# A Note on Thue Inequalities with Few Coefficients

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*This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday*

**Abstract** Let $F(X, Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \in \mathbb{Z}[X, Y]$ be a form of degree $r \geq 3$, irreducible over $\mathbb{Q}$, and having at most $s + 1$ nonzero coefficients. Mueller and Schmidt showed that the number of solutions of the Thue inequality

$$|F(X, Y)| \leq h$$

is $\ll s^2 h^{2/r}(1 + \log h^{1/r})$. They conjectured that $s^2$ may be replaced by $s$. In this note we show some instances when $s^2$ may be improved.

**Keywords** Thue equations · Thue inequalities
Large, medium and small solutions

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# 1 Introduction

Let $F(X, Y)$ be a form of degree $r \geq 3$ with integer coefficients, irreducible over $\mathbb{Q}$, and having at most $s+1$ nonzero coefficients. Write

$$F(X, Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \tag{1.1}$$

with $0 = r_0 < r_1 < \ldots < r_s = r$. Let $D$, $H$, and $M$ denote the discriminant, naive height, and Mahler height of $F(X, 1)$, respectively. For $h \geq 1$, consider the Thue inequality

$$|F(X, Y)| \leq h. \tag{1.2}$$

Let $N_F(h)$ denote the number of integer solutions $(x, y)$ of (1.2). Bombieri modified a conjecture of Siegel and asked if $N_F(h)$ could be bounded by a function depending only on $s$ and $h$. (See Mueller and Schmidt [8, p. 208]). Toward this, Schmidt [10] proved that

$$N_F(h) \ll \sqrt{rs}\, h^{2/r}(1 + \log h^{1/r}). \tag{1.3}$$

Throughout this note, the constants implied by $\ll$ are absolute. The modified Siegel's conjecture was shown to be true in the case $s = 1$ by Hyyrö [3], Evertse [1], and Mueller [6]. The case $s \geq 2$ was considered by Mueller and Schmidt in [7] and [8]. They proved that

$$N_F(h) \ll s^2 C(r, h) \tag{1.4}$$

where $C(r, h) = h^{2/r}(1 + \log h^{1/r})$. From a result of Mahler [5], it is known that the factor $h^{2/r}$ in $C(r, h)$ is unavoidable while the logarithmic factor was improved by Thunder when $h$ is large, see [11] and [12].

When $s$ is as large as $r$, (1.4) is weaker than (1.3). It was conjectured in [8] that it may possible to replace the factor $s^2$ above by $s$. In [9], some results were given where the factor $s^2$ was improved. For instance, the following results were proven.

(i) We always have $|r_i - r_w| \geq |i - w| \geq 1$ for $i \neq w$. Suppose $|r_i - r_w| \geq c_1|i - w|$ with $c_1 \geq 1$, an absolute constant. Then,

$$N_F(h) \ll s^{1+\frac{1}{c_1}}\, C(r, h).$$

Thus, the exponent of $s$ is $< 2$ whenever $c_1 > 1$.

(ii) Suppose $|r_i - r_w| \geq \frac{1}{3}|i - w| \log |i - w|$. Then,

$$N_F(h) \ll s \log^3 s\, C(r, h).$$

In another direction, it was shown that if the coefficients of $F(X, Y)$ satisfy

$$\left|\frac{a_i}{a_0}\right|^{1/r_i} \le \left|\frac{a_s}{a_0}\right|^{1/r_s} \quad \text{for } i = 1, \ldots, s-1 \tag{1.5}$$

and $r \ge \max(4s, s\log^3 s)$, then

$$N_F(h) \ll s(\log s)\, C(r, h). \tag{1.6}$$

If $r < \max(4s, s\log^3 s)$, then by (1.3), we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r, h).$$

Thus, under the condition (1.5), we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r, h).$$

In particular, the above estimate holds whenever $|a_0| = |a_s| = H$ where $H$ is the maximum of the absolute values of the coefficients of $F$. Hence, the estimate holds for forms $F$ with coefficients $\pm 1$.

Let $q_1$ be the smallest integer, $0 \le q_1 \le s$, with $|a_{q_1}| = H$ and let $q_2$ be the largest integer, $0 \le q_2 \le s$, with $|a_{q_2}| = H$. The condition (1.5) implies that

$$(q_1, q_2) \in \{(0, 0), (0, s), (s, s)\}.$$

In this note, we shall consider a few more cases of $(q_1, q_2)$. Throughout, we use the following assumptions A.

A: (a) $r \ge \max(4s, s\log^3 s)$.
(b) $|a_i|^{r_{q_1}} \le |a_{q_1}|^{r_i} |a_0|^{r_{q_1} - r_i}$ for $0 \le i \le q_1$.
(c) $|a_i|^{r - r_{q_2}} \le |a_{q_2}|^{r - r_i} |a_s|^{r_i - r_{q_2}}$ for $q_2 \le i \le s$.

Note that A(b) holds trivially if $q_1 = 0$ and A(c) holds trivially if $q_2 = s$. We prove the following result.

**Theorem 1.1.** *Suppose that the assumption A holds. Then,* (1.6) *is valid in the following three cases:*

*(i)* $q_1 = 0$, $0 < q_2 < s$ *and* $H \le |a_s|^{\frac{r}{\max(s, r q_2)}}$;
*(ii)* $q_2 = s$, $0 < q_1 < q_2$ *and* $H \le |a_0|^{\frac{r}{\max(s, r - r q_1)}}$;
*(iii)* $q_1 \ne 0$, $q_2 \ne s$ *and* $H \le \min\left(|a_0|^{\frac{r-s}{r}} |a_s|^{\frac{r q_1}{r}}, |a_0|^{\frac{r - r q_2}{r}} |a_s|^{\frac{r-s}{r}}\right)$.

**Remark.**

(a) When $r < \max(4s, s\log^3 s)$, we use (1.3) to obtain

$$N_F(h) \ll s(\log s)^{3/2}\, C(r,h).$$

Therefore, under the conditions of the theorem, we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r,h).$$

(b) We may assume that $s$ is large, as otherwise inequality (1.4) is sufficient.

## 2 Preliminaries

Let $F(X,Y)$ be given by (1.1). Let $X_1, X_2 > 0$. Divide the solutions $(x,y)$ of (1.2) into three sets as

$$\max(|x|,|y|) > X_1;\ \max(|x|,|y|) \le X_1 \text{ and } \min(|x|,|y|) \ge X_2;$$
$$\min(|x|,|y|) < X_2.$$

Denote the number of primitive solutions in these sets by $P_{lar}(X_1)$, $P_{med}(X_1, X_2)$, and $P_{sma}(X_2)$, respectively. If $X_2 > X_1$, put $P_{med}(X_1, X_2) = 0$. Let $P(h)$ be the number of primitive solutions of (1.2). Thus,

$$P(h) = P_{lar}(X_1) + P_{med}(X_1, X_2) + P_{sma}(X_2).$$

We can bound $N_F(h)$ by finding an upper estimate for $P(h)$. Choose numbers $a, b$ with $0 < a < b < 1$. Define

$$t = \sqrt{2/(r+a^2)},\ \lambda = 2/((1-b)t),$$

$$A = \frac{1}{a^2}\left(\log M + \frac{r}{2}\right).$$

Further, we put

$$B = \frac{2^r r^{r/2} M^r h}{\sqrt{|D|}},\ R = e^{800\log^3 r},$$

$$Y_E = (2B\sqrt{|D|})^{1/(r-\lambda)}(4e^A)^{\lambda/(r-\lambda)},\ Y_W = R^{1/(r-\lambda)} Y_E$$

$$Y_S = (4^r (rs)^{2s} R^s h)^{\frac{1}{r-2s}}.$$

We take

$$X_1 = Y_W \text{ and } X_2 = Y_S.$$

Mueller and Schmidt [8] had shown that

$$P_{lar}(Y_W) \ll s. \tag{2.1}$$

To estimate the number of small solutions, we use the following lemma from [8].

**Lemma 2.1.** *[8, Lemma 18]*
*Let $F(X, Y)$ be given by (1.1) and let $r \geq 4s$. Then for any $Y \geq 1$, we have*

$$P_{sma}(Y) \ll (rs^2)^{2s/r} h^{2/r} + s\, Y.$$

The next lemma is a consequence of the above lemma.

**Lemma 2.2.** *Let $F(X, Y)$ be given by (1.1). Then,*

$$P_{sma}(Y_S) \ll s\, h^{2/r} \textit{whenever } r \geq s \log^3 s.$$

For dealing with the medium solutions, we use the Archimedean Newton polygon of the polynomial $F(X, 1)$. This is the lower boundary of the convex hull of the points $P_i = (r_i, -\log|a_i|)$, $0 \leq i \leq s$. Let $L_{i,j}$ denote the line joining $P_i$ to $P_j$ with slope $\sigma(i, j)$, say. Further, let

$$\hat{F}(X, 1) = a_s + a_{s-1} X^{r_s - r_{s-1}} + \cdots + a_1 X^{r_s - r_1} + a_0 X^{r_s}$$

be the reciprocal polynomial of $F(X, 1)$. Let $Q_i = (r_s - r_i, -\log|a_i|)$ and let $L'_{i,j}$ denote the line joining $Q_i$ to $Q_j$ with slope $\sigma'(i, j)$.

**Lemma 2.3.** *Suppose that the coefficients of $F(X, Y)$ satisfy the assumptions A(b) and A(c). Then, the edges of the Archimedean Newton polygon of $F(X, 1)$ are $L_{0,q_1}$, $L_{q_1,q_2}$, and $L_{q_2,s}$. Further, $\sigma(q_1, q_2) = 0$ and every root $\alpha$ of $F(x, 1)$ satisfies*

$$\frac{1}{2} e^{\sigma(0,q_1)} < |\alpha| < 2e^{\sigma(q_2,s)}. \tag{2.2}$$

*Every root $\beta$ of $\hat{F}(X, 1)$ satisfies*

$$\frac{1}{2} e^{-\sigma(q_2,s)} < |\beta| < 2e^{-\sigma(0,q_1)}.$$

*Proof.* Put

$$\sigma_1 = \sigma(0, q_1) \text{ and } \sigma_2 = \sigma(q_2, s).$$

By A(b), we have

$$\sigma(0,i) \geq \sigma_1 \text{ for } 0 \leq i \leq q_1.$$

By A(c), we have

$$\sigma(q_2,i) \geq \sigma_2 \text{ for } q_2 \leq i \leq s.$$

Hence, the Archimedean Newton polygon consists of $L_{0,q_1}$ and $L_{q_2,s}$ as the left most edge and right most edge, respectively. Since the height of the polynomial is attained at $a_{q_1}$ and at $a_{q_2}$, we see that $\sigma(q_1,q_2)=0$ and $\sigma(q_1,i)\geq 0$ for $q_1 \leq i \leq q_2$. Thus, $L_{q_1,q_2}$ is the third edge. Now we prove (2.2). By the convexity of the Newton polygon, we have

$$\sigma_2 \geq \sigma(i,s) \text{ for } 0 \leq i \leq s. \tag{2.3}$$

Let $z = e^{\sigma_2 w}$ with $|w| \geq 2$. Then,

$$|F(z,1)| \geq |a_s|e^{\sigma_2 r_s}\left(|w|^{r_s} - \frac{|a_{s-1}|}{|a_s|}e^{\sigma_2(r_{s-1}-r_s)}|w|^{r_{s-1}} - \cdots - \frac{|a_0|}{|a_s|}e^{-\sigma_2 r_s}\right).$$

By (2.3), we have

$$\frac{|a_i|}{|a_s|}e^{\sigma_2(r_i-r_s)} \leq 1.$$

Hence,

$$|F(z,1)| \geq |w|^{r_s} - |w|^{r_{s-1}} - \cdots - 1 > 0$$

since $|w| \geq 2$. Thus, every root $\alpha$ of $F(X,1)$ has $|\alpha| < 2e^{\sigma_2}$.

To get the lower bound, we use the reciprocal polynomial $\hat{F}(X,1)$. The Archimedean Newton polygon of this polynomial has edges $L'_{s,q_2}$, $L'_{q_2,q_1}$, and $L'_{q_1,0}$. Arguing as above, we find that every root $\beta$ which is the inverse of some root $\alpha$ of $F(X,1)$ satisfies $|\beta| \leq 2e^{\sigma'(q_1,0)}$, where $\sigma'(q_1,0)$ is the slope of $L'_{q_1,0}$. Hence,

$$|\alpha| \geq \frac{1}{2}e^{-\sigma'(q_1,0)}.$$

Now the result follows on noticing that $\sigma'(q_1,0) = -\sigma_1$. □

The Archimedean Newton polygons when conditions (i), (ii), or (iii) of Theorem 1.1 hold, are shown in Figure 1.

Another tool needed to estimate the medium solutions is the Diophantine approximation property. Let $S$ be the set of roots $\alpha_1, \cdots, \alpha_r$ of $f(z) = F(z,1)$ and $S^*$ the set of roots of $F(1,z)$. Then, $S^* = \{\alpha_1^{-1}, \cdots, \alpha_r^{-1}\}$. Let $(x,y)$ be a solution of (1.2) with $y \neq 0$. Define

$$d\left(S, \frac{x}{y}\right) = \min_{1\leq i\leq r}\left|\alpha_i - \frac{x}{y}\right|.$$

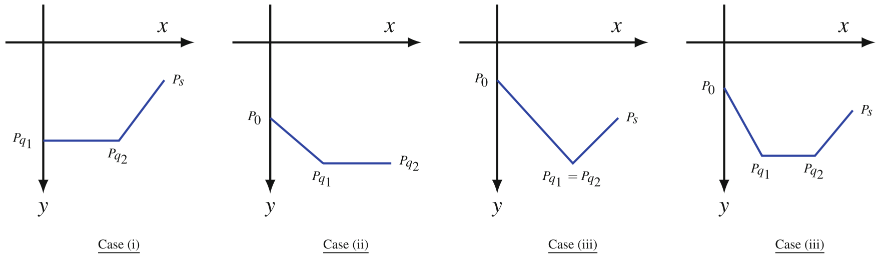


**Fig. 1** Archimedean Newton polygon.

It was shown in [8, Lemma 7], that there exists $S_1 \subseteq S$ with $|S_1| \ll s$ such that

$$d\left(S_1,\ \frac{x}{y}\right) \leq R\, d\left(S,\ \frac{x}{y}\right). \tag{2.4}$$

Suppose that $d(S, x/y) = |\alpha - x/y|$ for some $\alpha \in S$. If $f^{(u)}(\alpha) \neq 0$ for some $u$ with $1 \leq u \leq r$, then by [8, Lemma 10], we have

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{r}{2}\left(\frac{2^r h}{|f^{(u)}(\alpha)||y|^r}\right)^{1/u}. \tag{2.5}$$

Let $e, h$ be two nonnegative integers. Let $(e)_h$ be the Pochhammer symbol defined as

$$(e)_h = \begin{cases} 0 & \text{if } e = 0 \\ 1 & \text{if } h = 0 \\ e(e-1)\cdots(e-h+1) & \text{otherwise.} \end{cases}$$

Using the explanations given in [8, p. 223–231], we can obtain

$$\sum_{u=1}^{s} E_u^{(s)} \alpha^u f^{(u)}(\alpha) = a_s \alpha^{r_s} \prod_{0 \leq i < j \leq s} (r_i - r_j) \tag{2.6}$$

where

$$E_u^{(s)} = (-1)^{s+u} \det \begin{pmatrix} 1 & \cdots & 1 \\ (r_0)_1 & \cdots & (r_{s-1})_1 \\ \vdots & \vdots & \vdots \\ (r_0)_{u-1} & \cdots & (r_{s-1})_{u-1} \\ (r_0)_{u+1} & \cdots & (r_{s-1})_{u+1} \\ \vdots & \vdots & \vdots \\ (r_0)_s & \cdots & (r_{s-1})_s \end{pmatrix}.$$

From [8, Eqns (6.12) & (6.13)], we get that

$$|E_u^{(s)}| \leq 2^s (s^2 r)^{s-1} \prod_{0 \leq i < j \leq s} (r_j - r_i).$$

We also refer to [9] for more details. Using the above estimate for $|E_u^{(s)}|$ in (2.6), we find that there exists $u$ with $1 \leq u \leq s$ such that

$$|f^{(u)}(\alpha)| \geq |a_s||\alpha|^{r-u} 2^{-s} (s^2 r)^{-(s-1)} s^{-1}.$$

The following lemma is now immediate from (2.4) and (2.5).

**Lemma 2.4.** *There exists a set* $S_1 \subseteq S$ *with* $|S_1| \ll s$, *such that for some* $\alpha \in S_1$, *we have*

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{rR}{2} \left( \frac{s(rs^2)^{s-1} 2^{r+s} h}{|y|^r |a_s||\alpha|^{r-u}} \right)^{1/u}.$$

*A similar inequality holds with* $(x, y)$ *replaced by* $(y, x)$ *for some set* $S_2 \subseteq S^*$ *of roots with* $|S_2| \leq s$.

The following is a lemma on counting the number of elements in a set satisfying some gap conditions. (See [9, Lemma 2.1(i)]).

**Lemma 2.5.** *Let* $n \geq 2$ *and let* $U = \{u_1, \cdots, u_n\}$ *be a set together with a map* $T : U \to \mathbb{R}^*$ *such that*

$$A_1 \leq T(u_1) \leq T(u_2) \leq \cdots \leq T(u_n)$$

*and*

$$T(u_i) \geq \beta T(u_{i-1})^{\gamma} \ for\ 2 \leq i \leq n\ with\ \beta > 0, \gamma \geq 2.$$

*Let*

$$\kappa = \begin{cases} 2\ if\ \beta > 1 \\ 1\ if\ \beta \leq 1. \end{cases}$$

*Suppose that* $T(u_n) \leq B_1$ *and* $A_1 \beta^{1/(\kappa(\gamma-1))} > 1$. *Then,*

$$n \leq 1 + \frac{1}{\log \gamma} \log \left( \frac{\log B_1}{\log A_1 + (\log \beta)/(\kappa(\gamma-1))} \right).$$

*Proof.* By induction, we get

$$\begin{aligned} T(u_n) &\geq \beta^{1+\gamma+\cdots+\gamma^{n-2}} T(u_1)^{\gamma^{n-1}} \\ &\geq (\beta^{1/(\kappa(\gamma-1))} T(u_1))^{\gamma^{n-1}}. \end{aligned} \tag{2.7}$$

Since $T(u_n) \le B_1$, from (2.7), we get

$$(\beta^{1/(\kappa(\gamma-1))}T(u_1))^{\gamma^{n-1}} \le B_1.$$

Taking logarithms twice, we get the assertion of the lemma. □

## 3 Proof of Theorem 1.1

By (2.1) and Lemma 2.1, it is enough to estimate $P_{med}(Y_W, Y_S)$. We shall consider $S_1$ from Lemma 2.4. The argument for $S_2$ is similar. We claim that

$$|a_s\alpha^{r-u}| \ge H^{u/r}2^{-(r-u)}.$$

We prove this when condition (ii) of the theorem holds. The other cases are similar. By Lemma 2.3,

$$\begin{aligned} |a_s\alpha^{r-u}| &> |a_s|e^{(r-u)\sigma_1}2^{-(r-u)} \\ &= |a_s|e^{(r-u)\frac{(-\log H+\log|a_0|)}{rq_1}}2^{-(r-u)}. \end{aligned}$$

Thus, our claim is true if

$$|a_s||a_0|^{\frac{r-u}{rq_1}} \ge H^{\frac{r-u}{rq_1}+\frac{u}{r}}.$$

Since $q_2 = s$, we have $|a_s| = H$. Therefore, the above inequality holds if

$$|a_0| \ge H^{\frac{r-rq_1}{r}},$$

which is true by our assumption. Hence, the claim follows. Thus, by Lemma 2.4, for $y \ge Y_S$,

$$\left|\alpha - \frac{x}{y}\right| < \frac{rR}{2H^{1/r}}\left(\frac{s2^{2r}(rs^2)^{s-1}h}{y^r}\right)^{1/s}. \tag{3.1}$$

Let $U = \{(x_1, y_1), \cdots, (x_\nu, y_\nu)\}$ be the set of all solutions of (3.1) with $\gcd(x_i, y_i) = 1$ and

$$Y_S \le y_1 \le \cdots \le y_\nu \le Y_W.$$

Suppose $\nu \ge 2$. Then,

$$\frac{1}{y_iy_{i+1}} \le \left|\frac{x_i}{y_i} - \frac{x_{i+1}}{y_{i+1}}\right| \le \left|\alpha - \frac{x_i}{y_i}\right| + \left|\alpha - \frac{x_{i+1}}{y_{i+1}}\right|$$

$$\leq \frac{K}{2y_i^{r/s}} + \frac{K}{2y_{i+1}^{r/s}} \leq \frac{K}{y_i^{r/s}},$$

where

$$K = R(rs)^2 4^{r/s} h^{1/s} H^{-1/r}.$$

Thus, we have

$$y_{i+1} \geq K^{-1} y_i^{r/s-1}.$$

We apply Lemma 2.5 with $T((x_i, y_i)) = y_i$, $\beta = 1/K$, $\gamma = \frac{r-s}{s}$, $A_1 = Y_S$ and $B_1 = Y_W$. Note that $\gamma = r/s - 1 \geq \max(3, \log^3 s - 1)$. Also, $R \geq 4(rs)^4$. Further, $\log Y_W \ll \sqrt{r} + \log H + \log h^{1/r}$. Hence, by Lemma 2.5, we get

$$\begin{aligned}
\nu &\ll 1 + \frac{1}{\log \gamma} \log \left( \frac{\log Y_W}{\log Y_S + \frac{\log \beta}{\kappa(\gamma-1)}} \right) \\
&\ll \frac{1}{\log \gamma} \log \left( \frac{2(r-2s)(\sqrt{r} + \log H + \log h^{1/r})}{\log H} \right) \\
&\ll \frac{\log r + \log(1 + \log h^{1/r})}{\log \gamma}.
\end{aligned}$$

Suppose $r \ll s^3$. Then,

$$\nu \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s} \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s}.$$

If $r \gg s^3$, then

$$\nu \ll 1 + \frac{\log(1 + \log h^{1/r})}{\log r} \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log s}.$$

Thus,

$$P_{med}(Y_W, Y_S) \ll s \left( \frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s} \right).$$

We combine the above inequality with (2.1) and Lemma 2.2 to get

$$P(h) \ll s(\log s) h^{2/r}.$$

Using a partial summation argument, it was shown in [8, p. 212] that

$$N_F(h) \ll P(h) + h^{1/r} r^{-1} \sum_{n=1}^{h-1} P(n) n^{-1-(1/r)}.$$

Substituting our estimate for $P(h)$, we obtain the result of the theorem.

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