Chapter 1
Lie Groups and Lie Algebras: Basic Concepts

Gauge theories are field theories of physics involving *symmetry groups*. Symmetry groups are groups of transformations that act on something and leave something (possibly something else) invariant. For example, symmetry groups can act on geometric objects (by rotation, translation, etc.) and leave those objects invariant. For the symmetries relevant in field theories, the groups act on fields and leave the *Lagrangian* or the *action* (the spacetime integral over the Lagrangian) invariant.

Concerning symmetry groups or groups in general we can make a basic distinction, without being mathematically precise for the moment: groups can be *discrete* or *continuous*. Both types of symmetry groups already occur for elementary geometric objects. Equilateral triangles and squares, for example, appear to have discrete symmetry groups, while other objects, such as the circle S^1, the 2-sphere S^2 or the plane $\mathbb{R}^2$ with a Euclidean metric, have continuous symmetry groups. A deep and less obvious fact, that we want to understand over the course of this book, is that similar symmetry groups play a prominent role in the classical and quantum field theories describing nature.

From a mathematical point of view, continuous symmetry groups can be conceptualised as *Lie groups*. By definition, Lie groups are groups in an algebraic sense which are at the same time smooth manifolds, so that both structures – algebraic and differentiable – are compatible. As a mathematical object, Lie groups capture the idea of a continuous group that can be parametrized locally by coordinates, so that the group operations (multiplication and inversion) are smooth maps in those coordinates. Lie groups also cover the case of discrete, i.e. 0-dimensional groups, consisting of a set of isolated points.

In theoretical physics, Lie groups like the Lorentz and Poincaré groups, which are related to spacetime symmetries, and gauge groups, defining *internal* symmetries, are important cornerstones. The currently accepted *Standard Model of elementary particles*, for instance, is a gauge theory with Lie group

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

M.J.D. Hamilton, *Mathematical Gauge Theory*, Universitext,
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There are also *Grand Unified Theories* (GUTs) based on Lie groups like SU(5). We shall see in later chapters that the specific kind of Lie group in a gauge theory (its dimension, whether it is abelian or not, whether it is *simple* or splits as a product of several factors, and so on) is reflected in interesting ways in the physics. For example, in the case of the Standard Model, it turns out that:

- The fact that there are 8 *gluons*, 3 *weak gauge bosons* and 1 *photon* is related to the dimensions of the Lie groups SU(3) and $\mathrm{SU}(2) \times \mathrm{U}(1)$ (the SU(5) Grand Unified Theory has 12 additional gauge bosons).
- The fact that the strong, weak and electromagnetic interactions have different strengths (*coupling constants*) is related to the product structure of the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (GUTs built on simple Lie groups like SU(5) have only a single coupling constant).
- The fact that gluons interact directly with each other while photons do not is related to the fact that SU(3) is non-abelian while U(1) is abelian.

Our main mathematical tool to construct non-trivial Lie groups will be *Cartan's Theorem*, which shows that any subgroup (in the algebraic sense) of a Lie group, which is a closed set in the topology, is already an embedded Lie subgroup.

Besides Lie groups, *Lie algebras* play an important role in the theory of symmetries. Lie algebras are vector spaces with a bilinear, antisymmetric product, denoted by a bracket $[\cdot\,,\cdot]$, satisfying the *Jacobi identity*. As an algebraic object, Lie algebras can be defined independently of Lie groups, even though Lie groups and Lie algebras are closely related: the tangent space to the neutral element $e \in G$ of a Lie group G has a canonical structure of a Lie algebra. This means that Lie algebras are in some sense an infinitesimal, algebraic description of Lie groups. Depending on the situation, it is often easier to work with linear objects, such as Lie algebras, than with non-linear objects like Lie groups. Lie algebras are also important in gauge theories: *connections on principal bundles*, also known as *gauge boson fields*, are (locally) 1-forms on spacetime with values in the Lie algebra of the gauge group.

In this chapter we define Lie groups and Lie algebras and describe the relations between them. In the following chapter we will study some associated concepts, like representations (which are used to define the actions of Lie groups on fields) and invariant metrics (which are important in the construction of the gauge invariant *Yang–Mills Lagrangian*). We will also briefly discuss the structure of simple and semisimple Lie algebras.

Concerning symmetries, we will study in this chapter Lie groups as symmetry groups of vector spaces and certain structures (scalar products and volume forms) defined on vector spaces (in Chap. 3 on group actions we will study Lie groups as symmetry groups of manifolds). Symmetry groups of vector spaces are more generic than it may seem: it can be shown as a consequence of the *Peter–Weyl Theorem* that any compact Lie group can be realized as a group of rotations of some finite-dimensional Euclidean vector space $\mathbb{R}^m$ (i.e. as an embedded Lie subgroup of $\mathrm{SO}(m)$).

We can only cover a selection of topics on Lie groups. The main references for this and the following chapter are [24, 83, 142] and [153], where more extensive discussions of Lie groups and Lie algebras can be found. Additional references are [14, 34, 70, 77] and [129].

1.1 Topological Groups and Lie Groups

We begin with a first elementary mathematical concept that makes the idea of a continuous group precise.

Definition 1.1.1 A **topological group** G is a group which is at the same time a topological space so that the map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h^{-1} \end{aligned}$$

is continuous, where $G \times G$ has the canonical product topology determined by the topology of G.

Remark 1.1.2 Here and in the following we shall mean by a **group** just a group in the algebraic sense, without the additional structure of a topological space or smooth manifold.
We usually set e for the neutral element in G. An equivalent description of topological groups is the following.

Lemma 1.1.3 *A group G is a topological group if and only if it is at the same time a topological space so that both of the maps*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h \\ G &\longrightarrow G \\ g &\longmapsto g^{-1}, \end{aligned}$$

called **multiplication** *and* **inversion***, are continuous.*

Proof Suppose that multiplication and inversion are continuous maps. Then the map

$$\begin{aligned} G \times G &\longrightarrow G \times G \\ (g,h) &\longmapsto \left(g,h^{-1}\right) \end{aligned}$$

is continuous and hence also the composition of this map followed by multiplication. This shows that

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h^{-1} \end{aligned}$$

is continuous, hence the group G is a topological group.

Conversely, assume that G is a topological group. Then the map

$$\begin{aligned} G &\longrightarrow G \times G \longrightarrow G \\ g &\longmapsto (e,g) \longmapsto e \cdot g^{-1} = g^{-1} \end{aligned}$$

is continuous and hence the map

$$\begin{aligned} G \times G &\longrightarrow G \times G \longrightarrow G \\ (g,h) &\longmapsto \left(g,h^{-1}\right) \longmapsto g \cdot \left(h^{-1}\right)^{-1} = g \cdot h \end{aligned}$$

is also continuous. This proves the claim. □

The concept of topological groups is a bit too general to be useful for our purposes. In particular, general topological spaces can be very complicated and do not have to be, for example, locally Euclidean, like topological manifolds. We now turn to the definition of Lie groups, which is the type of continuous groups we are most interested in.

Definition 1.1.4 A **Lie group** G is a group which is at the same time a manifold so that the map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h^{-1} \end{aligned}$$

is smooth, where $G \times G$ has the canonical structure of a product manifold determined by the smooth structure of G.

Remark 1.1.5 Note that we only consider Lie groups of finite dimension.

Remark 1.1.6 Here and in the following we mean by a **manifold** a smooth manifold, unless stated otherwise.

Of course, every Lie group is also a topological group. We could define Lie groups equivalently as follows.

Lemma 1.1.7 *A group G is a Lie group if and only if it is at the same time a manifold so that both of the maps*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h \\ G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are smooth.

Proof The proof is very similar to the proof of Lemma 1.1.3. □

A Lie group is thus a second countable, Hausdorff, topological group that can be parametrized locally by finitely many coordinates in a smoothly compatible way, so that multiplication and inversion depend smoothly on the coordinates.

Remark 1.1.8 (Redundancy in the Definition of Lie Groups) A curious fact about Lie groups is that it suffices to check that multiplication is smooth, because then inversion is automatically smooth (see Exercise 1.9.5).

Remark 1.1.9 (Hilbert's Fifth Problem) It was shown by Gleason [65], Montgomery and Zippin [97] in 1952 that a topological group, which is also a topological manifold, has the structure of a Lie group. This is the solution to one interpretation of *Hilbert's fifth problem* (see [135] for more details). We will not prove this theorem (the existence of a smooth structure), but we will show in Corollary 1.8.17 that on a topological group, which is a topological manifold, there is at most one smooth structure that turns it into a Lie group.

We will see in Sect. 1.5 that there is a deeper reason why we want symmetry groups to be smooth manifolds: only in this situation can we canonically associate to a group a *Lie algebra*, which consists of certain *left-invariant* smooth vector fields on the group. Vector fields are only defined on smooth manifolds (they need tangent spaces and a tangent bundle to be defined). This explains why we are particularly interested in groups having a smooth structure.

We consider some simple examples of Lie groups (more examples will follow later, in particular, in Sect. 1.2.2).

Example 1.1.10 Euclidean space $\mathbb{R}^n$ with vector addition is an n-dimensional Lie group, since addition

$$(x, y) \longmapsto x + y$$

and inversion

$$x \longmapsto -x$$

for $x, y \in \mathbb{R}^n$ are linear and hence smooth. The Lie group $\mathbb{R}^n$ is connected, non-compact and abelian.

Remark 1.1.11 Euclidean spaces $\mathbb{R}^n$ can also carry other (non-abelian) Lie group structures besides the abelian structure coming from vector addition. For example,

$$\mathrm{Nil}^3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}(3 \times 3, \mathbb{R}) \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

with matrix multiplication is an example of a (so-called *nilpotent*), non-abelian Lie group structure on $\mathbb{R}^3$, also known as the **Heisenberg group**.

Example 1.1.12 Every countable group G with the manifold structure as a discrete space, i.e. a countable union of isolated points, is a 0-dimensional Lie group, because every map $G \times G \to G$ is smooth (locally constant). A discrete group is compact if and only if it is finite. In particular, the integers $\mathbb{Z}$ and the finite cyclic groups $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ for $k \in \mathbb{N}$ are discrete, abelian Lie groups. The Lie group $\mathbb{Z}_2$ can be identified with the 0-sphere

$$S^0 = \{x \in \mathbb{R} \mid |x| = 1\} = \{\pm 1\}$$

with multiplication induced from $\mathbb{R}$.

Example 1.1.13 The circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is a 1-dimensional Lie group with multiplication induced from $\mathbb{C}$: multiplication on $\mathbb{C}$ is quadratic and inversion on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a rational function in real and imaginary parts, thus smooth. Both maps restrict to smooth maps on the embedded submanifold $S^1 \subset \mathbb{C}$. The Lie group S^1 is connected, compact and abelian.

Example 1.1.14 The following set of matrices together with matrix multiplication is a Lie group:

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{R}) \,\middle|\, \alpha \in \mathbb{R} \right\}.$$

As a manifold $\mathrm{SO}(2) \cong \mathbb{R}/2\pi\mathbb{Z} \cong S^1$. We have

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

and

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix},$$

showing that multiplication and inversion are smooth maps and that SO(2) is closed under these operations. The Lie group SO(2) is one of the simplest examples of a whole class of Lie groups, known as **matrix** or **linear Lie groups** (the Heisenberg group is another example). The Lie group SO(2) is *isomorphic* to S^1.
We want to generalize the examples of the Lie groups S^0 and S^1 and show that the 3-sphere S^3 also has the structure of a Lie group. This is a good opportunity to introduce (or recall) in a short detour the skew field of **quaternions** $\mathbb{H}$.

1.1.1 Normed Division Algebras and the Quaternions

The real and complex numbers are known from high school and the first mathematics courses at university. There are, however, other types of "higher-dimensional" numbers, which are less familiar, but still occur in mathematics and physics. It is useful to consider the following, general, algebraic notions (a nice reference is [8]):

Definition 1.1.15

1. A real **algebra** is a finite-dimensional real vector space A with a bilinear map

$$A \times A \longrightarrow A$$
$$(a, b) \longmapsto a \cdot b$$

   and a unit element $1 \in A$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in A$. In particular, the multiplication on A is distributive, but in general not associative.
2. We call the algebra A **normed** if there is a norm $||\cdot||$ on the vector space A such that $||ab|| = ||a|| \cdot ||b||$.
3. We call the algebra A a **division algebra** if $ab = 0$ implies that either $a = 0$ or $b = 0$.
4. The algebra has **multiplicative inverses** if for any non-zero $a \in A$ there is an $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.

It follows that every normed algebra is a division algebra and every associative division algebra has multiplicative inverses (this is not true in general for non-associative division algebras).

The following is a classical theorem due to Hurwitz.

Theorem 1.1.16 (Hurwitz's Theorem on Normed Division Algebras) *There are only four normed real division algebras:*

1. *The real numbers* $\mathbb{R}$ *of dimension* 1.
2. *The complex numbers* $\mathbb{C}$ *of dimension* 2.
3. *The quaternions* $\mathbb{H}$ *of dimension* 4.
4. *The octonions* $\mathbb{O}$ *of dimension* 8.

We want to describe the quaternions in this section and leave the (non-associative) octonions to Exercise 3.12.15.

Recall that there is an algebra structure on the vector space $\mathbb{R}^2$, so that this vector space becomes a *field*, called the complex numbers $\mathbb{C}$: the multiplication is associative and commutative. Furthermore, every non-zero element of $\mathbb{C}$ has a multiplicative inverse.

The complex plane is spanned as a real vector space by the basis vectors 1 and i, with $i^2 = -1$. By distributivity this determines the product of any two complex numbers. We define the **conjugate** of a complex number $z = a + ib$ as $\bar{z} = a - ib$ and the **norm** squared as $||z||^2 = z\bar{z} = a^2 + b^2$. The multiplicative inverse of a non-zero complex number is then

$$z^{-1} = \frac{\bar{z}}{||z||^2}.$$

We also have

$$||uv|| = ||u|| \cdot ||v||$$

for all complex numbers $u, v \in \mathbb{C}$, so that $\mathbb{C}$ is a normed division algebra.

There is a similar construction of an algebra structure on the vector space $\mathbb{R}^4$, so that this vector space becomes a *skew field*, called the **quaternions** $\mathbb{H}$: the multiplication is associative and every non-zero element of $\mathbb{H}$ has a multiplicative inverse. However, the multiplication is not commutative. As a real vector space $\mathbb{H}$ is spanned by the basis vectors $1, i, j$ and k. The product satisfies

$$i^2 = j^2 = k^2 = ijk = -1.$$

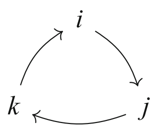
Using associativity of multiplication this determines all possible products among the basis elements i, j, k and thus by distributivity the product of any two quaternions. We have

$$ij = -ji = k,$$

$$jk = -kj = i,$$

$$ki = -ik = j,$$

showing in particular that the product is not commutative. The products among i, j and k can be memorized with the following diagram:



We define the **real quaternions** by

$$\mathrm{Re}\,\mathbb{H} = \{a = a1 \in \mathbb{H} \mid a \in \mathbb{R}\}$$

and the **imaginary quaternions** by

$$\mathrm{Im}\,\mathbb{H} = \{bi + cj + dk \in \mathbb{H} \mid b, c, d \in \mathbb{R}\}.$$

We also define the **conjugate** of a quaternion $w = a + bi + cj + dk$ as

$$\bar{w} = a - bi - cj - dk$$

and the **norm** squared as

$$||w||^2 = a^2 + b^2 + c^2 + d^2 = w\bar{w} = \bar{w}w.$$

The multiplicative inverse of a non-zero quaternion is then

$$w^{-1} = \frac{\bar{w}}{||w||^2}.$$

We also have

$$||wz|| = ||w|| \cdot ||z||$$

for all $w, z \in \mathbb{H}$, so that $\mathbb{H}$ is a normed division algebra.

Remark 1.1.17 One has to be careful with division of quaternions: the expression $\frac{z}{w}$ for $z, w \in \mathbb{H}$ is not well-defined, even for $w \neq 0$, since multiplication is not commutative. One rather has to write zw^{-1} or $w^{-1}z$.
Multiplication of quaternions defines a group structure on the 3-sphere:

Example 1.1.18 The 3-sphere

$$S^3 = \{w \in \mathbb{H} \mid ||w|| = 1\}$$

of unit quaternions is a 3-dimensional embedded submanifold of $\mathbb{H} \cong \mathbb{R}^4$ and a Lie group with multiplication induced from $\mathbb{H}$. As a Lie group S^3 is connected, compact and non-abelian (it contains, in particular, the elements $1, i, j, k \in \mathbb{H}$). The 3-sphere and thus the quaternions have an interesting relation to the **rotation group** SO(3) of $\mathbb{R}^3$, to be discussed in Example 1.3.8.

Remark 1.1.19 We shall see in Exercise 3.12.15 that there is an algebra structure on the vector space $\mathbb{R}^8$, so that this vector space becomes a normed division algebra, called the **octonions** $\mathbb{O}$. This multiplication induces a multiplication on S^7. However, the multiplication does not define a group structure on S^7, because it is not associative.

*1.1.2 *Quaternionic Matrices*

Certain properties of matrices with quaternionic entries cannot be proved in the same way as for matrices with real or complex entries, because the quaternions are only a skew field. In particular, it is not immediately clear how to make sense of the inverse and a determinant for quaternionic square matrices. Even if this is possible, it is not clear that such a determinant would have the nice properties we expect of it, like multiplicativity and the characterization of invertible matrices as those of non-zero determinant. Since we are going to consider groups of quaternionic matrices as examples of Lie groups, like the so-called **compact symplectic group** $\mathrm{Sp}(n)$, we would like to fill in some of the details in this section (we follow the exposition in [83, Sect. I.8] and [152]). Like any section with a star * this section can be skipped on a first reading.

Definition 1.1.20 We denote by $\mathrm{Mat}(m \times n, \mathbb{H})$ the set of all $m \times n$-matrices with entries in $\mathbb{H}$.
The set $\mathrm{Mat}(m \times n, \mathbb{H})$ is an abelian group with the standard addition of matrices. We can also define **right** and **left multiplication** with elements $q \in \mathbb{H}$:

$$\begin{aligned} \mathrm{Mat}(m \times n, \mathbb{H}) \times \mathbb{H} &\longrightarrow \mathrm{Mat}(m \times n, \mathbb{H}) \\ (A, q) &\longmapsto Aq \end{aligned}$$

and

$$\begin{aligned} \mathbb{H} \times \mathrm{Mat}(m \times n, \mathbb{H}) &\longrightarrow \mathrm{Mat}(m \times n, \mathbb{H}) \\ (q, A) &\longmapsto qA. \end{aligned}$$

This gives $\mathrm{Mat}(m \times n, \mathbb{H})$ the structure of a right or left module over the quaternions $\mathbb{H}$; we call this a **right** or **left quaternionic vector space** (since $\mathbb{H}$ is not commutative, left and right multiplication differ). In particular, the spaces of row and column vectors, denoted by $\mathbb{H}^n$, each have the structure of a right and left quaternionic vector space.

We can define matrix multiplication

$$\begin{aligned}\mathrm{Mat}(m \times n, \mathbb{H}) \times \mathrm{Mat}(n \times p, \mathbb{H}) &\longrightarrow \mathrm{Mat}(m \times p, \mathbb{H}) \\ (A, B) &\longmapsto A \cdot B\end{aligned}$$

in the standard way, where we have to be careful to preserve the ordering in the products of the entries. All of the constructions so far work in exactly the same way for matrices over any ring.

We now restrict to quaternionic square matrices $\mathrm{Mat}(n \times n, \mathbb{H})$. It is sometimes useful to have a different description of such matrices. The following is easy to see:

Lemma 1.1.21 *Let $q \in \mathbb{H}$ be a quaternion. Then there exist unique complex numbers $q_1, q_2 \in \mathbb{C}$ such that*

$$q = q_1 + jq_2.$$

Let $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ be a quaternionic square matrix. Then there exist unique complex square matrices $A_1, A_2 \in \mathrm{Mat}(n \times n, \mathbb{C})$ such that

$$A = A_1 + jA_2.$$

Definition 1.1.22 For a matrix $A = A_1 + jA_2 \in \mathrm{Mat}(n \times n, \mathbb{H})$, with A_1, A_2 complex, we define the **adjoint** to be the complex square matrix

$$\chi_A = \begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix} \in \mathrm{Mat}(2n \times 2n, \mathbb{C}).$$

Example 1.1.23 For the special case of a quaternion $q = q_1 + jq_2 \in \mathbb{H}$ with $q_1, q_2 \in \mathbb{C}$, considered as a 1×1-matrix, we get

$$\chi_q = \begin{pmatrix} q_1 & -\overline{q_2} \\ q_2 & \overline{q_1} \end{pmatrix}.$$

The quaternions $\mathbb{H}$ are sometimes *defined* as the subset of $\mathrm{Mat}(2 \times 2, \mathbb{C})$ consisting of matrices of this form.

Remark 1.1.24 Another definition of the adjoint, used in [152], is to write $A = A_1 + A_2 j$ with A_1, A_2 complex and define

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$

We continue to use our Definition 1.1.22, which seems to be more standard.

Using the adjoint we get the following identification of $\mathrm{Mat}(n \times n, \mathbb{H})$ with a subset of $\mathrm{Mat}(2n \times 2n, \mathbb{C})$.

Proposition 1.1.25 (Quaternionic Matrices as a Subspace of Complex Matrices) *We define an element* $J \in \mathrm{Mat}(2n \times 2n, \mathbb{C})$ *by*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $J^{-1} = -J$ *and the image of the injective map*

$$\begin{aligned} \chi\colon \mathrm{Mat}(n \times n, \mathbb{H}) &\longrightarrow \mathrm{Mat}(2n \times 2n, \mathbb{C}) \\ A &\longmapsto \chi_A \end{aligned}$$

consists of the set

$$\mathrm{im}\,\chi = \left\{ X \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid JXJ^{-1} = \overline{X} \right\}.$$

The proof is a simple calculation. The following proposition can also be verified by a direct calculation, where for the second property we use that $Cj = j\overline{C}$ for a complex matrix C:

Proposition 1.1.26 (Properties of the Adjoint) *The adjoints of quaternionic* $n \times n$*-matrices* A, B *satisfy*

$$\begin{aligned} \chi_{A+B} &= \chi_A + \chi_B \\ \chi_{AB} &= \chi_A \chi_B \\ \mathrm{tr}(\chi_A) &= 2\mathrm{Re}(\mathrm{tr}(A)). \end{aligned}$$

Corollary 1.1.27 *If quaternionic* $n \times n$*-matrices* A, B *satisfy* $AB = I$*, then* $BA = I$*.*

Proof If $AB = I_n$, then

$$\chi_A \chi_B = \chi_{AB} = I_{2n}.$$

Hence by a property of the inverse of complex matrices

$$\chi_{BA} = \chi_B \chi_A = I_{2n}$$

and thus $BA = I_n$. □

We can therefore define:

Definition 1.1.28 A matrix $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ is called **invertible** if there exists a matrix $B \in \mathrm{Mat}(n \times n, \mathbb{H})$ with $AB = I = BA$. The matrix B is called the **inverse** of A.

Corollary 1.1.29 *A matrix* $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ *is invertible if and only if its adjoint* χ_A *is invertible.*

Proof If A is invertible, then χ_A is invertible by Proposition 1.1.26. Conversely, assume that χ_A is invertible. We can write χ_A^{-1} in the form

$$\chi_A^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

for some complex $n \times n$-matrices $B_1, \ldots, B_4$. Then

$$A_1 B_1 - \overline{A_2} B_3 = I_n,$$
$$A_2 B_1 + \overline{A_1} B_3 = 0.$$

Setting

$$B = B_1 + jB_3 \in \text{Mat}(n \times n, \mathbb{H})$$

we get

$$\begin{aligned} AB &= A_1 B_1 + jA_2 B_1 + A_1 jB_3 + jA_2 jB_3 \\ &= (A_1 B_1 - \overline{A_2} B_3) + j(A_2 B_1 + \overline{A_1} B_3) \\ &= I_n. \end{aligned}$$

We conclude that A is invertible with inverse B. □

Definition 1.1.30 The **determinant** of a quaternionic matrix $A \in \text{Mat}(n \times n, \mathbb{H})$ is defined by

$$\det(A) = \det(\chi_A).$$

Remark 1.1.31 Note that the determinant is not defined for matrices over a general non-commutative ring.
The determinant on $\text{Mat}(n \times n, \mathbb{H})$ has the following property.

Proposition 1.1.32 (Quaternionic Determinant Is Real and Non-negative) *The determinant of a matrix* $A \in \text{Mat}(n \times n, \mathbb{H})$ *is real and non-negative,* $\det(A) \geq 0$.

Proof According to Proposition 1.1.25

$$J \chi_A J^{-1} = \overline{\chi_A}$$

for all $A \in \text{Mat}(n \times n, \mathbb{H})$. Since $\det(J) = 1$ we get

$$\det(\chi_A) = \det\left(J \chi_A J^{-1}\right) = \overline{\det(\chi_A)},$$

hence $\det(\chi_A) \in \mathbb{R}$.

The proof that $\det(A)$ is non-negative is more involved. Any quaternionic matrix $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ can be brought to the form

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

for some $k \leq n$ by applying elementary row and column operations. The corresponding elementary quaternionic matrices E act by left or right multiplication on A, thus the adjoint χ_E acts by left or right multiplication on χ_A. The elementary matrix E for switching two rows (columns) or adding a row (column) to another one are real, thus

$$\chi_E = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

and $\det(\chi_E) = \det(E)^2 = 1$. It remains to consider the elementary matrix E that multiplies a row (column) by a non-zero quaternion a. Writing $a = a_1 + ja_2$ with a_1, a_2 complex, the elementary matrix E is of the form $E = E_1 + jE_2$ with

$$E_1 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & a_1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & a_2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

and

$$\chi_E = \begin{pmatrix} E_1 & -\overline{E_2} \\ E_2 & \overline{E_1} \end{pmatrix}.$$

Interchanging twice a pair of rows and twice a pair of columns we can bring χ_E to the following form, without changing $\det(\chi_E)$:

$$\chi'_E = \begin{pmatrix} a_1 & -\bar{a}_2 & & & \\ a_2 & \bar{a}_1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

It follows that

$$\det(\chi_E) = \det(\chi'_E) = |a_1|^2 + |a_2|^2 > 0.$$

This proves the claim, since χ_A is the product of matrices of non-negative determinant. □

Remark 1.1.33 A second, independent proof of the second assertion in Proposition 1.1.32 uses that the subset $\mathrm{GL}(n, \mathbb{H})$ of invertible matrices in $\mathrm{Mat}(n \times n, \mathbb{H})$ is connected, cf. Theorem 1.2.22. Therefore its image under the determinant is contained in $\mathbb{R}_+$. The claim then follows, because $\mathrm{GL}(n, \mathbb{H})$ is dense in $\mathrm{Mat}(n \times n, \mathbb{H})$.

Example 1.1.34 According to Exercise 1.9.8 the determinant of a matrix

$$(a) \in \mathrm{Mat}(1 \times 1, \mathbb{H}), \quad \text{with } a \in \mathbb{H},$$

is equal to

$$\det(a) = ||a||^2.$$

The next corollary follows from what we have shown above.

Corollary 1.1.35 (Properties of the Quaternionic Determinant) *The determinant is a smooth map*

$$\det\colon \mathrm{Mat}(n \times n, \mathbb{H}) \longrightarrow \mathbb{R},$$

where $\mathrm{Mat}(n \times n, \mathbb{H})$ *has the manifold structure of* $\mathbb{R}^{4n^2}$. *For all matrices* $A, B \in \mathrm{Mat}(n \times n, \mathbb{H})$ *the following identity holds:*

$$\det(AB) = \det(A)\det(B).$$

A matrix $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ *is invertible if and only if* $\det(A) \neq 0$.

The determinant of quaternionic matrices, defined above via the adjoint, therefore has at least some of the expected properties.

Remark 1.1.36 Note that the determinant of quaternionic matrices does *not* define an $\mathbb{H}$-multilinear, alternating map on $\mathbb{H}^n \times \ldots \times \mathbb{H}^n$ (n factors).

1.1.3 Products and Lie Subgroups

There are certain well-known constructions that yield new groups (and manifolds) from given groups (manifolds). Some of these constructions are compatible for both groups and manifolds and can be employed to generate new Lie groups from known ones. We discuss two important examples of such constructions: **products** and **subgroups**.

Proposition 1.1.37 (Products of Lie Groups) *Let G and H be Lie groups. Then the product manifold $G \times H$ with the direct product structure as a group is a Lie group, called the* **product Lie group**.

Proof This follows, because the maps

$$\begin{aligned} G \times H &\longrightarrow G \times H \\ (g, h) &\longmapsto \left(g^{-1}, h^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} (G \times H) \times (G \times H) &\longrightarrow G \times H \\ (g_1, h_1, g_2, h_2) &\longmapsto (g_1 g_2, h_1 h_2) \end{aligned}$$

are smooth. □

Example 1.1.38 The **n-torus**

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is a compact, abelian Lie group. Similarly multiple (finite) products of copies of S^3 and S^1 are compact Lie groups. A particularly interesting case is $S^3 \times S^3$, because it can be identified with the Lie group Spin(4), to be defined in Chap. 6 (see Example 6.5.17).

Definition 1.1.39 Let G be a Lie group.

1. An **immersed Lie subgroup** of G is the image of an injective immersion $\phi: H \to G$ from a Lie group H to G such that ϕ is a group homomorphism.
2. An **embedded Lie subgroup** of G is the image of an injective immersion $\phi: H \to G$ from a Lie group H to G such that ϕ is a group homomorphism and a homeomorphism onto its image.

We call the map ϕ a **Lie group immersion** or **Lie group embedding**, respectively. In both cases, the set $\phi(H)$ is endowed with the topology, manifold structure and group structure such that $\phi: H \to \phi(H)$ is a diffeomorphism and a group isomorphism. Then $\phi(H)$ is a Lie group itself. The difference between embedded and immersed Lie subgroups $\phi(H) \subset G$ is whether the topology on $\phi(H)$ coincides with the subspace topology on $\phi(H)$ inherited from G or not. The group structure on $\phi(H)$ is in both cases the subgroup structure inherited from G.

An embedded Lie subgroup can be described equivalently as an embedded submanifold which is at the same time a subgroup. Most of the time we will consider embedded Lie subgroups (immersed Lie subgroups appear naturally in Sect. 1.6). Note the following:

Proposition 1.1.40 *If $\phi: H \to G$ is a Lie group immersion where H is compact, then ϕ is a Lie group embedding.*

Proof Since H is compact and G is Hausdorff, it follows that the injective immersion $\phi: H \to G$ is a closed map, hence a homeomorphism onto its image. □

Hence immersed Lie subgroups which are not embedded can only be non-compact.

Example 1.1.41 Consider the Lie group $G = S^1$ and an element $x = e^{2\pi i\alpha} \in S^1$ with $\alpha \in \mathbb{R}$. The number α is rational if and only if there exists an integer N such that $N\alpha$ is an integer. This happens if and only if $x^N = 1$. Hence if α is rational, then x generates an embedded Lie subgroup in S^1, isomorphic to the finite cyclic group $\mathbb{Z}_K$, where K is the smallest positive integer such that $K\alpha$ is an integer. If α is irrational, then x generates an immersed Lie subgroup in S^1, isomorphic to $\mathbb{Z}$.

Example 1.1.42 Similarly, consider the Lie group $G = T^2$, which we think of as being obtained by identifying opposite sides of a square $[-1, 1] \times [-1, 1]$. It can be shown that the straight lines on the square of rational slope through the neutral element $e = (0, 0)$ define embedded Lie subgroups, diffeomorphic to S^1, while the straight lines of irrational slope through $(0, 0)$ define immersed Lie subgroups, diffeomorphic to $\mathbb{R}$.

Example 1.1.43 The sets $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ of invertible elements, for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, together with multiplication are Lie groups of dimension $1, 2, 4$, respectively. The spheres S^0, S^1, S^3 are embedded Lie subgroups of codimension 1.

The proof of the following (non-trivial) theorem is deferred to Sect. 1.8.

Theorem 1.1.44 (Cartan's Theorem, Closed Subgroup Theorem) *Let G be a Lie group and suppose that $H \subset G$ is a subgroup in the algebraic sense. Then H is an embedded Lie subgroup if and only if H is a closed set in the topology of G.*

By a **closed subgroup** of a Lie group G we always mean a subgroup which is a closed set in the topology of G. The difficult part in the proof of this theorem is to show that a closed subgroup of a Lie group is an embedded submanifold. Cartan's Theorem allows us to construct many new interesting Lie groups by realizing them as closed subgroups of given Lie groups.

Remark 1.1.45 Groups can also be constructed by taking the **quotient** of a group G by a normal subgroup H. However, for a Lie group G and a subgroup H, the quotient space G/H will not be a smooth manifold in general (at least not canonically). We will show in Sect. 3.8.3 that if H is a *closed* subgroup of a Lie group G, then G/H is indeed a smooth manifold. In this case, if H is normal, then G/H will again be a Lie group.

1.2 Linear Groups and Symmetry Groups of Vector Spaces

The most famous class of examples of Lie groups are the **general linear groups** $\mathrm{GL}(n, \mathbb{K})$ over the real, complex and quaternionic numbers $\mathbb{K}$ as well as the following groups:

Definition 1.2.1 A closed subgroup of a general linear group is called a **linear group** or **matrix group**.
By Cartan's Theorem 1.1.44 linear groups are embedded Lie subgroups and, in particular, Lie groups themselves. We are especially going to study the following families of linear groups, which are called **classical Lie groups**:

- the **special linear groups** in the real, complex and quaternionic case
- the **(special) orthogonal group** in the real case
- the **(special) unitary group** in the complex case
- the **compact symplectic group** (also called the **quaternionic unitary group**) in the quaternionic case
- the real **pseudo-orthogonal groups for indefinite scalar products**, like the Lorentz group.

The general linear groups are the (maximal) symmetry groups of vector spaces. The families of linear groups above arise as automorphism groups of certain structures on vector spaces. They can also be understood as *isotropy groups* in certain *representations* of the general linear groups.

There are two classes of Lie groups we are interested in which are not (at least *a priori*) linear:

- the **exceptional compact Lie groups** $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ (we will discuss G_2 in detail in Sect. 3.10)
- the **spin groups**, which are certain double coverings of (pseudo-)orthogonal groups.

All Lie groups that we will consider belong to one of these classes or are products of such Lie groups. Most linear groups are non-abelian and certain classes of linear groups – the (special) orthogonal, (special) unitary and symplectic groups – are compact. Lie groups like the Lorentz group and its spin group are not compact.

There are several reasons why linear groups are important, in particular with regard to *compact* Lie groups:

1. First, it is possible to prove that any compact Lie group is *isomorphic* as a Lie group to a linear group, see Theorem 1.2.7. In particular, the compact exceptional Lie groups and the compact spin groups are isomorphic to linear groups.
2. Secondly, there is a classification theorem which shows that (up to finite coverings) any compact Lie group G is isomorphic to a product

$$G = G_1 \times \ldots \times G_r$$

of compact Lie groups, all of which belong to the classes mentioned above (classical linear, spin and exceptional Lie groups); see Theorem 2.4.23 and Theorem 2.4.29 for the classification of compact Lie groups.

Even though the classical linear groups look quite special, they are thus of general significance, in particular for gauge theories with compact gauge groups.

Spin groups, such as the universal covering of the Lorentz group and its higher-dimensional analogues, are also important in physics, because they are involved in the mathematical description of *fermions*.

Finally, the exceptional Lie groups appear in several places in physics: E_6, for example, is the gauge group of certain *Grand Unified Theories*, E_8 plays a role in *heterotic string theory* and G_2 is related to *M-theory compactifications*.

Remark 1.2.2 It is an interesting fact that there are non-compact Lie groups which are *not* isomorphic to linear groups. One example is the universal covering of the Lie group $\mathrm{SL}(2,\mathbb{R})$ (see [70, Sect. 5.8] for a proof).

1.2.1 Isomorphism Groups of Vector Spaces

The simplest and fundamental linear groups, perhaps already known from a course on linear algebra, are the general linear groups themselves. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 1.2.3 For $n \geq 1$ the **general linear group** is defined as the group of linear isomorphisms of $\mathbb{K}^n$:

$$\mathrm{GL}(n,\mathbb{K}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid A \text{ is invertible}\},$$

where $\mathrm{Mat}(n \times n, \mathbb{K})$ denotes the ring of $n \times n$-matrices with coefficients in $\mathbb{K}$. Group multiplication in $\mathrm{GL}(n,\mathbb{K})$ is matrix multiplication.
For $\mathbb{K} = \mathbb{H}$ we act with matrices on the left of the right vector space $\mathbb{H}^n$, so that the maps are indeed $\mathbb{H}$-linear:

$$A(vq) = (Av)q \quad \forall A \in \mathrm{Mat}(n \times n, \mathbb{H}), v \in \mathbb{H}^n, q \in \mathbb{H}.$$

The following alternative description of the general linear group follows immediately:

Proposition 1.2.4 *The general linear group is given by*

$$\mathrm{GL}(n,\mathbb{K}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \det(A) \neq 0\}.$$

Clearly, for $n = 1$, we have

$$\mathrm{GL}(1,\mathbb{K}) = \mathbb{K}^*.$$

We first consider the real general linear group.

Proposition 1.2.5 (The Real General Linear Group Is a Lie Group) $\mathrm{GL}(n,\mathbb{R})$ *is a non-compact* n^2*-dimensional Lie group. It is not abelian for* $n \geq 2$.

Proof It is clear by properties of the determinant that $\mathrm{GL}(n,\mathbb{R})$ is a group in the algebraic sense.

Note that by continuity of the determinant, $\mathrm{GL}(n,\mathbb{R})$ is an open subset of

$$\mathrm{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}.$$

In particular, $\mathrm{GL}(n,\mathbb{R})$ is a smooth manifold of dimension n^2. Multiplication of two matrices A, B is quadratic in their coordinates, hence a smooth map. According to Remark 1.1.8 this shows that $\mathrm{GL}(n,\mathbb{R})$ is indeed a Lie group (we can also see directly that inversion of a matrix is, by Cramer's rule, a rational map in the coordinates, hence smooth).

The manifold $\mathrm{GL}(n,\mathbb{R})$, as a subset of $\mathbb{R}^{n^2}$ with the Euclidean norm, is not bounded, because it contains the unbounded set of diagonal matrices of the form rI_n with $r \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ (these elements actually define a subgroup of the general linear group, isomorphic to $\mathbb{R}^*$). By Heine–Borel, $\mathrm{GL}(n,\mathbb{R})$ is not compact.

To show that $\mathrm{GL}(n,\mathbb{R})$ is not abelian for $n \geq 2$, note that $\mathrm{GL}(n,\mathbb{R})$ contains the subgroup H isomorphic to $\mathrm{GL}(2,\mathbb{R})$, consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix}, \quad A \in \mathrm{GL}(2,\mathbb{R}).$$

It therefore suffices to show that $\mathrm{GL}(2,\mathbb{R})$ is not abelian: it is easy to find two matrices in $\mathrm{GL}(2,\mathbb{R})$ which do not commute. □

Using similar arguments it can be shown:

Proposition 1.2.6 (Complex and Quaternionic General Linear Groups Are Lie Groups) *Over the complex numbers and quaternions we have:*

1. $\mathrm{GL}(n,\mathbb{C})$ *is a non-compact* $2n^2$*-dimensional Lie group. It is not abelian for* $n \geq 2$.
2. $\mathrm{GL}(n,\mathbb{H})$ *is a non-compact* $4n^2$*-dimensional Lie group. It is not abelian for* $n \geq 1$.

As an application of the *Peter–Weyl Theorem*, the following can be shown (for a proof, which is beyond the scope of this book, see [24, 83, 129]):

Theorem 1.2.7 (Compact Lie Groups Are Linear) *Let* G *be a compact Lie group. Then there exists a smooth, injective group homomorphism* ϕ *of* G *into a general linear group* $\mathrm{GL}(n,\mathbb{C})$ *for some* n.

According to Corollary 1.8.18 the map ϕ is a *Lie group isomorphism* onto a linear group (see Sect. 1.3 for the formal definition of Lie group homomorphisms and isomorphisms).

The Peter–Weyl Theorem shows that every compact Lie group can be considered as a linear group. If we assume this result, we shall see later as a consequence of Theorem 2.1.39 that a compact Lie group G can even be embedded as a closed subgroup in a unitary group $\mathrm{U}(n)$ for some n and thus in the rotation group $\mathrm{SO}(2n)$ by Exercise 1.9.10. As a consequence any compact Lie group G can be literally thought of as a group whose elements are rotations on some $\mathbb{R}^{2n}$.

Remark 1.2.8 By comparison, recall that *Cayley's Theorem* says that any finite group of order n can be embedded as a subgroup of the symmetric group S_n.

1.2.2 Automorphism Groups of Structures on Vector Spaces

We want to consider specific classes of linear groups that arise as automorphism groups of certain structures on vector spaces.

Definition 1.2.9 We define the following scalar products:

1. On $\mathbb{R}^n$ the **standard Euclidean scalar product**

$$\langle v, w\rangle = v^T w = \sum_{k=1}^{n} v_k w_k.$$

2. On $\mathbb{C}^n$ the **standard Hermitian scalar product**

$$\langle v, w\rangle = v^\dagger w = \sum_{k=1}^{n} \overline{v_k} w_k.$$

3. On $\mathbb{H}^n$ the **standard symplectic scalar product**

$$\langle v, w\rangle = v^\dagger w = \sum_{k=1}^{n} \overline{v_k} w_k.$$

Here

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

are column vectors in $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively.

Definition 1.2.10 Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We define the **standard volume form** vol on $V = \mathbb{K}^n$ by

$$\begin{aligned} \mathrm{vol}\colon V \times \ldots \times V &\longrightarrow \mathbb{K} \\ (v_1, \ldots, v_n) &\longmapsto \det(v_1, \ldots, v_n), \end{aligned}$$

where $(v_1, \ldots, v_n)$ is the $n \times n$-matrix with columns $v_1, \ldots, v_n$.

Remark 1.2.11 For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ this form is $\mathbb{K}$-multilinear and alternating, but not for $\mathbb{K} = \mathbb{H}$.

Definition 1.2.12 Let $n \geq 1$.

1. The **special linear groups** for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ are defined as the automorphism groups of the standard volume forms:
$$\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}) \mid \mathrm{vol}(Av_1, \ldots, Av_n) = \mathrm{vol}(v_1, \ldots, v_n) \quad \forall v_1, \ldots, v_n \in \mathbb{K}^n\}.$$
2. The **orthogonal**, **unitary** and **(compact) symplectic** (also called **quaternionic unitary**) **groups** are defined as the automorphism groups of the standard Euclidean, Hermitian and symplectic scalar products:
$$\begin{aligned} \mathrm{O}(n) &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{R}^n\}, \\ \mathrm{U}(n) &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{C}^n\}, \\ \mathrm{Sp}(n) &= \{A \in \mathrm{GL}(n, \mathbb{H}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{H}^n\}. \end{aligned}$$
3. The **special orthogonal** and **special unitary groups** are defined as
$$\begin{aligned} \mathrm{SO}(n) &= \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}), \\ \mathrm{SU}(n) &= \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}). \end{aligned}$$

These Lie groups are called **classical groups**.

Remark 1.2.13 There are additional classes of classical groups, called *(non-compact) symplectic groups*, which are defined as automorphism groups of skew-symmetric forms. We will not consider these Lie groups in the subsequent discussions.

Remark 1.2.14 In the physics literature the Lie group $\mathrm{Sp}(n)$ is sometimes denoted by $\mathrm{USp}(n)$ (and occasionally by $\mathrm{Sp}(2n)$ or $\mathrm{USp}(2n)$). For example, [149] uses the notation $\mathrm{Sp}(n)$, whereas [90] uses the notation $\mathrm{Sp}(2n)$ and [138] uses the notation $\mathrm{USp}(2n)$. We continue to use the notation $\mathrm{Sp}(n)$.
It is often useful to have the following alternative description of these groups.

Proposition 1.2.15 (Matrix Description of Classical Groups) *Let $n \geq 1$.*

1. *The special linear groups are given by*

$$\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \det(A) = 1\}.$$

2. *The orthogonal, unitary and symplectic groups are given by*

$$\mathrm{O}(n) = \left\{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^T = I\right\},$$
$$\mathrm{U}(n) = \left\{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^\dagger = I\right\},$$
$$\mathrm{Sp}(n) = \left\{A \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid A \cdot A^\dagger = I\right\},$$

where A^T denotes the transpose of A and $A^\dagger = (\bar{A})^T$.

3. *The special orthogonal and special unitary groups are given by*

$$\mathrm{SO}(n) = \left\{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^T = I, \det(A) = 1\right\},$$
$$\mathrm{SU}(n) = \left\{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^\dagger = I, \det(A) = 1\right\}.$$

Proof

1. For $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ and column vectors $v_1, \ldots, v_n \in \mathbb{K}^n$ the following identity holds:

$$\mathrm{vol}(Av_1, \ldots, Av_n) = \det(A) \cdot \mathrm{vol}(v_1, \ldots, v_n).$$

This follows because as matrices

$$(Av_1, \ldots, Av_n) = A \cdot (v_1, \ldots, v_n)$$

and the determinant is multiplicative: $\det(AB) = \det(A)\det(B)$. This implies the formula for $\mathrm{SL}(n, \mathbb{K})$.

2. A matrix $A \in \mathrm{GL}(n, \mathbb{R})$ satisfies

$$\langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{R}^n$$

if and only if

$$v^T \left(A^T A\right) w = v^T w \quad \forall v, w \in \mathbb{R}^n.$$

Choosing $v = e_i, w = e_j$ this happens if and only if $A^T A = I$ and thus $AA^T = I$. The complex and quaternionic case follow similarly.

3. This is clear by the results above.

□

We did not define a quaternionic *special* unitary group, because it turns out that there is no difference between such a group and the quaternionic unitary group. This follows from the next proposition.

Proposition 1.2.16 *Let* $A \in \mathrm{Sp}(n)$. *Then* $\det(A) = 1$.

Proof Let $A = A_1 + jA_2 \in \mathrm{Mat}(n \times n, \mathbb{H})$ with A_1, A_2 complex. According to Proposition 1.1.32 the determinant of A is real and non-negative. We have

$$A^\dagger = A_1^\dagger - jA_2^T$$

and

$$\chi_{A^\dagger} = \chi_A^\dagger.$$

Thus for $A \in \mathrm{Sp}(n)$

$$1 = \det(I) = \det(\chi_A)\det\left(\chi_A^\dagger\right) = |\det(\chi_A)|^2 = (\det(A))^2.$$

Therefore $\det(A) = 1$. □

We now want to prove the main result in this subsection.

Theorem 1.2.17 (Classical Groups Are Linear) *The special linear, (special) orthogonal, (special) unitary and symplectic groups are closed subgroups of general linear groups, i.e. linear groups. They have the following properties:*

1. *The special linear groups have dimension*

$$\dim \mathrm{SL}(n, \mathbb{R}) = n^2 - 1,$$
$$\dim \mathrm{SL}(n, \mathbb{C}) = 2n^2 - 2,$$
$$\dim \mathrm{SL}(n, \mathbb{H}) = 4n^2 - 1.$$

These Lie groups are not compact for $n \geq 2$.

2. *The orthogonal, unitary and symplectic groups have dimension*

$$\dim \mathrm{O}(n) = \frac{1}{2}n(n-1),$$
$$\dim \mathrm{U}(n) = n^2,$$
$$\dim \mathrm{Sp}(n) = 2n^2 + n.$$

(continued)

Theorem 1.2.17 (continued)

These Lie groups are compact for all $n \geq 1$.

3. *The special orthogonal and special unitary groups have dimension*

$$\dim \mathrm{SO}(n) = \frac{1}{2}n(n-1),$$
$$\dim \mathrm{SU}(n) = n^2 - 1.$$

These Lie groups are compact for all $n \geq 1$.

Proof

- **Closed subgroups:** It is clear that all of these subsets are subgroups of general linear groups. The maps

$$\det\colon \mathrm{Mat}(n \times n, \mathbb{K}) \longrightarrow \mathbb{K},$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and

$$\det\colon \mathrm{Mat}(n \times n, \mathbb{H}) \longrightarrow \mathbb{R}$$

are continuous (polynomial in the coordinates), hence the preimages $\det^{-1}(1)$ are closed subsets. This shows that $\mathrm{SL}(n, \mathbb{K})$ is a closed subgroup of the general linear group $\mathrm{GL}(n, \mathbb{K})$. Similarly, the map

$$\mathrm{Mat}(n \times n, \mathbb{R}) \longrightarrow \mathrm{Mat}(n \times n, \mathbb{R})$$
$$A \longmapsto A \cdot A^T$$

is continuous (quadratic in the coordinates), hence the preimage of I is a closed subset. This shows that $\mathrm{O}(n)$ and the intersection $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ are closed subgroups of the general linear group $\mathrm{GL}(n, \mathbb{R})$. Similarly for $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$.
- **Compactness:** To show that $\mathrm{SL}(n, \mathbb{K})$ is not compact for $n \geq 2$ it suffices to show that the subgroup $\mathrm{SL}(2, \mathbb{K})$ is not compact. This follows by considering the unbounded subset of matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{K}), \quad a \in \mathbb{R}.$$

To show that $\mathrm{O}(n)$ and hence $\mathrm{SO}(n)$ are compact it suffices to show by Heine–Borel that $\mathrm{O}(n)$ is a bounded subset of the Euclidean space

$$\mathrm{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}.$$

Let $A \in \mathrm{O}(n)$. For fixed $i = 1, \ldots, n$ we have

$$1 = \left(AA^T\right)_{ii} = A_{i1}^2 + A_{i2}^2 + \ldots + A_{in}^2,$$

hence $|A_{ij}| \leq 1$ for all indices i, j. This implies the claim. Compactness of $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ follows similarly.

- **Dimensions of special linear groups:** Finally we calculate the dimensions. We claim that the smooth map

$$\det\colon \mathrm{Mat}(n \times n, \mathbb{K}) \longrightarrow \mathbb{K}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ has 1 as a regular value. This implies again that $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ are smooth manifolds and also the formulas $n^2 - 1$ and $2n^2 - 2$ for the dimensions (the argument for $\mathrm{SL}(n, \mathbb{H})$ is slightly different and is given below).

To prove the claim, let $A \in \mathrm{SL}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and write A as

$$A = (v_1, v_2, \ldots, v_n),$$

where the v_i are column vectors in $\mathbb{K}^n$. For $a \in \mathbb{K}$ fixed consider the curve

$$C(t) = ((1 + at)v_1, v_2, \ldots, v_n)$$

in $\mathrm{Mat}(n \times n, \mathbb{K})$. Then $C(0) = A$ and since the determinant is multilinear

$$\det C(t) = (1 + at) \det A = (1 + at),$$

thus

$$\left.\frac{d}{dt}\right|_{t=0} \det C(t) = a.$$

This shows that the differential of the determinant is surjective in every $A \in \mathrm{SL}(n, \mathbb{K})$ and 1 is a regular value.

We now prove the formula for the dimension of $\mathrm{SL}(n, \mathbb{H})$. We want to show that

$$\det\colon \mathrm{Mat}(n \times n, \mathbb{H}) \longrightarrow \mathbb{R}$$

has 1 as a regular value. This implies that $\mathrm{SL}(n, \mathbb{H})$ is a smooth manifold of dimension $4n^2 - 1$. Let $A = A_1 + jA_2 \in \mathrm{SL}(n, \mathbb{H})$ with A_1, A_2 complex, written in terms of complex column vectors as

$$A_1 = (v_1, v_2, \ldots, v_n),$$
$$A_2 = (w_1, w_2, \ldots, w_n).$$

For $b \in \mathbb{R}$ consider the curve

$$D(t) = ((1+bt)v_1, v_2, \ldots, v_n) + j((1+bt)w_1, w_2, \ldots, w_n)$$

in $\mathrm{Mat}(n \times n, \mathbb{H})$. Then $D(0) = A$ and the adjoint of $D(t)$ is

$$\chi_{D(t)} = \begin{pmatrix} (1+bt)v_1 & v_2 & \ldots & v_n & -(1+bt)\bar{w}_1 & -\bar{w}_2 & \ldots & -\bar{w}_n \\ (1+bt)w_1 & w_2 & \ldots & w_n & (1+bt)\bar{v}_1 & \bar{v}_2 & \ldots & \bar{v}_n \end{pmatrix}$$

with determinant

$$\det D(t) = \det \chi_{D(t)} = (1+bt)^2 \det \chi_A = (1+bt)^2.$$

It follows that

$$\left.\frac{d}{dt}\right|_{t=0} \det D(t) = 2b.$$

This shows that the differential of the determinant is surjective in every $A \in \mathrm{SL}(n, \mathbb{H})$ and 1 is a regular value.

- **Dimensions of** $\mathrm{O}(n)$**,** $\mathrm{U}(n)$ **and** $\mathrm{Sp}(n)$**:** To calculate the dimension of the orthogonal group $\mathrm{O}(n)$ consider the map

$$f\colon \mathrm{Mat}(n \times n, \mathbb{R}) \longrightarrow \mathrm{Sym}(n, \mathbb{R})$$
$$A \longmapsto A \cdot A^T,$$

where $\mathrm{Sym}(n, \mathbb{R})$ denotes the space of symmetric, real $n \times n$-matrices. Then $\mathrm{O}(n) = f^{-1}(I)$. The differential of this map at a point $A \in \mathrm{O}(n)$ in the direction $X \in \mathrm{Mat}(n \times n, \mathbb{R})$ is

$$(D_A f)(X) = XA^T + AX^T.$$

Let $B \in \mathrm{Sym}(n, \mathbb{R})$ and set

$$X = \frac{1}{2} BA.$$

Then $(D_A f)(X) = B$ and thus I is a regular value of f. This shows that $\mathrm{O}(n)$ is a smooth manifold of dimension

$$\begin{aligned} \dim \mathrm{O}(n) &= \dim \mathrm{Mat}(n \times n, \mathbb{R}) - \dim \mathrm{Sym}(n, \mathbb{R}) \\ &= n^2 - \frac{1}{2} n(n+1) \\ &= \frac{1}{2} n(n-1). \end{aligned}$$

We can calculate the dimensions of U(n) and Sp(n) in a similar way, utilizing for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ the map

$$\begin{aligned} f\colon \mathrm{Mat}(n \times n, \mathbb{K}) &\longrightarrow \mathrm{Herm}(n, \mathbb{K}) \\ A &\longmapsto A \cdot A^{\dagger}, \end{aligned}$$

where Herm($n, \mathbb{K}$) denotes the space of Hermitian $n \times n$-matrices (the set of all matrices B with $B^{\dagger} = B$). Again I is a regular value and thus U(n) and Sp(n) are smooth manifolds of dimension

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{Mat}(n \times n, \mathbb{K}) - \dim_{\mathbb{R}} \mathrm{Herm}(n, \mathbb{K}) &= kn^2 - \left(\frac{1}{2}k(n-1)n + n\right) \\ &= \frac{1}{2}kn(n+1) - n, \end{aligned}$$

where $k = 2, 4$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$. This implies

$$\dim \mathrm{U}(n) = n^2$$

and

$$\dim \mathrm{Sp}(n) = 2n^2 + n.$$

- **Dimensions of** SO(n) **and** SU(n)**:** We claim that SO(n) is a submanifold of codimension zero in O(n). The determinant on O(n) has values in $\{+1, -1\}$,

$$\det\colon \mathrm{O}(n) \longrightarrow \{+1, -1\}.$$

This map obviously has 1 as a regular value.

Similarly, we claim that SU(n) is a submanifold of codimension one in U(n). The determinant on U(n) has values in S^1,

$$\det\colon \mathrm{U}(n) \longrightarrow S^1.$$

We claim that this map has 1 as a regular value. Let $A \in \mathrm{SU}(n)$, written in terms of complex column vectors as

$$A = (v_1, v_2, \ldots, v_n).$$

For $\alpha \in \mathbb{R}$ consider the curve

$$C(t) = \left(e^{i\alpha t} v_1, v_2, \ldots, v_n\right).$$

It is easy to check that $C(t)$ is a curve in $\mathrm{U}(n)$ and

$$\det C(t) = e^{i\alpha t} \det A = e^{i\alpha t}$$

with

$$\left.\frac{d}{dt}\right|_{t=0} \det C(t) = i\alpha.$$

This proves the claim.

□

Example 1.2.18 It follows directly from the matrix description that

$$\mathrm{SL}(1,\mathbb{R}) = \mathrm{SL}(1,\mathbb{C}) = \{1\},$$
$$\mathrm{SL}(1,\mathbb{H}) = S^3$$

and

$$\mathrm{O}(1) = S^0,$$
$$\mathrm{U}(1) = S^1,$$
$$\mathrm{Sp}(1) = S^3.$$

We also saw that

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{Mat}(2\times 2,\mathbb{R}) \,\middle|\, \alpha \in \mathbb{R} \right\}$$

and it is not difficult to check that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{Mat}(2\times 2,\mathbb{C}) \,\middle|\, a,b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

We will discuss less trivial identifications between Lie groups, like $\mathrm{SU}(2) \cong S^3$, after we have defined the notion of a Lie group isomorphism.

Remark 1.2.19 Some of the linear groups appear directly in gauge theories: For instance, as mentioned before, the gauge group of the current Standard Model of particle physics is the product

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

There are Grand Unified Theories based on Lie groups like SU(5) and SO(10) (or rather its *universal covering group* Spin(10)).

Remark 1.2.20 (Classical Linear Groups as Isotropy Groups) Note that $\mathrm{GL}(n, \mathbb{K})$ and its subgroups act canonically on the (column) vector space $\mathbb{K}^n$ by matrix multiplication from the left. This is an example of a representation, called the **fundamental representation**. We will consider representations in more detail in Sect. 2.1.

Representations are special classes of *group actions*, see Chap. 3. It is not difficult to see that the linear groups can be realized as *isotropy groups* of certain elements in suitable representation spaces of the general linear groups. Representations and isotropy groups will also be used in Sect. 3.10 to define the exceptional Lie group G_2 as an embedded Lie subgroup of $\mathrm{GL}(7, \mathbb{R})$.

1.2.3 Connectivity Properties of Linear Groups

Proposition 1.2.21 (Connected Components of Lie Groups) *Let G be a Lie group. Then all connected components of G are diffeomorphic to the connected component G_e of the neutral element $e \in G$. In particular, all connected components have the same dimension.*

Proof Let $g \in G$ and denote the connected component of G containing g by G_g. Consider the *left translation*, given by

$$\begin{aligned} L_g \colon G_e &\longrightarrow G \\ x &\longmapsto gx \end{aligned}$$

restricted to the connected component G_e containing e. The image of this smooth map is connected and contains g, therefore the image is contained in G_g. By the same argument

$$L_{g^{-1}} \colon G_g \longrightarrow G$$

has image contained in G_e. It follows that

$$L_g \colon G_e \longrightarrow G_g$$

is a diffeomorphism and thus all connected components of G are diffeomorphic. □

We want to understand how many connected components the classical Lie groups have.

Theorem 1.2.22 (Connected Components of Classical Groups) *Let $n \geq 1$.*

1. *The Lie group $\mathrm{GL}(n, \mathbb{R})$ has two connected components $\mathrm{GL}(n, \mathbb{R})_\pm$, determined by the sign of the determinant. The Lie groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{H})$ are connected.*
2. *The special linear groups $\mathrm{SL}(n, \mathbb{K})$ are connected for all $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.*

3. *The Lie group* $\mathrm{O}(n)$ *has two connected components* $\mathrm{O}(n)_{\pm}$*, determined by the sign of the determinant. The Lie group* $\mathrm{SO}(n) = \mathrm{O}(n)_{+}$ *is connected.*
4. *The Lie groups* $\mathrm{U}(n)$*,* $\mathrm{SU}(n)$ *and* $\mathrm{Sp}(n)$ *are connected.*

There are direct proofs for these assertions that the reader can find in the literature. We chose to give a different argument using *homogeneous spaces* in Sect. 3.8.3, which is conceptually clearer and simpler. The proof with homogeneous spaces utilizes certain actions of the classical groups on $\mathbb{K}^n \setminus \{0\}$ and spheres S^m. The assertions then follow by induction over n from the corresponding (trivial) statements for the groups with $n = 1$.

The number of connected components of a Lie group G can be identified with the number of elements of the homotopy group $\pi_0(G)$. In Sect. 2.6 we will discuss higher homotopy groups of Lie groups.

1.3 Homomorphisms of Lie Groups

Lie groups have two structures: the algebraic structure of a group and the smooth structure of a manifold. A homomorphism between Lie groups should be compatible with both structures.

Definition 1.3.1 Let G and H be Lie groups. A map $\phi\colon G \to H$ which is smooth and a group homomorphism, i.e.

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2) \quad \forall g_1, g_2 \in G,$$

is called a **Lie group homomorphism**. The map ϕ is called a **Lie group isomorphism** if it is a diffeomorphism and a homomorphism (hence an isomorphism) of groups. A Lie group isomorphism $\phi\colon G \to G$ is called a **Lie group automorphism** of G.

Remark 1.3.2 We will show in Theorem 1.8.14 as an application of Cartan's Theorem that *continuous* group homomorphisms $\phi\colon G \to H$ between Lie groups are automatically *smooth*, hence Lie group homomorphisms.

Remark 1.3.3 Occasionally we will call a Lie group homomorphism just a homomorphism if the meaning is clear from the context.
We consider some examples of Lie group homomorphisms.

Example 1.3.4 Let G and H be Lie groups. The constant map

$$\begin{aligned} \phi\colon G &\longrightarrow H \\ g &\longmapsto e \end{aligned}$$

is always a Lie group homomorphism, called the **trivial** homomorphism.

Example 1.3.5 Consider $\mathbb{R}$ with addition and the Lie group S^1. Then

$$\begin{aligned}\phi\colon \mathbb{R} &\longrightarrow S^1\\ x &\longmapsto e^{ix}\end{aligned}$$

is a surjective homomorphism of Lie groups. The kernel of this map is

$$\phi^{-1}(1) = 2\pi\mathbb{Z}.$$

Taking products, there is a similar surjective homomorphism $\mathbb{R}^n \to T^n$.

Example 1.3.6 It is easy to check that the map

$$\begin{aligned}\phi\colon S^1 &\longrightarrow \mathrm{SO}(2)\\ e^{i\alpha} &\longmapsto \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\end{aligned}$$

is an isomorphism of Lie groups.

Example 1.3.7 We have

$$\begin{aligned}S^3 &= \{w \in \mathbb{H} \mid ||w||^2 = w\bar{w} = 1\}\\ &= \{x + yi + uj + vk \in \mathbb{H} \mid x^2 + y^2 + u^2 + v^2 = 1\}.\end{aligned}$$

We also have

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b}\\ b & \bar{a}\end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{C}) \,\middle|\, |a|^2 + |b|^2 = 1 \right\}.$$

It can be checked that the map

$$\begin{aligned}\phi\colon S^3 &\longrightarrow \mathrm{SU}(2)\\ x + yi + uj + vk &\longmapsto \begin{pmatrix} x+iy & -u-iv\\ u-iv & x-iy\end{pmatrix}\end{aligned}$$

is an isomorphism of Lie groups. This follows from a direct calculation or from Proposition 1.1.26.

Example 1.3.8 (Universal Covering of SO(3)*)* We define a Lie group homomorphism

$$\phi\colon S^3 \longrightarrow \mathrm{SO}(3)$$

in the following way: Let $w \in \mathbb{H}$ be a unit quaternion, $||w|| = 1$, and consider the map

$$\begin{aligned}\tau_w \colon \mathbb{H} &\longrightarrow \mathbb{H}\\ z &\longmapsto wzw^{-1}.\end{aligned}$$

This is an $\mathbb{R}$-linear isomorphism of the 4-dimensional real vector space $\mathbb{H}$. Since

$$||wzw^{-1}|| = ||w|| \cdot ||z|| \cdot ||w||^{-1} = ||z||,$$

the map τ_w is orthogonal with respect to the standard Euclidean scalar product on $\mathbb{H} \cong \mathbb{R}^4$. The map τ_w clearly fixes

$$\mathrm{Re}\,\mathbb{H} = \{x \in \mathbb{H} \mid x \in \mathbb{R}\}$$

and therefore restricts to an orthogonal isomorphism

$$\phi(w) = \tau_w|_{\mathrm{Im}\,\mathbb{H}} \colon \mathrm{Im}\,\mathbb{H} \longrightarrow \mathrm{Im}\,\mathbb{H}$$

on the orthogonal complement

$$\mathrm{Im}\,\mathbb{H} = \mathrm{Re}\,\mathbb{H}^{\perp} = \{yi + uj + vk \in \mathbb{H} \mid y, u, v \in \mathbb{R}\}.$$

This shows that $\phi(w) \in \mathrm{O}(3)$. Since the map $\phi \colon S^3 \to \mathrm{O}(3)$ is continuous, S^3 is connected and $\phi(1) = I$, it follows that ϕ has image in the connected component $\mathrm{SO}(3)$ and hence defines a map

$$\phi \colon S^3 \longrightarrow \mathrm{SO}(3).$$

It can be checked that this map is a surjective homomorphism of Lie groups with kernel $\{+1, -1\}$, cf. Exercise 1.9.20. The homomorphism ϕ defines a connected double covering of $\mathrm{SO}(3)$ by S^3 (this is the *universal covering* of $\mathrm{SO}(3)$, since S^3 is simply connected).

1.4 Lie Algebras

Lie algebras are of similar importance for symmetries and gauge theories as Lie groups. We will begin with the general definition of Lie algebras and describe in the next section their relation to Lie groups.

Definition 1.4.1 A vector space V together with a map

$$[\cdot,\cdot]\colon V \times V \longrightarrow V$$

is called a **Lie algebra** if the following hold:

1. $[\cdot,\cdot]$ is bilinear.
2. $[\cdot,\cdot]$ is antisymmetric:

$$[v,w] = -[w,v] \quad \forall v,w \in V.$$

3. $[\cdot,\cdot]$ satisfies the **Jacobi identity**:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0 \quad \forall u,v,w \in V.$$

The map $[\cdot,\cdot]\colon V \times V \to V$ is called the **Lie bracket**. We will only consider Lie algebras defined on real or complex vector spaces. Unless stated otherwise the vector spaces underlying Lie algebras are finite-dimensional.

We collect some examples to show that Lie algebras occur quite naturally (we discuss many more examples in Sect. 1.5.5).

Example 1.4.2 (Abelian Lie Algebras) Every real or complex vector space with the trivial Lie bracket $[\cdot,\cdot] \equiv 0$ is a Lie algebra. Such Lie algebras are called **abelian**. Every 1-dimensional Lie algebra is abelian, because the Lie bracket is antisymmetric.

Example 1.4.3 (Lie Algebra of Matrices) The vector space $V = \mathrm{Mat}(n \times n, \mathbb{K})$ of square matrices with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a real or complex Lie algebra with bracket defined by the commutator of matrices A, B:

$$[A,B] = A \cdot B - B \cdot A.$$

The only axiom that has to checked is the Jacobi identity. This example is very important, because the Lie algebras of linear groups have the same Lie bracket, cf. Corollary 1.5.26. It even follows from *Ado's Theorem* 1.5.25 that any finite-dimensional Lie algebra can be embedded into such a matrix Lie algebra.

Example 1.4.4 (Lie Algebra of Endomorphisms) In the same way the vector space $V = \mathrm{End}(W)$ of endomorphisms (linear maps) on a real or complex vector space W is a real or complex Lie algebra with Lie bracket defined by the commutator of endomorphisms f, g:

$$[f,g] = f \circ g - g \circ f.$$

Example 1.4.5 (Lie Algebra Defined by an Associative Algebra) Even more generally, let A be any associative algebra with multiplication $\cdot$. Then the commutator

$$[a, b] = a \cdot b - b \cdot a$$

defines a Lie algebra structure on A.

Example 1.4.6 (Cross Product on $\mathbb{R}^3$) The vector space $\mathbb{R}^3$ is a Lie algebra with the bracket given by the cross product:

$$[v, w] = v \times w.$$

Again, the only axiom that has to be checked is the Jacobi identity. We will identify $(\mathbb{R}^3, \times)$ with a classical Lie algebra in Exercise 1.9.14.

Example 1.4.7 (Lie Algebra of Vector Fields on a Manifold) Let M be a differentiable manifold and $\mathfrak{X}(M)$ the real vector space of smooth vector fields on M. It follows from Theorem A.1.45 that $\mathfrak{X}(M)$ together with the commutator of vector fields is a real Lie algebra, which is infinite-dimensional if the dimension of M is at least one.

As in the case of Lie groups we have two constructions that yield new Lie algebras from given ones.

Definition 1.4.8 Let $(V, [\cdot\,,\cdot])$ be a Lie algebra. A vector subspace $W \subset V$ is called a **Lie subalgebra** if for all $w, w' \in W$ the Lie bracket $[w, w']$ is an element of W.

Example 1.4.9 Every 1-dimensional vector subspace of a Lie algebra V is an abelian subalgebra.

Example 1.4.10 From the geometric interpretation of the cross product it follows that $(\mathbb{R}^3, \times)$ does not have 2-dimensional Lie subalgebras.

Example 1.4.11 (Intersection of Lie Subalgebras) If $W_1, W_2 \subset V$ are Lie subalgebras, then the **intersection** $W_1 \cap W_2$ is again a Lie subalgebra of V.

Definition 1.4.12 Let $(V, [\cdot\,,\cdot]_V)$ and $(W, [\cdot\,,\cdot]_W)$ be Lie algebras over the same field. Then the **direct sum Lie algebra** is the vector space $V \oplus W$ with the Lie bracket

$$\left[v \oplus w, v' \oplus w'\right] = \left[v, v'\right]_V \oplus \left[w, w'\right]_W.$$

Remark 1.4.13 Note that if V, W are Lie subalgebras in a Lie algebra Q which are complementary as vector spaces, so that $Q = V \oplus W$, it does not follow in general that $Q = V \oplus W$ as Lie algebras. For $v, v' \in V, w, w' \in W$ we have

$$\left[v + w, v' + w'\right] = \left[v, v'\right] + \left[w, w'\right] + \left[v, w'\right] + \left[w, v'\right],$$

hence we need in addition

$$[V, W] = 0.$$

We finally want to define homomorphisms between Lie algebras.

Definition 1.4.14 Let $(V, [\cdot, \cdot]_V)$ and $(W, [\cdot, \cdot]_W)$ be Lie algebras. A linear map $\psi: V \to W$ is called **Lie algebra homomorphism** if

$$[\psi(x), \psi(y)]_W = \psi([x, y]_V) \quad \forall x, y \in V.$$

A **Lie algebra isomorphism** is a bijective homomorphism. An **automorphism** of a Lie algebra V is a Lie algebra isomorphism $\psi: V \to V$.

Example 1.4.15 Let V and W be Lie algebras over $\mathbb{K}$. The constant map

$$\begin{aligned} \psi: V &\longrightarrow W \\ X &\longmapsto 0 \end{aligned}$$

is always a Lie algebra homomorphism, called the **trivial** homomorphism.

Example 1.4.16 The injection $i: W \hookrightarrow V$ of a Lie subalgebra into a Lie algebra is of course a Lie algebra homomorphism.
The following notion appears, in particular, in physics:

Definition 1.4.17 Let V be a Lie algebra over $\mathbb{K}$ and $T_1, \ldots, T_n$ a vector space basis for V. Then we can write

$$[T_a, T_b] = \sum_{c=1}^{n} f_{abc} T_c,$$

where the coefficients $f_{abc} \in \mathbb{K}$ are called **structure constants** for the given basis $\{T_a\}$.
Because of bilinearity the structure constants determine all commutators between elements of V. The structure constants are antisymmetric in the first two indices

$$f_{abc} = -f_{bac} \quad \forall a, b, c$$

and satisfy the Jacobi identity

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0 \quad \forall a, b, c, e$$

(here we use the Einstein summation convention and sum over d). Conversely, every set of $n \times n \times n$ numbers $f_{abc} \in \mathbb{K}$ satisfying these two conditions define a Lie algebra structure on $V = \mathbb{K}^n$.

1.5 From Lie Groups to Lie Algebras

So far we have discussed Lie groups and Lie algebras as two independent notions. We now want to turn to a well-known construction that yields for every Lie group an associated Lie algebra, which can be thought of as an infinitesimal or linear description of the Lie group.

Recall that for every smooth manifold M, the set of smooth vector fields $\mathfrak{X}(M)$ on M with the commutator forms a Lie algebra, which is infinite-dimensional if $\dim M \geq 1$. We could associate to a Lie group G the Lie algebra $\mathfrak{X}(G)$ of all vector fields on G. However, as an infinite-dimensional Lie algebra this is somewhat difficult to handle. It turns out that for a Lie group G there exists a canonical *finite-dimensional* Lie subalgebra $\mathfrak{g}$ in $\mathfrak{X}(G)$ which has the same dimension as the Lie group G itself. This will be the Lie algebra associated to the Lie group G.

1.5.1 Vector Fields Invariant Under Diffeomorphisms

We first consider a very general situation. Let M be a smooth manifold and Γ an arbitrary set of diffeomorphisms from M to M.

Definition 1.5.1 We define the set of vector fields on M invariant under Γ by

$$A_\Gamma(M) = \{X \in \mathfrak{X}(M) \mid \phi_* X = X \quad \forall \phi \in \Gamma\}.$$

We have:

Proposition 1.5.2 *For every set Γ of diffeomorphisms of M, the set $A_\Gamma(M)$ is a Lie subalgebra in the Lie algebra $\mathfrak{X}(M)$ with the commutator.*

Proof Suppose $\Gamma = \{\phi\}$ consists of a single diffeomorphism. If $X, Y \in A_{\{\phi\}}(M)$ and $a, b \in \mathbb{R}$, then

$$\begin{aligned} \phi_*(aX + bY) &= a\phi_* X + b\phi_* Y \\ &= aX + bY \end{aligned}$$

and

$$\begin{aligned} \phi_*[X, Y] &= [\phi_* X, \phi_* Y] \\ &= [X, Y] \end{aligned}$$

according to Corollary A.1.51. Hence $A_{\{\phi\}}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$. The claim for a general set Γ of diffeomorphisms then follows from

$$A_\Gamma(M) = \bigcap_{\phi \in \Gamma} A_{\{\phi\}}(M).$$

□

1.5.2 Left-Invariant Vector Fields

We now consider the case of a Lie group G. There exist special diffeomorphisms on G that are defined by group elements $g \in G$.

Definition 1.5.3 For $g \in G$ we set:

$$\begin{aligned} L_g\colon G &\longrightarrow G \\ h &\longmapsto g \cdot h \\ R_g\colon G &\longrightarrow G \\ h &\longmapsto h \cdot g \\ c_g\colon G &\longrightarrow G \\ h &\longmapsto g \cdot h \cdot g^{-1}. \end{aligned}$$

These maps are called **left translation**, **right translation** and **conjugation** by g, respectively.

Example 1.5.4 For $G = \mathbb{R}^n$ with vector addition and $a \in \mathbb{R}^n$ we have

$$\begin{aligned} L_a\colon \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto a + x \\ R_a\colon \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto x + a. \end{aligned}$$

This explains the names left and right translation.
The following properties are easy to check:

Lemma 1.5.5 (Properties of Translations and Conjugation) *For all $g \in G$ we have:*

1. *The inverses of left and right translations are given by*

$$L_g^{-1} = L_{g^{-1}} \quad R_g^{-1} = R_{g^{-1}}.$$

   The inverse of conjugation is given by

$$c_g^{-1} = c_{g^{-1}}.$$

   In particular, L_g, R_g and c_g are diffeomorphisms of G.
2. *L_g and R_h commute for all $g, h \in G$.*

3. $c_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$.
4. *The conjugations c_g for $g \in G$ are Lie group automorphisms of G, called* **inner automorphisms**.

Remark 1.5.6 Note that left translation L_g and right translation R_g are not Lie group homomorphisms for $g \neq e$, because

$$L_g(e) = R_g(e) = g \neq e.$$

Example 1.5.7 If G is abelian, then $L_g = R_g$ for all $g \in G$ and $c_g = \mathrm{Id}_G$ for all $g \in G$. Each of these two properties characterizes abelian Lie groups.
We now set:

Definition 1.5.8 A vector field $X \in \mathfrak{X}(G)$ on a Lie group G is called **left-invariant** if $L_{g*}X = X$ for all $g \in G$. In other words, the set of left-invariant vector fields on G is $A_\Gamma(G)$, where Γ is the set of all left translations.
We get with Proposition 1.5.2:

Theorem 1.5.9 (The Lie Algebra of a Lie Group) *The set of left-invariant vector fields together with the commutator $[\cdot, \cdot]$ of vector fields on the Lie group G forms a Lie subalgebra*

$$L(G) = \mathfrak{g}$$

in the Lie algebra $\mathfrak{X}(G)$ of all vector fields on G. We call $\mathfrak{g}$ the **Lie algebra of** *(or* **associated to***) G.*

Remark 1.5.10 We could also define the Lie algebra of a Lie group with right-invariant vector fields. Using left-invariant vector fields is just the standard convention.

Remark 1.5.11 We defined Lie algebras in general on vector spaces over arbitrary fields. The Lie algebra of a Lie group, however, is always a *real* Lie algebra.
As mentioned before, vector fields, their flows and the commutator are only defined on smooth manifolds. This is the reason why only Lie groups have an associated Lie algebra and not other types of groups.

We want to show that there is a vector space isomorphism between the Lie algebra $\mathfrak{g}$ and the tangent space T_eG.

Definition 1.5.12 Let G be a Lie group with neutral element e and associated Lie algebra $\mathfrak{g}$. We define the **evaluation map**

$$\begin{aligned} \mathrm{ev}\colon \mathfrak{g} &\longrightarrow T_eG \\ X &\longmapsto X_e. \end{aligned}$$

Lemma 1.5.13 *The evaluation map is a vector space isomorphism.*

Proof The evaluation map is clearly linear. To construct the inverse of a vector $x \in T_eG$ under the map ev define a vector field X on G by

$$X_h = (D_eL_h)x, \quad h \in G.$$

To show that X is smooth, consider the multiplication map

$$\begin{aligned} \mu \colon G \times G &\longrightarrow G \\ (h, g) &\longmapsto hg \end{aligned}$$

with differential

$$\begin{aligned} D\mu \colon TG \times TG &\longrightarrow TG \\ ((h, Y), (g, X)) &\longmapsto (D_gL_h)(X) + (D_hR_g)(Y). \end{aligned}$$

Then the following map is smooth

$$\begin{aligned} G &\longrightarrow TG \\ h &\longmapsto D\mu((h, 0), (e, x)) = (D_eL_h)x \end{aligned}$$

which is just the vector field X.

The vector field X is also left-invariant, because

$$(D_hL_g)X_h = (D_e(L_g \circ L_h))x = (D_eL_{gh})x = X_{gh}$$

for all $g \in G$ and thus $L_{g*}X = X$. The map

$$\begin{aligned} T_eG &\longrightarrow \mathfrak{g} \\ x &\longmapsto X \end{aligned}$$

is the inverse of ev. □

We can therefore think of the tangent space T_eG of the Lie group G at the neutral element e as having the structure of the Lie algebra $\mathfrak{g}$. In particular, we get:

Corollary 1.5.14 *The Lie algebra of a Lie group is finite-dimensional with dimension equal to the dimension of the Lie group. Furthermore, a left-invariant (or right-invariant) vector field on a Lie group is completely determined by its value at one point.*

It is a non-trivial theorem that any abstract real Lie algebra can be realized by the construction above (for a proof, see [77, 83]):

Theorem 1.5.15 (Lie's Third Theorem) *Every finite-dimensional real Lie algebra is isomorphic to the Lie algebra of some connected Lie group.*
Lie's Third Theorem was proved in this form by Élie Cartan. Note that there may be different, non-isomorphic Lie groups with isomorphic Lie algebras: a trivial example is given by the Lie groups $(\mathbb{R}, +)$ and $(S^1, \cdot)$ whose Lie algebras are one-dimensional and hence abelian. The orthogonal and spin groups provide another example, to be discussed in Chap. 6.

1.5.3 Induced Homomorphisms

Just as we get for every Lie group an associated Lie algebra, we get for every homomorphism between Lie groups a homomorphism between the associated Lie algebras.

Definition 1.5.16 Let G, H be Lie groups and $\phi\colon G \to H$ a homomorphism of Lie groups. If X is a left-invariant vector field on G, we can uniquely define a left-invariant vector field $\phi_* X$ on H by

$$\mathrm{ev}(\phi_* X) = (\phi_* X)_e = (D_e\phi)(X_e).$$

This defines a map

$$\phi_*\colon \mathfrak{g} \longrightarrow \mathfrak{h},$$

called the **differential** or **induced homomorphism** of the homomorphism ϕ.

Remark 1.5.17 Here are two remarks concerning this definition:

1. Note that $\phi(e) = e$ for a homomorphism, so that $\phi_* X \in \mathfrak{h}$ is well-defined.
2. As the notation of the theorem indicates, the push-forward on vector fields is defined in the case of Lie groups not only for diffeomorphisms, but also for *Lie group homomorphisms* acting on *left-invariant* vector fields. This definition is possible, because left-invariant vector fields on Lie groups are determined by their value at one point.

Theorem 1.5.18 (The Differential Is a Lie Algebra Homomorphism) *The differential $\phi_*\colon \mathfrak{g} \to \mathfrak{h}$ of a Lie group homomorphism $\phi\colon G \to H$ is a homomorphism of Lie algebras.*

Proof We have to show that

$$[\phi_* X, \phi_* Y] = \phi_*[X, Y] \quad \forall X, Y \in \mathfrak{g}.$$

By Proposition A.1.49 this will follow if we can show that $\phi_* X$ is ϕ-related to X, i.e. that

$$(\phi_* X)_{\phi(g)} = (D_g \phi)(X_g) \quad \forall g \in G.$$

We have

$$\begin{aligned}(\phi_* X)_{\phi(g)} &= (D_e L_{\phi(g)} \circ D_e \phi)(X_e) \\ &= D_e(L_{\phi(g)} \circ \phi)(X_e) \\ &= D_e(\phi \circ L_g)(X_e) \\ &= (D_g \phi)(X_g),\end{aligned}$$

because $\phi_* X$ and X are left-invariant and ϕ is a homomorphism. This proves the claim. □

Note that it is essential for this argument that the map ϕ is a Lie group homomorphism.

Corollary 1.5.19 *Let $H \subset G$ be an immersed or embedded Lie subgroup. Then $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra.*

Proof The inclusion $i: H \hookrightarrow G$ is a homomorphism of Lie groups and an immersion. Thus the induced inclusion $i_*: \mathfrak{h} \hookrightarrow \mathfrak{g}$ is an injective homomorphism of Lie algebras. □

We can ask whether it is possible to reverse these relations:

- If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, does there exist a Lie subgroup H in G whose Lie algebra is $\mathfrak{h}$?
- If $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism between the Lie algebras of Lie groups G and H, does there exist a Lie group homomorphism $\psi: G \rightarrow H$ inducing ϕ on Lie algebras?

Both questions are related to the concept of *integration* from a linear object on the level of Lie algebras to a non-linear object on the level of Lie groups. We shall answer the first question in Sect. 1.6 and briefly comment here, without proof, on the second question. The following theorem specifies a sufficient condition for the existence of a Lie group homomorphism inducing a given Lie algebra homomorphism (for a proof, see [77, 142]):

Theorem 1.5.20 (Integrability Theorem for Lie Algebra Homomorphisms) *Let G be a connected and simply connected Lie group, H a Lie group and $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ a Lie algebra homomorphism. Then there exists a unique Lie group homomorphism $\psi: G \rightarrow H$ such that $\psi_* = \phi$.*

Example 1.5.21 Without the condition that G is simply connected this need not hold: every Lie algebra homomorphism $\phi: \mathfrak{so}(2) \rightarrow \mathfrak{h}$ induces a unique Lie group homomorphism $\psi: \mathbb{R} \rightarrow H$. However, ϕ does not always induce a Lie group homomorphism $\mathrm{SO}(2) \rightarrow H$ (see the discussion after Corollary 2.1.13).

Similarly there are homomorphisms $\mathfrak{so}(n) \to \mathfrak{h}$ for $n \geq 3$ (so-called *spinor representations*, see Sect. 6.5.2) that do not integrate to homomorphisms $\mathrm{SO}(n) \to H$ (it can be shown that $\mathrm{SO}(n)$ has fundamental group $\mathbb{Z}_2$ for $n \geq 3$).

1.5.4 The Lie Algebra of the General Linear Groups

We have defined the Lie algebra associated to a Lie group, but so far we have not seen any explicit examples of this construction. In this and the subsequent subsection we want to study the Lie algebra associated to the linear groups, i.e. closed subgroups of the general linear groups. We can understand the structure of the corresponding Lie algebras by Corollary 1.5.19 once we have understood the structure of the Lie algebra of the general linear groups.

Theorem 1.5.22 (Lie Algebra of General Linear Groups) *The Lie algebra of the general linear group* $\mathrm{GL}(n,\mathbb{R})$ *is* $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}(n\times n,\mathbb{R})$ *and the Lie bracket on* $\mathfrak{gl}(n,\mathbb{R})$ *is given by the standard commutator of matrices:*

$$[X,Y] = X\cdot Y - Y\cdot X \quad \forall X,Y \in \mathfrak{gl}(n,\mathbb{R}).$$

An analogous result holds for the Lie algebra $\mathfrak{gl}(n,\mathbb{K})$ *of* $\mathrm{GL}(n,\mathbb{K})$ *for* $\mathbb{K} = \mathbb{C},\mathbb{H}$.

We want to prove this theorem. The Lie group $G = \mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, therefore we can canonically identify the tangent space at the unit element I,

$$T_I\mathrm{GL}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R}) = \mathfrak{g},$$

with the vector space

$$\mathbb{R}^{n^2} = \mathrm{Mat}(n\times n,\mathbb{R}).$$

Lemma 1.5.23 *If* $X \in \mathrm{Mat}(n\times n,\mathbb{R}) = \mathfrak{g}$, *then the associated left-invariant vector field* $\tilde{X}$ *on* G *is given by*

$$\tilde{X}_A = A\cdot X, \quad \forall A \in G,$$

where $\cdot$ *denotes matrix multiplication.*

Proof To show this, let γ_X be an arbitrary curve in G through e and tangent to X. Then

$$\begin{aligned}\tilde{X}_A &= (D_eL_A)(X)\\ &= \left.\frac{d}{dt}\right|_{t=0} L_A(\gamma_X(t))\end{aligned}$$

$$= \left.\frac{d}{dt}\right|_{t=0} A \cdot \gamma_X(t)$$
$$= A \cdot X.$$

The last equality sign in this calculation can be understood by considering each entry of the time-dependent matrix $A \cdot \gamma_X(t)$ separately. □

Lemma 1.5.24 *Let $\tilde{X}, \tilde{Y}$ be vector fields on an open subset U of a Euclidean space $\mathbb{R}^N$ and $\gamma_{\tilde{X}}, \gamma_{\tilde{Y}}$ curves tangent to $\tilde{X}$ and $\tilde{Y}$ at a point $p \in U$. Then*

$$\left[\tilde{X}, \tilde{Y}\right]_p = \left.\frac{d}{dt}\right|_{t=0} \tilde{Y}_{\gamma_{\tilde{X}}(t)} - \left.\frac{d}{dt}\right|_{t=0} \tilde{X}_{\gamma_{\tilde{Y}}(t)}.$$

Proof Let $e_1, \ldots, e_N$ be the standard basis of the Euclidean space and write

$$\tilde{X} = \sum_{k=1}^{N} \tilde{X}_k e_k,$$
$$\tilde{Y} = \sum_{k=1}^{N} \tilde{Y}_k e_k.$$

Then, because of $[e_k, e_l] = 0$, we get

$$\begin{aligned}
\left[\tilde{X}, \tilde{Y}\right]_p &= \sum_{k,l=1}^{N} \left(\tilde{X}_k \cdot (L_k \tilde{Y}_l) - \tilde{Y}_k \cdot (L_k \tilde{X}_l)\right) e_l \\
&= \sum_{l=1}^{N} \left(L_{\tilde{X}} \tilde{Y}_l - L_{\tilde{Y}} \tilde{X}_l\right)(p) e_l \\
&= \left.\frac{d}{dt}\right|_{t=0} \tilde{Y}_{\gamma_{\tilde{X}}(t)} - \left.\frac{d}{dt}\right|_{t=0} \tilde{X}_{\gamma_{\tilde{Y}}(t)}.
\end{aligned}$$

□

We can now prove Theorem 1.5.22.

Proof Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of a Euclidean space, we can calculate the commutator of the vector fields $\tilde{X}, \tilde{Y}$ at the point I by

$$\begin{aligned}
\left[\tilde{X}, \tilde{Y}\right]_I &= \left.\frac{d}{dt}\right|_{t=0} \tilde{Y}_{\gamma_{\tilde{X}}(t)} - \left.\frac{d}{dt}\right|_{t=0} \tilde{X}_{\gamma_{\tilde{Y}}(t)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\gamma_{\tilde{X}}(t) \cdot Y\right) - \left.\frac{d}{dt}\right|_{t=0} \left(\gamma_{\tilde{Y}}(t) \cdot X\right) \\
&= X \cdot Y - Y \cdot X.
\end{aligned}$$

This proves the assertion. □

We would like to mention the following conceptually interesting theorem concerning Lie algebras (for the proof in a special case, see Proposition 2.4.4; the general proof can be found in [77, 83]).

Theorem 1.5.25 (Ado's Theorem) *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Then there exists an injective Lie algebra homomorphism of* $\mathfrak{g}$ *into* $\mathfrak{gl}(n, \mathbb{K})$ *for some* n.

As a consequence of Ado's Theorem every Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ for some n.

We will later show in Theorem 1.6.4 that for a given Lie group G, every Lie *subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of a connected Lie *subgroup* $H \subset G$. Therefore Lie's Third Theorem 1.5.15 follows from Ado's Theorem 1.5.25 and Theorem 1.6.4, applied to some general linear group $\mathrm{GL}(n, \mathbb{R})$.

1.5.5 The Lie Algebra of the Linear Groups

As a corollary to Theorem 1.5.22 and Corollary 1.5.19 we get:

Corollary 1.5.26 (Lie Algebra of Linear Groups) *If the Lie algebra of an embedded or immersed Lie subgroup of* $\mathrm{GL}(n, \mathbb{K})$ *is identified in the canonical way with a Lie subalgebra of* $\mathrm{Mat}(n \times n, \mathbb{K})$, *then the Lie bracket on the Lie subalgebra is the standard commutator of matrices.*

As simple as this corollary may seem, it is in fact very useful. In general it can be quite difficult to calculate the commutator of two vector fields on a given manifold. Corollary 1.5.26 shows that this is very easy for left-invariant vector fields on Lie subgroups of general linear groups.

Theorem 1.5.27 (Lie Algebras of Classical Groups) *We can identify the Lie algebras of the classical groups with the following real Lie subalgebras of the Lie algebra* $\mathrm{Mat}(n \times n, \mathbb{K})$.

1. *The Lie algebras of the special linear groups are:*

$$\mathfrak{sl}(n, \mathbb{R}) = \{M \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \mathrm{tr}(M) = 0\},$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{M \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \mathrm{tr}(M) = 0\},$$

$$\mathfrak{sl}(n, \mathbb{H}) = \{M \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid \mathrm{Re}(\mathrm{tr}(M)) = 0\}.$$

(continued)

Theorem 1.5.27 (continued)

2. *The Lie algebras of the orthogonal, unitary and symplectic groups are:*

$$\mathfrak{o}(n) = \{M \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid M + M^T = 0\},$$
$$\mathfrak{u}(n) = \{M \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid M + M^\dagger = 0\},$$
$$\mathrm{sp}(n) = \{M \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid M + M^\dagger = 0\}.$$

3. *The Lie algebras of the special orthogonal and special unitary groups are:*

$$\mathfrak{so}(n) = \mathfrak{o}(n),$$
$$\mathfrak{su}(n) = \{M \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid M + M^\dagger = 0, \mathrm{tr}(M) = 0\}.$$

Remark 1.5.28 We can check directly that these subsets of $\mathrm{Mat}(n \times n, \mathbb{K})$ are real vector subspaces and closed under the commutator; see Exercise 1.9.16.
We prove Theorem 1.5.27.

Proof

1. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and suppose that $A \in \mathfrak{sl}(n, \mathbb{K})$. We shall show in Sect. 1.7 (without using the results here) that $e^{tA} \in \mathrm{SL}(n, \mathbb{K})$ and that

$$1 = \det\left(e^{tA}\right) = e^{\mathrm{tr}(A)t}$$

for all $t \in \mathbb{R}$. We get

$$0 = \frac{d}{dt}\bigg|_{t=0} e^{\mathrm{tr}(A)t} = \mathrm{tr}(A),$$

hence

$$\mathfrak{sl}(n, \mathbb{K}) \subset \{M \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \mathrm{tr}(M) = 0\}.$$

Since we already know the dimension of $\mathrm{SL}(n, \mathbb{K})$ from Theorem 1.2.17, the assertion follows by calculating the dimension of the subspace on the right of this inclusion.

The claim for $\mathfrak{sl}(n, \mathbb{H})$ follows, because under the adjoint map χ, the group $\mathrm{SL}(n, \mathbb{H})$ gets identified according to Proposition 1.1.25 with the submanifold

$$\left\{X \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid JXJ^{-1} = \overline{X}, \det(X) = 1\right\}.$$

The tangent space to the neutral element I is contained in

$$\left\{A \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid JAJ^{-1} = \overline{A}, \mathrm{tr}(A) = 0\right\}$$

which corresponds under χ to

$$\{M \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid \mathrm{Re}(\mathrm{tr}(M)) = 0\},$$

since

$$\mathrm{tr}(\chi_M) = 2\mathrm{Re}(\mathrm{tr}(M))$$

by Proposition 1.1.26. The claim follows by a similar dimension argument as before.

2. If $A(t)$ is a curve in $\mathrm{O}(n)$ through I with $\dot{A}(0) = M$, then $A(t)A(t)^T = I$, hence

$$0 = \left.\frac{d}{dt}\right|_{t=0} A(t)A(t)^T = M + M^T,$$

i.e. M is skew-symmetric. The claim then follows by comparing the dimensions of $\mathrm{O}(n)$ and the vector space of skew-symmetric matrices. The cases of $\mathrm{u}(n)$ and $\mathrm{sp}(n)$ follow similarly.

3. The case of $\mathrm{su}(n)$ is clear by a similar dimension argument as before. The case of $\mathfrak{so}(n)$ follows, because if $M \in \mathfrak{o}(n)$, then automatically $\mathrm{tr}(M) = 0$.

□

Example 1.5.29 The Lie algebra $\mathfrak{u}(1)$ has dimension 1 and is equal to $\mathrm{Im}\,\mathbb{C}$, spanned by i.

Example 1.5.30 The Lie algebra $\mathfrak{so}(2)$ has dimension 1 and consists of the skew-symmetric 2×2-matrices. A basis is given by the *rotation matrix*

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{so}(2)$ is isomorphic to $\mathfrak{u}(1)$, because both are 1-dimensional and abelian.

Example 1.5.31 The Lie algebra $\mathfrak{so}(3)$ has dimension 3 and consists of skew-symmetric 3×3-matrices. A basis is given by the *rotation matrices*

$$r_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad r_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices satisfy

$$[r_a, r_b] = \epsilon_{abc} r_c,$$

where ϵ_{abc} is totally antisymmetric in a, b, c with $\epsilon_{123} = 1$.

Example 1.5.32 The Lie algebra $\mathfrak{su}(2)$ has dimension 3 and consists of the skew-Hermitian 2×2-matrices of trace zero. We consider the Hermitian **Pauli matrices**:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then a basis for $\mathfrak{su}(2)$ is given by the matrices

$$\tau_a = -\frac{i}{2}\sigma_a \quad a = 1, 2, 3.$$

The commutators of these matrices are

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau_c.$$

The map

$$\begin{aligned} \mathfrak{so}(3) &\longrightarrow \mathfrak{su}(2) \\ r_a &\longmapsto \tau_a \end{aligned}$$

is a Lie algebra isomorphism.

Example 1.5.33 The Lie algebra $\mathfrak{su}(3)$ has dimension 8 and consists of the skew-Hermitian 3×3-matrices of trace zero. We consider the Hermitian **Gell-Mann matrices**:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(continued)

Example 1.5.33 (continued)
Then a basis for $\mathfrak{su}(3)$ is given by the matrices $\frac{i\lambda_a}{2}$ for $a = 1, \ldots, 8$. The matrices $i\lambda_a$ for $a = 1, 2, 3$ span a Lie subalgebra, isomorphic to $\mathfrak{su}(2)$.

Example 1.5.34 The Lie algebra $\mathfrak{sp}(1)$ has dimension 3 and is equal to $\mathrm{Im}\,\mathbb{H}$, spanned by the imaginary quaternions i, j, k. If we set

$$e_1 = \frac{i}{2}, \quad e_2 = \frac{j}{2}, \quad e_3 = \frac{k}{2},$$

then

$$[e_a, e_b] = \epsilon_{abc} e_c.$$

The map

$$\begin{aligned} \mathfrak{sp}(1) &\longrightarrow \mathfrak{su}(2) \\ e_a &\longmapsto \tau_a \end{aligned}$$

is a Lie algebra isomorphism.

Example 1.5.35 The Lie algebra $\mathfrak{sl}(1, \mathbb{H})$ is equal to $\mathfrak{sp}(1)$.

Example 1.5.36 The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ has dimension 3 and consists of the real 2×2-matrices of trace zero. A basis is given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with commutators

$$\begin{aligned} [H, X] &= 2X, \\ [H, Y] &= -2Y, \\ [X, Y] &= H. \end{aligned}$$

Example 1.5.37 The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has dimension 6 and consists of the complex 2×2-matrices of trace zero. It is also a *complex* Lie algebra of complex dimension 3. A complex basis is given by the same matrices H, X, Y as above for $\mathfrak{sl}(2, \mathbb{R})$. In analogy to the *quantum angular momentum* and *quantum harmonic oscillator*, X is sometimes called the **raising operator** and Y the **lowering operator**. According to Exercise 1.9.18, as a complex Lie algebra, $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to the complex Lie algebra $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$.

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ plays a special role in physics, because as a real Lie algebra it is isomorphic to the Lie algebra of the *Lorentz group* of 4-dimensional spacetime (see Sect. 6.8.2).

Example 1.5.38 (The Heisenberg Lie Algebra) The Lie algebra of the Heisenberg group Nil^3 is

$$\mathfrak{nil}_3 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}(3 \times 3, \mathbb{R}) \,\middle|\, a, b, c \in \mathbb{R} \right\}.$$

A basis is given by the matrices

$$q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

satisfying

$$[q, p] = z,$$
$$[q, z] = 0,$$
$$[p, z] = 0.$$

We see that z commutes with every element in the Lie algebra $\mathfrak{nil}_3$, i.e. z is a *central element*. Furthermore,

$$[\mathfrak{nil}_3, [\mathfrak{nil}_3, \mathfrak{nil}_3]] = 0,$$

so that $\mathfrak{nil}_3$ is an example of a *nilpotent* Lie algebra.

1.6 *From Lie Subalgebras to Lie Subgroups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. In this section we want to show that there exists a 1-to-1 correspondence between Lie subalgebras of $\mathfrak{g}$ and connected (immersed or embedded) Lie subgroups of G (we follow [142]). We need some background on distributions and foliations that can be found in Sect. A.1.12.

Definition 1.6.1 Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. If we consider $\mathfrak{g}$ as the set of left-invariant vector fields on G, then $\mathfrak{h}$ is a distribution on G, denoted by $\mathscr{H}$. Equivalently, if we think of $\mathfrak{g}$ as the tangent space T_eG and $\mathfrak{h} \subset T_eG$ as a vector subspace, then the distribution $\mathscr{H}$ is defined by

$$\mathscr{H}_p = L_{p*}\mathfrak{h} \quad \forall p \in G.$$

Lemma 1.6.2 *The distribution $\mathscr{H}$ associated to a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is integrable.*

Proof Let $V_1, \ldots, V_d$ be left-invariant vector fields on G defined by a vector space basis for $\mathfrak{h}$. Then V_i is a section of $\mathscr{H}$ for all $i = 1, \ldots, d$ and since $\mathfrak{h}$ is a subalgebra, the commutators $[V_k, V_l]$ are again sections of $\mathscr{H}$. If X and Y are arbitrary sections of $\mathscr{H}$, then there exist functions f_i, g_i on G such that

$$X = \sum_{k=1}^{d} f_k V_k,$$

$$Y = \sum_{k=1}^{d} g_k V_k.$$

We get

$$\begin{aligned}[X, Y] &= \left[\sum_{k=1}^{d} f_k V_k, \sum_{l=1}^{d} g_l V_l \right] \\ &= \sum_{k,l=1}^{d} \left(f_k g_l [V_k, V_l] + f_k (L_{V_k} g_l) V_l - g_l (L_{V_l} f_k) V_k \right).\end{aligned}$$

This is a section of $\mathscr{H}$. Thus the distribution $\mathscr{H}$ is integrable. □

Definition 1.6.3 For a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let H denote the maximal connected leaf of the foliation $\mathscr{H}$ through the neutral element $e \in G$.

Theorem 1.6.4 (The Immersed Lie Subgroup Defined by a Lie Subalgebra) *The immersed submanifold H is the unique, connected, immersed Lie subgroup of G with Lie algebra $\mathfrak{h}$.*

Remark 1.6.5 The subgroup H is sometimes called the **integral subgroup** associated to the Lie algebra $\mathfrak{h}$.

Proof We first show that H is a subgroup in the algebraic sense: Let $g \in H$. Since $\mathscr{H}$ is left-invariant, we have

$$L_{g^{-1}*}\mathscr{H} = \mathscr{H},$$

hence $L_{g^{-1}}H$ is a connected leaf of $\mathscr{H}$, containing $g^{-1}g = e$. By maximality of H we have

$$L_{g^{-1}}H \subset H.$$

Hence if $g, h \in H$, then $g^{-1}h \in H$, showing that H is a subgroup.

We want to show that the group operations on H are smooth with respect to the manifold structure. The map

$$\begin{aligned} H \times H &\longrightarrow G \\ (g,h) &\longmapsto gh^{-1} \end{aligned}$$

is smooth and has image in H. Since H is the leaf of a foliation, it follows from Theorem A.1.56 that

$$\begin{aligned} H \times H &\longrightarrow H \\ (g,h) &\longmapsto gh^{-1} \end{aligned}$$

is smooth. We prove the remaining statement on the uniqueness of H below. □

It remains to show that H is the *unique* connected immersed Lie subgroup with Lie algebra $\mathfrak{h}$. We need the following:

Proposition 1.6.6 (Connected Lie Groups Are Generated by Any Open Neighbourhood of the Neutral Element) *Let G be a connected Lie group and $U \subset G$ an open neighbourhood of e. Then*

$$G = \bigcup_{n=1}^{\infty} U^n,$$

where

$$U^n = \underbrace{U \cdot U \cdots U}_{n \text{ factors}}.$$

Proof This follows from Exercise 1.9.4. □

Here is an immediate consequence.

Corollary 1.6.7 *Let K and K' be Lie groups, where K' is connected. Suppose that $\phi\colon K \to K'$ is a Lie group homomorphism, so that $\phi(K)$ contains an open neighbourhood of $e \in K'$. Then ϕ is surjective. In particular, if $K \subset K'$ is an open subgroup, then $K = K'$.*

We now prove the uniqueness part in Theorem 1.6.4.

Proof Let K be another connected, immersed Lie subgroup with Lie algebra $\mathfrak{h}$. Then K must also be a connected leaf of the foliation $\mathscr{H}$ through $e \in G$, hence by maximality of H we get $K \subset H$. The differential of the inclusion $i\colon K \hookrightarrow H$ is an isomorphism at every point, hence $K \subset H$ is an open subgroup. The assertion then follows from Corollary 1.6.7. □

Example 1.6.8 According to Example 1.1.42 the 1-dimensional Lie subalgebras of the Lie algebra of the torus T^2 define embedded Lie subgroups if they have rational slope in $\mathbb{R}^2$ and immersed Lie subgroups if they have irrational slope.

1.7 The Exponential Map

We saw above that the tangent space T_eG of a Lie group G at the neutral element $e \in G$ has the structure of a Lie algebra $\mathfrak{g}$. In this section we want to study the famous exponential map from $\mathfrak{g}$ to G, which is defined using integral curves of left-invariant vector fields (we follow [14, Sect. 1.2] for the construction).

1.7.1 The Exponential Map for General Lie Groups

For the following statements some background on integral curves and flows of vector fields can be found in Sect. A.1.9.

Theorem 1.7.1 (Integral Curves of Left-Invariant Vector Fields) *Let G be a Lie group and* $\mathfrak{g}$ *its Lie algebra. Let*

$$\begin{aligned}\phi_X\colon \mathbb{R} \supset I &\longrightarrow G\\ t &\longmapsto \phi_X(t)\end{aligned}$$

denote the maximal integral curve of a left-invariant vector field $X \in \mathfrak{g}$ *through the neutral element* $e \in G$*. Then the following holds:*

1. ϕ_X *is defined on all of* $\mathbb{R}$.
2. $\phi_X\colon \mathbb{R} \to G$ *is a homomorphism of Lie groups, i.e.*

$$\phi_X(s+t) = \phi_X(s) \cdot \phi_X(t) \quad \forall s, t \in \mathbb{R}.$$

3. $\phi_{sX}(t) = \phi_X(st)$ *for all* $s, t \in \mathbb{R}$.

Definition 1.7.2 The homomorphism $\phi_X\colon \mathbb{R} \to G$ is called the **one-parameter subgroup** of the Lie group G determined by the left-invariant vector field X.
We prove Theorem 1.7.1 in a sequence of steps. Let

$$\phi_X\colon \mathbb{R} \supset I = (t_{min}, t_{max}) \longrightarrow G$$

denote the maximal integral curve of the vector field X through e, satisfying

$$\phi_X(0) = e, \quad \dot{\phi}_X(t) = X_{\phi_X(t)}.$$

Lemma 1.7.3 *For all $s, t \in I$ with $s + t \in I$ the following identity holds:*

$$\phi_X(s + t) = \phi_X(s) \cdot \phi_X(t).$$

Proof Let $g = \phi_X(s) \in G$. Consider the smooth curves

$$\begin{aligned} \eta\colon I &\longrightarrow G \\ t &\longmapsto g \cdot \phi_X(t) \end{aligned}$$

and

$$\begin{aligned} \tilde{\eta}\colon (t_{min} - s, t_{max} - s) &\longrightarrow G \\ t &\longmapsto \phi_X(s + t). \end{aligned}$$

It is easy to show that both η and $\tilde{\eta}$ are integral curves of X with $\eta(0) = \tilde{\eta}(0) = g$. Hence by the uniqueness of integral curves (which is a theorem about the uniqueness of solutions to ordinary differential equations) we have

$$\phi_X(s) \cdot \phi_X(t) = \phi_X(s + t) \quad \forall t \in I \cap (t_{min} - s, t_{max} - s).$$

This implies the claim. □

Lemma 1.7.4 *We have $t_{max} = \infty$ and $t_{min} = -\infty$.*

Proof Suppose $t_{max} < \infty$ and set $\alpha = \min\{t_{max}, |t_{min}|\} < \infty$. Consider the curve

$$\begin{aligned} \gamma\colon \left(-\frac{\alpha}{2}, \frac{3\alpha}{2}\right) &\longrightarrow G \\ t &\longmapsto \phi_X\left(\frac{\alpha}{2}\right)\phi_X\left(t - \frac{\alpha}{2}\right). \end{aligned}$$

It is easy to check that γ is an integral curve of X with $\gamma(0) = e$. However,

$$\frac{3\alpha}{2} > t_{max}$$

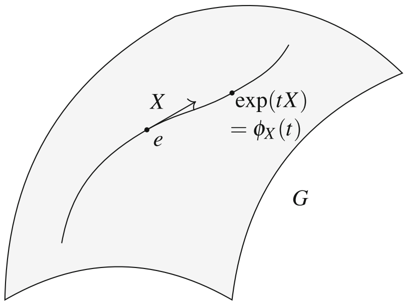
by construction, hence γ is an extension of ϕ_X, contradicting the choice of t_{max}. This shows that $t_{max} = \infty$ and similarly $t_{min} = -\infty$. □

Lemma 1.7.5 $\phi_{sX}(t) = \phi_X(st)$ *for all $s, t \in \mathbb{R}$.*

Proof Fix $s \in \mathbb{R}$ and consider the curve

$$\begin{aligned} \delta\colon \mathbb{R} &\longrightarrow G \\ t &\longmapsto \phi_X(st). \end{aligned}$$

Fig. 1.1 Exponential map



It is easy to show that δ is an integral curve of the vector field sX with $\delta(0) = e$. Hence by uniqueness $\phi_X(st) = \phi_{sX}(t)$. □

Definition 1.7.6 Let $\phi_X : \mathbb{R} \to G$ denote the integral curve through $e \in G$ for an element $X \in \mathfrak{g}$. Then we define the **exponential map**

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ X &\longmapsto \exp(X) = \exp X = \phi_X(1). \end{aligned}$$

See Fig. 1.1.

Remark 1.7.7 The reason for the name *exponential map* will become apparent in Sect. 1.7.3. Note that by definition the exponential map of the Lie group G maps the Lie algebra $\mathfrak{g}$ to the connected component G_e of the neutral element e. Elements in other connected components can never be in the image of the exponential map.

Example 1.7.8 The simplest example is the exponential map of the abelian Lie group $G = \mathbb{R}^n$ with vector addition. We can canonically identify the Lie algebra $\mathfrak{g}$ with $\mathbb{R}^n$. Then the exponential map is the identity map, since the left-invariant vector fields on G are the constant (parallel) vector fields. In this particular case, the exponential map is therefore a diffeomorphism between the Lie algebra and the Lie group.

Proposition 1.7.9 (Properties of the Exponential Map) *The exponential map has the following properties, explaining the name one-parameter subgroup:*

1. $\exp(0) = e$
2. $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$
3. $\exp(-X) = (\exp X)^{-1}$

for all $X \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$.

Proof This is an exercise. □

Remark 1.7.10 One-parameter subgroups are the immersed or embedded Lie subgroups determined as in Theorem 1.6.4 by 1-dimensional (abelian) subalgebras of $\mathfrak{g}$.

Definition 1.7.11 Let X be a left-invariant vector field on a Lie group G. Then we denote its **flow** through a point $p \in G$ by $\phi_t^X(p)$ or $\phi_t(p)$. It is characterized by

$$\phi_0(p) = p, \quad \left.\frac{d}{dt}\right|_{t=s} \phi_t(p) = X_{\phi_s(p)} \quad \forall s \in \mathbb{R}.$$

The one-parameter subgroup $\phi_X(t)$ determined by X is in this notation $\phi_t(e)$.

Proposition 1.7.12 (Relation Between the Flow and the Exponential Map) *Let G be a Lie group and X a left-invariant vector field. Then its flow $\phi_t(p)$ through a point $p \in G$ is defined for all $t \in \mathbb{R}$,*

$$\begin{aligned} \phi \colon \mathbb{R} \times G &\longrightarrow G \\ (t, p) &\longmapsto \phi_t(p), \end{aligned}$$

and given by

$$\phi_t(p) = p \cdot \exp tX = R_{\exp tX}(p) = L_p(\exp tX).$$

Proof Define $\phi_t(p)$ for all $t \in \mathbb{R}$ by the right-hand side. It is clear that

$$\phi_0(p) = p \cdot \exp(0) = p.$$

Furthermore,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=s} \phi_t(p) &= \left.\frac{d}{d\tau}\right|_{\tau=0} L_p(\exp sX \cdot \exp \tau X) \\ &= D_e L_{p \exp sX}(X_e) \\ &= X_{p \exp sX} \\ &= X_{\phi_s(p)}, \end{aligned}$$

since X is left-invariant. This implies the claim by uniqueness of solutions of ordinary differential equations. □

Remark 1.7.13 Integral curves of vector fields are defined for all times usually only on *compact* manifolds. It is a special property of Lie groups that integral curves of *left-invariant* vector fields are defined for all times, even on *non-compact* Lie groups.

We want to prove a property of the exponential map that is sometimes useful in applications.

Proposition 1.7.14 (Exponential Map Is a Local Diffeomorphism) *Under the canonical identifications*

$$T_0\mathfrak{g} \cong \mathfrak{g}, \quad T_eG \cong \mathfrak{g},$$

the differential of the exponential map

$$D_0 \exp \colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

is the identity map. In particular, there exist open neighbourhoods V of 0 in $\mathfrak{g}$ and U of e in G such that

$$\exp|_V \colon V \longrightarrow U$$

is a diffeomorphism.

Proof Let $X \in \mathfrak{g}$ and $\gamma(t) = tX$ the associated curve in $\mathfrak{g}$. Then $\dot{\gamma}(0) = X$ and

$$D_0 \exp(X) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X.$$

□

Remark 1.7.15 In general, the exponential map is neither injective nor surjective and hence not a *global* diffeomorphism between the Lie algebra and the Lie group (of course, the exponential map can only be a diffeomorphism if the Lie group itself is not compact and diffeomorphic to a vector space). See Corollary 1.7.20 for a situation when the exponential map is surjective. It is known that the exponential map is a diffeomorphism in the case of simply connected *nilpotent* Lie groups (see [83] for a proof).

Recall from Theorem 1.5.18 that every homomorphism between Lie groups induces a homomorphism between Lie algebras. The exponential map behaves nicely with respect to these homomorphisms.

Theorem 1.7.16 (Induced Homomorphisms on Lie Algebras and the Exponential Map) *Let $\psi \colon G \to H$ be a homomorphism between Lie groups and $\psi_* \colon \mathfrak{g} \to \mathfrak{h}$ the induced homomorphism on Lie algebras. Then*

$$\psi(\exp X) = \exp(\psi_* X) \quad \forall X \in \mathfrak{g},$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\psi} & H \end{array}$$

Proof Consider the curve $\gamma(t) = \psi(\exp tX)$ for $t \in \mathbb{R}$. Then

$$\gamma(0) = \psi(e) = e$$

and

$$\begin{aligned}\dot{\gamma}(t) &= D_{\exp tX}\psi\left(\left.\frac{d}{ds}\right|_{s=t}\exp sX\right)\\ &= D_{\exp tX}\psi\left(X_{\exp tX}\right)\\ &= D_e(\psi\circ L_{\exp tX})\,(X_e)\\ &= D_e(L_{\psi(\exp tX)}\circ\psi)\,(X_e)\\ &= (\psi_*X)_{\gamma(t)}.\end{aligned}$$

Here we used that ψ is a homomorphism and X left-invariant. We conclude that γ is the (unique) integral curve of the left-invariant vector field ψ_*X through $e\in H$ and therefore

$$\exp(\psi_*X) = \gamma(1) = \psi(\exp X).$$

□

Corollary 1.7.17 (Exponential Map for Embedded Lie Subgroups) *Let G be a Lie group and $H\subset G$ an embedded Lie subgroup with exponential maps*

$$\begin{aligned}\exp^G&\colon\mathfrak{g}\longrightarrow G,\\ \exp^H&\colon\mathfrak{h}\longrightarrow H.\end{aligned}$$

Then for $X\in\mathfrak{h}\subset\mathfrak{g}$ the following identity holds

$$\exp^G(X) = \exp^H(X).$$

Proof This follows from Theorem 1.7.16 with the embedding $i\colon H\hookrightarrow G$. □

The following generalization of Theorem 1.7.1 to time-dependent vector fields is sometimes useful in applications.

Theorem 1.7.18 (Integral Curves of Time-Dependent Vector Fields on Lie Groups) *Suppose that G is a Lie group and $x\colon[0,1]\rightarrow\mathfrak{g}$ a smooth map. Let $X(t)$ denote the associated left- (or right-)invariant time-dependent vector field on G. Then there exists a unique smooth integral curve $g\colon[0,1]\rightarrow G$ such that*

$$\begin{aligned}g(0) &= e,\\ \dot{g}(t) &= X(t)\quad\forall t\in[0,1].\end{aligned}$$

Proof We only indicate the idea of the proof in the case of a left-invariant vector field $X(t)$. Details can be found in [14] and are left as an exercise. Let Z be the vector field on $G \times \mathbb{R}$ defined by

$$Z_{(g,s)} = (X_g(s), 1) \in T_{(g,s)}(G \times \mathbb{R}).$$

On the interval $[0, \delta]$, for $\delta > 0$ small enough, there exist integral curves $(g(t), t)$ and $(h(t), t + \delta)$ of Z with $g(0) = h(0) = e$. Then

$$g(t) = g(\delta) \cdot h(t - \delta) \quad \forall t \in [\delta, 2\delta]$$

defines an extension of $g(t)$ to an integral curve of X on $[0, 2\delta]$. □

1.7.2 *The Exponential Map of Tori

It is important to understand the exponential map of the torus T^n, because compact Lie groups always contain embedded Lie subgroups isomorphic to tori (see Exercise 1.9.11 for explicit examples of tori contained in the classical groups).

Proposition 1.7.19 (Exponential Map of Tori) *The exponential map of every torus T^n is surjective.*

Proof This follows, because according to Example 1.7.8 the exponential map of $\mathbb{R}^n$ is surjective. □

Every element of a compact connected Lie group G is contained in some embedded torus subgroup (for a proof, see [24, Sect. IV.1]). Then Corollary 1.7.17 implies:

Corollary 1.7.20 (Exponential Map of Compact Connected Lie Groups) *The exponential map of a compact connected Lie group G is surjective.*

Remark 1.7.21 As Exercise 1.9.27 shows, this is not true in general for non-compact connected Lie groups like $\mathrm{SL}(2, \mathbb{R})$. See Exercise 1.9.28 for a statement which is true in the general case.

Example 1.7.22 Every one-parameter subgroup of SO(3) is given by the subgroup of rotations around a common axis in $\mathbb{R}^3$ (see Example 1.7.32). Corollary 1.7.20 then implies that any rotation of $\mathbb{R}^3$ can be obtained from the identity by rotating around a fixed axis by a certain angle.

The following assertion was discussed in Example 1.1.42.

Proposition 1.7.23 (Embedded and Immersed One-Parameter Subgroups) *A torus of dimension at least two has both embedded and immersed one-parameter subgroups.*

Corollary 1.7.24 *Every Lie group G that contains a torus of dimension at least two has both embedded and immersed one-parameter subgroups.*

Example 1.7.25 The Lie groups SO(4) and SU(3) contain embedded tori of dimension two and thus immersed one-parameter subgroups. The Lie groups SO(3) and SU(2), on the other hand, only contain embedded tori of dimension one. It can be shown that every one-parameter subgroup of SO(3) and SU(2) is closed, hence isomorphic to S^1 (cf. Exercise 1.9.25 for the case of SU(2)).

This is intuitively clear for SO(3), because the one-parameter subgroups are rotations around a fixed axis and thus return to the identity after a rotation by 2π. The result can also be interpreted for SO(4): we can define an embedded Lie subgroup $T^2 = \mathrm{SO}(2)\times\mathrm{SO}(2)$ in SO(4) as those rotations which preserve a splitting of $\mathbb{R}^4$ into two orthogonal planes $\mathbb{R}^2 \oplus \mathbb{R}^2$ and only rotate each plane in itself. If the velocities of the two rotations have an irrational ratio, then both rotations never return at the same time to the identity. This corresponds to an immersed one-parameter subgroup of SO(4).

*1.7.3 *The Matrix Exponential*

In this section we want to determine the exponential map for the linear groups. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 1.7.26 Let $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ be a square matrix. Then we set

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Lemma 1.7.27 (Convergence of Exponential Series) *For any square matrix* $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ *the series* $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ *converges in each entry.*

Proof Let $||\cdot||$ denote the Euclidean norm on $\mathbb{K}^n$. Define the operator norm of a matrix $M \in \mathrm{Mat}(n \times n, \mathbb{K})$ by

$$||M|| = \sup_{||x|| \leq 1} ||Mx||.$$

Then $||\cdot||$ is indeed a norm on the vector space of square matrices and satisfies $||MN|| \leq ||M|| \cdot ||N||$. Since the exponential series for real numbers converges, it follows that the exponential series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ is Cauchy and hence converges. □

Lemma 1.7.28 (Exponential of a Sum for Commuting Matrices) *If matrices* $A, B \in \mathrm{Mat}(n \times n, \mathbb{K})$ *commute,* $AB = BA$*, then*

$$e^{A+B} = e^A e^B.$$

In particular, e^{-A} *is the inverse of* e^A*, so that* $e^A \in \mathrm{GL}(n, \mathbb{K})$.

Proof This is Exercise 1.9.23. □

The following is immediate.

Theorem 1.7.29 *For every* $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ *the map*

$$\begin{aligned} \phi_A \colon \mathbb{R} &\longrightarrow \mathrm{GL}(n, \mathbb{K}) \\ t &\longmapsto e^{tA} \end{aligned}$$

is smooth and satisfies

$$\phi_A(0) = I, \qquad \left.\frac{d}{dt}\right|_{t=s} \phi_A(t) = \phi_A(s)A \quad \forall s \in \mathbb{R}.$$

We get:

Corollary 1.7.30 (Exponential Map of Linear Group Is Matrix Exponential) *Let*

$$A \in \mathfrak{gl}(n, \mathbb{K}) = \mathrm{Mat}(n \times n, \mathbb{K}).$$

Then

$$\exp(A) = e^A$$

where exp *on the left denotes the canonical exponential map from Lie algebra to Lie group. The same formula holds for the exponential map of any linear group.*

Proof For $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ let $\tilde{A}$ denote the associated left-invariant vector field on $\mathrm{GL}(n, \mathbb{K})$. According to Lemma 1.5.23, $\tilde{A}$ is given at a point $P \in \mathrm{GL}(n, \mathbb{K})$ by

$$\tilde{A}_P = PA.$$

This shows that the map

$$\phi_A \colon \mathbb{R} \longrightarrow \mathrm{GL}(n, \mathbb{K})$$

from Theorem 1.7.29 is the integral curve of the vector field $\tilde{A}$ through I. The first claim now follows by Definition 1.7.6 of the exponential map. The second claim concerning linear groups follows by Corollary 1.7.17. □

Example 1.7.31 The simplest non-trivial case of this theorem is the exponential map

$$\begin{aligned} \exp \colon \mathfrak{u}(1) \cong i\mathbb{R} &\longrightarrow \mathrm{U}(1) \cong S^1 \\ i\alpha &\longmapsto e^{i\alpha}. \end{aligned}$$

Example 1.7.32 A slightly less trivial example is the matrix exponential of tr, where $t \in \mathbb{R}$ and

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the generator of $\mathfrak{so}(2)$ from Example 1.5.30. It is easy to see that

$$r^{2n} = (-1)^n I$$

and thus

$$r^{2n+1} = (-1)^n r$$

for all $n \geq 0$. Hence

$$\begin{aligned} e^{tr} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} r^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2). \end{aligned}$$

This is just the matrix for a rotation in $\mathbb{R}^2$ by an angle t. Similarly the matrix exponential of tr_3, with

$$r_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

one of the generators of $\mathfrak{so}(3)$ from Example 1.5.31, is

$$e^{tr_3} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3),$$

which is the matrix for a rotation in $\mathbb{R}^3$ around the z-axis. Rotations around other axes in $\mathbb{R}^3$ are given by one-parameter subgroups conjugate to the one defined by r_3, showing that all one-parameter subgroups of $\mathrm{SO}(3)$ are closed.

The proof of the following well-known formula uses that the determinant is multilinear in the columns of a matrix and thus only holds for real and complex matrices.

Theorem 1.7.33 (Determinant of Matrix Exponential) *Let* $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ *where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *Then*

$$\det\left(e^{A}\right) = e^{\mathrm{tr}(A)}.$$

Proof We first calculate the differential $D_I \det$. Let $X = (x_1, \ldots, x_n)$ be an arbitrary $n \times n$-matrix with column vectors $x_i \in \mathbb{K}^n$. We have by multilinearity and antisymmetry of the determinant

$$\begin{aligned}\det(I + (0, \ldots, 0, x_i, 0 \ldots, 0)) &= \det \begin{pmatrix} 1 & & & x_{1i} & & & \\ & \ddots & & & & & \\ & & 1 & x_{(i-1)i} & & & \\ & & & (1 + x_{ii}) & & & \\ & & & x_{(i+1)i} & 1 & & \\ & & & & & \ddots & \\ & & & x_{ni} & & & 1 \end{pmatrix} \\ &= 1 + x_{ii}.\end{aligned}$$

Then

$$\begin{aligned}(D_I \det)(X) &= \sum_{i=1}^{n} (D_I \det)(0, \ldots, 0, x_i, 0 \ldots, 0)) \\ &= \sum_{i=1}^{n} \left.\frac{d}{dt}\right|_{t=0} \det(I + (0, \ldots, 0, tx_i, 0 \ldots, 0)) \\ &= \sum_{i=1}^{n} x_{ii} \\ &= \mathrm{tr}(X).\end{aligned}$$

Consider the curve

$$\begin{aligned}\gamma \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \det\left(e^{tX}\right).\end{aligned}$$

Then

$$\gamma(0) = 1$$

and for all $s \in \mathbb{R}$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=s} \gamma(t) &= \det\left(e^{sX}\right) \left.\frac{d}{d\tau}\right|_{\tau=0} \det\left(e^{\tau X}\right) \\ &= \det\left(e^{sX}\right) (D_I \det)(X) \\ &= \det\left(e^{sX}\right) \operatorname{tr}(X) \\ &= \gamma(s)\operatorname{tr}(X). \end{aligned}$$

The unique solution of this differential equation for $\gamma(t)$ is

$$\gamma(t) = e^{\operatorname{tr}(X)t}.$$

This implies the assertion with $t = 1$. □

Example 1.7.34 Let

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$$

be a real or complex diagonal matrix. Then

$$e^D = \begin{pmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{pmatrix}$$

and the equation

$$\det\left(e^D\right) = e^{d_1} \cdots e^{d_n} = e^{d_1 + \ldots + d_n} = e^{\operatorname{tr}(D)}$$

is trivially satisfied. The same argument works for upper triangular matrices.
Using Theorem 1.7.16 we can write the statement of Theorem 1.7.33 as follows:

Corollary 1.7.35 *The determinant*

$$\begin{aligned} \det\colon \mathrm{GL}(n, \mathbb{K}) &\longrightarrow \mathbb{K}^* \\ A &\longmapsto \det(A) \end{aligned}$$

is a group homomorphism with differential given by the trace

$$\det_* = \mathrm{tr}\colon \mathrm{Mat}(n \times n, \mathbb{K}) \longrightarrow \mathbb{K}$$
$$X \longmapsto \mathrm{tr}(X).$$

Notice that the trace is indeed a Lie algebra homomorphism to the abelian Lie algebra $\mathbb{K}$.

1.8 *Cartan's Theorem on Closed Subgroups

Our aim in this section is to prove Cartan's Theorem 1.1.44 (we follow [14, 24] and [142]). This theorem is important, because we used it, for example, to show that closed subgroups of the general linear groups are embedded Lie subgroups. We will also employ it later to show that isotropy groups of Lie group actions on manifolds are embedded Lie subgroups. The proof of Cartan's Theorem is one of the more difficult proofs in this book and follows from a sequence of propositions. One direction is quite easy.

Let G be a Lie group.

Definition 1.8.1 Let $H \subset G$ be a subset. A chart $\psi\colon U \to \mathbb{R}^n$ of G such that

$$\psi(U \cap H) = \psi(U) \cap \left(\{0\} \times \mathbb{R}^k\right)$$

for some $k < n$ is called a **submanifold chart** or **flattener** for H around p.

Proposition 1.8.2 *Let $H \subset G$ be an embedded Lie subgroup. Then H is a closed subset in the topology of G.*

Proof Suppose that $H \subset G$ is an embedded Lie subgroup. In a submanifold chart U of G around e for the embedded submanifold H, the set $H \cap U$ is closed in U.

Suppose y is a point in the closure $\bar{H}$ and let x_n be a sequence in H converging to y. Then $x_n^{-1}y$ is a sequence in G converging to e. Hence for sufficiently large index n, the element $x = x_n \in H$ satisfies $x^{-1}y \in U$ and thus $y \in xU$.

Since the group operations on G are continuous, the closure $\bar{H}$ is a subgroup of G. It follows that $y \in \bar{H} \cap xU$ and $x^{-1}y \in \bar{H} \cap U = H \cap U$. Thus $y \in H$ and H is closed. □

The converse statement is more difficult. Assume from now on that $H \subset G$ is a subgroup in the algebraic sense, which is a closed set in the topology of G. To show that H is a k-dimensional embedded Lie subgroup of the n-dimensional Lie group G, we have to find around every point $p \in H$ a chart $\psi\colon U \to \mathbb{R}^n$ of G which is a submanifold chart for H around p. The following argument shows that it suffices to find a submanifold chart for H around e.

Proposition 1.8.3 (Submanifold Charts) *Let $\phi\colon U \to \mathbb{R}^n$ be a submanifold chart for H around $e \in H$. Suppose $p \in H$. Then*

$$\phi_p = \phi \circ L_{p^{-1}}\colon L_p(U) \longrightarrow \mathbb{R}^n$$

is a submanifold chart for H around p. Here L_p denotes left translation on G by p.

Proof Note that

$$L_p(H) = H,$$

since H is a subgroup of G, hence

$$L_p(U) \cap H = L_p(U \cap H).$$

This implies

$$\phi_p(L_p(U) \cap H) = \phi(U \cap H)$$

and

$$\phi_p(L_p(U) \cap H) = \phi_p(L_p(U)) \cap \left(\{0\} \times \mathbb{R}^k\right).$$

□

We want to find a submanifold chart for H around e. It turns out that we first have to find a candidate for the Lie algebra of the subgroup H.

Proposition 1.8.4 (The Candidate for the Lie Algebra of H) *Let $H \subset G$ be a subgroup in the algebraic sense which is a closed set in the topology of G. Then*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp tX \in H \quad \forall t \in \mathbb{R}\}$$

is a vector subspace of $\mathfrak{g}$.

To prove Proposition 1.8.4 note that if $X \in \mathfrak{h}$, then $sX \in \mathfrak{h}$ for all $s \in \mathbb{R}$, since we can choose $t' = ts \in \mathbb{R}$. It remains to show that if $X, Y \in \mathfrak{h}$, then $X + Y \in \mathfrak{h}$. We have to understand terms of the form

$$\exp(t(X + Y)).$$

This is the purpose of the following proposition.

Proposition 1.8.5 (Special Case of the Baker–Campbell–Hausdorff Formula) *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then for arbitrary vectors $X, Y \in \mathfrak{g}$ we have*

$$\exp(tX) \cdot \exp(tY) = \exp\left(t(X + Y) + O\left(t^2\right)\right) \quad \forall |t| < \epsilon,$$

where $\epsilon > 0$ *is small enough and* $O(t^2)$ *is some function of* t *such that* $O(t^2)/t^2$ *stays finite as* $t \to 0$.

Proof According to Proposition 1.7.14 the exponential map is a diffeomorphism from an open neighbourhood V of 0 in $\mathfrak{g}$ onto an open neighbourhood W of e in G. We can thus introduce so-called *normal coordinates* on W: Choose a basis $(v_1, \ldots, v_n)$ for the vector space $\mathfrak{g}$. Then there is a unique chart (W, ϕ) of G around e with

$$\phi: G \supset W \longrightarrow \mathbb{R}^n$$

such that

$$\phi(\exp(x_1 v_1 + \ldots + x_n v_n)) = (x_1, \ldots, x_n).$$

Let

$$\begin{aligned} \phi(\exp(tX)) &= tx, \\ \phi(\exp(tY)) &= ty. \end{aligned}$$

Let μ denote group multiplication in G:

$$\begin{aligned} \mu: G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h. \end{aligned}$$

Utilizing the chart ϕ this induces a map

$$\tilde{\mu}: \mathbb{R}^n \times \mathbb{R}^n \supset \tilde{U} \longrightarrow \mathbb{R}^n,$$

where $\tilde{U}$ is a small open neighbourhood of $(0, 0)$. The map $\tilde{\mu}$ is defined by

$$\tilde{\mu} = \phi \circ \mu \circ \left(\phi^{-1} \times \phi^{-1}\right).$$

We then have to show that

$$\begin{aligned} \tilde{\mu}(tx, ty) &= \phi(\exp(tX) \cdot \exp(tY)) \\ &= t(x + y) + O(t^2). \end{aligned}$$

This follows from the Taylor formula for $\tilde{\mu}$ since

$$\mu(e, e) = e$$

hence

$$\tilde{\mu}(0,0) = 0$$

and

$$D_{(e,e)}\mu(u,0) = u = D_{(e,e)}\mu(0,u) \quad \forall u \in T_eG$$

hence

$$D_{(0,0)}\tilde{\mu}(w,0) = w = D_{(0,0)}\tilde{\mu}(0,w) \quad \forall w \in T_0\mathbb{R}^n.$$

□

We will use this proposition in the following special form.

Corollary 1.8.6 (Lie Product Formula) *For arbitrary vectors* $X, Y \in \mathfrak{g}$ *and all* $t \in \mathbb{R}$

$$\lim_{n\to\infty}\left(\exp\frac{tX}{n}\exp\frac{tY}{n}\right)^n = \exp(t(X+Y)).$$

Proof This follows from Proposition 1.8.5 with the general formula $\exp(Z)^n = \exp(nZ)$ for any $Z \in \mathfrak{g}$. □

We can now finish the proof of Proposition 1.8.4.

Proof If $X, Y \in \mathfrak{h}$, then

$$\left(\exp\frac{tX}{n}\exp\frac{tY}{n}\right)^n \in H \quad \forall n \in \mathbb{N}, t \in \mathbb{R},$$

since H is a subgroup of G. Corollary 1.8.6 together with the assumption that H is a closed subset implies $\exp(t(X+Y)) \in H$ for all $t \in \mathbb{R}$ and thus $X + Y \in \mathfrak{h}$. □

Let $\mathfrak{h}$ be the vector subspace of $\mathfrak{g}$ from Proposition 1.8.4 and $\mathfrak{m}$ an arbitrary complementary vector subspace of $\mathfrak{g}$, so that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

We now start to construct the submanifold chart for H around e. We fix an arbitrary norm $||\cdot||$ on the vector space $\mathfrak{g}$.

Proposition 1.8.7 (Choice of the Open Subset $V_\mathfrak{m} \subset \mathfrak{m}$) *There exists an open neighbourhood* $V_\mathfrak{m} \subset \mathfrak{m}$ *of* 0 *in* $\mathfrak{m}$, *so that*

$$\exp(V_\mathfrak{m} \setminus \{0\}) \cap H = \emptyset.$$

Proof Suppose that for every open neighbourhood $V_{\mathfrak{m}}$ of 0 in $\mathfrak{m}$ we have

$$\exp(V_{\mathfrak{m}} \setminus \{0\}) \cap H \neq \emptyset.$$

Then there exists a non-zero sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathfrak{m}$ converging to 0 so that $\exp(Z_n) \in H$. Let K denote the set

$$K = \{Z \in \mathfrak{m} \mid 1 \leq ||Z|| \leq 2\}$$

and choose for every $n \in \mathbb{N}$ a positive integer $c_n \in \mathbb{N}$ such that $c_n Z_n \in K$. Since K is a compact set, we can assume (after passing to a subsequence) that $(c_n Z_n)_{n\in\mathbb{N}}$ converges to some $Z \in K$. Then

$$\frac{Z_n}{||Z_n||} = \frac{c_n Z_n}{||c_n Z_n||} \overset{n\to\infty}{\longrightarrow} \frac{Z}{||Z||}.$$

Since $\exp(Z_n) \in H$ for all $n \in \mathbb{N}$ we get with Lemma 1.8.8 below that $Z \in \mathfrak{h}$. However, $Z \in K \subset \mathfrak{m}$ and $\mathfrak{m}$ is complementary to $\mathfrak{h}$, therefore $Z = 0$. This contradicts that $1 \leq ||Z|| \leq 2$. □

In the proof we used the following lemma.

Lemma 1.8.8 *Let* $(Z_n)_{n\in\mathbb{N}}$ *be a sequence of non-zero vectors in* $\mathfrak{g}$ *with* $\exp(Z_n) \in H$ *and* $Z_n \to 0$ *as* $n \to \infty$. *Suppose that the limit*

$$W = \lim_{n\to\infty} \frac{Z_n}{||Z_n||}$$

exists. Then

$$\exp(tW) \in H \quad \forall t \in \mathbb{R}$$

and thus $W \in \mathfrak{h}$.

Proof Let $t \in \mathbb{R}$ be fixed and define

$$c_n = \max\{k \in \mathbb{Z} \mid k||Z_n|| < t\}.$$

We claim that

$$\lim_{n\to\infty} c_n||Z_n|| = t.$$

Note that

$$c_n||Z_n|| < t \leq (c_n + 1)||Z_n|| = c_n||Z_n|| + ||Z_n||.$$

This implies the claim, because $Z_n \to 0$. We get

$$\exp\left(c_n||Z_n|| \cdot \frac{Z_n}{||Z_n||}\right) \longrightarrow \exp(tW).$$

However,

$$\exp\left(c_n||Z_n|| \cdot \frac{Z_n}{||Z_n||}\right) = \exp(c_n Z_n) = \exp(Z_n)^{c_n} \in H.$$

Since H is a closed subset it follows that $\exp(tW) \in H$. □

The following map will be the (inverse of the) submanifold chart.

Lemma 1.8.9 (Definition of the Map F) *The map*

$$\begin{aligned} F\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} &\longrightarrow G \\ X + Y &\longmapsto \exp X \cdot \exp Y \end{aligned}$$

is smooth and its differential at 0 *is the identity. Thus F is a local diffeomorphism on a neighbourhood of* 0.

Proof The differential of F maps

$$D_0F\colon T_0\mathfrak{g} \cong \mathfrak{g} \longrightarrow T_eG \cong \mathfrak{g}.$$

We will show that this map is the identity. For $X \in \mathfrak{h}$ we have

$$\begin{aligned} D_0F(X) &= \left.\frac{d}{dt}\right|_{t=0} F(tX) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot \exp(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \\ &= X. \end{aligned}$$

A similar argument applies for $X \in \mathfrak{m}$. □

Proposition 1.8.10 (The Map F Defines a Submanifold Chart) *Let*

$$\begin{aligned} F\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} &\longrightarrow G \\ X + Y &\longmapsto \exp X \cdot \exp Y \end{aligned}$$

be the map from Lemma 1.8.9. Then there exists a small neighbourhood V of 0 *in* $\mathfrak{g}$ *such that* $F(V) = U$ *is an open neighbourhood of e in G and*

$$F^{-1}|_U \colon U \longrightarrow V$$

is a submanifold chart for H around e.

Proof According to Lemma 1.8.9 we can choose an open neighbourhood

$$V = V_{\mathfrak{m}} \times V_{\mathfrak{h}} \subset \mathfrak{m} \oplus \mathfrak{h}$$

of 0 in $\mathfrak{g}$ so that

$$F|_V \colon V \longrightarrow U = F(V)$$

is a diffeomorphism. According to Proposition 1.8.7 we can choose $V_{\mathfrak{m}}$ small enough such that

$$\emptyset = \exp(V_{\mathfrak{m}} \setminus \{0\}) \cap H.$$

Note that $\exp(Y) \in H$ for all $Y \in \mathfrak{h}$, hence

$$\emptyset = F\left((V_{\mathfrak{m}} \setminus \{0\}) \times V_{\mathfrak{h}}\right) \cap H,$$

and

$$\emptyset = \left((V_{\mathfrak{m}} \setminus \{0\}) \times V_{\mathfrak{h}}\right) \cap F^{-1}(H).$$

We conclude that

$$F^{-1}(U \cap H) \subset \{0\} \times V_{\mathfrak{h}}$$

and thus

$$F^{-1}(U \cap H) = \{0\} \times V_{\mathfrak{h}}. \tag{1.1}$$

This implies the claim. □

With Proposition 1.8.3 this finishes the proof of Cartan's Theorem. From the proof we see:

Corollary 1.8.11 (The Lie Algebra of an Embedded Lie Subgroup) *Let* $H \subset G$ *be as in Cartan's Theorem 1.1.44. Then the Lie algebra of H is given by*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \quad \forall t \in \mathbb{R}\}.$$

Proof We denote the Lie algebra of H for the moment by $L(H)$. It is clear that $L(H) \subset \mathfrak{h}$. We know from Proposition 1.8.4 that $\mathfrak{h}$ is a vector subspace in $\mathfrak{g}$ and from Eq. (1.1) that $\mathfrak{h}$ has the same dimension as H. This implies the claim. □

We collect some consequences of Cartan's Theorem:

Theorem 1.8.12 (Kernel of Lie Group Homomorphism) *Let $\phi: G \to K$ be a Lie group homomorphism. Then $H = \ker \phi$ is an embedded Lie subgroup of G with Lie algebra $\mathfrak{h} = \ker \phi_*$.*

Proof It is clear that H is a subgroup of G and closed, because $H = \phi^{-1}(e)$. By Cartan's Theorem 1.1.44, H is an embedded Lie subgroup of G.

Let X be an element in the Lie algebra $\mathfrak{h}$ of H. Then $\exp tX \in H$ for all $t \in \mathbb{R}$, hence

$$\phi(\exp tX) = e \quad \forall t \in \mathbb{R}.$$

This implies that

$$\phi_* X = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp tX) = 0.$$

Conversely, let $X \in \mathfrak{g}$ with $\phi_* X = 0$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \phi(\exp tX) = 0.$$

This implies for all $s \in \mathbb{R}$

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=s} \phi(\exp tX) &= \left.\frac{d}{d\tau}\right|_{\tau=0} \phi(\exp sX)\phi(\exp \tau X) \\ &= D_e L_{\phi(\exp sX)} \left.\frac{d}{d\tau}\right|_{\tau=0} \phi(\exp \tau X) \\ &= 0.\end{aligned}$$

Therefore, the curve $\phi(\exp tX)$ is constant and equal to $\phi(e) = e$. This implies $\exp tX \in H$ for all $t \in \mathbb{R}$ and thus $X \in \mathfrak{h}$. □

Proposition 1.8.13 (Image of Compact Lie Group Under Homomorphism) *Let $\phi: G \to H$ be a Lie group homomorphism. If G is compact, then the image of ϕ is an embedded Lie subgroup of H.*

Proof This is clear, because the image of ϕ is compact, hence closed. □

Theorem 1.8.14 (Continuous Group Homomorphisms Between Lie Groups Are Smooth) *Let $\phi: G \to K$ be a continuous group homomorphism between Lie groups. Then ϕ is smooth and thus a Lie group homomorphism.*

We use the following lemma from topology, whose proof is left as an exercise.

Lemma 1.8.15 *Let X, Y be topological spaces and $f: X \to Y$ a map. Define the graph of f by*

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

If Y is Hausdorff and f continuous, then Γ_f is closed in the product topology of $X \times Y$.

We also need the following.

Lemma 1.8.16 *Let $\psi: K \to G$ be a Lie group homomorphism, which is a homeomorphism. Then ψ is a diffeomorphism, hence a Lie group isomorphism.*

Proof It suffices to show that $\psi_*: \mathfrak{k} \to \mathfrak{g}$ is injective, because K and G have the same dimension. Suppose $\mathfrak{h} \neq 0$ is the kernel of ψ_* and H the kernel of ψ. According to Theorem 1.8.12 the subalgebra $\mathfrak{h}$ is the Lie algebra of H and thus $H \neq \{e\}$. This shows that ψ is not injective and hence not a homeomorphism. □

We now prove Theorem 1.8.14.

Proof The graph $\Gamma_\phi \subset G \times H$ is a closed subgroup of the Lie group $G \times H$, thus an embedded Lie subgroup by Cartan's Theorem 1.1.44. The projection $\mathrm{pr}_1: G \times H \to G$ restricts to a smooth homeomorphism

$$p: \Gamma_\phi \longrightarrow G$$

on the embedded submanifold Γ_ϕ with continuous inverse

$$\begin{aligned} p^{-1}: G &\longrightarrow \Gamma_\phi \\ g &\longmapsto (g, \phi(g)). \end{aligned}$$

It follows by Lemma 1.8.16 that p is a diffeomorphism and thus

$$\phi = \mathrm{pr}_2 \circ p^{-1}$$

is a smooth map. □

Corollary 1.8.17 (Uniqueness of Smooth Lie Group Structure) *Let G be a topological manifold which is a topological group. Then there is at most one smooth structure on G so that G is a Lie group.*

Proof Suppose G' and G'' are smooth Lie group structures on G. The identity map

$$\mathrm{Id}_G: G' \longrightarrow G''$$

is a group isomorphism and a homeomorphism. By Theorem 1.8.14 this map is a diffeomorphism. □

Corollary 1.8.18 (Embeddings of Compact Lie Groups) *Let G, H be Lie groups, G compact, and $\phi: G \to H$ an injective Lie group homomorphism. Then ϕ is a Lie group embedding, i.e. a Lie group isomorphism onto its image, an embedded Lie subgroup of H.*

Proof Since G is compact, the image of ϕ is compact, hence closed in H. This shows that the image of ϕ is an embedded Lie subgroup by Cartan's Theorem 1.1.44. Moreover, $\phi: G \to \phi(G)$ is a closed map, hence a homeomorphism. Lemma 1.8.16 implies that ϕ is an isomorphism onto its image. □

1.9 Exercises for Chap. 1

1.9.1 Let G be a topological group and G_e the connected component containing the neutral element e. Prove that G_e is a normal subgroup of G.

1.9.2 (From [135]) Let G be a topological group which is locally Euclidean. Prove that G is Hausdorff.

1.9.3 Let G be a connected topological group and $H \subset G$ an open subgroup. Prove that $H = G$.

1.9.4 Let G be a connected topological group and $U \subset G$ an open neighbourhood of e. Prove that the set

$$W = \bigcup_{n=1}^{\infty} U^n,$$

where

$$U^n = \underbrace{U \cdot U \cdots U}_{n \text{ factors}},$$

contains an open subgroup of G. Deduce that $W = G$.

1.9.5 (From [129]) Let G be a group which is at the same time a manifold so that the multiplication map

$$\begin{aligned} \mu: G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h \end{aligned}$$

is smooth.

1. Show that the multiplication map μ is a submersion.
2. Prove that the map

$$G \longrightarrow G$$

$$g \longmapsto g^{-1}$$

is smooth and therefore G is a Lie group.

1.9.6 Prove Proposition 1.1.25 that realizes the space $\mathrm{Mat}(n \times n, \mathbb{H})$ of quaternionic matrices via the adjoint as a subspace of the space $\mathrm{Mat}(2n \times 2n, \mathbb{C})$ of complex matrices.

1.9.7 Prove Proposition 1.1.26 on the properties of the adjoint for quaternionic matrices.

1.9.8 Show that the determinant of a matrix $(a) \in \mathrm{Mat}(1 \times 1, \mathbb{H})$ with $a \in \mathbb{H}$ is equal to

$$\det(a) = ||a||^2.$$

1.9.9 Show that every Lie group homomorphism $\phi: S^1 \rightarrow \mathbb{R}$ between the Lie groups $(S^1, \cdot)$ and $(\mathbb{R}, +)$ is the constant map to $0 \in \mathbb{R}$.

1.9.10

1. Find an explicit Lie group embedding

$$\mathrm{O}(n) \hookrightarrow \mathrm{SO}(n+1).$$

2. Write $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ as $A = A_1 + iA_2$ with A_1, A_2 real matrices and find Lie group embeddings

$$\mathrm{GL}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}_+(2n, \mathbb{R})$$

and

$$\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n).$$

3. Identify the image of $\mathrm{Sp}(n)$ in $\mathrm{Mat}(2n \times 2n, \mathbb{C})$ under the adjoint map χ and find a Lie group embedding

$$\mathrm{Sp}(n) \hookrightarrow \mathrm{U}(2n).$$

1.9.11 Let T^n denote the torus of dimension n.

1. Find Lie group embeddings

$$T^n \hookrightarrow \mathrm{U}(n),$$
$$T^{n-1} \hookrightarrow \mathrm{SU}(n),$$
$$T^n \hookrightarrow \mathrm{Sp}(n).$$

2. Find Lie group embeddings

$$T^n \hookrightarrow \mathrm{SO}(2n),$$
$$T^n \hookrightarrow \mathrm{SO}(2n+1).$$

1.9.12 Find an explicit Lie group homomorphism

$$\phi \colon \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \longrightarrow \mathrm{SU}(5)$$

with discrete kernel.

1.9.13 Show that the Lie group homomorphisms from Example 1.3.7 and Example 1.3.8 together give a Lie group homomorphism $\psi \colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ equal to

$$\psi\begin{pmatrix} x+iy & -u-iv \\ u-iv & x-iy \end{pmatrix} = \begin{pmatrix} x^2+y^2-u^2-v^2 & -2xv+2yu & 2xu+2yv \\ 2xv+2yu & x^2-y^2+u^2-v^2 & -2xy+2uv \\ -2xu+2yv & 2xy+2uv & x^2-y^2-u^2+v^2 \end{pmatrix}.$$

Deduce that

$$\psi\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad \forall \alpha \in \mathbb{R}.$$

1.9.14 Recall that according to Example 1.4.6 the vector space $\mathbb{R}^3$ is a Lie algebra with bracket given by the cross product:

$$[v, w] = v \times w, \quad \forall v, w \in \mathbb{R}^3.$$

Find an explicit isomorphism of $(\mathbb{R}^3, \times)$ with the Lie algebra $\mathfrak{so}(3)$.

1.9.15 Prove that for $n \geq 1$ the sphere S^{2n} does not admit the structure of a Lie group.

1.9.16 (From [24]) Consider the Lie algebras $\mathfrak{g}$ of the classical linear groups G from Theorem 1.5.27.

1. Show directly that the subsets of $\mathrm{Mat}(n \times n, \mathbb{K})$ defined in Theorem 1.5.27 are real vector subspaces and closed under the commutator of matrices.
2. Show also by a direct calculation that these subsets are closed under the following map:

$$X \longmapsto g \cdot X \cdot g^{-1},$$

where $X \in \mathfrak{g}$ is an element of the Lie algebra and $g \in G$ an element of the corresponding linear group (we will identify this map with the adjoint representation in Sect. 2.1.5).

1.9.17

1. Prove that there are, up to isomorphism, only two 2-dimensional real Lie algebras.
2. Show that $\mathfrak{sl}(2, \mathbb{R})$ is not isomorphic to $\mathfrak{su}(2)$.

1.9.18 The Lie algebra $\mathfrak{su}(2)$ is spanned as a real vector space by the matrices τ_1, τ_2, τ_3 from Example 1.5.32. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is spanned as a complex vector space by the matrices H, X, Y from Example 1.5.36.

1. Show that the matrices τ_1, τ_2, τ_3 are a complex basis for $\mathfrak{sl}(2, \mathbb{C})$ and express this basis in terms of H, X, Y.
2. Show that as complex Lie algebras $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$, where the Lie bracket on the right is the complex linear extension of the Lie bracket of $\mathfrak{su}(2)$.

1.9.19 Consider the Lie group $\mathrm{U}(n)$ with Lie algebra $\mathfrak{u}(n)$.

1. Find an explicit Lie algebra isomorphism

$$\mathfrak{u}(n) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(n).$$

2. Find an explicit group isomorphism

$$\mathrm{U}(n) \cong (\mathrm{U}(1) \times \mathrm{SU}(n))/\mathbb{Z}_n,$$

where $\mathbb{Z}_n \subset \mathrm{U}(1) \times \mathrm{SU}(n)$ is a normal subgroup.

1.9.20 Recall that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

We identify quaternions $u + jv \in \mathbb{H}$, where $u, v \in \mathbb{C}$, with the following matrices:

$$\mathbb{H} \cong \left\{ \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \;\middle|\; u, v \in \mathbb{C} \right\}.$$

Consider the following isomorphism of real vector spaces:

$$\begin{aligned} \mathbb{R}^3 \cong i\mathbb{R} \times \mathbb{C} &\longrightarrow \operatorname{Im} \mathbb{H} \\ x = (ic, v) &\longmapsto X = \begin{pmatrix} ic & -\bar{v} \\ v & -ic \end{pmatrix}. \end{aligned}$$

The Euclidean norm of an element $x \in \mathbb{R}^3$ is given by $||x||^2 = \det X$. Under this identification we set:

$$\begin{aligned} \mathrm{SU}(2) \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (A, X) &\longmapsto AXA^{\dagger}. \end{aligned}$$

1. Prove that this map is well-defined and yields a homomorphism

$$\phi \colon \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

   of Lie groups.
2. Show that ϕ is surjective and that its kernel consists of $\{I, -I\}$.

1.9.21 (From [98]) We identify the Lie group SU(2) and the quaternions $\mathbb{H}$ with subsets of the complex 2×2-matrices as in Exercise 1.9.20. Consider the following isomorphism of real vector spaces:

$$\begin{aligned} \mathbb{R}^4 \cong \mathbb{C}^2 &\longrightarrow \mathbb{H} \\ x = (u, v) &\longmapsto X = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}. \end{aligned}$$

The Euclidean norm of an element $x \in \mathbb{R}^4 \cong \mathbb{C}^2$ is given by $||x||^2 = \det X$. Under this identification we set:

$$\begin{aligned} \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ (A_-, A_+, X) &\longmapsto A_- \cdot X \cdot A_+^{\dagger}, \end{aligned}$$

where $\cdot$ denotes matrix multiplication.

1. Prove that this map is well-defined and yields a homomorphism

$$\psi \colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$$

   of Lie groups.
2. Show that ψ is surjective with kernel $\{(I, I), (-I, -I)\}$.

1.9.22 Prove that every Lie group homomorphism $\rho: S^1 \to S^1$ is of the form

$$\rho(z) = z^k$$

for some $k \in \mathbb{Z}$.

1.9.23 Show that if matrices $A, B \in \mathrm{Mat}(n \times n, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ commute, $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

1.9.24 Calculate $\exp(s\tau_a) \in \mathrm{SU}(2)$ for $s \in \mathbb{R}$ and the basis τ_1, τ_2, τ_3 of the Lie algebra $\mathfrak{su}(2)$ from Example 1.5.32.

1.9.25 Consider the Lie group $\mathrm{SU}(2)$ with Lie algebra $\mathfrak{su}(2)$.

1. Show that every element $X \in \mathfrak{su}(2)$ can be written as

$$X = -2rA \cdot \tau_3 \cdot A^{-1}$$

with $r \in \mathbb{R}$, $A \in \mathrm{SU}(2)$ and

$$\tau_3 = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2).$$

2. Prove that every one-parameter subgroup of $\mathrm{SU}(2)$ is closed, i.e. its image is isomorphic to $\mathrm{U}(1)$.

1.9.26 Consider the Lie algebra $\mathfrak{su}(3)$ from Example 1.5.33 with the basis $v_1, \ldots, v_8$, where $v_a = \frac{i\lambda_a}{2}$ and λ_a are the Gell-Mann matrices.

1. Prove that the following three sets of basis vectors

$$\{v_1, v_2, v_3\},$$
$$\{v_4, v_5, \alpha v_3 + \beta v_8\},$$
$$\{v_6, v_7, \gamma v_3 + \delta v_8\},$$

where $\alpha, \beta, \gamma, \delta$ are certain real numbers, span Lie subalgebras of $\mathfrak{su}(3)$ isomorphic to $\mathfrak{su}(2)$. Determine $\alpha, \beta, \gamma, \delta$.
2. Prove that the one-parameter subgroups generated by each of the basis vectors $v_1, \ldots, v_8$ are closed.

1.9.27 (From [24]) Consider a matrix

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}).$$

1. Calculate e^A and $\operatorname{tr}\left(e^A\right)$.
2. Prove that the exponential map $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective.

1.9.28

1. Let G be a connected Lie group. Show that every group element $g \in G$ is of the form

$$g = \exp(X_1) \cdot \exp(X_2) \cdots \exp(X_n)$$

for finitely many vectors $X_1, \ldots, X_n$ in the Lie algebra $\mathfrak{g}$ of G.
2. Let $\phi\colon G \to H$ be a Lie group homomorphism, where G is connected. Suppose that the induced Lie algebra homomorphism $\phi_*\colon \mathfrak{g} \to \mathfrak{h}$ is trivial. Prove that ϕ is trivial.

1.9.29 (From [77])

1. Calculate the k-th power of the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdot & 0 & 1 & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & \cdots & & \cdots & 1 \\ 0 & & \cdots & & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}(n \times n, \mathbb{C}).$$

2. Calculate the k-th power of a Jordan block matrix $\lambda I_n + N$ with $\lambda \in \mathbb{C}$.
3. Calculate e^{tA} for a matrix A in Jordan normal form and $t \in \mathbb{R}$.

1.9.30 (From [77]) Let $A \in \mathrm{Mat}(n \times n, \mathbb{C})$.

1. Use Exercise 1.9.29 to show that the set

$$\left\{e^{tA} \mid t \in \mathbb{R}\right\}$$

is bounded in $\mathrm{Mat}(n \times n, \mathbb{C})$ if and only if A is diagonalizable with purely imaginary eigenvalues.
2. Show that $e^A = I$ if and only if A is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$.