperplane $\{x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$. We then project a point x in $U_N = S^n \setminus \{N\}$, where N is the north pole

$$N = (0, \ldots, 0, +1) \in S^n \subset \mathbb{R}^{n+1},$$

along the line through N and x onto the hyperplane $\mathbb{R}^n$. It is easy to check that this defines a map

$$\begin{aligned} \phi_N: U_N &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n). \end{aligned}$$

Similarly projection through the south pole

$$S = (0, \ldots, 0, -1) \in S^n \subset \mathbb{R}^{n+1}$$

defines a map on $U_S = S^n \setminus \{S\}$, given by

$$\begin{aligned} \phi_S: U_S &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n). \end{aligned}$$

We can check that ϕ_N and ϕ_S are bijective, continuous and have continuous inverses. Therefore they are homeomorphisms. They define two charts that cover S^n and hence the n-sphere is shown to be a topological manifold.

A.1.2 Differentiable Structures and Atlases

Suppose we have two topological manifolds M and N and a continuous map $f: M \to N$ between them. We want to define what it means that f is *differentiable*. To do so we first have to define a *differentiable (or smooth) structure* on both manifolds.

Definition A.1.4 Let M be a topological n-manifold. Suppose (U, ϕ) and (V, ψ) are two charts of M. We call these charts *compatible* if the **change of coordinates** (or **coordinate transformation**), given by the map

$$\psi \circ \phi^{-1}: \mathbb{R}^n \supset \phi(U \cap V) \longrightarrow \psi(U \cap V) \subset \mathbb{R}^n,$$

is a smooth diffeomorphism between open subsets of $\mathbb{R}^n$, i.e. the homeomorphism $\psi \circ \phi^{-1}$ and its inverse are infinitely differentiable.

Definition A.1.5 Let $\mathscr{A}$ be a set of charts that cover M. We call $\mathscr{A}$ an **atlas** if any two charts in $\mathscr{A}$ are compatible. We call $\mathscr{A}$ a **maximal atlas** (or **differentiable structure**) if the following holds: Any chart of M that is compatible with all charts in $\mathscr{A}$ belongs to $\mathscr{A}$. It can be checked that any given atlas for M is contained in a unique maximal atlas.

Definition A.1.6 A topological manifold M together with a maximal atlas is called a **differentiable** (or **smooth**) **manifold**.

Example A.1.7 The topological manifold $\mathbb{R}^n$ is a differentiable manifold: We have one chart $(\mathbb{R}^n, \mathrm{Id})$, where $\mathrm{Id}: \mathbb{R}^n \to \mathbb{R}^n$ is the identity. Since we only have a single chart, there are no non-trivial changes of coordinates. Therefore $\mathscr{A} = \{(\mathbb{R}^n, \mathrm{Id})\}$ forms an atlas that induces a unique differentiable structure on $\mathbb{R}^n$ (the *standard differentiable structure*).

Example A.1.8 Recall that we defined on the n-sphere S^n two charts (U_N, ϕ_N) and (U_S, ϕ_S). We want to show that these two charts are compatible and hence form an atlas. This atlas is contained in a unique maximal atlas that defines a differentiable structure on the n-sphere (the *standard structure*).

We first have to calculate the inverse of the chart mappings: We have

$$\begin{aligned}\phi_N^{-1}: \mathbb{R}^n &\longrightarrow U_N \\ y &\longmapsto \left(\frac{2y}{1 + ||y||^2}, \frac{||y||^2 - 1}{1 + ||y||^2}\right)\end{aligned}$$

and

$$\begin{aligned}\phi_S^{-1}: \mathbb{R}^n &\longrightarrow U_S \\ y &\longmapsto \left(\frac{2y}{1+||y||^2}, \frac{1-||y||^2}{1+||y||^2}\right).\end{aligned}$$

Since

$$\phi_N(U_N \cap U_S) = \phi_S(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$$

we get

$$\phi_S \circ \phi_N^{-1}: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$

with

$$\begin{aligned}\phi_S \circ \phi_N^{-1}(y) &= \phi_S\left(\frac{2y}{1+||y||^2}, \frac{||y||^2-1}{1+||y||^2}\right) \\ &= \frac{y}{||y||^2}.\end{aligned}$$

A similar calculation shows that

$$\phi_N \circ \phi_S^{-1}(y) = \frac{y}{||y||^2}.$$

Since these maps are infinitely differentiable, it follows that the charts (U_N, ϕ_N) and (U_S, ϕ_S) are compatible and define a smooth structure on the n-sphere S^n.

Remark A.1.9 In certain dimensions n there exist *exotic spheres*, which are differentiable structures on the topological manifold S^n not diffeomorphic to the standard structure. The first examples have been described by Milnor and Kervaire.

Remark A.1.10 From now we consider **only smooth manifolds**.

Example A.1.11 It is possible to extend the definition of smooth manifolds to include manifolds M with boundary ∂M. We usually consider only manifolds without boundary, even though most concepts in this book also make sense for manifolds with boundary.

Definition A.1.12 A manifold M is called **closed** if it is compact and without boundary.

Definition A.1.13 A manifold M is called **oriented** if it has an atlas $\mathscr{A}$ of charts $\{(U_i, \phi_i)\}$ such that the differential $D_{\phi_i(p)}(\phi_j \circ \phi_i^{-1})$ (represented by the Jacobi matrix) of any change of coordinates has positive determinant at each point.

A.1.3 Differentiable Mappings

We can now define the notion of a differentiable map between differentiable manifolds.

Definition A.1.14 Let M^m and N^n be differentiable manifolds and $f: M \to N$ a continuous map. Let $p \in M$ be a point and (V, ψ) a chart of N around $f(p)$. Since f is continuous, there exists a chart (U, ϕ) around p such that $f(U) \subset V$. We call f **differentiable at** p if the map

$$\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \supset \phi(U) \longrightarrow \psi(V) \subset \mathbb{R}^n$$

is infinitely differentiable (in the usual sense) at $\phi(p)$ as a map between open subsets of $\mathbb{R}^n$.

Remark A.1.15 The property of a map f being differentiable at a point p does not depend on the choice of charts, precisely because all changes of coordinates are diffeomorphisms: if f is differentiable at p for one pair of charts, then it is also differentiable for all other pairs.

Definition A.1.16 We call a continuous map $f: M \to N$ **differentiable** if it is differentiable at every $p \in M$. We call f a **diffeomorphism** if it is a homeomorphism such that f and f^{-1} are differentiable.

Remark A.1.17 All differentiable maps between manifolds in the following will be **infinitely differentiable (smooth)**, also called $\mathscr{C}^\infty$.

Example A.1.18 It is a nice exercise to show that the involution

$$\begin{aligned} i: S^n &\longrightarrow S^n \\ x &\longmapsto -x \end{aligned}$$

is a diffeomorphism.

A.1.4 Products of Manifolds

Let M^m and N^n be differentiable manifolds. Then the Cartesian product $X^{m+n} = M^m \times N^n$ canonically has the structure of a differentiable manifold of dimension $m + n$. We have to define charts for X: Let (U, ϕ) and (V, ψ) be local charts for M and N. Then $(U \times V, \phi \times \psi)$ is a local chart for X, where

$$\begin{aligned} \phi \times \psi: U \times V &\longrightarrow \mathbb{R}^m \times \mathbb{R}^n \\ (x, y) &\longmapsto (\phi(x), \psi(y)). \end{aligned}$$

It can easily be checked that with this definition the changes of coordinates are smooth.

A.1.5 Tangent Space

Suppose M^n is a differentiable manifold and $p \in M$ is a point. An important notion is that of the *tangent space* T_pM of the manifold at the point p. This is something that only exists on *smooth* manifolds and not on *topological* manifolds.

How can we define such a tangent space? To get some intuition, we can first consider the case of a *submanifold* $M \subset \mathbb{R}^d$ of some Euclidean space. The standard definition is that the tangent space in $p \in M$ is the *subspace* of $\mathbb{R}^d$ consisting of all tangent vectors to differentiable curves through p:

$$T_pM = \{\dot{\gamma}(0) \in \mathbb{R}^d \mid \gamma\colon (-\epsilon, \epsilon) \to M \text{ differentiable}, \gamma(0) = p\}.$$

The problem with general manifolds is that they are *a priori* not embedded in any surrounding space, so this notion of tangent vector does not work. However, what we can do, is that instead of taking the tangent vectors in the surrounding space, we take the full set of curves through p in the manifold M and define on this set an equivalence relation that identifies two of them, α and β, if *in a chart* $\phi\colon M \supset U \to \mathbb{R}^n$ they have the same tangent vector in p:

$$\alpha \sim \beta \Leftrightarrow (\phi \mathbin{\dot{\circ}} \alpha)(0) = (\phi \mathbin{\dot{\circ}} \beta)(0).$$

To be equivalent in this sense does not depend on the choice of charts: If we choose another chart $\psi\colon M \supset V \to \mathbb{R}^n$ around p, then the tangent vectors in the charts ϕ and ψ are related by a linear map, the differential $D_{\phi(p)}(\psi \circ \phi^{-1})$ of the change of coordinates. Since the tangent vectors of α and β in chart ϕ are identical, they will thus still be identical in chart ψ. With this equivalence relation we can therefore set:

Definition A.1.19 The **tangent space** of a smooth manifold M^n at a point $p \in M$ is defined by

$$T_pM = \{\gamma \mid \gamma\colon (-\epsilon, \epsilon) \to M \text{ differentiable}, \gamma(0) = p\}/_\sim.$$

For the equivalence class of the curve γ in M we write

$$[\gamma] = \dot{\gamma}(0) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)$$

and call this a **tangent vector**.

Proposition A.1.20 *At any point $p \in M^n$ the tangent space T_pM has the structure of a real n-dimensional vector space.*

Proof Let $\phi\colon U \to \mathbb{R}^n$ be a chart around p. We set

$$\begin{aligned} D_p\phi\colon T_pM &\longrightarrow \mathbb{R}^n \\ [\gamma] &\longmapsto (\phi \mathbin{\dot{\circ}} \gamma)(0). \end{aligned}$$

It can be shown that this is a bijection. We define the vector space structure on T_pM so that this map becomes a vector space isomorphism. This structure does not depend on the choice of chart: If $\psi\colon V \to \mathbb{R}^n$ is another chart around p, then the following diagram is commutative, where $D_{\phi(p)}(\psi \circ \phi^{-1})$ is a vector space isomorphism:

$$\begin{array}{ccc} T_pM & \xrightarrow{\;=\;} & T_pM \\ {\scriptstyle D_p\phi}\downarrow & & \downarrow{\scriptstyle D_p\psi} \\ \mathbb{R}^n & \xrightarrow[D_{\phi(p)}(\psi\circ\phi^{-1})]{} & \mathbb{R}^n \end{array}$$

Hence the identity between T_pM and T_pM defined with the respective vector space structures is a vector space isomorphism. □

Definition A.1.21 The set

$$TM = \bigcup_{p\in M} \{p\} \times T_pM$$

is called the **tangent bundle** of M.

In Sect. 4.5 it is shown that the tangent bundle is an example of a *vector bundle* over M with *fibres* T_pM.

A.1.6 Differential of a Smooth Map

Let $f\colon M \to N$ be a smooth map between differentiable manifolds. With the tangent space at hand, we can now define the differential of f.

Definition A.1.22 The **differential** D_pf of the map f at a point $p \in M$ is defined by

$$\begin{aligned} D_pf\colon T_pM &\longrightarrow T_{f(p)}N \\ [\gamma] &\longmapsto [f \circ \gamma]. \end{aligned}$$

Equivalently,

$$\begin{aligned} D_pf\colon T_pM &\longrightarrow T_{f(p)}N \\ \dot{\gamma}(0) &\longmapsto (f \dot{\circ} \gamma)(0). \end{aligned}$$

The differential is a *well-defined* (independent of choice of representatives for $[\gamma]$) *linear map* between the tangent spaces.

For a vector $X \in T_pM$ we sometimes write

$$f_*X = (D_pf)(X).$$

The differential satisfies the so-called chain rule.

Proposition A.1.23 *The following* **chain rule** *holds for the differential: If $f: X \to Y$ and $g: Y \to Z$ are differentiable maps, then $g \circ f$ is differentiable and at any point $p \in X$*

$$D_p(g \circ f) = D_{f(p)}g \circ D_pf.$$

Corollary A.1.24 *The differential D_pf of a diffeomorphism $f: M \to N$ is at every point $p \in M$ a linear isomorphism of tangent spaces.*

Definition A.1.25 Let $f: M \to N$ be a differentiable map between manifolds.

- A point $p \in M$ is called a **regular point of** f if the differential D_pf is surjective onto $T_{f(p)}N$.
- A point $q \in N$ is called a **regular value of** f if each point p in the preimage $f^{-1}(q) \subset M$ is a regular point.
- The map f is called a **submersion** if every point $p \in M$ is regular.
- The map f is called an **immersion** if the differential D_pf is injective at every point $p \in M$.

Remark A.1.26 Every point of N that is not in the image $f(M)$ is automatically a regular value, because the condition is empty.

Theorem A.1.27 (Sard's Theorem) *For any differentiable map $f: M \to N$ between smooth manifolds M and N the set of regular values is dense in N.*
The following theorem shows that a map f has a certain normal form in a neighbourhood of a regular point.

Theorem A.1.28 (Regular Point Theorem) *Let p be a regular point of the map f. Then there exist charts (U, ϕ) of M around p and (V, ψ) of N around $f(p)$ with*

- $\phi(p) = 0$
- $\psi(f(p)) = 0$
- $f(U) \subset V$

such that the map $\psi \circ f \circ \phi^{-1}$ has the form

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_{n+k}) = (x_1, \ldots, x_n),$$

where $\dim M = n + k$ *and* $\dim N = n$.

Remark A.1.29 The theorem says that in suitable charts the map f is given by the standard projection of $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$ onto $\mathbb{R}^n$.

A.1.7 Immersed and Embedded Submanifolds

There are two notions of submanifolds which need to be distinguished.

Definition A.1.30 Let M be a smooth manifold.

1. An **immersed submanifold** of M is the image of an injective immersion $f\colon N \to M$ from a manifold N to M.
2. An **embedded submanifold** of M is the image of an injective immersion $f\colon N \to M$ from a manifold N to M which is a homeomorphism onto its image.

In both cases, the set $f(N)$ is endowed with the topology and manifold structure making $f\colon N \to f(N)$ a diffeomorphism. The difference between embedded and immersed submanifolds $f(N) \subset M$ is whether the topology on $f(N)$ coincides with the subspace topology on $f(N)$ inherited from M or not.
An embedded submanifold can be characterized equivalently as follows:

Proposition A.1.31 *A subset K of an m-dimensional manifold M is an embedded submanifold of dimension k if and only if around each point $p \in K$ there exists a chart (U, ϕ) of M such that*

$$\phi|_{U\cap K}\colon U \cap K \longrightarrow \phi(U) \cap \left(\mathbb{R}^k \times \{0\}\right) \subset \mathbb{R}^m.$$

Such a chart is also called a ***submanifold chart*** *or* ***flattener*** *for K.*
The regular point theorem implies:

Theorem A.1.32 (Regular Value Theorem) *Let $q \in N$ be a regular value of a smooth map $f\colon M \to N$ and $L = f^{-1}(q)$ the preimage of q. Then L is an embedded submanifold of M of dimension*

$$\dim L = \dim M - \dim N.$$

A.1.8 Vector Fields

Let M^n be a smooth manifold. A vector field on M is a map X that assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$ in a smooth way. To make this precise let $\phi\colon M \supset U \to \phi(U) \subset \mathbb{R}^n$ be a chart. We set

$$TU = \bigcup_{p\in U} \{p\} \times T_pM$$

for the tangent bundle of U and define the map

$$\begin{aligned} D\phi\colon TU &\longrightarrow \phi(U) \times \mathbb{R}^n \\ (p, v) &\longmapsto (\phi(p), D_p\phi(v)). \end{aligned}$$

The map $D\phi$ is on each fibre $\{p\} \times T_pM$ of TU an isomorphism onto $\{\phi(p)\} \times \mathbb{R}^n$.

Definition A.1.33 A **vector field** X on M is a map $X: M \to TM$ such that:

1. $X_p = X(p) \in T_pM$ for all $p \in M$.
2. The map X is differentiable in the following sense: For any chart (U, ϕ) the lower horizontal map in the following diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad X \quad} & TU \\ \phi \downarrow & & \downarrow D\phi \\ \phi(U) & \xrightarrow[D\phi \circ X \circ \phi^{-1}]{} & \phi(U) \times \mathbb{R}^n \end{array}$$

   is differentiable (this is just a standard vector field on $\phi(U) \subset \mathbb{R}^n$).

A particularly important set of vector fields is defined by a chart.

Definition A.1.34 Let (U, ϕ) be a chart for M. Then we define at every point $p \in U$ the following vectors:

$$\frac{\partial}{\partial x_\mu}(p) = (D_p\phi)^{-1}(e_\mu), \quad \forall \mu = 1, \dots, n,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. We also write

$$\partial_\mu = \frac{\partial}{\partial x_\mu}.$$

For a fixed index μ, as p varies, the vectors $\partial_\mu(p)$ form a smooth vector field ∂_μ on U. We call the vector fields ∂_μ **basis vector fields** or **coordinate vector fields** on U.

Lemma A.1.35 *At each point $p \in U$ the vectors $\partial_1(p), \dots \partial_n(p)$ form a basis for the tangent space T_pM.*

Proof This is clear, because $D_p\phi: T_pM \to \mathbb{R}^n$ is an isomorphism of vector spaces. □

Proposition A.1.36 *Every smooth vector field X on M can be written on U as*

$$X|_U = \sum_{\mu=1}^{n} X^\mu \partial_\mu \equiv X^\mu \partial_\mu$$

*where $X^1, \dots, X^n: U \to \mathbb{R}$ are smooth real-valued functions on U, called the **components** of X with respect to the basis $\{\partial_\mu\}$.*

Remark A.1.37 The second equality in this proposition is an example of the so-called **Einstein summation convention**.

A.1.9 Integral Curves

Let M be a smooth manifold and X a smooth vector field on M.

Definition A.1.38 A curve $\gamma\colon I \to M$, where $I \subset \mathbb{R}$ is an open interval around 0, is called an **integral curve** for X through $p \in M$ if

$$\gamma(0) = p \quad \text{and} \quad \dot{\gamma}(t) = X_{\gamma(t)} \quad \forall t \in I.$$

The theory of ordinary differential equations (ODEs) applied in a chart for M shows that:

Theorem A.1.39 *For every point $q \in M$ there exists an interval I_q around 0 and a unique curve $\gamma_q\colon I_q \to M$ which is an integral curve for X.*
Using a theorem on the behaviour of solutions to ODEs under variation of the initial condition we get:

Theorem A.1.40 *For all $p \in M$ there exists an open neighbourhood U of p in M and an open interval I around 0 such that the integral curves γ_q are defined on I for all $q \in U$. The map*

$$\begin{aligned} \phi_U\colon U \times I &\longrightarrow M \\ (q,t) &\longmapsto \gamma_q(t) \end{aligned}$$

*is differentiable and is called the **local flow** of X.*

Theorem A.1.41 *Let M be a closed manifold (compact and without boundary). Then there exists a **global flow** of X which is a smooth map*

$$\begin{aligned} \phi\colon M \times \mathbb{R} &\longrightarrow M \\ (q,t) &\longmapsto \gamma_q(t). \end{aligned}$$

The map

$$\phi_t = \phi(\cdot, t)\colon M \longrightarrow M$$

is a diffeomorphism for all $t \in \mathbb{R}$.

A.1.10 The Commutator of Vector Fields

Let X be a smooth vector field on the manifold M.

Definition A.1.42 The **Lie derivative** L_X is the map

$$L_X\colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M),$$

defined by

$$(L_X f)(p) = (D_p f)(X_p)$$

for all $f \in \mathscr{C}^\infty(M)$ and $p \in M$.
The Lie derivative L_X is the directional derivative of a smooth function along the vector field X: If γ is a curve through p such that $\dot{\gamma}(0) = X_p$, then

$$(L_X f)(p) = (f \dot{\circ} \gamma)(0).$$

Proposition A.1.43 *The Lie derivative is a* ***derivation****, i.e.*

1. *L_X is $\mathbb{R}$-linear*
2. *L_X satisfies the Leibniz rule:*

$$L_X(f \cdot g) = (L_X f) \cdot g + f \cdot (L_X g) \quad \forall f, g \in \mathscr{C}^\infty(M).$$

Using the Lie derivative we can define the so-called commutator of vector fields.

Theorem A.1.44 *Let X and Y be smooth vector fields on M. Then there exists a unique vector field $[X, Y]$ on M, called the* ***commutator*** *of X and Y, such that*

$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X. \tag{A.1}$$

If in a local chart (U, ϕ) the vector fields are given by

$$X = X^\mu \partial_\mu \quad \textit{and} \quad Y = Y^\mu \partial_\mu,$$

then $[X, Y]$ is given by

$$[X, Y] = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) \partial_\mu. \tag{A.2}$$

Theorem A.1.45 *The set of vector field $\mathfrak{X}(M)$ together with the commutator is an (infinite-dimensional)* ***Lie algebra****, i.e. for all $X, Y, Z \in \mathfrak{X}(M)$ we have:*

- *antisymmetry:*

$$[Y, X] = -[X, Y]$$

- *$\mathbb{R}$-bilinearity:*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z] \quad \forall a, b \in \mathbb{R}$$

- *Jacobi identity:*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

We can calculate the commutator $[X, Y]$ using the flow of X:

Theorem A.1.46 *Let X and Y be smooth vector fields on M, ϕ_t the flow of X and $p \in M$ a point. Then*

$$[X,Y]_p = \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* Y_{\phi_t(p)}.$$

Note that $(\phi_{-t})_* Y_{\phi_t(p)}$ is a smooth curve in T_pM.

A.1.11 Vector Fields Related by a Smooth Map

Definition A.1.47 Let M and N be smooth manifolds and $\phi: M \to N$ a smooth map. Suppose that X is a vector field on M and Y a vector field on N. Then Y is said to be **ϕ-related** to X if

$$Y_{\phi(p)} = (D_p\phi)(X_p) \quad \forall p \in M.$$

Lemma A.1.48 *Let M and N be smooth manifolds, $\phi: M \to N$ a smooth map. Suppose that X and Y are vector fields on M and N and that Y is ϕ-related to X. Then*

$$(L_Y f) \circ \phi = L_X(f \circ \phi) \quad \forall f \in \mathscr{C}^\infty(N).$$

Proposition A.1.49 *Let M and N be smooth manifolds and $\phi: M \to N$ a smooth map. Suppose that X' is ϕ-related to X and Y' is ϕ-related to Y. Then $[X', Y']$ is ϕ-related to $[X, Y]$.*

Definition A.1.50 If $\phi: M \to N$ is a diffeomorphism and X is a smooth vector field on M, then we define a smooth vector field $\phi_* X$ on N, called the **pushforward** of X under ϕ, by

$$(\phi_* X)_{\phi(p)} = (D_p\phi)(X_p).$$

Note that $\phi_* X$ is the unique vector field on N that is ϕ-related to X.

Corollary A.1.51 *If $\phi: M \to N$ is a diffeomorphism, then*

$$[\phi_* X, \phi_* Y] = \phi_*[X, Y]$$

for all vector fields X and Y on M.

A.1.12 Distributions and Foliations

We consider some concepts related to distributions and foliations on manifolds (we follow [142] where proofs and more details can be found). Let M be a smooth manifold of dimension n.

Definition A.1.52 A **distribution** D of rank k on M is a collection of vector subspaces $D_p \subset T_pM$ of dimension k for all $p \in M$ which vary smoothly over M, i.e. each $p \in M$ has an open neighbourhood $U \subset M$ so that $D|_U$ is spanned by k smooth vector fields $X_1, \ldots, X_k$ on U.
An equivalent definition is that D is a *subbundle* of rank k of the tangent bundle TM.

Definition A.1.53 A distribution is called **involutive** or **integrable** if for all vector fields X, Y on M with $X_p, Y_p \in D_p$ for all $p \in M$, the vector field $[X, Y]$ on M again satisfies $[X, Y]_p \in D_p$ for all $p \in M$.

Definition A.1.54 A **foliation** $\mathscr{F}$ of rank k on M is a decomposition of M into k-dimensional immersed submanifolds, called **leaves**, which locally have the following structure: around each point $p \in M$ there exists a coordinate neighbourhood diffeomorphic to $\mathbb{R}^n$ such that the leaves of the foliation decompose $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$, with the leaves given by the affine subspaces $\mathbb{R}^k \times \{x\}$ for all $x \in \mathbb{R}^{n-k}$.
It is clear that the tangent spaces to the leaves of a foliation define a distribution. In fact, we have:

Theorem A.1.55 (Frobenius Theorem) *A distribution D defines a foliation $\mathscr{F}$ if and only if D is integrable.*
The following statement is Theorem 1.62 in [142].

Theorem A.1.56 *Let $f: N \to M$ be a smooth map between manifolds, $\mathscr{H}$ a foliation on M and $H \subset M$ a leaf of $\mathscr{H}$. Suppose that f has image in H. Then $f: N \to H$ is smooth.*
This theorem is clear if H is an embedded submanifold of M and only non-trivial if H is an immersed submanifold.

A.2 Tensors and Forms

A.2.1 Tensors and Exterior Algebra of Vector Spaces

We recall some notions from linear algebra. Let V denote an n-dimensional real vector space.

Definition A.2.1 We set

$$V^* = \{\lambda \mid \lambda: V \to \mathbb{R} \text{ is linear}\}$$

for the **dual space** of V. The dual space V^* is itself an n-dimensional real vector space. We call the elements $\lambda \in V^*$ **1-forms** on V.
If $\{e_\mu\}$ is a basis for V we get a **dual basis** $\{\omega^\nu\}$ for V^* defined by

$$\omega^\nu(e_\mu) = \delta^\nu_\mu, \quad \forall \mu, \nu = 1, \dots, n,$$

where δ^ν_μ is the standard Kronecker delta. Just as we decompose any vector $X \in V$ in the basis $\{e_\mu\}$ as

$$X = X^\mu e_\mu$$

we can decompose any 1-form $\lambda \in V^*$ as

$$\lambda = \lambda_\nu \omega^\nu.$$

(Note the Einstein summation convention in both cases.)

Definition A.2.2 A **tensor** of type (l, k) is a multilinear map

$$T: \underbrace{V^* \times \cdots \times V^*}_{l} \times \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}.$$

In particular, a $(0, 1)$-tensor is a 1-form and a $(1, 0)$-tensor is a vector. The set of all (l, k)-tensors forms a vector space.
We are interested in a particular class of tensors on a vector space V.

Definition A.2.3 We call a $(0, k)$-tensor

$$\lambda: \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}$$

a **k-form** on V if λ is **alternating**, i.e. totally antisymmetric:

$$\lambda(\dots, v, \dots, w, \dots) = -\lambda(\dots, w, \dots, v, \dots)$$

for all insertions of vectors into λ, where only the vectors v and w are interchanged. The set of k-forms on V forms a vector space denoted by $\Lambda^k V^*$.

Remark A.2.4 It follows that for k-forms λ

$$\lambda(\dots, v, \dots, v, \dots) = 0 \quad \forall v \in V$$

and

$$\lambda(v_1, v_2, \dots, v_k) = 0$$

whenever the vectors $v_1, v_2, \dots, v_k$ are linearly dependent. In particular, every k-form on V vanishes identically if k is larger than the dimension of V.

Definition A.2.5 Let λ be a k-form and μ an l-form. Then the **wedge product** of $\lambda \wedge \mu$ is the $(k+l)$-form defined by

$$(\lambda \wedge \mu)(X_1, \dots, X_{k+l}) \\ = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\lambda\left(X_{\sigma(1)}, \dots, X_{\sigma(k)}\right) \cdot \mu\left(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}\right).$$

Here S_{k+l} denotes the set of permutations of $\{1, 2, \dots, k+l\}$. It can be checked that $\lambda \wedge \mu$ is indeed a $k+l$-form.

Example A.2.6 Let α, β be 1-forms on V. Then

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$$

for all vectors $X, Y \in V$.

Lemma A.2.7 *Let V be a vector space of dimension n and $\{\omega^\nu\}$ a basis for V^*. Then the set of k-forms*

$$\omega^{\nu_1} \wedge \dots \wedge \omega^{\nu_k}, \textit{ with } 1 \leq \nu_1 < \nu_2 < \dots < \nu_k \leq n,$$

is a basis for the vector space of k-forms.

A.2.2 Tensors and Differential Forms on Manifolds

Let M be an n-dimensional smooth manifold. We want to extend the notion of tensors and forms on vector spaces to tensors and forms on M. One possibility is to first define certain *vector bundles* and then tensors and forms as smooth *sections* of these bundles. However, since we define vector bundles in Sect. 4.5, we use here another, equivalent definition for tensors.

Remark A.2.8 In the following all functions and vector fields on M are smooth.

Definition A.2.9 We denote by $\mathscr{C}^\infty(M)$ the ring of all smooth functions $f\colon M \to \mathbb{R}$. We also denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M. The set $\mathfrak{X}(M)$ is a real vector space and module over $\mathscr{C}^\infty(M)$ by point-wise multiplication.
We can now define:

Definition A.2.10 A **1-form** λ on the manifold M is a map

$$\lambda\colon \mathfrak{X}(M) \longrightarrow \mathscr{C}^\infty(M)$$

that is linear over $\mathscr{C}^\infty(M)$, i.e.

$$\lambda(X+Y) = \lambda(X) + \lambda(Y),$$
$$\lambda(f \cdot X) = f \cdot \lambda(X)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$ and functions $f \in \mathscr{C}^\infty(M)$. We denote the set of all 1-forms on M by $\Omega^1(M)$, which is a real vector space and module over $\mathscr{C}^\infty(M)$. The following can be proved:

Proposition A.2.11 *The value of $\lambda(X)(p)$ for a 1-form λ and vector field X at a point $p \in M$ depends only on X_p. Hence if Y is another vector field on M with $Y_p = X_p$, then $\lambda(X)(p) = \lambda(Y)(p)$.*
A proof of this proposition can be found in [142, p. 64]. Similarly we set:

Definition A.2.12 A **tensor** T of type (l, k) on M is a map

$$T: \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{l} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \longrightarrow \mathscr{C}^\infty(M)$$

that is $\mathscr{C}^\infty(M)$-linear in each entry. A k**-form** or **differential form** ω on M is a $(0, k)$-tensor

$$\omega: \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \longrightarrow \mathscr{C}^\infty(M)$$

that is in addition alternating (totally antisymmetric). We denote the set of k-forms on M by $\Omega^k(M)$.

Remark A.2.13 An argument similar to the proof of Proposition A.2.11 shows that tensors and k-forms on manifolds have well-defined values at every point $p \in M$. We can therefore insert, for example, in a k-form $\omega \in \Omega^k(M)$ vectors $X_1, \ldots, X_k$ in the tangent space T_pM at any point $p \in M$ and get a real number. We can also speak unambiguously of the value of a tensor or form at a point.

Remark A.2.14 We can define the wedge product $\wedge$ of forms as before by replacing in the definition vectors by vector fields on the manifold. The wedge product is then a map

$$\wedge: \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M).$$

A.2.3 Scalar Products and Metrics on Manifolds

We consider the following definition from linear algebra.

Definition A.2.15 A **scalar product** on the vector space V is a **symmetric non-degenerate** $(0, 2)$-tensor g on V:

$$g(v, w) = g(w, v) \quad \forall v, w \in V \quad \text{(symmetric)}$$
$$g(v, \cdot) \neq 0 \in V^* \quad \forall v \neq 0 \in V \quad \text{(non-degenerate)}.$$

The scalar product g is called **Euclidean** if it is positive definite

$$g(v, v) \geq 0 \quad \forall v \in V$$
$$g(v, v) > 0 \quad \forall v \neq 0$$

and **pseudo-Euclidean** otherwise.
We can do the same construction on manifolds.

Definition A.2.16 A **metric** on a smooth manifold M is a $(0, 2)$-tensor g which is a scalar product at each point $p \in M$. The metric is called **Riemannian** if the scalar products g_p are Euclidean and **pseudo-Riemannian** if the scalar products g_p are pseudo-Euclidean, for all $p \in M$.
It can be shown using partitions of unity that every smooth manifold admits a Riemannian metric (but not necessarily a pseudo-Riemannian metric).

A.2.4 The Levi-Civita Connection

Let (M, g) be a pseudo-Riemannian manifold. The **Levi-Civita connection** is a metric and torsion-free, covariant derivative on the tangent bundle of the manifold, i.e. a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$(X, Y) \longmapsto \nabla_X Y$$

with the following properties:

1. ∇ is $\mathbb{R}$-linear in both X and Y.
2. ∇ is $\mathscr{C}^\infty(M)$-linear in X and satisfies

$$\nabla_X (fY) = (L_X f)Y + f\nabla_X Y \quad \forall f \in \mathscr{C}^\infty(M), X, Y \in \mathfrak{X}(M).$$

3. ∇ is metric, i.e.

$$L_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

4. ∇ is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \mathfrak{X}(M).$$

The Levi-Civita connection can be calculated with the following **Koszul formula**:

$$\begin{aligned}2g(\nabla_X Y, Z) &= L_X g(Y,Z) + L_Y g(X,Z) - L_Z g(X,Y)\\ &\quad - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z).\end{aligned}$$

A.2.5 Coordinate Representations

We saw above that we can represent every vector field X on a chart neighbourhood U by $X|_U = X^\mu \partial_\mu$, where X^μ are certain functions on U, called components. We want to decompose in a similar way tensors and forms on U. In the physics literature tensors and forms are often given in terms of their components in coordinate systems.

Definition A.2.17 Let U be a chart neighbourhood. We define the set of **dual 1-forms** dx^μ, for $\mu = 1, \ldots, n$, by $dx^\mu(\partial_\nu) = \delta^\mu_\nu$ at each point $p \in U$.

Proposition A.2.18 *Let λ be a 1-form on M. Then we can decompose λ on U as $\lambda|_U = \lambda_\mu dx^\mu$ for certain smooth functions λ_μ on M. Similarly, we can decompose a k-form ω as*

$$\omega|_U = \sum_{1\le\nu_1<\cdots<\nu_k\le n} \omega_{\nu_1\ldots\nu_k} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_k},$$

with smooth functions $\omega_{\nu_1\ldots\nu_k}$.
Note that these functions, corresponding to the components, depend on the choice of the chart (U, ϕ), while the objects themselves (vectors fields, k-forms) are independent of charts.

A.2.6 The Pullback of Forms on Manifolds

Let $\omega \in \Omega^k(N)$ be a k-form on a manifold N and $f\colon M \to N$ a smooth map.

Definition A.2.19 The **pullback** of ω under f is the k-form $f^*\omega \in \Omega^k(M)$ on M defined by

$$(f^*\omega)(X_1, \ldots, X_k) = \omega(f_*X_1, \ldots, f_*X_k)$$

for all tangent vectors $X_1, \ldots, X_k \in T_pM$ and all $p \in M$.

Proposition A.2.20 *The pullback defines a map $f^*\colon \Omega^k(N) \longrightarrow \Omega^k(M)$. We have*

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$$

for all $\omega \in \Omega^k(N), \eta \in \Omega^l(N)$ *and*

$$(g \circ f)^* = f^* \circ g^*$$

for all smooth maps $f: M \to N, g: N \to Q$.
The second property follows from the chain rule for the differential of the map $g \circ f$.

A.2.7 The Differential of Forms on Manifolds

The differential is a very important map on forms on a manifold that raises the degree by one.

Theorem A.2.21 *Let M be a smooth manifold. Then there is a unique map*

$$d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

for every $k \geq 0$*, called the* ***differential*** *or* ***exterior derivative****, that satisfies the following properties:*

1. d *is* $\mathbb{R}$*-linear.*
2. *For a function* $f \in \Omega^0(M) = \mathscr{C}^\infty(M)$ *and a vector field* $X \in \mathfrak{X}(M)$ *we have* $df(X) = L_X f$.
3. $d^2 = d \circ d = 0: \Omega^k(M) \to \Omega^{k+2}(M)$.
4. d *satisfies the following Leibniz rule:*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

for all $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$.

The proof of this fundamental theorem can be found in any book on differential geometry. Let (U, ϕ) be a local chart. If we assume that the differential d has these properties, then it follows that the differential is given on functions f by

$$df = \frac{\partial f}{\partial x^\nu} dx^\nu$$

and on $\Omega^k(M)$ by

$$\begin{aligned} d\omega &= d \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq n} \omega_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \cdots dx^{\mu_k} \\ &= \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq n} \sum_{\nu=1}^{n} \frac{\partial \omega_{\mu_1 \ldots \mu_k}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \cdots dx^{\mu_k}. \end{aligned}$$

The defining properties of the differential d imply for 1-forms and 2-forms:

Proposition A.2.22 *1. Let $\alpha \in \Omega^1(M)$ be a 1-form. Then*

$$d\alpha(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]) \quad \forall X,Y \in \mathfrak{X}(M).$$

2. Let $\beta \in \Omega^2(M)$ be a 2-form. Then

$$\begin{aligned} d\beta(X,Y,Z) &= L_X(\beta(Y,Z)) + L_Y(\beta(Z,X)) + L_Z(\beta(X,Y)) \\ &\quad - \beta([X,Y],Z) - \beta([Y,Z],X) - \beta([Z,X],Y) \quad \forall X,Y,Z \in \mathfrak{X}(M). \end{aligned}$$

The differential is natural under pullback:

Proposition A.2.23 *If $f: M \to N$ is a smooth map and $\omega \in \Omega^k(N)$, then $d(f^*\omega) = f^*d\omega$.*

Let M be a compact oriented n-dimensional manifold and $\sigma \in \Omega^n(M)$ a form of top degree. Then there is a well-defined integral

$$\int_M \sigma \in \mathbb{R}.$$

The integral can also be defined if M is non-compact and σ has compact support.

Theorem A.2.24 (Stokes' Theorem)

1. *Let M be a compact n-dimensional oriented manifold with boundary ∂M and $\omega \in \Omega^{n-1}(M)$. Then (with a suitable orientation of the boundary)*

$$\int_M d\omega = \int_{\partial M} \omega.$$

2. *Let M be an n-dimensional oriented manifold (not necessarily compact) without boundary and $\omega \in \Omega^{n-1}(M)$ an $(n-1)$-form with compact support. Then*

$$\int_M d\omega = 0.$$