Three-Colored Partitions and Dilated Companions of Capparelli's Identities

Kathrin Bringmann and Karl Mahlburg

In honor of Krishna Alladi, who has been a great inspiration, for the celebration of his 60th birthday

Abstract Capparelli's partition identities state that certain gap restrictions on partitions into distinct parts are equinumerous with congruential restrictions modulo 12. Subsequently, generalizations to higher moduli were proven by Alladi, Andrews, and Gordon by means of hypergeometric q-series, and by Meurman and Primc using the vertex operator algebra program of Lepowsky and Wilson. Furthermore, these generalized families arise as specializations of underlying identities for three-colored partitions. In this paper, we continue our investigation of companions to Capparelli's identities, and prove two new general identities for three-colored partitions that specialize to Jacobi theta functions and false theta functions.

Keywords False theta functions · Integer partitions · Capparelli's identities

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K. Bringmann
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: kbringma@math.uni-koeln.de

K. Mahlburg (✉)
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
e-mail: mahlburg@math.lsu.edu

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1 Introduction and statement of results

This paper continues the study of Capparelli's partition identities [7] from the perspective of hypergeometric q-series, automorphic forms, and the combinatorial theory of integer partitions. Capparelli's identities are notable because they were the first new examples that were discovered using Lepowsky and Wilson's vertex operator algebras, which were famously introduced in [13] as a method for explicitly constructing affine Lie algebras. This framework was further developed in [14] to include Z-algebras. Our current investigation combines ideas from [1, 3, 5] in order to prove new generalized identities that relate Capparelli's work to false theta functions, Jacobi forms, and multi-colored partitions.

Capparelli's identities first arose conjecturally in [7] (see also his Ph.D. thesis [6]), where he used Lepowsky and Wilson's Z-algebra program [14] to construct the level 3 standard modules for $A_2^{(2)}$. The identities were proven shortly thereafter in a number of independent works; Andrews [3] and Andrews, Alladi, and Gordon [1] gave proofs using the theory of hypergeometric q-series, while the proofs of Tamba and Xie [20] and Capparelli himself [8] used Z-algebras. Subsequently, there has been a great deal of additional progress on the combinatorial implications of vertex-operator-theoretic techniques; for example, see [12, 15, 17] for a small sampling.

Indeed, as our present focus is on multi-parameter generalizations and/or dilated identities, we do not state Capparelli's identities as originally presented in [7], but rather a generalization due to Meurman and Primc [15], which also follows from Alladi, Andrews, and Gordon's results in [1]. For an integer partition λ, define indicator functions ψ_j such that $\psi_j(\lambda) = 1$ if j is a part of λ, and $\psi_j(\lambda) = 0$ otherwise. We frequently suppress the argument unless it is important to distinguish a particular partition. Suppose that $d \geq 3$ and $1 \leq \ell < d/2$. Loosely following the terminology and notation from [5, 7, 15], we say that a partition λ satisfies the (d, ℓ)-*dilated gap condition* if for all $j \in \mathbb{N}$,

$$\begin{aligned} \psi_{(j+1)d-\ell} + \psi_{jd} + \psi_{jd-\ell} &\leq 1, \\ \psi_{jd+\ell} + \psi_{jd} + \psi_{(j-1)d-\ell} &\leq 1, \\ \psi_{jd-\ell} + \psi_{(j-1)d+\ell} &\leq 1, \end{aligned} \tag{1.1}$$

and $\psi_n = 0$ if $n \not\equiv 0, \pm\ell \pmod{d}$. This system of inequalities is the special case $k = 1$, $s_0 = \ell$, and $s_1 = d - \ell$ of (11.2.6) in [15], which describes the partition ideals that arise from root lattices. Note that (11.2.6) of [15] is actually a system of four inequalities, but in the special case $k = 1$ it is overdetermined and reduces to the above. Capparelli's original identities correspond to $(d, \ell) = (3, 1)$, and in that case the conditions in (1.1) are equivalently characterized by requiring that the successive parts in a partition differ by at least 2, and two parts differ by 2 or 3 only if their sum is a multiple of 3.

In order to state the identities of Capparelli and Meurman–Primc, we also require enumeration functions for the partitions described above. For $\alpha, \beta \in \{0, 1\}$, let $c_{\alpha,\beta}^{d,\ell}(n)$ denote the number of partitions of n that satisfy the (d, ℓ)-gap condition with the further restriction that $\psi_\ell \le \alpha$ and $\psi_{d-\ell} \le \beta$. We write the corresponding generating functions as

$$\mathscr{C}_{\alpha,\beta}^{d,\ell}(q) := \sum_{n\ge 0} c_{\alpha,\beta}^{d,\ell}(n)q^n = \sum_{\substack{\lambda \text{ satisfies } (d,\ell)\text{-gap condition} \\ \lambda_\ell \le \alpha,\ \lambda_{d-\ell}\le\beta}} q^{|\lambda|}.$$

Here $|\lambda|$ denotes the *size* of a partition λ, which is the sum of its parts. Meurman and Primc's generalized identity is now stated as follows.

Theorem (Lemma 2 in [1]; equations (11.1.5)–(11.1.6) in [15]). For $d \ge 3$, we have

$$\mathscr{C}_{0,1}^{d,\ell}(q) = \frac{\prod_{n\ge 0}\left(1+q^{(2n+1)d-\ell}\right)\left(1+q^{(2n+1)d+\ell}\right)}{\prod_{n\ge 0}\left(1-q^{(2n+1)d}\right)}, \tag{1.2}$$

$$\mathscr{C}_{1,0}^{d,\ell}(q) = \frac{\prod_{n\ge 0}\left(1+q^{2(n+1)d-\ell}\right)\left(1+q^{2nd+\ell}\right)}{\prod_{n\ge 0}\left(1-q^{(2n+1)d}\right)}. \tag{1.3}$$

Remarks. 1. In fact, there is a version of Meurman and Primc's result that also holds for $d = 1$ or 2, although the partition combinatorics from (1.1) are no longer the correct formulation, and instead require multiple colors. This becomes clearer from the statement of our main results below.

2. The results in [1] are more general than the theorem statement, as they include additional parameters that distinguish between parts based on residue classes modulo d. Equation (1.2) follows from (5.2) in [1] by setting $q \mapsto q^d$, $a \mapsto q^{-d+\ell}$, and $b \mapsto q^{-d-\ell}$, and (1.3) from $q \mapsto q^d$, $a \mapsto q^{-\ell}$, and $b \mapsto q^{-2d+\ell}$. These parameters are discussed further in the sequel.

3. The above theorem is not the original combinatorial formulation of Capparelli's identities, but it is straightforward to show that the product expression for $\mathscr{C}_{0,1}^{3,1}(q)$ also enumerates the number of partitions of n into parts congruent to $\pm 2, \pm 3 \pmod{12}$, as in Theorem 21 A of [7]. Note that this product is also equivalently stated in the unnumbered equation following (5.2) in [1].

Our investigation in [5] was motivated by the observation that (1.2) and (1.3) are *modular* identities, in the sense that the right-hand sides are (essentially) weakly holomorphic modular forms. Furthermore, the refinements of Capparelli's results in [1] and [3] are of additional number-theoretic interest due to the presence of an additional parameter. In order to describe the refined identities, let $\nu_{d,j}(\lambda)$ be the number of parts of λ that are congruent to j modulo d, and define the generating functions

$$\mathscr{C}_{\alpha,\beta}^{d,\ell}(t;q) := \sum_{\substack{\lambda \text{ satisfies } (d,\ell)\text{-gap condition} \\ \lambda_\ell \le \alpha,\ \lambda_{d-\ell}\le\beta}} t^{\nu_{d,\ell}(\lambda)-\nu_{d,d-\ell}(\lambda)} q^{|\lambda|}. \tag{1.4}$$

Throughout the remainder of the paper, we adopt the standard q-factorial notation for $a \in \mathbb{C}$ and $n \in \mathbb{N}_0 \cup \{\infty\}$, namely $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. We also use the additional shorthand $(a_1,\ldots,a_r;q)_n := (a_1;q)_n \cdots (a_r;q)_n$. The *Jacobi theta function* is defined by

$$\theta(z;q) := \left(-z, -z^{-1}q, q; q\right)_\infty = \sum_{k\in\mathbb{Z}} z^k q^{\frac{k(k-1)}{2}}, \tag{1.5}$$

where the final equality follows from Jacobi's Triple Product identity ((2.2.10) in [4]). This function is essentially a holomorphic *Jacobi form*, as described in the seminal work of Eichler and Zagier [9]. Finally, define the shifted Dirichlet character $\chi_3(m) := \left(\frac{m+1}{3}\right)$, and let

$$\begin{aligned} T_1(t;q) &:= \sum_{n\geq 0} \chi_3(n) t^{-n} q^{\frac{n(n+2)}{3}}, \\ T_2(t;q) &:= \sum_{n\geq 0} \chi_3(n) t^{n} q^{\frac{n(n-1)}{3}}. \end{aligned}$$

There has been a great deal of recent work illuminating the connections between "false" theta functions such as the T_j and classical automorphic forms, particularly through the theory of *quantum modular forms*, as in [11] and [21].

The main result in [5] demonstrates the role of these functions in identities related to Capparelli's results.

Theorem 1.1 **([5], Theorem 3.1).** *If* $\alpha, \beta \in \{0,1\}$, *then*

$$\begin{aligned} \mathscr{C}^{3,1}_{\alpha,\beta}(t;q) =& (\alpha+\beta-1)\Big(-q^3;q^3\Big)_\infty \theta\left(-t^2q^2;q^6\right) \\ &+ \frac{\theta\left(tq^4;q^6\right)}{\left(q^3;q^3\right)_\infty}\Big(\beta + (1-\alpha-\beta)T_1\Big(tq;q^3\Big)\Big) \\ &+ \frac{\theta\left(tq;q^6\right)}{\left(q^3;q^3\right)_\infty}\Big(\alpha + (1-\alpha-\beta)T_2\Big(tq;q^3\Big)\Big). \end{aligned} \tag{1.6}$$

Remark. This theorem includes Capparelli's original identities [7], which correspond to the two cases where $\alpha+\beta=1$. In particular, in these cases (1.6) simplifies to the products (1.2) and (1.3) with $d=3$ and $\ell=1$.

Remark. We note that in [19] Sills proved a one-parameter generalization of an "analytic counterpart" to Capparelli's identities, using Bailey chains to obtain interesting hypergeometric q-series representations for infinite products related to the case $(d,\ell)=(3,1)$ in (1.2) and (1.3).

The main automorphic result of this paper extends (1.6) to an arbitrary modulus, providing a general family of identities that imply (1.2) and (1.3).

Theorem 1.2. *For* $\alpha, \beta \in \{0, 1\}$,

$$\mathscr{C}^{d,\ell}_{\alpha,\beta}(t;q) = (\alpha+\beta-1)\theta\left(-t^2q^{2\ell};q^{2d}\right)\left(-q^d;q^d\right)_\infty + \frac{\theta\left(tq^{d+\ell};q^{2d}\right)}{(q^d;q^d)_\infty}\left(\beta+(1-\alpha-\beta)T_1\left(tq^\ell;q^d\right)\right) + \frac{\theta\left(tq^{\ell};q^{2d}\right)}{(q^d;q^d)_\infty}\left(\alpha+(1-\alpha-\beta)T_2\left(tq^\ell;q^d\right)\right).$$

In fact, this theorem statement is a specialization of a more general result that we prove for three-colored partitions with gap restrictions; see Theorem 4.1. The general result is inspired by Section 5 of [1], where the authors studied three-colored partitions using the "method of weighted words" to obtain multi-parameter generalizations of (1.2) and (1.3). Our generalizations are of a different shape, as we instead use the analytic theory of q-difference equations and hypergeometric q-series. Identities for three-colored partitions also arise in [16], where the basic $A_2^{(1)}$-module is constructed using vertex operator methods.

The remainder of the paper is structured as follows. Section 2 consists of a brief review of classical results from the theory of hypergeometric q-series. This is followed by proofs of combinatorial results and finite generating series for three-colored partitions in Section 3. We conclude in Section 4 by evaluating the infinite limiting cases, thereby proving Theorem 1.2.

2 Hypergeometric q-series identities

In this section, we record a number of identities that are useful in the evaluation of the generating functions that are the main topic of this paper. If $0 \le m \le n$, the *q-binomial coefficient* is denoted by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

We also need the limiting case

$$\lim_{n\to\infty}\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1}{(q;q)_m}. \tag{2.1}$$

Next, we recall two identities due to Euler, which state (see (2.2.5) and (2.2.6) in [4])

$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n}, \tag{2.2}$$

$$(x;q)_\infty = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}. \tag{2.3}$$

A related summation formula is

$$\sum_{\substack{n\geq 0\\ n \text{ even}}} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} = \frac{1}{\left(q;q^2\right)_\infty} = (-q;q)_\infty\,; \tag{2.4}$$

the first equality follows from Cauchy's identity, which is (2.2.8) in [4].

We also need a result from Ramanujan's famous "Lost Notebook", which appears as (4.1) in [2]:

$$\sum_{n\geq 0} \frac{q^n}{(-aq;q)_n(-bq;q)_n} = \left(1+a^{-1}\right) \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}} \left(\frac{b}{a}\right)^n}{(-bq;q)_n} - \frac{a^{-1}\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}} \left(\frac{b}{a}\right)^n}{(-aq,-bq;q)_\infty}. \tag{2.5}$$

Finally, in order to derive expressions involving false theta functions, we recall a related identity of Rogers [18] (equation (3) on page 335), which states that

$$\sum_{n\geq 0} \frac{(-1)^n y^{2n} q^{\frac{n(n+1)}{2}}}{(yq;q)_n} = \sum_{n\geq 0} (-1)^n y^{3n} q^{\frac{n(3n+1)}{2}} \left(1-y^2q^{2n+1}\right).$$

In fact, we need a one-parameter generalization of Rogers' identity, which follows from Fine's systematic study of hypergeometric functions in [10].

Lemma 2.1. *We have*

$$\sum_{n\geq 0} \frac{(-1)^n (bx)^n q^{\frac{n(n+1)}{2}}}{(bq;q)_n} = \sum_{n\geq 0} \frac{(xq;q)_n}{(bq;q)_n} \left(-xb^2\right)^n q^{\frac{n(3n+1)}{2}} \left(1-bxq^{2n+1}\right).$$

Proof. We use Fine's notation for the basic hypergeometric series, namely

$$F(a,b;t) := \sum_{n\geq 0} \frac{(aq;q)_n t^n}{(bq;q)_n}.$$

The left-hand side of the lemma statement may be expressed as a limit of Fine's function, since

$$\sum_{n\geq 0}\frac{(-1)^n(bx)^nq^{\frac{n(n+1)}{2}}}{(bq;q)_n}=\lim_{a\to\infty}F\left(ab,b;\frac{x}{a}\right).$$

By (6.3) of [10], this expression transforms to

$$\lim_{a\to\infty}F\left(ab,b;\frac{x}{a}\right)=\lim_{a\to\infty}\frac{1-b}{1-\frac{x}{a}}F\left(x,\frac{x}{a};b\right)=(1-b)F\left(x,0;b\right). \tag{2.6}$$

By the Rogers-Fine identity (see (14.1) of [10]), (2.6) becomes

$$(1-b)\lim_{w\to 0}F(x,w;b)=\sum_{n\geq 0}\frac{(xq;q)_n}{(bq;q)_n}\left(1-xbq^{2n+1}\right)b^nq^{n^2}\lim_{w\to 0}\left(\frac{xbq}{w};q\right)_n w^n.$$

Evaluating the limit completes the proof.

3 Three-colored partitions and finite recurrences

In this section, we combine ideas from [1, 3, 5] and introduce certain three-colored partitions with gap restrictions that are related to generalizations of Capparelli's identities. The combinatorics of the partition colorings are inspired by Sections 5 and 6 of [1], where the method of weighted words was used in order to evaluate the corresponding generating functions. However, we instead use techniques from [3], which were further adapted in [5] in order to find hypergeometric q-series solutions to the appropriate q-difference equations.

3.1 Colored partitions

For an integer partition, we write the parts of a partition in nonincreasing order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. For the remainder of the paper we consider three-colored partitions into distinct parts with gap restrictions. In particular, if j is a part of a partition λ, then it is given one of three colors, a, b, or c; a part of size j and color k is denoted by j_k. When writing the parts of a partition they are ordered by both size and color, according to the sequence

$$1_a \prec 1_b \prec 1_c \prec 2_a \prec 2_b \prec 2_c \prec \dots. \tag{3.1}$$

Note that this is slightly different than the ordering in Section 5 of [1]; to compare the two, we have effectively shifted all of the parts with color b by 1.

We say that a three-colored partition satisfies the Capparelli *gap conditions* if it is in the subset

$$\mathscr{R} := \Big\{\lambda \vdash n \ \mid \ \text{distinct parts } \lambda_j, \ \text{with color } k_{\lambda_j} \in \{a, b, c\},$$
$$\text{and for consecutive integer parts } (j+1), \ j \in \lambda, \ k_{j+1}k_j \neq aa, bb, ac, \text{ or } bc\Big\}.$$

In other words, a 3-colored partition into distinct parts λ is in $\mathscr{R}$ if

$$\lambda_r - \lambda_{r+1} \geq A\Big(k_{\lambda_{r+1}}, k_{\lambda_r}\Big), \tag{3.2}$$

where A is the following matrix (indexed in order by rows and columns; note that λ_{r+1} is smaller than λ_r).

		k_{λ_r}		
	A	a	b	c
	a	2	1	1
$k_{\lambda_{r+1}}$	b	1	2	1
	c	2	2	1

For example, the second row implies that if $j_b \in \lambda$, then the next largest part cannot be j_c or $(j+1)_b$, but any of $(j+1)_a$, $(j+1)_c$, $(j+2)_a$, $(j+2)_b$, or larger, are allowed.

3.2 Finite generating functions

Extending the notation from Section 1 to include colored parts, we define indicator functions

$$\psi_{m_k}(\lambda) := \begin{cases} 1 & \text{if } m_k \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\nu_k(\lambda)$ count the number of parts of λ with color k; we see below that the $\nu_{d,\ell}$ from Section 1 (see (1.4)) corresponds to certain specializations.

For $\alpha, \beta \in \{0, 1\}$, $M \geq 1$, and $k \in \{a, b, c\}$, define the bounded generating functions

$$\begin{aligned} F(M_k) &= F_{\alpha,\beta}(M_k; A, B; q) \\ &:= \sum_{\substack{\lambda \in \mathscr{R} \\ \lambda_j \leq M_k \text{ for all } j}} \big(1 - (1-\alpha)\psi_{1_a}(\lambda)\big)\big(1 - (1-\beta)\psi_{1_b}(\lambda)\big) A^{\nu_a(\lambda)} B^{\nu_b(\lambda)} q^{|\lambda|}. \end{aligned} \tag{3.3}$$

Note that the indicator functions have the effect of limiting the number of occurrences of 1_a to at most α, and the occurrences of 1_b to at most β. It also important to note that there is no variable associated with the color c; to our knowledge, none of the known results related to Capparelli's identities generalize to this degree.

Following Sections 4 of [3] and [1], by conditioning on the largest part, we easily find the recurrences

$$F(n_b) = F(n_a) + Bq^n F((n-1)_a), \tag{3.4}$$
$$F(n_c) = F(n_b) + q^n F((n-1)_c), \tag{3.5}$$
$$F((n+1)_a) = F(n_c) + Aq^{n+1}\Big(F((n-1)_c) + Bq^n F((n-1)_a)\Big). \tag{3.6}$$

For example, on the right-side of (3.4), the first term corresponds to the case that n_b does not occur, so the largest part is at most n_a, and the second term the case that n_b does occur, so that the next part is at most $(n-1)_a$.

Moreover, by writing down the first several partitions in $\mathscr{R}$, we directly calculate the initial values

$$\begin{aligned} F(1_a) &= 1 + \alpha Aq, \\ F(1_b) &= 1 + \alpha Aq + \beta Bq, \\ F(1_c) &= 1 + \alpha Aq + \beta Bq + q, \\ F(2_a) &= 1 + \alpha Aq + \beta Bq + q + Aq^2(1 + \beta Bq). \end{aligned} \tag{3.7}$$

Furthermore, it is convenient to have a value for $F(0_a)$, which can be obtained in a consistent manner by plugging $n = 1$ in to (3.4) and working in reverse. In particular, combined with (3.7), this implies that $F(0_a) = \beta$.

We now manipulate the system (3.4)–(3.6) in order to obtain a recurrence involving only one color. Isolating $F(n_b)$ in (3.4) and (3.5) yields

$$F(n_c) - q^n F((n-1)_c) = F(n_a) + Bq^n F((n-1)_a), \tag{3.8}$$

and rearranging the terms of (3.6) gives

$$F(n_c) + Aq^{n+1} F((n-1)_c) = F((n+1)_a) - ABq^{2n+1} F((n-1)_a). \tag{3.9}$$

Taking Aq times (3.8) and adding it to (3.9) then results in

$$\begin{aligned} &(Aq+1)F(n_c) \\ &\quad = AqF(n_a) + ABq^{n+1}F((n-1)_a) + F((n+1)_a) - ABq^{2n+1}F((n-1)_a). \end{aligned} \tag{3.10}$$

Similarly, subtracting (3.8) from (3.9) gives

$$\begin{aligned} &q^n(Aq+1)F((n-1)_c) \\ &= F((n+1)_a) - ABq^{2n+1}F((n-1)_a) - F(n_a) - Bq^n F((n-1)_a). \end{aligned} \tag{3.11}$$

We now shift $n \mapsto n-1$ in (3.10) and plug this in to (3.11), which yields an equality involving only the color a, namely

$$q^n\Big(AqF\left((n-1)_a\right)+F\left(n_a\right)+ABq^n\Big(1-q^{n-1}\Big)F\left((n-2)_a\right)\Big)$$
$$=F\left((n+1)_a\right)-F\left(n_a\right)-Bq^n\Big(1+Aq^{n+1}\Big)F\left((n-1)_a\right).$$

Regrouping terms, we finally have the single recurrence

$$F\left((n+1)_a\right)=\left(1+q^n\right)F\left(n_a\right)+\Big(Aq^{n+1}+Bq^n+ABq^{2n+1}\Big)F\left((n-1)_a\right)$$
$$+ABq^{2n}\Big(1-q^{n-1}\Big)F\left((n-2)_a\right). \tag{3.12}$$

By introducing an auxiliary variable and constructing a generating function for the $F(n_a)$, the problem can now be translated to a q-difference equation, which is then amenable to techniques from the theory of hypergeometric q-series. The details are carried out in the sequel.

3.3 Hypergeometric q-series solution

We close this section by solving the recurrence (3.12), thereby finding a hypergeometric q-series expression for the three-colored partitions satisfying Capparelli's gap condition. We begin by setting

$$\gamma_n := \frac{F\left(n_a\right)}{(q;q)_n}, \tag{3.13}$$

and then shift $n \mapsto n-1$ in (3.12), which implies that, for $n \geq 3$,

$$\gamma_n=\frac{1+q^{n-1}}{1-q^n}\gamma_{n-1}+\frac{Aq^n+Bq^{n-1}+ABq^{2n-1}}{\left(1-q^{n-1}\right)\left(1-q^n\right)}\gamma_{n-2}+\frac{ABq^{2n-2}}{\left(1-q^{n-1}\right)\left(1-q^n\right)}\gamma_{n-3}. \tag{3.14}$$

Note that the initial conditions are given by

$$\gamma_0=\beta, \quad \gamma_1=\frac{1+\alpha Aq}{1-q}, \quad \gamma_2=\frac{1+\alpha Aq+\beta Bq+q+Aq^2\left(1+\beta Bq\right)}{\left(1-q\right)\left(1-q^2\right)}. \tag{3.15}$$

We then further rewrite (3.14) as

$$\left(1-q^{n-1}\right)\left(1-q^{n}\right)\gamma_n = \left(1-q^{2n-2}\right)\gamma_{n-1} + \left(Aq^{n} + Bq^{n-1} + ABq^{2n-1}\right)\gamma_{n-2} + ABq^{2n-2}\gamma_{n-3}, \tag{3.16}$$

from which we next derive a q-difference equation.

In order to convert the above recurrence to a series relation, for $m \in \mathbb{N}_0$ we define the (shifted) generating functions

$$G^{(m)}(z) := \sum_{n\geq m} \gamma_n z^n, \tag{3.17}$$

and we also set $G := G^{(0)}$. Multiplying (3.16) by z^n and summing over $n \geq 3$, we obtain

$$\begin{aligned} &G^{(3)}(z) - G^{(3)}(zq) - q^{-1}G^{(3)}(zq) + q^{-1}G^{(3)}\left(zq^2\right) \\ &= z\left(G^{(2)}(z) - G^{(2)}\left(zq^2\right)\right) + z^2\left(\left(Aq^2+Bq\right)G^{(1)}(zq) + ABq^3G^{(1)}\left(zq^2\right)\right) \\ &\qquad + z^3ABq^4G\left(zq^2\right). \end{aligned} \tag{3.18}$$

A short calculation shows that after adding back in the boundary terms and plugging in (3.15), (3.18) simplifies to

$$\begin{aligned} (1-z)G(z) =& \left(1+q^{-1}+z^2\left(Aq^2+Bq\right)\right)G(zq) \\ &- q^{-1}(1+zq)\left(1-z^2q^4AB\right)G\left(zq^2\right) + z^2q^2A(1-\alpha-\beta). \end{aligned}$$

The order of this q-difference equation is reduced by re-normalizing the generating function, so we set

$$H(z) := \frac{G(z)}{(-z;q)_\infty}. \tag{3.19}$$

Then we have

$$\begin{aligned} \left(1-z^2\right)H(z) =& \left(1+q^{-1}+Aq^2z^2+Bqz^2\right)H(zq) - q^{-1}\left(1-z^2q^4AB\right)H\left(zq^2\right) \\ &+ A\left(1-\alpha-\beta\right)\sum_{n\geq 0}\frac{(-1)^n z^{n+2}q^{n+2}}{(q;q)_n}, \end{aligned}$$

where the final term follows from (2.2).

We now find a hypergeometric solution to this q-difference equation by expanding the series as

$$H(z) =: \sum_{k\geq 0} \delta_k z^k. \tag{3.20}$$

Then for $k \geq 2$ we have

$$\delta_k - \delta_{k-2} = q^k\left(1+q^{-1}\right)\delta_k + q^{k-1}(Aq+B)\delta_{k-2} - q^{2k-1}\delta_k + q^{2k-1}AB\delta_{k-2} + \frac{A(1-\alpha-\beta)(-1)^k q^k}{(q;q)_{k-2}},$$

which can be rewritten as the (nonhomogeneous) recurrence

$$\delta_k = \frac{\left(1+Aq^k\right)\left(1+Bq^{k-1}\right)}{\left(1-q^{k-1}\right)\left(1-q^k\right)}\delta_{k-2} + \frac{A(1-\alpha-\beta)(-1)^k q^k}{(q;q)_k}. \tag{3.21}$$

The initial conditions are found by recalling (3.15), (3.17), and (3.20), which imply that

$$\delta_0 = \gamma_0 = \beta, \quad \delta_1 = \gamma_1 - \frac{\gamma_0}{1-q} = \frac{1-\beta+\alpha Aq}{1-q}.$$

A short proof by induction using (3.21) then gives the solutions

$$\delta_{2k} = \frac{\left(-Aq^2;q^2\right)_k\left(-Bq;q^2\right)_k}{(q;q)_{2k}}\left(A(1-\alpha-\beta)\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell\left(-Bq;q^2\right)_\ell}+\beta\right) \tag{3.22}$$

for the even indices, and similarly, for the odd indices,

$$\delta_{2k+1} = \frac{\left(-Aq^3;q^2\right)_k\left(-Bq^2;q^2\right)_k}{(q;q)_{2k+1}} \times\left(-A(1-\alpha-\beta)q\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell\left(-Bq^2;q^2\right)_\ell}+1-\beta+\alpha Aq\right). \tag{3.23}$$

It is now possible to combine (3.13), (3.17), (3.19), (3.20), (3.22), and (3.23) in order to write down closed-form expressions for $F(n_a)$ (cf. Lemma 3.2 in [5]). However, our primary interest is on the limiting case $n \to \infty$, as this then implies Theorem 1.2. We evaluate this limit in the next section.

4 Infinite series evaluation and the proof of Theorem 1.2

In this section, we take the infinite limits of the bounded expressions from the previous section in order to find the full generating functions for three-colored partitions satisfying Capparelli's gap condition. We denote the limiting functions by

$$F_{\alpha,\beta}(A, B; q) := \lim_{n\to\infty} F_{\alpha,\beta}(n_a; A, B; q), \qquad (4.1)$$
$$F_{\alpha,\beta}(t; q) := F_{\alpha,\beta}\left(tq^{-1}, t^{-1}; q\right).$$

The dilated identities in Theorem 1.2 are then immediate consequences of the following result for colored partitions.

Theorem 4.1. *For* $\alpha, \beta \in \{0, 1\}$, *we have*

$$F_{\alpha,\beta}(t; q) = \frac{\theta\left(tq; q^2\right)}{(q; q)_\infty}\left(\beta + (1-\alpha-\beta)T_1(t; q)\right)$$
$$+ \frac{\theta\left(t; q^2\right)}{(q; q)_\infty}\left(\alpha + (1-\alpha-\beta)T_2(t; q)\right) - (1-\alpha-\beta)\theta\Big(-t^2; q^2\Big)(-q; q)_\infty.$$

Proof of Theorem 1.2. The theorem statement follows from letting $q \mapsto q^d$ and $t \mapsto tq^\ell$ in Theorem 4.1. Recalling (1.4), (3.3), and (4.1), it is a short combinatorial exercise to show that under this specialization the colored gap condition (3.2) is equivalent to (1.1).

Remark. This specialization is slightly different from the one in [1], which we previously described in Remark 2 following (1.3). This is again due to the fact that the coloring we specify in (3.1) differs from that of Andrews, Alladi, and Gordon.

4.1 Limits separated by parity

The calculations are most convenient if the odd and even indices for the δ_n are separated. For $j \in \{0, 1\}$ we therefore set

$$H_j(z) := \sum_{n\equiv j \pmod 2} \delta_k z^k,$$
$$G_j(z) := (-z)_\infty H_j(z).$$

Then, by (2.3) and (3.22),

$$G_0(z) = \sum_{m\geq 0} \frac{z^m q^{\frac{m(m-1)}{2}}}{(q)_m} \sum_{k\geq 0} \delta_{2k} z^{2k}$$
$$= \sum_{n\geq 0} \frac{z^n}{(q)_n} \sum_{m+2k=n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \left(-Aq^2;q^2\right)_k \left(-Bq;q^2\right)_k$$
$$\times \left(\beta + (1-\alpha-\beta)A \sum_{\ell=1}^{k} \frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell \left(-Bq;q^2\right)_\ell}\right),$$

and similarly by (3.23),

$$G_1(z) = \sum_{n\geq 0} \frac{z^n}{(q)_n} \sum_{m+2k+1=n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \left(-Aq^3;q^2\right)_k \left(-Bq^2;q^2\right)_k$$
$$\times \left(1-\beta+\alpha Aq - q(1-\alpha-\beta)A \sum_{\ell=1}^{k} \frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell \left(-Bq^2;q^2\right)_\ell}\right).$$

Written in the above form we can finally isolate the generating functions from Section 3,

$$G_j(z) =: \sum_{n\geq 0} \frac{z^n}{(q)_n} C_{j,n}.$$

Then $F(n_a) = C_{0,n} + C_{1,n}$, and thus

$$F_{\alpha,\beta} = \lim_{n\to\infty} \left(C_{0,n} + C_{1,n}\right). \tag{4.2}$$

We begin with the even case, using (2.1) to calculate

$$C_0 := \lim_{\substack{n\to\infty \\ n \text{ even}}} C_{0,n} = \lim_{\substack{n\to\infty \\ n \text{ even}}} \sum_{\substack{m=0 \\ m \text{ even}}}^{n} q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(-Aq^2;q^2\right)_{\frac{n-m}{2}} \left(-Bq;q^2\right)_{\frac{n-m}{2}}$$
$$\times \left(\beta + (1-\alpha-\beta)A \sum_{\ell=1}^{\frac{n-m}{2}} \frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell \left(-Bq;q^2\right)_\ell}\right)$$
$$= \sum_{\substack{m\geq 0 \\ m \text{ even}}} \frac{q^{\frac{m(m-1)}{2}}}{(q;q)_m} \left(-Aq^2;q^2\right)_\infty \left(-Bq;q^2\right)_\infty \tag{4.3}$$
$$\times \left(\beta + (1-\alpha-\beta)A \sum_{\ell\geq 1} \frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell \left(-Bq;q^2\right)_\ell}\right).$$

The sum on m evaluates to $(-q;q)_\infty$ by (2.4), so it remains to compute the sum on ℓ. For this, we first shift the summation index to obtain

$$\frac{q^2}{\left(1+Aq^2\right)(1+Bq)}\sum_{\ell\geq 0}\frac{q^{2\ell}}{\left(-Aq^4;q^2\right)_\ell\left(-Bq^3;q^2\right)_\ell}.$$

Then we apply (2.5) with $q\mapsto q^2$, $a=Aq^2$, and $b=Bq$, yielding the equivalent expression

$$\frac{q^2}{\left(1+Aq^2\right)(1+Bq)}\left(\left(1+A^{-1}q^{-2}\right)\sum_{\ell\geq 0}\frac{(-1)^\ell q^{\ell(\ell+1)}\left(\frac{B}{Aq}\right)^\ell}{\left(-Bq^3;q^2\right)_\ell}\right.$$
$$\left. -A^{-1}q^{-2}\frac{\sum_{\ell\geq 0}(-1)^\ell q^{\ell(\ell+1)}\left(\frac{B}{Aq}\right)^\ell}{\left(-Aq^4,-Bq^3;q^2\right)_\infty}\right)$$
$$=A^{-1}\sum_{\ell\geq 0}\frac{(-1)^\ell q^{\ell^2}\left(\frac{B}{A}\right)^\ell}{\left(-Bq;q^2\right)_{\ell+1}}-\frac{A^{-1}\sum_{\ell\geq 0}(-1)^\ell q^{\ell^2}\left(\frac{B}{A}\right)^\ell}{\left(-Aq^2,-Bq;q^2\right)_\infty}. \tag{4.4}$$

Shifting the summation index of the first sum in (4.4), we have

$$\sum_{\ell\geq 0}\frac{(-1)^\ell q^{\ell^2}\left(\frac{B}{A}\right)^\ell}{\left(-Bq;q^2\right)_{\ell+1}}=-\sum_{\ell\geq 1}\frac{(-1)^\ell q^{\ell^2-2\ell+1}\left(\frac{B}{A}\right)^{\ell-1}}{\left(-Bq;q^2\right)_\ell}.$$

Applying Lemma 2.1 with $q\mapsto q^2$, $b=-Bq^{-1}$, and $x=-A^{-1}q^{-2}$, the above expression equals

$$\frac{Aq}{B}-\frac{Aq}{B}\sum_{\ell\geq 0}\frac{\left(-A^{-1};q^2\right)_\ell}{\left(-Bq;q^2\right)_\ell}\left(\frac{B^2}{A}\right)^\ell q^{3\ell(\ell-1)}\left(1-\frac{B}{A}q^{4\ell-1}\right). \tag{4.5}$$

Combining (4.4)–(4.5) and plugging in to (4.3), we, therefore, have

$$C_0=(-q;q)_\infty\left(-Aq^2;q^2\right)_\infty\left(-Bq;q^2\right)_\infty \tag{4.6}$$
$$\times\left[\beta+(1-\alpha-\beta)\left(\frac{Aq}{B}-\frac{Aq}{B}\sum_{\ell\geq 0}\frac{\left(-A^{-1};q^2\right)_\ell}{\left(-Bq;q^2\right)_\ell}\left(\frac{B^2}{A}\right)^\ell q^{3\ell(\ell-1)}\left(1-\frac{B}{A}q^{4\ell-1}\right)\right)\right]$$
$$-(1-\alpha-\beta)(-q;q)_\infty\sum_{\ell\geq 0}(-1)^\ell\left(\frac{B}{A}\right)^\ell q^{\ell^2}.$$

We now calculate the contribution from F_1, which proceeds similarly to the above. Again using (2.1) and (2.4), we have

$$C_1 := \lim_{\substack{n\to\infty\\ n \text{ odd}}} C_{1,n} = \lim_{\substack{n\to\infty\\ n \text{ odd}}} \sum_{\substack{m=0\\ m \text{ even}}}^{n-1} q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(-Aq^3;q^2\right)_{\frac{n-m-1}{2}} \left(-Bq^2;q^2\right)_{\frac{n-m-1}{2}}$$
$$\times \left(1-\beta+\alpha Aq - q(1-\alpha-\beta)A \sum_{\ell=1}^{\frac{n-m-1}{2}} \frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell \left(-Bq^2;q^2\right)_\ell}\right)$$
$$= (-q;q)_\infty \left(-Aq^3;q^2\right)_\infty \left(-Bq^2;q^2\right)_\infty \tag{4.7}$$
$$\times \left(1-\beta+\alpha Aq - q(1-\alpha-\beta)A \sum_{\ell\ge 1} \frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell \left(-Bq^2;q^2\right)_\ell}\right).$$

By Ramanujan's identity (2.5) with $q \mapsto q^2, b = Aq$, and $a = B$, the sum on ℓ becomes

$$-1 + \left(1+B^{-1}\right) \sum_{\ell\ge 0} \frac{(-1)^\ell q^{\ell(\ell+1)} \left(\frac{Aq}{B}\right)^\ell}{\left(-Aq^3;q^2\right)_\ell} - B^{-1} \frac{\sum_{\ell\ge 0} (-1)^\ell q^{\ell(\ell+1)} \left(\frac{Aq}{B}\right)^\ell}{\left(-Aq^3,-Bq^2;q^2\right)_\infty}. \tag{4.8}$$

Lemma 2.1 with $q \mapsto q^2, b = -Aq$, and $x = -B^{-1}$ then implies that the first sum from (4.8) equals

$$\sum_{\ell\ge 0} \frac{(-1)^\ell q^{\ell^2+2\ell} \left(\frac{A}{B}\right)^\ell}{\left(-Aq^3;q^2\right)_\ell} = \sum_{\ell\ge 0} \frac{\left(-B^{-1}q^2;q^2\right)_\ell}{\left(-Aq^3;q^2\right)_\ell} \left(\frac{A^2}{B}\right)^\ell q^{3\ell(\ell+1)} \left(1 - \frac{A}{B} q^{4\ell+3}\right). \tag{4.9}$$

Plugging in (4.8) and (4.9) to (4.7), we obtain

$$C_1 = (-q;q)_\infty \left(-Aq^3;q^2\right)_\infty \left(-Bq^2;q^2\right)_\infty \Bigg[1-\beta+\alpha Aq - qA(1-\alpha-\beta)$$
$$\left(-1 + \left(1+B^{-1}\right) \sum_{\ell\ge 0} \frac{\left(-B^{-1}q^2;q^2\right)_\ell}{\left(-Aq^3;q^2\right)_\ell} \left(\frac{A^2}{B}\right)^\ell q^{3\ell(\ell+1)} \left(1 - \frac{A}{B} q^{4\ell+3}\right)\right)\Bigg]$$
$$+ (1-\alpha-\beta)(-q;q)_\infty AB^{-1} \sum_{\ell\ge 0} (-1)^\ell q^{\ell(\ell+2)+1} \left(\frac{A}{B}\right)^\ell. \tag{4.10}$$

For a final simplification, note that the last term in (4.10) can be rewritten, since

$$-AB^{-1} \sum_{\ell\ge 0} (-1)^\ell q^{\ell(\ell+2)+1} \left(\frac{A}{B}\right)^\ell = \sum_{\ell\le -1} (-1)^\ell q^{\ell^2} \left(\frac{B}{A}\right)^\ell.$$

This combines with the last term from (4.6) to give the single summation

$$- (1 - \alpha - \beta)(-q; q)_\infty \sum_{\ell \in \mathbb{Z}} (-1)^\ell \left(\frac{B}{A}\right)^\ell q^{\ell^2}. \tag{4.11}$$

Our calculation is now complete, as (4.2), (4.6), and (4.10) give a hypergeometric formula for $F_{\alpha,\beta}$.

4.2 *The modular case and the proof of Theorem 4.1*

Many of the expressions from above simplify quite drastically under the specialization $A = tq^{-1}$ and $B = t^{-1}$, and in this case we can further identify components in terms of theta functions. Adding (4.6) and (4.10) (and recalling (4.11)), and writing $\omega := 1 - \alpha - \beta$ to save space, we have

$$\begin{aligned}\frac{C_0 + C_1}{(-q;q)_\infty} =& \left(-tq, -t^{-1}q; q^2\right)_\infty \Bigg[\beta + \omega\, t^2 \left(1 - \sum_{\ell \geq 0} t^{-3\ell} q^{3\ell^2 - 2\ell} \left(1 - t^{-2} q^{4\ell}\right)\right) \\ &+ \left(-t, -t^{-1}q^2; q^2\right)_\infty \left(1 - \beta - t\omega \sum_{\ell \geq 0} t^{3\ell} q^{3\ell^2 + \ell} \left(1 - t^2 q^{4\ell+2}\right)\right)\Bigg] \\ &- \omega \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\ell^2 + \ell} t^{-2\ell}.\end{aligned} \tag{4.12}$$

To obtain the theorem statement, we first apply the following cases of (1.5):

$$\begin{aligned}\left(-tq, -t^{-1}q; q^2\right)_\infty &= \frac{\theta\left(tq; q^2\right)}{\left(q^2; q^2\right)_\infty}, \\ \left(-t, -t^{-1}q^2; q^2\right)_\infty &= \frac{\theta\left(t; q^2\right)}{(1+t)\left(q^2; q^2\right)_\infty}, \\ \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\ell^2 + \ell} t^{-2\ell} &= \theta\left(-t^2; q^2\right).\end{aligned}$$

We further simplify the first sum in (4.12) as

$$\begin{aligned}&t^2 \left(1 - \sum_{\ell \geq 0} t^{-3\ell} q^{3\ell^2 - 2\ell} \left(1 - t^{-2} q^{4\ell}\right)\right) \\ &\qquad = \sum_{\ell \geq 0} \left(t^{-3\ell} q^{3\ell^2 + 2\ell} - t^{-3\ell - 1} q^{(\ell+1)(3\ell+1)}\right) = T_1(t; q).\end{aligned}$$

Moreover, the fact that $1 - \beta = \alpha + \omega$ implies that all of the terms in the inner parentheses from the second and third lines of (4.12) combine and simplify as follows:

$$\alpha + \omega - \omega\, t \sum_{\ell \geq 0} t^{3\ell} q^{3\ell^2+\ell} \left(1 - t^2 q^{4\ell+2}\right) = \alpha + (1 - \alpha - \beta) T_2(t; q).$$

Plugging back in to (4.12) completes the proof of Theorem 4.1.

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