Ramanujan's Tau Function

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This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Ramanujan's tau function is defined by

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1-q^n)$. It is known that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right),$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n. We give proofs of this relation for $p = 2,\ 3,\ 5,\ 7$ and 13. which rely on nothing more than Jacobi's triple product identity. I believe that the case $p = 11$ is intrinsically more difficult, and I do not attempt it here.

Keywords Ramanujan's tau function · Jacobi's triple product identity · Theta series

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1 Introduction

Ramanujan's tau function is defined by

$$\sum_{n\geq 1}\tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1-q^n)$.

The tau function has many fascinating properties.

One of these is that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right), \tag{1.1}$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n.

It follows easily from (1.1) that tau is multiplicative,

$$\tau(mn) = \tau(m)\tau(n) \tag{1.2}$$

provided m and n have no common divisor other than 1,
and that, at least formally,

$$\sum_{n\geq 1}\frac{\tau(n)}{n^s} = \prod_{p\ \text{prime}}\left(1 - \frac{\tau(p)}{p^s} + \frac{1}{p^{2s-11}}\right)^{-1}. \tag{1.3}$$

I have found proofs of (1.1) for $p = 2,\ 3,\ 5,\ 7$ and 13 which require nothing more than Jacobi's triple product identity,

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty}a^nq^{n^2}. \tag{1.4}$$

Completely different elementary proofs of (1.1) for $p = 2$ and 3 have recently been given by Kenneth S. Williams [6].

A modern proof of (1.1) may be found in [2].

2 $p = 2$

Let

$$\phi(q) = \sum_{-\infty}^{\infty}q^{n^2},\quad \psi(q) = \sum_{n\geq 0}q^{(n^2+n)/2} = \sum_{-\infty}^{\infty}q^{2n^2+n}.$$

It can be shown with, or even without, (1.4), [4, chap. 1] that

$$\phi(q)\phi(-q) = \phi(-q^2)^2 \text{ and } \phi(q)\psi(q^2) = \psi(q)^2.$$

Also, by (1.4),

$$\phi(-q) = \frac{E(q)^2}{E(q^2)} \text{ and } \psi(q) = \frac{E(q^2)^2}{E(q)}.$$

It is easy to see that

$$\phi(q) = \phi(q^4) + 2q\psi(q^8) \tag{2.1}$$

Put $-q$ for q in (2.1).

$$\phi(-q) = \phi(q^4) - 2q\psi(q^8). \tag{2.2}$$

Multiply (2.1) by (2.2).

$$\phi(-q^2)^2 = \phi(q^4)^2 - 4q^2\psi(q^8)^2. \tag{2.3}$$

Put q for q^2 in (2.3).

$$\phi(-q)^2 = \phi(q^2)^2 - 4q\psi(q^4)^2. \tag{2.4}$$

Put $-q$ for q in (2.4).

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{2.5}$$

Multiply (2.4) by (2.5).

$$\phi(-q^2)^4 = \phi(q^2)^4 - 16q^2\psi(q^4)^4. \tag{2.6}$$

Put q for q^2 in (2.6) and rearrange.

$$\phi(q)^4 - \phi(-q)^4 = 16q\psi(q^2)^4. \tag{2.7}$$

We have

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24} \tag{2.8}$$

$$= qE(q^2)^{12}\left(\frac{E(q)^2}{E(q^2)}\right)^{12}$$

$$= qE(q^2)^{12}\phi(-q)^{12}$$

$$= qE(q^2)^{12}\left(\phi(-q)^2\right)^6$$
$$= qE(q^2)^{12}\left(\phi(q^2)^2 - 4q\psi(q^4)^2\right)^6$$

$$\begin{aligned}= qE(q^2)^{12}\big(&\phi(q^2)^{12} - 24q\phi(q^2)^{10}\psi(q^4)^2 + 240q^2\phi(q^2)^8\psi(q^4)^4\\ &-1280q^3\phi(q^2)^6\psi(q^4)^6 + 3840q^4\phi(q^2)^4\psi(q^4)^8 - 6144q^5\phi(q^2)^2\psi(q^4)^4\\ &+4096q^6\psi(q^4)^6\big).\end{aligned}$$

If we extract the even powers and replace q^2 by q, we obtain

$$\sum_{n\geq 1}\tau(2n)q^n = -8qE(q)^{12}\phi(q)^2\psi(q^2)^2\left(3\phi(q)^8 + 160q\phi(q)^4\psi(q^2)^4 + 768q^2\psi(q^2)^8\right) \tag{2.9}$$

$$= -8qE(q)^{12}\left(\phi(q)\psi(q^2)\right)^2\left(3\left(\phi(q)^4 - 16q\psi(q^2)^4\right)^2 + 256q\phi(q)^4\psi(q^2)^4\right)$$

$$= -8qE(q)^{12}\psi(q)^4\left(3\left(\phi(-q)^4\right)^2 + 256q\psi(q)^8\right)$$

$$= -8qE(q)^{12}\left(\frac{E(q^2)^2}{E(q)}\right)^4\left(3\left(\frac{E(q)^2}{E(q^2)}\right)^8 + 256q\left(\frac{E(q^2)^2}{E(q)}\right)^8\right)$$
$$= -24qE(q)^{24} - 2^{11}q^2E(q^2)^{24}$$

$$= -24\sum_{n\geq 1}\tau(n)q^n - 2^{11}\sum_{n\geq 1}\tau(n)q^{2n}.$$

The term $n = 1$ in (2.9) gives

$$\tau(2) = -24\tau(1) = -24,$$

so (2.9) becomes

$$\sum_{n\geq 1}\tau(2n)q^n = \tau(2)\sum_{n\geq 1}\tau(n)q^n - 2^{11}\sum_{n\geq 1}\tau(n)q^{2n},$$

as claimed.

Aside: (2.7) can be written

$$\left(\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\right)^4 - \left(\prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n}\right)^4 \tag{2.10}$$
$$= 16q\left(\prod_{n\geq 1}\frac{(1-q^{4n})^2}{(1-q^{2n})}\right)^4.$$

If we divide (2.10) by $\prod_{n\geq 1}(1-q^{2n})^4$, we find

$$\prod_{n\geq 1}(1+q^{2n-1})^8 - \prod_{n\geq 1}(1-q^{2n-1})^8 = 16q\prod_{n\geq 1}(1+q^{2n})^8. \tag{2.11}$$

Jacobi described (2.11) as *"aequatio identica satis abstrusa"*. ("A fairly obscure identity".)

(2.11) can perhaps most strikingly be written [4, chap. 19]

$$\mathbf{O}\Big(\prod_{\substack{n\geq 1 \\ n\not\equiv 0 \pmod 4}}(1-q^n)^8\Big) = -8q.$$

Observe that

$$\prod_{\substack{n\geq 1 \\ n\not\equiv 0 \pmod 4}}(1-q^n)^8 = 1 - 8q + 20q^2 - 62q^4 + 216q^6 - 641q^8 \\ +1636q^{10} - 3778q^{12} + 8248q^{14} + \cdots .$$

3 $p = 3$

Using Jacobi's formula for the cube of Euler's product, which follows from (1.4), namely

$$E(q)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}, \tag{3.1}$$

we have the 3-dissection

$$\begin{aligned} E(q)^3 &= \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \\ &= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + \cdots \\ &= A(q^3) - 3qE(q^9)^3. \end{aligned} \tag{3.2}$$

So,

$$\begin{aligned} \sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24} \\ &= q\left(E(q)^3\right)^8 \\ &= q\left(A(q^3) - 3qE(q^9)^3\right)^8 \end{aligned} \tag{3.3}$$

$$= q\left(A(q^3)^8 - 24qA(q^3)^7E(q^9)^3 + 252q^2A(q^3)^6E(q^9)^6 - 1512q^3A(q^3)^5E(q^9)^9 \right.$$
$$+ 5670q^4A(q^3)^4E(q^9)^{12} - 13608q^5A(q^3)^3E(q^9)^{15} + 20412q^6A(q^3)^2E(q^9)^{18}$$
$$\left. - 17496q^7A(q^3)E(q^9)^{21} + 6561q^8E(q^9)^{24}\right).$$

If we extract those terms in which the power of q is a multiple of 3, and replace q^3 by q, we obtain

$$\sum_{n\geq 1}\tau(3n)q^n = 252qA(q)^6E(q^3)^6 - 13608q^2A(q)^3E(q^3)^{15} + 6561q^3E(q^3)^{24}. \tag{3.4}$$

If we put q, ωq, $\omega^2 q$ for q in (3.2) and multiply the three results, we find

$$E(q)^3E(\omega q)^3E(\omega^2 q)^3 = A(q^3)^3 - 27q^3E(q^9)^9, \tag{3.5}$$

or,

$$\left(\frac{E(q^3)^4}{E(q^9)}\right)^3 = A(q^3)^3 - 27q^3E(q^9)^9. \tag{3.6}$$

If in (3.6) we replace q^3 by q and rearrange, we obtain

$$A(q)^3 = \frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9. \tag{3.7}$$

If we substitute (3.7) into (3.4), we obtain

$$\begin{aligned}\sum_{n\geq 1}\tau(3n)q^n &= 252qE(q^3)^6\left(\frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9\right)^2 \\ &\quad - 13608q^2E(q^3)^{15}\left(\frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9\right) \\ &\quad + 6561q^3E(q^3)^{24} \\ &= 252qE(q)^{24} - 3^{11}q^3E(q^3)^{24} \\ &= 252\sum_{n\geq 1}\tau(n)q^n - 3^{11}\sum_{n\geq 1}\tau(n)q^{3n}.\end{aligned} \tag{3.8}$$

The term $n = 1$ in (3.8) gives

$$\tau(3) = 252\tau(1) = 252,$$

so (3.8) becomes

$$\sum_{n\geq 0}\tau(3n)q^n = \tau(3)\sum_{n\geq 1}\tau(n)q^n - 3^{11}\sum_{n\geq 1}\tau(n)q^{3n},$$

as claimed.

Aside: It can be shown [4, chap. 21] that

$$\begin{aligned} A(q) &= E(q)\left(1+6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \\ &= E(q)\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \end{aligned}$$

4 $p = 5$

We have Euler's pentagonal numbers theorem (this follows from (1.4))

$$\begin{aligned} E(q) &= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}--++\cdots \\ &= E_0+E_1+E_2 \end{aligned} \tag{4.1}$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 5. ($i = 0,\ 1,\ 2.$)

It is easy to prove that

$$E_1 = -qE(q^{25}) \tag{4.2}$$

and, using Jacobi's formula for the cube of Euler's product (3.1),

$$(E_0+E_1+E_2)^3 = \prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \tag{4.3}$$

that

$$E_0E_2 = -E_1^2. \tag{4.4}$$

If we write

$$\alpha = -\frac{E_0}{E_1} \text{ and } \beta = -\frac{E_2}{E_1} \tag{4.5}$$

then $\alpha\beta = -1$ and

$$E(q) = qE(q^{25})\,(\alpha - 1 + \beta)\,. \tag{4.6}$$

We have

$$\sum_{n\geq 1}\tau(n)q^n = qE(q)^{24} \tag{4.7}$$
$$= q^{25}E(q^{25})^{24}(\alpha - 1 + \beta)^{24}$$
$$= q^{25}E(q^{25})^{24}\big(\alpha^{24} - 24\alpha^{23} + 252\alpha^{22} - 1472\alpha^{21} + 4830\alpha^{20} - 6072\alpha^{19}$$
$$-16192\alpha^{18} + 78936\alpha^{17} - 82731\alpha^{16} - 212520\alpha^{15} + 649704\alpha^{14}$$
$$-73416\alpha^{13} - 1977862\alpha^{12} + 2034672\alpha^{11} + 3487260\alpha^{10}$$
$$-7072408\alpha^{9} - 3432198\alpha^{8} + 15343944\alpha^{7} + 134596\alpha^{6}$$
$$-25077360\alpha^{5} + 6067446\alpha^{4} + 33474936\alpha^{3} - 12286968\alpha^{2}$$
$$-38228232\alpha + 14903725 - 38228232\beta - 12286968\beta^{2} + 33474936\beta^{3}$$
$$+6067446\beta^{4} - 25077360\beta^{5} + 134596\beta^{6} + 15343944\beta^{7} - 3432198\beta^{8}$$
$$-7072408\beta^{9} + 3487260\beta^{10} + 2034672\beta^{11} - 1977862\beta^{12} - 73416\beta^{13}$$
$$+649704\beta^{14} - 212520\beta^{15} - 82731\beta^{16} + 78936\beta^{17} - 16192\beta^{18}$$
$$-6072\beta^{19} + 4830\beta^{20} - 1472\beta^{21} + 252\beta^{22} - 24\beta^{23} + \beta^{24}\big).$$

If we extract those terms in which the power of q is a multiple of 5, we obtain

$$\sum_{n\geq 1}\tau(5n)q^{5n} = q^{25}E(q^{25})^{24}\Big(4830\alpha^{20} - 212520\alpha^{15} + 3487260\alpha^{10} - 25077360\alpha^{5}$$
$$+14903725 - 25077360\beta^{5} + 3487260\beta^{10} - 212520\beta^{15} + 4830\beta^{20}\Big). \tag{4.8}$$

Miraculously, this can be written

$$\sum_{n\geq 1}\tau(5n)q^{5n} = q^{25}E(q^{25})^{24}\big(4830(\alpha^5 - 11 + \beta^5)^4 - 5^{11}\big). \tag{4.9}$$

If in (4.6) we replace q by q, ηq, $\eta^2 q$, $\eta^3 q$ and $\eta^4 q$ where η is a fifth root of unity other than 1, and multiply the five results, we obtain

$$E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) = q^5E(q^{25})^5(\alpha^5 - 11 + \beta^5). \tag{4.10}$$

or,

$$\alpha^5 - 11 + \beta^5 = \frac{E(q^5)^6}{q^5E(q^{25})^6}. \tag{4.11}$$

If we substitute (4.11) into (4.9) we find

$$\sum_{n\geq 1}\tau(5n)q^{5n} = q^{25}E(q^{25})^{24}\left(4830\left(\frac{E(q^5)^6}{q^5E(q^{25})^6}\right)^4 - 5^{11}\right)$$
$$= 4830q^5E(q^5)^{24} - 5^{11}q^{25}E(q^{25})^{24}. \tag{4.12}$$

If in (4.12) we replace q^5 by q, we obtain

$$\sum_{n\geq 1}\tau(5n)q^n = 4830qE(q)^{24} - 5^{11}q^5E(q^5)^{24} \tag{4.13}$$
$$= 4830\sum_{n\geq 1}\tau(n)q^n - 5^{11}\sum_{n\geq 1}\tau(n)q^{5n}.$$

The term $n = 1$ in (4.13) gives

$$\tau(5) = 4830\tau(1) = 4830,$$

so (4.13) becomes

$$\sum_{n\geq 1}\tau(5n)q^n = \tau(5)\sum_{n\geq 1}\tau(n)q^n - 5^{11}\sum_{n\geq 1}\tau(n)q^{5n},$$

as claimed.

Aside: It can be shown [4, chap. 8] that

$$\alpha = r(q^5)^{-1}, \quad \beta = -r(q^5),$$

where

$$r(q) = \cfrac{q^{\frac{1}{5}}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}}$$
$$= q^{\frac{1}{5}}\prod_{n\geq 0}\frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}.$$

5 $p = 7$

We can write

$$E(q) = E_0 + E_1 + E_2 + E_5 \tag{5.1}$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 7. ($i = 0,\ 1,\ 2,\ 5$.)

It is easy to show that

$$E_2 = -q^2E(q^{49}). \tag{5.2}$$

If we write

$$\alpha = -\frac{E_0}{E_2}, \quad \beta = -\frac{E_1}{E_2}, \quad \gamma = -\frac{E_5}{E_2}, \tag{5.3}$$

then

$$E(q) = q^2 E(q^{49})(\alpha + \beta - 1 + \gamma). \tag{5.4}$$

Jacobi's identity (3.1) yields

$$\alpha\beta\gamma = -1, \tag{5.5}$$

$$-\alpha^2 + \alpha\beta^2 + \gamma = 0, \tag{5.6}$$

$$\alpha - \beta^2 + \beta\gamma^2 = 0 \tag{5.7}$$

and

$$\alpha^2\gamma + \beta - \gamma^2 = 0. \tag{5.8}$$

We have

$$\begin{aligned} \sum_{n \geq 1} \tau(n) q^n &= q E(q)^{24} \\ &= q^{49} E(q^{49})^{24} (\alpha + \beta - 1 + \gamma)^{24}. \end{aligned} \tag{5.9}$$

We can expand the right side of (5.9) and extract those terms in which the power of q is a multiple of 7. Thus, if H is the Huffing operator modulo 7, given by

$$H\left(\sum_n a(n) q^n\right) = \sum_n a(7n) q^{7n}, \tag{5.10}$$

and if we apply H to (5.9), we find

$$\sum_{n \geq 1} \tau(7n) q^{7n} = q^{49} E(q^{49})^{24} H\left((\alpha + \beta - 1 + \gamma)^{24}\right). \tag{5.11}$$

Let

$$\zeta = \alpha + \beta - 1 + \gamma = \frac{E(q)}{q^2 E(q^{49})}. \tag{5.12}$$

Then (5.11) becomes

$$\sum_{n \geq 1} \tau(7n) q^{7n} = q^{49} E(q^{49})^{24} H(\zeta^{24}). \tag{5.13}$$

Now,

$$H(\zeta^0) = H(1) = 1, \tag{5.14}$$

$$H(\zeta) = H(\alpha + \beta - 1 + \gamma) = -1, \tag{5.15}$$

$$H(\zeta^2) = H(\alpha^2 + 2\alpha\beta + (\beta^2 - 2\alpha) - 2\beta + 1 + 2\alpha\gamma + 2\beta\gamma + 2\gamma + \gamma^2) = 1 \tag{5.16}$$

and from Jacobi's identity (3.1)

$$\begin{aligned} E(q)^3 &= \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \\ &= A(q^7) - 3qB(q^7) + 5q^3C(q^7) - 7q^6E(q^{49})^3 \end{aligned} \tag{5.17}$$

it follows that

$$H(\zeta^3) = -7. \tag{5.18}$$

It can also be shown (see [3] or [4, chap. 7]) that

$$H(\zeta^4) = -4T - 7, \tag{5.19}$$

$$H(\zeta^5) = 10T + 49 \tag{5.20}$$

and

$$H(\zeta^6) = 49 \tag{5.21}$$

where

$$T = \frac{E(q^7)^4}{q^7E(q^{49})^4}. \tag{5.22}$$

It can then be shown that ζ satisfies the so-called modular equation

$$\zeta^7 + 7\zeta^6 + 21\zeta^5 + 49\zeta^4 + (7T + 147)\zeta^3 + (35T + 343)\zeta^2 + (49T + 343)\zeta - T^2 = 0. \tag{5.23}$$

It follows that for $i \geq 0$,

$$\begin{aligned} &H(\zeta^{i+7}) + 7H(\zeta^{i+6}) + 21H(\zeta^{i+5}) + 49\zeta^{i+4}) + (7T + 147)H(\zeta^{i+3}) \\ &+(35T + 343)H(\zeta^{i+2}) + (49T + 343)H(\zeta^{i+1}) - T^2H(\zeta^i) = 0. \end{aligned} \tag{5.24}$$

Now write

$$u_i = H(\zeta^i). \tag{5.25}$$

Then

$$u_0 = 1, \quad u_1 = -1, \quad u_2 = 1, \quad u_3 = 7, \quad u_4 = -4T - 7, \quad u_5 = 10T + 49, \quad u_6 = 49 \tag{5.26}$$

and for $i \geq 0$,

$$u_{i+7}+7u_{i+6}+21u_{i+5}+49u_{i+4}+(7T+147)u_{i+3} \\ +(35T+343)u_{i+2}+(49T+343)u_{i+1}-T^2u_i=0. \tag{5.27}$$

If we write

$$U=\sum_{i\geq 0}u_iz^i \tag{5.28}$$

it follows from (5.26) and (5.27) that

$$(1+7z+21z^2+49z^3+(7T+147)z^4+(35T+343)z^5+(49T+343)z^6-T^2z^7)U \\ =1+6z+15z^2+28z^3+(3T+63)z^4+(10T+98)z^5+(7T+49)z^6 \tag{5.29}$$

and so

$$U=\frac{1+6z+15z^2+28z^3+(3T+63)z^4+(10T+98)z^5+(7T+49)z^6}{1+7z+21z^2+49z^3+(7T+147)z^4+(35T+343T)z^5+(49T+343T)z^6-T^2z^7}. \tag{5.30}$$

If we expand the right side of (5.30) as a series, we find

$$u_{24}=-16744T^6-7^{11}. \tag{5.31}$$

That is,

$$H(\zeta^{24})=-16744\left(\frac{E(q^7)^4}{q^7E(q^{49})^4}\right)^6-7^{11}. \tag{5.32}$$

If we substitute (5.32) into (5.13), we find

$$\sum_{n\geq 1}\tau(7n)q^{7n}=q^{49}E(q^{49})^{24}\left(-16744\left(\frac{E(q^7)^4}{q^7E(q^{49})^4}\right)^6-7^{11}\right) \\ =-16744q^7E(q^7)^{24}-7^{11}q^{49}E(q^{49})^{24}. \tag{5.33}$$

If in (5.33) we replace q^7 by q, we obtain

$$\sum_{n\geq 1}\tau(7n)q^n=-16744qE(q)^{24}-7^{11}q^7E(q^7)^{24} \tag{5.34}$$

$$=-16744\sum_{n\geq 1}\tau(n)q^n-7^{11}\sum_{n\geq 1}\tau(n)q^{7n}$$

The term $n=1$ in (5.34) gives

$$\tau(7)=-16744\tau(1)=-16744,$$

so (5.33) becomes

$$\sum_{n\geq 1} \tau(7n)q^n = \tau(7)\sum_{n\geq 1} \tau(n)q^n - 7^{11}\sum_{n\geq 1} \tau(n)q^{7n},$$

as claimed.

Aside: It can be shown [4, chap. 10], using the quintuple product identity, that

$$\alpha = q^{-2}\prod_{n\geq 0} \frac{(1-q^{49n+14})(1-q^{49n+35})}{(1-q^{49n+7})(1-q^{49n+42})},$$
$$\beta = -q^{-1}\prod_{n\geq 0} \frac{(1-q^{49n+21})(1-q^{49n+28})}{(1-q^{49n+14})(1-q^{49n+35})},$$
$$\gamma = q^{3}\prod_{n\geq 0} \frac{(1-q^{49n+7})(1-q^{49n+42})}{(1-q^{49n+21})(1-q^{49n+28})}.$$

6 $p = 13$

Define

$$\zeta = \frac{E(q)}{q^7 E(q^{169})}, \quad T = \frac{E(q^{13})^2}{q^{13}E(q^{169})^2}. \tag{6.1}$$

The following results may be proved in a fashion similar to the proofs of (5.15)–(5.21), using the work of O'Brien [5], Part 1, Sections 1–3 and Bilgici and Ekin [1]. We omit the details.

$$H(\zeta) = 1, \tag{6.2}$$
$$H(\zeta^2) = -2T - 1, \tag{6.3}$$
$$H(\zeta^3) = 13, \tag{6.4}$$
$$H(\zeta^4) = 2T^2 - 13, \tag{6.5}$$
$$H(\zeta^5) = -20T^2 - 10 \times 13T - 13^2, \tag{6.6}$$
$$H(\zeta^6) = 10T^3 - 13^2, \tag{6.7}$$
$$H(\zeta^7) = 98T^3 + 28 \times 13T^2 - 13^3, \tag{6.8}$$
$$H(\zeta^8) = -70T^4 - 13^3, \tag{6.9}$$
$$H(\zeta^9) = -162T^4 + 108 \times 13T^3 + 72 \times 13^2T^2 + 18 \times 13^3T + 13^4, \tag{6.10}$$
$$H(\zeta^{10}) = 238T^5 - 13^4, \tag{6.11}$$
$$H(\zeta^{11}) = -902T^5 - 1672 \times 13T^4 - 792 \times 13^2T^3 - 198 \times 13^3T^2 - 22 \times 13^4T - 13^5, \tag{6.12}$$
$$H(\zeta^{12}) = -418T^6 - 13^5. \tag{6.13}$$

For $0 \le i \le 12$ let

$$\zeta_i = \zeta(\eta^i q) \tag{6.14}$$

where η is a 13th root of unity other than 1.

Then

$$\sum_i \zeta_i = 13, \tag{6.15}$$

$$\sum_i \zeta_i^2 = -2 \times 13T - 13, \tag{6.16}$$

$$\sum_i \zeta_i^3 = 13^2, \tag{6.17}$$

$$\sum_i \zeta_i^4 = 2 \times 13T^2 - 13^2, \tag{6.18}$$

$$\sum_i \zeta_i^5 = -20 \times 13T^2 - 10 \times 13^2 T - 13^3, \tag{6.19}$$

$$\sum_i \zeta_i^6 = 10 \times 13T^3 - 13^3, \tag{6.20}$$

$$\sum_i \zeta_i^7 = 98 \times 13T^3 + 28 \times 13^2 T^2 - 13^4, \tag{6.21}$$

$$\sum_i \zeta_i^8 = -70 \times 13T^4 - 13^4, \tag{6.22}$$

$$\sum_i \zeta_i^9 = -162 \times 13T^4 + 108 \times 13^2 T^3 + 72 \times 13^3 T^2 + 18 \times 13^4 T + 13^5, \tag{6.23}$$

$$\sum_i \zeta_i^{10} = 238 \times 13T^5 - 13^5, \tag{6.24}$$

$$\sum_i \zeta_i^{11} = -902 \times 13T^5 - 1672 \times 13^2 T^4 - 792 \times 13^3 T^3 - 198 \times 13^4 T^2 - 22 \times 13^5 T - 13^6, \tag{6.25}$$

$$\sum_i \zeta_i^{12} = -418 \times 13T^6 - 13^6. \tag{6.26}$$

From these we obtain the symmetric functions,

$$\sigma_1 = \sum_i \zeta_i = 13, \tag{6.27}$$

$$\sigma_2 = \sum_{i<j} \zeta_i \zeta_j = 13T + 7 \times 13, \tag{6.28}$$

$$\sigma_3 = \sum_{i<j<k} \zeta_i \zeta_j \zeta_k = 13^2 T + 3 \times 13^2, \tag{6.29}$$

$$\sigma_4 = 6 \times 13T^2 + 7 \times 13^2 T + 15 \times 13^2, \tag{6.30}$$

$$\sigma_5 = 74 \times 13T^2 + 37 \times 13^2 T + 5 \times 13^3, \tag{6.31}$$

$$\sigma_6 = 20 \times 13T^3 + 38 \times 13^2 T^2 + 13^4 T + 19 \times 13^3, \tag{6.32}$$

$$\sigma_7 = 222 \times 13T^3 + 184 \times 13^2 T^2 + 51 \times 13^3 T + 5 \times 13^4, \tag{6.33}$$

$$\sigma_8 = 38 \times 13T^4 + 102 \times 13^2 T^3 + 56 \times 13^3 T^2 + 13^5 T + 15 \times 13^4, \tag{6.34}$$

$$\sigma_9 = 346 \times 13T^4 + 422 \times 13^2 T^3 + 184 \times 13^3 T^2 + 37 \times 13^4 T + 3 \times 13^5, \tag{6.35}$$

$$\sigma_{10} = 36 \times 13T^5 + 126 \times 13^2 T^4 + 102 \times 13^3 T^3 + 38 \times 13^4 T^2 + 7 \times 13^5 T + 7 \times 13^5, \tag{6.36}$$

$$\sigma_{11} = 204 \times 13T^5 + 346 \times 13^2 T^4 + 222 \times 13^3 T^3 + 74 \times 13^4 T^2 + 13^6 T + 13^6 \tag{6.37}$$

$$\sigma_{12} = 11 \times 13T^6 + 36 \times 13^2 T^5 + 38 \times 13^3 T^4 + 20 \times 13^4 T^3 + 6 \times 13^5 T^2 + 13^6 T + 13^6 \tag{6.38}$$

and

$$\sigma_{13} = \prod_i \zeta_i = \frac{E(q^{13})^{14}}{q^{91} E(q^{169})^{14}} = T^7. \tag{6.39}$$

It follows that the modular equation is

$$\begin{aligned}
&\zeta^{13} - 13\zeta^{12} + (13T + 7 \times 13)\zeta^{11} - (13^2 T + 3 \times 13^2)\zeta^{10} + (6 \times 13T^2 + 7 \times 13^2 T + 15 \times 13^2)\zeta^9 \\
&-(74 \times 13T^2 + 37 \times 13^2 T + 5 \times 13^3)\zeta^8 + (20 \times 13T^3 + 38 \times 13^2 T^2 + 13^4 T + 19 \times 13^3)\zeta^7 \\
&-(222 \times 13T^3 + 184 \times 13^2 T^2 + 51 \times 13^3 T + 5 \times 13^4)\zeta^6 \\
&+(38 \times 13T^4 + 102 \times 13^2 T^3 + 56 \times 13^3 T^2 + 13^5 T + 15 \times 13^4)\zeta^5 \\
&-(346 \times 13T^4 + 422 \times 13^2 T^3 + 184 \times 13^3 T^2 + 37 \times 13^4 T + 3 \times 13^5)\zeta^4 \\
&+(36 \times 13T^5 + 126 \times 13^2 T^4 + 102 \times 13^3 T^3 + 38 \times 13^4 T^2 + 7 \times 13^5 T + 7 \times 13^5)\zeta^3 \\
&-(204 \times 13T^5 + 346 \times 13^2 T^4 + 222 \times 13^3 T^3 + 74 \times 13^4 T^2 + 13^6 T + 13^6)\zeta^2 \\
&+(11 \times 13T^6 + 36 \times 13^2 T^5 + 38 \times 13^3 T^4 + 20 \times 13^4 T^3 + 6 \times 13^5 T^2 + 13^6 T + 13^6)\zeta - T^7 = 0.
\end{aligned} \tag{6.40}$$

If, as before, we let $u_i = H(\zeta^i)$ and $U = \sum_{i \geq 0} u_i z^i$, then

$$U = \frac{N}{D} \tag{6.41}$$

where

$$D = 1 - 13z + (13T + 7 \times 13)z^2 - (13^2T + 3 \times 13^2)z^3 + (6 \times 13T^2 + 7 \times 13^2T + 15 \times 13^2)z^4$$
$$-(74 \times 13T^2 + 37 \times 13^2T + 5 \times 13^3)z^5 + (20 \times 13T^3 + 38 \times 13^2T^2 + 13^4T + 19 \times 13^3)z^6$$
$$-(222 \times 13T^3 + 184 \times 13^2T^2 + 51 \times 13^3T + 5 \times 13^4)z^7$$
$$+(38 \times 13T^4 + 102 \times 13^2T^3 + 56 \times 13^3T^2 + 13^5T + 15 \times 13^4)z^8$$
$$-(346 \times 13T^4 + 422 \times 13^2T^3 + 184 \times 13^3T^2 + 37 \times 13^4T + 3 \times 13^5)z^9$$
$$+(36 \times 13T^5 + 126 \times 13^2T^4 + 102 \times 13^3T^3 + 38 \times 13^4T^2 + 7 \times 13^5T + 7 \times 13^5)z^{10}$$
$$-(204 \times 13T^5 + 346 \times 13^2T^4 + 222 \times 13^3T^3 + 74 \times 13^4T^2 + 13^6T + 13^6)z^{11}$$
$$+(11 \times 13T^6 + 36 \times 13^2T^5 + 38 \times 13^3T^4 + 20 \times 13^4T^3 + 6 \times 13^5T^2 + 13^6T + 13^6)z^{12} - T^7z^{13} \tag{6.42}$$

and

$$N = 1 - 12z + (11T + 77)z^2 - (10 \times 13T + 30 \times 13)z^3 + (54T^2 + 63 \times 13T + 135 \times 13)z^4$$
$$-(592T^2 + 296 \times 13T + 40 \times 13^2)z^5 + (140T^3 + 266 \times 13T^2 + 7 \times 13^3T + 133 \times 13^2)z^6$$
$$-(1332T^3 + 1104 \times 13T^2 + 306 \times 13^2T + 30 \times 13^3)z^7$$
$$+(190T^4 + 510 \times 13T^3 + 280 \times 13^2T^2 + 5 \times 13^4T + 75 \times 13^3)z^8$$
$$-(1384T^4 + 1688 \times 13T^3 + 736 \times 13^2T^2 + 148 \times 13^3T + 12 \times 13^4)z^9$$
$$+(108T^5 + 378 \times 13T^4 + 306 \times 13^2T^3 + 114 \times 13^3T^2 + 21 \times 13^4T + 21 \times 13^4)z^{10}$$
$$-(408T^5 + 692 \times 13T^4 + 444 \times 13^2T^3 + 148 \times 13^3T^2 + 2 \times 13^5T + 2 \times 13^5)z^{11}$$
$$+(11T^6 + 36 \times 13T^5 + 38 \times 13^2T^4 + 20 \times 13^3T^3 + 6 \times 13^4T^2 + 13^5T + 13^5)z^{12}. \tag{6.43}$$

If we expand U to the power 24, we find that

$$H(\zeta^{24}) = u_{24} = -577738T^{12} - 13^{11}. \tag{6.44}$$

We then have

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24} \tag{6.45}$$
$$= q^{169}E(q^{169})^{24}\left(\frac{E(q)}{q^7E(q^{169})}\right)^{24}$$
$$= q^{169}E(q^{169})^{24}\zeta^{24},$$

$$\sum_{n\geq 1} \tau(13n)q^{13n} = q^{169}E(q^{169})^{24}H(\zeta^{24}) \tag{6.46}$$
$$= q^{169}E(q^{169})^{24}\left(-577738T^{12} - 13^{11}\right)$$
$$= q^{169}E(q^{169})^{24}\left(-577738\left(\frac{E(q^{13})^2}{q^{13}E(q^{169})^2}\right)^{12} - 13^{11}\right)$$

$$= -577738q^{13}E(q^{13})^{24} - 13^{11}q^{169}E(q^{169})^{24}$$

and

$$\sum_{n\geq 1} \tau(13n)q^n = -577738qE(q)^{24} - 13^{11}q^{13}E(q^{13})^{24} \quad (6.47)$$
$$= -577738\sum_{n\geq 1} \tau(n)q^n - 13^{11}\sum_{n\geq 1} \tau(n)q^{13n}.$$

The term $n = 1$ gives

$$\tau(13) = -577738\tau(1) = -577738,$$

so (6.47) becomes

$$\sum_{n\geq 1} \tau(13n)q^n = \tau(13)\sum_{n\geq 1} \tau(n)q^n - 13^{11}\sum_{n\geq 1} \tau(n)q^{13n}, \quad (6.48)$$

as claimed.

Aside: It can be shown, using the quintuple product identity, that

$$\zeta = \alpha + \beta + \gamma + \delta + 1 + \varepsilon + \theta$$

where

$$\alpha = q^{-7}\prod_{n\geq 0} \frac{(1-q^{169n+52})(1-q^{169n+117})}{(1-q^{169n+26})(1-q^{169n+143})}$$
$$\beta = -q^{-6}\prod_{n\geq 0} \frac{(1-q^{169n+78})(1-q^{169n+91})}{(1-q^{169n+39})(1-q^{169n+130})}$$
$$\gamma = -q^{-5}\prod_{n\geq 0} \frac{(1-q^{169n+26})(1-q^{169n+143})}{(1-q^{169n+13})(1-q^{169n+156})}$$
$$\delta = q^{-2}\prod_{n\geq 0} \frac{(1-q^{169n+65})(1-q^{169n+104})}{(1-q^{169n+52})(1-q^{169n+117})}$$
$$\varepsilon = q^{5}\prod_{n\geq 0} \frac{(1-q^{169n+39})(1-q^{169n+130})}{(1-q^{169n+65})(1-q^{169n+104})}$$
$$\theta = q^{15}\prod_{n\geq 0} \frac{(1-q^{169n+13})(1-q^{169n+156})}{(1-q^{169n+78})(1-q^{169n+91})}$$

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