On an Additive Prime Divisor Function of Alladi and Erdős

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This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract This paper discusses the additive prime divisor function $A(n) := \sum_{p^\alpha || n} \alpha\ p$ which was introduced by Alladi and Erdős in 1977. It is shown that $A(n)$ is uniformly distributed (mod q) for any fixed integer $q > 1$ with an explicit bound for the error.

1 Introduction

Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the unique prime decomposition of a positive integer n. In 1977, Alladi and Erdős [1] introduced the additive function

$$A(n) := \sum_{i=1}^{r} a_i \cdot p_i.$$

Among several other things they proved that $A(n)$ is uniformly distributed modulo 2. This was obtained from the identity

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$$\sum_{n=1}^{\infty} \frac{(-1)^{A(n)}}{n^s} = \frac{2^s+1}{2^s-1} \cdot \frac{\zeta(2s)}{\zeta(s)} \tag{1}$$

together with the known zero-free region for the Riemann zeta function. As a consequence they proved that there exists a constant $c > 0$ such that

$$\sum_{n \le x} (-1)^{A(n)} = \mathscr{O}\left(x\, e^{-c\sqrt{\log x \log\log x}} \right),$$

for $x \to \infty$.

In 1969 Delange [3] gave a necessary and sufficient condition for uniform distribution in progressions for integral valued additive functions which easily implies that $A(n)$ is uniformly distributed (mod q) for all $q \ge 2$ (although without a bound for the error in the asymptotic formula). The main goal of this paper is to show that $A(n)$ is uniformly distributed modulo q for any integer $q \ge 2$ with an explicit bound for the error.

Unfortunately, it is not possible to obtain such a simple identity as in (1) for the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i \frac{hA(n)}{q}}}{n^s}$$

when $q > 2$ and h, q are coprime. Instead we require a representation involving a product of rational powers of Dirichlet L-functions which will have branch points at the zeros of the L-functions.

The uniform distribution of $A(n)$ is a consequence of the following theorem (1.1) which is proved in §3. To state the theorem we require some standard notation. Let μ denote the Mobius function and let ϕ denote Euler's function. For any Dirichlet character $\chi \pmod q$ (with $q > 1$) let $\tau(\chi) = \sum\limits_{\ell \pmod q} \chi(\ell) e^{\frac{2\pi i \ell}{q}}$ denote the associated Gauss sum and let $L(s, \chi)$ denote the Dirichlet L-function associated to χ.

Theorem 1.1. *Let h, q be fixed coprime integers with $q > 2$. Then for $x \to \infty$ we have the asymptotic formula*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \begin{cases} C_{h,q} \cdot x\, (\log x)^{-1+\frac{\mu(q)}{\phi(q)}} \left(1 + \mathscr{O}\left((\log x)^{-1}\right)\right) & \text{if } \mu(q) \ne 0, \\ \mathscr{O}\left(x\, e^{-c_0 \sqrt{\log x}} \right) & \text{if } \mu(q) = 0, \end{cases}$$

where $c_0 > 0$ is a constant depending at most on h, q,

$$C_{h,q} = \frac{V_{h,q} \cdot \sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi}\, \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \prod_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}},$$

and

$$V_{h,q} := \exp\left[-\frac{\mu(q)}{\phi(q)} \sum_{p|q} \sum_{k=1}^{\infty} \frac{1}{kp^k} + \sum_{p\,|\,q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^k} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^k} \right].$$

Theorem 1.1 has the following easily proved corollary.

Corollary 1.2. *Let $q > 1$ and let h be an arbitrary integer. Then*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \mathscr{O}\left(\frac{x}{\sqrt{\log x}} \right).$$

The above corollary can then be used to obtain the desired uniform distribution theorem.

Theorem 1.3. *Let h, q be fixed integers with $q > 2$. Then for $x \to \infty$, we have*

$$\sum_{\substack{n \le x \\ A(n) \equiv h \pmod q}} 1 = \frac{x}{q} + \mathscr{O}\left(\frac{x}{\sqrt{\log x}} \right).$$

We remark that the error term in theorem 1.3 can be replaced by a second order asymptotic term which is not uniformly distributed (mod q).

The proof of theorem (1.1) relies on explicitly constructing an L-function with coefficients of the form $e^{2\pi i \frac{hA(n)}{q}}$. It will turn out that this L-function will be a product of Dirichlet L-functions raised to complex powers. The techniques for obtaining asymptotic formulae and dealing with branch singularities arising from complex powers of ordinary L-series were first introduced by Selberg [7], and see also Tenenbaum [8] for a very nice exposition with different applications. In [4–6], one finds a larger class of additive functions where these methods can also be applied yielding similar results but with different constants.

2 On the function $L(s, \psi_{h/q})$

Let h, q be coprime integers with $q > 1$. In this paper we shall investigate the completely multiplicative function

$$\psi_{h/q}(n) := e^{\frac{2\pi i h A(n)}{q}}.$$

Then the L-function associated to $\psi_{h/q}$ is defined by the absolutely convergent series

$$L(s, \psi_{h/q}) := \sum_{n=1}^{\infty} \psi_{h/q}(n) n^{-s}, \tag{2}$$

in the region $\Re(s) > 1$, and has an Euler product representation (product over rational primes) of the form

$$L(s, \psi_{h/q}) := \prod_p \left(1 - \frac{e^{\frac{2\pi i h p}{q}}}{p^s}\right)^{-1}. \tag{3}$$

The Euler product (3) converges absolutely to a non-vanishing function for $\Re(s) > 1$. We would like to show it has analytic continuation to a larger region.

Lemma 2.1. *Let* $\Re(s) > 1$. *Then*

$$\log\left(L(s, \psi_{h/q})\right) = \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} \; + \; T_{h,q}(s)$$

where, for any $\varepsilon > 0$, *the function*

$$T_{h,q}(s) := \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}$$

is holomorphic for $\Re(s) > \frac{1}{2} + \varepsilon$ *and satisfies* $|T_{h,q}(s)| = \mathcal{O}_\varepsilon(1)$ *where the* $\mathcal{O}_\varepsilon$-*constant is independent of* q *and depends at most on* ε.

Proof. Taking log's, we obtain

$$\begin{aligned}\log\left(L(s, \psi_{h/q})\right) &= \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}}}{k\, p^{sk}} \\ &= \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}.\end{aligned}$$

Hence, we may take

$$T_{h,q}(s) = \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}},$$

which is easily seen to converge absolutely for $\Re(s) > \frac{1}{2}$. □

For $q > 2$, let χ denote a Dirichlet character $\pmod q$ with associated Gauss sum $\tau(\chi)$. We also let χ_0 be the trivial character $\pmod q$.

We require the following lemma.

Lemma 2.2. *Let $h, q \in \mathbf{Z}$ with $q > 2$ and $(h, q) = 1$. Then*

$$e^{\frac{2\pi i h}{q}} = \left(\frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi) \cdot \overline{\chi(h)} \right) + \frac{\mu(q)}{\phi(q)}.$$

Proof. Since $(h, q) = 1$, it follows that for $\chi \pmod q$ with $\chi \neq \chi_0$,

$$\tau(\chi)\, \overline{\chi(h)} = \sum_{\ell=1}^{q} \chi(\ell) e^{\frac{2\pi i \ell h}{q}}.$$

This implies that

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi)\, \overline{\chi(h)} = (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} + \sum_{\substack{\ell=2 \\ (\ell,q)=1}}^{q} \left(\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \chi(\ell) \right) e^{\frac{2\pi i \ell h}{q}}$$
$$= (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} - \sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} + e^{\frac{2\pi i h}{q}}.$$

The proof is completed upon noting that the Ramanujan sum on the right side above can be evaluated as

$$\sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} = \sum_{d \mid (q,h)} \mu\left(\frac{q}{d}\right) d = \mu(q).$$ □

Theorem 2.3. *Let $s \in \mathbf{C}$ with $\Re(s) > 1$. Then we have the representation*

$$L(s, \psi_{h/q}) = \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(s, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot \zeta(s)^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(s)},$$

where

$$U_{h,q}(s) := -\frac{\mu(q)}{\phi(q)} \sum_{p|q} \sum_{k=1}^{\infty} \frac{1}{kp^{sk}} + \sum_{p\,|\,q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}.$$

Proof. If we combine lemmas (2.1) and (2.2) it follows that for $\Re(s) > 1$,

$$\log\big(L(s,\psi_{h/q})\big) = \sum_p \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + T_{h,q}(s)$$

$$= \sum_{p\nmid q} \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + \sum_{p\mid q} \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + T_{h,q}(s)$$

$$= \sum_{p\nmid q} \sum_{k=1}^\infty \frac{\left(\frac{1}{\phi(q)} \sum\limits_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \tau(\chi)\cdot\overline{\chi(h\,p^k)} + \frac{\mu(q)}{\phi(q)}\right)}{k\,p^{sk}} + \sum_{p\mid q} \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + T_{h,q}(s).$$

Hence

$$\log\big(L(s,\psi_{h/q})\big) = \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \tau(\chi)\overline{\chi(h)}\,\log(L(s,\overline{\chi}) + \frac{\mu(q)}{\phi(q)} \log\big(\zeta(s)\big)$$

$$- \frac{\mu(q)}{\phi(q)} \sum_{p|q} \sum_{k=1}^\infty \frac{1}{kp^{sk}} + \sum_{p\mid q} \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + T_{h,q}(s).$$

The theorem immediately follows after taking exponentials. □

The representation of $L(s,\psi_{h/q})$ given in theorem 2.3 allows one to analytically continue the function $L(s,\psi_{h/q})$ to a larger region which lies to the left of the line $\Re(s) = 1+\varepsilon$ ($\varepsilon > 0$). This is a region which does not include the branch points of $L(s,\psi_{h/q})$ at the zeros and poles of $L(s,\chi)$, $\zeta(s)$.

Assume that $q > 1$ and $\chi \pmod q$. It is well known (see [2]) that the Dirichlet L-functions $L(\sigma+it,\chi))$ do not vanish in the region

$$\sigma \geq \begin{cases} 1 - \frac{c_1}{\log q|t|} & \text{if } |t|\geq 1, \\ 1 - \frac{c_2}{\log q} & \text{if } |t|\leq 1, \end{cases} \qquad \text{(for absolute constants } c_1, c_2 > 0), \qquad (4)$$

unless χ is the exceptional real character which has a simple real zero (Siegel zero) near $s = 1$.

Similarly, $\zeta(\sigma+it)$ does not vanish for

$$\sigma \geq 1 - \frac{c_3}{\log(|t|+2)}, \qquad \text{(for an absolute constant } c_3 > 0). \qquad (5)$$

Assume $q > 1$ and that there is no exceptional real character (mod q). It follows from (4) and (5) that $L(s,\psi_{h/q})$ is holomorphic in the region to the right of the contour $\mathscr{C}_q$ displayed in Figure 1.

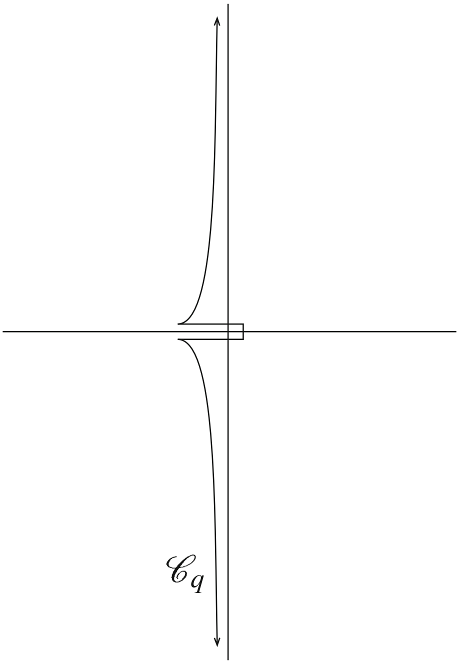

Fig. 1 The contour $\mathscr{C}_q$

To construct the contour $\mathscr{C}_q$ first take a slit along the real axis from $1 - \frac{c_2}{\log q}$ to 1 and construct a line just above and just below the slit. Then take two asymptotes to the line $\Re(s) = 1$ with the property that if $\sigma + it$ is on the asymptote and $|t| \geq 1$, then σ satisfies (4). If $q = 1$, we do a similar construction using (5).

3 Proof of theorem 1.1

The proof of theorem 1.1 is based on the following theorem.

Theorem 3.1. *Let h, q be fixed coprime integers with $q > 2$ and $\mu(q) \neq 0$. Then for $x \to \infty$ there exist absolute constants $c, c' > 0$ such that*

$$\sum_{n\le x} e^{2\pi i \frac{hA(n)}{q}}$$

$$= \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} L(\sigma,\overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{H_{h,q}(\sigma)} \, \frac{x^\sigma}{\sigma}\, d\sigma$$

$$+ \mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right).$$

On the other hand if $\mu(q) = 0$*, then* $\sum_{n\le x} e^{2\pi i \frac{hA(n)}{q}} = \mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right)$.

Proof. The proof of theorem 3.1 relies on the following lemma taken from [2].

Lemma 3.2. *Let*

$$\delta(x) := \begin{cases} 0, & \text{if } 0 < x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 1, & \text{if } x > 1, \end{cases}$$

then for $x, T > 0$*, we have*

$$\left| \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} \frac{x^s}{s}\, ds \; - \; \delta(x) \right| \; < \; \begin{cases} x^c \cdot \min\left(1, \frac{1}{T|\log x|}\right), & \text{if } x \ne 1, \\ cT^{-1}, & \text{if } x = 1. \end{cases}$$

It follows from lemma 3.2, for $x, T \gg 1$ and $c = 1 + \frac{1}{\log x}$, that

$$\frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} L\big(s, \psi_{h/q}\big)\, \frac{x^s}{s}\, ds = \sum_{n\le x} \psi_{h/q}(n) \; + \; \mathcal{O}\left(\frac{x\log x}{T}\right). \tag{6}$$

Fix large constants $c_1, c_2 > 0$. Next, shift the integral in (6) to the left and deform the line of integration to a contour

$$L^+ \; + \; \mathcal{C}_{T,x} \; + \; L^-$$

as in figure 2 below which contains two short horizontal lines:

$$L^{\pm} = \left\{ \sigma \pm iT \;\middle|\; 1 - \frac{c_1}{\log qT} \;\le\; \sigma \;\le\; 1 + \frac{1}{\log x} \right\},$$

together with the contour $C_{T,x}$ which is similar to C_q except that the two curves asymptotic to the line $\Re(s) = 1$ go from $1 - \frac{c_1}{\sqrt{\log qT}} + iT$ to $1 - \frac{c_2}{\sqrt{\log x}} + i\varepsilon$ and $1 - \frac{c_2}{\sqrt{\log x}} - i\varepsilon$ to $1 - \frac{c_1}{\sqrt{\log qT}} - iT$, respectively, for $0 < \varepsilon \to 0$.

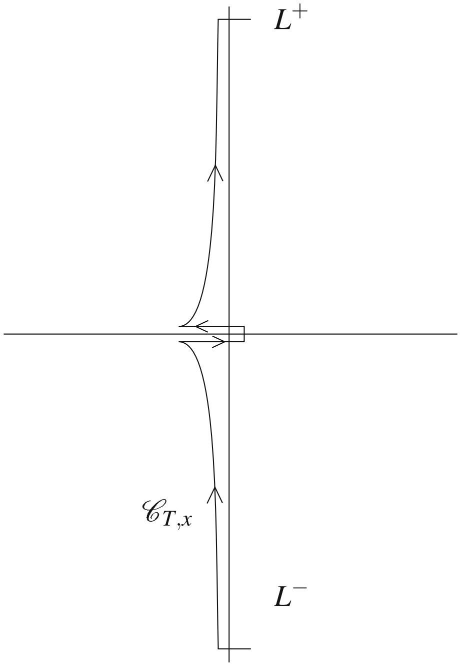

Fig. 2 The contour $\mathscr{C}_{T,x}$

Now, by the zero-free regions (4), (5), the region to the right of the contour $L^+ + \mathscr{C}_{T,x} + L^-$ does not contain any branch points or poles of the L-functions $L(s,\chi)$ for any $\chi \pmod q$. It follows that

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} L\big(s,\psi_{h/q}\big)\,\frac{x^s}{s}\,ds \;=\; \frac{1}{2\pi i}\left(\int_{L^+}+\int_{\mathscr{C}_\varepsilon}+\int_{L^-}\right) L\big(s,\psi_{h/q}\big)\,\frac{x^s}{s}\,ds. \quad (7)$$

The main contribution for the integral along $L^+ + \mathscr{C}_{T,x} + L^-$ in (7) comes from the integrals along the straight lines above and below the slit on the real axis $\left[1-\frac{c_2}{\sqrt{\log x}},\ 1\right]$. These integrals cancel if the function $L\big(s,\psi_{h/q}\big)$ has no branch points or poles on the slit. It follows from theorem 2.3 that this will be the case if $\mu(q)=0$. The remaining integrals in (7) can then be estimated as in the proof of the prime number theorem for arithmetic progressions (see [2]), yielding an error term of the form $\mathscr{O}\left(xe^{-c'\sqrt{\log x}}\right)$. This proves the second part of theorem 3.1.

Next, assume $\mu(q)\neq 0$. In this case $L(s,\psi_{h/q})$ has a branch point at $s=1$ coming from the Riemann zeta function, it is necessary to keep track of the change in argument. Let 0^+i denote the upper part of the slit and let 0^-i denote the lower part of the slit. Then we have $\log[\zeta(\sigma+0^+ i) = \log|\zeta(\sigma)| - i\pi$ and $\log[\zeta(\sigma+0^- i) = \log|\zeta(\sigma)| + i\pi$.

By the standard proof of the prime number theorem for arithmetic progressions it follows that (with an error $\mathscr{O}\big(e^{-c'\sqrt{\log x}}\big)$) the right hand side of (7) is asymptotic to

$$\mathscr{I}_{\text{slit}} := \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \Big[\exp\Big(\log\big(L\left(\sigma+0^+ i,\ \psi_{h/q}\right)\big)\Big) - \exp\Big(\log\big(L\left(\sigma-0^- i,\ \psi_{h/q}\right)\big)\Big)\Big] \frac{x^\sigma}{\sigma}\, d\sigma. \tag{8}$$

We may evaluate $\mathscr{I}_{\text{slit}}$ using theorem 2.3. This gives

$$\mathscr{I}_{\text{slit}} = \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot e^{U_{h,q}(\sigma)} \cdot \left[\exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| - i\pi\Big)\right) - \exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| + i\pi\Big)\right)\right] \frac{x^\sigma}{\sigma}\, d\sigma$$

$$= \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\ \frac{x^\sigma}{\sigma}\, d\sigma.$$

As in the previous case when $\mu(q)=0$, the remaining integrals in (7) can then be estimated as in the proof of the prime number theorem for arithmetic progressions, yielding an error term of the form $\mathscr{O}\left(xe^{-c'\sqrt{\log x}}\right)$. This completes the proof of theorem 3.1. □

The proof of theorem 1.1 follows from theorem 3.1 if we can obtain an asymptotic formula for the integral

$$\mathscr{I}_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\ \frac{x^\sigma}{\sigma}\, d\sigma. \tag{9}$$

Since we have assumed q is fixed, it immediately follows that for arbitrarily large $c \gg 1$ and $x\to\infty$, we have

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int_{1-\frac{c\log\log x}{\log x}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)} \frac{x^\sigma}{\sigma}\, d\sigma + \mathcal{O}\left(\frac{x}{(\log x)^c}\right).$$

Now, in the region $1 - \frac{c\log\log x}{\log x} \le \sigma \le 1$,

$$\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \cdot \frac{e^{H_{h,q}(\sigma)}}{\sigma} = \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} + \mathcal{O}\left(\frac{\log\log x}{\log x}\right).$$

Consequently,

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} \int_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}} x^\sigma \, d\sigma$$
$$+ \mathcal{O}\left(\frac{\log\log x}{\log x} \left| \int_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}} x^\sigma \, d\sigma \right| \right). \tag{10}$$

It remains to compute the integral of $|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}}$ occurring in (10). For σ very close to 1, we have

$$|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma - 1|} + \mathcal{O}(1)\right)^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma - 1|}\right)^{\frac{\mu(q)}{\phi(q)}} + \mathcal{O}\left(\left(\frac{1}{|\sigma - 1|}\right)^{\frac{\mu(q)}{\phi(q)}-1}\right).$$

It follows that

$$\int_{1-\frac{c\log\log x}{\log x}}^{1} |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} x^\sigma \, d\sigma = \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} + \mathcal{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right). \tag{11}$$

Combining equations (10) and (11) we obtain

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi}\,\Gamma\left(1-\frac{\mu(q)}{\phi(q)}\right)\prod_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} L(1,\ \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}}\, e^{U_{h,q}(1)}\,\frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} + \mathcal{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right).$$

Remark: As pointed out to me by Gérald Tenenbaum, it is also possible to deduce Corollary 1.2 directly from theorem 2.3 by using theorem II.5.2 of [8]. In this manner one can obtain an explicit asymptotic expansion which, furthermore, is valid for values of q tending to infinity with x.

4 Examples of equidistribution (mod 3) and (mod 9)

Equidistribution (mod 3): Theorem (1.1) says that for $h = 1,\ q = 3$:

$$\begin{aligned}\sum_{n\le x} e^{\frac{2\pi i A(n)}{3}} &= \frac{-V_{1,3}}{\pi}\,\Gamma\left(\frac{3}{2}\right)\prod_{\substack{\chi \pmod 3 \\ \chi\neq\chi_0}} L(1,\ \chi)^{\frac{G(\overline{\chi})}{2}}\,\frac{x}{(\log x)^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{\log x}\right)\right) \\ &\approx (-0.503073 + 0.24042\, i)\,\frac{x}{(\log x)^{\frac{3}{2}}}.\end{aligned}$$

We computed the above sum for $x = 10^7$ and obtained

$$\sum_{n\le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -98,423.00 + 55,650.79\, i.$$

Our theorem predicts that

$$\sum_{n\le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -88,870.8 + 42,471.7\, i.$$

Since $\log\left(10^7\right) \approx 16.1$ is small, this explains the discrepancy between the actual and predicted results.

As $x \to \infty$, we have

$$\sum_{\substack{n\le x \\ A(n)\equiv a \pmod 3}} = \frac{1}{3}\sum_{h=0}^{2}\sum_{n\le x} e^{\frac{2\pi i A(n) h}{3}} e^{-\frac{2\pi i h a}{3}} = \frac{x}{3} + c_a\,\frac{x}{(\log x)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{5}{2}}}\right)$$

where

$$c_0 = -0.335382, \qquad c_1 \approx 0.306498, \qquad c_2 \approx 0.0288842.$$

Equidistribution (mod 9):

Our theorem says that for $h \neq 3, 6$ $(1 \leq h < 9)$ and $q = 9$:

$$\sum_{n \leq x} e^{\frac{2\pi i h A(n)}{9}} = \mathscr{O}\left(x\, e^{-c_0\sqrt{\log x}}\right).$$

Surprisingly!! there is a huge amount of cancellation when $x = 10^7$:

$$\sum_{n \leq 10^7} e^{\frac{2\pi i h A(n)}{9}} \approx \begin{cases} -315.2 - 140.4\, i & \text{if } h = 1, \\ 282.2 - 543.4\, i & \text{if } h = 2, \\ 94.5 + 321.9\, i & \text{if } h = 4, \\ 94.5 - 321.9\, i & \text{if } h = 5, \\ 282.2 + 543.4\, i & \text{if } h = 7, \\ -315.2 + 140.4\, i & \text{if } h = 8. \end{cases}$$

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