Integrals Involving Rudin–Shapiro Polynomials and Sketch of a Proof of Saffari's Conjecture

Shalosh B. Ekhad and Doron Zeilberger

Dedicated to Krishnaswami "Krishna" Alladi, the tireless apostle of Srinivasa Ramanujan, yet a great mathematician in his own right

Abstract The Rudin–Shapiro polynomials $P_k(z)$, are defined by a certain linear functional recurrence equation and are of interest in signal processing due to their special autocorrelation properties. An algorithmic approach to computation of the moments of these polynomials is given. A proof sketch is given of Saffari's Conjecture for the asymptotic growth of these moments.

Keywords Rudin–Shapiro polynomials · Saffari's Conjecture

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Preface: Krishna Alladi

One of the greatest *disciples* of Srinivasa Ramanujan, who did so much to make him a household name in the mathematical community, and far beyond, is Krishnaswami "Krishna" Alladi. Among many other things, he founded and is still editor-in-chief, of the very successful *Ramanujan Journal* (very ably managed by managing editor Frank Garvan), and initiated the SASTRA Ramanujan prize given to promising young mathematicians.

But Krishna is not *just* a mathematical leader, he is also a great number-theorist with very broad interests, including analytic number theory and, inspired by Ramanujan, q-series and partitions. That's why it is not surprising that the conference to

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celebrate his 60th birthday, that took place last March, attracted attendees and speakers with very diverse interests, and enabled the participants to learn new things far afield from their own narrow specialty. That's how we found out, and got hooked on, *Rudin–Shapiro polynomials*.

Hugh Montgomery's Erdős's colloquium

One of the highlights of the conference was a fascinating talk by the eminent Michigan number theorist (and Krishna's former postdoc mentor) Hugh Montgomery, who talked about *Littlewood polynomials* of interest **both** in pure number theory and, surprisingly, in *signal processing*. These are polynomials whose coefficients are in $\{-1, 1\}$. Among these stand out the famous *Rudin–Shapiro* polynomials, introduced ([8, 9]) by Harold "Silent" Shapiro[1] and rediscovered by Walter Rudin ([7]).

The Rudin–Shapiro polynomials

The Rudin–Shapiro polynomials, $P_k(z)$, are best defined by the functional recurrence (see [10])

$$P_k(z) = P_{k-1}(z^2) + zP_{k-1}(-z^2), \qquad (DefiningRecurrence)$$

with the initial condition $P_0(z) = 1$.

As Hugh Montgomery described so well in his talk, these have amazing properties. Both number-theorists and signal-processors are very interested in the so-called *sequence of (even) moments*, whose definition usually involves the integral sign, but is better phrased entirely in terms of high-school algebra as follows.

$$M_n(k) := CT[P_k(z)^n P_k(z^{-1})^n],$$

where CT denotes the "constant term functional", that for any Laurent polynomial $f(z)$ of z, extracts the coefficient of z^0. For example $CT[4/z^2 + 11/z + 101 + 5z + 11z^{15}] = 101$.

Can we find closed-form expressions for $M_n(k)$, in k, for any given, specific, positive integer n? Failing this, can we find explicit expressions for the generating functions

[1] Harold S. Shapiro (S. originally stood for Seymour) was one of a brilliant cohort of students at City College, in the late 1940s, that included Leon Ehrenpreis, Donald Newman, Israel Aumann, and another Harold Shapiro, Harold N. Shapiro (N. originally stood for Nathaniel). But their friends, in order to distinguish between the two Harold Shapiros, called them "Silent" and "Noisy" respectively. It is ironic that Harold Silent Shapiro's son is the eminent, **but very loud**, MIT cosmologist, Max Tegmark.

$$R_n(t) := \sum_{k=0}^{\infty} M_n(k) t^k ?$$

The sequence $M_1(k)$ has a very nice closed-form, $M_1(k) = 2^k$. This is not very hard, even for humans. Indeed, using Eq. $(DefiningRecurrence)$, we get

$$P_k(z)P_k(z^{-1}) = \left(P_{k-1}(z^2) + zP_{k-1}(-z^2) \right) \cdot \left(P_{k-1}(z^{-2}) + z^{-1}P_{k-1}(-z^{-2}) \right)$$

$$= P_{k-1}(z^2)P_{k-1}(z^{-2}) \; + \; P_{k-1}(-z^2)P_{k-1}(-z^{-2}) \; + \; \left\{ zP_{k-1}(-z^2)P_{k-1}(z^{-2}) \right. \\ \left. + z^{-1}P_{k-1}(z^2)P_{k-1}(-z^{-2}) \right\}.$$

The quantity in the braces only has **odd** powers, so its constant term vanishes. Hence

$$M_1(k) = CT\,[\,P_k(z)P_k(z^{-1})\,] \; = \; CT\,[\,P_{k-1}(z^2)P_{k-1}(z^{-2})\,] \; + \; CT\,[\,P_{k-1}(-z^2)P_{k-1}(-z^{-2})\,].$$

Replacing z^2 by z in the first term on the right, and $-z^2$ by z in the second term, does not change the constant term, hence, we have the **linear recurrence equation with constant coefficients**

$$M_1(k) = 2M_1(k-1),$$

with the obvious initial condition $M_1(0) = 1$, that implies the explicit expression $M_1(k) = 2^k$. Equivalently, the generating function $R_1(t)$ is given by

$$R_1(t) = \frac{1}{1-2t}.$$

Let's move on to find an explicit formula for $M_2(k)$ and/or $R_2(t)$. That was already done by smart human John Littlewood ([5]) but let's do it again.

Once again, let's use the defining recurrence for the Rudin–Shapiro polynomials, but let's abbreviate

$$a(k)(z) = P_k(z) \quad , \quad b(k)(z) = P_k(-z) \quad , \quad A(k)(z) = P_k(z^{-1}) \quad , \quad B(k)(z) = P_k(-z^{-1}).$$

We have

$$P_k(z)^2 P_k(z^{-1})^2 = \left(P_{k-1}(z^2) + zP_{k-1}(-z^2) \right)^2 \cdot \left(P_{k-1}(z^{-2}) + z^{-1}P_{k-1}(-z^{-2}) \right)^2.$$

Expanding, discarding odd terms, replacing z^2 by z, and using trivial symmetries due to the fact that the functional CT is preserved under the dihedral group $\{z \to z, z \to -z, z \to z^{-1}, z \to -z^{-1}\}$, we get that

$$CT\,[a(k)^2A(k)^2] = 2CT[a(k-1)^2A(k-1)^2\,] - 2CT\,[za(k-1)^2B(k-1)^2] \\ + 4CT[a(k-1)A(k-1)b(k-1)B(k-1)\,].$$

The first term is an old friend, our quantity of interest with k replaced by $k-1$, but the other two are newcomers. So we do the same treatment to them. They in turn, may (and often do) introduce new quantities, but if all goes well, there would only be finitely many sequences, and we would get a **finite** system of first-order linear recurrences. This indeed happens, and one gets, for the generating functions of the encountered sequences, a system of six equations with six unknowns, and in particular, we get (in a split second, of course, we let Maple do it) that our desired object, the generating function of the sequence $CT\,[a(k)^2A(k)^2]$, alias, $R_2(t)$, is given by:

$$R_2(t) = \frac{4t+1}{(1+2t)(1-4t)} \quad = \frac{4}{3}\frac{1}{1-4t} - \frac{1}{3}\frac{1}{1+2t}.$$

By extracting the coefficient of t^k, we even get a nice explicit expression for $M_2(k)$, already known to Littlewood

$$M_2(k) = \frac{4}{3}4^k - \frac{1}{3}(-2)^k.$$

This can be done for *any* monomial

$$z^{\alpha_0}a(k)^{\alpha_1}A(k)^{\alpha_2}b(k)^{\alpha_3}B(k)^{\alpha_4}.$$

Define the sequence

$$E[\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4](k) := CT\,[\,z^{\alpha_0}a(k)^{\alpha_1}A(k)^{\alpha_2}b(k)^{\alpha_3}B(k)^{\alpha_4}\,].$$

Replacing $a(k), A(k), b(k), B(k)$ by their expressions in terms of $z, a(k-1)$, $A(k-1), b(k-1), B(k-1)$, expanding, discarding odd terms, replacing z^2 by z, and replacing each monomial by its *canonical form*, implied by the above-mentioned action of the dihedral group that preserves CT, we can express, each such $E[.]$, in terms of other $E[.]$'s evaluated at $k-1$. It is (presumably, they may some issues about powers of z) possible to show (and it has been done by Doche and Habsieger [DH], using a different approach) that this process terminates and eventually we will not get any new sequences, leaving us with a finite system of linear equations for the corresponding generating functions, that can be automatically solved, and lead to an expression in terms of a rational function, since we get a first-order system

$$\mathbf{F}(t) = \mathbf{v} + t\mathbf{A}\mathbf{F}(t),$$

(where $\mathbf{F}(t)$ is the vector of generating functions whose first component is our desired one), for some matrix $\mathbf{A}$, of integers that the computer finds automatically, and our object of desire is the first component of $\mathbf{F}(t) = (\mathbf{I} - t\mathbf{A})^{-1}\mathbf{v}$.

While it is painful for a human to do this, a computer does not mind, and the Maple package
`HaroldSilentShapiro.txt`
accompanying this article does it for any desired monomial in z, $P_k(z)$, $P_k(-z)$, $P_k(z^{-1})$, $P_k(-z^{-1})$. See the output files accompanying this article, that may be viewed from the front of this article
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hss.html`.

Unlike the beautiful approach of Doche and Habsieger, that uses clever human pre-processing to establish an *algorithm*, that was then hard-programmed by hand, our approach is naive "dynamical programming", where we don't make any *a priori* human analysis, and let the computer introduce new quantities as needed. To guarantee that it *halts*, we input a parameter, that we call `K`, and if the size of the system exceeds K it returns `FAIL`, leaving us the option to forget about it, or try again with a larger `K`.

Higher moments and Saffari's Conjecture

Now that we have reduced, for any specific positive integer n, the computation of the generating function of the sequence of moments $M_n(k)$, that we call $R_n(t)$, to a routine calculation, we can ask our beloved computer to crank-out as many of them as it can output in a reasonable amount of time. According to the output file
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oHaroldSilentShapiro1.txt`,
we get

$$R_1(t) = \frac{1}{1-2t} \quad ,$$

$$R_2(t) = \frac{1+4\,t}{(1\,+\,2\,t)\,(1\,-\,4\,t)} \quad ,$$

[both of which were already given above],

$$R_3(t) = \frac{1+16\,t}{(1+4\,t)\,(1-8\,t)} \quad ,$$

$$R_4(t) =$$

-(90194313216*t**11 - 15300820992*t**10 - 1979711488*t**9 - 292552704*t**8 - 22216704*t**7 + 10649600*t**6 - 1024*t**5 - 144384*t**4 + 7008*t**3 + 664*t**2 - 54*t - 1)/ ((8*t + 1)* (16*t - 1)*(1409286144*t**10 - 264241152*t**9 - 25690112*t**8 - 4128768*t**7 - 311296*t**6 + 170496*t**5 - 2624*t**4 - 2208*t**3 + 148*t**2 + 8*t - 1)),

$$R_5(t) =$$

- (369435906932736*t**11 - 32160715112448*t**10 - 2001454759936*t**9 - 145223581696*t**8 - 4454350848*t**7 + 1392508928*t**6 - 5865472*t**5 - 4599808*t**4 + 123648*t**3 + 4768*t**2 - 220*t - 1)/ ((1 + 16*t)*(32*t - 1)*(1443109011456*t**10 - 135291469824*t**9 - 6576668672*t**8 - 528482304* t **7 -19922944*t**6+5455872*t**5-41984*t**4-17664*t**3+592*t**2+16*t-1)).

To see $R_k(t)$ for $6 \leq k \leq 10$, look at the above-mentioned output file. Of course, one can easily go further. Note that these have already been computed in [3] (but their output is not easily accessible to the casual reader).

By looking at the smallest root of the denominator of $R_k(t)$ and computing the residue, one confirms for small (and not so small!) values of k (and one can easily go much further), the following conjecture of Bahman Saffari, as already done in [3] (for small k).

Saffari's Conjecture *For every positive integer n, as $k \to \infty$, the following asymptotic formula holds.*

$$M_n(k) \sim \frac{2^n}{n+1} \cdot (2^n)^k.$$

Saffari never published his conjecture, and it is mentioned as "private communication" in [3].

Sketch of a proof of Saffari's Conjecture

While for each *numeric* n, one can get an explicit expression, in *symbolic* t, for $R_n(t)$, these get more and more complicated as n gets larger, and there is (probably) no hope to get an explicit expression, in **symbolic** n, for $R_n(t)$, from which one can deduce that the smallest root (in absolute value) of the denominator is 2^{-n} and the residue is $\frac{2^n}{n+1}$.

But one can prove *rigorously* Saffari's conjectured asymptotic formula as follows.

Let n be a general (symbolic) positive integer. Recall that we are interested in the sequence

$$M_n(k) := CT[P_k(z)^n P_k(z^{-1})^n],$$

that we abbreviate

$$CT[\, a^n A^n \,],$$

under the convention

$$a = P_k(z), \quad b = P_k(-z), \quad A = P_k(z^{-1}), \quad B = P_k(-z^{-1}).$$

To get a scheme we use the *rewriting rules*, implied by the defining recurrence

$$a \to a + zb, \quad b \to a - zb, \quad A \to A + z^{-1}B, \quad B \to A - z^{-1}B,$$

where the discrete argument on the left is k and on the right $k - 1$, and the continuous argument on the left is z and on the right is z^2.

Using the binomial theorem, we have

$$a^n A^n \to (a + zb)^n (A + z^{-1}B)^n = \left(\sum_{i=0}^{n} \binom{n}{i} a^i (zb)^{n-i} \right) \left(\sum_{j=0}^{n} \binom{n}{j} A^j (z^{-1}B)^{n-j} \right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} a^i b^{n-i} A^j B^{n-j} z^{j-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i}^2 (aA)^i (bB)^{n-i} \quad + SmallChange,$$

where $SmallChange$ is a linear combination of unimportant monomials and we **define** an **important monomial** (in a, A, b, B, z) to be any member of the set of monomials

$$\{(aA)^m (bB)^{n-m} \,|\, 0 \leq m \leq n\}.$$

Let's try to find the "going down" evolution-step for the other important monomials.

We have

$$(aA)^m (bB)^{n-m} \to (a + zb)^m (A + z^{-1}B)^m (a - zb)^{n-m} (A - z^{-1}B)^{n-m}$$

$$= \left(\sum_{i_1=0}^{m} \binom{m}{i_1} a^{i_1} (zb)^{m-i_1} \right) \left(\sum_{i_2=0}^{m} \binom{m}{i_2} A^{i_2} (z^{-1}B)^{m-i_2} \right) \cdot$$

$$\left(\sum_{i_3=0}^{n-m} \binom{n-m}{i_3} a^{i_3} (-zb)^{n-m-i_3} \right) \left(\sum_{i_4=0}^{n-m} \binom{n-m}{i_4} A^{i_4} (-z^{-1}B)^{n-m-i_4} \right)$$

$$= \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{n-m} \sum_{i_4=0}^{n-m} \binom{m}{i_1} \binom{m}{i_2} \binom{n-m}{i_3} \binom{n-m}{i_4}$$
$$(-1)^{i_3+i_4} a^{i_1+i_3} A^{i_2+i_4} b^{n-i_1-i_3} B^{n-i_2-i_4} z^{i_2-i_1+i_4-i_3}.$$

The coefficient of a typical important monomial, $(aA)^r (bB)^{n-r}$ $(0 \leq r \leq n)$ in the above quadruple sum is

$$\sum_{i_1+i_3=r\,,\,i_2+i_4=r} (-1)^{i_3+i_4} \binom{m}{i_1}\binom{m}{i_2}\binom{n-m}{i_3}\binom{n-m}{i_4}$$

$$= \sum_{i_1=0}^{r}\sum_{i_2=0}^{r}(-1)^{i_1+i_2}\binom{m}{i_1}\binom{m}{i_2}\binom{n-m}{r-i_1}\binom{n-m}{r-i_2}$$

$$= \left(\sum_{i_1=0}^{r}(-1)^{i_1}\binom{m}{i_1}\binom{n-m}{r-i_1}\right)\left(\sum_{i_2=0}^{r}(-1)^{i_2}\binom{m}{i_2}\binom{n-m}{r-i_2}\right)$$

$$= \left(\sum_{i=0}^{r}(-1)^{i}\binom{m}{i}\binom{n-m}{r-i}\right)^2 .$$

This is an important quantity, so let's give it a name

$$K_n(m,r) := \left(\sum_{i=0}^{r}(-1)^{i}\binom{m}{i}\binom{n-m}{r-i}\right)^2 .$$

All the remaining monomials belong to *SmallChange*, and we have the general "evolution equation"

$$(aA)^m(bB)^{n-m} \to \sum_{r=0}^{n} K_n(m,r)(aA)^r(bB)^{n-r} + \quad SmallChange.$$

Assuming for now, that *SmallChange* is, asymptotically less than the "important monomials" (i.e. the rate of growth of a small change sequence divided by an "important monomial" sequence is $o(1)$), let α_n be the largest eigenvalue of the $n+1$ by $n+1$ matrix K_n (whose (m,r) entry is $K_n(m,r)$), then for $0 \le m \le n$

$$CT[(P_k(z)P_k(z^{-1}))^m(P_k(-z)P_k(-z^{-1}))^{n-m}] \sim c_m(\alpha_n)^k,$$

where $(c_0,\ldots,c_n)$ is an eigenvector corresponding to the largest eigenvalue, α_n.

We now need two elementary propositions that should be provable using the **Wilf–Zeilberger algorithmic proof theory**, (a suitable extension of [4, 12] for the first, [1, 11] for the second). They may be even provable by purely human means, but since we know *for sure* that they are both true, we do not bother.

Proposition 1. *The characteristic polynomial,* $\det(z\mathbf{I}-K_n)$*, of the* $(n+1)\times(n+1)$ *matrix* K_n *whose* (m,r) *entry is*

$$K_n(m,r) := \left(\sum_{i=0}^{r}(-1)^{i}\binom{m}{i}\binom{n-m}{r-i}\right)^2 ,$$

equals

$$\det(z\mathbf{I} - K_n) = z^{\lfloor (n+1)/2 \rfloor} \prod_{j=0}^{\lfloor n/4 \rfloor} \left(z - 2^{n-4j} \binom{4j}{2j} \right) \prod_{j=0}^{\lfloor (n-2)/4 \rfloor} \left(z + 2^{n-4j-2} \binom{4j+2}{2j+1} \right).$$

[To confirm this *shaloshable* determinant identity for $n \le N$, type, in the Maple package `HaroldSilentShapiro.txt`, `CheckCP(N);` . For example, `CheckCP(20);` returns `true` in one second, and `CheckCP(40);` returns `true` in 20 seconds.]

So the non-zero eigenvalues of the matrix K_n are

$$\{ \, 2^{n-4j} \binom{4j}{2j} \; ; \; 0 \le j \le \lfloor n/4 \rfloor \, \} \bigcup \{ \, -2^{n-4j-2} \binom{4j+2}{2j+1} \; ; \; 0 \le j \le \lfloor (n-2)/4 \rfloor \, \}.$$

In particular, the largest eigenvalue (in absolute value) is indeed 2^n. We also need the following *shaloshable* binomial coefficients identity.

Proposition 2. *The vector* $(c_0, \ldots, c_n)$ *defined by* $c_r = \binom{n}{r}^{-1}$ $(0 \le r \le n)$ *is an eigenvector of the matrix* K_n *corresponding to its largest eigenvalue* 2^n *(with multiplicity* 1*). In other words, for* $0 \le m \le n$

$$\sum_{r=0}^{n} K_n(m, r) c_r = 2^n c_m.$$

[To confirm this *shaloshable* binomial coefficient identity for $n \le N$, type, in the Maple package `HaroldSilentShapiro.txt`, `CheckEV(N);` . For example, `CheckEV(50);` returns `true` in two seconds, and `CheckCP(100);` returns `true` in 30 seconds.]

But an eigenvector is only determined up to a constant multiple. Let's find it (modulo the Small Change hypothesis). We know that

$$CT \, [\, (P_k(z) P_k(z^{-1}))^m (P_k(-z) P_k(-z^{-1}))^{n-m} \,] \sim \frac{C}{\binom{n}{m}} \cdot (2^n)^k,$$

for *some* constant C. To find it, we use the well-known, and easily proved identity (see [10])

$$P_k(z) P_k(z^{-1}) + P_k(-z) P_k(-z^{-1}) = 2^{k+1}.$$

Raising it to the n-th power, using the binomial theorem, and taking the constant term, we have

$$\sum_{m=0}^{n} \binom{n}{m} CT[(P_k(z)P_k(z^{-1}))^m (P_k(-z)P_k(-z^{-1}))^{n-m}] = 2^{(k+1)n}.$$

Hence

$$\sum_{m=0}^{n} \binom{n}{m} \frac{C}{\binom{n}{m}} \cdot (2^n)^k \quad = 2^{(k+1)n},$$

that implies that

$$C = \frac{2^n}{n+1}.$$

We just established

Proposition 3. *Modulo the Small Change Hypothesis, for* $0 \le m \le n < \infty$

$$CT[(P_k(z)P_k(z^{-1}))^m (P_k(-z)P_k(-z^{-1}))^{n-m}] \sim \frac{2^n}{(n+1)\binom{n}{m}} \cdot (2^n)^k.$$

In particular, taking $m = n$, *we get Saffari's conjecture (for even moments)*

$$M_n(k) = CT[P_k(z)^n P_k(z^{-1})^n] \sim \frac{2^n}{n+1} \cdot (2^n)^k.$$

Towards a Proof of the Small Change Hypothesis

It would have been great if the "children" of each unimportant monomial, in the evolution equation described above (implemented in procedure `GD` in our Maple package), would all be unimportant. Then we could have easily proved, by induction that, asymptotically, they are insignificant compared to the important monomials. It turns out that for *most* unimportant monomials, this is indeed the case, but there are a few, that we call *false pretenders* that do have important children.

It should not be hard to fully characterize these. In fact it turns out (empirically, for now) that for n even there are $(n/2)^2 - 1$ of them, and for n odd there are $(n^2 - 1)/4$. Then for those false pretenders one should be able to describe all their important children, and then prove that the leading terms of their contributions cancel out (using the inductive hypothesis, and Prop. 3).

This has been verified empirically up to $n \le 16$. See procedures `Medio` and `MedioP` in the Maple package `HaroldSilentShapiro.txt`.

Hugh Montgomery's Stronger Conjecture

In [6], Hugh Montgomery considered the more general sequences

$$M_{m,n}(k) := CT\ [\ P_k(z)^m P_k(z^{-1})^n\ ].$$

He conjectured that, for $m \neq n$,

$$M_{m,n}(k) = o(\ 2^{(m+n)k/2}\ ).$$

Once again, the generating function, for each specific m and n, is always a rational function, and our Maple package (procedure `RS(m,n,t,K)`) computes them, and procedure `MamarH(N,K,t)` prints out an article confirming Hugh Montgomery's conjecture, as well as giving the generating functions for $1 \leq m < n \leq N$. (`K` is a parameter that should be made large enough, say 1000).

To see the output for $1 \leq m < n \leq 7$, go to: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oHaroldSilentShapiro2.txt`, that contains the explicit expressions for all these cases, and confirms Montgomery's conjecture with a vengeance. Unlike the $m = n$ case, the smallest root (alias the reciprocal of the largest eigenvalue) is not "nice", and there are usually several roots with smallest absolute value, hence the sequences often oscillate. Nevertheless, Montgomery's conjecture is true for all $1 \leq m < n \leq 7$, and one could go much further.

Let's Generalize!

The same approach works for *any* sequence of Laurent polynomials defined by a recurrence of the form

$$P_k(z) = C_1(z)P_{k-1}(z^r) + C_2(z)P_{k-1}(-z^r) + C_3(z)P_{k-1}(z^{-r}) + C_4(z)P_{k-1}(-z^{-r}),$$

with the initial condition $P_0(z) = 1$, where $C_1(z)$, $C_2(z)$, $C_3(z)$, $C_4(z)$ are Laurent polynomials of degree less than r and low-degree larger than $-r$, for *any* positive integer r larger than 1.

One always gets a finite scheme (disclaimer: we don't have a rigorous proof, but we believe that such a proof exists, at any rate, it is true in all the cases that we tried out) and hence a rational generating function for the sequence

$$S[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4](k) := CT\left[z^{\alpha_0} P_k(z)^{\alpha_1} P_k(z^{-1})^{\alpha_2} P_k(-z)^{\alpha_3} P_k(-z^{-1})^{\alpha_4}\right],$$

for any non-negative $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$. This is implemented in the Maple package `ShapiroGeneral.txt` also available from the webpage of this article, or directly from http://www.math.rutgers.edu/~zeilberg/tokhniot/Shapiro General.txt.

Let's (not!) Generalize Even More!

The set $\{1, -1\}$ is a multiplicative subgroup of the field of complex numbers. For any finite multiplicative subgroup G of the field of complex numbers, and any positive integer r larger than 1, the same approach should be able to handle sequences of polynomials given by a recurrence

$$P_k(z) = \sum_{g \in G} \alpha_g(z) P_{k-1}(gz^r) + \sum_{g \in G} \beta_g(z) P_{k-1}(gz^{-r}), \quad P_0(z) = 1,$$

where $\alpha_g(z)$, $\beta_g(z)$ are $2|G|$ given Laurent polynomials in z of degree $< r$ and low-degree $> -r$.

This includes the case treated in [2], where G is a cyclotomic group.

We could go even further, with *higher order* recurrences (as opposed to only first order), several continuous variables (as opposed to only z), and, presumably, even several discrete variables (as opposed to only k), but *enough is enough!*

Added May 27, 2016: Brad Rodgers, independently, and simultaneously, found a (complete) proof of Saffari's conjecture, that he is writing up now and will soon post in the arxiv. Meanwhile, you can read his proof in a letter posted out in http://www.math.rutgers.edu/~zeilberg/mamarim/mamarim html/BradleyRodgersLetter.pdf.

Added June 7, 2016: Brad Rodgers' beautiful paper, that also proves the more general Montgomery conjecture, mentioned above, is now available here: http://arxiv.org/abs/1606.01637.

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