Springer Proceedings in Mathematics & Statistics

George E. Andrews
Frank Garvan *Editors*

Analytic Number Theory, Modular Forms and *q*-Hypergeometric Series

In Honor of Krishna Alladi's 60th Birthday, University of Florida, Gainesville, March 2016

Springer

Springer Proceedings in Mathematics & Statistics

Volume 221

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

George E. Andrews · Frank Garvan
Editors

Analytic Number Theory, Modular Forms and q-Hypergeometric Series

In Honor of Krishna Alladi's 60th Birthday, University of Florida, Gainesville, March 2016

Springer

Editors
George E. Andrews
Department of Mathematics
Pennsylvania State University
State College, PA
USA

Frank Garvan
Department of Mathematics
University of Florida
Gainesville, FL
USA

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-319-68375-1 ISBN 978-3-319-68376-8 (eBook)
https://doi.org/10.1007/978-3-319-68376-8

Library of Congress Control Number: 2017958728

Mathematics Subject Classification (2010): 05A17, 11B65, 11F30, 11K65, 11N05, 11P84, 33D15

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

We are extremely pleased to offer these proceedings of the Gainesville Number Theory Conference of 2016, more affectionately known as ALLADI60, honoring Krishna Alladi on his 60th birthday. Krishna has been a major contributor to number theory and mathematics in several ways. First, he is a first-class mathematician. We who have collaborated with him are most vividly aware of his insight and talent. Second, he is Editor-in-chief of the Ramanujan Journal and a Series Editor of Developments in Mathematics. Third, he instituted the internationally admired SASTRA Ramanujan Prize. Fourth, he has been the major actor in the creation of numerous important conferences.

In the past two decades, the University of Florida has been the main international venue for conferences in the areas of partitions, q-series, modular forms, and Ramanujan's work. The special feature of this conference was that in addition to these areas, analytic number theory, irrationality, and transcendence were also covered—areas that Alladi had worked on until 1990.

The conference attracted nearly 200 participants and was spread over 5 days to accommodate the nearly 100 speakers who had come from Australia, Austria, Canada, China, England, France, Germany, Hong Kong, Hungary, India, Israel, Korea, The Netherlands, New Zealand, Norway, Serbia, Switzerland, Tunisia, Turkey, and the USA. The conference was supported by grants from the National Science Foundation, the National Security Agency, the Number Theory Foundation, and by funds from The Pennsylvania State University. Local support was provided by the University of Florida, Mathematics Department and the College of Liberal Arts and Sciences, and the Alachua County Tourism Board. We are most grateful for all this support which helped make the conference a success.

Special Lectures

There were four special lectures at the conference:

Opening Lecture
Manjul Bhargava (Princeton University)
Squarefree values of polynomial discriminants
In 2003, Manjul Bhargava won the First SASTRA Ramanujan Prize, a prize that Krishna Alladi was instrumental in launching.

The Erdős Colloquium
Hugh L. Montgomery (University of Michigan)
Littlewood polynomials
This Special Colloquium, initiated by Krishna Alladi, has been given yearly at the University of Florida since 1999. Following the Colloquium, the participants were treated to a dinner party at the home of Krishna and Mathura Alladi, and our thanks to Mathura for graciously hosting this.

The Ramanujan Colloquium
James Maynard (University of Oxford)
Linear equations in primes
This Special Colloquium, initiated by Krishna Alladi, has been given yearly at the University of Florida since 2007, and is sponsored by The Pennsylvania State University and George Andrews. James Maynard also gave two other lectures.

Conference Closing Lecture and Math Colloquium
Wadim Zudilin (University of Newcastle)
Short random walks and Mahler measures
Wadim Zudilin also gave a lecture on certain irrational values of the logarithm, which was related to Krishna Alladi's work from 1979.

Other Conference Highlights

Piano Recital
Following Maynard's Colloquium, there was a Reception at the Keene Faculty Center where the participants were treated to a lovely piano concert by Christian Krattenthaler of the University of Vienna.

Awards
Ron Graham, former President of the AMS, presented cheques to James Maynard and Kevin Ford in recognition of their work for the resolution of a famous $10,000 problem of Erdős on large gaps between primes. Maynard received $5,000 for his solo paper, and Kevin Ford received a cheque for $5,000 made out to Kevin Ford, Ben Green, S. Konyagin, and Terence Tao for their joint work.

Soundararajan's Lecture
Another SASTRA Ramanujan prize winner, Kannan Soundararajan (Stanford), presented for the first time his recent work with his colleague Robert Lemke Olivera on a startling new result on a bias in the distribution of consecutive prime numbers.

The Man Who Knew Infinity
The movie *The Man Who Knew Infinity* on the remarkable life of Ramanujan was shown at the conference as a Preview before its official opening in theaters thanks to the efforts of Manjul Bhargava, one of the Associate Producers of the movie. Our thanks to Edward Pressman films for this kind gesture. See Bruce Berndt's paper in this volume for some background on an interesting scene from the movie.

Talks on Alladi's Work
Several talks at the conference dealt with Alladi's work—not just his recent work on partitions and q-series, but his early work as well. Dorian Goldfeld (Columbia University) spoke about extensions of results in Alladi's first paper (written when he was an undergraduate) with Paul Erdős on an additive arithmetic function—now called the Alladi-Erdős function. Doron Zeilberger (Rutgers University) and Wadim Zudilin (University of Newcastle, Australia) discussed extensions of results of Alladi-Robinson (1980) on irrationality measures.

Conference Banquet
A Conference Banquet was held at the Paramount Hotel. Many speeches honoring Krishna were given. The text of Elizabeth Loew's speech is given after this preface.

Mathematical Interests of Krishna Alladi

Over his career, Krishna Alladi has maintained an interest in Number Theory, Combinatorics, Discrete Mathematics, Analytic Number Theory, Sieve Methods, Probabilistic Number Theory, Diophantine Approximations, Partitions, and q-Series Identities, His research in mathematics began as an 18-year-old undergraduate in 1973. The first part of his mathematical career was in Analytic Number Theory. In particular, he wrote five joint papers with Paul Erdős. In 1987, the Ramanujan Centenary year, Krishna became interested in partitions and in the early 1990s, he began a fruitful collaboration with George Andrews, Alex Berkovich, and Basil Gordon, when they made impressive breakthroughs in discovering partition identities beyond those of Rogers-Ramanujan, Schur and Göllnitz. In 1993, he spent a sabbatical at Penn State with George Andrews. There he learned the importance of basic hypergeometric series, and modular forms for the theory of partitions.

Volume Contents

Below, we give a brief description of the papers in this volume and group them according to these topics: *Analytic Number Theory*, *Probabilistic Number Theory*, *Partitions*, *Basic Hypergeometric Functions*, and *Modular Forms*.

Analytic Number Theory

Benli, Elma, and Yidirim extend Conrey and Ghosh's results on zeros of derivatives of the Riemann zeta-function near the critical line, to Dirichlet L-functions.

Deshouillers and Grekos study the problem of the number of integral points on a convex curve in terms of length and curvature, and make improvements on previous results.

In 1977, Alladi and Erdős showed that a certain important additive function is uniformly distributed modulo 2. Goldfeld generalizes this result to an arbitrary modulus.

Montgomery's survey paper on Littlewood Polynomials is an expanded version of his talk given at the conference.

Nicolas obtains an effective version of Ramanujan's result for the difference between the logarithmic integral of Chebychev's function and $\pi(x)$.

Ono, Schneider, and Wagner prove a partition theoretic analog of Alladi's Möbius function identity.

Saradha and Sharma prove some conjectures of Mueller and Schmidt for the number of integer solutions of a Thue inequality for certain binary quadratic forms.

Tenenbaum extends a result of Mertens for the sum of the reciprocals of primes to the sum of the reciprocals of the product of k primes.

Inspired by Hugh Montgomery's talk on Littlewood polynomial, Zeilberger (with his computer collaborator Ekhad) gives an algorithmic approach to Saffari's conjecture on the asymptotic growth of moments of the Rudin–Shapiro polynomials.

Probabilistic Number Theory

Elliott's paper is survey of abstract multiplicative functions and their application to the study of the Fourier coefficients of automorphic forms, together with a discussion in the context of the theory of Probabilistic Number Theory.

Partitions

George Andrews recently gave a refinement of Krishna Alladi's variant of Schur's 1926 partition theorem. In his paper, Andrews develops a surprising factorization of the related polynomial generating functions.

Chen, Ji, and Zang previously proved a rank-crank inequality conjecture of Andrews, Dyson, and Rhoades. By using combinatorial methods, they show that there is a reordering of partitions that explains the very nearly equal distributions of the rank and the crank.

Motivated by recent research of Krishna Alladi, Berkovich, and Uncu give new weighted partition identities for partitions, overpartitions, and partitions with distinct even parts, using the theory of basic hypergeometric functions.

Dousse extends Krishna Alladi's method of weighted words to obtain generalizations and refinements for previous extensions of Schur's partition theorem to overpartitions due to Andrews, Corteel, and Lovejoy.

Kolitsch gives new partition interpretations of truncated forms of Euler's Pentagonal Number Theorem and Jacobi's Triple Product identity in terms of overpartitions.

Krattenthaler finds congruences mod 16 for the number of unique path partitions of n, which occur in the study of character values of finite symmetric groups, and which generalize results of Olsson, Bessendrodt, and Sellers.

Kanade, Kurşungöz, and Russel give combinatorial interpretations of overpartition variants of Andrews's H and J functions which occurred in the study of the Andrews-Gordon partition identities and their generalizations.

Lovejoy gives two overpartition extensions of Alladi and Gordons generalization of Schurs theorem.

Bringmann and Mahlburg present new companions to the Capparelli partition identities and two new general identities for three-color partitions that may be specialized to theta functions and false theta functions.

Seo and Yee give a combinatorial proof of a result of Andrews which is an overpartition analog of Rogers-Ramanujan type theorem related to restricted successive ranks.

q-Series and Basic Hypergeometric Functions

Gaurav Bhatnagar gives a marvelous bibasic version of Heine's basic hypergeometric transformation and uses it to prove and organize a raft of identities of Ramanujan, some of which are easy and some which are not.

Cooper, Wan, and Zudilin prove a number of Z.-W. Sun's conjectures for series for $1/\pi$ by relating them to known series using techniques of basic hypergeometric series and Zeilberger's algorithm for holonomic sequences.

Banerjee and Dixit obtain new identities for Ramanujan's function $\sigma(q)$ which is the generating function for the excess number of partitions of n into distinct parts with even rank over those of odd rank.

Hirschhorn gives elementary proofs of some well-known arithmetic properties of Ramanujan's tau function using nothing more than Jacobi's triple product identity.

Liu describes a method for finding certain series expansions of functions that satisfy a q-partial differential equation and, as an application, finds a generalization of Andrews's transformation formula for the q-Lauricella function.

Mc Laughlin gives a new approach using bilateral hypergeometric series to obtain identities for mock theta functions and finds radial limit formulas as an application.

Schlosser and Yoo employ a one-variable extension of q-rook theory to give combinatorial proofs of some basic hypergeometric summations formulas.

Sills gives an elementary approach for finding the sum side of Rogers-Ramanujan type identities from the product forms related to the standard modules of the Kac–Moody algebra $A_2^{(2)}$.

Modular Forms

Nicolas Andersen follows up on his previous work, "Vector-Valued Modular Forms and The Mock Theta Conjectures," where he gave a new proof of Ramanujan's fifth-order mock theta conjectures using the theory of vector-valued modular forms and harmonic Maass forms. In his new paper, he extends these ideas to give a new proof of Hickerson's seventh-order identities.

Jha and Kumar compute the adjoint (with respect to the Petersson inner product) of the linear map related to the Cohen–Rankin bracket, thus extending work of Kohnen and Herrero to half-integral weight modular forms.

Kimport obtains asymptotic expansions for weight $1/2$ and $3/2$ partial theta functions at roots of unity which generalizes results of Berndt and Kim and are important in the study of certain quantum modular forms.

McIntosh proves that Zweger's μ-function, which is important in the study of mock theta functions from a modular form view, is essentially no more general than the universal mock theta function g_2.

Paule and Radu derive a new type of modular function identity that implies Ramanujan's partition congruence mod 11.

Ramakrishnan, Sahu, and Singh use the theory of modular forms to find formulas for the number of representations of a positive integer by certain class of quadratic forms in eight variables.

Jon Borwein

Jon Borwein was unable to come to ALLADI60 due to his commitment to give a series of lectures as a Distinguished Scholar in Residence at Western University, London Ontario. Later, in May 2016, Jon and his wife Judi were able to visit us in Florida. He gave two talks—one at the University of Florida Brain Institute on *CARMA: A Model for Multi-Discipline and Multi-Institution Collaboration*. The other talk on *Seeing Things by Walking on Numbers* was given in the Math Department. Later, in July, I (Frank) was with Jon at the Lambert Conference in London, Ontario. It was a great shock to us that he died just a few days after the Lambert Conference, and we still feel the loss. Jon was the first person to submit a paper to our proceedings. The referee only required some minor revisions. Jon wanted to wait until he got back to Australia to complete these revisions but sadly this did not happen. I made the revisions myself and got David Bailey to check them over. Jon's paper is an expanded companion to a talk he gave at a workshop celebrating Tony Guttmann's 70th birthday. It describes his encounters over nearly 30 years with Sloane's (Online) Encyclopedia of Integer Sequences. We agree with the referee that it is a masterpiece, with beautiful math and beautiful exposition.

Alladi Ramakrishnan

Krishna's father, the late Prof. Alladi Ramakrishnan, the Founder-Director of MATSCIENCE, The Institute of Mathematical Sciences in Madras, India, was an inspiration to Krishna and supported all of Krishna's efforts. In an emotionally charged speech at the banquet, Krishna said that if his father were alive, he would have been the happiest person to see such an impressive gathering of mathematicians from around the world for the 60th birthday conference. Several speakers in their speeches at the banquet made references to Krishna's father. Kryuchkov, Lanfear, and Suslov have dedicated their paper to the memory of Krishna's father, the famous physicist Prof. Alladi Ramakrishnan, on the topic of a complex form of classical and quantum electrodynamics.

Thanks

We thank Marc Strauss and Elizabeth Loew for the Springer book exhibit and Rochelle Kronzek for the World Scientific book exhibit. In addition, we thank Marc Strauss for publishing these proceedings in the Springer Proceedings Series. We express special thanks to Margaret Somers, Cyndi Garvan, Ali Uncu, and Chris Jennings-Shaffer, the staff of the Math Department and the number theory graduate students for all aspects of preparing for and running a smooth conference. We thank Jis Joseph for his photography and help with other aspects of the conference.

We conclude by expressing again our thanks to Krishna for his monumental contributions. Our community has been greatly enriched by him, and we are deeply in his debt.

Gainesville, FL, USA — George E. Andrews
August 2017 — Frank Garvan

Introductory Speech

Speech Given by Elizabeth Loew at the ALLADI60 Banquet

I am especially honored to have been asked by George Andrews to give a short speech tonight. Thank you George. I also want to thank Frank Garvan for inviting me to share in this important celebration in honor of Krishna's 60th birthday.

***

While I don't quite remember the first time that I met Krishna, it must have been sometime in 2008, here at UFL, after I had become a senior editor with Springer. Our meeting was certainly during Krishna's tenure as Chair of the department. Among my responsibilities for journals and books were *The Ramanujan Journal* and the series *Developments in Mathematics*, both of which were founded by Krishna. Both became demanding, because they were intellectually rich and I wanted to ensure that I was doing my part to reinvigorate and help expand from publishing capabilities. This was enjoyable for me personally.

***

It was politely suggested to me that we weren't coming here tonight to roast Krishna. (I assume that everyone knows what I mean by ROAST). That is, I am not going to say anything particularly funny or amusing. The Krishna that I have come to know is a serious kind of guy. But I have to say, that in the midst of situations and discussions, he has often made me laugh. He has certain unique expressions and ways of politely conveying dissatisfaction. From my perspective, however, Krishna has always been above all forms of pettiness. Over the years, I have greatly appreciated his warmth and genuine friendship to me. This extends to his lovely wife, Mathura, and all of the Alladi children. I have been privileged to enjoy many evenings in their home and I wish the Alladi family many years of continued happiness and good health.

***

Krishna and I have worked closely and well together for these last 8 years and I would like to comment and recollect on a few important things about the publications we continue to work on together. Krishna is both the founder and

Editor-in-chief of *The Ramanujan Journal*. Frank Garvan is the journal's extremely capable managing editor. And many of you here today are active members of the board. The journal was established in 1997 with Kluwer and upon the Kluwer and Springer merger, RAMA became a Springer publication. There is no need to convey to everyone here the importance of Ramanujan himself and the enormous impact his genius had and still has on the mathematics community. **So** RAMA is devoted to those aspects of mathematics influenced by Ramanujan. Indeed during my years at Springer, these influences have grown substantially. The journal had always been strong in covering q-series and special functions, for example, and in the last few years modular forms has become a critical area of coverage for the journal as well. Various topics have expanded greatly, some have even 'exploded' (to use Krishna's wording). I just want to emphasize that the journal continues to reinforce connections and to establish links with new fields. And I would like to put it this way: the editorial board is **choc full** of valued mathematicians all of whom are experts in their areas of research. The journal publishes 9 issues per year. Special issues are published from time to time with respect to the Sastra Ramanujan lectures, covering the 125th anniversary of Ramanujan, memorial issues for such great figures as Erdős, Rankin, Gordon, and a forthcoming one for Marvin Knopp. Special issues have also been published for milestone birthdays such as for Richard Askey and George Andrews, to name just two. So, on behalf of Springer I would like to thank Krishna, Frank, and the entire board for their continued devotion to enhancing the quality and excellence of RAMA.

***

The book series **Developments in Mathematics** was founded by Krishna as previously mentioned. When I entered the picture Bob Guralnick was co-series editor and eventually stepped down. In 2009 Hershel Farkas joined as co-series editor. The 3 of us have been working effectively and diligently toward expanding the series. This too has been an enriching collaboration for me. In earlier years, most of the publications in DEVM included edited works, most often in the area of number theory, but in other areas as well. Nowadays, DEVM predominantly intends to publish quality monographs. The aim being to cover new topics in the forefront of mathematical research; topics that are perhaps a bit more specialized. While new topics in number theory have published in DEVM, new books in the series include those in fields such as differential equations, algebra, enumerative combinatorics, analysis, to name several. Quality edited volumes are still occasionally considered for publication. A selection of edited volumes that may be of interest to you are displayed here at the conference **"Partitions, *q*-Series, and Modular Forms" (Alladi-Garvan), "Quadratic and Higher Degree Forms" (Alladi, Barghava, Savitt, Tiep), "Surveys in Number Theory" (Alladi), and "Combinatory Analysis" (Alladi, Paule, Sellers, Yee)**.

***

A word about Krishna's father. I remember Krishna's father very well. My parents, Bert and Ann Kostant, also had the pleasure of meeting Alladi Ramakrishnan. With his kind face and beautiful smile, he was a very interesting person. Discussions with him took place in Krishna's home where I recall (and as Bert also told me) hearing a variety of thoughts and recollections that came to his father's mind. When Dr. Ramakrishnan passed away, I remember remarking to Krishna what a devoted son he had been to his father. The magnificent book that Krishna put together entitled **The Legacy of Alladi Ramakrishnan in the Mathematical Sciences** is on display here and is a beautiful intellectual and sentimental tribute to a highly respected mathematical physicist. I felt happy to be involved in publishing that Work.

***

The book **Ramanujan's Place in the World of Mathematics** was published in 2013 with my colleagues in Springer India. This book represents a compendium of Krishna's articles over time and has received great reviews from various sources. In it, the reader learns of Ramanujan and his mathematical work in an historical context and with comparisons to other great mathematicians throughout history. I hope that there are several copies of this book here at the conference.

***

Krishna is a creator of original mathematical research; a composer of books; a builder of two noteworthy publishing venues, journal and book series. And again and again, over and over, I have been enriched and delighted to be part of much of Krishna's publishing activities. I consider myself rather fortunate to have cultivated several strong and enriching relationships within the mathematics community. There is strong and enriching, and there is strong and enriching PLUS. I believe that my relationship with Krishna includes the PLUS. From Springer, from Joachim Heinze, from Marc Strauss, from myself, my son Max, and Ann and Bert Kostant, we wish Krishna healthy years ahead filled with as much joy as he can take in.

List of Participants

S. D. Adhikari, Harish-Chandra Institute
Scott Ahlgren, University of Illinois
Yildirim Akbal, Bilkent University
Elie Alhajjar, George Mason University, Virginia
Krishnaswami Alladi, University of Florida
Badria Alsulmi, Kansas State University
Nickolas Andersen, University of Illinois
George Andrews, Pennsylvania State University
Victor Manuel Aricheta, Emory University
Richard Askey, University of Wisconsin
Roger Baker, Brigham Young University
William Banks, University of Missouri
Olivia Beckwith, Emory University
Lea Beneish, Emory University
Kübra Benli, University of Georgia
Alexander Berkovich, University of Florida
Bruce Berndt, University of Illinois
Edward Bertram, University of Hawaii at Manoa
Manjul Bhargava, Princeton University
Gaurav Bhatnagar, University of Vienna
Matt Boylan, University of South Carolina
Kathrin Bringman, University of Cologne
Dale Brownawell, Pennsylvania State University
Hannah Burson, University of Illinois
Neil Calkin, Clemson University
Zhu Cao, Kennesaw State University
Song Heng Chan, Nanyang Technological University, Singapore
Diego Chaves, Pennsylvania State University
Youn-seo Choi, KIAS
Fatma Cicek, University of Rochester
Shaun Cooper, Massey University, New Zealand

Hédi Daboussi, Université de Picardie, Amiens,France
Colin Defant, University of Florida
Jean-Marc Deshouillers, Bordeaux INP
Harold Diamond, University of Illinois
Atul Dixit, Indian Institute of Technology, Gandhinagar
Jehanne Dousse, Universität Zürich
Dennis Eichhorn, University of California, Irvine
Ali-Bulent Ekin, Ankara University
Peter Elliott, University of Colorado, Boulder
Ertan Elma, University of Waterloo
Geremias Polanco Encarnaci, Hampshire College
Larry Ericksen, Millville, New Jersey
Hershel Farkas, The Hebrew University of Jerusalem
Endrit Fejzullahu, University of Florida
Amanda Folsom, Amherst College
Kevin Ford, University of Illinois
Craig Franze, The Ohio State University at Marion
Richard Frnka, Louisiana State University
Ayla Gafni, Pennsylvania State University
Cyndi Garvan, University of Florida
Frank Garvan, University of Florida
Malay Ghosh, University of Florida
Dorian Goldfeld, Columbia University
Dan Goldston, San Jose State University
Oscar Gonzalez, University of Puerto Rico at Rio Piedras
Ankush Goswami, University of Florida
Ron Graham, University of California San Diego
Michael Griffin, Princeton University
Pavel Guerzhoy, University of Hawaii
Mike Hirschhorn, University of New South Wales, Australia
Tim Huber, University of Texas Rio Grande Valley
Mourad Ismail, University of Central Florida
Aleksandar Ivic, Serbian Academy of Sciences
Marie Jameson, University of Tennessee
Min-Joo Jang, University of Cologne
Paul Jenkins, Brigham Young University
Chris Jennings-Shaffer, Oregon State University
Abhas Kuma Jha, NISER, Bhubaneswar, India
Jin Seokho, KIAS
Jason Johnson, University of Florida
Junsoo Ha, KIAS
Ben Kane, University of Hong Kong
K. Kannan, SASTRA University
Kevin Keating, University of Florida
William Keith, Michigan Technical University

Byungchan Kim, Seoul National University of Science and Technology
Susie Kimport, Stanford University
Sun Kim, University of Illinois
Louis Kolitsch, The University of Tennessee at Martin
Anant Kota, University of California, Berkeley
Christian Krattenthaler, University of Vienna
Brandt Kronholm, University Of Texas Rio Grande Valley
Rochelle Kronzek, World Scientific Publishers
Kağan Kurşungö, Sabanci University, Istanbul
Jeff Lagarias, University of Michigan
Robert Lemke Oliver, Stanford University
Jim Lepowsky, Rutgers University
Winnie Li, Pennsylvania State University
Zhi Guo Liu, East China Normal University, Shanghai
Maddie Locus, Emory University
Steffen Loebrich, University of Cologne
Elizabeth Loew, Springer
Lisa Lorentzen, Norwegian University of Science and Technology
Karl Mahlburg, Louisiana State University
Helmut Maier, Universität Ulm
Bibekananda Maji, Harish-Chandra Research Institute, Allahabad
Amita Malik, University of Illinois
Wenjun Ma, Shandong University
James Maynard, University of Oxford
Kamel Mazhouda, Universite de Monastir, Tunisia
Richard McIntosh, University of Regina
Jim McLaughlin, West Chester University of Pennsylvania
Stephen C Milne, Ohio State University
Adriana Morales, University of Puerto Rico
Todd Molnar, University of Florida
Hugh Montgomery, University of Michigan
Michael Mossinghoff, Davidson College
Anton Mosunov, University of Waterloo
K. A. Muttalib, University of Florida
Mel Nathanson, Lehman College, CUNY
Ken Ono, Emory Unversity
Donny Passary, The Pennsylvania State University
Frank Patane, Samford University

Peter Paule, RISC, J. Kepler University, Linz
John Pfeilsticker, University of Florida
Janos Pintz, MTA Rényi Institute, Budapest, Hungary
Paul Pollack, University of Georgia
B. Ramakrishnan, Harish-Chandra Institute
Samin Riasat, University of Waterloo
Michael Robinson, Center for Computing Sciences, Maryland
Sinai Robins, Brown University and ICERM
Larry Rolen, Pennsylvania State University
Mohan Rudravarapu, Government Polytechnic, Visakhapatnam, AP,India
Carla Savage, North Carolina State University
Michael Schlosser, University of Vienna
Robert P. Schneider, Emory University
James Sellers, Pennsylvania State University
Li Shen, University of Florida
Andrew Sills, Georgia Southern University
Peter Sin, University of Florida
Meera Sitharam, University of Florida
Jaebum Sohn, Yonsei University, Seoul
Kannan Soundararajan, Stanford University
Richard Stanley, University of Miami
Cameron Stewart, University of Waterloo
Armin Straub, University of South Alabama
Marc Strauss, Springer
Sergei Suslov, Arizona State University
Karen Taylor, CUNY
Gérald Tenenbaum, Université de Lorraine
Jesse Thorner, Emory University
Sarah Trebat-Leder, Emory University
Lee Troupe, University of Georgia
Alexandre Turull, University of Florida
Ali Uncu, University of Florida
Robert Vaughan, Pennsylvania State University
Christophe Vignat, Tulane University
Ian Wagner, Emory University
Tanay Wakhare, Quince Orchard High School, Gaithersburg, Maryland
Liuquan Wang, National University of Singapore
Ole Warnaar, The University of Queensland, Australia

Victor Weenink, Radboud University Nijmegen
Michael Woodbury, University of Cologne
Stanley Yao Xiao, University of Waterloo
Ruize Yang, University of Florida
Ae Ja Yee, Pennsylvania State University
Doron Zeilberger, Rutgers University
Wadim Zudilin, University of Newcastle, Australia

Conference Photographs

Krishna Alladi speaking at the Conference Banquet

International Conference on Number Theory in honor of Krishna Alladi's 60th Birthday, Gainesville, Florida, March 17–21, 2016

2003 SASTRA Ramanujan Prize Winner and 2014 Fields Medalist Manjul Bhargava (Princeton University) giving the Opening Lecture of the conference

A section of the audience at the Straughn Center in between lectures. Peter Paule (L) and George Andrews (R) are in the front row and Ken Ono (second row left) is right behind them

Krishna Alladi (center) flanked by the conference organizers Frank Garvan (L) and George Andrews (R)

Krishna Alladi with Doron Zeilberger (Rutgers University) after Doron’s exciting lecture

After 2014 SASTRA Ramanujan Prize Winner James Maynard (University of Oxford) delivered the Ramanujan Colloquium, he accepts a cheque for $5,000 from Ron Graham (University of California San Diego) for his work on the resolution of the problem posed by Paul Erdős on large gaps between primes

Christian Krattenthaler (University of Vienna) plays for the conference participants during the Welcome Reception at the Keene Faculty Center

Conference participants at the Welcome Reception at the elegant Keene Faculty Center

Richard Askey (University of Wisconsin) speaking at the Welcome Reception

Kevin Ford (University of Illinois, Urbana) about to accept a cheque for $5,000 from Ron Graham (University of California San Diego) on behalf of his co-authors Ben Green, S. Konyagin, and Terence Tao for their work on the resolution of the $10,000 problem of Paul Erdős on large gaps between primes

Hugh Montgomery (University of Michigan), Krishna's postdoctoral mentor, delivering the Erdős Colloquium

(L to R) Michael Hirschhorn (University of New South Wales, Australia) and James Sellers (Pennsylvania State University) discussing during a break between sessions at the Straughn Center

(L to R) Doron Zeilberger (Rutgers University) discussing with University of Florida students

Gaurav Bhatnagar (L) and Ole Warnaar (R) enjoying a point being made by Peter Paule (RISC, Austria)

Manjul Bhargava (Princeton University) with the Alladi family at the Alladi home during the conference party. From L to R: daughter Lalitha Alladi, V. R. Srinivasan, wife Mathura Alladi, Manjul Bhargava, Krishna's mother Lalitha Ramakrishnan, son-in-law Jis Joseph, and Krishna

Conference participants on the deck at Alladi's home during the Conference Party

(L to R) Ron Graham (University of California San Diego), Aleksandar Ivic (University of Belgrade), Jean-Marc Deshouillers (University of Bordeaux), and Mel Nathanson (Lehman College, CUNY) in a discussion as they enjoy an Indian dinner at the Alladi House

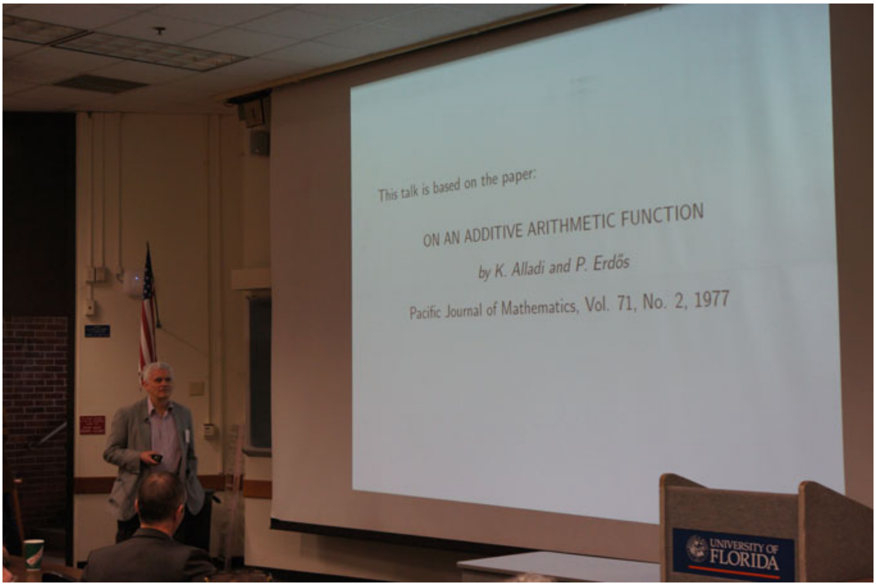
(L to R) Nick Diculeanu (Krishna’s colleague), Sinai Robins (Brown University), and Michael Schlosser (University of Vienna) relaxing at the Conference Party at the Alladi House

The conference attracted a number of graduate students, postdocs, and young researchers

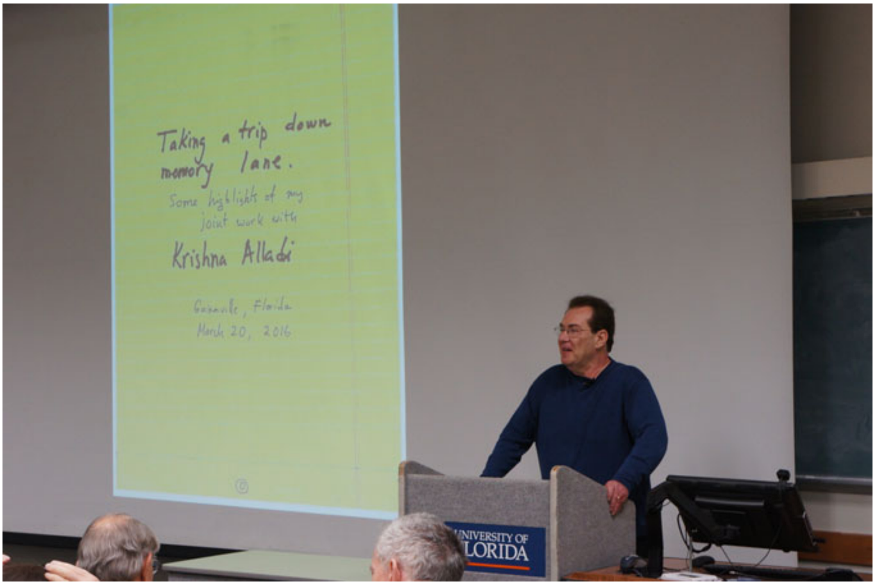
(L to R) Harold Diamond (University of Illinois, Urbana), and University of Florida graduate students Todd Molnar and Jason Johnson, with James Maynard (University of Oxford) at the Alladi House party

Cam Stewart (University of Waterloo), Roger Baker (Brigham Young University), Dale Brownawell (Pennsylvania State University) with Kevin Keating and Li Shen (Krishna's colleagues) in the background

Margaret Somers, who handled all conference arrangements so well, in discussion with Frank Garvan. With them is Krishna's colleague, Nick Dinculeanu



Dorian Goldfeld (Columbia University) giving his talk on extensions of Alladi's early work with Paul Erdős



Alex Berkovich (University of Florida) giving his "Walk Down Memory Lane" talk in honor of Krishna

The audience in Little Hall awaiting the screening of the movie "The Man Who Knew Infinity." George Andrews and Richard Askey share a light moment, while Bruce Berndt and Peter Paule behind them are deep in discussion

The Ramanujan movie "The Man Who Knew Infinity" was screened at the conference as a preview. Krishna Alladi and Manjul Bhargava introduce the movie

A section of the audience listening to the introduction of the movie on Ramanujan

Cyndi Garvan, who meticulously planned many of the conference events, was the emcee for the Conference Banquet. She presented a marvelous slideshow to start the banquet program

Elizabeth Loew (Executive Editor at Springer) giving a warm and amusing speech at the Conference Banquet

2003 SASTRA Ramanujan Prize Winner Kannan Soundararajan (Stanford) speaking at the banquet. Sound first met Alladi when he (Sound) was a high school student in Madras

S. D. Adhikari (Harish-Chandra Institute, Allahabad, India) draping Krishna with a shawl—as per the traditional way in India to honor someone

The audience applauding after Krishna gives his speech of thanks at the banquet

Hershel Farkas (Hebrew University, Jerusalem) and his wife expressing their appreciation to Krishna at the banquet. Farkas is co-editor of the book series Developments in Mathematics (Springer) with Krishna. In the background is Beverly Brechner, Krishna's colleague, speaking to Krishna's mother and Mathura

Krishna with his daughters, Lalitha (L) and Amritha (R), at the Banquet Reception

(L to R) Alice Bertram, Jean Elliott, Peter Elliott (University of Colorado), and Adi Srinivasan (Krishna's son-in-law) at the Banquet Reception

Graduate student Ali Uncu (center) who gave so much technical help for the conference, was ably assisted by Chris Jennings-Shaffer (Oregon State), a graduate of the University of Florida

Gerald Tenenbaum (Institut Élie Cartan, University of Lorraine, France) speaking at the conference

Bruce Berndt (University of Illinois, Urbana) about to start his talk

Krishna with Rochelle Kronzek of World Scientific Publishing Company (WSPC) at the WSPC book exhibit

Krishna with Marc Strauss, Editorial Director of Mathematics at Springer, at the Conference book exhibit

Mathematicians in discussion near the book exhibit. Clockwise from left: Hershel Farkas, Frank Garvan, Sergei Suslov, Ron Graham (standing), Doron Zeilberger and his wife (back to camera)

Wadim Zudilin (University of Newcastle, Australia) gave two talks at the conference, this one being a regular conference lecture, and the second being the Mathematics Colloquium which was the concluding talk of the conference

Krishna and Mathura with the light of their lives, their grandchildren Kamakshi and Keshav. who both enlivened the conference party at the Alladi house

Contents

Plus-Minus Weighted Zero-Sum Constants: A Survey

Sukumar Das Adhikari

Dedicated to Prof. Krishna Alladi on the occasion of his 60th birthday

Abstract A particular weighted generalization of some classical zero-sum constants was first considered about 10 years back. Since then, many people got interested in this generalization. Similar generalizations of other zero-sum constants were also considered and these gave rise to several conjectures and questions; some of these conjectures have been established, some of the questions have been answered. And most interestingly, some applications of this weighted generalization have also been found. There are already some expository articles on the classical results as well as on this particular generalization; here we consider the case with a special weight, namely $\{\pm 1\}$, and dwell mainly on some recent developments not covered in the earlier expository articles.

Keywords Zero-sum constants · Plus-minus weights

2010 Mathematics Subject Classification 11B30 · 11B50 · 05E15

S. D. Adhikari (✉)
Harish-Chandra Research Institute, Chhatnag Road,
Jhunsi, Allahabad 211 019, Uttar Pradesh, India
e-mail: adhikari@hri.res.in

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_1

1 Introduction

Given a finite abelian group G (written additively), a sequence over G is called a *zero-sum sequence* if the sum of its elements is zero (the identity element). For a finite abelian group G, the Davenport constant $\mathsf{D}(G)$ is the smallest natural number k such that any sequence of k elements in G has a non-empty zero-sum subsequence. Though attributed to Davenport, it was first studied by K. Rogers [28] in 1962; this reference was missed-out by most of the early authors working in this area. Initial motivations of Rogers and Davenport for defining this constant were keeping in mind the application in nonunique factorization in algebraic number theory. There have been other applications since then; most notable among them being those by Alford, Granville, and Pomerance [9], in their proof of the infinitude of Carmichael numbers.

Given a finite abelian group $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, with $n_1 | n_2 | \cdots | n_r$, we denote $n_r = \exp(G)$, the exponent of G, and $r(G) = r$ the rank of G. Writing $\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$, it is trivial to see that $\mathsf{D}^*(G) \leq \mathsf{D}(G) \leq |G|$.

One may look into the articles [1, 15, 18] or the book [19], for known results on upper bounds of $\mathsf{D}(G)$ and related open questions.

Another combinatorial invariant $\mathsf{E}(G)$ for a finite abelian group G of n elements is defined to be the smallest positive integer t such that for any sequence of t elements of G, there exists a zero-sum subsequence of length n.

The constants $\mathsf{D}(G)$ and $\mathsf{E}(G)$ were being studied independently until Gao [14] (see also [19], Proposition 5.7.9) established the following result connecting these two invariants.

Theorem 1.1. *If G is a finite abelian group of order n, then*

$$\mathsf{E}(G) = \mathsf{D}(G) + n - 1.$$

We define one more zero-sum constant here.

If G is a finite abelian group with $n = \exp(G)$, the exponent of G, then *the Erdős-Ginzburg-Ziv constant* $\mathsf{s}(G)$ is defined to be the least positive integer k such that any sequence S with length k of elements in G has a zero-sum subsequence of length n.

The above definition is motivated by the theorem of Erdős, Ginzburg, and Ziv [13] (known as the *EGZ theorem*), a prototype of zero-sum theorems, which says that $\mathsf{s}(\mathbb{Z}_n) \leq 2n - 1$. The inequality $\mathsf{s}(\mathbb{Z}_n) \geq 2n - 1$ in the other direction, is easy to observe. Regarding the corresponding question in dimension two, the *Kemnitz Conjecture* $\mathsf{s}(\mathbb{Z}_n^2) = 4n - 3$ has now been settled by Reiher [27]. Once again, we refer to the article [15] of Gao and Geroldinger and the book [19] of Geroldinger and Halter-Koch, for further information about known results and open questions on the Erdős-Ginzburg-Ziv constant.

The following weighted version of the above zero-sum constants was initiated by Adhikari, Chen, Friedlander, Konyagin and Pappalardi [4], Adhikari and Rath [8], Thangadurai [29], Adhikari and Chen [3] and Adhikari, Balasubramanian, Pappalardi and Rath [2].

For a finite abelian group G and a non-empty subset A of $[1, \exp(G) - 1]$, the *Davenport constant of G with weight A*, denoted by $\mathsf{D}_A(G)$, is defined to be the least natural number k such that for any sequence $(x_1, \cdots, x_k)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $a_1, \ldots, a_l \in A$ such that

$$\sum_{i=1}^{l} a_i x_{j_i} = 0.$$

Similarly, for any such weight set A, for a finite abelian group G of order n, the constant $\mathsf{E}_A(G)$ is defined to be the least $t \in \mathbb{N}$ such that for any sequence $(x_1, \cdots, x_t)$ of t elements with $x_i \in G$, there exists an *A-weighted zero-sum subsequence* of length n, that is, there exist indices $j_1, \ldots, j_n \in \mathbb{N}$, $1 \leq j_1 < \cdots < j_n \leq t$, and $\vartheta_1, \ldots, \vartheta_n \in A$ with $\sum_{i=1}^{n} \vartheta_i x_{j_i} = 0$.

The case $A = \{1\}$ corresponds to $\mathsf{D}(G)$ and $\mathsf{E}(G)$.

The generalizations $\mathsf{D}_A(G)$ and $\mathsf{E}_A(G)$ were considered in [4] and [8] for the particular group $\mathbb{Z}/n\mathbb{Z}$; later in [29] and [3], they were introduced for an arbitrary finite abelian group G.

For a finite abelian group G and a non-empty subset A of $[1, \exp(G) - 1]$, one defines $\mathsf{s}_A(G)$ (as introduced in [2]; the notation used here being the standard one at present) to be the least integer k such that any sequence S with length k of elements in G has an *A-weighted zero-sum subsequence* of length $\exp(G)$. Once again, taking $A = \{1\}$, one recovers the classical Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$.

One observes that for the cyclic group $\mathbb{Z}_n$, $\mathsf{s}_A(\mathbb{Z}_n) = \mathsf{E}_A(\mathbb{Z}_n)$.

In the present article, we are mainly concerned with the constants $\mathsf{D}_A(G)$, $\mathsf{E}_A(G)$ and $\mathsf{s}_A(G)$ with the weight set $A = \{\pm 1\}$. In Section 4, we shall have a brief discussion on the plus-minus weighted Harborth constant. For various results with general weight sets, apart from the research papers mentioned in the text, we would like to mention the recent book by Grynkiewicz [20]; Chapter "From Ramanujan to Groups of Rationals: A Personal History of Abstract Multiplicative Functions" of this book is devoted to weighted zero-sum problems.

2 Plus-minus weighted zero-sum constants: $\mathsf{D}_{\{\pm 1\}}(G)$, $\mathsf{E}_{\{\pm 1\}}(G)$

The very first paper [4] introducing $\mathsf{E}_A(G)$, for the particular group $G = \mathbb{Z}_n$, considers the case $A = \{\pm 1\}$ and evaluates $\mathsf{E}_{\{\pm 1\}}(\mathbb{Z}_n)$. More precisely, the following result was proved in [4].

Theorem 2.1. *We have*

$$\mathsf{E}_{\{\pm 1\}}(\mathbb{Z}_n) = n + \lfloor \log_2 n \rfloor.$$

On the way to prove the above result, it was observed in [4] that $\mathsf{D}_{\{\pm 1\}}(\mathbb{Z}_n) = \lfloor \log_2 n \rfloor + 1$; while by the pigeonhole principle it follows that $\mathsf{D}_A(n) \leq \lfloor \log_2 n \rfloor + 1$, considering the sequence $(1, 2, \cdots, 2^r)$, where r is defined by $2^{r+1} \leq n < 2^{r+2}$, one has $\mathsf{D}_A(n) \geq \lfloor \log_2 n \rfloor + 1$.

From the above, the statement in Theorem 2.1 follows rather easily, when n is even. When n is odd, one had to argue differently depending on whether a given sequence is 'complete' or not. Here completeness with respect to some positive integer m requires that the number of elements not divisible by d is at least $d - 1$ for every positive divisor d of m.

The following particular weighted generalization of a result of Bollobás and Leader [11] (see Yu [30] for a simpler proof of the result in [11]) was done in [5].

Theorem 2.2. *Let G be a finite abelian group of order n and let it be of the form $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, where $1 < n_1 \mid \cdots \mid n_r$. Let $A = \{1, -1\}$ and k be a natural number satisfying $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, where $r' = |\{i \in \{1, 2, \cdots, r\} : n_i \text{ is even}\}|$. Then, given a sequence $S = (x_1, x_2, \cdots, x_{n+k})$, with $x_i \in G$, if S has no A-weighted zero-sum subsequence of length n, then there are at least $2^{k+1} - \delta$ distinct A-weighted n-sums, where $\delta = 1$ if $2 \mid n$ and $\delta = 0$ otherwise.*

A result of Yuan and Zeng [31] on the existence of zero-smooth subsequences and a theorem of DeVos, Goddyn, and Mohar [12] which generalizes Kneser's addition theorem [23], were used in the proof of Theorem 2.2.

For a finite abelian group G with $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, $1 < n_1 \mid \cdots \mid n_r$, satisfying $|G| > 2^{(2^{r'-1}-1+\frac{r'}{2})}$, where $r' = |\{i \in \{1, 2, \cdots, r\} : 2 \mid n_i\}|$, and $A = \{1, -1\}$, one has

$$|G| + \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor \leq \mathsf{E}_A(G) \leq |G| + \lfloor \log_2 |G| \rfloor. \tag{1}$$

While the upper bound follows from Theorem 2.2, the lower bound in the above follows from some counterexamples like those given in [4] (see also [6]).

The result (1) gives the exact value of $\mathsf{E}_A(G)$ when G is cyclic (thus giving another proof of the main result in [4]) and unconditional bounds in many cases.

However, we mention that when $A = \{1, -1\}$, finding the corresponding bounds for $\mathsf{D}_A(G)$ for a finite abelian group G and the exact value of $\mathsf{D}_A(G)$ when G is cyclic, is not so difficult (see [4, 6]). Therefore, from the relation

$$\mathsf{E}_A(G) = \mathsf{D}_A(G) + n - 1,$$

for an abelian group G with $|G| = n$ and a non-empty subset A of $\{1, \ldots, n-1\}$, the main result in [4], mentioned in Theorem 2.1, follows.

The above relation, which is a weighted generalization of the result of Gao, stated in Theorem 1, had been expected by Adhikari and Rath [8] and conjectured by Thangadurai [29]; it was established for the group $\mathbb{Z}_p$ by Adhikari and Rath [8] (a conditional general result of Adhikari and Chen [3] also implies the result for the group $\mathbb{Z}_p$), for general cyclic groups by Yuan and Zeng [32] and for general finite abelian groups by Grynkiewicz, Marchan, and Ordaz [21].

For a general finite abelian group, the following bounds were observed in [6].

Theorem 2.3. *Let G be a finite abelian group with $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, where $1 < n_1 | \ldots | n_r$. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + 1 \leq \mathsf{D}_{\{\pm 1\}}(G) \leq \lfloor \log_2 |G| \rfloor + 1.$$

Marchan, Ordaz, and Schmid [26] observed that for the plus-minus weighted Davenport constant, different types of decompositions into direct sums of cyclic groups can be optimal in yielding lower bounds. Defining

$$\mathsf{D}^*_{\{\pm 1\}}(G) = \max \left\{ \sum_{i=1}^{t} \lfloor \log_2 n_i \rfloor + 1 : G \cong \oplus_{i=1}^{t} \mathbb{Z}_{n_i} \text{for positive integers } \ t, n_i \right\},$$

as a corollary of Theorem 2.3, it was first observed in [26] that

$$\mathsf{D}^*_{\{\pm 1\}}(G) \leq \mathsf{D}_{\{\pm 1\}}(G) \leq \mathsf{D}^*_{\{\pm 1\}}(G) + r(G) - 1.$$

By careful analysis, the authors in [26], were able to obtain exact values of $\mathsf{D}_{\{\pm 1\}}(G)$ for certain groups G; examples of groups were also given where the actual value of $\mathsf{D}_{\{\pm 1\}}(G)$ is strictly bigger than $\mathsf{D}^*_{\{\pm 1\}}(G)$.

3 Plus-minus weighted zero-sum constant $\mathsf{s}_{\{\pm 1\}}(G)$

Here, when G is cyclic, Theorem 2.1 takes care of the problem and we have

$$\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_n) = \mathsf{E}_{\{\pm 1\}}(\mathbb{Z}_n) = n + \lfloor \log_2 n \rfloor.$$

In [2] it was proved that

$$\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_n^2) = 2n - 1,$$

when n is odd.

Whereas from the above result, for $G = \mathbb{Z}_n^2$, we have $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_n^2) = 2\ \exp(G) - 1$ if n is odd, for the group $H = \mathbb{Z}_2^2$, the Kemnitz-Reiher Theorem [27] says that (it can also be checked directly) $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_2^2) = 5 = 2\ \exp(H) + 1$.

However, in contrast to these results, in [6], the following asymptotic behavior of $\mathsf{s}_{\{\pm 1\}}(G)$ when $\exp(G)$ is even, was established. It was shown in [6] that, for finite abelian groups of even exponent and fixed rank,

$$\mathsf{s}_{\{\pm 1\}}(G) = \exp(G) + \log_2|G| + O(\log_2\log_2|G|) \quad \text{as} \quad \exp(G) \to \infty.$$

We remark that for the classical question corresponding to the weight $A = \{1\}$, the general upper bound of Alon and Dubiner [10] says that there is an absolute constant $c > 0$ so that

$$\mathsf{s}(\mathbb{Z}_n^d) \le (cd\log_2 d)^d n, \text{ for all } n,$$

which shows that the growth of $\mathsf{s}(\mathbb{Z}_n^d)$ is linear in n. However, this result of Alon and Dubiner is far from the expected one; it has been conjectured [10] that there is an absolute constant c such that

$$\mathsf{s}(\mathbb{Z}_n^d) \le c^d n, \text{ for all } n \text{ and } d.$$

For the plus-minus weight, the question of determining $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_n^r)$, for odd n remains open for $r \ge 3$.

However, in [7], it has been proved that a sequence of length $\frac{(9p-3)}{2}$ over elements of the group $\mathbb{Z}_p^3$, must have a plus-minus weighted zero-sum subsequence of length $3p$; here one expects a sequence of much smaller length to guarantee such a subsequence.

4 Plus-minus weighted Harborth constant

If G is a finite abelian group with $n = \exp(G)$, the exponent of G, then *the Harborth constant* $\mathsf{g}(G)$ is defined to be the least integer k such that any subset of G of cardinality k has a subset of cardinality n whose terms sum to zero.

For the cyclic group, $\mathsf{g}(\mathbb{Z}_n)$, it is n or $n+1$, according to n is odd or even; in the second case when n is even, that the constant is $n+1$ means that there is no set with the desired property at all. More generally, it is known that $\mathsf{g}(G) = |G| + 1$, if and only if G is an elementary 2-group or a cyclic group of even order.

Gao and Thangadurai [17] have shown that $\mathsf{g}(\mathbb{Z}_p^2) = 2p - 1$ for $p \ge 67$ (later in [16], it was shown to hold for $p \ge 47$) and $\mathsf{g}(\mathbb{Z}_4^2) = 9$ and have conjectured that $\mathsf{g}(\mathbb{Z}_n^2)$ equals $2n-1$ or $2n+1$, according as n is odd or even.

Recently, Marchan et al. [24] have shown that for a positive integer n, one has

$$\mathsf{g}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = \begin{cases} 2n+3, & \text{for } n \text{ odd} \\ 2n+2, & \text{for } n \text{ even.} \end{cases}$$

Coming to the corresponding weighted generalization, taking up the case where the weight set $A = \{\pm 1\}$, Marchan et al. [24] have obtained the following results.

Theorem 4.1. *For a positive integer n,*

$$\mathsf{g}_{\{\pm 1\}}(\mathbb{Z}_n) = \begin{cases} n+1, & \text{for } n \equiv 2 \pmod 4 \\ n, & \text{otherwise.} \end{cases}$$

Theorem 4.2. *For a positive integer $n \geq 3$,*

$$\mathsf{g}_{\{\pm 1\}}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + 2.$$

Moreover,

$$\mathsf{g}_{\{\pm 1\}}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) = \mathsf{g}_{\{\pm 1\}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = 5.$$

5 Applications

Halter-Koch [22] has shown that the plus-minus weighted Davenport constant is related to questions on the norms of principal ideals in quadratic number fields. More precisely, he has proved the following.

Theorem 5.1. *Let K be a quadratic algebraic number field and we denote by $\mathcal{O}_K$ its ring of integers and by $\mathcal{C}_K$ its ideal class group. Then $D_{\{\pm 1\}}(\mathcal{C}_K)$ is the smallest positive integer l with the following property:*

If $q_1, q_2, \ldots q_l$ are pairwise coprime positive integers such that their product $q = q_1 \cdots q_l$ is the norm of an ideal of $\mathcal{O}_K$, then some divisor $d > 1$ of q is the norm of a principal ideal of $\mathcal{O}_K$.

The above result related to the plus-minus weighted Davenport constant follows from a more general result, for which we refer to the original paper [22]. The paper of Halter-Koch [22] also has an interpretation of the plus-minus weighted Davenport constant in terms of binary quadratic forms.

For interactions of some weighted zero-sum constants and coding theory, one may look into the recent article of Marchan et al. [25].

References

1. S.D. Adhikari, R. Balasubramanian, P. Rath, Some combinatorial group invariants and their generalizations with weights, in *Additive Combinatorics*. CRM Proceedings and Lecture Notes, vol. 43, ed. by Granville, Nathanson, Solymosi (American Mathematical Society, Providence, RI, 2007), pp. 327–335
2. S.D. Adhikari, R. Balasubramanian, F. Pappalardi, P. Rath, Some zero-sum constants with weights. Proc. Indian Acad. Sci. (Math. Sci.) **118**, 183–188 (2008)
3. S.D. Adhikari, Y.G. Chen, Davenport constant with weights and some related questions - II. J. Comb. Theory. Ser. A. **115**, 178–184 (2008)

4. S.D. Adhikari, Y.G. Chen, J.B. Friedlander, S.V. Konyagin, F. Pappalardi, Contributions to zero-sum problems. Discret. Math. **306**, 1–10 (2006)
5. S.D. Adhikari, M.N. Chintamani, Number of weighted subsequence sums with weights in $\{1, -1\}$. Integers **11**, paper A 36 (2011)
6. S.D. Adhikari, D.J. Grynkiewicz, Z.-W. Sun, On weighted zero-sum sequences. Adv. Appl. Math. **48**, 506–527 (2012)
7. S.D. Adhikari, E. Mazumdar, Modifications of some methods in the study of zero-sum constants. Integers **14**, paper A 25 (2014)
8. S.D. Adhikari, P. Rath, Davenport constant with weights and some related questions, in *Integers*, vol.6, A 30 (2006)
9. W.R. Alford, A. Granville, C. Pomerance, There are infinitely many Carmichael numbers. Ann. Math. **139**(2) no. 3, 703–722 (1994)
10. N. Alon, M. Dubiner, A lattice point problem and additive number theory. Combinatorica **15**, 301–309 (1995)
11. B. Bollobás, I. Leader, The number of k-sums modulo k. J. Number Theory **78**(1), 27–35 (1999)
12. M. DeVos, L. Goddyn, B. Mohar, A generalization of Kneser's addition theorem. Adv. Math. **220**, 1531–1548 (2009)
13. P. Erdős, A. Ginzburg, A. Ziv, Theorem in the additive number theory. Bull. Res. Counc. Isr. **10F**, 41–43 (1961)
14. W.D. Gao, A combinatorial problem on finite abelian groups. J. Number Theory **58**, 100–103 (1996)
15. W.D. Gao, A. Geroldinger, Zero-sum problems in finite abelian groups: a survey. Expo. Math. **24**, 337–369 (2006)
16. W.D. Gao, A. Geroldinger, W.A. Schmid, Inverse zero-sum problems. Acta Arith. **128**, 245–279 (2007)
17. W.D. Gao, R. Thangadurai, A variant of Kemnitz conjecture. J. Comb. Theory. Ser. A. **107**, 69–86 (2004)
18. A. Geroldinger, Additive group theory and non-unique factorizations, in *Combinatorial Number Theory and Additive Group Theory*, ed. by A. Geroldinger, I. Ruzsa. Advanced Courses in Mathematics - CRM Barcelona, (Birkhäuser Verlag, Basel, 2009)
19. A. Geroldinger, F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*. Pure and Applied Mathematics (Boca Raton), 278. (Chapman & Hall/CRC, Boca Raton, 2006)
20. D.J. Grynkiewicz, *Structural Additive Theory*, (Springer, Berlin 2013)
21. D.J. Grynkiewicz, L.E. Marchan, O. Ordaz, A weighted generalization of two theorems of Gao. Ramanujan J. **28**(3), 323–340 (2012)
22. F. Halter-Koch, Arithmetical interpretation of weighted Davenport constants. Arch. Math. (Basel) **103**(2), 125–131 (2014)
23. M. Kneser, Abschätzung der asymptotischen Dichte von Summenmengen. Math. Z. **58**, 459–484 (1953)
24. L.E. Marchan, O. Ordaz, D. Ramos, W.A. Schmid, Some exact values of the Harborth constant and its plus-minus weighted analogue. Arch. Math. (Basel) **101**(6), 501–512 (2013)
25. L.E. Marchan, O. Ordaz, I. Santos, W.A. Schmid, Multi-wise and constrained fully weighted Davenport constants and interactions with coding theory. J. Comb. Theory Ser. A **135**, 237–267 (2015)
26. L.E. Marchan, O. Ordaz, W.A. Schmid, Remarks on the plus-minus weighted Davenport constant. Int. J. Number Theory **10**(5), 1219–1239 (2014)
27. C. Reiher, On Kemnitz's conjecture concerning lattice points in the plane. Ramanujan J. **13**, 333–337 (2007)
28. K. Rogers, A Combinatorial problem in Abelian groups. Proc. Camb. Phil. Soc. **59**, 559–562 (1963)
29. R. Thangadurai, A variant of Davenport's constant. Proc. Indian Acad. Sci. (Math. Sci.) **117**(2), 147–158 (2007)

30. H.B. Yu, A simple proof of a theorem of Bollobás and Leader. Proc. Amer. Math. Soc. **131**(9), 2639–2640 (2003)
31. P. Yuan, X. Zeng, A new result on Davenport Constant. J. Number Theory **129**, 3026–3028 (2009)
32. P. Yuan, X. Zeng, Davenport constant with weights. Eur. J. Comb. **31**, 677–680 (2010)

Vector-valued Modular Forms and the Seventh Order Mock Theta Functions

Nickolas Andersen

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract In 1988, Hickerson proved the mock theta conjectures (identities involving Ramanujan's fifth order mock theta functions) using q-series methods. In a follow-up paper he proved three analogous identities which involve Ramanujan's seventh order mock theta functions. Recently, the author gave a unified proof of the mock theta conjectures using the theory of vector-valued modular forms which transform according to the Weil representation. Here we apply the method to Hickerson's seventh order identities.

Keywords Mock theta functions · Vector-valued modular forms · Weil representation

2010 Mathematics Subject Classification 11F37

1 Introduction

In his last letter to Hardy, Ramanujan introduced a new class of functions which he called mock theta functions, and he listed 17 examples [3, p. 220]. Each of these he labeled third order, fifth order, or seventh order. The seventh order mock theta functions are

N. Andersen (✉)
UCLA Mathematics Department, Box 951555,
Los Angeles, CA 90095-1555, USA
e-mail: nandersen@math.ucla.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_2

$$\mathscr{F}_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n},$$
$$\mathscr{F}_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}},$$
$$\mathscr{F}_2(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1},q)_{n+1}}.$$

Here we have used the standard q-Pochhammer notation $(a;q)_n := \prod_{m=0}^{n-1}(1-aq^m)$. In Ramanujan's lost notebook there are many identities which relate linear combinations of mock theta functions to modular forms. Andrews and Garvan [2] named ten of these identities, those which involved the fifth order mock theta functions, the *mock theta conjectures*. Hickerson proved two of these identities in [11]; his proof, together with the work of Andrews and Garvan [2], established the truth of the mock theta conjectures. In a companion paper [10] immediately following [11], Hickerson proved analogous identities for the seventh order mock theta functions, namely

$$\mathscr{F}_0(q) = 2qM\left(\frac{1}{7},q^7\right) + 2 - \frac{j(q^3,q^7)^2}{(q,q)_\infty}, \tag{1.1}$$
$$\mathscr{F}_1(q) = 2qM\left(\frac{2}{7},q^7\right) + q\frac{j(q,q^7)^2}{(q,q)_\infty}, \tag{1.2}$$
$$\mathscr{F}_2(q) = 2qM\left(\frac{3}{7},q^7\right) + \frac{j(q^2,q^7)^2}{(q,q)_\infty}. \tag{1.3}$$

Here (following the notation of [9])

$$M(r,q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^r;q)_n(q^{1-r};q)_n}$$

and

$$j(q^\rho,q^7) := (q^\rho,q^7)_\infty(q^{7-\rho},q^7)_\infty(q^7,q^7)_\infty.$$

We will refer to (1.1)–(1.3) as the *seventh order mock theta conjectures*.

Zwegers [14] showed that the mock theta functions can be completed to real analytic modular forms of weight 1/2 by multiplying by a suitable rational power of q and adding nonholomorphic integrals of certain unary theta series of weight 3/2. This allows the mock theta functions to be studied using the theory of modular forms. Recently the author [1], building on Zwegers' work and work of Bringmann–Ono [5], proved the mock theta conjectures using the theory of vector-valued modular forms. The purpose of this paper is to apply this method to prove the seventh order mock theta conjectures.

We begin by defining two nonholomorphic vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ corresponding to the left-hand and right-hand sides of (1.1)–(1.3), respectively, and we establish their transformation properties using the results of [5, 8, 14]. Next, we construct a holomorphic vector-valued modular form $\mathscr{H}$ from the components of $\mathbf{F} - \mathbf{G}$ which transforms according to the Weil representation (see Lemma 4 below). There is a natural isomorphism between the space of such forms and the space $J_{1,42}$ of Jacobi forms of weight 1 and index 42. The seventh order mock theta conjectures follow from the result of Skoruppa that $J_{1,m} = \{0\}$ for all $m \geq 1$.

2 Definitions and Transformations

In this section, we describe the transformation behavior for the functions $M(\frac{a}{7}, q)$ and $j(q^\rho, q^7)$ and the mock theta functions under the generators

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{Z})$. We employ the usual $\big|_k$ notation, defined for $k \in \mathbb{R}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by

$$(f\big|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

We always take $\arg z \in (-\pi, \pi]$. It is not always the case that $f\big|_k AB = f\big|_k A\big|_k B$, but for $k \in \frac{1}{2}\mathbb{Z}$ we have

$$f\big|_k AB = \pm f\big|_k A\big|_k B, \tag{2.1}$$

(see [12, §2.6]). Much of the arithmetic here and throughout the paper takes place in the splitting field of the polynomial $x^6 - 7x^4 + 14x^2 - 7$, which has roots $\pm\kappa$, $\pm\lambda$, $\pm\mu$, where

$$\kappa := 2\sin\frac{\pi}{7}, \qquad \lambda := 2\sin\frac{2\pi}{7}, \qquad \mu := 2\sin\frac{3\pi}{7}. \tag{2.2}$$

The modular transformations satisfied by the mock theta functions $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{F}_2$ are given in Section 4.3 of [14]. The nonholomorphic completions are written in terms of the nonholomorphic Eichler integral (see [14, Proposition 4.2])

$$R_{a,b}(z) := -i\int_{-\bar{z}}^{i\infty} \frac{g_{a,-b}(\tau)}{\sqrt{-i(\tau+z)}}d\tau,$$

where $g_{a,b}$ (see [14, §1.5]) is the unary theta function

$$g_{a,b}(z) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2 z+2\pi i\nu b}.$$

Let $q := \exp(2\pi iz)$ and $\zeta_m := \exp(2\pi i/m)$. Following §4.3 of [14] we define

$$\widetilde{\mathscr{F}}_0(z) := q^{-\frac{1}{168}}\mathscr{F}_0(q) + \zeta_{14}\left(\zeta_{12}^{-1}R_{-\frac{1}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{13}{42},\frac{1}{2}}\right)(21z), \tag{2.3}$$

$$\widetilde{\mathscr{F}}_1(z) := q^{-\frac{25}{168}}\mathscr{F}_1(q) + \zeta_{7}\left(\zeta_{12}^{-1}R_{\frac{5}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{19}{42},\frac{1}{2}}\right)(21z), \tag{2.4}$$

$$\widetilde{\mathscr{F}}_2(z) := q^{\frac{47}{168}}\mathscr{F}_2(q) + \zeta_{14}^3\left(\zeta_{12}^{-1}R_{\frac{11}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{25}{42},\frac{1}{2}}\right)(21z). \tag{2.5}$$

Note that we have used Proposition 1.5 of [14] to slightly modify the components of $G_7(\tau)$ on p. 75 of [14]. The following is Proposition 4.5 of [14] (we have rearranged the order of the components of the vector F_7 in that proposition).

Proposition 1. *The vector*

$$\boldsymbol{F}(z) := \left(\widetilde{\mathscr{F}}_0(z), \widetilde{\mathscr{F}}_1(z), \widetilde{\mathscr{F}}_2(z)\right)^{\mathsf{T}} \tag{2.6}$$

satisfies the transformations

$$\boldsymbol{F}\big|_{\frac{1}{2}}T = M_T\,\boldsymbol{F} \quad \text{and} \quad \boldsymbol{F}\big|_{\frac{1}{2}}S = \zeta_8^{-1}\frac{1}{\sqrt{7}}M_S\,\boldsymbol{F},$$

where

$$M_T = \begin{pmatrix} \zeta_{168}^{-1} & 0 & 0 \\ 0 & \zeta_{168}^{-25} & 0 \\ 0 & 0 & \zeta_{168}^{47} \end{pmatrix} \quad \text{and} \quad M_S = \begin{pmatrix} \kappa & \lambda & \mu \\ \lambda & -\mu & \kappa \\ \mu & \kappa & -\lambda \end{pmatrix}.$$

Following [5, 9], we define, for $1 \le a \le 6$, the functions

$$M\left(\frac{a}{7}, z\right) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{\frac{a}{7}}; q)_n (q^{1-\frac{a}{7}}; q)_n}, \tag{2.7}$$

$$N\left(\frac{a}{7}, z\right) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta_7^a q; q)_n (\zeta_7^{-a} q; q)_n}. \tag{2.8}$$

Clearly, we have $M(1-\frac{a}{7}, z) = M(\frac{a}{7}, z)$ and $N(1-\frac{a}{7}, z) = N(\frac{a}{7}, z)$. Bringmann and Ono [5] also define auxiliary functions $M(a, b, 7, z)$ and $N(a, b, 7, z)$ for $0 \le a \le 6$ and $1 \le b \le 6$. Together, the completed versions of these functions form a set that is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$ (see [5, Theorem 3.4]). Garvan [8] corrected the definitions of these functions and wrote their transformation formulas more explicitly, so in what follows we reference his paper.

The nonholomorphic completions for $M(\frac{a}{7}, z)$ and $N(\frac{a}{7}, z)$ are given in terms of integrals of weight 3/2 theta functions $\Theta_1(\frac{a}{7}, z)$ and $\Theta_1(0, -a, 7, z)$ (defined in Section 2 of [8]). A straightforward computation shows that

$$\Theta_1(0, -a, 7, z) = 21\sqrt{3}\,\zeta_{14}^a \left(\zeta_{12}^{-1}\, g_{\frac{6a-7}{42}, -\frac{1}{2}}(3z) + \zeta_{12}\, g_{\frac{6a+7}{42}, -\frac{1}{2}}(3z)\right).$$

Following (2.5), (2.6), (3.5), and (3.6) of [8], we define

$$\widetilde{M}\left(\frac{a}{7}, z\right) := 2q^{\frac{3a}{14}(1-\frac{a}{7})-\frac{1}{24}}\, M\left(\frac{a}{7}, z\right) + \zeta_{14}^a \left(\zeta_{12}^{-1} R_{\frac{6a-7}{42}, \frac{1}{2}} + \zeta_{12} R_{\frac{6a+7}{42}, \frac{1}{2}}\right)(3z) + \begin{cases} 2q^{-\frac{1}{1176}} & \text{if } a = 1, \\ 0 & \text{if } a = 2, 3, \end{cases} \tag{2.9}$$

$$\widetilde{N}\left(\frac{a}{7}, z\right) := \csc\left(\frac{a\pi}{7}\right) q^{-\frac{1}{24}} N\left(\frac{a}{7}, z\right) + \frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(\frac{a}{7}, \tau)}{\sqrt{-i(\tau + z)}}\, d\tau. \tag{2.10}$$

The completed functions $\widetilde{M}(a, b, z) := \mathscr{G}_2(a, b, 7; z)$ and $\widetilde{N}(a, b, z) := \mathscr{G}_1(a, b, 7; z)$ are defined in (3.7) and (3.8) of that paper. By Theorem 3.1 of [8] we have

$$\widetilde{M}\left(\frac{a}{7}, z\right)\Big|_{\frac{1}{2}} T^7 = \widetilde{M}\left(\frac{a}{7}, z\right) \times \begin{cases} \zeta_{168}^{-1} & \text{if } a = 1, \\ \zeta_{168}^{-25} & \text{if } a = 2, \\ \zeta_{168}^{47} & \text{if } a = 3, \end{cases} \tag{2.11}$$

$$\widetilde{N}\left(\frac{a}{7}, z\right)\Big|_{\frac{1}{2}} T = \zeta_{24}^{-1} \widetilde{N}\left(\frac{a}{7}, z\right), \tag{2.12}$$

and

$$\widetilde{M}\left(\frac{a}{7}, z\right)\Big|_{\frac{1}{2}} S = \zeta_8^{-1} \widetilde{N}\left(\frac{a}{7}, z\right). \tag{2.13}$$

The functions $j(q^\rho, q^7)$ are essentially theta functions of weight $1/2$. It will be more convenient to work with (following [4])

$$f_\rho(z) = f_{7,\rho}(z) := q^{\frac{(7-2\rho)^2}{56}}\, j(q^\rho, q^7). \tag{2.14}$$

The transformation properties of theta functions are well-known; for $f_\rho(z)$ we have (see e.g. [9, pp. 217-218])

$$\left(f_1, f_2, f_3\right)^{\mathsf{T}}\Big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}} \begin{pmatrix} \lambda & -\mu & \kappa \\ -\mu & -\kappa & \lambda \\ \kappa & \lambda & \mu \end{pmatrix} \left(f_1, f_2, f_3\right)^{\mathsf{T}}. \tag{2.15}$$

The mock theta conjectures (1.1)–(1.3) are implied by the corresponding completed versions:

$$\widetilde{\mathscr{F}}_0(z) = \widetilde{M}\left(\frac{1}{7}, 7z\right) - \frac{f_3^2(z)}{\eta(z)}, \tag{2.16}$$

$$\widetilde{\mathscr{F}}_1(z) = \widetilde{M}\left(\frac{2}{7}, 7z\right) + \frac{f_1^2(z)}{\eta(z)}, \tag{2.17}$$

$$\widetilde{\mathscr{F}}_2(z) = \widetilde{M}\left(\frac{3}{7}, 7z\right) + \frac{f_2^2(z)}{\eta(z)}. \tag{2.18}$$

Motivated by (2.6) and (2.16)–(2.18), we define the vector

$$\boldsymbol{G}(z) := \begin{pmatrix} \widetilde{M}\left(\frac{1}{7}, 7z\right) - \dfrac{f_3^2(z)}{\eta(z)} \\ \widetilde{M}\left(\frac{2}{7}, 7z\right) + \dfrac{f_1^2(z)}{\eta(z)} \\ \widetilde{M}\left(\frac{3}{7}, 7z\right) + \dfrac{f_2^2(z)}{\eta(z)} \end{pmatrix}. \tag{2.19}$$

To prove that $\boldsymbol{F} = \boldsymbol{G}$ we first show that they transform in the same way.

Proposition 2. *The vector $\boldsymbol{G}(z)$ defined in (2.19) satisfies the transformations*

$$\boldsymbol{G}\big|_{\frac{1}{2}} T = M_T\, \boldsymbol{G} \quad \textit{and} \quad \boldsymbol{G}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}} M_S\, \boldsymbol{G}, \tag{2.20}$$

where M_T and M_S are as in Proposition 1.

In order to prove Proposition 2 we require the following three identities (equivalent identities can be found on p. 220 of [9] without proof).

Lemma 1. *Let κ, λ, and μ be as in (2.2). Then*

$$\begin{aligned} \widetilde{N}\left(\frac{1}{7}, z\right) &- \left(\kappa\, \widetilde{M}\left(\frac{1}{7}, 49z\right) + \lambda\, \widetilde{M}\left(\frac{2}{7}, 49z\right) + \mu\, \widetilde{M}\left(\frac{3}{7}, 49z\right)\right) \\ &= \frac{1}{\eta(7z)}\left[\frac{1}{\sqrt{7}}\big(\kappa\, f_1(7z) + \lambda\, f_2(7z) + \mu\, f_3(7z)\big)^2 \right. \\ &\qquad \left. - \kappa\, f_3^2(7z) + \lambda\, f_1^2(7z) + \mu\, f_2^2(7z)\right]. \end{aligned} \tag{2.21}$$

We defer the proof of Lemma 1 to Section 5; here we deduce two immediate consequences. Note that the right-hand side of (2.21) is holomorphic; this implies that the nonholomorphic completion terms on the left-hand side sum to zero. By (2.7), the coefficients of $N(\frac{a}{7}, z)$ lie in $\mathbb{Q}(\zeta_7 + \zeta_7^{-1}) = \mathbb{Q}(\kappa^2)$, and the automorphisms $\kappa^2 \mapsto \lambda^2$ and $\kappa^2 \mapsto \mu^2$ map $N(\frac{1}{7}, z)$ to $N(\frac{2}{7}, z)$ and $N(\frac{3}{7}, z)$, respectively. By (2.10) it follows that the coefficients of both sides of (2.21) lie in $\mathbb{Q}(\kappa)$. Let τ_1 and τ_2 be the automorphisms

$$\tau_1 = (\kappa \mapsto \lambda,\ \lambda \mapsto -\mu,\ \mu \mapsto \kappa),$$
$$\tau_2 = (\kappa \mapsto \mu,\ \lambda \mapsto \kappa,\ \mu \mapsto -\lambda).$$

Since $\sqrt{7} = \kappa\lambda\mu$, we have $\tau_1(\sqrt{7}) = \tau_2(\sqrt{7}) = -\sqrt{7}$. Applying τ_1 and τ_2 to Lemma 1 gives the following identities.

Lemma 2. *Let* κ*,* λ*, and* μ *be as in* (2.2). *Then*

$$\widetilde{N}\left(\frac{2}{7}, z\right) - \left(\lambda\, \widetilde{M}\left(\frac{1}{7}, 49z\right) - \mu\, \widetilde{M}\left(\frac{2}{7}, 49z\right) + \kappa\, \widetilde{M}\left(\frac{3}{7}, 49z\right)\right)$$
$$= \frac{1}{\eta(7z)}\left[-\frac{1}{\sqrt{7}}\big(\lambda\, f_1(7z) - \mu\, f_2(7z) + \kappa\, f_3(7z)\big)^2 - \lambda\, f_3^2(7z) - \mu\, f_1^2(7z) + \kappa\, f_2^2(7z)\right]. \tag{2.22}$$

Lemma 3. *Let* κ*,* λ*, and* μ *be as in* (2.2). *Then*

$$\widetilde{N}\left(\frac{3}{7}, z\right) - \left(\mu\, \widetilde{M}\left(\frac{1}{7}, 49z\right) + \kappa\, \widetilde{M}\left(\frac{2}{7}, 49z\right) - \lambda\, \widetilde{M}\left(\frac{3}{7}, 49z\right)\right)$$
$$= \frac{1}{\eta(7z)}\left[-\frac{1}{\sqrt{7}}\big(\mu\, f_1(7z) + \kappa\, f_2(7z) - \lambda\, f_3(7z)\big)^2 - \mu\, f_3^2(7z) + \kappa\, f_1^2(7z) - \lambda\, f_2^2(7z)\right]. \tag{2.23}$$

Proof of Proposition 2. The transformation with respect to T follows immediately from (2.11). Let $G_j(z)$ denote the j-th component of $\boldsymbol{G}(z)$. By (2.13), (2.15), and the fact that $\eta\big|_{\frac{1}{2}} S = \zeta_8^{-1}\eta$, we have

$$G_1(z)\big|_{\frac{1}{2}} S = \zeta_8^{-1}\frac{1}{\sqrt{7}}\left[\widetilde{N}\left(\frac{1}{7}, \frac{z}{7}\right) - \frac{1}{\sqrt{7}}\frac{\big(\kappa f_1(z) + \lambda f_2(z) + \mu f_3(z)\big)^2}{\eta(z)}\right].$$

Applying Lemma 1 with z replaced by $\frac{z}{7}$, we find that

$$G_1(z)\big|_{\frac{1}{2}} S = \zeta_8^{-1}\frac{1}{\sqrt{7}}\left[\kappa\, \widetilde{M}\left(\frac{1}{7}, 7z\right) + \lambda\, \widetilde{M}\left(\frac{2}{7}, 7z\right) + \mu\, \widetilde{M}\left(\frac{3}{7}, 7z\right) - \frac{\kappa f_3^2(z) - \lambda f_1^2(z) - \mu f_2^2(z)}{\eta(z)}\right]$$
$$= \zeta_8^{-1}\frac{1}{\sqrt{7}}\big(\kappa G_1(z) + \lambda G_2(z) + \mu G_3(z)\big).$$

The transformations for G_2 and G_3 are similarly obtained using Lemmas 2 and 3, respectively.

3 Vector-valued Modular Forms and the Weil Representation

In this section, we define vector-valued modular forms which transform according to the Weil representation, and we construct such a form from the components of $\boldsymbol{F} - \boldsymbol{G}$. A good reference for this material is [6, Section 1.1].

Let $L = \mathbb{Z}$ be the lattice with associated bilinear form $(x, y) = -84xy$ and quadratic form $q(x) = -42x^2$. The dual lattice is $L' = \frac{1}{84}\mathbb{Z}$. Let $\{\mathfrak{e}_h : \frac{h}{84} \in \frac{1}{84}\mathbb{Z}/\mathbb{Z}\}$ denote the standard basis for $\mathbb{C}[L'/L]$. Let $\mathrm{Mp}_2(\mathbb{R})$ denote the metaplectic twofold cover of $\mathrm{SL}_2(\mathbb{R})$; the elements of this group are pairs (M, ϕ), where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi^2(z) = cz + d$. Let $\mathrm{Mp}_2(\mathbb{Z})$ denote the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map; this group is generated by

$$\widetilde{T} := (T, 1) \quad \text{and} \quad \widetilde{S} := (S, \sqrt{z}).$$

The Weil representation can be defined by its action on these generators, namely

$$\rho_L(T, 1)\mathfrak{e}_h := \zeta_{168}^{-h^2}\,\mathfrak{e}_h, \tag{3.1}$$

$$\rho_L(S, \sqrt{z})\mathfrak{e}_h := \frac{1}{\sqrt{-84i}} \sum_{h'(84)} \zeta_{84}^{hh'}\,\mathfrak{e}_{h'}. \tag{3.2}$$

A holomorphic function $\mathscr{F} : \mathbb{H} \to \mathbb{C}[L'/L]$ is a vector-valued modular form of weight $1/2$ and representation ρ_L if

$$\mathscr{F}(\gamma z) = \phi(z)\rho_L(\gamma, \phi)\mathscr{F}(z) \qquad \text{for all } (\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z}) \tag{3.3}$$

and if $\mathscr{F}$ is holomorphic at ∞ (i.e. if the components of $\mathscr{F}$ are holomorphic at ∞ in the usual sense). The following lemma shows how to construct such forms from vectors that transform as in Propositions 1 and 2.

Lemma 4. *Suppose that* $\boldsymbol{H} = (H_1, H_2, H_3)$ *satisfies*

$$\boldsymbol{H}\big|_{\frac{1}{2}} T = M_T\, \boldsymbol{H} \quad \textit{and} \quad \boldsymbol{H}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}}\, M_S\, \boldsymbol{H},$$

where M_T *and* M_S *are as in Proposition 1, and define*

$$\mathscr{H}(z) := \sum_{h=1,13,29,41} a(h) H_1(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{h=5,19,23,37} H_2(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{h=11,17,25,31} H_3(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}),$$

where

$$a_h = \begin{cases} +1 & \text{if } h = 1, 41, \\ -1 & \text{if } h = 13, 29. \end{cases}$$

Then $\mathscr{H}(z)$ *satisfies* (3.3).

Proof. The proof is a straightforward but tedious verification involving (3.1) and (3.2) that is best carried out with the aid of a computer algebra system; the author used MATHEMATICA.

4 Proof of the Mock Theta Conjectures

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be as in Section 2. To prove (2.16)–(2.18) we will prove that $\boldsymbol{H} := \boldsymbol{F} - \boldsymbol{G} = 0$. It is easy to see that the nonholomorphic parts of $\boldsymbol{F}$ and $\boldsymbol{G}$ agree, as do the terms in the Fourier expansion involving negative powers of q. It follows that the function $\mathscr{H}$ defined in Lemma 4 is a vector-valued modular form of weight $1/2$ with representation ρ_L. By Theorem 5.1 of [7], the space of such forms is canonically isomorphic to the space $J_{1,42}$ of Jacobi forms of weight 1 and index 42. By a theorem of Skoruppa [13, Satz 6.1] (see also [7, Theorem 5.7]), we have $J_{1,m} = \{0\}$ for all m; therefore $\mathscr{H} = 0$. The seventh order mock theta conjectures (1.1)–(1.3) follow. □

5 Proof of Lemma 1

We begin with a lemma which describes the modular transformation properties of $f_\rho(z)$. Let ν_η denote the multiplier system for the eta function (see [4, (2.5)]). For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define

$$\gamma_n := \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix}.$$

Lemma 5. *Let* $\rho \in \{1, 2, 3\}$. *If*

$$\gamma \in \Gamma(7) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod 7 \right\},$$

then

$$f_\rho(\gamma z) = \nu_\eta^3(\gamma_7)\sqrt{cz + d}\, f_\rho(z). \tag{5.1}$$

Proof. Suppose that $\rho \in \{1, 2, 3\}$ and that $\gamma \in \Gamma(7)$. Lemma 2.1 of [4] gives

$$f_\rho(\gamma z) = (-1)^{\rho b + \lfloor \rho a/7 \rfloor}\, \zeta_{14}^{\rho^2 ab}\, \nu_\eta^3(\gamma_7)\sqrt{cz + d}\, f_\rho(z).$$

Writing $a = 1 + 7r$ and $b = 7b'$, we find that

$$(-1)^{\rho b+\lfloor \rho a/7\rfloor} \zeta_{14}^{\rho^2 ab} = (-1)^{\rho(b+r+\rho br+\rho b')}. \tag{5.2}$$

Using the fact that $br + r \equiv 0 \pmod 2$ we find that, in each case, the right-hand side of (5.2) equals 1. This completes the proof.

We are now ready to prove Lemma 1. Let $L(z)$ and $R(z)$ denote the left-hand and right-hand sides of (2.21), respectively. Let Γ denote the congruence subgroup

$$\Gamma = \Gamma_0(49) \cap \Gamma_1(7) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod 49 \text{ and } a, d \equiv 1 \bmod 7 \right\}.$$

We claim that

$$\eta(z)L(z),\ \eta(z)R(z) \in M_1(\Gamma), \tag{5.3}$$

where $M_k(G)$ (resp. $M_k^!(G)$) denotes the space of holomorphic (resp. weakly holomorphic) modular forms of weight k on $G \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\frac{1}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 14,$$

so once (5.3) is established it suffices to check that the first 15 coefficients of $\eta(z)L(z)$ and $\eta(z)R(z)$ agree. A computation shows that the Fourier expansion of each function begins

$$2\left(\frac{1}{\kappa} - \kappa\right) + 2\kappa q - 2\mu q^3 - 2\left(\frac{2}{\mu} - \mu - \frac{1}{\lambda}\right) q^4 - 2\kappa q^5 + 2\lambda q^6 + 2\left(\frac{2}{\kappa} - 2\kappa + \frac{1}{\mu}\right) q^7$$
$$+ 4\kappa q^8 + 2\left(k - \frac{2}{\kappa} + 2\mu - \frac{1}{\mu}\right) q^9 - 2\mu q^{10} + 2(\mu + \lambda - 2\kappa) q^{14} + \dots.$$

To prove (5.3), we first note that Theorem 5.1 of [8] shows that $\eta(49z)L(z) \in M_1^!(\Gamma)$; since $\eta(z)/\eta(49z) \in M_0^!(\Gamma_0(49))$ it follows that $\eta(z)L(z) \in M_1^!(\Gamma)$. Using Lemma 5 we find that $\eta(z)R(z) \in M_1^!(\Gamma)$ provided that

$$\frac{\nu_\eta(\gamma)\nu_\eta^6(\gamma_{49})}{\nu_\eta(\gamma_7)} = 1. \tag{5.4}$$

This follows from a computation involving the definition of ν_η [4, (2.5)].

It remains to show that $\eta(z)L(z)$ and $\eta(z)R(z)$ are holomorphic at the cusps. Using MAGMA we compute a set of Γ-inequivalent cusp representatives:

$$\left\{\infty,\ 0,\ \frac{1}{7},\ \frac{3}{17},\ \frac{5}{28},\ \frac{3}{16},\ \frac{4}{21},\ \frac{3}{14},\ \frac{8}{35},\ \frac{5}{21},\ \frac{2}{7},\ \frac{5}{14},\ \frac{18}{49},\ \frac{13}{35},\ \frac{8}{21},\ \frac{19}{49},\ \frac{11}{28},\ \frac{3}{7},\ \frac{4}{7},\ \frac{13}{21},\ \frac{9}{14},\ \frac{5}{7},\ \frac{11}{14},\ \frac{6}{7}\right\}. \tag{5.5}$$

Given a cusp $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ and a meromorphic modular form f of weight k with Fourier expansion $f(z) = \sum_{n\in\mathbb{Q}} a(n)q^n$, the invariant order of f at $\mathfrak{a}$ is defined as

$$\mathrm{ord}(f, \infty) := \min\{n : a(n) \neq 0\},$$
$$\mathrm{ord}(f, \mathfrak{a}) := \mathrm{ord}(f\big|_k \delta_{\mathfrak{a}}, \infty),$$

where $\delta_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$ sends ∞ to $\mathfrak{a}$. For $N \in \mathbb{N}$, we have the relation (see e.g. [4, (1.7)])

$$\mathrm{ord}(f(Nz), \tfrac{r}{s}) = \frac{(N,s)^2}{N}\,\mathrm{ord}(f, \tfrac{Nr}{s}). \tag{5.6}$$

We extend this definition to functions f in the set

$$S := \left\{\widetilde{M}(\tfrac{a}{7}, z),\ \widetilde{N}(\tfrac{a}{7}, z) : a = 1, 2, 3\right\} \cup \left\{\widetilde{M}(a, b, z),\ \widetilde{N}(a, b, z) : 0 \leq a \leq 6,\ 1 \leq b \leq 6\right\}$$

by defining the orders of these functions at ∞ to be the orders of their holomorphic parts at ∞ (see Sects. 6.2, (2.1)–(2.4), and (3.5)–(3.8) of [8]); that is,

$$\mathrm{ord}\left(\widetilde{M}\left(\frac{a}{7}, z\right), \infty\right) = \mathrm{ord}\left(\widetilde{M}(a, b, z), \infty\right) := \begin{cases} -\frac{1}{24} & \text{if } a = 0, \\ -\frac{1}{1176} & \text{if } a = 1, 6, \\ \frac{3a}{14}\left(1 - \frac{a}{7}\right) - \frac{1}{24} & \text{otherwise,} \end{cases} \tag{5.7}$$

$$\mathrm{ord}\left(\widetilde{N}\left(\frac{a}{7}, z\right), \infty\right) := -\frac{1}{24}, \tag{5.8}$$

$$\mathrm{ord}\left(\widetilde{N}(a, b, z), \infty\right) := \frac{b}{7}\left(\frac{1}{2} + k(b, 7)\right) - \frac{3b^2}{98} - \frac{1}{24}, \tag{5.9}$$

where

$$k(b, 7) := \begin{cases} 0 & \text{if } b = 1, \\ 1 & \text{if } b = 2, 3, \\ 2 & \text{if } b = 4, 5, \\ 3 & \text{if } b = 6. \end{cases}$$

Lastly, for $f \in S$ we define

$$\mathrm{ord}\,(f, \mathfrak{a}) := \mathrm{ord}(f\big|_{\frac{1}{2}} \delta_{\mathfrak{a}}, \infty). \tag{5.10}$$

This is well-defined since S is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$. By this same fact, we have

$$\min_{\text{cusps } \mathfrak{a}} \operatorname{ord}(f, \mathfrak{a}) \geq \min_{g \in S} \operatorname{ord}(g, \infty) = -\frac{1}{24} \tag{5.11}$$

for all $f \in S$, from which it follows that

$$\operatorname{ord}(\eta f, \mathfrak{a}) \geq 0$$

for all cusps $\mathfrak{a}$ and for all $f \in S$.

To determine the order of $\eta(z)\widetilde{M}(\frac{a}{7}, 49z)$ at the cusps of Γ, we write

$$\eta(z)\widetilde{M}(\tfrac{a}{7}, 49z) = \frac{\eta(z)}{\eta(49)} m(49z), \qquad \text{where } m(z) = \eta(z)\widetilde{M}(\tfrac{a}{7}, z).$$

The cusps of $\Gamma_0(49)$ are ∞ and $\frac{r}{7}$, $0 \leq r \leq 6$. By (5.6) the function $\eta(z)/\eta(49z)$ is holomorphic at every cusp except for those which are $\Gamma_0(49)$-equivalent to ∞ (the latter are ∞, $\frac{18}{49}$, and $\frac{19}{49}$ in (5.5)); there we have $\operatorname{ord}(\eta(z)/\eta(49z), \infty) = -2$. By (5.11), to show that $\eta(z)\widetilde{M}(\frac{a}{7}, 49z)$ is holomorphic at every cusp, it suffices to verify that $\operatorname{ord}(m(49z), \frac{r}{49}) \geq 2$ for $r = 18, 19$. By (5.6), [8, Theorems 3.1 and 3.2], the fact that $\left(\begin{smallmatrix} r & r-1 \\ 1 & 1 \end{smallmatrix}\right) = T^r ST$, and (2.1), we have

$$\begin{aligned} \operatorname{ord}\left(m(49z), \frac{18}{49}\right) &= 49 \operatorname{ord}(m(z), 18) \\ &= 49\left(\frac{1}{24} + \operatorname{ord}\left(\widetilde{M}(4a \bmod 7, 4a \bmod 7, z), \infty\right)\right) \\ &= \begin{cases} 46 & \text{if } a = 1, \\ 2 & \text{if } a = 2, \\ 50 & \text{if } a = 3. \end{cases} \end{aligned}$$

A similar computation shows that $\operatorname{ord}(m(49z), \frac{19}{49}) \geq 2$. Since $L(z)$ is holomorphic on $\mathbb{H}$, we have, for each cusp $\mathfrak{a}$, the inequality

$$\begin{aligned} &\operatorname{ord}(\eta(z)L(z), \mathfrak{a}) \\ &\geq \min\left\{\operatorname{ord}(\eta(z)f(z), \mathfrak{a}) : f(z) = \widetilde{N}(\tfrac{1}{7}, z) \text{ or } f(z) = \widetilde{M}(\tfrac{a}{7}, 49z), 1 \leq a \leq 3\right\} \geq 0. \end{aligned}$$

We turn to $\eta(z)R(z)$. Using Lemma 3.2 of [4], we find that

$$\mathrm{ord}(f_\rho, \tfrac{r}{s}) = \begin{cases} \frac{25}{56} & \text{if } 7 \mid s \text{ and } \rho\, r \equiv \pm 2 \pmod 7, \\ \frac{9}{56} & \text{if } 7 \mid s \text{ and } \rho\, r \equiv \pm 3 \pmod 7, \\ \frac{1}{56} & \text{otherwise.} \end{cases} \tag{5.12}$$

By (5.6) and (5.12) we have

$$\mathrm{ord}\left(\eta(z)R(z), \tfrac{r}{s}\right) \geq \frac{1}{24} - \frac{(7,s)^2}{168} + 2 \min_{\rho=1,2,3} \mathrm{ord}\left(f_\rho(7z), \tfrac{r}{s}\right) \geq 0.$$

Therefore $\eta(z)R(z) \in M_1(\Gamma)$, which proves (5.3) and completes the proof of Lemma 1.

References

1. N. Andersen, Vector-valued modular forms and the mock theta conjectures. Res. Number Theory **2**, 32 (2016)
2. G.E. Andrews, F.G. Garvan, Ramanujan's "lost" notebook: the sixth order mock theta conjectures. Adv. Math. **73**(2), 242–255 (1989)
3. B.C. Berndt, R.A. Rankin, *Ramanujan: Letters and Commentary*, History of Mathematics, vol. 9 (American Mathematical Society, Providence, 1995)
4. A.J.F. Biagioli, A proof of some identities of Ramanujan using modular forms. Glasg. Math. J. **31**(3), 271–295 (1989)
5. K. Bringmann, K. Ono, Dyson's ranks and Maass forms. Ann. Math. (2) **171**(1), 419–449 (2010)
6. J.H. Bruinier, *Borcherds Products on O(2, l) and Chern Classes of Heegner Divisors*. Lecture Notes in Mathematics, vol. 1780 (Springer, Berlin, 2002)
7. M. Eichler, D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, vol. 55 (Birkhäuser Boston Inc, Boston, 1985)
8. F. Garvan, Transformation properties for Dyson's rank function. Preprint, arXiv:1601.05781v2
9. B. Gordon, R.J. McIntosh, Modular transformations of Ramanujan's fifth and seventh order mock theta functions. Ramanujan J. **7**(1–3), 193–222 (2003)
10. D. Hickerson, On the seventh order mock theta functions. Invent. Math. **94**(3), 661–677 (1988)
11. D. Hickerson, A proof of the mock theta conjectures. Invent. Math. **94**(3), 639–660 (1988)
12. H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, vol. 17 (American Mathematical Society, Providence, 1997)
13. N.P. Skoruppa, Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts. Bonner Mathematische Schriften [Bonn Mathematical Publications], 159. Universität Bonn, Mathematisches Institut, Bonn (1985)
14. S.P. Zwegers, *Mock theta functions*, Ph.D. thesis, University of Utrecht, Utrecht, The Netherlands (2002)

The Alladi–Schur Polynomials and Their Factorization

George E. Andrews

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract K. Alladi first observed the following variant of I. Schur's 1926 partition theorem. Namely, the number of partitions of n in which all parts are odd and none appears more than twice equals the number of partitions of n in which all parts differ by at least 3 and more than 3 if one of the parts is a multiple of 3. Subsequently, the theorem was refined to count also the number of parts in the relevant partitions. In this paper, a surprising factorization of the related polynomial generating functions is developed.

Keywords Schur's 1926 Theorem · Partitions · The Alladi–Schur theorem

2010 Mathematics Subject Classification 11P83 · 05A19

1 Introduction

In 1926, I. Schur [6] proved the following result:

Partially supported by NSA Grant: #98230-12-1-0205.

G. E. Andrews (✉)
Department of Mathematics, Pennsylvania State University, University Park,
State College 16802, PA, USA
e-mail: gea1@psu.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_3

Theorem. *Let $A(n)$ denote the number of partitions of n into parts congruent to* ± 1 (mod 6). *Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ where $b_i - b_{i+1} \geq 3$ with strict inequality if $3|b_i$. Then*

$$A(n) = B(n) = D(n).$$

K. Alladi [1] has pointed out (cf. [2, p. 46, eq. (1.3)]) that if we define $C(n)$ to be the number of partitions of n into odd parts with none appearing more than twice, then also

$$C(n) = D(n).$$

Recently [3] it was shown that a refinement (in the spirit of Gleissberg's refinement [4] of Schur's original theorem [6]) is valid:

Theorem. *Let $C(m,n)$ denote the number of partitions of n into m parts, all odd and none appearing more than twice. Let $D(m,n)$ denote the number of partitions of n into parts of the type enumerated by $D(n)$ with the added condition that the total number of parts plus the number of even parts is m (i.e., m is the weighted count of parts where each even is counted twice). Then*

$$C(m,n) = D(m,n).$$

The proof relied on a study of the generating function of $D_N(m,n)$ the number of partitions of the type enumerated by $D(m,n)$ with the added restriction that each part be $\leq N$. Thus,

$$d_N(x) = \sum_{n,m \geq 0} D_N(m,n) x^m q^n.$$

In fact, the above theorem was directly deduced from the functional equations

$$d_{6n+2}(x) = (1 + xq + x^2q^2) d_{6n-1}(xq^2), \tag{1.1}$$

$$d_{6n-1}(x) = (1 + xq + x^2q^2)\{d_{6n-4}(xq^2) + xq^{6n-1}(1 - qx) d_{6n-7}(xq^2)\}, \tag{1.2}$$

where $d_{-1}(x)$ is defined to be 1.

It turns out that much more than this is true.

Theorem 1.1. *For $n \geq 3$, with $d_{-1}(x) = 1$,*

$$d_{2n}(x) = (1 + xq + x^2q^2) d_{2n-3}(xq^2), \tag{1.3}$$

$$d_{2n-1}(x) = (1 + xq + x^2q^2)\{d_{2n-4}(xq^2) + xq^{2n-1}(1 - qx) d_{2n-7}(xq^2)\}. \tag{1.4}$$

From Theorem 1.1, it is possible to provide a factorization of the $d_n(x)$. We define

$$p_n(x) = \prod_{j=1}^{n}(1 + xq^{2j-1} + x^2q^{4j-2}). \tag{1.5}$$

Theorem 1.2. *If* $n \not\equiv 3 \pmod 6$*, then* $p_{\lfloor \frac{n+4}{6} \rfloor}(x)$ *divides* $d_n(x)$*. If* $n \equiv 3 \pmod 6$*, then* $p_{\lfloor \frac{n-2}{6} \rfloor}(x)$ *divides* $d_n(x)$*.*

Finally, it is possible to give a full account of the quotient arising in the division given in Theorem 1.2.

Theorem 1.3.

$$d_{6n-1}(x) = p_n(x) \sum_{j=0}^{n} c(n, j)x^j, \tag{1.6}$$

where

$$c(n, j) = \sum_{r=0}^{j} \sum_{0 \le 3i \le r} \frac{(-1)^i q^{4nj-2nr+j+3i(i-1)}(q^2; q^2)_n}{(q^2; q^2)_{n-j}(q^2; q^2)_{j-r}(q^2; q^2)_{r-3i}(q^6; q^6)_i}. \tag{1.7}$$

From Theorem 1.3, one can deduce explicit formulas for the other $d_{6n-i}(x)$, and we will discuss this in the conclusion.

The paper concludes with a discussion of other possible factorization theorems in the theory of partitions.

It should be emphasized that, in some real sense, the intrinsic theorem is the Alladi–Schur theorem. Not only do we see that

$$d_\infty(x) = p_\infty(x),$$

but also the partial products of $p_\infty(x)$ as revealed in Theorems 1.2 and 1.3 are naturally arising as n increases. None of the other variants of Schur's theorem reveals the successive appearance of the relevant partial products.

2 Proof of Theorem 1.1

Theorem 1.1 is actually an extension of Lemma 3 in [3] to its full generality, and the proof builds upon what was proved there.

Proof of Theorem 1.1 Let

$$\chi(n) = \begin{cases} 1 & \text{if } 3|n \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

then the recurrences (2.2)–(2.4) of [3] can be rewritten as

$$d_{2n}(x) = d_{2n-1}(x) + x^2q^{2n}d_{2n-3-\chi(2n)}(x), \tag{2.2}$$
$$d_{2n-1}(x) = d_{2n-2}(x) + xq^{2n-1}d_{2n-4-\chi(2n-1)}(x). \tag{2.3}$$

Next we define

$$\mathscr{F}(n) = d_{2n+2}(x) - (1 + xq + x^2q^2)d_{2n-1}(xq^2), \tag{2.4}$$

and

$$\mathscr{G}(n) = d_{2n-1}(x) - (1 + xq + x^2q^2)(d_{2n-4}(xq^2) + xq^{2n-1}(1 - xq)d_{2n-7}(xq^2)). \tag{2.5}$$

To prove (1.3) and (1.4), we only need to show that for $n \geq 3$,

$$\mathscr{F}(n) = \mathscr{G}(n) = 0.$$

Now

$$\begin{aligned}
\mathscr{F}(3) =& d_8(x) - (1 + xq + xq^2)d_5(xq^2)\\
=&(1 + xq^5 + xq^7)(1 + xq + x^2q^2)(1 + xq^3 + x^2q^6)\\
&- (1 + xq + x^2q^2)\{(1 + xq^3 + x^2q^6)(1 + xq^5 + xq^7)\}\\
=&0,
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{G}(3) =& d_5(x) - (1 + xq + xq^2)\{d_2(xq^2) + xq^5(1 - xq)d_{-1}(xq^2)\}\\
=&1 + xq + x^2q^2 + xq^3 + x^2q^4 + xq^5 + x^3q^5\\
&+ x^2q^6 + x^3q^7 - (1 + xq + xq^2)(1 + x^2q^3 + x^2q^6)\\
&+ xq^5(1 - xq)(1 + xq + xq^2)\\
=&0.
\end{aligned}$$

In the following, we write for simplicity

$$\lambda(x) = 1 + xq + x^2q^2.$$

Now Lemma 3 of [3] asserts that for $n \geq 1$,

$$\mathscr{F}(3n) = \mathscr{G}(3n) = 0.$$

Hence by (2.4) and (2.5)

$$\begin{aligned}\mathscr{F}(3n-1) =&\mathscr{F}(3n-1)-\mathscr{G}(3n)-x^2q^{6n}\mathscr{F}(3n-3)\\ =&(d_{6n}(x)-\lambda(x)d_{6n-3}(xq^2))\\ &-(d_{6n-1}(x)-\lambda(x)d_{6n-4}(xq^2)-\lambda(x)xq^{6n-1}(1-xq)d_{6n-7}(xq^2))\\ &-x^2q^{6n}(d_{6n-4}(x)-\lambda(x)d_{6n-7}(xq^2))\\ =&(d_{6n}(x)-d_{6n-1}(x)-x^2q^{6n}d_{6n-4}(x))\\ &-\lambda(x)(d_{6n-3}(xq^2)-d_{6n-4}(xq^2)-xq^{6n-1}d_{6n-7}(xq^2))\\ =&0,\end{aligned}$$

by (2.2) and (2.3). So $\mathscr{F}(3n-1)$ is identically 0 for $n \geq 2$.

Next

$$\begin{aligned}\mathscr{G}(3n+2) =&\mathscr{G}(3n+2)-\mathscr{F}(3n)-xq^{6n+3}\mathscr{G}(3n)\\ =&(d_{6n+3}(x)-\lambda(x)d_{6n}(xq^2)-xq^{6n+3}(1-xq)\lambda(x)d_{6n-3}(xq^2))\\ &-(d_{6n+2}(x)-\lambda(x)d_{6n-1}(xq^2))\\ &-xq^{6n+3}(d_{6n-1}(x)-\lambda(x)d_{6n-4}(xq^2)-\lambda(x)xq^{6n-1}(1-xq)d_{6n-7}(xq^2))\\ =&(d_{6n+3}(x)-d_{6n+2}(x)-xq^{6n+3}d_{6n-1}(x))\\ &-\lambda(x)(d_{6n}(xq^2)-d_{6n-1}(xq^2)-x^2q^{6n+4}d_{6n-4}(xq^2))\\ &-xq^{6n+3}(1-xq)\lambda(x)(d_{6n-3}(xq^2)-d_{6n-4}(xq^2)\\ &-xq^{6n-1}d_{6n-7}(xq^2))\\ =&0,\end{aligned}$$

by (2.2) and (2.3). So $\mathscr{G}(3n+2)$ is identically 0, for $n \geq 1$.

Next,

$$\begin{aligned}\mathscr{F}(3n-2) =&-(\mathscr{G}(3n)-\mathscr{F}(3n-2)-xq^{6n-1}\mathscr{F}(3n-3))\\ =&-(d_{6n-1}(x)-\lambda(x)d_{6n-4}(xq^2)-xq^{6n-1}(1-xq)\lambda(x)d_{6n-7}(xq^2))\\ &+(d_{6n-2}(x)-\lambda(x)d_{6n-5}(xq^2))\\ &+xq^{6n-1}(d_{6n-4}(x)-\lambda(x)d_{6n-7}(xq^2))\\ =&-(d_{6n-1}(x)-d_{6n-2}(x)-xq^{6n-1}d_{6n-4}(x))\\ &+\lambda(x)(d_{6n-4}(xq^2)-d_{6n-5}(xq^2)-x^2q^{6n}d_{6n-7}(xq^2))\\ =&0,\end{aligned}$$

by (2.2) and (2.3). Thus $\mathscr{F}(3n-2)$ is identically 0, for $n \geq 2$.

Finally,

$$\begin{aligned}
\mathscr{G}(3n+1) =& -(\mathscr{F}(3n) - \mathscr{G}(3n+1) - x^2q^{6n+2}\mathscr{G}(3n))\\
=& -(d_{6n+2}(x) - \lambda(x)d_{6n-1}(xq^2))\\
&+ (d_{6n+1}(x) - \lambda(x)d_{6n-2}(xq^2) - xq^{6n+1}(1-xq)\lambda(x)d_{6n-5}(xq^2))\\
&+ x^2q^{6n+2}(d_{6n-1}(x) - \lambda(x)d_{6n-4}(xq^2) - xq^{6n-1}(1-xq)\lambda(x)d_{6n-7}(xq^2))\\
=& -(d_{6n+2}(x) - d_{6n+1}(x) - x^2q^{6n+2}d_{6n-1}(x))\\
&+ \lambda(x)(d_{6n-1}(xq^2) - d_{6n-2}(xq^2) - xq^{6n+1}d_{6n-4}(xq^2))\\
&+ xq^{6n+1}\lambda(x)(1-xq)(d_{6n-4}(xq^2) - d_{6n-5}(xq^2) - x^2q^{6n}d_{6n-7}(xq^2))\\
=&0,
\end{aligned}$$

by (2.2) and (2.3). Thus $\mathscr{G}(3n+1)$ is identically 0 for $n \geq 1$, and Theorem 1.1 is proved. □

3 Proof of Theorem 1.2

This result is essentially a corollary of Theorem 1.1, but is of major significance in Theorem 1.3 and is the factorization referred to in the title.

Proof of Theorem 1.2 The succinct assertion of Theorem 1.3 may be stated more comprehensibly as follows. We are to prove that there exist polynomials

$$\triangle(i,n) = \triangle(i,n,x,q),$$

such that

$$d_{6n+1}(x) = p_n(x)\triangle(-1,n) \tag{3.1}$$
$$d_{6n}(x) = p_n(x)\triangle(0,n) \tag{3.2}$$
$$d_{6n-1}(x) = p_n(x)\triangle(1,n) \tag{3.3}$$
$$d_{6n-2}(x) = p_n(x)\triangle(2,n) \tag{3.4}$$
$$d_{6n-3}(x) = p_{n-1}(x)\triangle(3,n) \tag{3.5}$$
$$d_{6n-4}(x) = p_n(x)\triangle(4,n) \tag{3.6}$$

Now by (1.3), note

$$d_7(x) = p_1(x)(1 + x(q^3+q^5+q^7) + x^2(q^6+q^{10})) \tag{3.7}$$
$$d_6(x) = p_1(x)(1 + x(q^3+xq^5) + x^2q^6) \tag{3.8}$$
$$d_5(x) = p_1(x)(1 + xq^3 + xq^5) \tag{3.9}$$
$$d_4(x) = p_1(x)(1 + xq^3) \tag{3.10}$$
$$d_3(x) = p_0(x)(1 + x(q+q^3) + x^2q^2) \tag{3.11}$$
$$d_2(x) = p_1(x) \tag{3.12}$$

so the case $n = 1$ is established.

Now assume (3.1)–(3.6) are proved up to but not including a given n. Then by (2.2) and (2.3),

$$\begin{aligned} d_{6n+1}(x) &= d_{6n}(x) + xq^{6n+1} d_{6n-2}(x) \\ &= p_n(x)(\triangle(0,n) + xq^{6n+1}\triangle(2,n)), \end{aligned}$$

$$\begin{aligned} d_{6n}(x) &= d_{6n-1}(x) + x^2q^{6n} d_{6n-4}(x) \\ &= p_n(x)(\triangle(1,n) + x^2q^{6n}\triangle(4,n)) \end{aligned}$$

$$\begin{aligned} d_{6n-1}(x) &= d_{6n-2}(x) + xq^{6n-1} d_{6n-4}(x) \\ &= p_n(x)(\triangle(2,n) + xq^{6n-1}\triangle(4,n)) \end{aligned}$$

$$\begin{aligned} d_{6n-2}(x) &= (1 + xq + x^2q^2) d_{6n-5}(xq^2) \\ &= p_n(x)\triangle(-1, n-1, xq^2, q), \end{aligned}$$

$$\begin{aligned} d_{6n-3}(x) &= d_{6n-2}(x) + xq^{6n-3} d_{6n-7}(x) \\ &= p_n(x)\triangle(2,n) + xq^{6n-3} p(n-1)\triangle(1, n-1) \\ &= p_{n-1}(x)((1 + xq^{2n-1} + x^2q^{4n-2})\triangle(2,n) + xq^{6n-3}\triangle(1, n-1)), \end{aligned}$$

and finally by (1.3),

$$\begin{aligned} d_{6n-4}(x) &= (1 + xq + xq^2) d_{6n-5}(xq^2) \\ &= p_n(x)\triangle(-1, n-1, xq^2, q), \end{aligned}$$

and our theorem is proved. □

4 Proof of Theorem 1.3

This result seems to require a rather elaborate proof. In order to make Theorem 1.3 comprehensible, we shall prove a number of preliminary lemmas.

We begin by defining

$$\overline{c}(n, j) := \sum_{r=0}^{j} \sum_{0 \le 3i \le r} \frac{(-1)^i q^{4nj-2nr+j+3i(i-1)} (q^2; q^2)_n}{(q^2; q^2)_{n-j} (q^2; q^2)_{j-r} (q^2; q^2)_{r-3j} (q^6; q^6)_i}. \tag{4.1}$$

Clearly, Theorem 1.3 reduces to proving that, in fact, $c(n, j) = \overline{c}(n, j)$.

We note that, of the partitions enumerated by $d_{2n-1}(x)$, the one that provides the largest x-exponents is

$$4 + 7 + 10 + \cdots + 6n - 2,$$

yielding x^{3n}. Furthermore, by (3.3), and noting that $p(n)$ is of degree $2n$ in x, we must have $\triangle(1, n)$ of degree n. So $c(n, j) = 0$ if $j < 0$ or $j > n$.

Lemma 4.1.

$$c(n, j) = \begin{cases} 1 & \text{if } n = j = 0 \\ 0 & \text{if } n < 0,\ j \leq 0;\ n \leq 0,\ j < 0,\ j > n \text{ and for } n > 0 \end{cases} \tag{4.2}$$

$$\begin{aligned} c(n, j) = q^{4j} c(n-1, j) &+ (q^{2n+4j-3} + q^{6n+2j-3}) c(n-1, j-1) \\ &+ (q^{4j+4n-6} - q^{6n+2j-4}) c(n-1, j-2). \end{aligned} \tag{4.3}$$

Proof. By (1.3) with n replaced by $3n - 2$

$$\begin{aligned} d_{6n-4}(x) &= p_n(x) \triangle(4, n, x, q) \\ &= p_n(x) \triangle(1, n-1, xq^2, q). \end{aligned} \tag{4.4}$$

Therefore, by (1.4) with n replaced by $3n$

$$d_{6n-1}(x) = \lambda(x)\{d_{6n-4}(xq^2) xq^{6n-1}(1 - qx) d_{6n-7}(xq^2)\}. \tag{4.5}$$

So

$$\begin{aligned} \sum_{j=0}^{n} c(n, j) x^j = &\sum_{j=0}^{n} c(n-1, j)(xq^4)^j (1 + xq^{2n+1} + x^2 q^{4n+2}) \\ &+ xq^{6n-1}(1 - xq) \sum_{j=0}^{n} c(n-1, j) x^j q^{2j}. \end{aligned}$$

Hence

$$\begin{aligned} c(n, j) = &c(n-1, j) q^{4j} + (q^{2n-3+4j} + q^{6n+2j-3}) c(n-1, j-1) \\ &+ (q^{4j+4n-6} - q^{6n+2j-4}) c(n-1, j-2) \end{aligned} \tag{4.6}$$

as desired. □

Lemma 4.2. *For $n \geq 0$,*

$$\begin{aligned} c(n, j)(1 - q^{2j}) = &c(n, j-1) q^{2n+2j-1}(1 - q^{4n-2j+2}) \\ &+ c(n, j-2) q^{4n+2j-2}(1 - q^{2n-2j+4}) \\ &- c(n-1, j-2) q^{6n+2j-4}(1 - q^{6n}). \end{aligned} \tag{4.7}$$

Proof. By (2.7) of [3] rewritten twice, first with n replaced by $n+1$, we see that

$$p_{n+1}(x)\sum_{j=0}^{n}c(n,j)(xq^2)^j = (1+xq^{6n+1}+x^2q^{6n+2})p_n(x)\sum_{j=0}^{n}c(n,j)x^j + x^2q^{6n}(1-q^{6n})p_n(x)\sum_{j=0}^{n-1}c(n-1,j)(xq^2)^j.$$

Now noting that $p_{n+1}(x)/p_n(x) = 1+xq^{2n+1}+x^2q^{4n+2}$, and dividing this last equation by $p_n(x)$, we obtain

$$\begin{aligned}(1+xq^{2n+1}+x^2q^{4n+2})&\sum_{j=0}^{n}c(n,j)(xq^2)^j\\ &=(1+xq^{6n+1}+x^2q^{6n+2})\sum_{j=0}^{n}c(n,j)x^j\\ &\quad+x^2q^{6n}(1-q^{6n})\sum_{j=0}^{n-1}c(n-1,j)(xq^2)^j.\end{aligned}$$

Finally, comparing coefficients of x^j on both sides, we deduce

$$\begin{aligned}q^{2j}c(n,j)&+q^{2n+2j-1}c(n,j-1)+q^{4n+2j-2}c(n,j-2)\\ &=c(n,j)+q^{6n+1}c(n,j-1)+q^{6n+2}c(n,j-2)\\ &\quad+q^{6n+2j-4}(1-q^{6n})c(n-1,j-2),\end{aligned}$$

which is equivalent to (4.7). □

From Lemma 4.2, we shall deduce a recurrence for $c(n,j)$ where only j varies.

Lemma 4.3.

$$\begin{aligned}0=&(q^{6j}-q^{4j})c(n-1,j)\\ &+(q^{2n+6j-3}-q^{6n+2j-3}+q^{2n+6j-5}+q^{2n+4j-3})c(n-1,j-1)\\ &+(q^{4n+6j-6}-q^{6n+4j-6}-q^{6n+4j-4}\\ &+q^{4n+6j-8}+q^{4n+6j-10}-q^{4n+4j-6})c(n-1,j-2)\\ &+q^{8n+2j-16}(q^{2n+4}-q^{2j})(q^{2n+6}-q^{2j})c(n-1,j-4).\end{aligned} \tag{4.8}$$

Proof. By (4.3), we see that $c(n,j)$ is equal to a combination of $c(n-1,j-i)$, $i=0,1,2$. Thus substitute the expressions for $c(n,j)$, $c(n,j-1)$ and $c(n,j-2)$ arising from (4.5) into the recurrence (4.7). Collect terms and simplify to obtain (4.8).

□

We now require a more succinct recurrence for $c(n, j)$.

Lemma 4.4.

$$0 =(q^{2j} - q^{2n})c(n, j) - q^{4j}(1 - q^{2n})c(n - 1, j) \tag{4.9}$$

$$- q^{4n+2j-3}(q^{2n} - q^{2j})(1 - q^{2n})c(n - 1, j - 1).$$

Proof. Let us substitute for $c(n, j)$ in the right-hand side of (4.9), the expression for $c(n, j)$ given in (4.3). Hence our assertion is equivalent to proving that

$$\begin{aligned}0 =&(q^{6j} - q^{4j})c(n - 1, j) + (q^{2n+6j-3} - q^{6n+2j-3})c(n - 1, j - 1) \\ &+ q^{4n+2j-6}(q^{2n} - q^{2j})(q^{2n+2} - q^{2j})c(n - 1, j - 2).\end{aligned} \tag{4.10}$$

Let us denote the right-hand side of (4.10) by $T(n, j)$. Now direct substitution reveals that (4.8) may be rewritten as

$$0 = T(n, j) - q^{2n+1}T(n, j - 1) - q^{4n+2}T(n, j - 2). \tag{4.11}$$

Furthermore, we see from (4.10) that

$$T(n, 0) = 0.$$

Now from (4.7)

$$c(n, 1) = \frac{q^{2n+1}(1 - q^{4n})}{1 - q^2}.$$

Hence

$$\begin{aligned}T(n, 1) &= (q^6 - q^4)\frac{q^{2n+1}(1 - q^{4n-4})}{1 - q^2} + q^{2n+3} - q^{6n-1} \\ &= -q^{2n+3}(1 - q^{4n-4}) + q^{2n+3} - q^{6n-1} \\ &= 0.\end{aligned}$$

Therefore, by (4.11),

$$T(n, j) = 0$$

for all n and j. Thus (4.10) is proved, and (4.10) is equivalent to (4.9). □

Finally, we need a "diagonal" recurrence for the $c(n, n)$.

Lemma 4.5. *For $n \geq -1$,*

$$\begin{aligned}0 =&c(n + 2, n + 2) - (q^{8n+11} + q^{8n+13})c(n + 1, n + 1) \\ &- (q^{10n+10} - q^{16n+16})c(n, n).\end{aligned} \tag{4.12}$$

Proof. Setting $j = n$ in (4.3), we find

$$c(n-1,n-2) = \frac{c(n,n) - (q^{6n-3} + q^{8n-3})c(n-1,n-1)}{q^{8n-6}(1-q^2)}. \tag{4.13}$$

Next set $j = n-1$ in (4.9)

$$\begin{aligned} 0 =& (q^{2n-2} - q^{2n})c(n,n-1) - q^{4n-4}(1-q^{2n})c(n-1,n-1) \\ & - q^{6n-3}(q^{2n} - q^{2n-2})(1-q^{2n})c(n-1,n-2). \end{aligned} \tag{4.14}$$

Now use (4.13) twice (first with n replaced by $n+1$), to reduce (4.14) to an expression that only involves instances of $c(n-i, n-i)$. Thus, after simplification, we find

$$\begin{aligned} 0 =& c(n+2,n+2) - (q^{8n+11} + q^{8n+13})c(n+1,n+1) \\ & - (q^{10n+10} - q^{16n+16})c(n,n), \end{aligned} \tag{4.15}$$

as desired. □

Lemma 4.6. *For $n \geq -2$,*

$$\begin{aligned} 0 =& - c(n+3,n+3) \\ & + q^{4n+11}(1 - q^{2n+2} - q^{2n+4} + q^{4n+6} + q^{4n+8} + q^{4n+10})c(n+2,n+2) \\ & + q^{10n+20}(1-q^{2n+4})(1 - q^{2n+2} + q^{4n+4} + q^{4n+6} + q^{4n+8})c(n+1,n+1) \\ & - q^{14n+21}(1-q^{2n+2})(1-q^{2n+4})(1-q^{6n+6})c(n,n). \end{aligned} \tag{4.16}$$

Proof. Let us denote the right-hand side of (4.12) by $U(n)$. Then it is easily verified by algebraic simplification that the expression on the right side of (4.16) is

$$-U(n+1) + q^{4n+11}(1-q^{2n+2})(1-q^{2n+4})U(n),$$

we see by Lemma 8, that (4.16) is established for $n \geq -1$, and inspection reveals the truth for $n = -2$. □

We now move to recurrences for $\overline{c}(n, j)$ as defined by (4.1).

Lemma 4.7.

$$\begin{aligned} 0 =& (q^{2j} - q^{2n})\overline{c}(n,j) - q^{4j}(1-q^{2n})\overline{c}(n-1,j) \\ & - q^{4n+2j-3}(q^{2n} - q^{2j})(1-q^{2n})\overline{c}(n-1,j-1). \end{aligned} \tag{4.17}$$

Proof. Here we require the assistance of qMultiSum [5]:

```
In[20]= qFindRecurrence[
    qPochhammer[q^2,q^2,n] *(-1)^i*q^(2*n*(2*j-r)+j+3*i*(i-1))/
    (qPochhammer[q^2,q^2,n-j] *qPochhammer[q^2,q^2,j-r]*
    qPochhammer[q^2,q^2,r-3*i] *qPochhammer[q^6,q^6,i]),
    {n,j},{i,r},{1,1},{0,0},{0,0}]//qSR[#,2] &//Timing
```

$$\begin{aligned}\text{Out[20]} =&\{0.093601, \{q^{3+2j+4n}(-q^j + q^n)\\ &(q^j + q^n)(-1 + q^{1+n})(1 + q^{1+n})\text{SUM}[n, j]+\\ &q^{2+4j}(-1 + q^{1+n})(1 + q^{1+n})\text{SUM}[n, 1 + j] - (-q^j + q^n)\\ &(q^j + q^n)\text{SUM}[1 + n, 1 + j] = 0, ...\end{aligned}$$

This is precisely the recurrence (4.17) with n replaced by $n + 1$. □

Lemma 4.8.

$$\begin{aligned}0 =& - \overline{c}(n + 3, n + 3)\\ &+ q^{4n+11}(1 - q^{2n+2} - q^{2n+4} + q^{4n+6} + q^{4n+8} + q^{4n+10})\overline{c}(n + 2, n + 2)\\ &+ q^{10n+20}(1 - q^{2n+4})(1 - q^{2n+2} + q^{4n+4} + q^{4n+6} + q^{4n+8})\overline{c}(n + 1, n + 1)\\ &- q^{14n+21}(1 - q^{2n+2})(1 - q^{2n+4})(1 - q^{6n+6})\overline{c}(n, n).\end{aligned} \tag{4.18}$$

Proof. Again we employ qMultiSum:

```
In[40]= qFindRecurrence[qPochhammer[q^2,q^2,n]*(-1)^i*
       q^(2*n*(2*n-r)+n+3*i*(i-1))  /(qPochhammer[q^2,q^2,n-r]*
       qPochhammer[q^2,q^2,r-3*i]*qPochhammer[q^6,q^6,i]),
       {n},{r,i},{2},{1,1}]//qSR[#,1] &// Timing
```

$$\begin{aligned}\text{Out[40]} =&\{1.060807,\\ &\{q^{21+14n}(-1 + q^{1+n})^2(1 + q^{1+n})^2(-1 + q^{2+n})\\ &(1 + q^{2+n})(1 - q^{1+n} + q^{2+2n})(1 + q^{1+n} + q^{2+2n})\text{SUM}[n]-\\ &q^{20+10n}(-1 + q^{2+n})\\ &(1 + q^{2+n})(1 - q^{2+2n} + q^{4+4n} + q^{6+4n} + q^{8+4n})\text{SUM}[1 + n]+\\ &q^{11+4n}(1 - q^{2+2n} - q^{4+2n} + q^{6+4n}\\ &+ q^{8+4n} + q^{10+4n})\text{SUM}[2 + n] - \text{SUM}[3 + n] = 0, ...\end{aligned}$$

This is precisely the recurrence (4.18). □

Finally, we are ready to deduce Theorem 1.3.

Proof of Theorem 3. First, it is easy to check by hand (tedious) or by computer algebra system (rapid) that Theorem 1.3 is valid for each $n \leq 3$. The fact that (4.18) and (4.16) are identical fourth order linear recurrences then allows us to establish by mathematical induction that for all n,

$$c(n, n) = \overline{c}(n, n). \tag{4.19}$$

Finally, the identity of the recurrences (4.9) and (4.17) allows us to establish by mathematical induction on n that

$$c(n, j) = \overline{c}(n, j). \tag{4.20}$$

We should note that (4.19) is necessarily established independently because (4.9) and (4.17) reduce to $0 = 0$ when $j = n$. □

5 Conclusion

It should be noted that while Theorem 1.3 only provides an exact formula for $\triangle(1, n)$, formulas for the other $\triangle(i, n)$ can easily be obtained from the recurrences for the $d_n(x)$. Indeed, (2.3) implies immediately with n replaced by $3n - 2$

$$\triangle(4, n, x, q) = \triangle(1, n, xq^2, q), \tag{5.1}$$

and the remaining $\triangle$'s are produced from the original recurrences for the $d_n(x)$ given in (2.2) and (2.3).

It is not obvious from Theorem 1.3 that $c(n, j)$ has nonnegative coefficients, but there is adequate numerical evidence to suggest the following:

Conjecture. For all n and j, $c(n, j)$ has nonnegative coefficients.

If the conjecture is true, it is natural to ask for a partition-theoretic interpretation of them. Also if that could be accomplished, it would be truly interesting to have a bijective proof of Theorem 1.3. Yee's bijective, related work [7] suggests this may well be possible.

Finally, we note that the Alladi–Schur version of Schur's theorem seems most fundamental in that the generating polynomials factor into increasing partial products of the product side of the limiting identity. It is natural to ask whether this phenomenon holds for other either classical or new partition identities.

Acknowledgements Quite obviously the proof of Theorem 1.3 relied crucially on Lemmas 4.7 and 4.8. These were achieved only through Axel Riese's qMultiSum package. I am grateful to Peter Paule and RISC for providing my access to the qMultiSum package and to my son, Derek Andrews, who skillfully ported the package to my computer.

References

1. K. Alladi, Personal Communication
2. G.E. Andrews, Schur's theorem, partitions with odd parts and the Al–Salam-Carlitz polynomials. Am. Math. Soc. Contemp. Math. **254**, 45–53 (2000)

3. G.E. Andrews, A refinement of the Alladi–Schur theorem (to appear)
4. W. Gleissberg, Über einen Sats von Herrn I. Schur. Math. Zeit. **28**, 372–382 (1928)
5. A. Riese, qMultiSum - a package for providing q-hypergeometric multiple summation identities. J. Symb. Comp. **35**, 349–376 (2003)
6. I. Schur, Zur additiven Zahlentheorie, S.-B. Preuss Akad. Wiss. Phys.-Math. Kl., (1926), pp. 488–495
7. A.J. Yee, A combinatorial proof of Andrews' partition functions related to Schur's partition theorem. Proc. Am. Math. Soc. **130**, 2229–2235 (2002)

New Representations for $\sigma(q)$ via Reciprocity Theorems

Koustav Banerjee and Atul Dixit

Dedicated to our friend Krishnaswami Alladi on the occasion of his 60th birthday

Abstract Two new representations for Ramanujan's function $\sigma(q)$ are obtained. The proof of the first one uses the three-variable reciprocity theorem due to Soon-Yi Kang and a transformation due to R.P. Agarwal while that of the second uses the four-variable reciprocity theorem due to George Andrews and a generalization of a recent transformation of Andrews, Schultz, Yee, and the second author. The advantage of these representations is that they involve free complex parameters—one in the first representation, and two in the second. In the course of obtaining these results, we arrive at one- and two-variable generalizations of $\sigma(q)$.

Keywords Reciprocity theorem · Quantum modular form · Basic hypergeometric series

2010 Mathematics Subject Classification Primary 33D15

1 Introduction

One of the celebrated functions of Ramanujan is the function $\sigma(q)$ defined by

K. Banerjee
Department of Mathematics, Ramakrishna Mission Vivekananda University,
PO Belur Math, Howrah 711202, West Bengal, India
e-mail: banerjeekoustav98@gmail.com

A. Dixit (✉)
Department of Mathematics, Indian Institute of Technology Gandhinagar,
Palaj, Gandhinagar 382355, Gujarat, India
e-mail: adixit@iitgn.ac.in

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_4

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q)_n}.$$

It is the generating function for the excess number of partitions of n into distinct parts with an even rank over those of odd rank [6]. Note that the rank of a partition is the largest part minus the number of parts.

On page 14 of the Lost Notebook [35], Ramanujan gave two surprising identities involving $\sigma(q)$:

$$\sum_{n=0}^{\infty} (S(q) - (-q)_n) = S(q)D(q) + \frac{1}{2}\sigma(q), \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \left(S(q) - \frac{1}{(q;q^2)_{n+1}} \right) = S(q)D(q^2) + \frac{1}{2}\sigma(q), \tag{1.2}$$

where

$$S(q) := (-q;q)_\infty,$$
$$D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

Here, and throughout the sequel, we assume $|q| < 1$ and use the standard q-series notation

$$(A)_0 := (A;q)_0 = 1,$$
$$(A)_n := (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}) \text{ for any positive integer } n,$$
$$(A)_\infty := (A;q)_\infty = \lim_{n\to\infty} (A;q)_n, \qquad |q| < 1,$$
$$(A)_n := (A)_\infty/(Aq^n)_\infty \text{ for any integer } n.$$

Since the base of almost all of the q-shifted factorials occurring in our paper is q, for simplicity, we also use the following notation:

$$(A_1, A_2, \cdots, A_m)_n := (A_1, A_2, \cdots, A_m; q)_n = (A_1)_n(A_2)_n \cdots (A_m)_n,$$
$$(A_1, A_2, \cdots, A_m)_\infty := (A_1, A_2, \cdots, A_m; q)_\infty = (A_1)_\infty(A_2)_\infty \cdots (A_m)_\infty.$$

We provide the associated base wherever there is a possibility of confusion.

The aforementioned identities involving $\sigma(q)$ were first proved by Andrews in [6]. The function $\sigma(q)$ enjoys many nice properties relevant to various fields of number theory, namely, the theory of partitions, algebraic number theory, Maass waveforms, quantum modular forms etc. We review these properties below.

In [6], and later more explicitly in [7], Andrews conjectured that infinitely many coefficients in the power series expansion of $\sigma(q)$ are zero but that the coefficients are unbounded. These two conjectures were later proved by Andrews, Dyson, and Hickerson in a beautiful paper [9], where they found that the coefficients of $\sigma(q)$ have multiplicative properties determined by a certain Hecke character associated to the real quadratic field $\mathbb{Q}(\sqrt{6})$. Results similar to these were later found by [14], Corson, Favero, Liesinger, and Zubairy [16], Lovejoy [27], [28], Lovejoy and Osburn [29], Patkowski [31], and more recently by Xiong [37].

Cohen [20] showed that if we set

$$\begin{aligned}\varphi(q) &:= q^{1/24}\sigma(q) + q^{-1/24}\sigma^*(q) \\ &= \sum_{\substack{n\in\mathbb{Z} \\ n\equiv 1 \pmod{24}}} T(n)q^{|n|/24},\end{aligned}$$

where

$$\sigma^*(q) := 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n},$$

then $T(n)$ are the coefficients of a Maass waveform of eigenvalue $1/4$. For another example of such a Maass waveform associated with the pair $(W_1(q), W_2(q))$ studied in [16], we refer the reader to Section 2 of a recent paper of Li, Ngo, and Rhoades [25]. At the end of [25], the authors posed an open problem of relating 10 other pairs of q-series to Maass waveforms or indefinite quadratic forms, which was recently solved by Krauel, Rolen, and Woodbury [24]. The function $\sigma(q)$ also occurs in one of the first examples of quantum modular forms given by Zagier [40], that is, $q^{1/24}\sigma(q)$, where $q = e^{2\pi i x}$, $x \in \mathbb{Q}$, is a quantum modular form.

The identities of the type (1.1) and (1.2) are known as 'sum of tails' identities. After Ramanujan, Zagier [39, Theorem 2] was the next mathematician to discover a 'sum of tails' identity. This is associated with the Dedekind eta-function and occurs in his work on Vassiliev invariants. Using a new Abel-type lemma, Andrews, Jiménez-Urroz, and Ono [10] obtained two general theorems involving q-series obtained by summing the iterated differences between an infinite product and its truncated products, and used them not only to prove (1.1) and (1.2) and similar other identities but also to determine the values at negative integers of certain L-functions. Chan [17, p. 78] gave a multiparameter 'sum of tails' identity which consists, as special cases, the two general theorems in [10]. More 'sum of tails' identities were obtained by Andrews and Freitas [13], Bringmann and Kane [15], and Patkowski [32], [33], [34].

Andrews [6] asked for a 'near bijection' between the weighted counts of partitions given by the left-hand sides of (1.1) and (1.2), and the coefficients of the corresponding first expressions obtained by the convolutions of the associated partition functions. Such a proof was supplied by Chen and Ji [18]. In [11, Theorem 3.3], the function $\sigma(q)$ was found to be related to the generating function of the number of partitions of n such that all even parts are less than or equal to twice the smallest part.

As mentioned before, identities (1.1) and (1.2) were proved by Andrews in [6]. His proof was based on an application of a beautiful q-series identity of Ramanujan [35, p. 40], [5, Equation (3.8)], now known as Ramanujan's reciprocity theorem, which was, in turn, proved earlier by Andrews himself in [5]. In [13], it was remarked that the proofs of (1.1) and (1.2) in [6] are nearly as odd as the identities themselves. In [8, p. 149] as well, it was remarked that 'the proofs provide no significant insight into the reasons for their existence'. While this may be true, the goal of this paper is to show that the underlying idea in these proofs can be adapted to obtain new representations for $\sigma(q)$, which are of a type completely different than those previously known, for example, [9, Equations (6.3), (6.4)] or (1.1) and (1.2). These two new representations involve natural generalizations of $\sigma(q)$ in one and two variables respectively.

These representations result from applying Andrews' idea in [6] to the three-variable reciprocity theorem of Kang [23, Theorem 4.1] which is equivalent to Ramanujan's ${}_1\psi_1$ summation formula, and to the four-variable reciprocity theorem [23, Theorem 1.2] which is equivalent to a formula of Andrews [5, Theorem 6].

For $|c| < |a| < 1$ and $|c| < |b| < 1$, Kang [23, Theorem 4.1] obtained the following three-variable reciprocity theorem:

$$\rho_3(a,b,c) - \rho_3(b,a,c) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c, aq/b, bq/a, q)_\infty}{(-c/a, -c/b, -aq, -bq)_\infty}, \tag{1.3}$$

where

$$\rho_3(a,b,c) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}.$$

Ramanujan's reciprocity theorem is a special case $c = 0$ of the above theorem.

Using (1.3), we obtain the following new representation for $\sigma(q)$. The surprising thing about this representation is that it is valid for any complex c such that $|c| < 1$.

Theorem 1.1. *For any complex c such that $|c| < 1$, we have*

$$\sigma(q) = (-c)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n (1 - cq^n)} - 2 \sum_{m,n=0}^{\infty} \frac{(-q)_m}{(q)_m (q)_n} \frac{(-1)^n q^{n(n+1)/2} c^{m+n+1}}{(1 - q^{n+m+1})}. \tag{1.4}$$

For $|c|, |d| < |a|, |b| < 1$, the four-variable reciprocity theorem is given by [23, Theorem 1.2]

$$\rho_4(a,b,c,d) - \rho_4(b,a,c,d) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(d, c, cd/(ab), aq/b, bq/a, q)_\infty}{(-d/a, -d/b, -c/a, -c/b, -aq, -bq)_\infty}, \tag{1.5}$$

where

$$\rho_4(a,b,c,d) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(d,c,cd/(ab))_n(1+cdq^{2n}/b)(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n(-c/b,-d/b)_{n+1}}.$$

Using (1.5), we obtain the following new representation for $\sigma(q)$ which consists of two free complex parameters c and d:

Theorem 1.2. *Let* $|c| < 1$ *and* $|d| < 1$. *Then*

$$\sigma(q) = \frac{(-c,-d)_\infty}{(-cd)_\infty}\sum_{n=0}^{\infty}\frac{(-cd)_n(1-cdq^{2n})q^{n(n+1)/2}}{(-q)_n(1-cq^n)(1-dq^n)} + \Lambda(c,d,q), \tag{1.6}$$

where

$$\begin{aligned}\Lambda(c,d,q) = 1 &- \frac{3cd(-cq,-dq)_\infty}{(-cd,-q)_\infty}\sum_{n=0}^{\infty}\frac{(c,d,-cd)_n}{(-cq,-dq,q)_n}q^{\frac{n(n+1)}{2}+2n}\\
&- \frac{(-dq,c)_\infty}{(-cd,-q)_\infty}\sum_{p=0}^{\infty}\frac{(-q)_p(-1)_p c^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q^{p+1})_k c^k}{(q^{p+1})_k}\\
&\times\sum_{n=0}^{\infty}\frac{(-cd,d)_n}{(-dq,q)_n}q^{\frac{n(n+1)}{2}+(p+k)n}(1+cdq^{2n}(1+q^p)(1+q^{p+1}))\\
&- \frac{(d,c)_\infty}{(-cd,-q)_\infty}\sum_{p=1}^{\infty}\frac{(-q)_p(-1)_p d^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q)_k(-q^p)_k c^k}{(q)_k}\sum_{j=0}^{\infty}\frac{(-q^{p+1})_j d^j}{(q^{p+1})_j}\\
&\times\sum_{n=0}^{\infty}\frac{(-cd)_n}{(q)_n}q^{\frac{n(n+1)}{2}+(p+k+j)n}(1+cdq^{2n}(1+q^{p+k})(1+q^{p+k+1}))\\
&- 2(d,c)_\infty\sum_{p=1}^{\infty}\frac{(-cd)^p}{(q)_p}\sum_{j=0}^{\infty}\frac{(-q)_j d^j}{(q)_j}\sum_{k=0}^{\infty}\frac{(-q)_k c^k}{(q)_k}\\
&\times\sum_{m=0}^{\infty}\frac{(-cd)^m}{(q^{p+1})_m(-q^{p+k+j})_{m+1}}\left(1+cd\frac{(1+q^{p+k+j})(1+q^{p+k+j+1})}{(1+q^{p+k+j+m+1})(1+q^{p+k+j+m+2})}\right).\end{aligned} \tag{1.7}$$

As in the case of Ramanujan's reciprocity theorem [23, p. 18], the conditions for the validity of (1.3) and (1.5) can be relaxed to allow the parameters a and b to be equal to 1.

It will be shown later that letting $d = 0$ in Theorem 1.2 results in Theorem 1.1. Still, pedagogically it is sound to first give a proof of Theorem 1.1 and then proceed to that of Theorem 1.2, especially since the complexity involved in the former is much lesser than that in the latter.

In order to obtain (1.6), we derive a new nine-parameter transformation contained in the following theorem which generalizes previous transformations due to Agarwal [1] (see Equation (2.5) below), and due to Andrews, Dixit, Schultz, and Yee [12] (see Equation (2.6) below).

Theorem 1.3. *For $\beta, \delta, f, h, t \neq q^{-j}$, $j \geq 0$, the following identity is true:*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma)_n(e)_n(g)_n}{(\beta)_n(\delta)_n(f)_n(h)_n} t^n \\
&= \frac{(g, e, \gamma, \frac{\beta}{\alpha}, q, \alpha t, \frac{q}{\alpha t}, \frac{\delta q}{\beta}, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{q}{\alpha}, \beta, \frac{\beta}{\alpha t}, \frac{\alpha t q}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} {}_4\phi_3\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{\gamma q}{\beta}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{\delta q}{\beta}, & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix} ; q, t\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{\delta q}{\beta}, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} {}_4\phi_3\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{\gamma q}{\beta}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{\delta q}{\beta}, & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix} ; q, t\right) \left({}_2\phi_1\left(\begin{matrix} q, & \frac{q}{t} \\ & \frac{\beta q}{\alpha t} \end{matrix} ; \frac{q}{\alpha}\right) - 1\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} \sum_{p=0}^{\infty} \frac{(\frac{\delta}{\gamma})_p (\frac{\alpha t}{\beta})_p \gamma^p}{(t)_p (q)_p} \sum_{k=0}^{\infty} \frac{(\frac{\delta q^p}{\gamma})_k (\frac{\gamma q}{\beta})^k}{(q^{p+1})_k} {}_3\phi_2\left(\begin{matrix} \frac{\alpha q}{b}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix} ; q, tq^{p+k}\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{gq}{\beta})_\infty} \\
&\quad \times \sum_{p=1}^{\infty} \frac{(\frac{f}{e})_p (\frac{\alpha t}{\beta})_p e^p}{(t)_p (q)_p} \sum_{k=0}^{\infty} \frac{(\frac{\delta}{\gamma})_k (\frac{\alpha t q^p}{\beta})_k \gamma^k}{(q)_k (tq^p)_k} \sum_{j=0}^{\infty} \frac{(\frac{fq^p}{e})_j (\frac{eq}{\beta})^j}{(q^{p+1})_j} {}_2\phi_1\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{gq}{\beta} \\ & \frac{hq}{\beta} \end{matrix} ; q, tq^{p+k+j}\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma)_\infty}{(h, f, \delta)_\infty} \sum_{p=1}^{\infty} \frac{(\frac{h}{g})_p g^p}{(q)_p} \sum_{j=0}^{\infty} \frac{(\frac{f}{e})_j e^j}{(q)_j} \sum_{k=0}^{\infty} \frac{(\frac{\delta}{\gamma})_k \gamma^k}{(q)_k} \sum_{m=0}^{\infty} \frac{(\frac{hq^p}{g})_m (tq^{p+k+j})_m}{(q^{p+1})_m (\frac{\alpha t q^{p+k+j}}{\beta})_{m+1}} \left(\frac{gq}{\beta}\right)^m.
\end{aligned}
\tag{1.8}
$$

A version of the above formula, and also of (2.6), in terms of q-Lauricella functions, was obtained by Gupta [22, p. 53] in his PhD thesis. However, his versions are not as explicit as the ones in (1.8) and (2.6). We remark that Gupta has obtained a general transformation of these results, with r q-shifted factorials in the numerator and r in the denominator, in terms of q-Lauricella functions. However, one can easily anticipate such general transformation by observing the pattern occurring in Agarwal's identity (2.5), (2.6) and (1.8). To avoid digression, we do not pursue it here.

This paper is organized as follows. In Section 2, we collect formulas from the literature that are used in the sequel. Section 3 is devoted to proving Theorem 1.1 while Section 4 to proving Theorem 1.2, and for deriving Theorem 1.1 from Theorem 1.2. We conclude this paper with Section 5 consisting of some remarks and directions for further research.

2 Preliminaries

The q-binomial theorem [4, p. 17, Equation (2.2.1)] states that for $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a; q)_n z^n}{(q; q)_n} = \frac{(az; q)_\infty}{(z; q)_\infty}. \tag{2.1}$$

For $|z| < 1$ and $|b| < 1$, Heine's transformation [4, p. 38] is given by

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) = \frac{(b, at)_\infty}{(c, t)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, t \\ at \end{matrix}; q, b\right), \tag{2.2} $$

whereas its second iterate [4, p. 38, last line] is

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) = \frac{(c/b, bt)_\infty}{(c, t)_\infty} {}_2\phi_1\left(\begin{matrix} b, abt/c \\ bt \end{matrix}; q, c/b\right). \tag{2.3} $$

Here ${}_{r+1}\phi_r$ is the basic hypergeometric series defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} z^n. $$

We need the following identity [21, p. 17, Equation (15.51)]:

$$ \sum_{n=0}^{\infty} \frac{t^n}{(bq)_n} = \frac{(1-b)}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(-t)^n q^{n(n+1)/2}}{(q)_n (1 - bq^n)}. \tag{2.4} $$

Agarwal [1, Equation (3.1)] obtained the following 'mild' extension/generalization of an important identity of Andrews [5, Theorem 1] in the sense that we get Andrews' identity from the following result when $t = q$:

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(\alpha)_n (\gamma)_n}{(\beta)_n (\delta)_n} t^n \\ &= \frac{(q/(\alpha t), \gamma, \alpha t, \beta/\alpha, q)_\infty}{(\beta/(\alpha t), \delta, t, q/\alpha, \beta)_\infty} {}_2\phi_1\left(\begin{matrix} \delta/\gamma, & t \\ & q\alpha t/\beta \end{matrix}; q, \gamma q/\beta\right) \\ &\quad + \frac{(\gamma)_\infty}{(\delta)_\infty}\left(1 - \frac{q}{\beta}\right) \sum_{m=0}^{\infty} \frac{(\delta/\gamma)_m (t)_m}{(q)_m (\alpha t/\beta)_{m+1}} (q\gamma/\beta)^m \left({}_2\phi_1\left(\begin{matrix} q, & q/t \\ & q\beta/(\alpha t) \end{matrix}; q, q/\alpha\right) - 1\right) \\ &\quad + \frac{(\gamma)_\infty}{(\delta)_\infty}\left(1 - \frac{q}{\beta}\right) \sum_{p=0}^{\infty} \frac{\gamma^p (\delta/\gamma)_p}{(q)_p} \sum_{m=0}^{\infty} \frac{(\delta q^p/\gamma)_m (tq^p)_m}{(q^{1+p})_m (\alpha t q^p/\beta)_{m+1}} (q\gamma/\beta)^m. \end{aligned} \tag{2.5} $$

The following generalization of the above identity of Agarwal was recently obtained in [12, Theorem 3.1] for $\beta, \delta, f, t \neq q^{-j}$, $j \geq 0$:

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(\alpha)_n (\gamma)_n (e)_n}{(\beta)_n (\delta)_n (f)_n} t^n \\ &= \frac{(e, \gamma, \frac{\beta}{\alpha}, q, \alpha t, \frac{q}{\alpha t}, \frac{\delta q}{\beta}, \frac{fq}{\beta})_\infty}{(f, \delta, \frac{q}{\alpha}, \beta, \frac{\beta}{\alpha t}, \frac{\alpha t q}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta})_\infty} {}_3\phi_2\left(\begin{matrix} \frac{\alpha q}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta} \\ \frac{\delta q}{\beta}, \frac{fq}{\beta} \end{matrix}; q, t\right) \\ &\quad + \left(1 - \frac{q}{\beta}\right) \frac{(e, \gamma, t, \frac{\delta q}{\beta}, \frac{fq}{\beta})_\infty}{(f, \delta, \frac{\alpha t}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta})_\infty} {}_3\phi_2\left(\begin{matrix} \frac{\alpha q}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta} \\ \frac{\delta q}{\beta}, \frac{fq}{\beta} \end{matrix}; q, t\right)\left({}_2\phi_1\left(\begin{matrix} q, \frac{q}{t} \\ \frac{\beta q}{\alpha t} \end{matrix}; \frac{q}{\alpha}\right) - 1\right) \end{aligned} $$

$$+\left(1-\frac{q}{\beta}\right)\frac{(e,\gamma,t,\frac{fq}{\beta})_\infty}{(f,\delta,\frac{\alpha t}{\beta},\frac{eq}{\beta})_\infty}\sum_{p=0}^{\infty}\frac{(\frac{\delta}{\gamma})_p(\frac{\alpha t}{\beta})_p\gamma^p}{(t)_p(q)_p}\sum_{k=0}^{\infty}\frac{(\frac{\delta q^p}{\gamma})_k(\frac{q\gamma}{\beta})^k}{(q^{1+p})_k}{}_2\phi_1\left(\begin{matrix}\frac{\alpha q}{\beta},\ \frac{eq}{\beta}\\ \frac{fq}{\beta}\end{matrix};q,tq^{k+p}\right)$$

$$+\left(1-\frac{q}{\beta}\right)\frac{(e,\gamma)_\infty}{(f,\delta)_\infty}\sum_{p=1}^{\infty}\frac{(\frac{f}{e})_pe^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(\frac{\delta}{\gamma})_k\gamma^k}{(q)_k}\sum_{m=0}^{\infty}\frac{(\frac{fq^p}{e})_m(tq^{p+k})_m}{(q^{1+p})_m(\frac{\alpha tq^{p+k}}{\beta})_{m+1}}\left(\frac{eq}{\beta}\right)^m. \tag{2.6}$$

We will also make use of the ε-operator acting on a differentiable function f by [6]

$$\varepsilon(f(z)) = f'(1).$$

3 The three-variable case

We prove Theorem 1.1 here. Letting $a = -z$ and $b = 1$ in (1.3) gives

$$\rho_3(1,-z,c) = \rho_3(-z,1,c) - \frac{(c,-zq,-z^{-1},q)_\infty}{(cz^{-1},-c,zq,-q)_\infty}. \tag{3.1}$$

Divide both sides by $(1-z^{-1})$ and let $z \to 1$. It is easy to see that the left side becomes $\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q)_n(1-cq^n)}$, which we denote by $\sigma(c,q)$. Denote the right-hand side of the above equation by $f(z)$. Note that

$$\lim_{z\to1} f(z) = 2\sum_{n=0}^{\infty}\frac{(c)_nq^{n(n+1)/2}}{(q)_n(-c)_{n+1}} - 2\frac{(-q)_\infty}{(-c)_\infty} = 0,$$

since replacing c by $-cq$, substituting $a = c, b = -q/\tau, t = \tau$, and then letting $\tau \to 0$ in (2.3) gives

$$\sum_{n=0}^{\infty}\frac{(c)_nq^{n(n+1)/2}}{(q)_n(-cq)_n} = \lim_{\tau\to0}{}_2\phi_1\left(\begin{matrix}c,\ -q/\tau\\ -cq\end{matrix};q,\tau\right) = \lim_{\tau\to0}\frac{(c\tau,-q)_\infty}{(-cq,\tau)_\infty}{}_2\phi_1\left(\begin{matrix}-q/\tau,\ 1\\ -q\end{matrix};q,c\tau\right)$$
$$= \lim_{\tau\to0}\frac{(c\tau,-q)_\infty}{(-cq,\tau)_\infty} = \frac{(-q)_\infty}{(-cq)_\infty}.$$

This result can also be found in [23, Corollary 7.5].

Hence using L'Hopital's rule, we see that

$$\lim_{z\to1}\frac{f(z)}{1-z^{-1}} = f'(1) = \varepsilon(f(z)),$$

so that from (3.1),

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n(1-cq^n)} = \varepsilon\left(\rho_3(-z,1,c) - \frac{(c,-zq,-z^{-1},q)_\infty}{(cz^{-1},-c,zq,-q)_\infty}\right). \tag{3.2}$$

The idea now is to rightly transform $\rho_3(-z,1,c) = 2\sum_{n=0}^{\infty} \frac{(c)_n z^n q^{n(n+1)/2}}{(zq)_n(-c)_{n+1}}$ into an expression which is amenable to the ε-operator. To that end, we invoke (2.5), the reasons for which will be clear soon. Let $\alpha = -q/\tau$, $\gamma = c$, $\beta = zq$, $\delta = -cq$ and $t = \tau z$ in (2.5). Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-q/\tau)_n(c)_n}{(zq)_n(-cq)_n}(\tau z)^n \\
&= \frac{(-z^{-1},c,-zq,-\tau z,q)_\infty}{(-1,-cq,\tau z,-\tau,zq)_\infty}\sum_{m=0}^{\infty}\frac{(\tau z)_m}{(q)_m}\left(\frac{c}{z}\right)^m + \frac{(c)_\infty}{2(-cq)_\infty}\left(1-\frac{1}{z}\right)\sum_{m=0}^{\infty}\frac{(\tau z)_m(\frac{c}{z})^m}{(q)_m}\sum_{j=1}^{\infty}\frac{\left(\frac{q}{\tau z}\right)_j}{(-q)_j}(-\tau)^j \\
&\quad + \frac{(c)_\infty}{(-cq)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=0}^{\infty}\frac{(-q)_p c^p}{(q)_p}\sum_{m=0}^{\infty}\frac{(\tau z q^p)_m}{(q^{p+1})_m}\frac{(c/z)^m}{(1+q^p)}.
\end{aligned}$$

Now use (2.1) to evaluate sums over m in the first two expressions, then let $\tau \to 0$ on both sides, separate the term corresponding to $p = 0$ in the double sum followed by another application of (2.1), and finally multiply throughout by $2/(1+c)$ to obtain

$$\begin{aligned}
&2\sum_{n=0}^{\infty}\frac{(c)_n z^n q^{n(n+1)/2}}{(zq)_n(-c)_{n+1}} \\
&= \frac{(-z^{-1},c,-zq,q)_\infty}{(-q,-c,zq,c/z)_\infty} + \frac{(c)_\infty}{(-c,c/z)_\infty}\left(1-\frac{1}{z}\right)\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j} \\
&\quad + \frac{(c)_\infty}{(-c,c/z)_\infty}\left(1-\frac{1}{z}\right) + 2\frac{(c)_\infty}{(-c)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_p c^p}{(q)_p(1+q^p)}\sum_{m=0}^{\infty}\frac{(c/z)^m}{(q^{p+1})_m} \\
&= \frac{(-z^{-1},c,-zq,q)_\infty}{(-q,-c,zq,c/z)_\infty} + \frac{(c)_\infty}{(-c,c/z)_\infty}\left(1-\frac{1}{z}\right)\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j} \\
&\quad + \frac{(c)_\infty}{(-c,c/z)_\infty}\left(1-\frac{1}{z}\right) + 2\frac{(c)_\infty}{(-c,c/z)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_{p-1}c^p}{(q)_{p-1}}\sum_{n=0}^{\infty}\frac{(-c/z)^n q^{n(n+1)/2}}{(q)_n(1-q^{n+p})},
\end{aligned} \tag{3.3}$$

where in the last step we applied (2.4) with $t = c/z$ and $b = q^p$.

Now substitute (3.3) in (3.2) and then apply the ε-operator to deduce that

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q)_n(1-cq^n)}$$

$$= \frac{(\sigma(q)-1)}{(-c)_\infty} + \frac{1}{(-c)_\infty} + \frac{2}{(-c)_\infty} \sum_{p=1}^{\infty} \frac{(-q)_{p-1} c^p}{(q)_{p-1}} \sum_{n=0}^{\infty} \frac{(-c)^n q^{n(n+1)/2}}{(q)_n (1-q^{n+p})},$$

which is nothing but (1.4). This completes the proof.

Remark 1. If we explicitly evaluate $\varepsilon\left(\frac{(c,-zq,-z^{-1},q)_\infty}{(cz^{-1},-c,zq,-q)_\infty}\right)$ using the Jacobi triple product identity [4, p. 28, Theorem 2.8], then, from (3.2), we obtain upon simplification

$$\sigma(c,q) = 2\varepsilon\left(\sum_{n=0}^{\infty} \frac{(c)_n z^n q^{n(n+1)/2}}{(zq)_n (-c)_{n+1}}\right) + S(c,q) + 2S(c,q)\left(\sum_{n=0}^{\infty} \frac{cq^n}{1-cq^n} - \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\right),$$

where $S(c,q) = (-q)_\infty/(-c)_\infty$. This is a one-variable generalization of [6, Equation (3.5)], as can be easily seen with the help of (1.1).

4 The four-variable case

We begin with a lemma that is used several times in the sequel.

Lemma 4.1. *For $|c| < 1$, $|d| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(c,d,-cd)_n}{(-cq,-dq,q)_n} (1+cdq^{2n}) q^{n(n+1)/2} = \frac{(-cd,-q)_\infty}{(-cq,-dq)_\infty}.$$

Proof. By Proposition 8 in [19], which is, in fact, equal to (1.5), we see that

$$y \sum_{n=0}^{\infty} (1-q^{2n+1}y/x) \frac{\left(\frac{q}{bx}, \frac{q}{cx}, \frac{q}{dx}\right)_n}{(by,dy)_{n+1}(cyq)_n} (-bcdxy^2/q)^n q^{n(n+1)/2}$$
$$- x(1-cy) \sum_{n=0}^{\infty} (1-q^{2n+1}x/y) \frac{\left(\frac{q}{by}, \frac{q}{cy}, \frac{q}{dy}\right)_n}{(bx,cx,dx)_{n+1}} (-bcdx^2y/q)^n q^{n(n+1)/2}$$
$$= (y-x) \frac{(q, \frac{qy}{x}, \frac{qx}{y}, bcxy, cdxy, bdxy)_\infty}{(bx,by,cx,cyq,dx,dy)_\infty}.$$

Now let $d = q/(ux)$, $b = q/(vx)$, $c = 1/y$ and $y = -uvx/q$ in the above identity to obtain upon simplification

$$\sum_{n=0}^{\infty} \frac{(u,v,-uv)_n}{(-uq,-vq,q)_n} (1+uvq^{2n}) q^{n(n+1)/2} = \frac{(-uv,-q)_\infty}{(-uq,-vq)_\infty}.$$

This completes the proof. □

We now first prove Theorem 1.3 and then use it to prove Theorem 1.2. Since the underlying idea in the proof of Theorem 1.3 is similar to that involved in the proof of (2.5) (see [1]) and in the proof of (2.6) (see [12]), we will be very brief here.

Let S denote the sum on the left side of (1.8). Writing $(g)_n/(h)_n = ((g)_\infty/(h)_\infty)\cdot((hq^n)_\infty/(gq^n)_\infty)$, then representing $((hq^n)_\infty/(gq^n)_\infty)$ as a sum using (2.1), interchanging the order of summation, and then employing (2.6), we find that

$$S = X + Y, \tag{4.1}$$

where

$$X := \frac{(g, e, \frac{\beta}{\alpha}, q, \frac{\delta q}{\beta}, \frac{fq}{\beta})_\infty}{(h, f, \delta, \frac{q}{\alpha}, \beta, \frac{\gamma q}{\beta}, \frac{eq}{\beta})_\infty} \sum_{m=0}^{\infty} \frac{(\frac{h}{g})_m (\alpha t q^m, \frac{q^{1-m}}{\alpha t})_\infty g^m}{(q)_m (\frac{\beta q^{-m}}{\alpha t}, \frac{\alpha t q^{m+1}}{\beta})_\infty} {}_3\phi_2\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{\gamma q}{\beta}, & \frac{eq}{\beta} \\ & \frac{\delta q}{\beta}, & \frac{fq}{\beta} \end{matrix}; q, tq^m\right),$$

$$Y := \frac{(g, e, \gamma)_\infty (1-\frac{q}{\beta})}{(h, f, \delta)_\infty} \sum_{m=0}^{\infty} \frac{(\frac{h}{g})_m g^m}{(q)_m} \sum_{j=0}^{\infty} \frac{(\frac{f}{e})_j e^j}{(q)_j (1-\frac{\alpha t q^{m+j}}{\beta})} \sum_{k=0}^{\infty} \frac{(\frac{\delta}{\gamma}, \frac{\alpha t q^{m+j}}{\beta})_k \gamma^k}{(q, \frac{\alpha t q^{j+1+m}}{\beta})_k} \sum_{r=0}^{\infty} \frac{(\frac{q^{1-k-j-m}}{t})_r (\frac{q}{\alpha})^r}{(\frac{\beta q^{1-k-j-m}}{\alpha t})_r}.$$

To evaluate X, we write the ${}_3\phi_2$ in the form of series, interchange the order of summation, make use of the identity $(\beta q^{-m}/(\alpha t))_\infty = (-\beta/(\alpha t))^m q^{-m(m+1)/2} (\beta/(\alpha t))_\infty (\alpha t q/\beta)_m$, and then use (2.1) again to deduce

$$X = \frac{(g, e, \gamma, \frac{\beta}{\alpha}, q, \alpha t, \frac{q}{\alpha t}, \frac{\delta q}{\beta}, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{q}{\alpha}, \beta, \frac{\beta}{\alpha t}, \frac{\alpha t q}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} {}_4\phi_3\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{\gamma q}{\beta}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{\delta q}{\beta}, & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix}; q, t\right). \tag{4.2}$$

Since

$$\sum_{r=0}^{\infty} \frac{(\frac{q^{1-k-m-j}}{t})_r}{(\frac{\beta q^{1-k-m-j}}{\alpha t})_r} \left(\frac{q}{\alpha}\right)^r = \frac{(t)_{m+j+k}}{\left(\frac{\alpha t}{\beta}\right)_{m+j+k}} \left(\frac{q}{\beta}\right)^{m+j+k} \left(\sum_{p=1}^{\infty} \frac{\left(\frac{q}{t}\right)_p}{\left(\frac{\beta q}{\alpha t}\right)_p} \left(\frac{q}{\alpha}\right)^p + \sum_{p=0}^{m+j+k} \frac{(\frac{\alpha t}{\beta})_p}{(t)_p} \left(\frac{\beta}{q}\right)^p\right), \tag{4.3}$$

we observe that

$$Y = Y_1 + Y_2, \tag{4.4}$$

where Y_1 is associated with the infinite sum on the right of (4.3) and Y_2 is associated with the finite sum. Even though the calculations for evaluating Y_1 and Y_2 are quite tedious, they are fairly straightforward. Using (2.2), it can be seen that

$$Y_1 = \left(1-\frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{\delta q}{\beta}, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{\gamma q}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} {}_4\phi_3\left(\begin{matrix} \frac{\alpha q}{\beta}, & \frac{\gamma q}{\beta}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{\delta q}{\beta}, & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix}; q, t\right) \left({}_2\phi_1\left(\begin{matrix} q, & \frac{q}{t} \\ & \frac{\beta q}{\alpha t} \end{matrix}; \frac{q}{\alpha}\right) - 1\right). \tag{4.5}$$

Now write the finite sum on p in Y_2 as

$$\sum_{p=0}^{m+j+k} = \sum_{p=0}^{k} + \sum_{p=k+1}^{k+j} + \sum_{p=k+j+1}^{m+j+k},$$

and let Y_{21}, Y_{22} and Y_{23} denote the expressions associated with the first, second, and third finite sums in the above equation respectively so that

$$Y_2 = Y_{21} + Y_{22} + Y_{23}. \tag{4.6}$$

Now using (2.2) repeatedly, it can be see that

$$Y_{21} = \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{fq}{\beta}, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{eq}{\beta}, \frac{gq}{\beta})_\infty} \sum_{p=0}^{\infty} \frac{(\frac{\delta}{\gamma})_p (\frac{\alpha t}{\beta})_p \gamma^p}{(t)_p (q)_p} \sum_{k=0}^{\infty} \frac{(\frac{\delta q^p}{\gamma})_k (\frac{\gamma q}{\beta})^k}{(q^{p+1})_k} {}_3\phi_2\left(\begin{matrix} \frac{\alpha q}{b}, & \frac{eq}{\beta}, & \frac{gq}{\beta} \\ & \frac{fq}{\beta}, & \frac{hq}{\beta} \end{matrix} ; q, tq^{p+k} \right),$$

$$Y_{22} = \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma, t, \frac{hq}{\beta})_\infty}{(h, f, \delta, \frac{\alpha t}{\beta}, \frac{gq}{\beta})_\infty}$$
$$\times \sum_{p=1}^{\infty} \frac{(\frac{f}{e})_p (\frac{\alpha t}{\beta})_p e^p}{(t)_p (q)_p} \sum_{k=0}^{\infty} \frac{(\frac{\delta}{\gamma})_k (\frac{\alpha t q^p}{\beta})_k \gamma^k}{(q)_k (tq^p)_k} \sum_{j=0}^{\infty} \frac{(\frac{fq^p}{e})_j (\frac{eq}{\beta})^j}{(q^{p+1})_j} {}_2\phi_1\left(\begin{matrix} \frac{\alpha q}{b}, & \frac{gq}{\beta} \\ & \frac{hq}{\beta} \end{matrix} ; q, tq^{p+k+j} \right),$$

$$Y_{23} = \left(1 - \frac{q}{\beta}\right) \frac{(g, e, \gamma)_\infty}{(h, f, \delta)_\infty} \sum_{p=1}^{\infty} \frac{(\frac{h}{g})_p g^p}{(q)_p} \sum_{j=0}^{\infty} \frac{(\frac{f}{e})_j e^j}{(q)_j} \sum_{k=0}^{\infty} \frac{(\frac{\delta}{\gamma})_k \gamma^k}{(q)_k} \sum_{m=0}^{\infty} \frac{(\frac{hq^p}{g})_m (tq^{p+k+j})_m}{(q^{p+1})_m (\frac{\alpha t q^{p+k+j}}{\beta})_{m+1}} \left(\frac{gq}{\beta}\right)^m. \tag{4.7}$$

Finally, from (4.1), (4.2), (4.4), (4.5), (4.6), and (4.7), we arrive at (1.8).

Proof. (Theorem 1.2) Let $a = -z$ and $b = 1$ in (1.5) to obtain

$$\rho_4(1, -z, c, d) = \rho_4(-z, 1, c, d) - \frac{(d, c, -cd/z, -zq, -z^{-1}, q)_\infty}{(d/z, -d, c/z, -c, zq, -q)_\infty}.$$

Divide both sides by $(1 - z^{-1})$ and let $z \to 1$. Observe that using Lemma 4.1, the resulting right side is of the form $0/0$; hence employing L'Hopital's rule, we see that

$$\sum_{n=0}^{\infty} \frac{(-cd)_n (1 - cdq^{2n}) q^{n(n+1)/2}}{(-q)_n (1 - cq^n)(1 - dq^n)}$$
$$= \varepsilon \left(2 \sum_{n=0}^{\infty} \frac{(d, c, -cd/z)_n}{(zq)_n (-c, -d)_{n+1}} (1 + cdq^{2n}) z^n q^{n(n+1)/2} - \frac{(d, c, -cd/z, -zq, -z^{-1}, q)_\infty}{(d/z, -d, c/z, -c, zq, -q)_\infty} \right) \tag{4.8}$$

The big task now is to transform the first series on the right side before applying the ε-operator. Note that

$$
\begin{aligned}
&2\sum_{n=0}^{\infty}\frac{(d,c,-cd/z)_n}{(zq)_n(-c,-d)_{n+1}}(1+cdq^{2n})z^nq^{n(n+1)/2}\\
&=\frac{2}{(1+c)(1+d)}\lim_{\tau\to 0}\sum_{n=0}^{\infty}\frac{(-\frac{q}{\tau},d,c,-cd/z)_n}{(\tau,zq,-cq,-dq)_n}(1+cdq^{2n})(\tau z)^n\\
&=\frac{2}{(1+c)(1+d)}\left\{\lim_{\tau\to 0}\sum_{n=0}^{\infty}\frac{(-\frac{q}{\tau},d,c,-cd/z)_n}{(\tau,zq,-cq,-dq)_n}(\tau z)^n+cd\lim_{\tau\to 0}\sum_{n=0}^{\infty}\frac{(-\frac{q}{\tau},d,c,-cd/z)_n}{(\tau,zq,-cq,-dq)_n}(\tau zq^2)^n\right\}\\
&=:\frac{2}{(1+c)(1+d)}(L_1+cd\ L_2).
\end{aligned}\tag{4.9}
$$

To evaluate L_1, let $\alpha=-q/\tau,\beta=zq,\gamma=c,\delta=-cq,e=d,f=-dq,g=-cd/z,h=\tau$ and $t=\tau z$ in Theorem 1.3. This results in

$$
\begin{aligned}
L_1&=\frac{(-\frac{cd}{z},d,c,q,-\frac{cq}{z},-\frac{dq}{z},-zq,-\frac{1}{z})_\infty}{(-dq,-cq,zq,-1,-q,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}}\\
&+\frac{(-\frac{cd}{z},d,c,-\frac{cq}{z},-\frac{dq}{z})_\infty}{(-dq,-cq,-1,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}}\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j}\\
&+\frac{(-\frac{cd}{z},d,c,-\frac{dq}{z})_\infty}{(-dq,-cq,-1,\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=0}^{\infty}\frac{(-q)_p(-1)_pc^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q^{p+1})_k(c/z)^k}{(q^{p+1})_k}\\
&\times\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2},\frac{d}{z})_n}{(-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+(p+k)n}\\
&+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq,-1,-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_p(-1)_pd^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q)_k(-q^p)_kc^k}{(q)_k}\\
&\times\sum_{j=0}^{\infty}\frac{(-q^{p+1})_j(d/z)^j}{(q^{p+1})_j}\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2})_n}{(q)_n}q^{\frac{n(n+1)}{2}+(p+k+j)n}\\
&+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-\frac{cd}{z})^p}{(q)_p}\sum_{j=0}^{\infty}\frac{(-q)_jd^j}{(q)_j}\sum_{k=0}^{\infty}\frac{(-q)_kc^k}{(q)_k}\sum_{m=0}^{\infty}\frac{(-cd/z^2)^m}{(q^{p+1})_m(-q^{p+k+j})_{m+1}}.
\end{aligned}\tag{4.10}
$$

Now let $\alpha=-q/\tau,\beta=zq,\gamma=c,\delta=-cq,e=d,f=-dq,g=-cd/z,h=\tau$ and $t=\tau zq^2$ in Theorem 1.3. This gives

$$
\begin{aligned}
L_2&=\frac{(-\frac{cd}{z},d,c,q,-\frac{cq}{z},-\frac{dq}{z},-zq^3,-\frac{1}{zq^2})_\infty}{(-dq,-cq,zq,-\frac{1}{q^2},-q^3,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+2n}\\
&+\frac{(-\frac{cd}{z},d,c,-\frac{cq}{z},-\frac{dq}{z})_\infty}{(-dq,-cq,-q^2,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+2n}\sum_{j=1}^{\infty}\frac{q^{j(j-3)/2}z^{-j}}{(-1/q)_j}\\
&+\frac{(-\frac{cd}{z},d,c,-\frac{dq}{z})_\infty}{(-dq,-cq,-q^2,\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=0}^{\infty}\frac{(-q)_p(-q^2)_pc^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q^{p+1})_k(c/z)^k}{(q^{p+1})_k}\\
&\times\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2},\frac{d}{z})_n}{(-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+(p+k+2)n}\\
&+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq,-q^2,-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_p(-q^2)_pd^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q)_k(-q^{p+2})_kc^k}{(q)_k}
\end{aligned}
$$

$$\times\sum_{j=0}^{\infty}\frac{(-q^{p+1})_j(d/z)^j}{(q^{p+1})_j}\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2})_n}{(q)_n}q^{\frac{n(n+1)}{2}+(p+k+j+2)n}$$
$$+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-\frac{cd}{z})^p}{(q)_p}\sum_{j=0}^{\infty}\frac{(-q)_jd^j}{(q)_j}\sum_{k=0}^{\infty}\frac{(-q)_kc^k}{(q)_k}\sum_{m=0}^{\infty}\frac{(-cd/z^2)^m}{(q^{p+1})_m(-q^{p+k+j+2})_{m+1}}. \tag{4.11}$$

Substituting (4.10) and (4.11) in (4.9), we find that

$$2\sum_{n=0}^{\infty}\frac{(d,c,-cd/z)_n}{(zq)_n(-c,-d)_{n+1}}(1+cdq^{2n})z^nq^{n(n+1)/2}$$
$$=\frac{2}{(1+c)(1+d)}\Bigg\{\frac{(-\frac{cd}{z},d,c,q,-\frac{cq}{z},-\frac{dq}{z},-zq,-\frac{1}{z})_\infty}{(-dq,-cq,zq,-1,-q,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}\left(1+\frac{cdq^{2n}}{z^2}\right)q^{\frac{n(n+1)}{2}}$$
$$+\frac{(-\frac{cd}{z},d,c,-\frac{cq}{z},-\frac{dq}{z})_\infty}{(-dq,-cq,-1,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}}$$
$$\times\left\{\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j}+2(1+q)cdq^{2n}\sum_{j=1}^{\infty}\frac{q^{j(j-3)/2}z^{-j}}{(-1/q)_j}\right\}$$
$$+\frac{(-\frac{cd}{z},d,c,-\frac{dq}{z})_\infty}{(-dq,-cq,-1,\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=0}^{\infty}\frac{(-q)_p(-1)_pc^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q^{p+1})_k(c/z)^k}{(q^{p+1})_k}$$
$$\times\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2},\frac{d}{z})_n}{(-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+(p+k)n}\left(1+cdq^{2n}(1+q^p)(1+q^{p+1})\right)$$
$$+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq,-1,-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_p(-1)_pd^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q)_k(-q^p)_kc^k}{(q)_k}$$
$$\times\sum_{j=0}^{\infty}\frac{(-q^{p+1})_j(d/z)^j}{(q^{p+1})_j}\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2})_n}{(q)_n}q^{\frac{n(n+1)}{2}+(p+k+j)n}\left(1+cdq^{2n}(1+q^{p+k})(1+q^{p+k+1})\right)$$
$$+\frac{(-\frac{cd}{z},d,c)_\infty}{(-dq,-cq)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-\frac{cd}{z})^p}{(q)_p}\sum_{j=0}^{\infty}\frac{(-q)_jd^j}{(q)_j}\sum_{k=0}^{\infty}\frac{(-q)_kc^k}{(q)_k}$$
$$\times\sum_{m=0}^{\infty}\frac{(-cd/z^2)^m}{(q^{p+1})_m(-q^{p+k+j})_{m+1}}\left(1+cd\frac{(1+q^{p+k+j})(1+q^{p+k+j+1})}{(1+q^{p+k+j+m+1})(1+q^{p+k+j+m+2})}\right)\Bigg\}.$$

Since

$$\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}\left(1+\frac{cdq^{2n}}{z^2}\right)q^{\frac{n(n+1)}{2}}=\frac{\left(-\frac{cd}{z^2},-q\right)_\infty}{\left(-\frac{cq}{z},-\frac{dq}{z}\right)_\infty},$$

by Lemma 4.1, and

$$\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j}+2(1+q)cdq^{2n}\sum_{j=1}^{\infty}\frac{q^{j(j-3)/2}z^{-j}}{(-1/q)_j}$$

$$= \left(1 + \frac{cdq^{2n}}{z^2}\right) \sum_{j=1}^{\infty} \frac{q^{j(j+1)/2} z^{-j}}{(-q)_j} + \frac{cdq^{2n}}{z^2}(1+2z),$$

we see that

$$\begin{aligned}
&2\sum_{n=0}^{\infty} \frac{(d,c,-cd/z)_n}{(zq)_n(-c,-d)_{n+1}}(1+cdq^{2n})z^n q^{n(n+1)/2} \\
&= \frac{(-\frac{cd}{z},d,c,q,-zq,-\frac{1}{z})_\infty}{(-d,-c,zq,-q,\frac{c}{z},\frac{d}{z})_\infty} + \frac{(-\frac{cd}{z},d,c)_\infty}{(-d,-c,\frac{c}{z},\frac{d}{z})_\infty}\left(1-\frac{1}{z}\right)\sum_{j=1}^{\infty}\frac{q^{j(j+1)/2}z^{-j}}{(-q)_j} \\
&\quad + \frac{cd(1+2z)}{z^2}\left(1-\frac{1}{z}\right)\frac{(-\frac{cd}{z},d,c,-\frac{cq}{z},-\frac{dq}{z})_\infty}{(-d,-c,-q,\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{z},\frac{d}{z},-\frac{cd}{z^2})_n}{(-\frac{cq}{z},-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+2n} \\
&\quad + \frac{(-\frac{cd}{z},d,c,-\frac{dq}{z})_\infty}{(-d,-c,-q,\frac{d}{z},-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=0}^{\infty}\frac{(-q)_p(-1)_p c^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q^{p+1})_k(c/z)^k}{(q^{p+1})_k} \\
&\quad \times \sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2},\frac{d}{z})_n}{(-\frac{dq}{z},q)_n}q^{\frac{n(n+1)}{2}+(p+k)n}\left(1+cdq^{2n}(1+q^p)(1+q^{p+1})\right) \\
&\quad + \frac{(-\frac{cd}{z},d,c)_\infty}{(-d,-c,-q,-\frac{cd}{z^2})_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-q)_p(-1)_p d^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(-q)_k(-q^p)_k c^k}{(q)_k} \\
&\quad \times \sum_{j=0}^{\infty}\frac{(-q^{p+1})_j(d/z)^j}{(q^{p+1})_j}\sum_{n=0}^{\infty}\frac{(-\frac{cd}{z^2})_n}{(q)_n}q^{\frac{n(n+1)}{2}+(p+k+j)n}\left(1+cdq^{2n}(1+q^{p+k})(1+q^{p+k+1})\right) \\
&\quad + 2\frac{(-\frac{cd}{z},d,c)_\infty}{(-d,-c)_\infty}\left(1-\frac{1}{z}\right)\sum_{p=1}^{\infty}\frac{(-\frac{cd}{z})^p}{(q)_p}\sum_{j=0}^{\infty}\frac{(-q)_j d^j}{(q)_j}\sum_{k=0}^{\infty}\frac{(-q)_k c^k}{(q)_k} \\
&\quad \times \sum_{m=0}^{\infty}\frac{(-cd/z^2)^m}{(q^{p+1})_m(-q^{p+k+j})_{m+1}}\left(1+cd\frac{(1+q^{p+k+j})(1+q^{p+k+j+1})}{(1+q^{p+k+j+m+1})(1+q^{p+k+j+m+2})}\right).
\end{aligned} \tag{4.12}$$

Finally, we substitute (4.12) in (4.8) and then apply the ε-operator to obtain (1.6) after simplification. This completes the proof. □

Remark 2. Let $S(c,d,q) := (-cd,-q)_\infty/(-d,-c)_\infty$ and denote the left-hand side of (4.8) by $\sigma(c,d,q)$. If we explicitly evaluate $\varepsilon\left(\frac{(-cd/z,d,c,q,-zq,-z^{-1},q)_\infty}{(-d,-c,zq,-q,c/z,d/z)_\infty}\right)$ using the Jacobi triple product identity, then (4.8) leads us to

$$\begin{aligned}
\sigma(c,d,q) &= \varepsilon\left(2\sum_{n=0}^{\infty}\frac{(d,c,-cd/z)_n(1+cdq^{2n})}{(zq)_n(-c,-d)_{n+1}}z^n q^{\frac{n(n+1)}{2}}\right) + S(c,d,q)\left(1+2\sum_{n=0}^{\infty}\frac{cdq^n}{1+cdq^n}\right) \\
&\quad + 2S(c,d,q)\left(\sum_{n=0}^{\infty}\frac{dq^n}{1-dq^n}+\sum_{n=0}^{\infty}\frac{cq^n}{1-cq^n}-\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right).
\end{aligned}$$

This gives a two-variable generalization of [6, Equation (3.5)], as can be seen with the help of (1.1).

4.1 Theorem 1.1 as a special case of Theorem 1.2

In this subsection, we deduce Theorem 1.1 from Theorem 1.2.

Let $d = 0$ in (1.6). Then

$$\sigma(q) = (-c)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n(1-cq^n)} + \Lambda(c,0,q).$$

Thus we need only show that

$$\Lambda(c,0,q) = -2\sum_{m,n=0}^{\infty} \frac{(-q)_m}{(q)_m(q)_n} \frac{(-1)^n q^{n(n+1)/2} c^{m+n+1}}{(1-q^{n+m+1})}.$$

To that end, note that the two quadruple sums in (1.7) just collapse to 0 so that

$$\Lambda(c,0,q) = 1 - \frac{(c)_\infty}{(-q)_\infty} \sum_{p=0}^{\infty} \frac{(-q)_p(-1)_p c^p}{(q)_p} \sum_{k=0}^{\infty} \frac{(-q^{p+1})_k c^k}{(q^{p+1})_k} \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}+(p+k)n}}{(q)_n}.$$

Now use Euler's formula [4, p. 19, Corollary 2.2]

$$\sum_{n=0}^{\infty} \frac{w^n q^{\frac{n(n-1)}{2}}}{(q)_n} = (-w)_\infty, \quad |w| < \infty,$$

to evaluate the sum over n in the above triple sum so that

$$\begin{aligned}\Lambda(c,0,q) &= 1 - \frac{(c)_\infty}{(-q)_\infty} \sum_{p=0}^{\infty} \frac{(-q)_p(-1)_p c^p}{(q)_p} \sum_{k=0}^{\infty} \frac{(-q^{p+1})_k}{(q^{p+1})_k} c^k (-q^{p+k+1})_\infty \\ &= 1 - (c)_\infty \sum_{k=0}^{\infty} \frac{c^k}{(q)_k} - (c)_\infty \sum_{p=1}^{\infty} \frac{(-1)_p}{(q)_p} c^p \sum_{k=0}^{\infty} \frac{c^k}{(q^{p+1})_k} \\ &= -2\sum_{p,n=0}^{\infty} \frac{(-q)_p}{(q)_p(q)_n} \frac{(-1)^n q^{n(n+1)/2} c^{p+n+1}}{(1-q^{p+n+1})},\end{aligned}$$

where we used (2.1) to evaluate the first sum in the penultimate expression, and (2.4 to evaluate the sum over k in the double sum over p and k. This completes the proof.

5 Concluding Remarks

The two series, namely

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n(1-cq^n)}, \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-cd)_n(1-cdq^{2n})q^{n(n+1)/2}}{(-q)_n(1-cq^n)(1-dq^n)},$$

occurring in Theorems 1.1 and 1.2, are respectively one- and two-variable generalizations of $\sigma(q)$ as can be seen from remarks 1 and 2 after the proofs of Theorems 1.1 and 1.2 respectively. It may be fruitful to see which properties of $\sigma(q)$ hold for these generalizations as well. Also, it may be important to see if there are any partition-theoretic interpretations of these generalizations of $\sigma(q)$.

As demonstrated in this paper, there are a number of advantages of using Agarwal's identity (2.5) and its generalization (1.8) for transforming $\rho_3(-z, 1, c)$ and $\rho_4(-z, 1, c, d)$ respectively. First of all, the infinite product expressions occurring in the specializations of the three- and the four-variable reciprocity theorems used in our proofs get canceled completely. Secondly, these identities contain ${}_2\phi_1\left(\begin{matrix} q, \ q/t \\ q\beta/(\alpha t) \end{matrix}; q, q/\alpha\right)$, which is what leads to $\sigma(q)$ after appropriately specializing the parameters. Thirdly, all of the other expressions in these identities contain the factor $1 - q/\beta$, or after letting $\beta = zq$, the factor $1 - 1/z$, which is extremely useful since all other factors involving z in an expression which contains $1 - 1/z$ get annihilated when we differentiate them with respect to z and then let $z \to 1$.

There are further generalizations of Ramanujan's reciprocity theorem, namely, the five-variable generalization due to Chu and Zhang [19] and Ma [30, Theorem 1.3], the six-variable generalization given in [30], the seven-variable generalization due to Wei, Wang and Yan [38, Theorem 3, Corollary 4] and a different one by Liu [26, Theorem 1.9], and finally the multiparameter generalization in [38, Theorem 7]. While there is no reason a priori why the ideas used in this paper may not be applicable to obtain further identities of the type we have established, the complexity of the computations involved in the proof of Theorem 1.2 suggests that the computations involved while applying the reciprocity theorems in more than four variables may be quite unwieldy.

That being said, we believe that one can further simplify $\Lambda(c, d, q)$ to the effect of at least having the 1 on the right-hand side of (1.7) canceled. First of all, note that the second expression in (1.7) admits further simplification, namely,

$$\begin{aligned}
&\frac{3cd(-cq, -dq)_\infty}{(-cd, -q)_\infty} \sum_{n=0}^{\infty} \frac{(c, d, -cd)_n}{(-cq, -dq, q)_n} q^{\frac{n(n+1)}{2}+2n} \\
&= \frac{3(-cq, -dq)_\infty}{(-cd, -q)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{(c, d, -cd)_n}{(-cq, -dq, q)_n}(1+cdq^{2n}) q^{\frac{n(n+1)}{2}} - \sum_{n=0}^{\infty} \frac{(c, d, -cd)_n}{(-cq, -dq, q)_n} q^{\frac{n(n+1)}{2}} \right\} \\
&= 3 - 3\frac{(-cq, -dq)_\infty}{(-cd, -q)_\infty} \sum_{n=0}^{\infty} \frac{(c, d, -cd)_n}{(-cq, -dq, q)_n} q^{\frac{n(n+1)}{2}},
\end{aligned}$$

by another application of Lemma 4.1. However, we are unable to represent the last series or the other multi-sums occurring in (1.7) in a convenient form. Note that the following special case of the q-analog of Kummer's theorem [3, Equation (1.7)], known as Lebesgue's identity, is well-known [4, Corollary 2.7]:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} q^{n(n+1)/2} = (-q)_\infty (aq; q^2)_\infty.$$

This prompts us to ask if there are higher level analogs of Lebesgue's identity which could possibly be used to represent the sum $\sum_{n=0}^{\infty} \frac{(c, d, -cd)_n}{(-cq, -dq, q)_n} q^{\frac{n(n+1)}{2}}$. A generalization of Lebesgue's identity in a different direction is given by Alladi [2, Equation (2.10), Section 4]. More importantly, does there exist a simpler representation for $\Lambda(c, d, q)$ as a whole?

The finite forms of Ramanujan's reciprocity theorem and its three- and four-variable generalizations are obtained in [36]. It may be of interest to see if something along the lines of (1.1), (1.2), and Theorems 1.1 and 1.2 could be obtained starting with these finite analogs.

Acknowledgements This work was done when the first author was visiting Indian Institute of Technology Gandhinagar as a summer intern under the Summer Research Internship Program-2016 (SRIP-2016) program. He would like to thank the institute for its hospitality and support.

References

1. R.P. Agarwal, On the paper a 'Lost' notebook of Ramanujan. Adv. Math. **53**, 291–300 (1984)
2. K. Alladi, Partitions with non-repeating odd parts and q-hypergeometric identities, in *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, ed. by K. Alladi, J. Klauder, C.R. Rao (Springer, New York, 2010), pp. 169–182
3. G.E. Andrews, On the q-analog of Kummer's theorem and applications. Duke Math. J. **40**, 525–528 (1973)
4. G.E. Andrews, *The Theory of Partitions* (Addison-Wesley Publishing Company, New York, 1998), p. 300. Reissued, Cambridge University Press, New York, 1976
5. G.E. Andrews, Ramanujan's "Lost" notebook: I. partial theta functions. Adv. Math. **41**, 137–172 (1981)
6. G.E. Andrews, Ramanujan's "Lost" Notebook V: Euler's partition identity. Adv. Math. **61**, 156–164 (1986)
7. G.E. Andrews, Questions and conjectures in partition theory. Am. Math. Mon. **93**, 708–711 (1986)
8. G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook. Part II* (Springer Science and Business Media, Berlin, 2009)
9. G.E. Andrews, F.J. Dyson, D. Hickerson, Partitions and indefinite quadratic forms. Invent. Math. **91**, 391–407 (1988)
10. G.E. Andrews, J. Jiménez-Urroz, K. Ono, q-Series identities and values of certain L-functions. Duke Math. J. **108**(3), 395–419 (2001)

11. G.E. Andrews, A. Dixit, A.J. Yee, Partitions associated with the Ramanujan/Watson mock theta functions $\omega(q)$, $\nu(q)$ and $\phi(q)$. Res. Number Theory **1**(1), 1–25 (2015)
12. G.E. Andrews, A. Dixit, D. Schultz, A.J. Yee, *Overpartitions related to the mock theta function* $\omega(q)$
13. G.E. Andrews, P. Freitas, Extension of Abel's Lemma with q-series implications. Ramanujan J. **10**, 137–152 (2005)
14. K. Bringmann, B. Kane, Multiplicative q-hypergeometric series arising from real quadratic fields. Trans. Am. Math. Soc. **363**(4), 2191–2209 (2011)
15. K. Bringmann, B. Kane, New identities involving sums of the tails related to real quadratic fields. Ramanujan J. **23**, 243–251 (2010). (Special issue in honor of George E. Andrews' 70th birthday)
16. D. Corson, D. Favero, K. Liesinger, S. Zubairy, Characters and q-series in $\mathbb{Q}(\sqrt{2})$. J. Number Theory **107**, 392–405 (2004)
17. S.H. Chan, On Sears's general transformation formula for basic hypergeometric series. Ramanujan J. **20**, 69–79 (2009)
18. W.Y.C. Chen, K.Q. Ji, Weighted forms of Euler's theorem. J. Comb. Theory Ser. A **114**(2), 360–372 (2007)
19. W. Chu, W. Zhang, Bilateral q-series identities and reciprocal formulae. Funct. Approx. Comment Math. **42**(2), 153–162 (2010)
20. H. Cohen, q-Identities for Maass waveforms. Invent. Math. **91**, 409–422 (1988)
21. N.J. Fine, *Basic Hypergeometric Series and Applications* (American Mathematical Society, Providence, 1988)
22. M. Gupta, A Study of Certain Basic Hypergeometric Identities and their Applications, Doctoral thesis, University of Rajasthan, Jaipur, 1989
23. S.-Y. Kang, Generalizations of Ramanujan's reciprocity theorem and their applications. J. Lond. Math. Soc. **75**(2), 18–34 (2007)
24. M. Krauel, L. Rolen, M. Woodbury, On a relation between certain q-hypergeometric series and Maass waveforms. Proc. Amer. Math. Soc. **145**(2), 543–557 (2017)
25. Y. Li, H.T. Ngo, R.C. Rhoades, Renormalization and quantum modular forms, Part I: Maass wave forms, Nov 13, 2013, arXiv: 1311.3043v1
26. Z.-G. Liu, Extensions of Ramanujan's reciprocity theorem and the Andrews-Askey integral. J. Math. Anal. Appl. **443**, 1110–1129 (2016)
27. J. Lovejoy, Lacunary partition functions. Math. Res. Lett. **9**, 191–198 (2002)
28. J. Lovejoy, Overpartitions and real quadratic fields. J. Number Theory **106**, 178–186 (2004)
29. J. Lovejoy, R. Osburn, Real quadratic double sums. Indag. Math. **26**, 697–712 (2015)
30. X.R. Ma, Six-variable generalization of Ramanujan's reciprocity theorem. J. Math. Anal. Appl. **353**, 320–328 (2009)
31. A. Patkowski, A family of lacunary partition functions. N. Z. J. Math. **38**, 87–91 (2008)
32. A. Patkowski, On curious generating functions for values of L-functions. Int. J. Number Theory **6**(7), 1531–1540 (2010)
33. A. Patkowski, An observation on the extension of Abel's lemma. Integers **10**, 793–800 (2010)
34. A. Patkowski, *On the q-Pell sequences and sums of tails*. Czechoslovak. Math. J. **67**(1), 279–288
35. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers* (Narosa, New Delhi, 1988)
36. D.D. Somashekara, K. Narasimha Murthy, Finite forms of reciprocity theorem of Ramanujan and its generalizations. Int. J. Math. Comb. **4**, 1–14 (2013)
37. X. Xiong, *Small values of coefficients of a half Lerch sum*. Int. J. Number. Theory. **13**(9), 2461–2470 (2017)
38. C. Wei, X. Wang, Q. Yan, Generalizations of Ramanujan's reciprocity formula and the Askey-Wilson integral. Ramanujan J. **37**(1), 203–217 (2015)
39. D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function. Topology **40**, 945–960 (2001)
40. D. Zagier, Quantum modular forms, *Quantas of Maths*, Clay Mathematics Proceedings (American Mathematical Society, Providence, 2010), pp. 659–675

Mean Values of the Functional Equation Factors at the Zeros of Derivatives of the Riemann Zeta Function and Dirichlet L-Functions

Kübra Benli, Ertan Elma and Cem Yalçın Yıldırım

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract In this work, average values of the functional equation factors of the Riemann zeta-function and Dirichlet L-functions at the zeros of derivatives of these functions are given with the intention of shedding a little light on the interaction between two such functions.

Keywords The Riemann zeta-function · Dirichlet L-functions
Derivatives · Zeros

2010 Mathematics Subject Classification 11M26 · 11M06

K. Benli (✉)
Department of Mathematics, University of Georgia, Athens, GA, USA
e-mail: kubra.benli25@uga.edu

E. Elma
Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada
e-mail: eelma@uwaterloo.ca

C. Y. Yildirim
Department of Mathematics, Boğaziçi University, İstanbul, Turkey
e-mail: yalciny@boun.edu.tr

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_5

1 Introduction

The Riemann zeta-function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\sigma > 1),$$

where $s = \sigma + it$ and σ and t are real numbers, and then it can be continued analytically to the whole complex plane (with a simple pole at $s = 1$ with residue 1), satisfying the functional equation

$$\zeta(s) = \chi_\zeta(s)\zeta(1 - s),$$

where

$$\chi_\zeta(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Apart from the trivial zeros at $s = -2, -4, \ldots$, $\zeta(s)$ has non-real (nontrivial) zeros in the critical strip $0 < \sigma < 1$. The number of nontrivial zeros with $0 < t < T$ is $\frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$, as $T \to \infty$. For such basic knowledge about the Riemann zeta-function and Dirichlet L-functions, we refer the reader to [4] or [9].

Let $\rho_{\zeta,k}$ denote a non-real zero of the k^{th} derivative $\zeta^{(k)}(s)$ of $\zeta(s)$ and $\gamma_{\zeta,k} := \Im\rho_{\zeta,k}$. Berndt [2] showed that the number of $\rho_{\zeta,k}$ with $\gamma_{\zeta,k} \in (0, T)$ is $\frac{T}{2\pi} \log \frac{T}{4\pi e} + O_k(\log T)$, as $T \to \infty$.

Similarly, a Dirichlet L-function is defined as

$$L(s, \psi) := \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}, \quad (\sigma > 1),$$

where ψ is a Dirichlet character modulo q. We will take q fixed (i.e., not depending on T) and for the results below we will consider only odd prime values of q. In case ψ is a primitive character, $L(s, \psi)$ satisfies the functional equation

$$L(s, \psi) = \chi_\psi(s) L(1 - s, \overline{\psi}),$$

where

$$\chi_\psi(s) := \begin{cases} \frac{\tau(\psi)}{q^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} & \text{if } \psi(-1) = 1, \\ \frac{\tau(\psi)}{iq^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} & \text{if } \psi(-1) = -1, \end{cases}$$

and

$$\tau(\psi) = \sum_{m=1}^{q} \psi(m) e^{\frac{2\pi i m}{q}}$$

is the Gaussian sum associated with ψ. Apart from the trivial zeros on the nonpositive real axis, $L(s, \psi)$ has nontrivial zeros in the critical strip. The number of nontrivial zeros of $L(s, \psi)$ in $0 < \sigma < 1$, $|t| < T$ is $\frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT)$, as $T \to \infty$.

Let $\rho_{\psi,k}$ denote a nontrivial zero of $L^{(k)}(s, \psi)$ and $\gamma_{\psi,k} := \Im \rho_{\psi,k}$. Yıldırım [10] showed that the number of zeros of $L^{(k)}(s, \psi)$ with $|\gamma_{\psi,k}| < T$ in a wide enough strip outside of which there are no nontrivial zeros is $\frac{T}{\pi} \log \frac{qT}{2\pi e m} + O_q(\log T)$, where m is the smallest prime not dividing q (so, for an odd prime q, $m = 2$).

It is known after the works of Conrey and Ghosh [3] (assuming RH, the Riemann Hypothesis) and Karabulut and Yıldırım [7] (unconditionally) that, for $k \in \mathbb{Z}^+$,

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k}) = \mathscr{A}_k \frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right), \quad (T \to \infty), \tag{1}$$

where

$$\begin{aligned} \mathscr{A}_k &:= -\sum_{u=0}^{\infty} (-1)^u \sum_{v=1}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1, \ldots i_k} \\ &\quad \times \prod_{w=1}^{k} (-1)^w w! \binom{k}{w}^{i_w} \frac{(-1)^v (v+1)!}{(i_1 + 2i_2 + \ldots + k i_k + v)!} \\ &= \sum_{r=1}^{k} e^{-z_r} - k - 1, \qquad (k \geq 1), \end{aligned} \tag{2}$$

with the z_r $(r = 1, \ldots, k)$ being the zeros of $P_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!}$.

One of the aims of the paper of Conrey and Ghosh was to prove that for any $\varepsilon > 0$ there are $\gg_\varepsilon T$ zeros of $\zeta^{(k)}(s)$ in the region $\frac{1}{2} \leq \sigma < \frac{1}{2} + \frac{(1+\varepsilon)\log\log T}{\log T}$, $0 < t < T$, and they used (1) for this purpose. That Karabulut and Yıldırım made (1) unconditional did not change the fact that this result of Conrey and Ghosh is dependent on RH because along the way one needs to know that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in $\sigma < \frac{1}{2}$ and that depends on RH essentially as the work of Levinson and Montgomery [8] showed.

2 Statement of the Results

In this paper, assuming the Generalized Riemann Hypothesis (GRH) for the Riemann zeta-function and Dirichlet L-functions, we give results of the calculation of some generalizations of the sum in (1).

Let $k \geq 0$ be a fixed integer, $\mathscr{A}_0 := -1$, q_1, q_2 be fixed odd prime numbers. Let $\psi_1(\text{mod } q_1)$ and $\psi_2(\text{mod } q_2)$ be non-principal Dirichlet characters.

If $q_1 = q_2$, then we have

$$\sum_{0<\gamma_{\psi_1,k}\leq T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{\overline{\tau\left(\overline{\psi_2}\right)}\tau(\psi_1)}{\varphi(q_1)} \mathscr{A}_k \frac{T}{2\pi} + O_{k,q_1}\left(\frac{T}{\log T}\right), \quad (T \to \infty). \tag{3}$$

On the other hand, if $q_1 \neq q_2$, then

$$\sum_{0<\gamma_{\psi_1,k}\leq T} \chi_{\psi_2}(\rho_{\psi_1,k}) \ll_{k,q_1,q_2} T^{1-\frac{1}{\log\log T}} \log T \log\log T, \quad (T \to \infty). \tag{4}$$

Furthermore, for a fixed odd prime q, let ψ be a non-principal Dirichlet character (mod q). We have

$$\sum_{0<\gamma_{\psi,k}\leq T} \chi_{\zeta}(\rho_{\psi,k}) \ll_{k,q} T^{1-\frac{1}{\log\log qT}} \log T \log\log T, \quad (T \to \infty). \tag{5}$$

and

$$\sum_{0<\gamma_{\zeta,k}\leq T} \chi_{\psi}(\rho_{\zeta,k}) = -\frac{\overline{\tau(\overline{\psi})}}{\varphi(q)} \mathscr{A}_k \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right), \quad (T \to \infty). \tag{6}$$

3 Remarks about the Proofs

For the purpose of gaining some insight into the interaction between two Dirichlet L-functions, we were interested in the value of the sums (3)–(6) above. We assumed the GRH to keep the calculations shorter, but as in [7], the results may be obtained without this assumption. The restriction of the Dirichlet characters to fixed prime moduli (in which case all non-principal characters are primitive) was made in order to avoid some minor complications in the calculations.

The proofs begin by expressing the sum under consideration as a contour integral of the form

$$\sum_{A<\gamma\le B} f(\rho) = \frac{1}{2\pi i}\int_{R} f(s)\frac{g^{(k+1)}(s)}{g^{(k)}(s)}\,ds,$$

where ρ is a zero of $g^{(k)}(s)$ with imaginary part γ, and R is an appropriate rectangular contour with vertices at $\sigma_k + iA$, $\sigma_k + iB$, $-\delta + iB$ and $-\delta + iA$, with σ_k and $\delta > 0$ suitably chosen so as to avoid having any poles of the integrand on the contour. The existence of such suitable contours is given by well-known results on the zeros of derivatives of $\zeta(s)$ and $L(s,\psi)$. For such a contour R, the integrals along the horizontal parts and the right side of the contour R can be bounded easily. The results are obtained by careful consideration of the integral along the left vertical segment of the rectangle R, from $-\delta + iB$ to $-\delta + iA$. As an example, here we give a sketch of the proof of (3) for $q := q_1 = q_2$. Upon bounding the integrals over the three sides of R (for which we take $A = \frac{T}{2},\ B = T$), except for the left vertical side, we have

$$\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\le T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{1}{2\pi i}\int_{-\delta+iT}^{-\delta+i\frac{T}{2}} \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds \;+\; O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T). \tag{7}$$

The integral here can be re-expressed as

$$\overline{-\frac{1}{2\pi i}\int_{1+\delta+i\frac{T}{2}}^{1+\delta+iT} \chi_{\overline{\psi_2}}(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1})\,ds}. \tag{8}$$

Now, using

$$\frac{\chi_{\psi}^{(m)}}{\chi_{\psi}}(s) = \left(-\ell + O\left(\frac{1}{|t|}\right)\right)^{m},\ (\ell = \log\frac{q|t|}{2\pi},\ m\in\mathbb{N},\ |t|\ge 1),$$

for s lying in a fixed vertical strip, k-fold differentiation of the functional equation of $L(s,\psi_1)$ gives

$$L^{(k)}(s,\psi_1) = \chi_{\psi_1}(s)\left(1 + O\left(\frac{1}{|t|}\right)\right)\left(-\ell + \frac{d}{ds}\right)^{k} L(1-s,\overline{\psi_1}).$$

From this we have

$$\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1}) = -\left(\ell + \frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{9}$$

for $\sigma' \leq \sigma \leq \sigma''$ (with σ', σ'' fixed real numbers), where

$$G_k(s, z, \psi_1) := \left(z + \frac{d}{ds}\right)^k L(s, \psi_1) = z^k L(s, \psi_1) + kz^k L'(s, \psi_1) + \ldots + L^{(k)}(s, \psi_1) \tag{10}$$

and the differentiation in G_k' is with respect to s. We will substitute (8) and (9) in (7), and use the following generalization of Lemma 5 of [6] and Lemma 2.2 of [7]:

For $m \in \mathbb{Z}$ with $|m| = o(\log T)$ as $T \to \infty$, we have

$$\frac{1}{2\pi} \int_{T/2}^{T} \chi_\psi(1 - (a + it)) \left(\log \frac{qt}{2\pi}\right)^m \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}}\, dt = \frac{\tau(\psi)}{q} \sum_{\frac{qT}{4\pi} < n \leqslant \frac{qT}{2\pi}} b_n (\log n)^m e^{-\frac{2\pi i n}{q}} + O_q(K_0^{|m|} T^{a-\frac{1}{2}} (\log T)^m). \tag{11}$$

Here $a > 1$ is fixed, ψ is a primitive Dirichlet character modulo $q \geq 3$, $(b_n)_{n\geq 1}$ is a sequence of complex numbers such that $b_n \ll n^\varepsilon$ for any $\varepsilon > 0$, and K_0 is any fixed number > 1.

But in order to apply this result to our calculation, we have to approximate $\frac{G_k'}{G_k}(s, z, \overline{\psi_1})$, which is not a Dirichlet series for $k \geq 1$, by a Dirichlet series (for $k = 0$, we just take the Dirichlet series and the calculation is simpler). From (10) we have

$$\frac{G_k'}{G_k}(s, z, \overline{\psi_1}) = \frac{\displaystyle\sum_{v=0}^{k} \binom{k}{v} \frac{1}{z^v} \frac{L^{(v+1)}}{L}(s, \overline{\psi_1})}{\displaystyle 1 + \sum_{w=1}^{k} \binom{k}{w} \frac{1}{z^w} \frac{L^{(w)}}{L}(s, \overline{\psi_1})}. \tag{12}$$

Since $\frac{L^{(w)}}{L}(s, \overline{\psi_1}) \ll_w 1$ for $\sigma \geq 1 + \delta$, we see

$$\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s, \overline{\psi_1}) \ll_k \frac{1}{\log T},$$

so that we can expand the denominator of the right-hand side of (12) in a geometric series and write

$$\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{-1} \tag{13}$$
$$=\sum_{u\le\frac{\log T}{\log\log T}}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u+O\left(\frac{1}{T}\right).$$

Thus we obtain

$$\overline{\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\le T}\chi_{\psi_2}(\rho_{\psi_1,k})}=\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times\frac{\tau\left(\overline{\psi_2}\right)}{q}\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})e^{-\frac{2\pi in}{q}}}{(\log n)^K}+O_{k,q}(T^{\frac{1}{2}+\delta+\varepsilon}), \tag{14}$$

where $K:=i_1+2i_2+\ldots+ki_k+v$, and

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})}{n^s}:=\frac{L^{(v+1)}}{L}(s,\overline{\psi_1})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}. \tag{15}$$

The sum over n in (14) can be re-expressed as

$$\frac{1}{\varphi(q)}\sum_{\substack{a\,(\mathrm{mod}\ q)\\(a,q)=1}}e^{-\frac{2\pi ia}{q}}\sum_{\psi\,(\mathrm{mod}\ q)}\overline{\psi}(a)\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K}.$$

Here the sum over ψ is split into two parts. The term $\psi=\psi_1$ (which will lead to the main term), and the terms $\psi\neq\psi_1$ the contribution of which will be denoted by $E_{\psi\neq\psi_1}$ (which turns out to be an error term). So now we have

$$\overline{\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\le T}\chi_{\psi_2}(\rho_{\psi_1,k})}= \tag{16}$$
$$\frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{q\varphi(q)}\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi_1(n)}{(\log n)^K}+E_{\psi\neq\psi_1}+O_{k,q}(T^{\frac{1}{2}+\delta+\varepsilon}).$$

The proof is then completed upon using (2) and the following result:

Assume GRH. Let q be a fixed odd prime number and ψ_1 be a primitive Dirichlet character modulo q; $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$. If ψ is a Dirichlet character modulo q such that $\psi \neq \psi_1$, then we have

$$\sum_{n \le x} b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi(n) = O(x^{1-\frac{1}{\log\log q(x+4)}}(\log x)(A(k)\log\log q(x+4))^{K+1}). \tag{17}$$

If also $i_1 + ... + i_k \leqslant \frac{\log x}{\log\log x}$, then we have

$$\sum_{n \le x} b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi_1(n) = S(i_1, ..., i_k; v)x(\log x)^K + E_b(i_1, ..., i_k; v), \tag{18}$$

where

$$S(i_1, ..., i_k; v) := \frac{(-1)^{K+1}(v+1)! \prod_{w=1}^{k}(w!)^{i_w}}{K!}, \tag{19}$$

and

$$E_b(i_1, ..., i_k; v) := O_q\left((A(k)^K)\left((\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x\,(\log x)^{(\frac{2}{3}+\varepsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\varepsilon}}}\right)\right). \tag{20}$$

The other results, (4)–(6), are proved along similar lines. The detailed proofs of the above results can be found in [1] and in [5].

Acknowledgements We thank Dr. Yunus Karabulut for a helpful conversation.

References

1. K. Benli, The mean values of the functional equation χ_ψ at the zeros of $\zeta^{(k)}(s)$ and $L^{(k)}(s, \psi_2)$, M.S. thesis, Boğaziçi University, 2015
2. B.C. Berndt, The number of zeros for $\zeta^{(k)}(s)$. J. Lond. Math. Soc. **2**(2), 577–580 (1970)
3. J.B. Conrey, A. Ghosh, Zeros of derivatives of the Riemann zeta-function near the critical line, *Analytic Number Theory (Allerton Park, IL 1989)*, Progress in Mathematics, vol. 85 (Birkhäuser Boston, Boston, 1990), pp. 95–110
4. H. Davenport, *Multiplicative Number Theory*, 3rd edn. revised by H.L. Montgomery (Springer-Verlag, New York, 2000)
5. E. Elma, Average values of the functional equation factors at the zeros of derivatives of Dirichlet L-functions, M.S. thesis, Boğaziçi University, 2015
6. S.M. Gonek, Mean values of the Riemann zeta-function and its derivatives. Invent. Math. **75**(1), 123–141 (1984)

7. Y. Karabulut, C.Y. Yıldırım, On some averages at the zeros of derivatives of the Riemann zeta-function. J. Number Theory **131**(10), 1939–1961 (2011)
8. N. Levinson, H.L. Montgomery, Zeros of the derivatives of Riemann's zeta-function. Acta Math. **133**, 49–65 (1974)
9. H.L. Montgomery, R. Vaughan, *Multiplicative Number Theory I. Classical Theory* (Cambridge University Press, Cambridge, 2006)
10. C.Y. Yıldırım, Zeros of derivatives of Dirichlet L-functions. Turk. J. Math. **20**(4), 521–534 (1996)

New Weighted Partition Theorems with the Emphasis on the Smallest Part of Partitions

Alexander Berkovich and Ali Kemal Uncu

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract We use the q-binomial theorem, the q-Gauss sum, and the ${}_2\phi_1 \to {}_2\phi_2$ transformation of Jackson to discover and prove many new weighted partition identities. These identities involve unrestricted partitions, overpartitions, and partitions with distinct even parts. The smallest part of a partition plays an important role in our analysis. This work was motivated in part by the research of Krishna Alladi.

Keywords q-hypergeometric identities · Partition identities · Smallest part of partitions · Overpartitions · Ramanujan's lost notebooks · Jackson's Transformation

2010 Mathematics Subject Classification 05A15 · 05A17 · 05A19 · 11B34 11B75 · 11P81 · 11P84 · 33D15

1 Introduction

A *partition*, $\pi = (\lambda_1, \lambda_2, \dots)$, is a finite sequence of nonincreasing positive integers. The empty sequence is conventionally considered to be the unique partition of zero.

A. Berkovich (✉) · A. K. Uncu
Department of Mathematics, University of Florida,
358 Little Hall, Gainesville, FL 32611, USA
e-mail: alexb@ufl.edu

A. K. Uncu
e-mail: akuncu@ufl.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_6

The elements λ_i that appear in the list π are called *parts* of the partition π. The sum of all parts of a partition is called the *norm* of a partition π. We call a partition π *a partition of* n if its norm is n.

We list some useful statistics/notations that will be used in the paper. Given a partition π,

$$\begin{aligned}|\pi| &:= \text{norm of the partition } \pi,\\ s(\pi) &:= \text{smallest part of the partition } \pi,\\ \nu_e(\pi) &:= \text{number of even parts in } \pi,\\ \nu_o(\pi) &:= \text{number of odd parts in } \pi,\\ \nu(\pi) &:= \nu_e(\pi)+\nu_o(\pi) = \text{number of parts in} \pi,\\ \nu_d(\pi) &:= \text{number of different parts in } \pi.\end{aligned}$$

For example, $\pi = (10, 9, 5, 5, 4, 1, 1)$ is a partition of 35 with $s(\pi) = 1$, $\nu(\pi) = 7$, and $\nu_d(\pi) = 5$.

Alladi studied the weighted partition identities methodically. In 1997, among many interesting results, he noted the general identity:

Theorem 1.1 **(Alladi** [1]**, 1997).** *Let a, b and q be variables.*

$$\frac{(a(1-b)q;q)_n}{(aq;q)_n} = 1 + \sum_{\pi\in\mathcal{U}_n} a^{\nu(\pi)} b^{\nu_d(\pi)} q^{|\pi|}, \tag{1.1}$$

where $\mathcal{U}_n$ is the set of non-empty ordinary partitions into parts $\leq n$.

In (1.1) and in the rest of the paper we use the standard q-Pochhammer symbol notations defined in [4], [12]. Let L be a nonnegative integer, then

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i) \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

We now discuss a special case of the Theorem 1.1 that plays an important role in the study of overpartitions. We define an overpartition to be a partition where the last appearance of a part may come with a mark (usually put as an overhead bar on the part, hence the name). Any partition is an oof the same number. One nontrivial example is $\bar{\pi} = (10, \bar{9}, 5, 5, 4, 1, \bar{1})$. All the statistics defined above translate in the obvious manner to overpartitions. The definition, the interpretation of overpartitions and the generating function for the number of overpartitions are given by Corteel and Lovejoy in their influential paper [11].

We would like to define the following sets:

$$\begin{aligned}\mathcal{U} &:= \text{the set of all non-empty ordinary partitions},\\ \mathcal{O} &:= \text{the set of all non-empty overpartitions}.\end{aligned}$$

The connection of the identity (1.1) and overpartitions can be seen by setting $a = 1$ and $b = 2$,

$$\frac{(-q;q)_n}{(q;q)_n} = 1 + \sum_{\pi \in \mathscr{U}_n} 2^{\nu_d(\pi)} q^{|\pi|}. \tag{1.2}$$

The left side of the identity (1.2) is interpreted as the generating function for the number of overpartitions into parts $\leq n$. The right-hand side of (1.2) is the weighted connection of overpartitions with ordinary partitions. We can write the weighted connection between ordinary partitions and overpartitions abstractly

$$\sum_{\pi \in \mathscr{O}_n} q^{|\pi|} = \sum_{\pi \in \mathscr{U}_n} 2^{\nu_d(\pi)} q^{|\pi|}, \tag{1.3}$$

where $\mathscr{O}_n$ is the set of non-empty overpartitions into parts $\leq n$.

The equation (1.3) is an example of a weighted partition identity between sets of partitions. In this paper, we prove new weighted partition identities involving statistics other than $2^{\nu_d(\pi)}$.

Section 2 has necessary definitions and identities to follow the results in the paper. The weighted partition identities for ordinary partitions and overpartitions with emphasis on the smallest part, $s(\pi)$, are given in Section 3. A weighted count of overpartitions' relation with the number of representations of a number as a sum of two squares will be given in Section 4. Section 5 has weighted partition identities related to partitions with distinct even parts. In Section 6 we provide a more involved weighted identity involving overpartitions into parts not divisible by 3.

2 Definition and Background Information

Partitions can be represented in the *frequency notation* $\pi = (1^{f_1}, 2^{f_2}, \dots)$ by writing parts of π in a finite sequence format with exponents, where the exponents $f_i(\pi)$ of the natural numbers denote the number of appearances of that part in π. We abuse the notation and write f_i, frequency of i, when the partition is understood from the context. Similarly, we drop the zero frequencies in our notation to keep the notations neat. A zero frequency may still be used to indicate and stress an integer not being a part of a partition. For example, the partition $\pi = (10, 9, 5, 5, 4, 1, 1)$ can be represented in the frequency notation as $(1^2, 2^0, 3^0, 4^1, 5^2, 6^0, 7^0, 8^0, 9^1, 10^1, 11^0, \dots) = (1^2, 4, 5^2, 7^0, 9, 10)$. Here π is a partition of 35 where the frequency of 1, $f_1(\pi) = f_1 = 2$, $f_4 = 1$, $f_5 = 2 \dots$ and the integer 7 is not a part of π.

One can also extend the frequency notation to overpartitions by allowing sequence elements with a positive frequency to have an overhead bar meaning that the first appearance of that part is marked. The norm of overpartitions and ordinary partitions are defined the same way. In the frequency notation, we can represent $\bar{\pi}$ as $(\bar{1}^2, 4, 5^2, \bar{9}, 10)$.

Other representations of partitions include the *Ferrers diagrams* and *2-modular Ferrers diagrams* [4, §1.3]. The Ferrers diagram and the 2-modular Ferrers diagram are formed by drawing rows of boxes where the row sum (count of boxes or the sum of the contents, respectively) adds up to the corresponding part of the partition. It should be reminded to the reader that in the 2-modular diagrams, only the boxes at the end of a row may be filled by 1 or 2; all the other boxes are filled with 2's. An example of the Ferrers diagram and a 2-modular Ferrers diagram is given in Figure. 1.

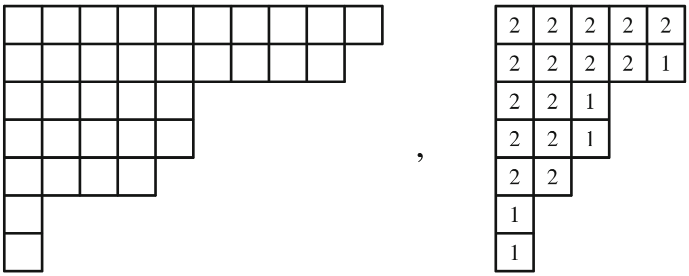


Fig. 1 The Ferrers Diagram and the 2-modular Ferrers Diagram of the partition $\pi = (10, 9, 5, 5, 4, 1, 1)$

Note that the *conjugate* of a Ferrers diagram (drawing a Ferrers diagram column-wise and reading it row-wise) is also a partition. The conjugate of π is $(7, 5, 5, 5, 4, 2, 2, 2, 2, 1)$. Conjugation does not extend to 2-modular graphs directly. A partition's 2-modular diagram yields another 2-modular diagram under conjugation only when the original partition has distinct odd parts. The conjugate of the 2-modular graph of the example in Figure 1 does not yield a permissible 2-modular diagram.

Ferrers diagrams can be extended to overpartitions. One can easily mark the rows of the Ferrers diagrams by coloring the box at the end of the row to indicate that the related part of the partition is overlined. Conjugation of the ordinary Ferrers diagrams carry over for overpartitions without a hitch. It is easy to check that the conjugate of $\bar{\pi} = (10, \bar{9}, 5, 5, 4, 1, \bar{1})$ is $(\bar{7}, 5, 5, 5, 4, 2, 2, 2, \bar{2}, 1)$.

We define the basic q-hypergeometric series as they appear in [12]. Let r and s be nonnegative integers and $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, q$, and z be variables. Then

$$
{}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1-r+s} z^n.
$$

Let a, b, c, q, and z be variables. The q-binomial theorem [12, II.4, p. 236] is

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.1} $$

and the q-Gauss sum [12, II.8, p. 236] is

$$ {}_2\phi_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}. \tag{2.2} $$

The Jackson ${}_2\phi_1$ to ${}_2\phi_2$ transformation [12, III.4, p. 241] is

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right). \tag{2.3} $$

We would also like to recall the definition of the classical theta functions φ and ψ

$$ \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \text{ and } \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. $$

The Gauss identities [4, Cor 2.10, p. 23] for these functions will be of use:

$$ \varphi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{2.4} $$

$$ \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.5} $$

3 Weighted Identities with respect to the Smallest Part of a Partition

Let $\mathscr{U}^*$ be the subset of $\mathscr{U}$ such that for every $\pi \in \mathscr{U}^*$, $f_1(\pi) \equiv 1 \mod 2$. Next, we introduce a new partition statistic $t(\pi)$ to be the number defined by the properties

i. $f_i \equiv 1 \mod 2$, for $1 \leq i \leq t(\pi)$,
ii. and $f_{t(\pi)+1} \equiv 0 \mod 2$.

Note that for any $\pi \in \mathscr{U}$ with an even frequency of 1 (which could be 0) we have $t(\pi) = 0$. Then we have the weighted partition identity between the set of ordinary partitions and its subset $\mathscr{U}^*$ as follows.

Theorem 3.1.

$$ \sum_{\pi \in \mathscr{U}} (-1)^{s(\pi)+1} q^{|\pi|} = \sum_{\pi \in \mathscr{U}^*} t(\pi) q^{|\pi|}. \tag{3.1} $$

The left side identity is the weighted count of partitions of a given norm n where every partition with an odd smallest part gets counted with $+1$ and the partitions of

n with an even smallest part gets counted with -1. There are 42 partitions of 10 in total. From this number, 9 partitions, (2^5), $(2^3, 4)$, $(2^2, 3^2)$, $(2^2, 6)$, $(2, 3, 5)$, $(2, 4^2)$, $(2, 8)$, $(4, 6)$, (10), have an even smallest part. Therefore, from the count of the left-hand side of (3.1), the coefficient of the q^{10} is $24 = 42 - 2 \cdot 9$. The right-hand side count and the weights can be found in Table 1.

Table 1 Example of Theorem 3.1 with $|\pi| = 10$

$\pi \in \mathcal{U}^*$	$t(\pi)$	$\pi \in \mathcal{U}^*$	$t(\pi)$
$(1, 2, 3, 4)$	4	$(1, 4, 5)$	1
$(1, 2^3, 3)$	3	$(1, 2^2, 5)$	1
$(1^5, 2, 3)$	3	$(1^5, 5)$	1
$(1, 2, 7)$	2	$(1^3, 3, 4)$	1
$(1^3, 2, 5)$	2	$(1, 3^3)$	1
$(1, 9)$	1	$(1^3, 2^2, 3)$	1
$(1^3, 7)$	1	$(1^7, 3)$	1
$(1, 3, 6)$	1		

The sum of the weights is 24, which is the same as the count of partitions with the altering sign with respect to their smallest part's parity.

The proof of Theorem 3.1 will be given as the combinatorial interpretation of the following analytic identity.

Theorem 3.2.

$$\sum_{n\geq 1} \frac{q^n}{1+q^n} \frac{1}{(q;q)_{n-1}} = \sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n (q^{n+1};q)_\infty}. \tag{3.2}$$

Proof. Recall that $(0;q)_n = 1$ for any integer $n \geq 0$. Also, note that

$$\frac{1+q}{1+q^n} = \frac{(-q;q)_{n-1}}{(-q^2;q)_{n-1}}, \tag{3.3}$$

for positive n. We start by writing the left–hand side of (3.2) as a q-hypergeometric function. Multiplying and dividing by $1+q$ and using (3.3), shifting the sum with $n \mapsto n+1$, and finally factoring out $q/(1+q)$ yields

$$\sum_{n\geq 1} \frac{q^n}{1+q^n} \frac{1}{(q;q)_{n-1}} = \frac{q}{1+q} {}_2\phi_1 \left(\begin{matrix} 0, -q \\ -q^2 \end{matrix} ; q, q \right). \tag{3.4}$$

We now apply Jackson's transformation (2.3) to (3.4). This gives us

$$\frac{q}{1+q} {}_2\phi_1 \left(\begin{matrix} 0, -q \\ -q^2 \end{matrix} ; q, q \right) = \frac{q}{1+q} \frac{1}{(q;q)_\infty} {}_2\phi_2 \left(\begin{matrix} 0, \ q \\ -q^2, \ 0 \end{matrix} ; q, -q^2 \right). \tag{3.5}$$

Distributing the front factor to each summand on the right-hand side of (3.5), doing the necessary simplifications, and finally shifting the summation index $n \mapsto n-1$ finishes the proof. □

Theorem 3.2 is the analytical version of Theorem 3.1. We will now move on to the generating function interpretations of both sides of (3.2). This study will in-turn prove Theorem 3.1.

We start with the left-hand side sum

$$\sum_{n\geq 1} \frac{q^n}{1+q^n} \frac{1}{(q;q)_{n-1}}, \tag{3.6}$$

of (3.2). For a positive integer n, the summand

$$\frac{q^n}{1+q^n} = \sum_{k\geq 1} (-1)^{k+1} q^{nk} = q^n - q^{2n} + q^{3n} \dots \tag{3.7}$$

is the generating function for the number of partitions of the form (k^n) where the partition gets counted with the weight $+1$ if the part k is odd and it gets counted with the weight -1 if the part k is even. The factor

$$\frac{1}{(q;q)_{n-1}} \tag{3.8}$$

is the generating function for the number of partitions into parts less than n. With conjugation in mind, another equivalent interpretation of (3.8) is that it is the generating function for the number of partitions into less than n parts.

We put the partitions counted by the factors in the summand into a single partition bijectively by part-by-part addition. For the same positive integer n, let π_1 be a partition counted by (3.7) and a partition π_2 counted by (3.8). We know that $\pi_1 = (k^n)$ for some positive integer k. Starting from the largest part of π_2, we add a part of π_2 to a part of π_1 and put the outcome as a part of a new partition π. Recall that a part of a partition is a positive integer that is an element of that partition. The partition π_2 has less than n parts. Therefore, there is at least one part of π_1 that does not get anything added to it. We add these leftover parts of π_1 to π after the additions. This way we know that the new partition π has exactly n parts, where the smallest part is exactly k. This can be easily demonstrated using Ferrers diagrams in Figure 2.

Moreover, the partition π gets counted with the weight $+1$ if the smallest part is odd and it gets counted with the weight -1 if the smallest part is even. The sum of all these terms gives us the generating function for the weighted count of ordinary partitions from $\mathcal{U}$. Hence,

$$\sum_{n\geq 1} \frac{q^n}{1+q^n} \frac{1}{(q;q)_{n-1}} = \sum_{\pi \in \mathcal{U}} (-1)^{s(\pi)+1} q^{|\pi|}, \tag{3.9}$$

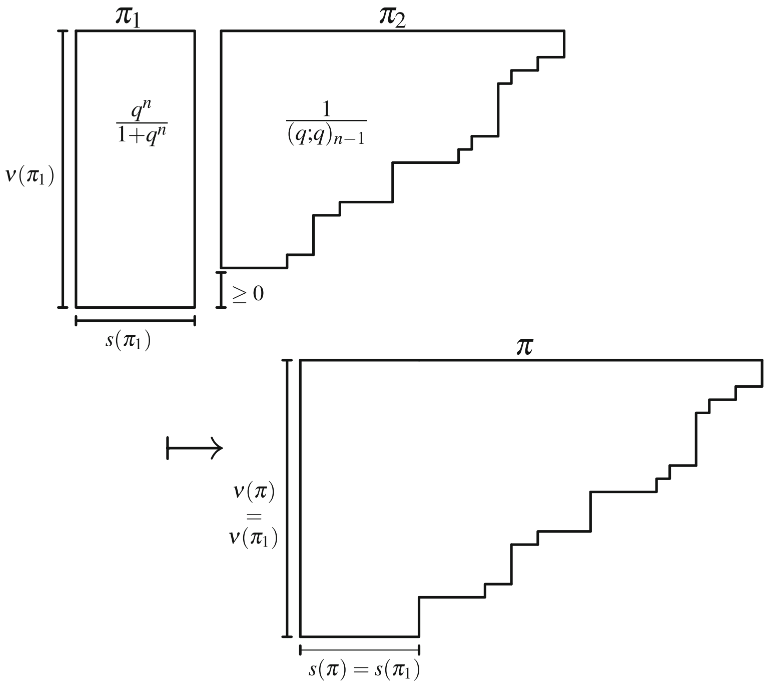


Fig. 2 Demonstration of putting together partitions in the summand of (3.6)

where $s(\pi)$ is the smallest part of the partition π.

The right-hand side summation

$$\sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n(q^{n+1};q)_\infty} \tag{3.10}$$

of (3.2) can also be interpreted as a weighted count of partitions. For some positive integer n, the term $q^{n(n+1)/2}$ can be thought of as the generating function of the partition $\pi_1^* = (1, 2, 3, 4, \ldots, n)$ where every part less than or equal to n appears exactly one time. The factor

$$\frac{1}{(q^2;q^2)_n} \tag{3.11}$$

is the generating function for partitions into parts $\leq n$ where every part appears with an even frequency. Let π_2^* be a partition counted by (3.11). By adding the frequencies of π_1^* and π_2^* we get another partition

$$\pi^* = (1^{f_1}, 2^{f_2}, \ldots, n^{f_n}),$$

where all $f_i \equiv 1 \mod 2$. The quotient

$$\frac{1}{(q^{n+1};q)_\infty} \tag{3.12}$$

is the generating function for the number of partitions into parts $> n$. Therefore, for a partition π' that is counted by (3.12) one can put together π^* and π' without the need of adding any frequencies. Call the outcome partition of merging π^* and π', π.

With this interpretation, the partitions counted by (3.10) have the frequency restriction that $f_1(\pi) \equiv 1 \mod 2$. Also, let i be the first positive integer where $f_i(\pi)$ is even (maybe zero). It is obvious that the partition π might be the final outcome of the merging procedure explained above for any summand in (3.10) as long as the index of the summand is $< i$. Therefore, the partition π is weighted by the number of the parts in its initial chain of odd frequencies of parts. This proves

$$\sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n (q^{n+1};q)_\infty} = \sum_{\pi \in \mathscr{U}^*} t(\pi) q^{|\pi|}, \tag{3.13}$$

where $t(\pi)$ is as defined in Theorem 3.1. The identities (3.9) and (3.13) together prove Theorem 3.1.

Now we move on to another analytical identity similar to (3.2). This identity will later prove a weighted partition identity for overpartitions.

Theorem 3.3.

$$\sum_{n\geq 1} \frac{2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n} \frac{2q^{n+1}}{1-q^{2(n+1)}} \frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} \tag{3.14}$$

Proof. Multiply and divide the left-hand side of (3.14) by $(1+q)$, use (3.3), and write it as a q-hypergeometric series:

$$\sum_{n\geq 1} \frac{2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \frac{2q}{1+q} {}_2\phi_1 \left(\begin{matrix} -q, -q \\ -q^2 \end{matrix} ; q, q \right). \tag{3.15}$$

Now we apply the transformation (2.3) to (3.15). This yields,

$$\frac{2q}{1+q} {}_2\phi_1 \left(\begin{matrix} -q, -q \\ -q^2 \end{matrix} ; q, q \right) = \frac{2q}{1+q} \frac{(-q^2;q)_\infty}{(q;q)_\infty} {}_2\phi_2 \left(\begin{matrix} -q, q \\ -q^2, -q^2 \end{matrix} ; q, -q^2 \right). \tag{3.16}$$

Distributing the front factor to each summand, doing the necessary simplifications, and regrouping terms show that the right-hand sides of identities (3.14) and (3.16) are equal. □

Identities (3.2) and (3.14) are $z = 1$ and 2 special cases of the more general result, respectively.

Theorem 3.4.

$$\sum_{n\geq 1}\frac{q^n}{1+q^n}\frac{((1-z)q;q)_{n-1}}{(q;q)_{n-1}}=\frac{((1-z)q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{q^{n(n+1)/2}}{(-q)_n(1-(1-z)q^n)}.$$

This identity can be proven using the same Jackson transformation (2.3) with $(a, b, c, q, z)\mapsto((1-z)q, -q, -q^2, q, q)$.

The combinatorial interpretation of (3.14) is similar to the one of (3.2). Consider the left-hand side sum

$$\sum_{n\geq 1}\frac{2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}$$

of (3.14). For a given n the summand factor

$$\frac{2q^n}{1+q^n}$$

is the generating function of the number of overpartitions into exactly n parts of the same size, where the partitions are counted with weight $+1$ if the part is odd and with -1 if the part is even. In other words, it is the generating function for the number of partitions (k^n) and $(\bar{k}^n)$ for any integer $k\geq 1$, where these partitions are counted with the weight $(-1)^{k+1}$. The other factor

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}, \tag{3.17}$$

(by (1.2)) is the generating function for the number of overpartitions with strictly less than n parts. As we did in the proof of Theorem 3.1, we put the parts of these partitions together. This part-by-part addition gives an overpartition in exactly n parts with the smallest part k. And coming from the first factor we count these partitions with weight $+1$ if the smallest part k is odd and with weight -1 if k is even. Hence,

$$\sum_{n\geq 1}\frac{2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}=\sum_{\pi\in\mathcal{O}}(-1)^{s(\pi)+1}q^{|\pi|}. \tag{3.18}$$

The right-hand side of (3.14) can be interpreted in a way similar to that of (3.10). For some nonnegative integer n, the factor

$$\frac{q^{n(n+1)/2}}{(q;q)_n} \tag{3.19}$$

is the generating function for a number of partitions of the type $(1^{f_1}, 2^{f_2}, \ldots, n^{f_n})$, where $f_i\geq 1$ for all $1\leq i\leq n$, as $n(n+1)/2=1+2+\cdots+n$. The rest of the factors

$$\frac{2q^{n+1}}{1-q^{2(n+1)}}\frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} \tag{3.20}$$

can be interpreted as the generating function for the number of overpartitions where the smallest part (which definitely appears in the partition) is $n+1$ and that part has an odd frequency.

There is no overlapping in the size of the parts in the partitions counted by (3.19) and (3.20) for a fixed n. One can merge these partitions into a single partition without any need of nontrivial addition of frequencies. On the other hand, an outcome overpartition may be coming from different merged couples of partitions/overpartitions. Given an outcome overpartition, there is no clean cut point that would indicate where the overpartition counted by (3.20) started. The only indication is the odd frequency of the smallest part of overpartitions. Also, we know that every part below the smallest part of overpartition in the combined partition is coming from a partition counted by the generating function (3.19). In particular, 1 appears as a part in any outcome of this merging process. Therefore, we need to keep account of all these possible connection points when we are finding the count of a partition coming from the right-hand side of (3.14). By going through only the odd frequencies in a given partition and counting the number of larger parts with the overpartition weights, we can find the total count of combinations that would yield the same merged overpartition images.

Given a partition π, let $m(\pi)$ be the smallest positive integer that is not a part of π. Let $\nu_d(\pi,n)$ be the number of different parts $\geq n$ in partition π. Let

$$\chi(statement) = \begin{cases} 1, & \text{if the } statement \text{ is true,} \\ 0, & \text{otherwise,} \end{cases} \tag{3.21}$$

be the *truth* function.

Then, the right-hand side of (3.14) can be written as a weighted count of partitions as

$$\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q;q)_n}\frac{2q^{n+1}}{1-q^{2(n+1)}}\frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} = \sum_{\pi\in\mathcal{U}}\tau(\pi)q^{|\pi|}, \tag{3.22}$$

where

$$\tau(\pi) = \sum_{i=1}^{m(\pi)}\chi(f_i\equiv 1 (\text{mod } 2))2^{\nu_d(\pi,i)}. \tag{3.23}$$

This study proves the combinatorial version of Theorem 3.3. We put (3.18) and (3.22) together, and get the following theorem.

Theorem 3.5.

$$\sum_{\pi\in\mathcal{O}}(-1)^{s(\pi)+1}q^{|\pi|} = \sum_{\pi\in\mathcal{U}}\tau(\pi)q^{|\pi|}, \tag{3.24}$$

where $\tau(\pi)$ is defined as in (3.23).

There are 100 overpartitions of 8. There are 18 overpartitions of 8 with an even smallest part. Hence, in the weighted count of the left-hand side of (3.24) the coefficient of q^8 term is $100 - 2 \cdot 18 = 64$. We exemplify the right-hand side weights of Theorem 3.5 for the same norm in Table 2.

Table 2 Example of Theorem 3.5 with $|\pi| = 8$.

$\pi \in \mathcal{U}$	$\tau(\pi)$	$\pi \in \mathcal{U}$	$\tau(\pi)$
$(1^3, 2, 3)$	$8+4+2=14$	$(1^2, 2, 4)$	4
$(1, 2, 5)$	$8+4=12$	$(1^3, 5)$	4
$(1, 2^2, 3)$	$8+2=10$	$(1, 7)$	4
$(1, 3, 4)$	8	$(1^6, 2)$	2
$(1^5, 3)$	4	$(1^2, 2^3)$	2

The sum of the weights is 64, which is the same as the count of overpartitions with the alternating sign with respect to their smallest part's parity.

4 A Weighted Identity with respect to the Smallest Part and the Number of Parts of a Partition in relation with Sums of Squares

We start with a short proof of an analytic identity.

Lemma 4.1.
$$\sum_{n\geq 1} \frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n} = \sum_{n\geq 1} (-1)^n q^{n^2}. \tag{4.1}$$

Proof. It is easy to see that

$$1 + 2\sum_{n\geq 1} \frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n} = \lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix}-1, & \rho q \\ & -q\end{matrix}; q, 1/\rho\right) = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

where we used q-Gauss sum (2.2). Rewiriting the sum in (2.4) as

$$1 + 2\sum_{n\geq 1} (-1)^n q^{n^2} \tag{4.2}$$

proves the claim. □

The identity (4.1) is a special case of a more general identity of Ramanujan [6, E. 1.6.2, p. 25] which even has a combinatorial proof [9]. But, more relevant to this paper, Alladi [2, Thm 2, p. 330] is the first one to give a combinatorial interpretation to the left-hand side of Lemma 4.1 in the spirit of the Euler pentagonal number theorem. In his study, he interpreted the left-hand side sum as the number of partitions into distinct parts with the smallest part being odd weighted with $+1$ or -1 depending on the number of parts of the partition being even or odd, respectively. In our notations:

Theorem 4.1 **(Alladi** [2], **2009).** *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi\in\mathscr{D}_o,\\ |\pi|=N}} (-1)^{\nu(\pi)} = (-1)^N \chi(N=\square),$$

where $\mathscr{D}_o$ is the set of non-empty partitions into distinct parts where the smallest part is odd, χ is as defined in (3.21)*, and $\square$ represents the statement "a perfect integer square."*

It is easy to check that

$$|\pi| \equiv \nu_o(\pi) \mod 2,$$

for any partition π. Hence,

$$\nu(\pi) - |\pi| \equiv \nu_e(\pi) \mod 2. \tag{4.3}$$

This enables us to rewrite Theorem 4.1 as in [10].

Theorem 4.2 **(Bessenrodt, Pak** [10], **2004).** *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi\in\mathscr{D}_o,\\ |\pi|=N}} (-1)^{\nu_e(\pi)} = \chi(N=\square).$$

There, they also discussed a refinement of Theorem 4.2.

Theorem 4.3 **(Bessenrodt, Pak** [10], **2004).**

$$\sum_{\substack{\pi\in\mathscr{D}_o,\\ |\pi|=N,\\ \nu_o(\pi)=k}} (-1)^{\nu_e(\pi)} = \chi(N=k^2).$$

Theorems 4.1 – 4.3 connect the weighted count of the partitions into distinct parts, where the smallest part is necessarily odd and the number of representations of an integer as a perfect square. Our next theorem will be connecting the weighted count of partitions and the number of representations of a number as a sum of two squares.

Theorem 4.4.

$$\sum_{n\geq 1} \frac{(-1)^n 2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \varphi(-q)^2 - \varphi(-q) \tag{4.4}$$

Proof. Similar to the proof of Theorem 3.2, we would like to write the left-hand side of (4.4) as a hypergeometric function first. On the left-hand side of (4.4) we multiply and divide the summand by $(1+q)$, use (3.3), factor out the terms $-2q/(1+q)$, and finally shift the summation variable $n \mapsto n+1$ to write the expression as a ${}_2\phi_1$ hypergeometric series. Applying the Jackson's transformation (2.3) to this expression yields

$$\frac{-2q}{1+q} {}_2\phi_1\left(\begin{matrix}-q,-q\\-q^2\end{matrix};q,-q\right) = \frac{-2q}{1+q}\frac{(q^2;q)_\infty}{(-q;q)_\infty} {}_2\phi_2\left(\begin{matrix}-q,q\\-q^2,q^2\end{matrix};q,q^2\right).$$

Writing the ${}_2\phi_2$ explicitly, distributing the factor $q/(1+q)$, performing the simple cancelations, shifting the summation variable $n \mapsto n-1$ and multiplying and dividing with $1-q$ we get

$$\frac{-2q}{1+q}\frac{(q^2;q)_\infty}{(-q;q)_\infty} {}_2\phi_2\left(\begin{matrix}-q,q\\-q^2,q^2\end{matrix};q,q^2\right) = 2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n}. \tag{4.5}$$

Applying Lemma 4.1 to the right-hand side of (4.5) and rewriting the identity we see that

$$\sum_{n\geq 1} \frac{(-1)^n 2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = 2\frac{(q;q)_\infty}{(-q;q)_\infty}\left(-1+\sum_{n\geq 0}(-1)^n q^{n^2}\right). \tag{4.6}$$

Using observations (4.2) and (2.4) on the right-hand side of (4.6) we complete the proof. □

The combinatorial interpretation of Theorem 4.4 combines a weighted partition count with a representation of numbers by the sum of two squares. Moreover, we can provide an explicit formula for the weighted count of partitions with respect to the norm.

Let n be a positive integer. The summand

$$\frac{(-1)^n 2q^n}{1+q^n}$$

of (4.4) is the generating function for the number of partitions of the form (k^n) (keeping (3.7) in mind) gets counted with the weight $(-1)^{k+n+1}2$. Here it should be noted that k is the smallest part and n is the number of parts of this partition. After the needed addition of partitions (similar to the ones we did for Theorems (3.1 and (3.5)) these two variables are going to stay the same for the outcome partition. To have a uniform notation, recall that $\nu(\pi)$ denotes the number of parts, and $\nu_d(\pi)$ is the number of different parts of a partition π. The second summand

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}} \tag{3.17}$$

that appears in (4.4) is the generating function for the number of overpartitions into strictly less than n parts as mentioned before. We know that this is the same as counting the number of ordinary partitions π in less than n parts counted with the weight $2^{\nu_d(\pi)}$ by (1.3).

Putting together the partition $\pi_1 = (k^n)$ and a partition π_2 counted by (3.17) (similar to the way we did in Figure 2) gives us an outcome overpartition π, which we will treat as a partition and count with the related weight at first. The partition π has the properties $s(\pi) = k$, $\nu(\pi) = n$, and $\nu_d(\pi) = \nu_d(\pi_1) + \nu_d(\pi_2) = \nu_d(\pi_2) + 1$. This partition is counted with the weight

$$\omega(\pi) := (-1)^{s(\pi)+\nu(\pi)+1} 2^{\nu_d(\pi)}, \tag{4.7}$$

(the multiplication of weights of π's generators) by the right-hand side of (4.4). This proves

$$\sum_{n\geq 1} \frac{(-1)^n 2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \sum_{\pi\in\mathcal{U}} \omega(\pi) q^{|\pi|}. \tag{4.8}$$

On the other side of the equation (4.4) we have the difference of two theta series. The summation of (2.4) is enough to see that

$$\varphi(-q)^2 - \varphi(-q) = \sum_{x,y\in\mathbb{Z}} (-1)^{x+y} q^{x^2+y^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{4.9}$$

Let $r_2(N)$ be the number of representations of N as a sum of two squares. Any positive integer N has the unique prime factorization

$$N = 2^e \prod_{i\geq 1} p_i^{v_i} \prod_{j\geq 1} q_j^{w_j},$$

where e, v_i, and w_j are nonnegative integers, and p_i and q_j are primes 1 and 3 mod 4, respectively. It is known [13, Thm 14.13, p. 572] that

$$r_2(N) = 4 \prod_{i\geq 1} (1+v_i) \prod_{j\geq 1} \frac{1+(-1)^{w_j}}{2}.$$

Writing the first series organized with respect to r_2, rewriting the second series, and finally canceling the constant terms of both series we get

$$\sum_{x,y\in\mathbb{Z}}(-1)^{x+y}q^{x^2+y^2} - \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \sum_{N\geq 1}(-1)^N r_2(N)q^N - 2\sum_{n\geq 1}(-1)^n q^{n^2}. \tag{4.10}$$

On the right-hand side of (4.10) one can collect the terms with respect to the exponents of q. Writing the two series together with the use of a truth function and comparing (4.9) and (4.10) yields the identity

$$\varphi(-q)^2 - \varphi(-q) = \sum_{N\geq 1}(-1)^N (r_2(N) - 2\chi(N = \square))q^N, \tag{4.11}$$

where χ is defined as in (3.21) and $\square$ represents "a perfect integer square."

Now we put the right-hand sides of (4.7), (4.8) and (4.11) together and get an explicit expression for the sum of weights $\omega(\pi)$ of partitions for a fixed positive norm N:

$$\sum_{\substack{\pi\in\mathcal{U},\\ |\pi|=N}} (-1)^{s(\pi)+\nu(\pi)+1}2^{\nu_d(\pi)} = (-1)^N (r_2(N) - 2\chi(N = \square)). \tag{4.12}$$

We can employ the observation (4.3) to simplify (4.12).

Theorem 4.5.

$$\sum_{\substack{\pi\in\mathcal{U},\\ |\pi|=N}} \omega^*(\pi) = r_2(N) - 2\chi(N = \square),$$

where

$$\omega^*(\pi) = (-1)^{s(\pi)+\nu_e(\pi)+1}2^{\nu_d(\pi)}.$$

Two examples of Theorem 4.5 are given in Table 3.

Another equivalent statement of Theorem 4.5 can be given over the set of overpartitions by evaluating (4.7) and (1.3).

Table 3 Examples of Theorem 4.5 with $|\pi| = 4$ and 5.

	$\pi\in\mathcal{U},\ |\pi|=4$	$\omega^*(\pi)$	$\pi\in\mathcal{U},\ |\pi|=5$	$\omega^*(\pi)$
	(4)	2	(5)	2
	(2^2)	-2	$(2,3)$	2^2
	$(1,3)$	2^2	$(1,4)$	-2^2
	$(1^2,2)$	-2^2	$(1^2,3)$	2^2
	(1^4)	2	$(1,2^2)$	2^2
			$(1^3,2)$	-2^2
			(1^5)	2
Total:		2		8

and the explicit formula of (4.12) suggests:

$$(r_2(4) - 2\cdot 1) = 4 - 2 = 2, \quad (r_2(5) - 2\cdot 0) = 8.$$

Theorem 4.6.

$$\sum_{\substack{\pi\in\mathscr{O},\\ |\pi|=N}} (-1)^{s(\pi)+\nu_e(\pi)+1} = r_2(N) - 2\chi(N=\Box).$$

5 Some Weighted Identities for Partitions with Distinct Even Parts

Let $\mathscr{P}$ denote the set of non-empty partitions with distinct even parts. A partition $\pi \in \mathscr{P}$ may still have repeated odd parts. This set has been studied before in [3, § 5], [5] and [7].

We start with the analytic identity:

Theorem 5.1.

$$\sum_{n\geq 1} \frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1} = \psi(-q) - \frac{1}{1+q}. \tag{5.1}$$

Proof. We multiply both sides of (5.1) with $1+q$ and add 1. The resulting identity becomes a special case of the q-binomial theorem (2.1) with $(a, q, z) = (-1/q, q^2, -q^3)$ provided that we use (2.5) with $q \mapsto -q$. ◻

The combinatorial interpretation of the left-hand side summand,

$$\frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1}, \tag{5.2}$$

for some positive n is really similar to the previous constructions. The main difference is the use of 2-modular Ferrers diagrams, which has been introduced in Section 2, instead. We will be following similar steps that we followed in finding the combinatorial interpretation of Theorem 3.2.

Let n be a fixed positive integer. The factor

$$\frac{(-1)^n q^{2n}}{1-q^{2n}} = \sum_{k\geq 1} (-1)^n q^{2kn}$$

is the generating function of partitions of the type $\pi_1 = (\,(2k)^n\,)$ for some positive integer k, where these partitions get counted with a weight $+1$ if the number of parts of the partition n is even and with -1 if n is odd. The second factor

$$\frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}}$$

is the generating function for the number of partitions with distinct odd parts $\leq 2n-2$. We can express these partitions in 2-modular Ferrers diagrams and take their conjugates. The outcome would show that the same factor is the generating function for the number of partitions π_2 with distinct odd parts where the number of parts is $< n$. Finally, the term q^{n-1} can be thought as the generating function of the partitions $\pi_3 = (1^{n-1})$.

We would like to add the partitions π_1, π_2, and π_3 to make up a new partition. This will be done similar to the example of Figure 2. We start by putting partitions π_1, π_2, and π_3 and add them up row-wise. When doing so, the possible boxes filled with 1's coming from π_2 are combined with the 1's of π_3 and turned into a row ending of a box with a 2 in it. There being $n-1$ parts in π_3 and the row-wise addition of these partitions also makes sure that the outcome partition is a partition π with distinct even parts where the smallest part is necessarily even. An illustration is given in Figure 3.

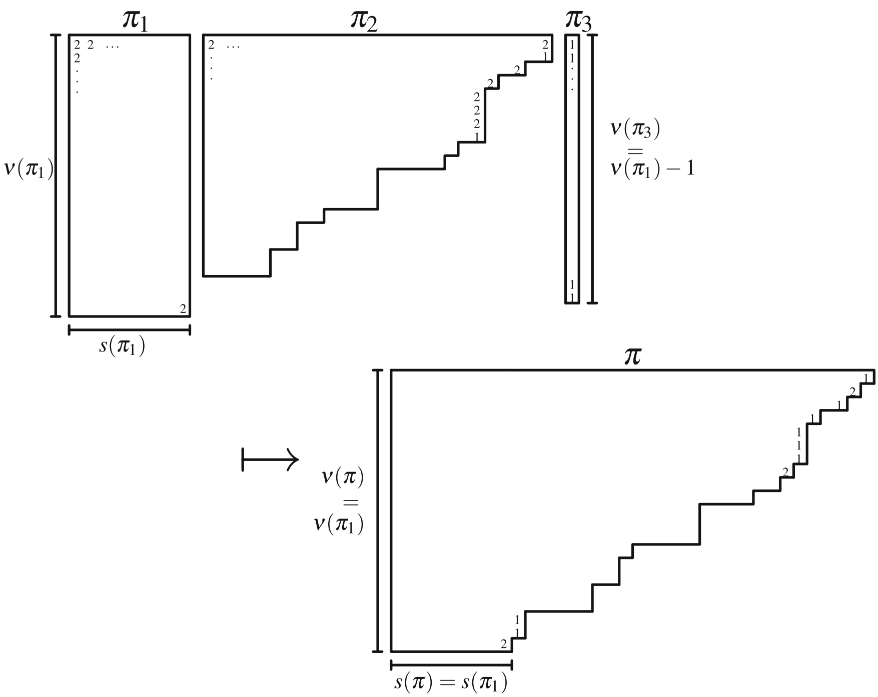


Fig. 3 Demonstration of putting together partitions in the summand of (5.2)

Let $\mathscr{P}_e$ be the subset of $\mathscr{P}$ where the smallest part is necessarily a positive even integer. The above construction proves that the left-hand side of (5.1) is the generating function for the weighted count of partitions from $\mathscr{P}_e$ counted by the weight $+1$ or -1 depending on the number of parts in the partition being even or odd, respectively:

$$\sum_{\pi \in \mathscr{P}_e} (-1)^{\nu(\pi)} q^{|\pi|} = \sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1}. \tag{5.3}$$

The right-hand side of (5.1), by looking at the geometric series, can easily be interpreted combinatorially. This study proves

$$\sum_{\substack{\pi \in \mathscr{P}_e, \\ |\pi|=N}} (-1)^{\nu(\pi)} = (-1)^{N+1} \chi(N \neq \triangle). \tag{5.4}$$

where N is a positive integer and $\triangle$ represents “a triangular number.” The simple observation (4.3) can be used on (5.4) to simplify the equation.

Theorem 5.2. *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi \in \mathscr{P}_e, \\ |\pi|=N}} (-1)^{\nu_e(\pi)+1} = \chi(N \neq \triangle).$$

where $\triangle$ represents “a triangular number.”

Moreover, it is easy to see that the generating function for the weighted count of partitions from $\mathscr{P}$ counted by the weight $+1$ or -1 depending on the number of parts is clearly

$$\sum_{\pi \in \mathscr{P}} (-1)^{\nu(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} - 1 = \psi(-q) - 1. \tag{5.5}$$

Hence, (5.1), (5.3), and (5.5) together yields

$$\sum_{\pi \in \mathscr{P}_o} (-1)^{\nu(\pi)} q^{|\pi|} = \frac{1}{1+q} - 1,$$

where $\mathscr{P}_o$ is the subset of $\mathscr{P}$ where the smallest part is necessarily a positive odd integer.

We note that the above study can be easily generalized by inserting an extra parameter z. The identities (5.1) and (5.3) turn into

$$\sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q/z;q^2)_{n-1}}{(q^2,q^2)_{n-1}} (zq)^{n-1} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - \frac{1}{1+zq}$$

and

$$\sum_{\pi \in \mathscr{P}_e} (-1)^{\nu(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - \frac{1}{1+zq}, \tag{5.6}$$

respectively. We also get the generalization of (5.5)

$$\sum_{\pi\in\mathscr{P}} (-1)^{\nu(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - 1. \tag{5.7}$$

Combining (5.6) and (5.7) and replacing z by $-z$ we get the result

$$\sum_{\pi\in\mathscr{P}_o} (-1)^{\nu_e(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{1}{1-zq} - 1, \tag{5.8}$$

which can also be found in [10, Cor 4, p.1146]. The equation (5.8) implies

Theorem 5.3.

$$\sum_{\substack{\pi\in\mathscr{P}_o,\\ |\pi|=N,\\ \nu_o(\pi)=k}} (-1)^{\nu_e(\pi)} = \chi(N=k).$$

We can step up our study on the set $\mathscr{P}$ by putting more restrictive conditions on the smallest part. Let $\mathscr{P}_{2,4}$ be the subset of $\mathscr{P}_e$ where the smallest part of a partition is necessarily 2 mod 4. Knowing the argument behind the generating function interpretation for $\mathscr{P}$, the generating function of $\mathscr{P}_{2,4}$ with the ± 1 weight with respect to the number of parts can easily be written as

$$\sum_{\pi\in\mathscr{P}_{2,4}} (-1)^{\nu(\pi)} q^{|\pi|} = \sum_{n\geq 1} \frac{(-1)^n q^{2n}}{1-q^{4n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1}. \tag{5.9}$$

We write the related analytic equality.

Theorem 5.4.

$$\sum_{n\geq 1} \frac{(-1)^n q^{2n}}{1-q^{4n}} \frac{(-q;q^2)_{n-1}}{(q^2;q^2)_{n-1}} q^{n-1} = \frac{1}{1-q} \sum_{n\geq 0} (-1)^n q^{n^2} - \frac{1}{1-q^2}. \tag{5.10}$$

Proof. By multiplying both sides of (5.10) with $2(1+q)$ and adding 1 to both sides, we see that one can apply the q-Gauss sum (2.2) where $(a, b, c, q, z) = (-1, -1/q, -q^2, q^2, -q^3)$ to the left-hand side. Showing the equality of the right-hand side to the outcome product of the q-Gauss sum is a simple task of combining like terms and using the Gauss identity (2.4). □

The right-hand side of (5.10) can be studied further to get exact formulas.

$$\begin{aligned}\frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n^2} &= \sum_{k\geq 0} q^k \sum_{n\geq 0}(-1)^n q^{n^2}\\ &= 1+q^4+q^5+q^6+q^7+q^8+q^{16}+q^{17}+q^{18}+q^{19}\dots\\ &= 1+\sum_{N\geq 1}\sum_{j\geq 1}\chi((2j)^2\leq N<(2j+1)^2)q^N, \end{aligned} \tag{5.11}$$

where χ is as defined in (3.21). Also from the geometric series

$$\frac{1}{1-q^2} = 1+q^2+q^4+q^6+q^8+q^{10}+\dots. \tag{5.12}$$

Therefore, combining (5.11) and (5.12), we get

$$\begin{aligned}\frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n^2} - \frac{1}{1-q^2} &= -q^2+q^5+q^7-q^{10}-q^{12}-q^{14}+q^{17}+\dots\\ = \sum_{N\geq 1}\sum_{j\geq 1}(\chi(N \text{ is odd})\chi((2j)^2<N<(2j+1)^2)& \\ -\chi(N \text{ is even})\chi((2j-1)^2<N<(2j)^2))q^N.& \end{aligned} \tag{5.13}$$

Combining (5.9), (5.10), and (5.13) we get the interesting explicit formula for the weighted count of partitions from the set $\mathscr{P}_{2,4}$.

Theorem 5.5.

$$\begin{aligned}\sum_{\substack{\pi\in\mathscr{P}_{2,4},\\ |\pi|=N}}(-1)^{\nu(\pi)} &= \sum_{j\geq 1}(\chi(N \textit{ is odd})\chi((2j)^2<N<(2j+1)^2)\\ &\qquad -\chi(N \textit{ is even})\chi((2j-1)^2<N<(2j)^2))\\ &= \begin{cases} 1, & \textit{if } N \textit{ is odd and in between an even square}\\ & \qquad \textit{and the following odd square,}\\ -1, & \textit{if } N \textit{ is even and in between an odd square}\\ & \qquad \textit{and the following even square,}\\ 0, & \textit{otherwise.}\end{cases}\end{aligned}$$

Let $\mathscr{P}_{3,4}$, similar to $P_{2,4}$, be the subset of $\mathscr{P}_o$ where the smallest part of a partition is necessarily 3 mod 4. Adding a single 1 to the smallest part of a partition from $\mathscr{P}_{2,4}$ is a bijective map from the set $\mathscr{P}_{2,4}$ to $\mathscr{P}_{3,4}$. Therefore, writing the analogous generating function of weighted count of partitions from $\mathscr{P}_{3,4}$ is rather easy and only requires multiplying (5.9) with and extra q. This proves the following theorem.

Theorem 5.6.

$$\sum_{\substack{\pi\in\mathscr{P}_{3,4},\\ |\pi|=N}} (-1)^{\nu(\pi)} = \sum_{j\geq 1}\Big(\chi(N \textit{ is even})\chi((2j)^2 < N < (2j+1)^2) - \chi(N \textit{ is odd})\chi((2j-1)^2 < N < (2j)^2)\Big)$$

$$= \begin{cases} 1, & \textit{if } N \textit{ is even and in between an even square and the following odd square,} \\ -1, & \textit{if } N \textit{ is odd and in between an odd square and the following even square}, \\ 0, & \textit{otherwise.} \end{cases}$$

The combination of the weighted generating functions accounts for every number that is not a perfect square. This interesting relation can be represented as follows.

Theorem 5.7.

$$\sum_{\substack{\pi\in\mathscr{P}_{3,4},\\ |\pi|=N}} (-1)^{\nu(\pi)} - \sum_{\substack{\pi\in\mathscr{P}_{2,4},\\ |\pi|=N}} (-1)^{\nu(\pi)} = (-1)^N \chi(N \neq \square). \tag{5.14}$$

This result, in a sense, is complementary to Alladi's identity, Theorem 4.1. Also, the right-hand side formula also appears in the recent study of Andrews and Yee [8, Thm 3.2, p.10] as the same weighted count with respect to the number of parts of bottom-heavy partitions (a specific subset of overpartitions). The interested reader is invited to examine the relation between the set of bottom-heavy partitions, $\mathscr{P}_{2,4}$, and $\mathscr{P}_{3,4}$.

Once again one can simplify the argument of (5.14) with the observation (4.3).

Theorem 5.8.

$$\sum_{\substack{\pi\in\mathscr{P}_{3,4},\\ |\pi|=N}} (-1)^{\nu_e(\pi)} - \sum_{\substack{\pi\in\mathscr{P}_{2,4},\\ |\pi|=N}} (-1)^{\nu_e(\pi)} = \chi(N \neq \square).$$

6 Overpartitions with no parts divisible by 3

In this section, we treat the weighted interpretation of an identity of Ramanujan [6, E. 4.2.8, p. 85]. We write this identity in an equivalent form for the ease of interpretation purposes.

Theorem 6.1 (**Ramanujan** [6]).

$$\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} - 1 = \sum_{n\geq 1} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} \frac{2q^n}{1-q^n} \frac{q^{n^2-n}}{(q;q^2)_n}. \tag{6.1}$$

Identity (6.1) also appears in the Slater's list [14, 6, p. 152] with a misplaced exponent type typo.

It is clear that the left-hand side of (6.1) is the generating function for the number of overpartitions where no part is 0 mod 3. Let $\mathscr{C}$ be the set of all non-empty partitions with no parts divisible by 3. We focus our interest in the combinatorial interpretation of the right-hand side of (6.1). Let n be a fixed positive integer. The factors

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}\frac{2q^n}{1-q^n}$$

of the right-hand side of (6.1) is the generating function for the number of overpartitions, $\bar{\pi}_1$, into parts $\leq n$ where the part n appears at least once. When counting the total number of overpartitions of this type of partitions, we can instead count the partitions π_1 into parts $\leq n$ where the part n appears at least once with the weight $2^{\nu_d(\pi_1)}$ as in (1.3). The remaining factor

$$\frac{q^{n^2-n}}{(q;q^2)_n}$$

can be split into two in the interpretation. The term q^{n^2-n} is the generating function of the partitions of type $\pi_2 = (2, 4, 6, \ldots, 2(n-1))$ in frequency notation as $n^2 - n$ is double a triangle number. The term $(q;q^2)_n^{-1}$ is the generating function for the number of partitions, π_3, into odd parts $\leq 2n-1$.

It is clear that among the parts of π_1, π_2, and π_3 the largest possible part-size is $2n-1$. Even if $2n-1$ is not a part of π_3, the second largest possible part $2n-2$ is a part of π_2. Therefore, given π_1, π_2, and π_3 we can directly find the respective n. We merge (add the parts' frequencies of) these three partitions into a new partition and look at the number of possible sources for different part sizes. The partition π has one appearance of all the even parts $\leq 2n-2$ coming from π_2, any extra appearance of an even number (which is necessarily $\leq n$) must be coming from the partition π_1 and should be counted with the overpartition weights. The odd parts $\leq n$ can either be coming from the partition π_1 or π_3. These parts need to be counted with both the overpartition weights and normally to account for both possibilities. All the other parts' source partitions can uniquely be identified so they would be counted with trivial weight 1.

Let $\mathscr{R}$ be the set of partitions, where

i. all parts $\leq 2n-1$ for some integer $n > 0$,
ii. all even integers $\leq 2n-2$ appears as parts,
iii. n appears with the frequency $f_n \geq 1 + \chi(n \text{ is even})$,
iv. no even part $> n$ repeats.

Clearly,

$$n := n(\pi) = \frac{\text{largest even part of } \pi}{2} + 1.$$

Define the statistics

$$\delta(\pi) = \sum_{j=1}^{n-1} \chi(f_{2j} > 1),$$
$$\gamma(\pi) = (\chi(n \text{ is even}) + 2 \cdot f_n \cdot \chi(n \text{ is odd})) \prod_{2j+1<n} (2f_{2j+1} + 1),$$

and

$$\mu(\pi) = 2^{\delta(\pi)} \cdot \gamma(\pi),$$

for $\pi \in \mathscr{R}$. We have the following identity.

Theorem 6.2.
$$\sum_{\pi \in \mathscr{C}} 2^{\nu_d(\pi)} q^{|\pi|} = \sum_{\pi \in \mathscr{R}} \mu(\pi) q^{|\pi|}.$$

One example of Theorem 6.2 will be given in Table 4.

In Ramanujan's entry [6, E. 4.2.9, p. 86],

$$\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} = \sum_{n\geq 0} \frac{q^{n^2}(-q;q)_n}{(q;q)_n(q;q^2)_{n+1}}, \tag{6.2}$$

we see the same product of (6.1). The sum on the right-hand side of (6.2) can also be interpreted as a weighted partition count for a special subset of partitions. This is rather analogous to $\mathscr{R}$. Let the set $\mathscr{Q}$ be the set of partitions π, where

i. the largest part is $= 2n - 1$ for some integer $n > 0$,
ii. all odd integers $\leq 2n - 1$ appear as a part,
iii. and no even parts $> n$ appear.

Clearly here
$$n := \frac{\text{largest part of } \pi + 1}{2}.$$

A similar weight to μ can be defined on $\mathscr{Q}$ as follows

$$\eta(\pi) := 2^{\nu_{d,e}(\pi)}(\chi(n \text{ is even}) \cdot (1 + \chi(f_n = 0)) + 2f_n \cdot \chi(n \text{ is odd})) \prod_{2j+1<n} (2f_{2j+1} - 1),$$

where $\nu_{d,e}(\pi)$ is the number of different even parts of π. Hence, we have the identity

Theorem 6.3.

$$\sum_{\pi \in \mathscr{C}} 2^{\nu_d(\pi)} q^{|\pi|} = \sum_{\pi \in \mathscr{R}} \mu(\pi) q^{|\pi|} = \sum_{\pi \in \mathscr{Q}} \eta(\pi) q^{|\pi|}.$$

The example of this result is included in Table 4. From that table, it appears that there exists a weight, norm, and n-value preserving bijection from $\mathscr{R}$ to $\mathscr{Q}$. We would like to leave the discovery of this bijection for a motivated reader.

Table 4 Example of Theorem 6.3 with $|\pi| = 7$.

s	$\pi \in \mathscr{C}$	$2^{\nu_d(\pi)}$	$\pi \in \mathscr{R}$	n	$\mu(\pi)$	$\pi \in \mathscr{Q}$	n	$\eta(\pi)$
	$(1, 2, 4)$	2^3	(1^7)	1	14	(1^7)	1	14
	$(2, 5)$	2^2	$(1^3, 2^2)$	2	14	$(1^4, 3)$	2	14
	$(1^2, 5)$	2^2	$(1, 2^3)$	2	6	$(1^2, 2, 3)$	2	6
	$(1^3, 4)$	2^2	$(2^2, 3)$	2	2	$(1, 3^2)$	2	2
	$(1^5, 2)$	2^2						
	$(1^3, 2^2)$	2^2						
	$(1, 2^3)$	2^2						
	(7)	2						
	(1^7)	2						
Total:		36			36			36

Acknowledgements The authors would like to thank George Andrews for his kind interest, and Jeramiah Hocutt for his careful reading of the manuscript. We are grateful to the anonymous referee for the thoughtful comments and suggestions.

References

1. K. Alladi, Partition identities involving gaps and weights. Trans. Am. Math. Soc. **349**(12), 5001–5019 (1997)
2. K. Alladi, A partial theta identity of Ramanujan and its number-theoretic interpretation. Ramanujan J. **20**, 329–339 (2009)
3. K. Alladi, Analysis of a generalized Lebesgue identity in Ramanujan's Lost Notebook. Ramanujan J. **29**, 339–358 (2012)
4. G.E. Andrews, *The Theory of Partitions* (Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998). Reprint of the 1976 original. MR1634067 (99c:11126)
5. G.E. Andrews, *Partitions with Distinct Evens*, Advances in combinatorial mathematics (Springer, Berlin, 2009), pp. 31–37
6. G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook Part II* (Springer, New York, 2008)
7. G.E. Andrews, M.D. Hirschhorn, J.A. Sellers, Arithmetic properties of partitions with even parts distinct. Ramanujan J. **23**(1), 169–181 (2010)
8. G.E. Andrews, A.J. Yee, Legendre theorems for subclasses of overpartitions. J. Comb. Theory Ser. A **144**, 16–36 (2016)
9. B.C. Berndt, B. Kim, A.J. Yee, Ramanujan's lost notebook: combinatorial proofs of identities associated with Heine's transformation or partial theta functions. J. Comb. Theory Ser. A **117**(7), 857–973 (2010)
10. C. Bessenrodt, I. Pak, Partition congruences by involutions. Euro. J. Comb. **25**, 1139–1149 (2004)

11. S. Corteel, J. Lovejoy, Overpartitions. Trans. Am. Math. Soc. **356**(4), 1623–1635 (2004)
12. G. Gasper, M. Rahman, *Basic Hypergeometric Series* (Cambridge university press, Cambridge, 2004)
13. K.H. Rosen, *Elementary Number Theory and its Applications*, 5th edn. (Addison-Wesley, 2004)
14. L.J. Slater, Further identities of the Rogers-Ramanujan type. Proc. Lond. Math. Soc. Ser. **2**(54), 147–167 (1952)

The Appearance of H.F. Baker and E.W. Hobson in “The Man Who Knew Infinity”

Bruce C. Berndt

Dedicated to Krishnaswami Alladi on his 60th birthday

Abstract Those attending the meeting at the University of Florida celebrating Krishna Alladi’s 60th birthday had the privilege of enjoying an advanced showing of “The Man Who Knew Infinity,” a screen adaptation of Robert Kanigel’s biography of Srinivasa Ramanujan with the same title. We explain the background of a brief scene in which H. F. Baker and E. W. Hobson verbally cast their votes for Ramanujan’s election as a Fellow of the Royal Society.

Keywords Ramanujan’s stay at Cambridge · The Man Who Knew Infinity Royal Society

2010 Mathematics Subject Classification 01A70

G. H. Hardy was not the first English mathematician to receive a letter from Ramanujan before he departed for England in March 1914. We know that Ramanujan wrote to Sir Francis Spring’s mathematics teacher, M. J. M. Hill, at the University of London in late 1912 [1, pp. 13–15]. Ramanujan also wrote to two Cambridge mathematicians before writing to Hardy, but their identities were kept secret for many years to save them embarrassment. However, Kanigel [2, pp. 106–107] offers their names, H. F. Baker and E. W. Hobson. In preparing to write his book, Kanigel visited the present author for a couple days in the mid-1980s. Neither he nor I can recall if I had informed him of these names, or if he had discovered them himself. But at any

B. C. Berndt (✉)
Department of Mathematics, University of Illinois at Urbana–Champaign,
1409 W. Green St, Urbana, IL 61801, USA
e-mail: berndt@illinois.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_7

rate, to the best of my knowledge, there is only one available source for these names, which I now relate.

Sometimes, we learn in unusual ways. One evening in the early 1980s, my wife Helen and I were invited to dinner at the home of close friends, who had also invited a friend from their graduate student days at Indiana University. Since my mind had been totally immersed in the mathematics of Ramanujan since February 1974, our early conversation focused on Ramanujan. Our host's friend told me that he recently read a psychological study of Ramanujan in the journal, *Psychoanalytic Review* [3]. On the following day, I located the article in the University of Illinois Library and discovered that the author, Ashis Nandy, had incorporated his article in a lengthier study of Ramanujan in a book [4], published 1 year later. In preparing to write his article, Nandy had fortuitously interviewed J. E. Littlewood shortly before he died in 1977. In a footnote [4, pp. 146–147], Nandy writes, "Littlewood says with some relish that these two mathematicians [who had received letters from Ramanujan], whom he identifies only as Baker and Popson, felt rather foolish afterwards." It is clear that Littlewood was speaking of Baker and Hobson and that Nandy had misunderstood Littlewood's pronunciation of Hobson.

Since Hobson and Baker were famous Cambridge mathematicians, it was natural for Ramanujan to write them about his mathematical discoveries. Hobson was Sadleirian Professor at Cambridge and in 1913 had published a very popular book, *Squaring the Circle*, which possibly had reached Madras before Ramanujan wrote his letter to him. Baker was Cayley Lecturer at Cambridge and in 1907 had published a book, *Abel's Theorem and the Allied Theory of Theta Functions*. Ramanujan made many contributions to the theory of theta functions, but his work is not in the same spirit as that of Baker. We do not know if Ramanujan was acquainted with Baker's treatise, but if he had been, it seems doubtful that it would have influenced Ramanujan to write Baker.

Ramanujan became the second Indian to be elected Fellow of the Royal Society, and in the film, "The Man Who Knew Infinity," in a very short scene, Baker and Hobson are shown voicing their approval for Ramanujan's election. The film score does not divulge why these two Royal Society Fellows were chosen for their votes, but their choice should now be clear.

Nandy is not a mathematician, and so many of his statements about Ramanujan's mathematics in his book [4] are without merit. As an example, he writes [4, p. 149], "One wonders that if it ever struck Ramanujan that out of the roughly eight areas in which he worked (hypergeometric series, partitions, definite integrals, elliptical integrals, highly composite numbers, fractional differentiation, and number theory) it was his work on fractional differentiation which perhaps came closest to being a major breakthrough in mathematics." (Indeed, Nandy recorded *seven*, NOT *eight*, areas of Ramanujan's interests.) We would not choose these compartments for Ramanujan's mathematics, and moreover, Ramanujan did not make any earthshaking discoveries in fractional differentiation. But nonetheless, we can thank Nandy for his conversation with Littlewood, and we can thank Littlewood for "letting his guard down."

References

1. B.C. Berndt, R.A. Rankin, *Ramanujan: Letters and Commentary* (American Mathematical Society, Providence, 1995); (London Mathematical Society, London, 1995)
2. R. Kanigel, *The Man Who Knew Infinity* (Charles Scribner's Sons, New York, 1991)
3. A. Nandy, Ramanujan in England: an east–west collaboration in science. Psychoanal. Rev. fall **66**, 423–442 (1979)
4. A. Nandy, *Alternative Sciences: Creativity and Authenticity in Two Indian Scientists* (Allied Publisher, New Delhi, 1980)

A Bibasic Heine Transformation Formula and Ramanujan's ${}_2\phi_1$ Transformations

Gaurav Bhatnagar

Dedicated to Krishnaswami Alladi on his 60th birthday

Abstract We study Andrews and Berndt's organization of Ramanujan's transformation formulas in Chapter 1 of their book *Ramanujan's Lost Notebook, Part II*. In the process, we rediscover a bibasic Heine's transformation, which follows from a Fundamental Lemma given by Andrews in 1966, and obtain identities proximal to Ramanujan's entries. We also provide a multibasic generalization of Andrews' 1972 theorem concerning a q-analog of the Lauricella function. Our results only require the q-binomial theorem, and are an application of what Andrews and Berndt call 'Heine's Method'.

Keywords Heine transformation · Bibasic and multibasic series Ramanujan's theta functions · Lost Notebook · Heine's method · q-Lauricella function

2010 Mathematics Subject Classification 33D15 · 33D65

1 Introduction

In Chapter 1, Part II of their edited version of Ramanujan's [15] Lost Notebook, Andrews and Berndt [5] have organized Ramanujan's transformation formulas

---

Research supported in part by the Austrian Science Fund (FWF): F 5008-N15.

G. Bhatnagar (✉)
Fakultät für Mathematik, Universität Wien,
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
e-mail: bhatnagarg@gmail.com

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_8

related to Heine's ${}_2\phi_1$ transformations. While studying their work, we discovered a large number of formulas that are proximal to Ramanujan's own entries.

For example, one of Ramanujan's formulas is [5, Entry 1.6.6]: for $|q| < 1$,

$$\frac{1}{1-q} + \sum_{j=1}^{\infty} \frac{(-1)^j q^{j^2+j}}{\left(1-q^{2j+1}\right)\left[(1-q^2)(1-q^4)(1-q^6)\cdots(1-q^{2j})\right]}$$
$$= \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}$$
$$= 1 + q^1 + q^3 + q^6 + q^{10} + q^{15} + \cdots .$$

The right-hand side is the well-known theta function which Ramanujan denoted as $\psi(q)$. It has the product representation (see Berndt [7, p. 11])

$$\psi(q) := \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = \prod_{k=0}^{\infty} \frac{(1-q^{2k+2})}{(1-q^{2k+1})}.$$

We recover Ramanujan's Entry 1.6.6 and, in the same breath, obtain the formula

$$\frac{1}{1+q} + \sum_{j=i}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{\left(1+q^{j+1}\right)\left[(1-q)(1-q^2)(1-q^3)\cdots(1-q^{j})\right]}$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}$$
$$= 1 - 2q + 2q^4 - 2q^9 + \cdots .$$

Now the right-hand side is (in Ramanujan's notation) $\phi(-q)$, with product representation given by [5, Eq. (1.4.10)]

$$\phi(-q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \prod_{k=1}^{\infty} \frac{(1-q^k)}{(1+q^k)}.$$

For our next example, we require some notation. The q-rising factorial is defined as $(A;q)_0 := 1$, and when k is a positive integer,

$$(A;q)_k := (1-A)(1-Aq)\cdots(1-Aq^{k-1}).$$

Notice that it is a product of k terms. The parameter q is called the 'base'. The infinite q-rising factorial is defined, for $|q| < 1$, as

$$(A;q)_\infty := \prod_{r=0}^{\infty}(1-Aq^r).$$

Observe that, for $|q|<1$ [10, Eq. (I.5)],

$$(A;q)_k = \frac{(A;q)_\infty}{\left(Aq^k;q\right)_\infty}. \tag{1.1}$$

This is used to define q-rising factorials when k is a complex number.

With this notation, consider Ramanujan's formula [5, Entry 1.4.17]

$$\begin{aligned}(-aq;q)_\infty \sum_{j=0}^{\infty} & \frac{b^j q^{\binom{j+1}{2}}}{(q;q)_j(-aq;q)_{tj}} \\ &=(-bq;q)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{\binom{k+1}{2}}}{(q;q)_k(-bq;q)_{tk}}\end{aligned}$$

and compare with the identity

$$\begin{aligned}\left(-aq^h;q^h\right)_\infty \sum_{j=0}^{\infty} & \frac{b^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j\left(-aq^h;q^h\right)_{tj}} \\ &=\left(-bq^t;q^t\right)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{h\binom{k+1}{2}}}{\left(q^h;q^h\right)_k(-bq^t;q^t)_{hk}},\end{aligned}$$

obtained in our study. Here $|q|<1$ and $|q^t|<1$ in the first formula, and $|q^h|<1$, $|q^t|<1$ and $|q^{ht}|<1$ in the second. The reader may enjoy recovering Entry 1.4.17 from this formula.

The objective of this paper is to report on our study of [5, Ch. 1]. We are able to obtain 14 of Ramanujan's entries as immediate special cases of a particular transformation formula, and a large number of identities that are proximal to Ramanujan's own entries. In addition, we give a multibasic generalization of Andrews' 1972 formula for a q-Lauricella function and obtain a few interesting special cases, which again extend formulas of Ramanujan.

During the course of our study, we stumbled upon the transformation formula

$$\sum_{k=0}^{\infty} \frac{\left(a;q^h\right)_k}{\left(q^h;q^h\right)_k} \frac{\left(b;q^t\right)_{hk}}{(c;q^t)_{hk}} z^k = \frac{\left(b;q^t\right)_\infty}{(c;q^t)_\infty} \frac{\left(az;q^h\right)_\infty}{\left(z;q^h\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(c/b;q^t\right)_j}{(q^t;q^t)_j} \frac{\left(z;q^h\right)_{tj}}{\left(az;q^h\right)_{tj}} b^j, \tag{1.2}$$

where $|z|<1$, $|b|<1$, and h and t are complex numbers such that $|q^h|<1$, $|q^t|<1$ and $|q^{ht}|<1$. Andrews and Berndt [5] use the $t=1$ case of this result (a formula due to Andrews [2, Lemma 1]) often combined with the $h=1$ and $t=1$ case (a famous transformation of Heine, see Gasper and Rahman [10, Eq. 1.4.1]). But these authors seem to have missed writing down (1.2) explicitly, even though it can be proved in the same manner as Heine's result, and indeed follows from a very general approach to Heine's ideas, which Andrews [1] calls his 'Fundamental Lemma'. This useful and simple identity may be a special case of a 50-year-old identity, but it has not shown up in the standard textbook by Gasper and Rahman [10], and perhaps deserves to be highlighted. And so, in §2, we attempt a brief introduction.

The plan for the rest of the paper is as follows. In §3 and §4, we report on our study of Ramanujan's transformation formulas. This part of our work can be considered to be an addendum to Chapter 1 of Andrews and Berndt [5]. In §5, we closely follow ideas from Andrews [3] to extend our work to multiple series that extend q-analogs of the Lauricella functions. We give a multibasic generalization of Andrews' formula [3, Eq. (4.1)], and give several generalizations of two of Ramanujan's identities.

Before proceeding to Ramanujan's ${}_2\phi_1$ transformations, we consider (1.2) again from the perspective of Heine's original ideas, an approach that Andrews and Berndt [5] have dubbed 'Heine's method'.

2 Heine's method: Transformations of Heine, Ramanujan, and Andrews

This section is an introduction to Identity (1.2). We begin with a famous transformation formula of Heine that he found in 1847. Heine's transformation formula [13, Eq. 78] is

$$\frac{(cx;q)_\infty}{(bx;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k(bx;q)_k}{(q;q)_k(cx;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}\sum_{j=0}^{\infty}\frac{(c/b;q)_j(z;q)_j}{(q;q)_j(az;q)_j}(bx)^j. \tag{2.1}$$

This is almost as Heine himself wrote it, except that he wrote q^α, q^β and q^γ in place of a, b and c. Usually, this formula is stated with $x = 1$, see Gasper and Rahman [10, eq. (1.4.1)].

Heine's formula was rediscovered by Ramanujan. It appears as Entry 6 in Chapter 16 of his second notebook, see Berndt [6, p. 15]. In addition, there is another transformation formula of Ramanujan resembling (2.1). It appeared on Page 3 of the famous Lost Notebook [15] (see [5, Entry 1.4.1]), and is dated circa 1919, going by Andrews and Berndt's [4, p. 4] remarks on the likely timing of work presented in the Lost Notebook.

$$\begin{aligned}\frac{(aq;q)_\infty\left(cq;q^2\right)_\infty}{(-bq;q)_\infty\left(dq^2;q^2\right)_\infty}&\sum_{j=0}^{\infty}\frac{(-bq/a;q)_j}{(q;q)_j}\frac{\left(dq^2;q^2\right)_j}{\left(cq;q^2\right)_{j+1}}(aq)^j\\ &=\sum_{k=0}^{\infty}\frac{\left(cq/d;q^2\right)_k}{\left(q^2;q^2\right)_k}\frac{(aq;q)_{2k}}{(-bq;q)_{2k+1}}(dq^2)^k.\end{aligned} \tag{2.2}$$

This is Ramanujan's Entry 1.4.1 and it resembles Heine's transformation (coincidentally, eq. (1.4.1) of [10]). Both the series have two products each in the numerator and denominator, and there are four infinite products outside the sums. However, some of the factors in the sums have base q^2 rather than q, and the number of terms in some of the factors of the summands are different. For example, notice the product $(aq;q)_{2k}$, a product of $2k$ factors in the summand on the right-hand side of (2.2).

Andrews and Berndt [5] study many of Ramanujan's transformation formulas (in particular (2.2)) in Chapter 1, Part II of their series of books on Ramanujan's Lost Notebook. A key component of their study is Andrews' 1966 transformation formula [2, Lemma 1]:

$$\sum_{k=0}^{\infty} \frac{\left(a;q^h\right)_k}{\left(q^h;q^h\right)_k} \frac{(b;q)_{hk}}{(c;q)_{hk}} z^k = \frac{(b;q)_\infty}{(c;q)_\infty} \frac{\left(az;q^h\right)_\infty}{\left(z;q^h\right)_\infty} \sum_{j=0}^{\infty} \frac{(c/b;q)_j}{(q;q)_j} \frac{\left(z;q^h\right)_j}{\left(az;q^h\right)_j} b^j, \quad (2.3)$$

where $h = 1, 2, 3, \ldots$. Andrews' formula contains both (2.1) and (2.2). This can be seen by taking $h = 1$ and $h = 2$, respectively. Andrews' transformation can also be found in [5, Th. 1.2.1, p. 6] and [10, Ex. 3.35, p. 111].

Now, inspired by Heine's formulation (2.1), we write Andrews' transformation more symmetrically as follows.

$$\frac{(bw;q)_\infty}{(w;q)_\infty} \sum_{k=0}^{\infty} \frac{\left(a;q^h\right)_k}{\left(q^h;q^h\right)_k} \frac{(w;q)_{hk}}{(bw;q)_{hk}} z^k = \frac{\left(az;q^h\right)_\infty}{\left(z;q^h\right)_\infty} \sum_{j=0}^{\infty} \frac{(b;q)_j}{(q;q)_j} \frac{\left(z;q^h\right)_j}{\left(az;q^h\right)_j} w^j.$$

This form suggests a further generalization of (2.3), where now we have terms involving two bases q^h and q^t (and hence the adjective *bibasic*).

Theorem 2.1 (A bibasic Heine transformation) *Let q, a, b, h, and t be complex numbers such that $|q^h| < 1$, $|q^t| < 1$, and $|q^{ht}| < 1$, and suppose that the denominators in (2.4) are not zero. Then for $|w| < 1$ and $|z| < 1$,*

$$\frac{\left(bw;q^t\right)_\infty}{(w;q^t)_\infty} \sum_{k=0}^{\infty} \frac{\left(a;q^h\right)_k}{\left(q^h;q^h\right)_k} \frac{\left(w;q^t\right)_{hk}}{(bw;q^t)_{hk}} z^k = \frac{\left(az;q^h\right)_\infty}{\left(z;q^h\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(b;q^t\right)_j}{(q^t;q^t)_j} \frac{\left(z;q^h\right)_{tj}}{\left(az;q^h\right)_{tj}} w^j. \quad (2.4)$$

Remark Replace w by b and b by c/b in (2.4) to obtain the form (1.2) of the identity.

Before heading into the proof of Theorem 2.1, we make a few comments on the convergence of the series and products appearing in this identity.

Observe that we require the conditions $|q^t| < 1$ and $|q^h| < 1$ for the convergence of the infinite products $\left(w;q^t\right)_\infty$ and $\left(z;q^h\right)_\infty$. In view of (1.1), we require these conditions for the definition of products such as $\left(w;q^t\right)_{hk}$ too.

Next, note that the function $f(w) := (w;q)_\infty$ is a continuous function of w in a neighborhood of $w = 0$, and $f(0) = 1$. This follows from the fact that for fixed q, with $0 < |q| < 1$, the sequence of partial products

$$f_k(w) = \prod_{r=0}^{k-1} \left(1 - wq^r\right)$$

converges absolutely to $f(w)$, and the convergence is uniform in a closed disk around $w = 0$ contained in the unit disk $\{w \in \mathbb{C} : |w| < 1\}$.

Now we consider a factor such as $\left(w; q^t\right)_{hk}$, and show that if $|q^{ht}| < 1$, then for large enough k, $|\left(w; q^t\right)_{hk}|$ is approximately equal to $|\left(w; q^t\right)_\infty|$.

By definition, we have

$$\left(w; q^t\right)_{hk} = \frac{\left(w; q^t\right)_\infty}{\left(wq^{thk}; q^t\right)_\infty}.$$

Now since $|q^{ht}| < 1$, we must have $|q^{htk}| \to 0$ as $k \to \infty$, and thus, by the continuity of $f(w)$, $\left(wq^{thk}; q^t\right)_\infty \to \left(0; q^t\right)_\infty = 1$. Thus for large enough k, $|\left(w; q^t\right)_{hk}|$ is approximately $|\left(w; q^t\right)_\infty|$.

Using the above remarks, we can consider the absolute convergence of the series appearing on either side of (2.4). Consider first the left-hand side of (2.4). We replace all the q-rising factorials in the summand by ratios of infinite products, using (1.1). Then we find that for large enough k, the absolute value of the summand is bounded by a constant times the factor $|z|^k$. Since the geometric series

$$\sum_{k=0}^{\infty} z^k$$

converges absolutely for $|z| < 1$, the sum on the left-hand side of (2.4) converges absolutely for $|z| < 1$. Similarly, the sum on the right-hand side converges absolutely for $|w| < 1$.

To summarize, we have the conditions $|q^h| < 1$, $|q^t| < 1$, $|q^{ht}| < 1$, $|z| < 1$ and $|w| < 1$ for the convergence of the products and series.

We now proceed with the proof of the theorem. Theorem 2.1 can be obtained as a very special case of Andrews' [1] Fundamental Lemma (see our remark below). But we prove it on the lines of the proof of Heine's own proof of his transformation formula, which Andrews and Berndt [5] call Heine's method. We only require the identity (1.1) and the q-binomial theorem [10, eq. (1.3.2)]: For $|z| < 1$, $|q| < 1$

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k. \tag{2.5}$$

Proof (*Proof of Theorem* 2.1) We begin with the left-hand side of (2.4).

$$\begin{aligned}\frac{\left(bw; q^t\right)_\infty}{(w; q^t)_\infty} \sum_{k=0}^{\infty} \frac{\left(a; q^h\right)_k}{\left(q^h; q^h\right)_k} \frac{\left(w; q^t\right)_{hk}}{(bw; q^t)_{hk}} z^k \\ = \sum_{k=0}^{\infty} \frac{\left(a; q^h\right)_k}{\left(q^h; q^h\right)_k} z^k \frac{\left(bwq^{htk}; q^t\right)_\infty}{\left(wq^{htk}; q^t\right)_\infty} \quad \text{(using (1.1))}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\left(a; q^h\right)_k}{\left(q^h; q^h\right)_k} z^k \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} (wq^{htk})^j \quad \text{(using (2.5))} \\
&= \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} w^j \sum_{k=0}^{\infty} \frac{\left(a; q^h\right)_k}{\left(q^h; q^h\right)_k} (zq^{htj})^k \\
&= \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} w^j \frac{\left(azq^{htj}; q^h\right)_\infty}{\left(zq^{htj}; q^h\right)_\infty} \quad \text{(using (2.5) again)} \\
&= \frac{\left(az; q^h\right)_\infty}{\left(z; q^h\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} \frac{\left(z; q^h\right)_{tj}}{\left(az; q^h\right)_{tj}} w^j.
\end{aligned}
$$

Observe that $|w| < 1$ and $|q^{ht}| < 1$ implies $|wq^{htk}| < 1$. Similarly, we must have $|zq^{htj}| < 1$. These conditions are required for the absolute convergence of the q-binomial series used here, and to justify the interchange of summation. □

Remark Andrews [2, Lemma 1] (see also [10, Ex. 3.35]) mentions that the formula (2.3) is valid when $h = 1, 2, 3, \ldots$. However, as we have seen, with sufficient conditions, we can take h to be a complex number in (2.3).

Observe that the $b = c$ case of Heine's transformation is (2.5), the q-binomial theorem. When $b = c$, the summand contains the factor $(1; q)_j$ that is 1 when $j = 0$, and 0 when $j > 0$. Thus the sum on the right-hand side of (2.1) reduces to 1, and we obtain (2.5).

A key property of Heine's transformation is that it can be iterated, and the process of iteration leads to symmetries of the sum which are useful in many contexts. See Gasper and Rahman [10, eqs. (1.4.2) and (1.4.5)]. Unfortunately, equation (2.4) cannot be iterated, making it less useful than Heine's transformation. However, there is a bibasic version of a special case of Heine's second iterate due to Guo and Zeng [11, Th. 2.2].

There are also bibasic transformation formulas due to Gasper [9, eq. (1.12)] (reproduced in [10, Ex. 3.20]). These consist of four sums that are equal to each other. By equating the second and fourth sum, we get a formula equivalent to Andrews' transformation formula. Replace p by q^h in Gasper's transformation to obtain an equivalent form of (2.3).

Remark Andrews stated and used (2.3) in [2], and derived it using Theorem A of [1], which in turn is derived from his 'Fundamental Lemma'. This lemma is really a most general approach to Heine's method, and should be better known. Andrews' [1] Fundamental Lemma can be stated as:

$$\sum_{k=0}^{\infty} \frac{(a;q)_{rk+s}}{(q;q)_{rk+s}} \frac{(b;p)_{uk+v}}{(c;p)_{uk+v}} z^k = \frac{1}{r} \frac{(b;p)_\infty}{(c;p)_\infty} \sum_{t=0}^{r-1} \omega_r^{-st} z^{-s/r}$$
$$\times \sum_{j=0}^{\infty} \frac{(c/b;p)_j}{(p;p)_j} \frac{\left(a\omega_r^t z^{1/r} p^{uj/r};q\right)_\infty}{\left(\omega_r^t z^{1/r} p^{uj/r};q\right)_\infty} \left(bp^{v-us/r}\right)^j, \tag{2.6}$$

where $\omega_r = e^{2\pi i/r}$ or some other primitive rth root of unity, and we assume the parameters satisfy suitable conditions to guarantee convergence of the two series.

Equation (1.2) can be obtained as a special case of (2.6). Take $r = 1$, $u = h$, $s = 0 = v$, $q \mapsto q^h$, and $p \mapsto q^t$ to obtain the second last step (suitably re-labeled) in our proof of Theorem 2.1. Professor Krattenthaler has remarked that, in fact, (2.6) follows from (1.2) by 'sectioning' the series on the left (a process described in our remark in §4). In other words, Andrews Fundamental Lemma is equivalent to (1.2). This involves recognizing that we can write the factors in the sums with two independent bases q and p, since h and t are complex numbers. Indeed, with these considerations, we can rewrite (1.2) as the $r = 1$ case of (2.6).

The reader may enjoy proving (2.6) directly using Heine's method and sectioning.

This completes our introduction to (1.2). We now consider special cases related to Ramanujan's transformations. In the rest of the paper, when stating special cases of (2.4), we do not always explicitly state all the applicable convergence conditions mentioned in Theorem 2.1.

3 Special cases inspired by Ramanujan's ${}_2\phi_1$ transformations

While studying Andrews and Berndt [5, ch. 1], we realized that many of Ramanujan's transformations in [5, §1.4] are immediate special cases of Ramanujan's transformation (2.2), where one takes limits or special cases such as $a \to 0$, $b = 0$, $c = 0$, and $d \to 0$ and combinations of these. So we first rewrite the bibasic Heine transformation in the form of Ramanujan's Entry 1.4.1, with a view to study its special cases. We will find that several of Ramanujan's entries in Chapter 1 of [5] are immediate special cases. In addition, we note new identities that resemble Ramanujan's formulas.

Entry 1.4.1

First, we write (1.2) in the form of Ramanujan's formula, by taking $a \mapsto cq/d$, $b \mapsto aq^t$, $c \mapsto -bq^{t+1}$, and $z \mapsto dq^h$. Now divide both sides by $1 + bq$ and multiply and divide the RHS by $1 - cq$ and interchange the sides to obtain

$$\frac{(aq^t;q^t)_\infty (cq;q^h)_\infty}{(-bq;q^t)_\infty (dq^h;q^h)_\infty} \sum_{j=0}^{\infty} \frac{(-bq/a;q^t)_j}{(q^t;q^t)_j} \frac{(dq^h;q^h)_{tj}}{(cq;q^h)_{tj+1}} (aq^t)^j$$
$$= \sum_{k=0}^{\infty} \frac{(cq/d;q^h)_k}{(q^h;q^h)_k} \frac{(aq^t;q^t)_{hk}}{(-bq;q^t)_{hk+1}} (dq^h)^k. \tag{3.1}$$

Again, h and t are complex numbers, and we have the conditions $|q^h| < 1$, $|q^t| < 1$ and $|q^{ht}| < 1$. Further, for the series to converge, we require $|aq^t| < 1$ and $|dq^h| < 1$.

Note that when $h = 2$ and $t = 1$, this reduces to (2.2), Ramanujan's Entry 1.4.1. The rest of Ramanujan's entries presented below are also special cases of (3.1).

Entry 1.4.2

In equation (3.1) take $a = d = 1$, replace c by a, and bring the product $(-bq;q^t)_\infty$ to the other side. In this manner, we obtain a generalization of Entry 1.4.2:

$$\frac{(q^t;q^t)_\infty (aq;q^h)_\infty}{(q^h;q^h)_\infty} \sum_{j=0}^{\infty} \frac{(-bq;q^t)_j}{(q^t;q^t)_j} \frac{(q^h;q^h)_{tj}}{(aq;q^h)_{tj+1}} q^{tj}$$
$$= (-bq;q^t)_\infty \sum_{k=0}^{\infty} \frac{(aq;q^h)_k}{(q^h;q^h)_k} \frac{(q^t;q^t)_{hk}}{(-bq;q^t)_{hk+1}} q^{hk}. \tag{3.2}$$

To take special cases, we use the following elementary identities from Gasper and Rahman [10, eq. (I.27)]:

$$(a;q)_{rk} = (a, aq, aq^2, \ldots, aq^{r-1}; q^r)_k, \tag{3.3}$$

and [10, eq. (I.30)]

$$(a^r;q^r)_k = (a, a\omega_r, a\omega_r^2, \ldots, a\omega_r^{r-1}; q)_k, \tag{3.4}$$

where $\omega_r = e^{2\pi i/r}$ or some other primitive rth root of unity; here, we use the shorthand notation

$$(a_1, a_2, \ldots, a_n; q)_k = (a_1;q)_k (a_2;q)_k \cdots (a_n;q)_k.$$

When $t = 1$, and h is a natural number bigger than 1, then (3.2) reduces to

$$(q, q^2, \ldots, q^{h-1}; q^h)_\infty (aq;q^h)_\infty \sum_{j=0}^{\infty} \frac{(-bq;q)_j}{(aq;q^h)_{j+1}} (q\omega_h, q\omega_h^2, \ldots, q\omega_h^{h-1}; q)_j q^j$$
$$= (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{(aq;q^h)_k}{(-bq;q)_{hk+1}} (q, q^2, \ldots, q^{h-1}; q^h)_k q^{hk}, \tag{3.5}$$

where $\omega_h = e^{2\pi i/h}$ or some other primitive hth root of unity. We have used (3.3) and (3.4) to write this expression.

When $h = 2$, the primitive hth root of unity reduces to -1, and we obtain Ramanujan's transformation [5, Entry 1.4.2]:

$$\left(q;q^2\right)_\infty \left(aq;q^2\right)_\infty \sum_{j=0}^{\infty} \frac{(-bq;q)_j(-q;q)_j}{\left(aq;q^2\right)_{j+1}} q^j = (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{\left(aq;q^2\right)_k (q;q^2)_k}{(-bq;q)_{2k+1}} q^{2k}. \tag{3.6}$$

Observe that the denominator of the sum on the left does not contain the (usually) mandatory term $(q;q)_j$. This term is required to terminate the series naturally from below, because

$$\frac{1}{(q;q)_j} = 0 \text{ whenever } j < 0.$$

The same is true for the right-hand side. This seems to be the motive for considering this special case. See also (3.14), (3.16), and Entry 1.6.5 (and related identities) below.

Entry 1.4.5

If we set $a = 0$ in (3.2), replace b by a, and bring all the infinite products to the right, we obtain a generalization of Entry 1.4.5:

$$\sum_{j=0}^{\infty} \frac{\left(-aq;q^t\right)_j \left(q^h;q^h\right)_{tj}}{(q^t;q^t)_j} q^{tj} = \frac{\left(-aq;q^t\right)_\infty \left(q^h;q^h\right)_\infty}{(q^t;q^t)_\infty} \sum_{k=0}^{\infty} \frac{\left(q^t;q^t\right)_{hk}}{\left(q^h;q^h\right)_k (-aq;q^t)_{hk+1}} q^{hk}. \tag{3.7}$$

When $t = 1$, and $h > 1$ is a positive integer, this reduces to

$$\sum_{j=0}^{\infty} (-aq;q)_j (q\omega_h, q\omega_h^2, \ldots, q\omega_h^{h-1};q)_j q^j = (q\omega_h, q\omega_h^2, \ldots, q\omega_h^{h-1};q)_\infty (-aq;q)_\infty \times \sum_{k=0}^{\infty} \frac{1}{(-aq;q)_{hk+1}} (q, q^2, \ldots, q^{h-1};q^h)_k q^{hk}, \tag{3.8}$$

where $\omega_h = e^{2\pi i/h}$ or some other primitive hth root of unity. Further take $h = 2$ to obtain Ramanujan's formula [5, Entry 1.4.5]:

$$\sum_{j=0}^{\infty}(-aq;q)_j(-q;q)_jq^j=(-q;q)_\infty(-aq;q)_\infty\sum_{k=0}^{\infty}\frac{(q;q^2)_k}{(-aq;q)_{2k+1}}q^{2k}. \tag{3.9}$$

Instead, when $t=2$ and $h=1$ in (3.7), we obtain the formula

$$\sum_{j=0}^{\infty}\left(-aq;q^2\right)_j\left(q;q^2\right)_jq^{2j}=\left(q;q^2\right)_\infty\left(-aq;q^2\right)_\infty\sum_{k=0}^{\infty}\frac{(-q;q)_k}{\left(-aq;q^2\right)_{k+1}}q^k. \tag{3.10}$$

Entry 1.4.3 and Entry 1.4.4

There is a common generalization of Entry 1.4.3 and Entry 1.4.4. Take $b=0, d\to 0$ in (3.1), replace c by b/q^h and cancel $1-b/q^{h-1}$ on the LHS. Bring the products to the RHS to obtain, for $|aq^t|<1$,

$$\sum_{j=0}^{\infty}\frac{\left(aq^t\right)^j}{(q^t;q^t)_j\left(bq;q^h\right)_{tj}}=\frac{1}{(aq^t;q^t)_\infty\left(bq;q^h\right)_\infty}\sum_{k=0}^{\infty}\frac{\left(aq^t;q^t\right)_{hk}}{\left(q^h;q^h\right)_k}(-bq)^kq^{h\binom{k}{2}}. \tag{3.11}$$

When $h=2$ and $t=1$, (3.11) reduces to [5, Entry 1.4.3]:

$$\sum_{j=0}^{\infty}\frac{a^jq^j}{(q;q)_j\left(bq;q^2\right)_j}=\frac{1}{(aq;q)_\infty\left(bq;q^2\right)_\infty}\sum_{k=0}^{\infty}\frac{(aq;q)_{2k}}{\left(q^2;q^2\right)_k}(-b)^kq^{k^2}. \tag{3.12}$$

When $h=1$ and $t=2$, (3.11) reduces to [5, Entry 1.4.4]:

$$\sum_{j=0}^{\infty}\frac{a^jq^{2j}}{\left(q^2;q^2\right)_j(bq;q)_{2j}}=\frac{1}{\left(aq^2;q^2\right)_\infty(bq;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(aq^2;q^2\right)_k}{(q;q)_k}(-b)^kq^{\binom{k+1}{2}}. \tag{3.13}$$

The case $a\to 0$ and $c=0$ case of (3.1) is equivalent to (3.11), up to re-labeling of parameters.

Entry 1.4.10 and Entry 1.4.11

Both these entries immediately follow from $h=1=t$ case of (3.11). In this case, when $a=1=b$ in (3.11), we obtain [5, Entry 1.4.10]

$$\sum_{j=0}^{\infty}\frac{q^j}{(q;q)_j^2}=\frac{1}{(q;q)_\infty^2}\sum_{k=0}^{\infty}(-1)^kq^{\binom{k+1}{2}}. \tag{3.14}$$

Next, take $a=q^t$ and $b=q^{h-1}$ in (3.11) to obtain

$$\sum_{j=0}^{\infty}\frac{q^{2tj}}{(q^t;q^t)_j\left(q^h;q^h\right)_{tj}}=\frac{1}{(q^t;q^t)_\infty\left(q^h;q^h\right)_\infty}\sum_{k=0}^{\infty}\frac{\left(q^t;q^t\right)_{hk+1}}{\left(q^h;q^h\right)_k}(-1)^kq^{h\binom{k+1}{2}}. \tag{3.15}$$

Now take $h = 1 = t$ and simplify as follows.

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{q^{2j}}{(q;q)_j^2} &= \frac{1}{(q;q)_\infty^2} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \left(1 - q^{k+1}\right) \\ &= \frac{1}{(q;q)_\infty^2} \left[\sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} + \sum_{k=0}^{\infty} (-1)^{k+1} q^{\binom{k+2}{2}} \right] \\ &= \frac{1}{(q;q)_\infty^2} \left[1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{\binom{k+1}{2}} \right].\end{aligned}$$

Ramanujan's Entry 1.4.11 is the first sum equated with the last in this chain of equalities [5, Entry 1.4.11]:

$$\sum_{j=0}^{\infty} \frac{q^{2j}}{(q;q)_j^2} = \frac{1}{(q;q)_\infty^2} \left[1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{\binom{k+1}{2}} \right]. \tag{3.16}$$

Entry 1.4.12, Entry 1.4.17, Entry 1.4.9, and a part of Entry 1.5.1

Consider the case $a \to 0$, $d \to 0$ of (3.1). In the resulting identity, bring the infinite product $\left(-bq; q^t\right)_\infty$ to the other side, and cancel $(1 - cq)$ on the LHS and $(1 + bq)$ on the RHS. Then replace c by $-a/q$ and b by b/q to obtain the appealing identity

$$\begin{aligned}\left(-aq^h; q^h\right)_\infty \sum_{j=0}^{\infty} & \frac{b^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j \left(-aq^h; q^h\right)_{tj}} \\ &= \left(-bq^t; q^t\right)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{h\binom{k+1}{2}}}{\left(q^h; q^h\right)_k (-bq^t; q^t)_{hk}}.\end{aligned} \tag{3.17}$$

Many special cases of this symmetric identity have been found useful, some noted below, and one considered in §4.

When $h = 1$, (3.17) reduces to [5, Entry 1.4.12]

$$\begin{aligned}(-aq;q)_\infty \sum_{j=0}^{\infty} & \frac{b^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j (-aq;q)_{tj}} \\ &= \left(-bq^t; q^t\right)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{\binom{k+1}{2}}}{(q;q)_k (-bq^t; q^t)_k}.\end{aligned} \tag{3.18}$$

Take $h = t$ in (3.17), and then replace q by $q^{1/t}$ to obtain [5, Entry 1.4.17]

$$(-aq;q)_\infty \sum_{j=0}^{\infty} \frac{b^j q^{\binom{j+1}{2}}}{(q;q)_j(-aq;q)_{tj}} = (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{\binom{k+1}{2}}}{(q;q)_k(-bq;q)_{tk}}, \tag{3.19}$$

one of Ramanujan's formulas highlighted in the introduction. Take $a = -1$ and $b = 1$ in Ramanujan's Entry 1.4.17 (eq. (3.19)) to obtain

$$\sum_{j=0}^{\infty} \frac{q^{\binom{j+1}{2}}}{(q;q)_j(q;q)_{tj}} = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k(-q;q)_{tk}}. \tag{3.20}$$

This further reduces to [5, Entry 1.4.9] when $t = 1$:

$$\sum_{j=0}^{\infty} \frac{q^{\binom{j+1}{2}}}{(q;q)_j^2} = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q^2;q^2)_k}, \tag{3.21}$$

where we use [10, eq. (I.28)]

$$(a;q)_k(-a;q)_k = \left(a^2;q^2\right)_k$$

in the denominator of the RHS.

In Entry 1.4.17 (eq. (3.19)) replace q by q^2 and take $t = 1$ to obtain an assertion equivalent to an observation of M. Soros (see [5, eq. (1.5.1)]):

$$\left(-aq^2;q^2\right)_\infty \sum_{j=0}^{\infty} \frac{b^j q^{j^2+j}}{\left(q^2;q^2\right)_j\left(-aq^2;q^2\right)_j} = \left(-bq^2;q^2\right)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2+k}}{\left(q^2;q^2\right)_k\left(-bq^2;q^2\right)_k}. \tag{3.22}$$

When we take $a \mapsto a/q$ and $b \mapsto b/q$ in (3.22), we obtain [5, eq. (1.5.1)]. Instead, if we take the special case $b \mapsto a/q$ in (3.22), we obtain the second equality of [5, Entry 1.5.1]:

$$\left(-aq^2;q^2\right)_\infty \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{\left(q^2;q^2\right)_j\left(-aq^2;q^2\right)_j} = \left(-aq;q^2\right)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2+k}}{\left(q^2;q^2\right)_k\left(-aq;q^2\right)_k}. \tag{3.23}$$

Entry 1.4.18

Entry 1.4.18 is due to Andrews, Berndt, and Ramanujan [5], and follows from the $a \to 0$ case of Ramanujan's transformation (2.2).

Take $a \to 0$ in (3.1), cancel the factor $(1 - cq)$ from the LHS, and $(1 + bq)$ from both sides. In the resulting identity take $b \mapsto b/q$, $c \mapsto a/q$, and $d \mapsto -c/q^h$, and bring the infinite products to the other side, we obtain: for $|c| < 1$,

$$\sum_{j=0}^{\infty} \frac{\left(-c; q^h\right)_{tj}}{(q^t; q^t)_j \left(aq^h; q^h\right)_{tj}} b^j q^{t\binom{j+1}{2}} = \frac{\left(-bq^t; q^t\right)_\infty \left(-c; q^h\right)_\infty}{\left(aq^h; q^h\right)_\infty} \sum_{k=0}^{\infty} \frac{\left(-aq^h/c; q^h\right)_k}{\left(q^h; q^h\right)_k (-bq^t; q^t)_{hk}} (-c)^k. \tag{3.24}$$

When $c = a/b$, $h = 2$ and $t = 1$, this reduces to [5, Entry 1.4.18]:

$$\sum_{j=0}^{\infty} \frac{\left(-a/b; q^2\right)_j}{(q; q)_j \left(aq^2; q^2\right)_j} b^j q^{\binom{j+1}{2}} = \frac{(-bq; q)_\infty \left(-a/b; q^2\right)_\infty}{\left(aq^2; q^2\right)_\infty} \sum_{k=0}^{\infty} \frac{\left(-bq^2; q^2\right)_k}{\left(q^2; q^2\right)_k (-bq; q)_{2k}} \left(-\frac{a}{b}\right)^k. \tag{3.25}$$

Perhaps the $c = a/b$, $h = 1$ and $t = 2$ case of (3.24) is equally pretty.

$$\sum_{j=0}^{\infty} \frac{(-a/b; q)_{2j}}{\left(q^2; q^2\right)_j (aq; q)_{2j}} b^j q^{j^2+j} = \frac{\left(-bq^2; q^2\right)_\infty (-a/b; q)_\infty}{(aq; q)_\infty} \sum_{k=0}^{\infty} \frac{(-bq; q)_k}{(q; q)_k \left(-bq^2; q^2\right)_k} \left(-\frac{a}{b}\right)^k. \tag{3.26}$$

To get other transformations of a similar nature, consider the $b = 0$ case of (3.1).

Take $b = 0$ in (3.1), cancel the factor $(1 - cq)$ from the LHS. In the resulting identity take $a \mapsto -b$, $c \mapsto a/q$, and $d \mapsto -c/q^h$, and bring the infinite products to the other side. We obtain: for $|bq^t| < 1$, $|c| < 1$,

$$\sum_{j=0}^{\infty} \frac{\left(-c; q^h\right)_{tj}}{(q^t; q^t)_j \left(aq^h; q^h\right)_{tj}} (-bq^t)^j = \frac{\left(-c; q^h\right)_\infty}{\left(aq^h; q^h\right)_\infty (-bq^t; q^t)_\infty} \sum_{k=0}^{\infty} \frac{\left(-aq^h/c; q^h\right)_k \left(-bq^t; q^t\right)_{hk}}{\left(q^h; q^h\right)_k} (-c)^k. \tag{3.27}$$

Take the $c = a/b$, $h = 2$, $t = 1$ case of (3.27) to obtain

$$\sum_{j=0}^{\infty}\frac{\left(-a/b;q^2\right)_j}{(q;q)_j\left(aq^2;q^2\right)_j}(-bq)^j$$
$$=\frac{\left(-a/b;q^2\right)_\infty}{\left(aq^2;q^2\right)_\infty(-bq;q)_\infty}\sum_{k=0}^{\infty}\frac{\left(-bq^2;q^2\right)_k(-bq;q)_{2k}}{\left(q^2;q^2\right)_k}\left(-\frac{a}{b}\right)^k. \tag{3.28}$$

An equivalent form of Entry 1.6.5

Consider the case $a\to 0$ of (3.1). Replace c by dq^{h-1}, and then take $b\mapsto b/q$ and $d\mapsto -a$ to obtain, for $|aq^h|<1$,

$$\sum_{j=0}^{\infty}\frac{b^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j\left(1+aq^{h(tj+1)}\right)}=\left(-bq^t;q^t\right)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k\left(aq^h\right)^k}{(-bq^t;q^t)_{hk}}. \tag{3.29}$$

In the case that $h=1$, $t=2$, and $b=a$ this reduces to an equivalent form of Entry 1.6.5, the second last equation in the proof of [5, Entry 1.6.5]:

$$\sum_{j=0}^{\infty}\frac{a^j q^{j^2+j}}{\left(q^2;q^2\right)_j\left(1+aq^{2j+1}\right)}=\left(-aq^2;q^2\right)_\infty\sum_{k=0}^{\infty}\frac{(-aq)^k}{\left(-aq^2;q^2\right)_k}. \tag{3.30}$$

This identity has a combinatorial proof, given by Berndt, Kim and Yee [8, Th. 5.7]. A very similar identity is obtained when $h=2$, $t=1$, $a\mapsto a/q$, and $b=a$ in (3.29):

$$\sum_{j=0}^{\infty}\frac{a^j q^{\binom{j+1}{2}}}{(q;q)_j\left(1+aq^{2j+1}\right)}=(-aq;q)_\infty\sum_{k=0}^{\infty}\frac{(-aq)^k}{(-aq;q)_{2k}}. \tag{3.31}$$

Similar identities are obtained when $b=0$ in (3.1). Set $d=c/q^{h-1}$ and in the resulting identity, relabel parameters by replacing c by $-aq^{h-1}$ and a by $-b$, to obtain, for $|aq^h|<1$, $|bq^t|<1$,

$$\sum_{j=0}^{\infty}\frac{\left(-bq^t\right)^j}{(q^t;q^t)_j\left(1+aq^{h(tj+1)}\right)}=\frac{1}{(-bq^t;q^t)_\infty}\sum_{k=0}^{\infty}\left(-bq^t;q^t\right)_{hk}\left(-aq^h\right)^k. \tag{3.32}$$

Now take $b=a$, $h=1$ and $t=2$ to find that

$$\sum_{j=0}^{\infty}\frac{(-aq^2)^j}{\left(q^2;q^2\right)_j\left(1+aq^{2j+1}\right)}=\frac{1}{\left(-aq^2;q^2\right)_\infty}\sum_{k=0}^{\infty}\left(-aq^2;q^2\right)_k(-aq)^k. \tag{3.33}$$

The $b=0=c$ case of (3.1)

Take $b=0=c$ in (3.1), replace d by b and take $\left(aq^t;q^t\right)_\infty$ on the other side to obtain an extremely symmetric transformation formula; for $|aq^t|<1$ and $|bq^h|<1$,

$$\frac{1}{\left(bq^h;q^h\right)_\infty}\sum_{j=0}^{\infty}\frac{\left(bq^h;q^h\right)_{tj}}{(q^t;q^t)_j}(aq^t)^j=\frac{1}{(aq^t;q^t)_\infty}\sum_{k=0}^{\infty}\frac{\left(aq^t;q^t\right)_{hk}}{\left(q^h;q^h\right)_k}(bq^h)^k. \quad (3.34)$$

There is no corresponding formula in Ramanujan's list appearing in Chapter 1 of [5], but it is related to [5, Cor. 1.2.2], a result originally due to Andrews [2]. When $h=2$ and $t=1$, then (3.34) reduces to

$$\frac{1}{\left(bq^2;q^2\right)_\infty}\sum_{j=0}^{\infty}\frac{\left(bq^2;q^2\right)_j}{(q;q)_j}(aq)^j=\frac{1}{(aq;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq;q)_{2k}}{\left(q^2;q^2\right)_k}(bq^2)^k. \quad (3.35)$$

Compare the sum on the left-hand side with that of [5, Cor. 1.2.2]. To obtain Cor. 1.2.2, Andrews and Berndt apply Heine's transformation once again on the right-hand side of (3.35).

Summary of special cases

So far, we have listed 13 entries that are immediate special or limiting cases of (3.1). One more will appear in §4. The main special case is Entry 1.4.1 (eq. (2.2)) which is the $h=2$ and $t=1$ case of (3.1). The others are:

1. The case $a=1=d$. This leads to Entry 1.4.2 and Entry 1.4.5.
2. The case $b=0$ and $d\to 0$ of (3.1). This leads to Entry 1.4.3 and Entry 1.4.4. Note that the case $a\to 0$ and $c=0$ leads to the same identities. Other special cases include Entry 1.4.10 and Entry 1.4.11.
3. Taking $a\to 0$ (without changing c) leads to Entry 1.4.18, and an equivalent form of Entry 1.6.5. See also Entry 1.6.6 in §4 below. We have also taken $b=0$ for the sake of completeness. (The $d\to 0$ and $c=0$ cases are equivalent due to the symmetry of (3.1).)
4. The case $a\to 0$ and $d\to 0$. This leads to Entry 1.4.12, Entry 1.4.17, Entry 1.4.9, and a part of Entry 1.5.1.
5. The case $b=0=c$. This leads to a new transformation formula. A special case is closely related to a useful transformation formula of Andrews in [5, Cor. 1.2.2].

By examining the above summary carefully, one can ask about the cases when $b=0$ followed by $a=0$, or $c=0$ followed by $d=0$ in (3.1). However, in both these cases, the resulting identity reduces to the q-binomial theorem.

It is apparent that most of Ramanujan's identities considered here are simple limiting cases of (3.1) where one or more parameters go to 0. However, there are a few that are motivated by getting a q-series (such as Entry 1.4.10 and Entry 1.4.11), or in getting an 'unnatural' identity, where the factor that naturally terminates the series from below is missing. See Entry 1.4.2 (eq. (3.6)), Entry 1.4.5 (eq. (3.9)) and (3.30).

4 Entry 1.6.6 and related summations

Entry 1.6.6 is also a special case of (3.29), our generalization of Entry 1.6.5 above, and so of (3.1). What is different here is that one can employ a special case of the q-binomial theorem to sum one of the series. The special case we need is [10, eq. (II.1)]: for $|z| < 1$,

$$\frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}. \tag{4.1}$$

Observe that when $b = -1$ and $h = 1$ in (3.29), then using (4.1), we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(-1)^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j\left(1+aq^{tj+1}\right)} &= \left(q^t;q^t\right)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k (aq)^k}{(q^t;q^t)_k} \\ &= \frac{\left(q^t;q^t\right)_\infty}{(-aq;q^t)_\infty}.\end{aligned}$$

Replace a by aq^{s-1} to rewrite this identity in the form

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j\left(1+aq^{tj+s}\right)} = \frac{\left(q^t;q^t\right)_\infty}{(-aq^s;q^t)_\infty}. \tag{4.2}$$

In the case where $a = -1$, $s = 1$ and $t = 2$, this reduces to Ramanujan's [5, Entry 1.6.6], an identity we highlighted in the introduction:

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+j}}{\left(q^2;q^2\right)_j\left(1-q^{2j+1}\right)} = \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}, \tag{4.3}$$

where the ratio of infinite products on the right-hand side is equal to Ramanujan's theta function $\psi(q)$, defined as

$$\psi(q) := \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}.$$

However, if we take $a = 1$, $t = 1$ and $s = 1$ in (4.2), we obtain

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{(q;q)_j\left(1+q^{j+1}\right)} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{4.4}$$

where now the products on the right-hand side are (in Ramanujan's notation) $\phi(-q)$, defined by

$$\phi(-q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}.$$

So in (4.2) we have a common generalization of (4.3) and (4.4).

More generally, when $b = -1$, $a \mapsto -a^h$, then (3.29) reduces to

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j \left(1-a^h q^{h(tj+1)}\right)} = \left(q^t;q^t\right)_\infty \sum_{k=0}^{\infty} \frac{(aq)^{hk}}{(q^t;q^t)_{hk}}.$$

When h is a positive integer, the sum on the right consists of every hth term of the summand in (4.1). There is a simple trick to compute such a sum. It uses the fact that

$$\frac{1}{h}\sum_{r=0}^{h-1} \omega_h^{rk} = \begin{cases} 1 & \text{if } h|k \\ 0 & \text{otherwise} \end{cases},$$

for $\omega_h = e^{2\pi i/h}$ or some other primitive hth root of unity. Using this trick, we find that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(aq)^{hk}}{(q^t;q^t)_{hk}} &= \sum_{k=0}^{\infty} \frac{(aq)^k}{(q^t;q^t)_k} \frac{1}{h}\sum_{r=0}^{h-1}\omega_h^{rk} \\ &= \frac{1}{h}\sum_{r=0}^{h-1}\sum_{k=0}^{\infty} \frac{\left(aq\omega_h^r\right)^k}{(q^t;q^t)_k} \\ &= \frac{1}{h}\sum_{r=0}^{h-1} \frac{1}{\left(aq\omega_h^r;q^t\right)_\infty}. \end{aligned}$$

So we obtain

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j \left(1-a^h q^{h(tj+1)}\right)} = \frac{1}{h}\sum_{r=0}^{h-1} \frac{\left(q^t;q^t\right)_\infty}{\left(aq\omega_h^r;q^t\right)_\infty}. \tag{4.5}$$

In particular when $h = 2$, then $\omega_h = -1$, and

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{t\binom{j+1}{2}}}{(q^t;q^t)_j \left(1-a^2 q^{2(tj+1)}\right)} = \frac{1}{2}\left(\frac{\left(q^t;q^t\right)_\infty}{(aq;q^t)_\infty} + \frac{\left(q^t;q^t\right)_\infty}{(-aq;q^t)_\infty}\right). \tag{4.6}$$

When $a = 1$ and $t = 2$, this reduces to

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+j}}{\left(q^2;q^2\right)_j \left(1-q^{2(2j+1)}\right)} = \frac{1}{2}\left(\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty} + \frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\right)$$
$$= \frac{1}{2}\left(\psi(q)+\psi(-q)\right). \tag{4.7}$$

The right-hand side is the even part of $\psi(q)$.

Remark Let $\omega_r = e^{2\pi i/r}$ or some other primitive rth root of unity. Then we can show that, formally,

$$\frac{1}{r}\sum_{\nu=0}^{r-1}\omega_r^{-\nu s}\sum_{n=0}^{\infty} f(n)\omega_r^{\nu n} = \sum_{k=0}^{\infty} f(rk+s)\,\frac{1}{r}\sum_{\nu=0}^{r-1}\omega_r^{\nu rk} = \sum_{k=0}^{\infty} f(rk+s).$$

This 'sectioning' process allows us to compute the sum $\sum_{k=0}^{\infty} f(rk+s)z^{rk+s}$, if we know the sum $\sum_{k=0}^{\infty} f(k)z^k$.

This trick is used in the proof of Andrews' Fundamental Lemma, given in equation (2.6).

We have seen an example where we can sum a series after applying Heine's method. Next, we obtain a result for multiple series by iterating Heine's method.

5 Multibasic Andrews q-Lauricella transformation

We now apply Heine's method to obtain a multibasic generalization of Andrews' [3, (eq. (4.1)] transformation formula for the q-Lauricella function. As special cases, we obtain some generalizations of Entry 1.4.10 and of equation (3.17). These results transform a multiple series to a multiple of a single series.

For $\boldsymbol{h}$ and $\boldsymbol{k}$ vectors, we use the following notations. The notation $|\boldsymbol{k}|$ is used to denote the sum of the components of the vector $k_1 + k_2 + \cdots + k_m$. We use the symbol for the dot product

$$\boldsymbol{h}\cdot\boldsymbol{k} = h_1k_1 + h_2k_2 + \cdots + h_mk_m.$$

We also use the vector $\boldsymbol{\delta}$ to denote the vector $(1, 2, \ldots, m)$. Thus,

$$\boldsymbol{\delta}\cdot\boldsymbol{k} = \sum_{r=1}^{m} rk_r.$$

Theorem 5.1 *Suppose $m = 1, 2, \ldots$ is a nonnegative integer. Let $a_1, a_2, \ldots, a_m$ and b be complex numbers, and suppose that the denominators in (5.1) are not zero. Further, let q, t, $h_1, h_2, \ldots h_m$ be complex numbers, satisfying $|q^t| < 1$,*

$|q^{h_r}| < 1$ *and* $|q^{th_r}| < 1$, *for* $r = 1, 2, \ldots, m$. *Then, for* $|w| < 1$, *and* $|z_r| < 1$ *(for* $r = 1, 2, \ldots, m$*),*

$$\sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m}} \prod_{r=1}^{m} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \frac{\left(w; q^t\right)_{\boldsymbol{h}\cdot\boldsymbol{k}}}{\left(bw; q^t\right)_{\boldsymbol{h}\cdot\boldsymbol{k}}} \prod_{r=1}^{m} z_r^{k_r} \tag{5.1}$$

$$= \frac{\left(w; q^t\right)_\infty}{\left(bw; q^t\right)_\infty} \prod_{r=1}^{m} \frac{\left(a_r z_r; q^{h_r}\right)_\infty}{\left(z_r; q^{h_r}\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} \prod_{r=1}^{m} \frac{\left(z_r; q^{h_r}\right)_{tj}}{\left(a_r z_r; q^{h_r}\right)_{tj}} w^j .$$

Remark When $m = 1$, then (5.1) reduces to (2.4). When $a_r \mapsto b_r, b \mapsto c/a, w \mapsto a$, and $h_1 = h_2 = \cdots h_m = 1 = t$, then (5.1) reduces to a transformation of Andrews for q-Lauricella functions [3, (eq. (4.1)].

Proof The proof of (5.1) is a direct extension of the proof of (2.4). For $m = 1$, it reduces to Theorem (2.1). When $m > 1$, we apply Heine's method m times. Expand the relevant products using the q-binomial theorem and interchange the sums one at a time. The first few steps of the proof are as follows.

$$\sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m}} \prod_{r=1}^{m} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \frac{\left(w; q^t\right)_{\boldsymbol{h}\cdot\boldsymbol{k}}}{\left(bw; q^t\right)_{\boldsymbol{h}\cdot\boldsymbol{k}}} \prod_{r=1}^{m} z_r^{k_r}$$

$$= \frac{\left(w; q^t\right)_\infty}{\left(bw; q^t\right)_\infty} \sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m}} \prod_{r=1}^{m} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \frac{\left(bwq^{t(h_1k_1+\cdots+h_mk_m)}; q^t\right)_\infty}{\left(wq^{t(h_1k_1+\cdots+h_mk_m)}; q^t\right)_\infty} \prod_{r=1}^{m} z_r^{k_r}$$

$$= \frac{\left(w; q^t\right)_\infty}{\left(bw; q^t\right)_\infty} \sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m}} \prod_{r=1}^{m} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \prod_{r=1}^{m} z_r^{k_r} \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} w^j q^{tj(h_1k_1+\cdots+h_mk_m)}$$

$$= \frac{\left(w; q^t\right)_\infty}{\left(bw; q^t\right)_\infty} \sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m-1}} \prod_{r=1}^{m-1} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \prod_{r=1}^{m-1} z_r^{k_r}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} w^j q^{tj(h_1k_1+\cdots+h_{m-1}k_{m-1})} \sum_{k_m \geq 0} \frac{\left(a_m; q^{h_m}\right)_{k_m}}{\left(q^{h_m}; q^{h_m}\right)_{k_m}} \left(z_m q^{tjh_m}\right)^{k_m}$$

$$= \frac{\left(w; q^t\right)_\infty}{\left(bw; q^t\right)_\infty} \sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,m-1}} \prod_{r=1}^{m-1} \frac{\left(a_r; q^{h_r}\right)_{k_r}}{\left(q^{h_r}; q^{h_r}\right)_{k_r}} \prod_{r=1}^{m-1} z_r^{k_r}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(b; q^t\right)_j}{\left(q^t; q^t\right)_j} w^j q^{tj(h_1k_1+\cdots+h_{m-1}k_{m-1})} \frac{\left(a_m z_m q^{tjh_m}; q^{h_m}\right)_\infty}{\left(z_m q^{tjh_m}; q^{h_m}\right)_\infty}$$

$$= \frac{(w;q^t)_\infty}{(bw;q^t)_\infty}\frac{(a_mz_m;q^{h_m})_\infty}{(z_m;q^{h_m})_\infty} \sum_{\substack{k_r\geq 0\\ r=1,2,\dots,m-1}} \prod_{r=1}^{m-1}\frac{(a_r;q^{h_r})_{k_r}}{(q^{h_r};q^{h_r})_{k_r}}\prod_{r=1}^{m-1} z_r^{k_r}$$
$$\times \sum_{j=0}^{\infty}\frac{(b;q^t)_j}{(q^t;q^t)_j}\frac{(z_m;q^{h_m})_{tj}}{(a_mz_m;q^{h_m})_{tj}} w^j q^{tj(h_1k_1+\dots+h_{m-1}k_{m-1})}.$$

So far, the steps are identical to the proof of Theorem 2.1, with the sum indexed by k_m replacing the sum indexed by k in the earlier proof. Repeating these steps $m-1$ times, we obtain the required single sum indexed by j on the right-hand side of (5.1).

The convergence considerations in §2 extend to both the series in this theorem, and to the interchange of summations required in the proof. □

Next, we indicate generalizations of a few special cases of results from §3, to hint at the many possibilities available.

First take $a_r \mapsto c_rq/d_r, b \mapsto -bq/a, w \mapsto aq^t$ and $z_r \mapsto d_rq^{h_r}$ in (5.1) to obtain a generalization of (3.1):

$$\frac{(aq^t;q^t)_\infty}{(-bq^{t+1};q^t)_\infty}\prod_{r=1}^{m}\frac{(c_rq^{h_r+1};q^{h_r})_\infty}{(d_rq^{h_r};q^{h_r})_\infty}\sum_{j=0}^{\infty}\frac{(-bq/a;q^t)_j}{(q^t;q^t)_j}\prod_{r=1}^{m}\frac{(d_rq^{h_r};q^{h_r})_{tj}}{(c_rq^{h_r+1};q^{h_r})_{tj}}(aq^t)^j$$
$$= \sum_{\substack{k_r\geq 0\\ r=1,2,\dots,m}}\prod_{r=1}^{m}\frac{(c_rq/d_r;q^{h_r})_{k_r}}{(q^{h_r};q^{h_r})_{k_r}}\frac{(aq^t;q^t)_{\boldsymbol{h}\cdot\boldsymbol{k}}}{(-bq^{t+1};q^t)_{\boldsymbol{h}\cdot\boldsymbol{k}}}q^{\boldsymbol{h}\cdot\boldsymbol{k}}\prod_{r=1}^{m}d_r^{k_r}. \tag{5.2}$$

Next, we obtain four generalizations of Ramanujan's Entry 1.4.10, equation (3.14).

First take $b=0$ and $d_r\to 0$, for $r=1,2,\dots,m$ in (5.2). Further, replace c_r by $1/q$ for each r, and take $a=1$. In the resulting identity, take $t=1$, and $h_r=r$ for $r=1,2,\dots,m$ to obtain

$$\sum_{j=0}^{\infty}\frac{q^j}{(q;q)_j}\prod_{r=1}^{m}\frac{1}{(q^r;q^r)_j} = \frac{1}{(q;q)_\infty}\prod_{r=1}^{m}\frac{1}{(q^r;q^r)_\infty}$$
$$\times\sum_{\substack{k_r\geq 0\\ r=1,2,\dots,m}}\prod_{r=1}^{m}\frac{1}{(q^r;q^r)_{k_r}}(q;q)_{\boldsymbol{\delta}\cdot\boldsymbol{k}}(-1)^{|\boldsymbol{k}|}q^{\sum_{r=1}^{m} r\binom{k_r+1}{2}}. \tag{5.3}$$

Next, again take $b=0$ and $d_r\to 0$, for $r=1,2,\dots,m$ in (5.2). But now take $m=n$, and replace h_r by n for all r. Further, set $c_r=cq^{r-2}$ for $r=1,2,\dots,n$, and invoke (3.3) to simplify some of the products. Finally, take $a=c=1$ to obtain the following generalization of Entry 1.4.10:

$$\sum_{j=0}^{\infty}\frac{q^{tj}}{(q^t;q^t)_j(q^n;q)_{ntj}}=\frac{1}{(q^t;q^t)_\infty(q^n;q)_\infty}$$
$$\times\sum_{\substack{k_r\geq 0\\ r=1,2,\ldots,n}}\prod_{r=1}^{n}\frac{1}{(q^n;q^n)_{k_r}}\left(q^t;q^t\right)_{n|\boldsymbol{k}|}(-1)^{|\boldsymbol{k}|}q^{\sum\limits_{r=1}^{n}(r-1)k_r+n\sum\limits_{r=1}^{n}\binom{k_r+1}{2}}. \tag{5.4}$$

The third generalization of Entry 1.4.10 is obtained as follows. In equation (5.2), take $a\to 0$ and $c_r=0$ for $r=1,2,\ldots,m$. Now take $b\mapsto -1/q$, $d_r=1$, for $r=1,2,\ldots,m$. In the resulting transformation, once again take $t=1$, and $h_r=r$, for $r=1,2,\ldots,m$, and obtain

$$\sum_{\substack{k_r\geq 0\\ r=1,2,\ldots,m}}\frac{1}{(q;q)_{\boldsymbol{\delta}\cdot\boldsymbol{k}}}\prod_{r=1}^{m}\frac{1}{(q^r;q^r)_{k_r}}q^{\boldsymbol{\delta}\cdot\boldsymbol{k}}$$
$$=\frac{1}{(q;q)_\infty}\prod_{r=1}^{m}\frac{1}{(q^r;q^r)_\infty}\sum_{j=0}^{\infty}(-1)^jq^{\binom{j+1}{2}}\prod_{r=2}^{m}\left(q^r;q^r\right)_j. \tag{5.5}$$

The fourth generalization of Entry 1.4.10 is as follows. Again, take $a\to 0$ and $c_r=0$ for $r=1,2,\ldots,m$. But now take $m=n$, and replace h_r by n for all r. Further, replace b by $-b/q$, and set $d_r=dq^{r-1}$ for $r=1,2,\ldots,n$. Again, we invoke (3.3) to simplify some of the products, and take $b=d=1$ to obtain the following generalization of Entry 1.4.10.

$$\sum_{\substack{k_r\geq 0\\ r=1,2,\ldots,n}}\frac{1}{(q^t;q^t)_{n|\boldsymbol{k}|}}\prod_{r=1}^{n}\frac{1}{(q^n;q^n)_{k_r}}q^{n|\mathbf{k}|+\sum\limits_{r=1}^{n}(r-1)k_r}$$
$$=\frac{1}{(q^t;q^t)_\infty(q^n;q)_\infty}\sum_{j=0}^{\infty}\frac{(q^n;q)_{ntj}}{(q^t;q^t)_j}(-1)^jq^{t\binom{j+1}{2}}. \tag{5.6}$$

Next, we give a generalization of the generalization of Entry 1.4.12 given in equation (3.17). To this end take the limit as $a\to 0$ and $d_r\to 0$, for $r=1,2,\ldots,m$ in (5.2). Now replace c_r by $-a_r/q$ and b by b/q to obtain

$$\prod_{r=1}^{m}\left(-a_rq^{h_r};q^{h_r}\right)_\infty\sum_{j=0}^{\infty}\frac{1}{(q^t;q^t)_j}\prod_{r=1}^{m}\frac{1}{\left(-a_rq^{h_r};q^{h_r}\right)_{tj}}b^jq^{t\binom{j+1}{2}}$$
$$=\left(-bq^t;q^t\right)_\infty\sum_{\substack{k_r\geq 0\\ r=1,2,\ldots,m}}\prod_{r=1}^{m}\frac{a_r^{k_r}}{\left(q^{h_r};q^{h_r}\right)_{k_r}}\frac{1}{(-bq^t;q^t)_{\boldsymbol{h}\cdot\boldsymbol{k}}}q^{\sum\limits_{r=1}^{m}h_r\binom{k_r+1}{2}}. \tag{5.7}$$

Finally, we present a special case of (5.7) with $m = n$ and where $h_r = n$, for all r. Take $a_r = aq^{r-1}$ for all r and simplify some of the products using (3.3) to obtain the following generalization of Entry 1.4.12:

$$\left(-aq^n;q\right)_\infty \sum_{j=0}^{\infty} \frac{b^j}{(q^t;q^t)_j(-aq^n;q)_{ntj}} q^{t\binom{j+1}{2}} = \left(-bq^t;q^t\right)_\infty$$
$$\times \sum_{\substack{k_r \geq 0 \\ r=1,2,\ldots,n}} \prod_{r=1}^{n} \frac{1}{(q^n;q^n)_{k_r}} \frac{a^{|\boldsymbol{k}|}}{(-bq^t;q^t)_{n|\boldsymbol{k}|}} q^{\sum\limits_{r=1}^{n}(r-1)k_r + n\sum\limits_{r=1}^{n}\binom{k_r+1}{2}}. \tag{5.8}$$

Perhaps this is a suitable place to close our study, at equation number 60 of this paper.

6 Closing remarks

We have seen that a minor modification of Andrews' earlier identity led to so many identities similar to Ramanujan's entries. Clearly, it is a good idea to study Ramanujan's Notebooks, edited by Berndt, and the Lost Notebook, edited by Andrews and Berndt. We conclude with a few remarks regarding Ramanujan's transformations and possible directions of further study.

Entry 1.4.1 is a key identity of Ramanujan, and deserves more importance than given in [5]. Recall that Entry 1.4.1 is the $h = 2$ and $t = 1$ case of (3.1). Many of Ramanujan's transformations considered here are immediate corollaries of Entry 1.4.1. These include Entries 1.4.2, 1.4.3, 1.4.4, 1.4.5, and 1.4.18. The special cases considered are the obvious ones, by letting one or more parameters equal to 0, or if necessary, taking limits to 0. Even the equivalent case of Entry 1.6.5 can be derived from Entry 1.4.1, by taking $d \to 0$.

Entries 1.4.9, 1.4.10, 1.4.11, and 1.5.1 follow from the $h = 1 = t$ case of (1.2) or in other words, from Heine's transformation (2.1). Since Heine's transformation formula appears in earlier notebooks of Ramanujan, why do these formulas show up here, in his later work? An explanation is that Ramanujan was searching for identities for series that look like or involve theta functions. So these entries, and Entry 1.6.6 fit in well here.

Next note that Entry 1.4.12 is obtained from the $h = 1$ case of (1.2), while Entry 1.4.17 requires the the $h = t$ case of (1.2). Of the entries studied here, these two are the only ones that require something more than (2.1) and (2.2) (the two identities noted by Ramanujan in his notebooks).

Many of Ramanujan's identities have been studied from a partition theoretic perspective by Berndt, Kim and Yee [8], including (3.18), (3.19), (3.21), (3.23) and (3.30). We expect that many of the identities considered here have a similar interpretation.

Finally, we note that Heine's method generalizes to multiple series related to root systems. We can combine the multidimensional q-binomial theorems (given by, for example, Milne [14] and Gustafson and Krattenthaler [12]) to obtain extensions of (1.2) and (5.1). We hope to present these elsewhere.

Acknowledgements We thank Professor George Andrews and Professor Christian Krattenthaler for many suggestions, pointers to useful references, and helpful discussions.

References

1. G.E. Andrews, On basic hypergeometric series, mock theta functions, and partitions. I. Quart. J. Math. Oxford Ser. **2**(17), 64–80 (1966)
2. G.E. Andrews, q-identities of Auluck, Carlitz, and Rogers. Duke Math. J. **33**, 575–581 (1966)
3. G.E. Andrews, Summations and transformations for basic Appell series. J. London Math. Soc. **2**(4), 618–622 (1972)
4. G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook Part I* (Springer, New York, 2005)
5. G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook Part II* (Springer, New York, 2009)
6. B.C. Berndt, *Ramanujan's notebooks. Part III* (Springer, New York, 1991). https://doi.org/10.1007/978-1-4612-0965-2
7. B.C. Berndt, *Number Theory In The Spirit Of Ramanujan*, Student Mathematical Library (American Mathematical Society, Providence, 2006). https://doi.org/10.1090/stml/034
8. B.C. Berndt, B. Kim, A.J. Yee, Ramanujan's lost notebook: combinatorial proofs of identities associated with Heine's transformation or partial theta functions. J. Comb. Theory Series A **117**(7), 857–973 (2010). https://doi.org/10.1016/j.jcta.2009.07.005
9. G. Gasper, Summation, transformation, and expansion formulas for bibasic series. Trans. Am. Math. Soc. **312**(1), 257–277 (1989). https://doi.org/10.2307/2001216
10. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edn., Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, 2004). https://doi.org/10.1017/CBO9780511526251. With a foreword by Richard Askey
11. V.J.W. Guo, J. Zeng, Basic and bibasic identities related to divisor functions. J. Math. Anal. Appl. **431**(2), 1197–1209 (2015). https://doi.org/10.1016/j.jmaa.2015.06.018
12. R.A. Gustafson, C. Krattenthaler, Heine transformations for a new kind of basic hypergeometric series in $U(n)$. J. Comput. Appl. Math. **68**(1–2), 151–158 (1996). https://doi.org/10.1016/0377-0427(95)00260-X
13. E. Heine, Untersuchungen Über die Reihe $1+\frac{(q^{\alpha}-1)(q^{\beta}-1)}{(q-1)(q^{\gamma}-1)}x+\frac{(q^{\alpha}-1)(q^{\alpha+1}-1)(q^{\beta}-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^{\gamma}-1)(q^{\gamma+1}-1)}+\cdots$. J. Reine Angew. Math. **34**, 285–328 (1847). https://doi.org/10.1515/crll.1847.34.285
14. S.C. Milne, Balanced ${}_3\phi_2$ summation theorems for U(n) basic hypergeometric series. Adv. Math. **131**(1), 93–187 (1997). https://doi.org/10.1006/aima.1997.1658
15. S. Ramanujan, *The lost notebook and other unpublished papers*. (Springer, Berlin, 1988); (Narosa Publishing House, New Delhi, 1988). With an introduction by George E. Andrews

Adventures with the OEIS

Jonathan M. Borwein

Abstract This paper is a somewhat expanded companion to a talk (Available at http://www.carma.newcastle.edu.au/jon/OEIStalk.pdf) with the same title presented in December 2015 at a 2015 workshop celebrating Tony Guttmann's seventieth birthday. My main intention is to further advertise the wonderful resource that the Online Encyclopedia of Integer Sequences (OEIS) has become.

Keywords Experimental mathematics · Integer sequences · Sloane's encyclopedia

2010 Mathematics Subject Classification 97I30 · 33C90

1 Introduction

What began in 1964 with a small set of personal file cards has grown over half a century into the current wonderful online resource: the *Online Encyclopedia of Integer Sequences* (OEIS).

1.1 Introduction to Sloane's online and off-line encyclopedia

I shall describe five encounters over nearly 30 years with Neil Sloane's (Online) Encyclopedia of Integer Sequences. Its brief chronology is as follows:

- In **1973** a published book (Sloane) with 2, 372 entries appeared. This was based on file cards kept since 1964.

Editorial Note: Jon Borwein passed away August 2, 2016.

J. M. Borwein
Centre for Computer-Assisted Research Mathematics and its Applications (CARMA), University of Newcastle, Callaghan, NSW 2308, Australia

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_9

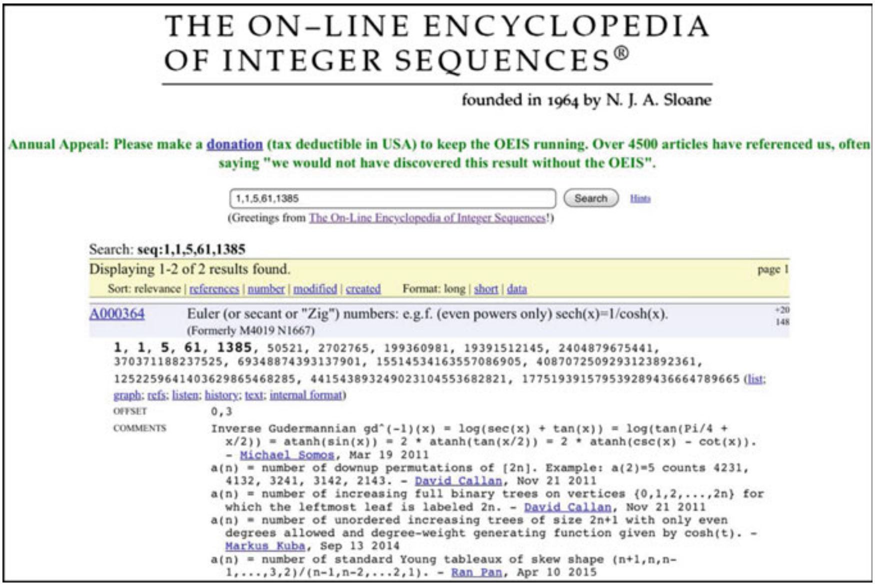


Fig. 1 The OIES in action.

- In **1995** a revised and expanded book (by Sloane & Simon Plouffe) with 5, 488 entries appeared.

—See the book review in *SIAM Review* by Rob Corless and me of the 1995 book at https://carma.newcastle.edu.au/jon/sloane/sloane.htm.

- Soon after the World Wide Web went public, between **1994–1996**, the OEIS went online with approximately 16, 000 entries.
- As of **Nov 15 21:28 EST 2015** OEIS had 263,957 entries

—all sequences used in this paper/talk were accessed between Nov 15–22, 2015.

1.2 *The OEIS in action*

As illustrated in Figure 1 taken from https://oeis.org/, the OEIS is easy to use, entering an integer sequence which it recognizes, one is rewarded with meanings, generating functions, computer code, links and references, and other delights.

1.3 *OEIS has some little known features*

The OEIS also now usefully recognizes numbers: entering 1.4331274267223117 583... yields the following answer.

Answer 1.1 (A060997). *Decimal representation of continued fraction*
1, 2, 3, 4, 5, 6, 7, ...
(as a ratio of Bessel functions $I_0(2)/I_1(2)$).

The OEIS currently has excellent search facilities, by topic or author, and so on. For instance, entering “Bell numbers” returned over 850 results while entering “Alladi” yielded 23 sequences. The third sequence listed on the page is:

Answer 1.2 (A000700). *Expansion of product* $(1+x^{2k+1})$, $k=0..\infty$*; number of partitions of n into distinct odd parts; number of self-conjugate partitions; number of symmetric Ferrers graphs with n nodes.*

The sequence begins

1, 1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 8, 8, 9, 11, 12, 12, 14, 16, 17, 18, 20

In the page we are told Krishna Alladi showed this is also the number of partitions of n into parts $\neq 2$ and differing by $\geqslant 6$ with strict inequality if a part is even.

Alladi’s paper “A variation on a theme of Sylvester—a smoother road to Göllnitz’s (Big) theorem”, *Discrete Math.*, **196** (1999), 1–11, through a link to http://www.sciencedirect.com/science/article/pii/S0012365X98001939 is also provided.

The OEIS also has an email-based “super-seeker” facility.

1.4 Stefan Banach (1892–1945) ... the OEIS notices analogies

The MacTutor website, see www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html, quotes Banach (Fig. 2) as saying:

> A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.

In a profound way the OEIS helps us—greater or lesser mathematicians—find analogies between theories.

2 1988: James Gregory (1638–1675) (Fig. 3) & Leonard Euler (1707–1783)

Sequence 2.1 (A000364 (1/2)).

$$2, -2, 10, -122, 2770\ldots$$

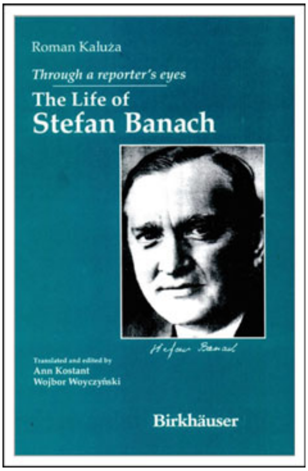


Fig. 2 A fine biography of Banach. Roman Kaluza, *Through a Reporter's Eyes: The Life of Stefan Banach*, Birkhäuser 1995.

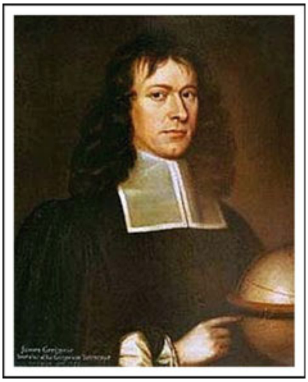


Fig. 3 James Gregory (1638–1675).

Answer 2.2 (A011248). *Twice* `A000364`[1] *Euler (or secant or "Zig") numbers: e.g.f. (even powers only)* $\operatorname{sech}(x) = 1/\cosh(x)$.

[1]Two sequences are found which we flag via (1/2). It is interesting to see how many terms are needed to uniquely define well-known sequences. We indicate the same information in the next two examples.

Story 2.3. In 1988 Roy North observed that Gregory's series for π,

$$\pi = 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \ = \ 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{1}$$

when truncated to 5,000,000 terms, gives a value differing strangely from the true value of π. Here is the truncated Gregory value and the true value of π:

3.141592**4**5358979323846464338327950**2**7**8**41971693993873058209749**4**18**22**30781640 . . .

3.14159265358979323846**2**643383279502884197169399**375**1058209749445923078164 0 . . .

Errors: 2 − 2 10 − 122 2770

The series value differs, as one might expect from a series truncated to 5, 000, 000 terms, in the seventh decimal place—a "4" where there should be a "6". But the next 13 digits are correct!

Then, following another erroneous digit, the sequence is once again correct for an additional 12 digits. In fact, of the first 46 digits, only four differ from the corresponding decimal digits of π.

Further, the "error" digits appear to occur in positions that have a period of 14, as shown above.

We note that each integer is *even*; dividing by two, we obtain $(1, -1, 5, -122, 1385)$. Sloane has told us we have the *Euler numbers* defined in terms of Taylor's series for $\sec x$:

$$\sec x = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} x^{2k}}{(2k)!}. \tag{2}$$

Indeed, we see the *asymptotic expansion* base 10 on the screen:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \approx \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}} \tag{3}$$

This works in hex (!!), and $\log 2$ instead of π yields the *tangent numbers*.

In 1988 we only had recourse to the original printed book and had to decide to divide the sequence by two before finding it. Now this sort of preprocessing and other such transformations are typically done for one by the OEIS. But it does not hurt to look for variants of one's sequence—such as considering the odd or square-free parts—if the original is not found.

Nico Temme's 1995 Wiley book *Special Functions: An Introduction to the Classical Functions of Mathematical Physics* starts with this motivating example.

References 2.4. The key references are

1. J.M. Borwein, P.B. Borwein, and K. Dilcher, "Euler numbers, asymptotic expansions and pi," *MAA Monthly*, **96** (1989), 681–687.
2. See also *Mathematics by Experiment* [1, §2.10] and "I prefer Pi," *MAA Monthly*, March 2015.

3 1999: Siméon Poisson (1781–1840) & E.T. Bell (1883–1960)

Sequence 3.1 (A000110 (1/10)).

$$1, 1, 2, 5, 15, 52, 203, 877, 4140 \ldots$$

Answer 3.2. *Bell or exponential numbers: number of ways to partition a set of n labeled elements.*

Story 3.3 (MAA Unsolved Problem). For $t > 0$, let

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t)\, \frac{t^k}{k!}$$

be the n-th moment of a *Poisson distribution* (Fig. 4) with parameter t. Let $c_n(t) = m_n(t)/n!$. Show

(a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex for all $t > 0$.

(b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.

(c*) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geqslant 1$.

Proof. (b) As

$$m_{n+1}(t) = t \sum_{k=0}^{\infty} (k+1)^n \exp(-t)\, \frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that

$$m_{n+1}(t) = t \sum_{k=0}^{n} \binom{n}{k} m_k(t), \qquad m_0(t) = 1.$$

Fig. 4 Siméon Poisson (1781–1840).

In particular for $t = 1$, we obtain the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$$

These we have learned are the *Bell numbers*.

The OEIS `A001861` also tells us that for $t = 2$, we have *generalized Bell numbers*, and gives us the exponential generating functions. [The Bell numbers—as with many other discoveries—were known earlier to Ramanujan.]

Now an explicit computation shows that

$$t\,\frac{1+t}{2} = c_0(t)\,c_2(t) \leqslant c_1(t)^2 = t^2$$

exactly if $t \geqslant 1$. Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n\geqslant 0} c_n(t)u^n = \exp\big(t(e^u - 1)\big)\,. \tag{4}$$

(c*) (The * indicates this was unsolved.) We appeal to a then recent theorem due to Canfield. A search in 2001 on *MathSciNet* for "Bell numbers" since 1995 turned up 18 items. Canfield showed up as paper #10. Later, *Google* found the paper immediately!

Theorem 3.4 (Canfield). *If a sequence* $1, b_1, b_2, \cdots$ *is nonnegative and log-concave, then so is* $1, c_1, c_2, \cdots$ *determined by the generating function equation*

$$\sum_{n\geqslant 0} c_n u^n = \exp\left(\sum_{j\geqslant 1} b_j \frac{u^j}{j}\right).$$

Our desired application has $b_j \equiv 1$ for $j \geqslant 1$. Can the theorem be adapted to deal with eventually log-concave sequences?

References 3.5. The key references are

1. *Experimentation in Mathematics* [2, §1.11].
2. E. A. Bender and R. E. Canfield, "Log-concavity and related properties of the cycle index polynomials," *J. Combin. Theory Ser. A* **74** (1996), 57–70.
3. Solution to "Unsolved Problem 10738." posed by Radu Theodorescu in the 1999 *American Mathematical Monthly.*

4 2000: Erwin Madelung (1881–1972) & Richard Crandall (1947–2012)

Sequence 4.1 (A055745 (1/3)).

$$1, 2, 6, 10, 22, 30, 42, 58, 70, 78, 102, 130190, 210, 330, 462 \ldots$$

Answer 4.2. *Square-free numbers not of form $ab + bc + ca$ for $1 \leqslant a \leqslant b \leqslant c$ (probably the list is complete).*

`A034168` *Disjoint discriminants (one form per genus) of type 2 (doubled).*

Story 4.3. A lovely 1986 formula for $\theta_4^3(q)$ due to Andrews is

$$\theta_4^3(q) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^n} - 2 \sum_{n=1,|j|<n}^{\infty} (-1)^j q^{n^2-j^2} \frac{1-q^n}{1+q^n}. \tag{5}$$

From (5) Crandall obtained

$$\sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s} = -4 \sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(nm+mp+pn)^s} - 6\alpha^2(s). \tag{6}$$

Here $\alpha(s) = \left(1 - 2^{1-s}\right) \zeta(s)$ is the alternating zeta function.

Crandall used Andrew's formula (6) to find representations for *Madelung's constant*, $M_3(1)$, where

$$M_3(2s) := \sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s}.$$

The nicest integral consequence of (6) is

$$M_3(1) = -\frac{1}{\pi}\int_0^1\int_0^{2\pi} \frac{1+3\,r^{\sin(2\theta)-1}}{\left(1+r^{\sin(2\theta)-1}\right)\left(1+r^{\cos^2\theta}\right)\left(1+r^{\sin^2\theta}\right)}\,\mathrm{d}\theta\,\mathrm{d}r.$$

A beautiful evaluation due to Tyagi also follows:

$$M_3(1) = -\frac{1}{8} - \frac{\log 2}{4\pi} - \frac{4\pi}{3} + \frac{1}{2\sqrt{2}} + \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\pi^{3/2}\sqrt{2}} \tag{7}$$

$$-2\sum\nolimits'_{m,n,p} \frac{(-1)^{m+n+p}(m^2+n^2+p^2)^{-1/2}}{\exp\left[8\pi\sqrt{m^2+n^2+p^2}\right]-1}, \tag{8}$$

Here the "closed form" part (7)—absent the rapidly convergent series (8)—is already correct to ten places of the total: $-1.74756459463318219063621\ldots$. No fully closed form for $M_3(1)$ is known.

Although not needed for his work, the ever curious Crandall then asked me what natural numbers were not of the form

$$ab + bc + ca.$$

It was bedtime in Vancouver so I asked my ex-postdoctoral fellow Roland Girgensohn in Munich. When I woke up, Roland had used MATLAB to send all 18 solutions up to 50, 000. Also 4, 18 are the only non-square-free solutions.

I recognized the square-free numbers as exactly the *singular values* of type II (Dickson), discussed in [3, §9.2]. One more 19th solution $s > 10^{11}$ might exist but only without GRH.

4.0.1 Ignorance can be bliss

Luckily, we only looked at the OEIS *after* the paper was written. In this unusual case, the entry was based only on a comment supplied by two correspondents. Had we seen it originally, we should have told Crandall and left the subject alone. As it is, two other independent proofs appeared around the time of our paper.

4.1 The Newcastle connection

> …Born decided to investigate the simple ionic crystal-rock salt (sodium chloride)—using a ring model. He asked Lande to collaborate with him in calculating the forces between the lattice points that would determine the structure and stability of the crystal. Try as they might, the mathematical expression that Born and Lande derived contained a summation of terms that would not converge. Sitting across from Born and watching his frustration, Madelung offered a solution. His interest in the problem stemmed from his own research in Goettingen on lattice energies that, 6 years earlier, had been a catalyst for Born and von Karman's article on specific heat.

> The new mathematical method he provided for convergence allowed Born and Lande to calculate the electrostatic energy between neighboring atoms (a value now known as the Madelung constant). Their result for lattice constants of ionic solids made up of light metal halides (such as sodium and potassium chloride), and the compressibility of these crystals agreed with experimental results.

Actually, soon after, Born and Lande discovered that they had forgotten to divide by two in the compressibility analysis. This ultimately led to the abandonment of the Bohr–Sommerfeld planar model of the atom.

Max Born was singer-and-actress Olivia Newton John's maternal grandfather. Newton John's father Brinley (1914–1992) was the first Provost of the University of Newcastle. He was a fluent German speaker who interrogated Hess after his mad flight to Scotland in 1941. So Olivia has a fine academic background.

References 4.4. The key references are

1. J. M. Borwein and K-K. S. Choi, "On the representations of $xy + yz + zx$," *Experimental Mathematics*, **9** (2000), 153–158.
2. J. M. Borwein, L. Glasser, R. McPhedran, J. Wan, and J. Zucker, *Lattice Sums: Then and Now.* Encyclopedia of Mathematics and its Applications, **150**, Cambridge University Press, 2013.

5 2015: Cyril Domb (1920–2012) & Karl Pearson (1857–1936)

Sequence 5.1 (A002895 & A253095).

$$1, 4, 28, 256, 2716, 31504, 387136, 4951552\ldots$$

and

$$1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276\ldots$$

Answer 5.2. *Respectively:*

(a) Domb numbers: number of $2n$-step polygons on diamond lattice.
(b) Moments of 4-step random walk in two and four dimensions.

Story 5.3. We developed the following expression for the even moments. It is only entirely integer for $d = 2$ and $d = 4$.

In two dimensions, it counts *abelian squares*. What does it count in four space?

Theorem 5.4 (Multinomial sum for the moments). *The even moments of an n-step random walk in dimension $d = 2\nu + 2$ are given by*

$$W_n(\nu;2k)=\frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\ldots,k_n}\binom{k+n\nu}{k_1+\nu,\ldots,k_n+\nu}.$$

Story 5.5 **(Generating function for three steps in four dimensions).** For $d = 4$, so $\nu = 1$, the moments are sequence `A103370`. The OEIS also records a hypergeometric form of the generating function, as the linear combination of a hypergeometric function and its derivative, added by Mark van Hoeij. On using linear transformations of hypergeometric functions, we have more simply that

$$\sum_{k=0}^{\infty} W_3(1;2k)x^k = \frac{1}{2x^2}-\frac{1}{x}-\frac{(1-x)^2}{2x^2(1+3x)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 2\end{matrix}\,\right|\frac{27x(1-x)^2}{(1+3x)^3}\right),$$

which we are able to generalize (the planar o.g.f has the same "form")—note the Laurent polynomial.

Theorem 5.6 **(Generating function for even moments with three steps).** *For integers* $\nu \geqslant 0$ *and* $|x| < 1/9$, *we have*

$$\sum_{k=0}^{\infty} W_3(\nu;2k)x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}}\frac{(1-1/x)^{2\nu}}{1+3x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1+\nu\end{matrix}\,\right|\frac{27x(1-x)^2}{(1+3x)^3}\right) - q_\nu\left(\frac{1}{x}\right), \tag{9}$$

where $q_\nu(x)$ *is a polynomial (that is,* $q_\nu(1/x)$ *is the principal part of the hypergeometric term on the right-hand side). In particular,*

$$\sum_{k=0}^{\infty} W_3(0;2k)x^k = \frac{1}{1+3x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\,\right|\frac{27x(1-x)^2}{(1+3x)^3}\right).$$

References 5.7 The key references are

1. J. M. Borwein, A. Straub and C. Vignat, "Densities of short uniform random walks in higher dimensions," *JMAA*, to appear 2016. See http://www.carma.newcastle.edu.au/jon/dwalks.pdf.
2. J. Borwein, A. Straub, J. Wan and W. Zudilin, with an Appendix by Don Zagier, "Densities of short uniform random walks," *Canadian. J. Math.* **64** (5), (2012), 961–990.

We finish with another recent example that again illustrates Richard Crandall's nimble mind.

6 2015: Poisson (1781–1840) & Crandall (1947–2012)

Sequence 6.1 (A218147).
2,2,4,4,12,8,18,8,30,16,36,24,32,32,64,36,90,32,96,60,132,64,100,72...

Notice that this is the first non-monotonic positive sequence we have studied.

Answer 6.2. *We are told it is the:*

(a) *Conjectured degree of polynomial satisfied by*

$$m(n) := \exp(8\pi\phi_2(1/n, 1/n)$$

(as defined in (10) *below).*

(b) A079458*:* $4\,m(n)$ *is the number of Gaussian integers in a reduced system modulo* n*.*

Story 6.3. The lattice sums in question are defined by

$$\phi_2(x, y) := \frac{1}{\pi^2} \sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2 + n^2}. \tag{10}$$

Crandall conjectured while developing a deblurring algorithm—and I then proved—that when x, y are rational

$$\phi_2(x, y) = \frac{1}{\pi} \log A, \tag{11}$$

where A is *algebraic*. Again, this number-theoretic discovery plays no role in the performance of the algorithm. Both computation and proof exploited the Jacobian theta-function representation [3, §2.7]:

$$\phi_2(x, y) = \frac{1}{2\pi} \log \left| \frac{\theta_2(z, q)\theta_4(z, q)}{\theta_1(z, q)\theta_3(z, q)} \right|, \tag{12}$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y + ix)$.

In Table 1 we display the recovered polynomial for $x = y = 35$. Note how much structure the picture reveals and how far from "random" it is.

Story 6.4. Remarkably, in **2012**, Jason Kimberley (University of Newcastle) observed that the degree $m(s)$ of the minimal polynomial for $x = y = 1/s$ appears to be as follows. Set $m(2) = 1/2$. For primes p congruent to 1 mod 4, set $m(p) = \text{int}^2(p/2)$, where int denotes greatest integer, and for p congruent to 3 mod 4, set $m(p) = \text{int}\,(p/2)(\text{int}\,(p/2) + 1)$. Then with prime factorization $s = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$,

Table 1 Visualizing big data: 192-degree minimal polynomial with up to 85-digit coefficients, found by the multipair PSLQ algorithm for the case $x = y = 1/35$.

$$m(s) = 4^{r-1} \prod_{i=1}^{r} p_i^{2(e_i-1)}\, m(p_i). \tag{13}$$

- **2015**. By employing large-scale computations with precision levels as high as 64,000 digits, (13) was shown to hold numerically for all tested cases where s ranges up to 50 (except $s = 41, 43, 47, 49$, which were too costly to test) and also for $s = 60$ and 64.
- **2016**. After a Google search for 387221579866, a coefficient of the polynomial P_{11} for $s = 11$, we learned that Gordan Savin and David Quarfoot (2010) had defined a sequence of polynomials $\psi_s(x, y)$ with $\psi_0 = \psi_1 = 1$, while $\psi_2 = 2y$, $\psi_3 = 3x^4 + 6x^2 + 1$, $\psi_4 = 2y(2x^6 + 10x^4 - 10x^2 - 2)$, and

$$\psi_{2n+1} = \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3 \qquad (n \geqslant 2) \tag{14}$$
$$2y\psi_{2n} = \psi_n \left(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2\right) \qquad (n \geqslant 3). \tag{15}$$

This led Kimberly to the following:

Conjecture 6.5 (Kimberley).

(a) For each integer $s \geqslant 1$, $P_s(-x^2)$ is a prime factor of $\psi_s(x)$. In fact, it is the unique prime factor of degree $2 \times A218147(s)$.
(b) The algebraic quantity is the largest real root of P_s.
(c) (Divisibility) For integer $m, n > 1$ when $m \mid n$, then $\psi_m \mid \psi_n$.
(d) (Irreducibility) For primes of form $4n + 3$, $\psi_s(x)$ is irreducible over $Q(i)$.

- **2016**. Conjecture (a) was *confirmed* for $s = 52$ and (b) was checked up to $s = 40$. Parts (c) and (d) have been confirmed for $n \leqslant 120$.
- **2016**. In March 2016, David H. Bailey presented a summary of these results in a number theory seminar at the University of California, Berkeley. After listening to the presentation, Watson Ladd, a Berkeley graduate student, contacted Bailey and myself saying that he believed that he could prove Kimberley's conjecture (13) on the degree of the polynomials. Subsequently he sent a proof (involving theta functions, ideals, and Galois theory) and also a proof of the empirically observed fact that when s is even, the resulting polynomial is palindromic. These results were added to the published paper—see reference 2 immediately below.

References 6.6 The key references are

1. D. H. Bailey, J. M. Borwein, R. E. Crandall and I. J. Zucker, "Lattice sums arising from the Poisson equation," *Journal of Physics A*, **46** (2013) #115201 (31pp).
2. D. H. Bailey, J. M. Borwein, J. Kimberley and W. Ladd, "Computer discovery and analysis of large Poisson polynomials," *Experimental Mathematics*, 27 August 2016, http://www.tandfonline.com/doi/abs/10.1080/10586458.2016.1180565.
3. G. Savin and D. Quarfoot, "On attaching coordinates of Gaussian prime torsion points of $y^2 = x^3 + x$ to $Q(i)$," March 2010. http://www.math.utah.edu/~savin/EllipticCurvesPaper.pdf.

7 Conclusion

When I started showing the OEIS in talks 20 years ago, only a few hands would go up when asked who had heard of it. Now often half the audience will claim some familiarity. So there has been much progress but there is still work to be done to further advertise the OEIS.

- The OEIS is an amazing *instrumental* resource. I recommend everyone read Sloane's 2015 interview in *Quanta*
  - https://www.quantamagazine.org/20150806-neil-sloane-oeis-interview/

  It is now a fifty year old model both for *curation* and for *moderation* of a web resource.
  - Since Neil Sloane retired from ATT, the OEIS has moved to an edited and wiki-based resource run by the OEIS foundation.
- As with all tools, the OEIS can help (very often) as in the examples of Section 2 and Section 3, and it can hinder (much less often) as in the Example of Section 4.
- If a useful sequence occurs in your work, please contribute to the OEID as we did with the examples of Section 4 and Section 6.
  - Many of the underlying issues of technology and mathematics are discussed in [4] and more fully in: J. Monaghan, L. Troché and JMB, *Tools and Mathematics*, Springer (Mathematical Education), 2015.

We finish with another quotation.

> Algebra is generous; she often gives more than is asked of her. (Jean d'Alembert, 1717–1783).

As generous as algebra is, the OEIS usually has something more to add.

Acknowledgements The author wishes to thank all of his coauthors, living and dead, who worked on one or more of these examples.

References

1. J.M. Borwein, D.H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century* (A.K. Peters Ltd., Massachusetts, 2004). ISBN: 1-56881-136-5. Combined Interactive CD version 2006. Expanded Second Edition, 2008
2. J.M. Borwein, D.H. Bailey, R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery* (A.K. Peters Ltd., Massachusetts, 2004). ISBN 1-56881-211-6. Combined Interactive CD version 2006
3. J.M. Borwein, P.B. Borwein, *Pi and the AGM*: *A Study in Analytic Number Theory and Computational Complexity*, (Wiley, New York, 1987); reprinted 1988, 1996, Chinese edition 1995, paperback 1998

4. J.M. Borwein, K. Devlin, *The Computer as Crucible: An Introduction to Experimental Mathematics*, A.K. Peters, Massachusetts, 2008
5. G.H. Hardy, *A Mathematician's Apology* (Cambridge University Press, Cambridge, 1941)

Three-Colored Partitions and Dilated Companions of Capparelli's Identities

Kathrin Bringmann and Karl Mahlburg

In honor of Krishna Alladi, who has been a great inspiration, for the celebration of his 60th birthday

Abstract Capparelli's partition identities state that certain gap restrictions on partitions into distinct parts are equinumerous with congruential restrictions modulo 12. Subsequently, generalizations to higher moduli were proven by Alladi, Andrews, and Gordon by means of hypergeometric q-series, and by Meurman and Primc using the vertex operator algebra program of Lepowsky and Wilson. Furthermore, these generalized families arise as specializations of underlying identities for three-colored partitions. In this paper, we continue our investigation of companions to Capparelli's identities, and prove two new general identities for three-colored partitions that specialize to Jacobi theta functions and false theta functions.

Keywords False theta functions · Integer partitions · Capparelli's identities

2010 Mathematics Subject Classification 05A17 · 11P84 · 17B10

Kathrin Bringmann was supported by the Alfried Krupp Prize for Young University Teachers of the Krupp Foundation and the research leading to these results received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant agreement n. 335220 - AQSER.
Karl Mahlburg was supported by NSF Grant DMS-1201435.

K. Bringmann
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: kbringma@math.uni-koeln.de

K. Mahlburg (✉)
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
e-mail: mahlburg@math.lsu.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_10

1 Introduction and statement of results

This paper continues the study of Capparelli's partition identities [7] from the perspective of hypergeometric q-series, automorphic forms, and the combinatorial theory of integer partitions. Capparelli's identities are notable because they were the first new examples that were discovered using Lepowsky and Wilson's vertex operator algebras, which were famously introduced in [13] as a method for explicitly constructing affine Lie algebras. This framework was further developed in [14] to include Z-algebras. Our current investigation combines ideas from [1, 3, 5] in order to prove new generalized identities that relate Capparelli's work to false theta functions, Jacobi forms, and multi-colored partitions.

Capparelli's identities first arose conjecturally in [7] (see also his Ph.D. thesis [6]), where he used Lepowsky and Wilson's Z-algebra program [14] to construct the level 3 standard modules for $A_2^{(2)}$. The identities were proven shortly thereafter in a number of independent works; Andrews [3] and Andrews, Alladi, and Gordon [1] gave proofs using the theory of hypergeometric q-series, while the proofs of Tamba and Xie [20] and Capparelli himself [8] used Z-algebras. Subsequently, there has been a great deal of additional progress on the combinatorial implications of vertex-operator-theoretic techniques; for example, see [12, 15, 17] for a small sampling.

Indeed, as our present focus is on multi-parameter generalizations and/or dilated identities, we do not state Capparelli's identities as originally presented in [7], but rather a generalization due to Meurman and Primc [15], which also follows from Alladi, Andrews, and Gordon's results in [1]. For an integer partition λ, define indicator functions ψ_j such that $\psi_j(\lambda) = 1$ if j is a part of λ, and $\psi_j(\lambda) = 0$ otherwise. We frequently suppress the argument unless it is important to distinguish a particular partition. Suppose that $d \geq 3$ and $1 \leq \ell < d/2$. Loosely following the terminology and notation from [5, 7, 15], we say that a partition λ satisfies the (d, ℓ)-*dilated gap condition* if for all $j \in \mathbb{N}$,

$$\begin{aligned} \psi_{(j+1)d-\ell} + \psi_{jd} + \psi_{jd-\ell} &\leq 1, \\ \psi_{jd+\ell} + \psi_{jd} + \psi_{(j-1)d-\ell} &\leq 1, \\ \psi_{jd-\ell} + \psi_{(j-1)d+\ell} &\leq 1, \end{aligned} \tag{1.1}$$

and $\psi_n = 0$ if $n \not\equiv 0, \pm\ell \pmod{d}$. This system of inequalities is the special case $k = 1$, $s_0 = \ell$, and $s_1 = d - \ell$ of (11.2.6) in [15], which describes the partition ideals that arise from root lattices. Note that (11.2.6) of [15] is actually a system of four inequalities, but in the special case $k = 1$ it is overdetermined and reduces to the above. Capparelli's original identities correspond to $(d, \ell) = (3, 1)$, and in that case the conditions in (1.1) are equivalently characterized by requiring that the successive parts in a partition differ by at least 2, and two parts differ by 2 or 3 only if their sum is a multiple of 3.

In order to state the identities of Capparelli and Meurman–Primc, we also require enumeration functions for the partitions described above. For $\alpha, \beta \in \{0, 1\}$, let $c_{\alpha,\beta}^{d,\ell}(n)$ denote the number of partitions of n that satisfy the (d, ℓ)-gap condition with the further restriction that $\psi_\ell \leq \alpha$ and $\psi_{d-\ell} \leq \beta$. We write the corresponding generating functions as

$$\mathscr{C}_{\alpha,\beta}^{d,\ell}(q) := \sum_{n\geq 0} c_{\alpha,\beta}^{d,\ell}(n) q^n = \sum_{\substack{\lambda \text{ satisfies } (d,\ell)\text{-gap condition} \\ \lambda_\ell \leq \alpha, \ \lambda_{d-\ell} \leq \beta}} q^{|\lambda|}.$$

Here $|\lambda|$ denotes the *size* of a partition λ, which is the sum of its parts. Meurman and Primc's generalized identity is now stated as follows.

Theorem (Lemma 2 in [1]; equations (11.1.5)–(11.1.6) in [15]). For $d \geq 3$, we have

$$\mathscr{C}_{0,1}^{d,\ell}(q) = \frac{\prod_{n\geq 0}\left(1+q^{(2n+1)d-\ell}\right)\left(1+q^{(2n+1)d+\ell}\right)}{\prod_{n\geq 0}\left(1-q^{(2n+1)d}\right)}, \tag{1.2}$$

$$\mathscr{C}_{1,0}^{d,\ell}(q) = \frac{\prod_{n\geq 0}\left(1+q^{2(n+1)d-\ell}\right)\left(1+q^{2nd+\ell}\right)}{\prod_{n\geq 0}\left(1-q^{(2n+1)d}\right)}. \tag{1.3}$$

Remarks. 1. In fact, there is a version of Meurman and Primc's result that also holds for $d = 1$ or 2, although the partition combinatorics from (1.1) are no longer the correct formulation, and instead require multiple colors. This becomes clearer from the statement of our main results below.

2. The results in [1] are more general than the theorem statement, as they include additional parameters that distinguish between parts based on residue classes modulo d. Equation (1.2) follows from (5.2) in [1] by setting $q \mapsto q^d$, $a \mapsto q^{-d+\ell}$, and $b \mapsto q^{-d-\ell}$, and (1.3) from $q \mapsto q^d$, $a \mapsto q^{-\ell}$, and $b \mapsto q^{-2d+\ell}$. These parameters are discussed further in the sequel.

3. The above theorem is not the original combinatorial formulation of Capparelli's identities, but it is straightforward to show that the product expression for $\mathscr{C}_{0,1}^{3,1}(q)$ also enumerates the number of partitions of n into parts congruent to $\pm 2, \pm 3 \pmod{12}$, as in Theorem 21 A of [7]. Note that this product is also equivalently stated in the unnumbered equation following (5.2) in [1].

Our investigation in [5] was motivated by the observation that (1.2) and (1.3) are *modular* identities, in the sense that the right-hand sides are (essentially) weakly holomorphic modular forms. Furthermore, the refinements of Capparelli's results in [1] and [3] are of additional number-theoretic interest due to the presence of an additional parameter. In order to describe the refined identities, let $\nu_{d,j}(\lambda)$ be the number of parts of λ that are congruent to j modulo d, and define the generating functions

$$\mathscr{C}_{\alpha,\beta}^{d,\ell}(t;q) := \sum_{\substack{\lambda \text{ satisfies } (d,\ell)\text{-gap condition} \\ \lambda_\ell \leq \alpha, \ \lambda_{d-\ell} \leq \beta}} t^{\nu_{d,\ell}(\lambda)-\nu_{d,d-\ell}(\lambda)} q^{|\lambda|}. \tag{1.4}$$

Throughout the remainder of the paper, we adopt the standard q-factorial notation for $a \in \mathbb{C}$ and $n \in \mathbb{N}_0 \cup \{\infty\}$, namely $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. We also use the additional shorthand $(a_1,\ldots,a_r;q)_n := (a_1;q)_n \cdots (a_r;q)_n$. The *Jacobi theta function* is defined by

$$\theta(z;q) := \left(-z, -z^{-1}q, q; q\right)_\infty = \sum_{k\in\mathbb{Z}} z^k q^{\frac{k(k-1)}{2}}, \tag{1.5}$$

where the final equality follows from Jacobi's Triple Product identity ((2.2.10) in [4]). This function is essentially a holomorphic *Jacobi form*, as described in the seminal work of Eichler and Zagier [9]. Finally, define the shifted Dirichlet character $\chi_3(m) := (\frac{m+1}{3})$, and let

$$\begin{aligned} T_1(t;q) &:= \sum_{n\geq 0} \chi_3(n) t^{-n} q^{\frac{n(n+2)}{3}}, \\ T_2(t;q) &:= \sum_{n\geq 0} \chi_3(n) t^{n} q^{\frac{n(n-1)}{3}}. \end{aligned}$$

There has been a great deal of recent work illuminating the connections between "false" theta functions such as the T_j and classical automorphic forms, particularly through the theory of *quantum modular forms*, as in [11] and [21].

The main result in [5] demonstrates the role of these functions in identities related to Capparelli's results.

Theorem 1.1 **([5], Theorem 3.1).** *If* $\alpha, \beta \in \{0,1\}$*, then*

$$\begin{aligned} \mathscr{C}^{3,1}_{\alpha,\beta}(t;q) =& (\alpha+\beta-1)\Big(-q^3;q^3\Big)_\infty \theta\left(-t^2q^2;q^6\right) \\ &+ \frac{\theta\left(tq^4;q^6\right)}{\left(q^3;q^3\right)_\infty}\Big(\beta + (1-\alpha-\beta)T_1\Big(tq;q^3\Big)\Big) \\ &+ \frac{\theta\left(tq;q^6\right)}{\left(q^3;q^3\right)_\infty}\Big(\alpha + (1-\alpha-\beta)T_2\Big(tq;q^3\Big)\Big). \end{aligned} \tag{1.6}$$

Remark. This theorem includes Capparelli's original identities [7], which correspond to the two cases where $\alpha+\beta=1$. In particular, in these cases (1.6) simplifies to the products (1.2) and (1.3) with $d=3$ and $\ell=1$.

Remark. We note that in [19] Sills proved a one-parameter generalization of an "analytic counterpart" to Capparelli's identities, using Bailey chains to obtain interesting hypergeometric q-series representations for infinite products related to the case $(d,\ell)=(3,1)$ in (1.2) and (1.3).

The main automorphic result of this paper extends (1.6) to an arbitrary modulus, providing a general family of identities that imply (1.2) and (1.3).

Theorem 1.2. *For* $\alpha, \beta \in \{0, 1\}$,

$$\begin{aligned}\mathscr{C}_{\alpha,\beta}^{d,\ell}(t;q) =&(\alpha+\beta-1)\theta\left(-t^2q^{2\ell};q^{2d}\right)\left(-q^d;q^d\right)_\infty\\&+\frac{\theta\left(tq^{d+\ell};q^{2d}\right)}{(q^d;q^d)_\infty}\left(\beta+(1-\alpha-\beta)T_1\left(tq^\ell;q^d\right)\right)\\&+\frac{\theta\left(tq^{\ell};q^{2d}\right)}{(q^d;q^d)_\infty}\left(\alpha+(1-\alpha-\beta)T_2\left(tq^\ell;q^d\right)\right).\end{aligned}$$

In fact, this theorem statement is a specialization of a more general result that we prove for three-colored partitions with gap restrictions; see Theorem 4.1. The general result is inspired by Section 5 of [1], where the authors studied three-colored partitions using the "method of weighted words" to obtain multi-parameter generalizations of (1.2) and (1.3). Our generalizations are of a different shape, as we instead use the analytic theory of q-difference equations and hypergeometric q-series. Identities for three-colored partitions also arise in [16], where the basic $A_2^{(1)}$-module is constructed using vertex operator methods.

The remainder of the paper is structured as follows. Section 2 consists of a brief review of classical results from the theory of hypergeometric q-series. This is followed by proofs of combinatorial results and finite generating series for three-colored partitions in Section 3. We conclude in Section 4 by evaluating the infinite limiting cases, thereby proving Theorem 1.2.

2 Hypergeometric q-series identities

In this section, we record a number of identities that are useful in the evaluation of the generating functions that are the main topic of this paper. If $0 \le m \le n$, the *q-binomial coefficient* is denoted by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

We also need the limiting case

$$\lim_{n\to\infty}\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1}{(q;q)_m}. \tag{2.1}$$

Next, we recall two identities due to Euler, which state (see (2.2.5) and (2.2.6) in [4])

$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n}, \tag{2.2}$$

$$(x;q)_\infty = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}. \tag{2.3}$$

A related summation formula is

$$\sum_{\substack{n\geq 0\\ n \text{ even}}} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} = \frac{1}{\left(q;q^2\right)_\infty} = (-q;q)_\infty\,; \tag{2.4}$$

the first equality follows from Cauchy's identity, which is (2.2.8) in [4].

We also need a result from Ramanujan's famous "Lost Notebook", which appears as (4.1) in [2]:

$$\sum_{n\geq 0} \frac{q^n}{(-aq;q)_n(-bq;q)_n} = \left(1+a^{-1}\right)\sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}}\left(\frac{b}{a}\right)^n}{(-bq;q)_n} - \frac{a^{-1}\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}}\left(\frac{b}{a}\right)^n}{(-aq,-bq;q)_\infty}. \tag{2.5}$$

Finally, in order to derive expressions involving false theta functions, we recall a related identity of Rogers [18] (equation (3) on page 335), which states that

$$\sum_{n\geq 0} \frac{(-1)^n y^{2n} q^{\frac{n(n+1)}{2}}}{(yq;q)_n} = \sum_{n\geq 0} (-1)^n y^{3n} q^{\frac{n(3n+1)}{2}}\left(1-y^2q^{2n+1}\right).$$

In fact, we need a one-parameter generalization of Rogers' identity, which follows from Fine's systematic study of hypergeometric functions in [10].

Lemma 2.1. *We have*

$$\sum_{n\geq 0} \frac{(-1)^n (bx)^n q^{\frac{n(n+1)}{2}}}{(bq;q)_n} = \sum_{n\geq 0} \frac{(xq;q)_n}{(bq;q)_n}\Big(-xb^2\Big)^n q^{\frac{n(3n+1)}{2}}\left(1-bxq^{2n+1}\right).$$

Proof. We use Fine's notation for the basic hypergeometric series, namely

$$F(a,b;t) := \sum_{n\geq 0} \frac{(aq;q)_n t^n}{(bq;q)_n}.$$

The left-hand side of the lemma statement may be expressed as a limit of Fine's function, since

$$\sum_{n\geq 0} \frac{(-1)^n (bx)^n q^{\frac{n(n+1)}{2}}}{(bq;q)_n} = \lim_{a\to\infty} F\left(ab, b; \frac{x}{a}\right).$$

By (6.3) of [10], this expression transforms to

$$\lim_{a\to\infty} F\left(ab, b; \frac{x}{a}\right) = \lim_{a\to\infty} \frac{1-b}{1-\frac{x}{a}} F\left(x, \frac{x}{a}; b\right) = (1-b)F\left(x, 0; b\right). \tag{2.6}$$

By the Rogers-Fine identity (see (14.1) of [10]), (2.6) becomes

$$(1-b)\lim_{w\to 0} F(x,w;b) = \sum_{n\geq 0} \frac{(xq;q)_n}{(bq;q)_n}\left(1-xbq^{2n+1}\right) b^n q^{n^2} \lim_{w\to 0}\left(\frac{xbq}{w};q\right)_n w^n.$$

Evaluating the limit completes the proof.

3 Three-colored partitions and finite recurrences

In this section, we combine ideas from [1, 3, 5] and introduce certain three-colored partitions with gap restrictions that are related to generalizations of Capparelli's identities. The combinatorics of the partition colorings are inspired by Sections 5 and 6 of [1], where the method of weighted words was used in order to evaluate the corresponding generating functions. However, we instead use techniques from [3], which were further adapted in [5] in order to find hypergeometric q-series solutions to the appropriate q-difference equations.

3.1 Colored partitions

For an integer partition, we write the parts of a partition in nonincreasing order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. For the remainder of the paper we consider three-colored partitions into distinct parts with gap restrictions. In particular, if j is a part of a partition λ, then it is given one of three colors, a, b, or c; a part of size j and color k is denoted by j_k. When writing the parts of a partition they are ordered by both size and color, according to the sequence

$$1_a \prec 1_b \prec 1_c \prec 2_a \prec 2_b \prec 2_c \prec \ldots. \tag{3.1}$$

Note that this is slightly different than the ordering in Section 5 of [1]; to compare the two, we have effectively shifted all of the parts with color b by 1.

We say that a three-colored partition satisfies the Capparelli *gap conditions* if it is in the subset

$$\mathscr{R} := \Big\{\lambda \vdash n \ \mid \ \text{distinct parts } \lambda_j, \ \text{with color } k_{\lambda_j} \in \{a, b, c\},$$
$$\text{and for consecutive integer parts } (j+1), \, j \in \lambda, \ k_{j+1}k_j \neq aa, bb, ac, \ \text{or } bc\Big\}.$$

In other words, a 3-colored partition into distinct parts λ is in $\mathscr{R}$ if

$$\lambda_r - \lambda_{r+1} \geq A\Big(k_{\lambda_{r+1}}, k_{\lambda_r}\Big), \tag{3.2}$$

where A is the following matrix (indexed in order by rows and columns; note that λ_{r+1} is smaller than λ_r).

	A	a	b	c
		k_{λ_r}		
	a	2	1	1
$k_{\lambda_{r+1}}$	b	1	2	1
	c	2	2	1

For example, the second row implies that if $j_b \in \lambda$, then the next largest part cannot be j_c or $(j+1)_b$, but any of $(j+1)_a$, $(j+1)_c$, $(j+2)_a$, $(j+2)_b$, or larger, are allowed.

3.2 Finite generating functions

Extending the notation from Section 1 to include colored parts, we define indicator functions

$$\psi_{m_k}(\lambda) := \begin{cases} 1 & \text{if } m_k \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\nu_k(\lambda)$ count the number of parts of λ with color k; we see below that the $\nu_{d,\ell}$ from Section 1 (see (1.4)) corresponds to certain specializations.

For $\alpha, \beta \in \{0, 1\}$, $M \geq 1$, and $k \in \{a, b, c\}$, define the bounded generating functions

$$\begin{aligned} F(M_k) &= F_{\alpha,\beta}\left(M_k; A, B; q\right) \\ &:= \sum_{\substack{\lambda \in \mathscr{R} \\ \lambda_j \leq M_k \text{ for all } j}} \left(1 - (1-\alpha)\psi_{1_a}(\lambda)\right)\left(1 - (1-\beta)\psi_{1_b}(\lambda)\right) A^{\nu_a(\lambda)} B^{\nu_b(\lambda)} q^{|\lambda|}. \end{aligned} \tag{3.3}$$

Note that the indicator functions have the effect of limiting the number of occurrences of 1_a to at most α, and the occurrences of 1_b to at most β. It also important to note that there is no variable associated with the color c; to our knowledge, none of the known results related to Capparelli's identities generalize to this degree.

Following Sections 4 of [3] and [1], by conditioning on the largest part, we easily find the recurrences

$$F(n_b) = F(n_a) + Bq^n F((n-1)_a), \tag{3.4}$$
$$F(n_c) = F(n_b) + q^n F((n-1)_c), \tag{3.5}$$
$$F((n+1)_a) = F(n_c) + Aq^{n+1}\Big(F((n-1)_c) + Bq^n F((n-1)_a)\Big). \tag{3.6}$$

For example, on the right-side of (3.4), the first term corresponds to the case that n_b does not occur, so the largest part is at most n_a, and the second term the case that n_b does occur, so that the next part is at most $(n-1)_a$.

Moreover, by writing down the first several partitions in $\mathscr{R}$, we directly calculate the initial values

$$\begin{aligned} F(1_a) &= 1 + \alpha Aq, \\ F(1_b) &= 1 + \alpha Aq + \beta Bq, \\ F(1_c) &= 1 + \alpha Aq + \beta Bq + q, \\ F(2_a) &= 1 + \alpha Aq + \beta Bq + q + Aq^2(1 + \beta Bq). \end{aligned} \tag{3.7}$$

Furthermore, it is convenient to have a value for $F(0_a)$, which can be obtained in a consistent manner by plugging $n = 1$ in to (3.4) and working in reverse. In particular, combined with (3.7), this implies that $F(0_a) = \beta$.

We now manipulate the system (3.4)–(3.6) in order to obtain a recurrence involving only one color. Isolating $F(n_b)$ in (3.4) and (3.5) yields

$$F(n_c) - q^n F((n-1)_c) = F(n_a) + Bq^n F((n-1)_a), \tag{3.8}$$

and rearranging the terms of (3.6) gives

$$F(n_c) + Aq^{n+1} F((n-1)_c) = F((n+1)_a) - ABq^{2n+1} F((n-1)_a). \tag{3.9}$$

Taking Aq times (3.8) and adding it to (3.9) then results in

$$\begin{aligned} &(Aq+1)F(n_c) \\ &\quad = AqF(n_a) + ABq^{n+1}F((n-1)_a) + F((n+1)_a) - ABq^{2n+1}F((n-1)_a). \end{aligned} \tag{3.10}$$

Similarly, subtracting (3.8) from (3.9) gives

$$\begin{aligned} &q^n(Aq+1)F((n-1)_c) \\ &= F((n+1)_a) - ABq^{2n+1}F((n-1)_a) - F(n_a) - Bq^n F((n-1)_a). \end{aligned} \tag{3.11}$$

We now shift $n \mapsto n-1$ in (3.10) and plug this in to (3.11), which yields an equality involving only the color a, namely

$$q^n\Big(AqF\left((n-1)_a\right)+F\left(n_a\right)+ABq^n\Big(1-q^{n-1}\Big)F\left((n-2)_a\right)\Big)$$
$$=F\left((n+1)_a\right)-F\left(n_a\right)-Bq^n\Big(1+Aq^{n+1}\Big)F\left((n-1)_a\right).$$

Regrouping terms, we finally have the single recurrence

$$F\left((n+1)_a\right)=\left(1+q^n\right)F\left(n_a\right)+\Big(Aq^{n+1}+Bq^n+ABq^{2n+1}\Big)F\left((n-1)_a\right)$$
$$+ABq^{2n}\Big(1-q^{n-1}\Big)F\left((n-2)_a\right). \tag{3.12}$$

By introducing an auxiliary variable and constructing a generating function for the $F(n_a)$, the problem can now be translated to a q-difference equation, which is then amenable to techniques from the theory of hypergeometric q-series. The details are carried out in the sequel.

3.3 Hypergeometric q-series solution

We close this section by solving the recurrence (3.12), thereby finding a hypergeometric q-series expression for the three-colored partitions satisfying Capparelli's gap condition. We begin by setting

$$\gamma_n := \frac{F\left(n_a\right)}{(q;q)_n}, \tag{3.13}$$

and then shift $n \mapsto n-1$ in (3.12), which implies that, for $n \geq 3$,

$$\gamma_n=\frac{1+q^{n-1}}{1-q^n}\gamma_{n-1}+\frac{Aq^n+Bq^{n-1}+ABq^{2n-1}}{\left(1-q^{n-1}\right)\left(1-q^n\right)}\gamma_{n-2}+\frac{ABq^{2n-2}}{\left(1-q^{n-1}\right)\left(1-q^n\right)}\gamma_{n-3}. \tag{3.14}$$

Note that the initial conditions are given by

$$\gamma_0=\beta, \quad \gamma_1=\frac{1+\alpha Aq}{1-q}, \quad \gamma_2=\frac{1+\alpha Aq+\beta Bq+q+Aq^2\left(1+\beta Bq\right)}{\left(1-q\right)\left(1-q^2\right)}. \tag{3.15}$$

We then further rewrite (3.14) as

$$\left(1-q^{n-1}\right)\left(1-q^{n}\right)\gamma_n = \left(1-q^{2n-2}\right)\gamma_{n-1}+\left(Aq^{n}+Bq^{n-1}+ABq^{2n-1}\right)\gamma_{n-2}+ABq^{2n-2}\gamma_{n-3}, \tag{3.16}$$

from which we next derive a q-difference equation.

In order to convert the above recurrence to a series relation, for $m \in \mathbb{N}_0$ we define the (shifted) generating functions

$$G^{(m)}(z) := \sum_{n\geq m} \gamma_n z^n, \tag{3.17}$$

and we also set $G := G^{(0)}$. Multiplying (3.16) by z^n and summing over $n \geq 3$, we obtain

$$G^{(3)}(z)-G^{(3)}(zq)-q^{-1}G^{(3)}(zq)+q^{-1}G^{(3)}\left(zq^2\right) = z\left(G^{(2)}(z)-G^{(2)}\left(zq^2\right)\right)+z^2\left(\left(Aq^2+Bq\right)G^{(1)}(zq)+ABq^3G^{(1)}\left(zq^2\right)\right) + z^3ABq^4G\left(zq^2\right). \tag{3.18}$$

A short calculation shows that after adding back in the boundary terms and plugging in (3.15), (3.18) simplifies to

$$(1-z)G(z) = \left(1+q^{-1}+z^2\left(Aq^2+Bq\right)\right)G(zq) - q^{-1}(1+zq)\left(1-z^2q^4AB\right)G\left(zq^2\right)+z^2q^2A(1-\alpha-\beta).$$

The order of this q-difference equation is reduced by re-normalizing the generating function, so we set

$$H(z) := \frac{G(z)}{(-z;q)_\infty}. \tag{3.19}$$

Then we have

$$\left(1-z^2\right)H(z) = \left(1+q^{-1}+Aq^2z^2+Bqz^2\right)H(zq)-q^{-1}\left(1-z^2q^4AB\right)H\left(zq^2\right) + A\left(1-\alpha-\beta\right)\sum_{n\geq 0}\frac{(-1)^n z^{n+2}q^{n+2}}{(q;q)_n},$$

where the final term follows from (2.2).

We now find a hypergeometric solution to this q-difference equation by expanding the series as

$$H(z) =: \sum_{k\geq 0} \delta_k z^k. \tag{3.20}$$

Then for $k \geq 2$ we have

$$\delta_k - \delta_{k-2} = q^k\left(1+q^{-1}\right)\delta_k + q^{k-1}(Aq+B)\delta_{k-2} - q^{2k-1}\delta_k + q^{2k-1}AB\delta_{k-2} + \frac{A(1-\alpha-\beta)(-1)^k q^k}{(q;q)_{k-2}},$$

which can be rewritten as the (nonhomogeneous) recurrence

$$\delta_k = \frac{\left(1+Aq^k\right)\left(1+Bq^{k-1}\right)}{\left(1-q^{k-1}\right)\left(1-q^k\right)}\delta_{k-2} + \frac{A(1-\alpha-\beta)(-1)^k q^k}{(q;q)_k}. \tag{3.21}$$

The initial conditions are found by recalling (3.15), (3.17), and (3.20), which imply that

$$\delta_0 = \gamma_0 = \beta, \quad \delta_1 = \gamma_1 - \frac{\gamma_0}{1-q} = \frac{1-\beta+\alpha Aq}{1-q}.$$

A short proof by induction using (3.21) then gives the solutions

$$\delta_{2k} = \frac{\left(-Aq^2;q^2\right)_k\left(-Bq;q^2\right)_k}{(q;q)_{2k}}\left(A(1-\alpha-\beta)\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell\left(-Bq;q^2\right)_\ell}+\beta\right) \tag{3.22}$$

for the even indices, and similarly, for the odd indices,

$$\delta_{2k+1} = \frac{\left(-Aq^3;q^2\right)_k\left(-Bq^2;q^2\right)_k}{(q;q)_{2k+1}} \times\left(-A(1-\alpha-\beta)q\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell\left(-Bq^2;q^2\right)_\ell}+1-\beta+\alpha Aq\right). \tag{3.23}$$

It is now possible to combine (3.13), (3.17), (3.19), (3.20), (3.22), and (3.23) in order to write down closed-form expressions for $F(n_a)$ (cf. Lemma 3.2 in [5]). However, our primary interest is on the limiting case $n \to \infty$, as this then implies Theorem 1.2. We evaluate this limit in the next section.

4 Infinite series evaluation and the proof of Theorem 1.2

In this section, we take the infinite limits of the bounded expressions from the previous section in order to find the full generating functions for three-colored partitions satisfying Capparelli's gap condition. We denote the limiting functions by

$$\begin{aligned} F_{\alpha,\beta}(A,B;q) &:= \lim_{n\to\infty} F_{\alpha,\beta}(n_a;A,B;q), \\ F_{\alpha,\beta}(t;q) &:= F_{\alpha,\beta}\left(tq^{-1},t^{-1};q\right). \end{aligned} \tag{4.1}$$

The dilated identities in Theorem 1.2 are then immediate consequences of the following result for colored partitions.

Theorem 4.1. *For $\alpha, \beta \in \{0, 1\}$, we have*

$$\begin{aligned} F_{\alpha,\beta}(t;q) = &\frac{\theta\left(tq;q^2\right)}{(q;q)_\infty}\left(\beta+(1-\alpha-\beta)T_1(t;q)\right) \\ &+\frac{\theta\left(t;q^2\right)}{(q;q)_\infty}\left(\alpha+(1-\alpha-\beta)T_2(t;q)\right)-(1-\alpha-\beta)\theta\Big(-t^2;q^2\Big)(-q;q)_\infty. \end{aligned}$$

Proof of Theorem 1.2. The theorem statement follows from letting $q \mapsto q^d$ and $t \mapsto tq^\ell$ in Theorem 4.1. Recalling (1.4), (3.3), and (4.1), it is a short combinatorial exercise to show that under this specialization the colored gap condition (3.2) is equivalent to (1.1).

Remark. This specialization is slightly different from the one in [1], which we previously described in Remark 2 following (1.3). This is again due to the fact that the coloring we specify in (3.1) differs from that of Andrews, Alladi, and Gordon.

4.1 Limits separated by parity

The calculations are most convenient if the odd and even indices for the δ_n are separated. For $j \in \{0, 1\}$ we therefore set

$$\begin{aligned} H_j(z) &:= \sum_{n\equiv j \pmod 2} \delta_k z^k, \\ G_j(z) &:= (-z)_\infty H_j(z). \end{aligned}$$

Then, by (2.3) and (3.22),

$$\begin{aligned}G_0(z) &= \sum_{m\geq 0}\frac{z^m q^{\frac{m(m-1)}{2}}}{(q)_m}\sum_{k\geq 0}\delta_{2k}z^{2k}\\ &= \sum_{n\geq 0}\frac{z^n}{(q)_n}\sum_{m+2k=n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{\frac{m(m-1)}{2}}\left(-Aq^2;q^2\right)_k\left(-Bq;q^2\right)_k\\ &\quad\times\left(\beta+(1-\alpha-\beta)A\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell\left(-Bq;q^2\right)_\ell}\right),\end{aligned}$$

and similarly by (3.23),

$$\begin{aligned}G_1(z) = \sum_{n\geq 0}\frac{z^n}{(q)_n}\sum_{m+2k+1=n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{\frac{m(m-1)}{2}}\left(-Aq^3;q^2\right)_k\left(-Bq^2;q^2\right)_k\\ \times\left(1-\beta+\alpha Aq-q(1-\alpha-\beta)A\sum_{\ell=1}^{k}\frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell\left(-Bq^2;q^2\right)_\ell}\right).\end{aligned}$$

Written in the above form we can finally isolate the generating functions from Section 3,

$$G_j(z) =: \sum_{n\geq 0}\frac{z^n}{(q)_n}C_{j,n}.$$

Then $F(n_a) = C_{0,n} + C_{1,n}$, and thus

$$F_{\alpha,\beta} = \lim_{n\to\infty}\left(C_{0,n}+C_{1,n}\right). \tag{4.2}$$

We begin with the even case, using (2.1) to calculate

$$\begin{aligned}C_0 := \lim_{\substack{n\to\infty\\ n \text{ even}}} C_{0,n} &= \lim_{\substack{n\to\infty\\ n \text{ even}}}\sum_{\substack{m=0\\ m\text{ even}}}^{n} q^{\frac{m(m-1)}{2}}\begin{bmatrix}n\\m\end{bmatrix}_q\left(-Aq^2;q^2\right)_{\frac{n-m}{2}}\left(-Bq;q^2\right)_{\frac{n-m}{2}}\\ &\quad\times\left(\beta+(1-\alpha-\beta)A\sum_{\ell=1}^{\frac{n-m}{2}}\frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell\left(-Bq;q^2\right)_\ell}\right)\\ &= \sum_{\substack{m\geq 0\\ m\text{ even}}}\frac{q^{\frac{m(m-1)}{2}}}{(q;q)_m}\left(-Aq^2;q^2\right)_\infty\left(-Bq;q^2\right)_\infty \qquad (4.3)\\ &\quad\times\left(\beta+(1-\alpha-\beta)A\sum_{\ell\geq 1}\frac{q^{2\ell}}{\left(-Aq^2;q^2\right)_\ell\left(-Bq;q^2\right)_\ell}\right).\end{aligned}$$

The sum on m evaluates to $(-q;q)_\infty$ by (2.4), so it remains to compute the sum on ℓ. For this, we first shift the summation index to obtain

$$\frac{q^2}{\left(1+Aq^2\right)(1+Bq)} \sum_{\ell \geq 0} \frac{q^{2\ell}}{\left(-Aq^4;q^2\right)_\ell \left(-Bq^3;q^2\right)_\ell}.$$

Then we apply (2.5) with $q \mapsto q^2$, $a = Aq^2$, and $b = Bq$, yielding the equivalent expression

$$\frac{q^2}{\left(1+Aq^2\right)(1+Bq)} \left(\left(1 + A^{-1}q^{-2}\right) \sum_{\ell \geq 0} \frac{(-1)^\ell q^{\ell(\ell+1)} \left(\frac{B}{Aq}\right)^\ell}{\left(-Bq^3;q^2\right)_\ell} \right.$$
$$\left. - A^{-1}q^{-2} \frac{\sum_{\ell \geq 0} (-1)^\ell q^{\ell(\ell+1)} \left(\frac{B}{Aq}\right)^\ell}{\left(-Aq^4, -Bq^3; q^2\right)_\infty} \right)$$
$$= A^{-1} \sum_{\ell \geq 0} \frac{(-1)^\ell q^{\ell^2} \left(\frac{B}{A}\right)^\ell}{\left(-Bq;q^2\right)_{\ell+1}} - \frac{A^{-1} \sum_{\ell \geq 0} (-1)^\ell q^{\ell^2} \left(\frac{B}{A}\right)^\ell}{\left(-Aq^2, -Bq; q^2\right)_\infty}. \tag{4.4}$$

Shifting the summation index of the first sum in (4.4), we have

$$\sum_{\ell \geq 0} \frac{(-1)^\ell q^{\ell^2} \left(\frac{B}{A}\right)^\ell}{\left(-Bq;q^2\right)_{\ell+1}} = -\sum_{\ell \geq 1} \frac{(-1)^\ell q^{\ell^2 - 2\ell + 1} \left(\frac{B}{A}\right)^{\ell-1}}{\left(-Bq;q^2\right)_\ell}.$$

Applying Lemma 2.1 with $q \mapsto q^2$, $b = -Bq^{-1}$, and $x = -A^{-1}q^{-2}$, the above expression equals

$$\frac{Aq}{B} - \frac{Aq}{B} \sum_{\ell \geq 0} \frac{\left(-A^{-1};q^2\right)_\ell}{\left(-Bq;q^2\right)_\ell} \left(\frac{B^2}{A}\right)^\ell q^{3\ell(\ell-1)} \left(1 - \frac{B}{A} q^{4\ell-1}\right). \tag{4.5}$$

Combining (4.4)–(4.5) and plugging in to (4.3), we, therefore, have

$$C_0 = (-q;q)_\infty \left(-Aq^2;q^2\right)_\infty \left(-Bq;q^2\right)_\infty \tag{4.6}$$
$$\times \left[\beta + (1-\alpha-\beta) \left(\frac{Aq}{B} - \frac{Aq}{B} \sum_{\ell \geq 0} \frac{\left(-A^{-1};q^2\right)_\ell}{\left(-Bq;q^2\right)_\ell} \left(\frac{B^2}{A}\right)^\ell q^{3\ell(\ell-1)} \left(1 - \frac{B}{A} q^{4\ell-1}\right)\right)\right]$$
$$- (1-\alpha-\beta)(-q;q)_\infty \sum_{\ell \geq 0} (-1)^\ell \left(\frac{B}{A}\right)^\ell q^{\ell^2}.$$

We now calculate the contribution from F_1, which proceeds similarly to the above. Again using (2.1) and (2.4), we have

$$C_1 := \lim_{\substack{n\to\infty \\ n \text{ odd}}} C_{1,n} = \lim_{\substack{n\to\infty \\ n \text{ odd}}} \sum_{\substack{m=0 \\ m \text{ even}}}^{n-1} q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(-Aq^3;q^2\right)_{\frac{n-m-1}{2}} \left(-Bq^2;q^2\right)_{\frac{n-m-1}{2}}$$
$$\times \left(1-\beta+\alpha Aq - q(1-\alpha-\beta)A \sum_{\ell=1}^{\frac{n-m-1}{2}} \frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell \left(-Bq^2;q^2\right)_\ell}\right)$$
$$= (-q;q)_\infty \left(-Aq^3;q^2\right)_\infty \left(-Bq^2;q^2\right)_\infty \tag{4.7}$$
$$\times \left(1-\beta+\alpha Aq - q(1-\alpha-\beta)A \sum_{\ell\geq 1} \frac{q^{2\ell}}{\left(-Aq^3;q^2\right)_\ell \left(-Bq^2;q^2\right)_\ell}\right).$$

By Ramanujan's identity (2.5) with $q \mapsto q^2$, $b = Aq$, and $a = B$, the sum on ℓ becomes

$$-1 + \left(1+B^{-1}\right) \sum_{\ell\geq 0} \frac{(-1)^\ell q^{\ell(\ell+1)} \left(\frac{Aq}{B}\right)^\ell}{\left(-Aq^3;q^2\right)_\ell} - B^{-1} \frac{\sum_{\ell\geq 0} (-1)^\ell q^{\ell(\ell+1)} \left(\frac{Aq}{B}\right)^\ell}{\left(-Aq^3,-Bq^2;q^2\right)_\infty}. \tag{4.8}$$

Lemma 2.1 with $q \mapsto q^2$, $b = -Aq$, and $x = -B^{-1}$ then implies that the first sum from (4.8) equals

$$\sum_{\ell\geq 0} \frac{(-1)^\ell q^{\ell^2+2\ell} \left(\frac{A}{B}\right)^\ell}{\left(-Aq^3;q^2\right)_\ell} = \sum_{\ell\geq 0} \frac{\left(-B^{-1}q^2;q^2\right)_\ell}{\left(-Aq^3;q^2\right)_\ell} \left(\frac{A^2}{B}\right)^\ell q^{3\ell(\ell+1)} \left(1-\frac{A}{B}q^{4\ell+3}\right). \tag{4.9}$$

Plugging in (4.8) and (4.9) to (4.7), we obtain

$$C_1 = (-q;q)_\infty \left(-Aq^3;q^2\right)_\infty \left(-Bq^2;q^2\right)_\infty \Bigg[1-\beta+\alpha Aq - qA(1-\alpha-\beta)$$
$$\left(-1+\left(1+B^{-1}\right)\sum_{\ell\geq 0} \frac{\left(-B^{-1}q^2;q^2\right)_\ell}{\left(-Aq^3;q^2\right)_\ell} \left(\frac{A^2}{B}\right)^\ell q^{3\ell(\ell+1)} \left(1-\frac{A}{B}q^{4\ell+3}\right)\right)\Bigg]$$
$$+ (1-\alpha-\beta)(-q;q)_\infty AB^{-1} \sum_{\ell\geq 0} (-1)^\ell q^{\ell(\ell+2)+1} \left(\frac{A}{B}\right)^\ell. \tag{4.10}$$

For a final simplification, note that the last term in (4.10) can be rewritten, since

$$-AB^{-1} \sum_{\ell\geq 0} (-1)^\ell q^{\ell(\ell+2)+1} \left(\frac{A}{B}\right)^\ell = \sum_{\ell\leq -1} (-1)^\ell q^{\ell^2} \left(\frac{B}{A}\right)^\ell.$$

This combines with the last term from (4.6) to give the single summation

$$-(1-\alpha-\beta)(-q;q)_\infty \sum_{\ell\in\mathbb{Z}}(-1)^\ell \left(\frac{B}{A}\right)^\ell q^{\ell^2}. \tag{4.11}$$

Our calculation is now complete, as (4.2), (4.6), and (4.10) give a hypergeometric formula for $F_{\alpha,\beta}$.

4.2 The modular case and the proof of Theorem 4.1

Many of the expressions from above simplify quite drastically under the specialization $A = tq^{-1}$ and $B = t^{-1}$, and in this case we can further identify components in terms of theta functions. Adding (4.6) and (4.10) (and recalling (4.11)), and writing $\omega := 1-\alpha-\beta$ to save space, we have

$$\begin{aligned}\frac{C_0+C_1}{(-q;q)_\infty} =& \left(-tq,-t^{-1}q;q^2\right)_\infty \Bigg[\beta+\omega\, t^2\left(1-\sum_{\ell\ge 0} t^{-3\ell}q^{3\ell^2-2\ell}\left(1-t^{-2}q^{4\ell}\right)\right)\\ &+\left(-t,-t^{-1}q^2;q^2\right)_\infty\left(1-\beta-t\omega\sum_{\ell\ge 0}t^{3\ell}q^{3\ell^2+\ell}\left(1-t^2q^{4\ell+2}\right)\right)\Bigg]\\ &-\omega\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{\ell^2+\ell}t^{-2\ell}.\end{aligned} \tag{4.12}$$

To obtain the theorem statement, we first apply the following cases of (1.5):

$$\begin{aligned}\left(-tq,-t^{-1}q;q^2\right)_\infty &= \frac{\theta\left(tq;q^2\right)}{\left(q^2;q^2\right)_\infty},\\ \left(-t,-t^{-1}q^2;q^2\right)_\infty &= \frac{\theta\left(t;q^2\right)}{(1+t)\left(q^2;q^2\right)_\infty},\\ \sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{\ell^2+\ell}t^{-2\ell} &= \theta\left(-t^2;q^2\right).\end{aligned}$$

We further simplify the first sum in (4.12) as

$$\begin{aligned}&t^2\left(1-\sum_{\ell\ge 0}t^{-3\ell}q^{3\ell^2-2\ell}\left(1-t^{-2}q^{4\ell}\right)\right)\\ &\quad=\sum_{\ell\ge 0}\left(t^{-3\ell}q^{3\ell^2+2\ell}-t^{-3\ell-1}q^{(\ell+1)(3\ell+1)}\right)=T_1(t;q).\end{aligned}$$

Moreover, the fact that $1-\beta=\alpha+\omega$ implies that all of the terms in the inner parentheses from the second and third lines of (4.12) combine and simplify as follows:

$$\alpha + \omega - \omega\, t \sum_{\ell \geq 0} t^{3\ell} q^{3\ell^2+\ell} \left(1 - t^2 q^{4\ell+2}\right) = \alpha + (1 - \alpha - \beta) T_2(t; q).$$

Plugging back in to (4.12) completes the proof of Theorem 4.1.

References

1. K. Alladi, G. Andrews, B. Gordon, Refinements and generalizations of Capparelli's conjecture on partitions. J. Algebra **174**, 636–658 (1995)
2. G. Andrews, An introduction to Ramanujan's "lost" notebook. Am. Math. Monthly **86**, 89–108 (1979)
3. G. Andrews, *Schur's theorem, Capparelli's conjecture and q-trinomial coefficients*, The Rademacher legacy to mathematics, Contemp. Math. **166** (American Mathematical Society, Providence, RI, 1994), pp. 141–154
4. G. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, 1998)
5. K. Bringmann, K. Mahlburg, False theta functions and companions to Capparelli's identities. Adv. Math. **278**, 121–136 (2015)
6. S. Capparelli, Vertex operator relations for affine algebras and combinatorial identities, Ph.D. Thesis, Rutgers University, 1988
7. S. Capparelli, On some representations of twisted affine Lie algebras and combinatorial identities. J. Algebra **154**, 335–355 (1993)
8. S. Capparelli, A construction of the level 3 modules for the affine Lie algebra $A_2^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type. Trans. Am. Math. Soc. **348**, 481–501 (1996)
9. M. Eichler, D. Zagier, *The theory of Jacobi forms*, Progress in Math. 55 (Birkhäuser Boston, MA, 1985)
10. N. Fine, *Basic hypergeometric series and applications*, Math. Surveys and Monographs 27 (American Mathematical Society, Providence, RI, 1988)
11. A. Folsom, K. Ono, Rob Rhoades, Mock theta functions and quantum modular forms. Forum Math. Pi **1**, e2 (2013)
12. I. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster, Pure and Applied Mathematics 134* (Academic Press, Boston, MA, 1988)
13. J. Lepowsky, R. Wilson, Construction of the affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. **62**, 43–53 (1978)
14. J. Lepowsky, R. Wilson, A new family of algebras underlying the Rogers-Ramanujan identities and generalizations. Proc. Nat. Acad. Sci. **78**, 7254–7258 (1981)
15. A. Meurman, M. Primc, Annihilating fields of standard modules of $\mathfrak{sl}(2, C)^{\sim}$ and combinatorial identities. Mem. Amer. Math. Soc. **137**(652), viii+89 (1999)
16. A. Meurman, M. Primc, A basis of the basic $\mathfrak{sl}(3, C)^{\sim}$-module. Commun. Contemp. Math. **3**, 593–614 (2001)
17. M. Primc, Vertex algebras and combinatorial identities. Acta Appl. Math. **73**, 221–238 (2002)
18. L. Rogers, On two theorems of combinatory analysis and some allied identities. Proc. Lond. Math. Soc. **16**, 315–336 (1917)
19. A. Sills, On series expansions of Capparelli's infinite product. Adv. Appl. Math. **33**, 397–408 (2004)
20. M. Tamba, C. Xie, Level three standard modules for $A_2^{(2)}$ and combinatorial identities. J. Pure Appl. Algebra **105**, 53–92 (1995)
21. D. Zagier, *Quantum modular forms*, Clay Mathematics Proceedings 11 (AMS and Clay Mathematics Institute, 2010), pp. 659–675

Nearly Equal Distributions of the Rank and the Crank of Partitions

William Y. C. Chen, Kathy Q. Ji and Wenston J. T. Zang

Dedicated to Professor Krishna Alladi on the occasion of his sixtieth birthday

Abstract Let $N(\leq m, n)$ denote the number of partitions of n with rank not greater than m, and let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Bringmann and Mahlburg observed that $N(\leq m, n) \leq M(\leq m, n) \leq N(\leq m+1, n)$ for $m < 0$ and $1 \leq n \leq 100$. They also pointed out that these inequalities can be restated as the existence of a reordering τ_n on the set of partitions of n such that $|\operatorname{crank}(\lambda)| - |\operatorname{rank}(\tau_n(\lambda))| = 0$ or 1 for all partitions λ of n, that is, the rank and the crank are nearly equal distributions over partitions of n. In the study of the spt-function, Andrews, Dyson, and Rhoades proposed a conjecture on the unimodality of the spt-crank, and they showed that it is equivalent to the inequality $N(\leq m, n) \leq M(\leq m, n)$ for $m < 0$ and $n \geq 1$. We proved this conjecture by combinatorial arguments. In this paper, we show that the inequality $M(\leq m, n) \leq N(\leq m+1, n)$ is true for $m < 0$ and $n \geq 1$. Furthermore, we provide a description of such a reordering τ_n and show that it leads to nearly equal distributions of the rank and the crank. Using this reordering, we give an interpretation of the function $\operatorname{ospt}(n)$ defined by Andrews, Chan, and Kim, which yields an upper bound for $\operatorname{ospt}(n)$ due to Chan and Mao.

W. Y. C. Chen · K. Q. Ji (✉)
Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China
e-mail: kathyji@tju.edu.cn

W. Y. C. Chen
e-mail: chenyc@tju.edu.cn

W. J. T. Zang
Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Heilongjiang 150001, P. R. China
e-mail: zang@hit.edu.cn

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_11

Keywords Partition · Rank · Crank · Andrews' spt-function · ospt-Function

2010 Mathematics Subject Classification 11P81 · 05A17 · 05A20

1 Introduction

The objective of this paper is to confirm an observation of Bringmann and Mahlburg [9] on the nearly equal distributions of the rank and the crank of partitions. Recall that the rank of a partition was introduced by Dyson [12] as the largest part minus the number of parts. The crank of a partition was defined by Andrews and Garvan [5] as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones.

Let m be an integer. For $n \geq 1$, let $N(m, n)$ denote the number of partitions of n with rank m, and for $n > 1$, let $M(m, n)$ denote the number of partitions of n with crank m. For $n = 1$, set

$$M(0, 1) = -1,\ M(1, 1) = M(-1, 1) = 1,$$

and for $n = 1$ and $m \neq -1, 0, 1$, set

$$M(m, 1) = 0.$$

Define the rank and the crank cumulation functions by

$$N(\leq m, n) = \sum_{r \leq m} N(r, n), \tag{1.1}$$

and

$$M(\leq m, n) = \sum_{r \leq m} M(r, n). \tag{1.2}$$

Bringmann and Mahlburg [9] observed that for $m < 0$ and $1 \leq n \leq 100$,

$$N(\leq m, n) \leq M(\leq m, n) \leq N(\leq m + 1, n). \tag{1.3}$$

For $m = -1$, an equivalent form of the inequality $N(\leq -1, n) \leq M(\leq -1, n)$ for $n \geq 1$ was conjectured by Kaavya [17]. Bringmann and Mahlburg [9] pointed out that this observation may also be stated as follows. For $1 \leq n \leq 100$, there must be some reordering τ_n of partitions λ of n such that

$$|\operatorname{crank}(\lambda)| - |\operatorname{rank}(\tau_n(\lambda))| = 0 \text{ or } 1. \tag{1.4}$$

Moreover, they noticed that using (1.4), one can deduce the following inequality on the spt-function $\mathrm{spt}(n)$:

$$\mathrm{spt}(n) \leq \sqrt{2n}\, p(n), \tag{1.5}$$

where $\mathrm{spt}(n)$ is the spt-function defined by Andrews [2] as the total number of smallest parts in all partitions of n and $p(n)$ is the number of partitions of n.

In the study of the spt-crank, Andrews, Dyson, and Rhoades [4] conjectured that the sequence $\{N_S(m,n)\}_m$ is unimodal for $n \geq 1$, where $N_S(m,n)$ is the number of S-partitions of size n with spt-crank m, see Andrews, Garvan and Liang [6]. They showed that this conjecture is equivalent to the inequality $N(\leq m,n) \leq M(\leq m,n)$ for $m < 0$ and $n \geq 1$. They obtained the following asymptotic formula for $M(\leq m,n) - N(\leq m,n)$, which implies that the inequality holds for fixed $m < 0$ and sufficiently large n.

Theorem 1.1 (Andrews, Dyson, and Rhoades). *For any given $m < 0$,*

$$M(\leq m,n) - N(\leq m,n) \sim -\frac{(1+2m)\pi^2}{96n}p(n) \quad \text{as} \quad n \to \infty. \tag{1.6}$$

By constructing a series of injections [11], we proved the conjecture of Andrews, Dyson, and Rhoades.

Theorem 1.2. *For $m < 0$ and $n \geq 1$,*

$$N(\leq m,n) \leq M(\leq m,n). \tag{1.7}$$

Mao [18] obtained an asymptotic formula for $N(\leq m+1,n) - M(\leq m,n)$, which implies that the inequality $M(\leq m,n) \leq N(\leq m+1,n)$ holds for any fixed $m < 0$ and sufficiently large n.

Theorem 1.3. (Mao). *For any given $m < 0$,*

$$N(\leq m+1,n) - M(\leq m,n) \sim \frac{\pi}{4\sqrt{6n}}p(n) \quad \text{as} \quad n \to \infty. \tag{1.8}$$

It turns out that our constructive approach in [11] can also be used to deduce the following assertion.

Theorem 1.4. *For $m < 0$ and $n \geq 1$,*

$$M(\leq m,n) \leq N(\leq m+1,n). \tag{1.9}$$

If we list the set of partitions of n in two ways, one by the ranks, and the other by the cranks, then we are led to a reordering τ_n of the partitions of n. Using the inequalities (1.3) for $m < 0$ and $n \geq 1$, we show that the rank and the crank are nearly equidistributed over partitions of n. Since there may be more than one partition with the same rank or crank, the aforementioned listings may not be unique. Nevertheless,

Table 1 The reordering τ_4

λ	crank(λ)	$\tau_4(\lambda)$	rank($\tau_4(\lambda)$)	crank(λ) − rank($\tau_4(\lambda)$)
(1, 1, 1, 1)	−4	(1,1,1,1)	−3	−1
(2,1,1)	−2	(2,1,1)	−1	−1
(3,1)	0	(2,2)	0	0
(2,2)	2	(3,1)	1	1
(4)	4	(4)	3	1

this does not affect the required property of the reordering τ_n. It should be noted that the above description of τ_n relies on the two orderings of partitions of n, it would be interesting to find a definition of τ_n explicitly on a partition λ of n.

Theorem 1.5. *For $n \geq 1$, let τ_n be a reordering on the set of partitions of n as defined above. Then for any partition λ of n,*

$$\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = \begin{cases} 0, & \text{if } \mathrm{crank}(\lambda) = 0, \\ 0 \text{ or } 1, & \text{if } \mathrm{crank}(\lambda) > 0, \\ 0 \text{ or } -1, & \text{if } \mathrm{crank}(\lambda) < 0. \end{cases} \tag{1.10}$$

Clearly, the above theorem implies relation (1.4). For example, for $n = 4$, the reordering τ_4 is illustrated in Table 1.

We find that the map τ_n is related to the function ospt(n) defined by Andrews, Chan, and Kim [3] as the difference between the first positive crank moment and the first positive rank moment, namely,

$$\mathrm{ospt}(n) = \sum_{m\geq 0} mM(m,n) - \sum_{m\geq 0} mN(m,n). \tag{1.11}$$

Andrews, Chan, and Kim [3] derived the following generating function of ospt(n).

Theorem 1.6 (Andrews, Chan, and Kim). *We have*

$$\begin{aligned}\sum_{n\geq 0} \mathrm{ospt}(n) q^n &= \frac{1}{(q;q)_\infty} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2} (1-q^{4i+2})(1-q^{4i+2j+3}) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+1} (1-q^{2i+1})(1-q^{4i+2j+2}) \right).\end{aligned}$$

Based on the above generating function, Andrews, Chan, and Kim [3] proved the positivity of $\mathrm{ospt}(n)$.

Theorem 1.7 (Andrews, Chan, and Kim). *For $n \geq 1$, $\mathrm{ospt}(n) > 0$.*

They also found a combinatorial interpretation of $\mathrm{ospt}(n)$ in terms of even strings and odd strings of a partition. The following theorem shows that the function $\mathrm{ospt}(n)$ is related to the reordering τ_n.

Theorem 1.8. *For $n > 1$, $\mathrm{ospt}(n)$ equals the number of partitions λ of n such that $crank(\lambda) - rank(\tau_n(\lambda)) = 1$.*

It can be seen that $\tau_n((n)) = (n)$ for $n > 1$, since the partition (n) has the largest rank and the largest crank among all partitions of n. It follows that $\mathrm{crank}((n)) - \mathrm{rank}(\tau_n((n))) = 1$ when $n > 1$. Thus Theorem 1.8 implies that $\mathrm{ospt}(n) > 0$ for $n > 1$.

The following upper bound for $\mathrm{ospt}(n)$ can be derived from Theorem 1.5 and Theorem 1.8.

Theorem 1.9. *For $n > 1$,*

$$\mathrm{ospt}(n) \leq \frac{p(n)}{2} - \frac{M(0,n)}{2}. \tag{1.12}$$

It is easily seen that $M(0,n) \geq 1$ for $n \geq 3$ since $\mathrm{crank}((n-1,1)) = 0$ when $n \geq 3$. Hence Theorem 1.9 implies the following inequality due to Chan and Mao [10]: For $n \geq 3$,

$$\mathrm{ospt}(n) < \frac{p(n)}{2}. \tag{1.13}$$

This paper is organized as follows. In Section 2, we give a combinatorial proof of Theorem 1.4 with the aid of m-Durfee rectangle symbols as introduced in [11]. In Section 3, we demonstrate that Theorem 1.5 follows from Theorem 1.4. Proofs of Theorem 1.8 and Theorem 1.9 are given in Section 4. For completeness, we include a derivation of inequality (1.5).

2 Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To this end, we first reformulate the inequality $M(\leq m, n) \leq N(\leq m+1, n)$ for $m < 0$ and $n \geq 1$ in terms of the rank-set. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. Recall that the rank-set of λ introduced by Dyson [14] is the infinite sequence

$$[-\lambda_1, 1-\lambda_2, \ldots, j-\lambda_{j+1}, \ldots, \ell-1-\lambda_\ell, \ell, \ell+1, \ldots].$$

Let $q(m,n)$ denote the number of partitions λ of n such that m appears in the rank-set of λ. Dyson [14] established the following relation: For $n \geq 1$,

$$M(\leq m,n) = q(m,n), \tag{2.1}$$

see also Berkovich and Garvan [8, (3.5)].

Let $p(m,n)$ denote the number of partitions of n with rank at least m, namely,

$$p(m,n) = \sum_{r=m}^{\infty} N(r,n).$$

By establishing the relation

$$M(\leq m,n) - N(\leq m,n) = q(m,n) - p(-m,n), \tag{2.2}$$

for $m < 0$ and $n \geq 1$, we see that $M(\leq m,n) \geq N(\leq m,n)$ is equivalent to the inequality $q(m,n) \geq p(-m,n)$. This was justified by a number of injections in [11].

Similarly, to prove $N(\leq m+1,n) \geq M(\leq m,n)$ for $m < 0$ and $n \geq 1$, we need the following relation.

Theorem 2.1. *For $m < 0$ and $n \geq 1$,*

$$N(\leq m+1,n) - M(\leq m,n) = q(-m-1,n) - p(m+2,n). \tag{2.3}$$

Proof. Since

$$N(\leq m+1,n) = \sum_{r=-\infty}^{m+1} N(r,n)$$

and

$$p(m+2,n) = \sum_{r=m+2}^{\infty} N(r,n),$$

we get

$$N(\leq m+1,n) = \sum_{r=-\infty}^{\infty} N(r,n) - p(m+2,n). \tag{2.4}$$

In fact,

$$\sum_{r=-\infty}^{\infty} N(r,n) = p(n),$$

so that (2.4) takes the form

$$N(\leq m+1, n) = p(n) - p(m+2, n). \tag{2.5}$$

On the other hand, owing to the symmetry

$$M(m, n) = M(-m, n),$$

due to Dyson [14], (2.1) becomes

$$q(-m-1, n) = \sum_{r=m+1}^{\infty} M(r, n).$$

Hence

$$M(\leq m, n) = \sum_{r=-\infty}^{m} M(r, n) = \sum_{r=-\infty}^{\infty} M(r, n) - q(-m-1, n). \tag{2.6}$$

But

$$\sum_{r=-\infty}^{\infty} M(r, n) = p(n),$$

so we arrive at

$$M(\leq m, n) = p(n) - q(-m-1, n). \tag{2.7}$$

Subtracting (2.7) from (2.5) gives (2.3). This completes the proof. □

In view of Theorem 2.1, we see that Theorem 1.4 is equivalent to the following assertion.

Theorem 2.2. *For $m \geq 0$ and $n \geq 1$,*

$$q(m, n) \geq p(-m+1, n). \tag{2.8}$$

Let $P(-m+1, n)$ denote the set of partitions counted by $p(-m+1, n)$, that is, the set of partitions of n with rank at least $-m+1$, and let $Q(m, n)$ denote the set of partitions counted by $q(m, n)$, that is, the set of partitions λ of n such that m appears in the rank-set of λ. Then Theorem 2.2 can be interpreted as the existence of an injection Θ from the set $P(-m+1, n)$ to the set $Q(m, n)$ for $m \geq 0$ and $n \geq 1$.

In [11], we have constructed an injection Φ from the set $Q(m, n)$ to $P(-m, n)$ for $m \geq 0$ and $n \geq 1$. It turns out that the injection Θ in this paper is less involved than the injection Φ in [11]. More specifically, to construct the injection Φ, the set $Q(m, n)$ is divided into six disjoint subsets $Q_i(m, n)$ $(1 \leq i \leq 6)$ and the set $P(-m, n)$ is divided into eight disjoint subsets $P_i(-m, n)$ $(1 \leq i \leq 8)$. For $m \geq 1$, the injection Φ consists of six injections ϕ_i from the set $Q_i(m, n)$ to the set $P_i(-m, n)$, where $1 \leq i \leq 6$. When $m = 0$, the injection Φ requires considerations of more cases. For the purpose of this paper, the set $P(-m+1, n)$ will be divided into three disjoint subsets $P_i(-m+1, n)$ $(1 \le i \le 3)$ and the set $Q(m, n)$ will be divided into three disjoint subsets $Q_i(m, n)$ $(1 \le i \le 3)$. For $m \ge 0$, the injection Θ consists of three injections θ_1, θ_2 and θ_3, where θ_1 is the identity map, and for $i = 2, 3, \theta_i$ is an injection from $P_i(-m+1, n)$ to $Q_i(m, n)$.

To describe the injection Θ, we shall represent the partitions in $Q(m, n)$ and $P(-m+1, n)$ in terms of m-Durfee rectangle symbols. As a generalization of a Durfee symbol defined by Andrews [1], an m-Durfee rectangle symbol of a partition is defined in [11]. Let λ be a partition of n and let $\ell(\lambda)$ denote the number of parts of λ. The m-Durfee rectangle symbol of λ is defined as follows:

$$(\alpha, \beta)_{(m+j)\times j} = \begin{pmatrix} \alpha_1, \ \alpha_2, \ \ldots, \ \alpha_s \\ \beta_1, \ \beta_2, \ \ldots, \ \beta_t \end{pmatrix}_{(m+j)\times j}, \tag{2.9}$$

where $(m+j) \times j$ is the m-Durfee rectangle of the Ferrers diagram of λ and α consists of columns to the right of the m-Durfee rectangle and β consists of rows below the m-Durfee rectangle, see Fig. 1. For the partition $\lambda = (7, 7, 6, 4, 3, 3, 2, 2, 2)$, the 2-Durfee rectangle symbol of λ is

$$\begin{pmatrix} 4, \ 3, \ 3, \ 2 \\ 3, \ 2, \ 2, \ 2 \end{pmatrix}_{5\times 3}.$$

Clearly, we have

$$m + j \ge \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_s, \quad j \ge \beta_1 \ge \beta_2 \ge \cdots \ge \beta_t,$$

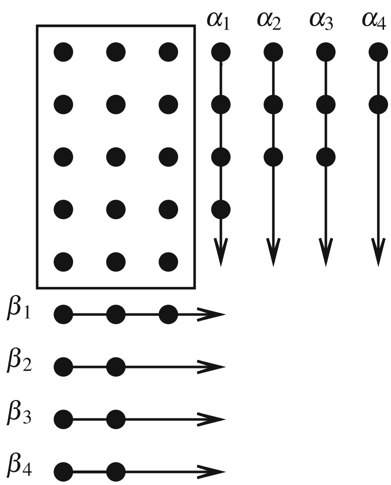


Fig. 1 The 2-Durfee rectangle representation of (7, 7, 6, 4, 3, 3, 2, 2, 2).

and

$$n = j(m + j) + \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i.$$

When $m = 0$, an m-Durfee rectangle symbol reduces to a Durfee symbol.

Notice that for a partition λ with $\ell(\lambda) \leq m$, it has no m-Durfee rectangle. In this case, we adopt the convention that the m-Durfee rectangle has no columns, that is, $j = 0$, and so the m-Durfee rectangle symbol of λ is defined to be $(\lambda', \emptyset)_{m\times 0}$, where λ' is the conjugate of λ. For example, the 3-Durfee rectangle symbol of $\lambda = (5, 5, 1)$ is

$$\begin{pmatrix} 3,\ 2,\ 2,\ 2,\ 2 \\ \end{pmatrix}_{3\times 0}.$$

The partitions in $P(-m+1, n)$ can be characterized in terms of m-Durfee rectangle symbols.

Proposition 2.3. *Assume that $m \geq 0$ and $n \geq 1$. Let λ be a partition of n and let $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then the rank of λ is at least $-m + 1$ if and only if either $j = 0$ or $j \geq 1$ and $\ell(\beta) + 1 \leq \ell(\alpha)$.*

Proof. The proof is substantially the same as that of [11, Proposition 3.2]. Assume that the rank of λ is at least $-m + 1$. We are going to show that either $j = 0$ or $j \geq 1$ and $\ell(\beta) + 1 \leq \ell(\alpha)$. There are two cases:
Case 1: $\ell(\lambda) \leq m$. We have $j = 0$.
Case 2: $\ell(\lambda) \geq m + 1$. We have $j \geq 1$, $\lambda_1 = j + \ell(\alpha)$ and $\ell(\lambda) = m + j + \ell(\beta)$. It follows that

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = \ell(\alpha) - \ell(\beta) - m.$$

Under the assumption that $\lambda_1 - \ell(\lambda) \geq -m + 1$, we see that $\ell(\alpha) - \ell(\beta) \geq 1$, that is, $\ell(\beta) + 1 \leq \ell(\alpha)$.

Conversely, we assume that $j = 0$ or $j \geq 1$ and $\ell(\beta) + 1 \leq \ell(\alpha)$. We proceed to show that the rank of λ is at least $-m + 1$. There are two cases:
Case 1: $j = 0$. Clearly, $\ell(\lambda) \leq m$, which implies that the rank of λ is at least $-m + 1$.
Case 2: $j \geq 1$ and $\ell(\beta) + 1 \leq \ell(\alpha)$. Note that $\lambda_1 = j + \ell(\alpha)$ and $\ell(\lambda) = j + m + \ell(\beta)$. It follows that

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = -m + \ell(\alpha) - \ell(\beta). \tag{2.10}$$

Under the assumption that $\ell(\alpha) - \ell(\beta) \geq 1$, (2.10) implies that $\lambda_1 - \ell(\lambda) \geq -m + 1$. This completes the proof. □

The following proposition will be used to describe the partitions in $Q(m, n)$ in terms of m-Durfee rectangle symbols.

Proposition 2.4. [11, Proposition 3.1] *Assume that $m \geq 0$ and $n \geq 1$. Let λ be a partition of n and let $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then m appears in the rank-set of λ if and only if either $j = 0$ or $j \geq 1$ and $\beta_1 = j$.*

If no confusion arises, we do not distinguish a partition λ and its m-Durfee rectangle symbol representation. We shall divide the set of the m-Durfee rectangle symbols $(\alpha, \beta)_{(m+j)\times j}$ in $P(-m+1, n)$ into three disjoint subsets $P_1(-m+1, n)$, $P_2(-m+1, n)$ and $P_3(-m+1, n)$. More precisely,

(1) $P_1(-m+1, n)$ is the set of m-Durfee rectangle symbols $(\alpha, \beta)_{(m+j)\times j}$ in $P(-m+1, n)$ for which either of the following conditions holds:
(i) $j = 0$;
(ii) $j \geq 1$ and $\beta_1 = j$;
(2) $P_2(-m+1, n)$ is the set of m-Durfee rectangle symbols $(\alpha, \beta)_{(m+j)\times j}$ in $P(-m+1, n)$ such that $j \geq 1$ and $\beta_1 = j-1$;
(3) $P_3(-m+1, n)$ is the set of m-Durfee rectangle symbols $(\alpha, \beta)_{(m+j)\times j}$ in $P(-m+1, n)$ such that $j \geq 2$ and $\beta_1 \leq j-2$.

The set $Q(m, n)$ will be divided into the following three subsets $Q_1(m, n)$, $Q_2(m, n)$ and $Q_3(m, n)$:

(1) $Q_1(m, n)$ is the set of m-Durfee rectangle symbols $(\gamma, \delta)_{(m+j')\times j'}$ in $Q(m, n)$ such that either of the following conditions holds:
(i) $j' = 0$;
(ii) $j' \geq 1$ and $\ell(\delta) - \ell(\gamma) \leq -1$;
(2) $Q_2(m, n)$ is the set of m-Durfee rectangle symbols $(\gamma, \delta)_{(m+j')\times j'}$ in $Q(m, n)$ such that $j' \geq 1$, $\ell(\delta) - \ell(\gamma) \geq 0$ and $\gamma_1 < m + j'$;
(3) $Q_3(m, n)$ is the set of m-Durfee rectangle symbols $(\gamma, \delta)_{(m+j')\times j'}$ in $Q(m, n)$ such that $j' \geq 1$, $\ell(\delta) - \ell(\gamma) \geq 0$ and $\gamma_1 = m + j'$.

We are now ready to define the injections θ_i from the set $P_i(-m+1, n)$ to the set $Q_i(m, n)$, where $1 \leq i \leq 3$. Since $P_1(-m+1, n)$ coincides with $Q_1(m, n)$, we set θ_1 to be the identity map. The following lemma gives an injection θ_2 from $P_2(-m+1, n)$ to $Q_2(m, n)$.

Lemma 2.5. *For $m \geq 0$ and $n > 1$, there is an injection θ_2 from $P_2(-m+1, n)$ to $Q_2(m, n)$.*

Proof. To define the map θ_2, let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_s \\ \beta_1, \beta_2, \ldots, \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $P_2(-m+1, n)$. From the definition of $P_2(-m+1, n)$, we see that $s - t \geq 1$, $j \geq 1$, $\alpha_1 \leq m + j$ and $\beta_1 = j - 1$.

Set

$$\theta_2(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \alpha_1 - 1,\ \alpha_2 - 1,\ \ldots,\ \alpha_s - 1 \\ \beta_1 + 1,\ \beta_2 + 1,\ \ldots,\ \beta_t + 1,\ 1^{s-t} \end{pmatrix}_{(m+j)\times j}.$$

Clearly, $\theta_2(\lambda)$ is an m-Durfee rectangle symbol of n. Furthermore, $j' = j$, $\ell(\delta) - \ell(\gamma) \geq 0$. Since $\alpha_1 \leq m + j$, we see that $\gamma_1 = \alpha_1 - 1 \leq m + j - 1 < m + j'$. Noting that $\beta_1 = j - 1$, we get $\delta_1 = \beta_1 + 1 = j = j'$. Moreover, $\delta_s = 1$ since $s - t \geq 1$. This proves that $\theta_2(\lambda)$ is in $Q_2(m, n)$.

To prove that θ_2 is an injection, define

$$H(m, n) = \{\theta_2(\lambda)\colon \lambda \in P_2(-m + 1, n)\}.$$

Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{s'} \\ \delta_1,\ \delta_2,\ \ldots,\ \delta_{t'} \end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $H(m, n)$. Since $\mu \in Q_2(m, n)$, we have $t' \geq s'$, $\gamma_1 < m + j'$ and $\delta_1 = j'$. According to the construction of θ_2, $\delta_{t'} = 1$. Define

$$\sigma(\mu) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \gamma_1 + 1,\ \gamma_2 + 1,\ \ldots,\ \gamma_{s'} + 1,\ 1^{t'-s'} \\ \delta_1 - 1,\ \delta_2 - 1,\ \ldots,\ \delta_{t'} - 1 \end{pmatrix}_{(m+j')\times j'}.$$

Clearly, $\ell(\beta) < t'$ since $\delta_{t'} = 1$, so that $\ell(\alpha) - \ell(\beta) \geq 1$. Moreover, since $\delta_1 = j'$ and $j' = j$, we see that $\beta_1 = \delta_1 - 1 = j' - 1 = j - 1$. It is easily checked that $\sigma(\theta_2(\lambda)) = \lambda$ for any λ in $P_2(-m + 1, n)$. Hence the map θ_2 is an injection from $P_2(-m + 1, n)$ to $Q_2(m, n)$. This completes the proof. $\square$

For example, for $m = 2$ and $n = 35$, consider the following 2-Durfee rectangle symbol in $P_2(-1, 35)$:

$$\lambda = \begin{pmatrix} 5,\ 5,\ 3,\ 1,\ 1 \\ 2,\ 2,\ 1 \end{pmatrix}_{5\times 3}.$$

Applying the injection θ_2 to λ, we obtain

$$\mu = \theta_2(\lambda) = \begin{pmatrix} 4,\ 4,\ 2 \\ 3,\ 3,\ 2,\ 1,\ 1 \end{pmatrix}_{5\times 3},$$

which is in $Q_2(2, 35)$. Applying σ to μ, we recover λ.

The following lemma gives an injection θ_3 from $P_3(-m + 1, n)$ to $Q_3(m, n)$.

Lemma 2.6. *For $m \geq 0$ and $n > 1$, there is an injection θ_3 from $P_3(-m + 1, n)$ to $Q_3(m, n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_s \\ \beta_1,\ \beta_2,\ \ldots,\ \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $P_3(-m+1,n)$. By definition, $s-t\geq 1$, $j\geq 2$ and $\beta_1\leq j-2$.

Define

$$\theta_3(\lambda)=\begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'}=\begin{pmatrix}m+j-1, & \alpha_1-1, & \alpha_2-1, & \ldots, & \alpha_s-1 & \\ j-1, & \beta_1+1, & \beta_2+1, & \ldots, & \beta_t+1 & , 1^{s-t+1}\end{pmatrix}_{(m+j-1)\times(j-1)}.$$

Evidently, $\ell(\delta)=s+2$ and $\ell(\gamma)\leq s+1$, and so $\ell(\delta)-\ell(\gamma)\geq 1$. Moreover, we have $\gamma_1=m+j-1=m+j'$, $\delta_1=j-1=j'$ and

$$\begin{aligned} & j'(m+j')+\sum_{i=1}^{s+1}\gamma_i+\sum_{i=1}^{s+2}\delta_i \\ &\quad =(m+j-1)(j-1)+\left(m+j-1+\sum_{i=1}^{s}(\alpha_i-1)\right) \\ &\qquad +\left(j-1+s-t+1+\sum_{i=1}^{t}(\beta_i+1)\right) \\ &\quad =j(m+j)+\sum_{i=1}^{s}\alpha_i+\sum_{i=1}^{t}\beta_i=n. \end{aligned}$$

This yields that $\theta_3(\lambda)$ is in $Q_3(m,n)$. In particular, since $s-t\geq 1$, we see that

$$\delta_{s+2}=\delta_{s+1}=1. \tag{2.11}$$

To prove that the map θ_3 is an injection, define

$$I(m,n)=\{\theta_3(\lambda)\colon \lambda\in P_3(-m+1,n)\}.$$

Let

$$\mu=\begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'}=\begin{pmatrix}\gamma_1, & \gamma_2, & \ldots, & \gamma_{s'}\\ \delta_1, & \delta_2, & \ldots, & \delta_{t'}\end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $I(m,n)$. Since $\mu\in Q_3(m,n)$, we have $t'\geq s'$, $\gamma_1=m+j'$ and $\delta_1=j'$. By the construction of θ_3, $t'-s'\geq 1$. Define

$$\pi(\mu)=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{(m+j)\times j}=\begin{pmatrix}\gamma_2+1, & \ldots, & \gamma_{s'}+1, & 1^{t'-s'-1}\\ \delta_2-1, & \ldots, & \delta_{t'}-1 & \end{pmatrix}_{(m+j'+1)\times(j'+1)}.$$

It follows from (2.11) that $\ell(\beta)\leq t'-3$ and $\ell(\alpha)=t'-2$. Therefore, $\ell(\alpha)\geq\ell(\beta)+1$ and $\beta_1=\delta_2-1\leq j'-1=j-2$, so that $\pi(\mu)$ is in $P_3(-m+1,n)$.

Moreover, it can be checked that $\pi(\theta_3(\lambda)) = \lambda$ for any λ in $P_3(-m+1, n)$. This proves that the map θ_3 is an injection from $P_3(-m+1, n)$ to $Q_3(m, n)$. □

For example, for $m = 3$ and $n = 63$, consider the following 3-Durfee rectangle symbol in $P_3(-2, 63)$:

$$\lambda = \begin{pmatrix} 7, \ 7, \ 4, \ 3, \ 3, \ 2, \ 1 \\ 2, \ 2, \ 2, \ 1, \ \ 1 \end{pmatrix}_{7\times 4}.$$

Applying the injection θ_3 to λ, we obtain

$$\mu = \theta_3(\lambda) = \begin{pmatrix} 6, \ 6, \ 6, \ 3, \ 2, \ 2, \ 1 \\ 3, \ 3, \ 3, \ 3, \ 2, \ 2, \ 1, \ 1, \ 1 \end{pmatrix}_{6\times 3},$$

which is in $Q_3(3, 63)$. Applying π to μ, we recover λ.

Combining the bijection θ_1 and the injections θ_2 and θ_3, we are led to an injection Θ from $P(-m+1, n)$ to $Q(m, n)$, and hence the proof of Theorem 2.2 is complete. More precisely, for a partition λ,

$$\Theta(\lambda) = \begin{cases} \theta_1(\lambda), & \text{if} \quad \lambda \in P_1(-m+1, n), \\ \theta_2(\lambda), & \text{if} \quad \lambda \in P_2(-m+1, n), \\ \theta_3(\lambda), & \text{if} \quad \lambda \in P_3(-m+1, n). \end{cases}$$

3 Proof of Theorem 1.5

In this section, we show that it is indeed the case that the reordering τ_n leads to the nearly equal distributions of the rank and the crank, with the aid of the inequalities in Theorem 1.2 and Theorem 1.4. For the sake of presentation, the inequalities in Theorem 1.2 and Theorem 1.4 for $m < 0$ can be recast for $m \geq 0$.

Theorem 3.1. *For $m \geq 0$ and $n \geq 1$,*

$$N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n). \tag{3.1}$$

To see that the inequalities (3.1) for $m \geq 0$ can be derived from (1.7) in Theorem 1.2 and (1.9) in Theorem 1.4 for $m < 0$, we assume that $m \geq 0$, so that (1.7) and (1.9) can be stated as follows:

$$N(\leq -m-1, n) \leq M(\leq -m-1, n) \leq N(\leq -m, n), \tag{3.2}$$

and hence

$$p(n) - N(\leq -m-1, n) \geq p(n) - M(\leq -m-1, n) \geq p(n) - N(\leq -m, n). \tag{3.3}$$

It follows that

$$\sum_{r=-m}^{\infty} N(r,n) \geq \sum_{r=-m}^{\infty} M(r,n) \geq \sum_{r=-m+1}^{\infty} N(r,n). \tag{3.4}$$

Now, by the symmetry $N(m,n) = N(-m,n)$, see [13], we have

$$\sum_{r=-m}^{\infty} N(r,n) = N(\leq m, n) \quad \text{and} \quad \sum_{r=-m+1}^{\infty} N(r,n) = N(\leq m-1, n). \tag{3.5}$$

Similarly, the symmetry $M(m,n) = M(-m,n)$, see [14], leads to

$$\sum_{r=-m}^{\infty} M(r,n) = M(\leq m, n). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain (3.1). Conversely, one can reverse the above steps to derive (1.7) and (1.9) for $m < 0$ from (3.1) for $m \geq 0$. This means that the inequalities (3.1) for $m \geq 0$ are equivalent to the inequalities (1.7) and (1.9) for $m < 0$.

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Let λ be a partition of n, and let $\tau_n(\lambda) = \mu$. Suppose that λ is the i-th partition of n when the partitions of n are listed in the increasing order of cranks used in the definition of τ_n. Meanwhile, μ is also the i-th partition in the list of partitions of n in the increasing order of ranks used in the definition of τ_n. Let $\text{crank}(\lambda) = a$ and $\text{rank}(\mu) = b$, so that

$$M(\leq a, n) \geq i > M(\leq a-1, n), \tag{3.7}$$

and

$$N(\leq b, n) \geq i > N(\leq b-1, n). \tag{3.8}$$

We now consider three cases:

Case 1: $a = 0$. We aim to show that $b = 0$. Assume to the contrary that $b \neq 0$. There are two subcases:

Subcase 1.1: $b < 0$. From (3.7) and (3.8), we have

$$N(\leq -1, n) \geq N(\leq b, n) \geq i > M(\leq -1, n),$$

which contradicts the inequality $N(\leq m, n) \leq M(\leq m, n)$ in Theorem 1.2 with $m = -1$.

Subcase 1.2: $b > 0$. From (3.7) and (3.8), we see that

$$M(\leq 0, n) \geq i > N(\leq b-1, n) \geq N(\leq 0, n),$$

which contradicts the inequality $M(\leq m, n) \leq N(\leq m, n)$ in (3.1) with $m = 0$. This completes the proof of Case 1.
Case 2: $a < 0$. We proceed to show that $b = a$ or $a + 1$. By (3.7) and the inequality $M(\leq m, n) \leq N(\leq m+1, n)$ in Theorem 1.4 with $m = a$, we see that

$$N(\leq a+1, n) \geq i. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that

$$N(\leq a+1, n) > N(\leq b-1, n),$$

and thus

$$a + 1 \geq b. \tag{3.10}$$

On the other hand, by (3.7) and the inequality $N(\leq m, n) \leq M(\leq m, n)$ in Theorem 1.2 with $m = a - 1$, we find that

$$N(\leq a-1, n) < i.$$

Together with (3.8), this gives

$$N(\leq a-1, n) < N(\leq b, n),$$

so that $a \leq b$. In view of (3.10), we obtain that $b = a$ or $a + 1$. This completes the proof of Case 2.
Case 3: $a > 0$. We claim that $b = a$ or $a - 1$. Combining the inequality $M(\leq m, n) \geq N(\leq m-1, n)$ in (3.1) with $m = a - 1$ and the inequality $M(\leq a-1, n) < i$ in (3.7), we get

$$N(\leq a-2, n) < i. \tag{3.11}$$

By means of (3.8) and (3.11), we find that

$$N(\leq b, n) > N(\leq a-2, n),$$

whence

$$a - 1 \leq b. \tag{3.12}$$

On the other hand, combining the inequality $N(\leq m, n) \geq M(\leq m, n)$ in (3.1) with $m = a$ and the inequality $M(\leq a, n) \geq i$ in (3.7), we are led to

$$N(\leq a, n) \geq i, \tag{3.13}$$

which together with (3.8) yields that

$$N(\leq a, n) > N(\leq b-1, n),$$

and hence $a \geq b$. But it has been shown that $b \geq a-1$, whence $b = a-1$ or a. This completes the proof of Case 3. □

4 Proofs of Theorem 1.8 and Theorem 1.9

In this section, we give a proof of Theorem 1.8 concerning an interpretation of the ospt-function in terms of the reordering τ_n. Then we use Theorem 1.8 to deduce Theorem 1.9, which gives an upper bound of the ospt-function. Finally, for completeness, we include a derivation of (1.5) from (1.4).

Proof of Theorem 1.8. Let $\mathscr{P}(n)$ denote the set of partitions of n. By the definition (1.11) of $\text{ospt}(n)$, we see that

$$\text{ospt}(n) = \sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{crank}(\lambda)>0}} \text{crank}(\lambda) - \sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{rank}(\lambda)>0}} \text{rank}(\lambda). \tag{4.1}$$

We claim that

$$\sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{rank}(\lambda)>0}} \text{rank}(\lambda) = \sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{crank}(\lambda)>0}} \text{rank}(\tau_n(\lambda)). \tag{4.2}$$

From Theorem 1.5, we see that if $\text{crank}(\lambda) > 0$, then $\text{rank}(\tau_n(\lambda)) \geq 0$. This implies that

$$\{\lambda \in \mathscr{P}(n) \colon \text{crank}(\lambda) > 0\} \subseteq \{\lambda \in \mathscr{P}(n) \colon \text{rank}(\tau_n(\lambda)) \geq 0\}. \tag{4.3}$$

Therefore,

$$\sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{crank}(\lambda)>0}} \text{rank}(\tau_n(\lambda)) \leq \sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{rank}(\tau_n(\lambda))\geq 0}} \text{rank}(\tau_n(\lambda)). \tag{4.4}$$

From Theorem 1.5, we also see that if $\text{crank}(\lambda) = 0$, then $\text{rank}(\tau_n(\lambda)) = 0$, and if $\text{crank}(\lambda) < 0$, then $\text{rank}(\tau_n(\lambda)) \leq 0$. Now,

$$\{\lambda \in \mathscr{P}(n) \colon \text{rank}(\tau_n(\lambda)) > 0\} \subseteq \{\lambda \in \mathscr{P}(n) \colon \text{crank}(\lambda) > 0\}. \tag{4.5}$$

Hence by (4.3),

$$\sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{rank}(\tau_n(\lambda))>0}} \text{rank}(\tau_n(\lambda)) \leq \sum_{\substack{\lambda \in \mathscr{P}(n) \\ \text{crank}(\lambda)>0}} \text{rank}(\tau_n(\lambda)). \tag{4.6}$$

Since

$$\sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{rank}(\tau_n(\lambda))\geq 0}} \mathrm{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{rank}(\tau_n(\lambda))>0}} \mathrm{rank}(\tau_n(\lambda)),$$

from (4.4) and (4.6), we infer that

$$\sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{rank}(\tau_n(\lambda))>0}} \mathrm{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{crank}(\lambda)>0}} \mathrm{rank}(\tau_n(\lambda)). \tag{4.7}$$

But

$$\sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{rank}(\tau_n(\lambda))>0}} \mathrm{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{rank}(\lambda)>0}} \mathrm{rank}(\lambda), \tag{4.8}$$

thus we arrive at (4.2), and so the claim is justified.

Substituting (4.2) into (4.1), we get

$$\begin{aligned}\mathrm{ospt}(n) &= \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{crank}(\lambda)>0}} \mathrm{crank}(\lambda) - \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{crank}(\lambda)>0}} \mathrm{rank}(\tau_n(\lambda)) \\ &= \sum_{\substack{\lambda\in\mathscr{P}(n)\\ \mathrm{crank}(\lambda)>0}} (\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda))). \end{aligned} \tag{4.9}$$

Appealing to Theorem 1.5, we see that if $\mathrm{crank}(\lambda) > 0$, then

$$\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 0 \quad \text{or} \quad 1.$$

By (4.9),

$$\mathrm{ospt}(n) = \#\{\lambda \in \mathscr{P}(n) \colon \mathrm{crank}(\lambda) > 0 \text{ and } \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\}. \tag{4.10}$$

Also, by Theorem 1.5, we see that if $\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1$, then $\mathrm{crank}(\lambda) > 0$. Consequently,

$$\mathrm{ospt}(n) = \#\{\lambda \in \mathscr{P}(n) \colon \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\}, \tag{4.11}$$

as required. □

Theorem 1.9 can be easily deduced from Theorem 1.5 and Theorem 1.8.

Proof of Theorem 1.9. From the symmetry $M(m, n) = M(-m, n)$, we see that

$$p(n) = \sum_{m=-\infty}^{\infty} M(m, n) = M(0, n) + 2\sum_{m\geq 1} M(m, n). \tag{4.12}$$

Hence

$$\sum_{m\geq 1} M(m,n) = \frac{p(n)}{2} - \frac{M(0,n)}{2}. \tag{4.13}$$

In virtue of Theorem 1.5, if $\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1$, then $\mathrm{crank}(\lambda) > 0$, and hence

$$\#\{\lambda \in \mathscr{P}(n) \colon \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\} \leq \#\{\lambda \in \mathscr{P}(n) \colon \mathrm{crank}(\lambda) > 0\}. \tag{4.14}$$

This, combined with Theorem 1.8, leads to

$$\mathrm{ospt}(n) \leq \#\{\lambda \in \mathscr{P}(n) \colon \mathrm{crank}(\lambda) > 0\} = \sum_{m\geq 1} M(m,n). \tag{4.15}$$

Substituting (4.13) into (4.15), we obtain that

$$\mathrm{ospt}(n) \leq \frac{p(n)}{2} - \frac{M(0,n)}{2},$$

as required. □

We conclude this paper with a derivation of inequality (1.5), that is, $\mathrm{spt}(n) \leq \sqrt{2n}\,p(n)$. Recall that the k-th moment $N_k(n)$ of ranks and the k-th moment $M_k(n)$ of cranks were defined by Atkin and Garvan [7] as follows:

$$N_k(n) = \sum_{m=-\infty}^{\infty} m^k N(m,n), \tag{4.16}$$

$$M_k(n) = \sum_{m=-\infty}^{\infty} m^k M(m,n). \tag{4.17}$$

Andrews [2] showed that the spt-function can be expressed in terms of the second moment $N_2(n)$ of ranks,

$$\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n). \tag{4.18}$$

Employing the following relation due to Dyson [14],

$$M_2(n) = 2np(n), \tag{4.19}$$

Garvan [15] observed that the following expression

$$\mathrm{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n) \tag{4.20}$$

which implies that $M_2(n) > N_2(n)$ for $n \geq 1$. In general, he conjectured and later proved that $M_{2k}(n) > N_{2k}(n)$ for $k \geq 1$ and $n \geq 1$, see [16].

Bringmann and Mahlburg [9] pointed out that inequality (1.5) can be derived by combining the reordering τ_n and the Cauchy–Schwarz inequality. By (4.20), we see that

$$\begin{aligned} 2\,\mathrm{spt}(n) &= \sum_{m=-\infty}^{\infty} m^2 M(m,n) - \sum_{m=-\infty}^{\infty} m^2 N(m,n) \\ &= \sum_{\lambda \in \mathscr{P}(n)} \mathrm{crank}^2(\lambda) - \sum_{\lambda \in \mathscr{P}(n)} \mathrm{rank}^2(\lambda). \end{aligned} \tag{4.21}$$

Since

$$\sum_{\lambda \in \mathscr{P}(n)} \mathrm{rank}^2(\lambda) = \sum_{\lambda \in \mathscr{P}(n)} \mathrm{rank}^2(\tau_n(\lambda)),$$

(4.21) can be rewritten as

$$\begin{aligned} 2\,\mathrm{spt}(n) &= \sum_{\lambda \in \mathscr{P}(n)} \big(\mathrm{crank}^2(\lambda) - \mathrm{rank}^2(\tau_n(\lambda))\big) \\ &= \sum_{\lambda \in \mathscr{P}(n)} (\mid \mathrm{crank}(\lambda) \mid - \mid \mathrm{rank}(\tau_n(\lambda)) \mid) \cdot (\mid \mathrm{crank}(\lambda) \mid + \mid \mathrm{rank}(\tau_n(\lambda)) \mid). \end{aligned} \tag{4.22}$$

By (1.4), we find that

$$\mid \mathrm{crank}(\lambda) \mid + \mid \mathrm{rank}(\tau_n(\lambda)) \mid \leq 2 \mid \mathrm{crank}(\lambda) \mid$$

and

$$0 \leq \mid \mathrm{crank}(\lambda) \mid - \mid \mathrm{rank}(\tau_n(\lambda)) \mid \leq 1.$$

Thus (4.22) gives

$$\mathrm{spt}(n) \leq \sum_{\lambda \in \mathscr{P}(n)} \mid \mathrm{crank}(\lambda) \mid . \tag{4.23}$$

Applying the inequality on the arithmetic and quadratic means

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \tag{4.24}$$

for nonnegative real numbers to the numbers $\mid \mathrm{crank}(\lambda) \mid$, where λ ranges over partitions of n, we are led to

$$\frac{\sum_{\lambda \in \mathscr{P}(n)} \mid \text{crank}(\lambda) \mid}{p(n)} \leq \sqrt{\frac{\sum_{\lambda \in \mathscr{P}(n)} \mid \text{crank}(\lambda) \mid^2}{p(n)}} = \sqrt{\frac{M_2(n)}{p(n)}}. \tag{4.25}$$

In light of Dyson's identity (4.19), this becomes

$$\sum_{\lambda \in \mathscr{P}(n)} \mid \text{crank}(\lambda) \mid \leq \sqrt{2n}\, p(n). \tag{4.26}$$

Combining (4.23) and (4.26) completes the proof. □

Acknowledgements This work was supported by the National Science Foundation of China. We wish to thank Kathrin Bringmann, Karl Mahlburg and the referee for their valuable comments and suggestions.

References

1. G.E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks. Invent. Math. **169**, 37–73 (2007)
2. G.E. Andrews, The number of smallest parts in the partitions of n. J. Reine Angew. Math. **624**, 133–142 (2008)
3. G.E. Andrews, S.H. Chan, B. Kim, The odd moments of ranks and cranks. J. Comb. Theory Ser. A **120**, 77–91 (2013)
4. G.E. Andrews, F.J. Dyson, R.C. Rhoades, On the distribution of the *spt* -crank. Mathematics **1**, 76–88 (2013)
5. G.E. Andrews, F.G. Garvan, Dyson's crank of a partition. Bull. Am. Math. Soc. **18**, 167–171 (1988)
6. G.E. Andrews, F.G. Garvan, J.L. Liang, Combinatorial interpretations of congruences for the spt-function. Ramanujan J. **29**, 321–338 (2012)
7. A.O.L. Atkin, F.G. Garvan, Relations between the ranks and cranks of partitions. Ramanujan J. **7**, 343–366 (2003)
8. A. Berkovich, F.G. Garvan, Some observations on Dyson's new symmetries of partitions. J. Comb. Theory Ser. A **100**, 61–93 (2002)
9. K. Bringmann, K. Mahlburg, Inequalities between ranks and cranks. Proc. Am. Math. Soc. **137**, 2567–2574 (2009)
10. S.H. Chan, R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions. Adv. Math. **258**, 414–437 (2014)
11. W.Y.C. Chen, K.Q. Ji, W.J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank. Adv. Math. **270**, 60–96 (2015)
12. F.J. Dyson, Some guesses in the theory of partitions. Eureka (Cambridge) **8**, 10–15 (1944)
13. F.J. Dyson, A new symmetry of partitions. J. Comb. Theory Ser. A **7**, 56–61 (1969)
14. F.J. Dyson, Mappings and symmetries of partitions. J. Comb. Theory Ser. A **51**, 169–180 (1989)
15. F.G. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank. Int. J. Number Theory **6**, 281–309 (2010)

16. F.G. Garvan, Higher order spt-functions. Adv. Math. **228**, 241–265 (2011)
17. S.J. Kaavya, Crank 0 partitions and the parity of the partition function. Int. J. Number Theory **7**, 793–801 (2011)
18. R. Mao, Asymptotic inequalities for k -ranks and their cumulation functions. J. Math. Anal. Appl. **409**, 729–741 (2014)

Holonomic Alchemy and Series for $1/\pi$

Shaun Cooper, James G. Wan and Wadim Zudilin

To Krishna Alladi,
on his smooth transition from the sixth decade

Abstract We adopt the "translation" as well as other techniques to express several identities conjectured by Z.-W. Sun by means of known formulas for $1/\pi$ involving Domb and other Apéry-like sequences.

Keywords Apéry-like sequence · Domb numbers · Eisenstein series · Holonomic function · Modular form · Modular parameterization · Ramanujan's series for $1/\pi$ · Sun's conjectures · Translation technique · Zeilberger's algorithm.

2010 Mathematics Subject Classification Primary 11Y60 · 33C20 · Secondary 11B65 · 11F11 · 11Y55 · 65B10

S. Cooper (✉)
Institute of Natural and Mathematical Sciences, Massey University — Albany, Private Bag 102904, North Shore Mail Centre, Auckland 0745, New Zealand
e-mail: s.cooper@massey.ac.nz

J. G. Wan
Engineering Systems and Design, Singapore University of Technology and Design, 8 Somapah Rd., Singapore 487372, Singapore
e-mail: james_wan@sutd.edu.sg

W. Zudilin
School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan NSW 2308, Australia
e-mail: wadim.zudilin@newcastle.edu.au

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_12

1 Introduction

The theory of Ramanujan's series for $1/\pi$ received a boost with the announcement of a large number of conjectures by Z.-W. Sun [19]. That work, which was first published on `arXiv.org` on Feb. 28, 2011, has been expanded through 47 versions at the time of writing. The conjectures have stimulated the development of new ideas, e.g., see [6, 11, 17, 21, 22, 24]. Despite the strong interest, a large number of conjectures remain open.

One of the goals of this work is to use a variety of methods to prove many of Sun's conjectures. In particular, we use translation techniques to convert several of the conjectures into known series that have already been classified. We also offer short and alternative proofs for some of the conjectures that have already been resolved, e.g., the "$520 challenge" [19, Eq. (3.24)] that was first proved by M. Rogers and A. Straub [17].

All of the underlying generating functions that we shall encounter are holonomic. That is, they are solutions of linear differential equations with polynomial coefficients. We provide fairly full detail for the examples in the next two sections. In subsequent sections, we are more brief and just communicate the main results, as it is a matter of routine to verify the computational details. In particular, we make frequent use, normally without explanation, of the standard algorithms for holonomic functions and their computer implementations, e.g., Maple's `gfun` package and the Wilf–Zeilberger algorithm.

Another goal is to classify the conjectures. Although our work provides an identification of several of the conjectured series with known series, a full classification remains elusive. Table 2 provides a summary of the underlying series. As a degree of mystery is still present, the topic is somewhat "alchemical" in nature.

2 Conjectures (5.1)–(5.8): Level 10

Conjectures (5.1)–(5.8) in [19] involve series of the form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(an+b)\,h(n,k)\,x^{n+k} \tag{2.1}$$

for particular values of a, b, and x, where

$$h(n,k)=\binom{2n}{n}\binom{2k}{k}\binom{n+2k}{n}\binom{n}{k}.$$

Observe that $h(n,k)=0$ if $k>n$. We will prove three lemmas and use them to convert the series (2.1) to an equivalent series that can be parameterized by level 10 modular forms.

Lemma 2.1. *The following identity holds:*

$$\sum_{k=0}^{n}\binom{n}{k}^4=\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{n-k}\binom{n-k}{k}.$$

Proof. It is routine to use a computer algebra system and apply (for instance) Zeilberger's algorithm [13] to show that each sequence satisfies the same three-term recurrence relation and initial conditions. Alternatively, make the specialization

$$b=c=-n,\quad d=-\frac{n}{2},\quad e=\frac{1}{2}-\frac{n}{2}$$

in Whipple's identity [1, Theorem 3.4.5]

$$\begin{aligned}&{}_7F_6\left(\begin{matrix}a,\ 1+\frac{a}{2},\ b,\ c,\ d,\ e,\ -n\\ \frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+n\end{matrix};1\right)\\ &\quad=\frac{(1+a-d-e)_n(1+a)_n}{(1+a-d)_n(1+a-e)_n}\,{}_4F_3\left(\begin{matrix}1+a-b-c,\ d,\ e,\ -n\\ 1+a-b,\ 1+a-c,\ d+e-a-n\end{matrix};1\right)\end{aligned}$$

and then take the limit as $a\to -n$. □

The next result will be used to compute derivatives. We call it a *satellite identity*—the term we coin from [24]; for details of why such identities exist and how to find them in a general situation see Remark 3.3 below.

Lemma 2.2. *The following identity holds in a neighborhood of $x=0$:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}h(n,k)x^{n+k}\left\{4x-2n(1-x)+3k(1+4x)\right\}=0.$$

Proof. Each of the power series

$$\sum_n\sum_k h(n,k)x^{n+k},\quad \sum_n\sum_k nh(n,k)x^{n+k}\quad\text{and}\quad \sum_n\sum_k kh(n,k)x^{n+k}$$

satisfies a fourth-order linear differential equation with coefficients from $\mathbb{Z}[x]$. Such a differential equation can be produced by the multiple Wilf–Zeilberger algorithm. It is routine to use a computer algebra system to verify that the desired number of leading coefficients in the x-expansion of the left-hand side of the required equality are zero, thus giving the result. □

Lemma 2.3. *Let*

$$f(x)=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}^4\right\}x^n,$$

and let D be the differential operator $D = x\,\dfrac{\mathrm{d}}{\mathrm{d}x}$. *Then,*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k} = f(x),$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\,h(n,k)x^{n+k} = \frac{1}{5(1+2x)}\left(4xf(x) + 3(1+4x)\,Df(x)\right).$$

Proof. On putting $n+k=m$ and applying Lemma 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k} &= \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m} h(m-k,k)\right\}x^m \\ &= \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{2k}{k}\binom{2m-2k}{m-k}\binom{m+k}{m-k}\binom{m-k}{k}\right\}x^m \\ &= \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{m}{k}^4\right\}x^m \\ &= f(x).\end{aligned}$$

To prove the second result of this lemma, start with the satellite identity in Lemma 2.2 in the form

$$\begin{aligned}&2(1-x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\,h(n,k)x^{n+k} \\ &\qquad = 4x\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k} + 3(1+4x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k\,h(n,k)x^{n+k}.\end{aligned}$$

Add $3(1+4x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\,h(n,k)x^{n+k}$ to both sides, and then apply the first result of this lemma to get

$$\begin{aligned}&5(1+2x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\,h(n,k)x^{n+k} \\ &\qquad = 4x\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k} + 3(1+4x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(n+k)\,h(n,k)x^{n+k}\end{aligned}$$

$$= 4x \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k} + 3(1+4x)\, D \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)x^{n+k}$$
$$= 4xf(x) + 3(1+4x)Df(x).$$

Divide both sides by $5(1+2x)$ to complete the proof. □

Theorem 2.4. *The identities* (5.1)–(5.8) *in Sun's Conjecture* 5 *in* [19] *are equivalent to the eight series for* $1/\pi$ *in Theorem* 5.3 *of* [8].

Proof. From Lemma 2.3, we deduce that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(an+b)\, h(n,k)x^{n+k} = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}^4\right\}(An+B)x^n,$$

where

$$A = \frac{3a(1+4x)}{5(1+2x)} \quad \text{and} \quad B = \frac{4ax}{5(1+2x)} + b.$$

For example, taking $(a,b,x) = (95, 13, 1/36)$ gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(95n+13)\, h(n,k)\,\frac{1}{36^{n+k}} = 60\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}^4\right\}\left(n+\frac{1}{4}\right)\frac{1}{36^n}. \tag{2.2}$$

The series on the left occurs in [19, Conjecture (5.3)], whereas the series on the right is due to Y. Yang and its value is known to be (e.g., see [5, Eq. (2.2)])

$$60 \times \frac{3\sqrt{15}}{10\pi} = \frac{18\sqrt{15}}{\pi}.$$

This proves Conjecture (5.3) in [19].

The series on the right-hand side of (2.2) corresponds to the data associated with $y_A = 1/36$ in [8, Table 1]. In fact, the arguments of $s_k(x)$ in each of Conjectures (5.1)–(5.8) are in one to one correspondence with the eight values[1] of $1/y_A$ in [8, Table 1]. In the case of Conjecture (5.1), the series corresponding to $y_A = -1/9$ in [8, Table 1] diverges. This can be handled by using the value of y_C in that table and the associated convergent series given by [8, Eq. (63)]. This accounts for all of the Conjectures (5.1)–(5.8) in [19]. □

Remark 2.5. Conjectures (5.2)–(5.8) in [19] were first proved in the second named author's PhD dissertation [20, Sections 12.3.4, 12.4.1, and 12.4.2], using the techniques outlined here.

[1]The entry $4/196$ in [8, Table 1] is a misprint and should be $1/196$.

3 Conjectures (3.1)–(3.10): Level 24

The Conjectures (3.1)–(3.10) in [19] are based on series of the form

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}^2\binom{2m-2k}{m-k}x^{m+k}=\sum_{n=0}^{\infty}t(n)x^n, \tag{3.1}$$

where

$$t(n)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}^2\binom{2n-4k}{n-2k}. \tag{3.2}$$

Zeilberger's algorithm can be used to show that the sequence $\{t(n)\}$ satisfies the four-term recurrence relation

$$(n+1)^3t(n+1)=2(2n+1)(2n^2+2n+1)t(n)+4n(4n^2+1)t(n-1)$$
$$-64n(n-1)(2n-1)t(n-2). \tag{3.3}$$

The single initial condition $t(0)=1$ is enough to start the sequence.

There is a modular parameterization of the series $\{t(n)\}$. To state it, we will need Dedekind's eta function $\eta(\tau)$ and the weight two Eisenstein series $P(q)$; they are defined by

$$\eta(\tau)=q^{1/24}\prod_{j=1}^{\infty}(1-q^j),\quad \text{where}\quad q=\exp(2\pi i\tau)\quad\text{and}\quad \operatorname{Im}\tau>0,$$

and

$$P(q)=24\,q\,\frac{\mathrm{d}}{\mathrm{d}q}\log\eta(\tau)=1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}.$$

Theorem 3.1. *Let*

$$z=\frac{1}{4}\left(6P(q^{12})-3P(q^6)+2P(q^4)-P(q^2)\right)+2\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$$

and

$$x=\frac{\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)}{z}.$$

Let $\{t(n)\}$ be the sequence defined by equation (3.2). Then in a neighborhood of $x=0$,

$$z=\sum_{n=0}^{\infty}t(n)x^n. \tag{3.4}$$

Proof. Consider the level 6 functions Z and X defined by

$$Z = \frac{1}{4}\left(6P(q^6) - 3P(q^3) + 2P(q^2) - P(q)\right)$$

and

$$X = \left(\frac{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}{Z}\right)^2.$$

It is known, e.g., [5, Theorem 3.1], that in a neighborhood of $X = 0$,

$$Z = \sum_{n=0}^{\infty} \binom{2n}{n} u(n) X^n,$$

where the coefficients $u(n)$ are given by the formula

$$u(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j},$$

or equivalently by the three-term recurrence relation

$$(n+1)^2 u(n+1) = (10n^2 + 10n + 3)u(n) - 9n^2 u(n-1)$$

and initial condition $u(0) = 1$. It follows from the recurrence relation that Z satisfies a third-order linear differential equation

$$X^2(1-4X)(1-36X)\frac{d^3Z}{dX^3} + 3X(1-60X+288X^2)\frac{d^2Z}{dX^2} + (1-132X+972X^2)\frac{dZ}{dX} = 6(1-18X)Z. \tag{3.5}$$

By using the definitions of z, x, Z, and X given above, it is routine to check that

$$x = \left.\frac{\sqrt{X}}{1+2\sqrt{X}}\right|_{q\to q^2} \quad \text{and} \quad z = \left.\left(1+2\sqrt{X}\right)Z\right|_{q\to q^2}. \tag{3.6}$$

On making this change of variables in the differential equation (3.5), we find that

$$x^2(1+4x)(1-4x)(1-8x)\frac{d^3z}{dx^3} + 3x(1-12x-32x^2+320x^3)\frac{d^2z}{dx^2} + (1-28x-116x^2+1536x^3)\frac{dz}{dx} = 2(1+10x-192x^2)z. \tag{3.7}$$

Substitution of the series expansion (3.4) into this differential equation produces the recurrence relation (3.3). □

We also have the following differentiation formula.

Theorem 3.2. *Let x and z be as for Theorem 3.1. Then,*

$$q\frac{\mathrm{d}x}{\mathrm{d}q} = z\,x\,\sqrt{(1+4x)(1-4x)(1-8x)}. \tag{3.8}$$

Proof. With X and Z as in the proof of Theorem 3.1, it is known, e.g., [5, Section 5.2], that

$$q\frac{\mathrm{d}X}{\mathrm{d}q} = Z\,X\,\sqrt{(1-4X)(1-36X)}.$$

The required formula follows by the change of variables given by (3.6). □

The differential equation (3.7) and the differentiation formula (3.8) were obtained independently using a different method by D. Ye [23].

Theorems 3.1 and 3.2 can be used in a theorem of H. H. Chan, S. H. Chan, and Z.-G. Liu [3, Theorem 2.1] to produce a family of series for $1/\pi$ of the form

$$\frac{1}{2\pi}\times\sqrt{\frac{24}{N}} = \sqrt{(1+4x_N)(1-4x_N)(1-8x_N)}\sum_{n=0}^{\infty}(n+\lambda_N)\,t(n)\,x_N^n, \tag{3.9}$$

where N is a positive integer and

$$x_N = x\left(\pm e^{-2\pi\sqrt{N/24}}\right).$$

The formula for λ_N is given in [3] but it is more complicated so we do not reproduce it here. In practice, since λ_N is an algebraic number, its value can be recovered symbolically by computing a sufficiently precise approximation. A list of values for which x_N is rational, together with the corresponding values of N and λ_N, is given in Table 1. The obvious symmetry in the table between $x(q)$ and $x(-q)$ is explained by the identity

$$\frac{1}{x(q)} + \frac{1}{x(-q)} = 4,$$

which is a trivial consequence of the definition of $x(q)$ and properties of even and odd functions.

The values in Table 1 appear to be the only positive integers N that give rise to rational values of x. Other algebraic values can be determined, e.g., $N = 11$ and $q = \exp(-2\pi\sqrt{11/24})$ gives $x_{11} = 1/(38+6\sqrt{33})$ and $\lambda_{11} = 58/(165+19\sqrt{33})$. The values in the table corresponding to $N = 1$ give rise to divergent series and are not part of the conjectures.

Table 1 Data to accompany the series (3.9)

N	$q = \exp(-2\pi\sqrt{N/24})$		$q = -\exp(-2\pi\sqrt{N/24})$	
	x_N	λ_N	x_N	λ_N
1	1/8	does not converge	−1/4	does not converge
3	1/12	1/4	−1/8	1/2
5	1/20	1/4	−1/16	2/5
7	1/32	5/21	−1/28	1/3
13	1/104	1/5	−1/100	3/13
17	1/200	143/238	−1/196	67/340

Conjectures (3.1)–(3.10) in [19] can be explained by the values corresponding to $N = 3, 5, 7, 13$, and 17 in Table 1 and the series (3.9). To complete the proof of these conjectures, we require the satellite identity

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}^2\binom{2m-2k}{m-k}x^{m+k}\left[x+k(1+x)+m\left(x-\frac{1}{2}\right)\right]=0 \tag{3.10}$$

that holds in a neighborhood of $x = 0$, to produce an analog of Lemma 2.3. We omit the details, as they are similar.

Remark 3.3. We note that satellite identities such (3.10) may, in fact, be first guessed, then proved using multiple Wilf–Zeilberger. The idea is to assume that the function inside the square brackets takes the form

$$(a_0 + a_1x + a_2x^2 + \cdots) + k(b_0 + b_1x + b_2x^2 + \cdots) + m(c_0 + c_1x + c_2x^2 + \cdots),$$

where a_i, b_i, c_i are undetermined rational coefficients (for i less than some chosen M). The coefficients a_i, b_i, c_i can be found by expanding in powers of x and equating coefficients to obtain a linear system. Alternatively, by replacing x with a sufficiently small irrational number, evaluating the sum to high precision and equating it to 0, a_i, b_i, c_i may then be determined using an integer relations algorithm, such as PSLQ. See [20, Section 12.4.2].

4 Conjectures (2.4)–(2.9): Level 4

Conjectures (2.4)–(2.9) in [19] are based on series of the form

$$\sum_{n=0}^{\infty}\binom{2n}{n}z^n\sum_{k=0}^{\infty}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k.$$

The numerical data given in [19] suggest that $z = x/(1+4x)^2$ for Conjectures (2.4), (2.7), and (2.8), and $z = -x/(1-8x)$ for Conjectures (2.5), (2.6), and (2.9). Expanding in powers of x leads to:

Theorem 4.1. *The following identities hold in a neighborhood of $x = 0$:*

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12\\ 1, \; 1\end{matrix}; 64x^2\right) &= \sum_{n=0}^{\infty}\binom{2n}{n}^3 x^{2n}\\
&= \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+4x)^{2n+1}}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k\\
&= \sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n x^n}{(1-8x)^{n+1/2}}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k.
\end{aligned}$$

The first equality in Theorem 4.1 is trivial. The substance of the theorem is in the other equalities, which first appeared in [20, Theorem 12.3], and proved using the multiple Wilf–Zeilberger algorithm.

The corresponding satellite identities can be determined, and these give rise to:

Theorem 4.2. *The following identities hold in a neighborhood of $x = 0$:*

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{2n}{n}(an+b)\frac{x^n}{(1+4x)^{2n}}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k&\\
= \sum_{n=0}^{\infty}\binom{2n}{n}^3(An+B)x^{2n}& \qquad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{2n}{n}(cn+d)\frac{(-x)^n}{(1-8x)^{n}}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k&\\
= \sum_{n=0}^{\infty}\binom{2n}{n}^3(Cn+D)x^{2n},& \qquad (4.2)
\end{aligned}$$

where A, B, C, and D are given by

$$A = \frac{3a}{2}\frac{(1+4x)^2}{1-4x}, \qquad B = (1+4x)\left(b + \frac{4ax}{1-4x}\right),$$

$$C = \frac{3c}{2}\frac{(1-8x)^{3/2}}{(1-16x^2)} \quad \textit{and} \quad D = \sqrt{1-8x}\left(d - \frac{4cx(1-2x)}{1-16x^2}\right).$$

Conjectures (2.4), (2.7), and (2.8) in [19] are obtained by taking

$$(a, b, x) = \left(12, 1, \frac{1}{16}\right), \quad \left(476, 103, \frac{-1}{64}\right), \quad \left(140, 19, \frac{1}{64}\right)$$

respectively, in (4.1). The first set of parameter values produces a constant multiple of the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \left(n + \frac{1}{6}\right) \frac{1}{256^n} = \frac{2}{3\pi}, \tag{4.3}$$

which is originally due to Ramanujan [14, Eq. (28)]. The other two sets of parameter values give constant multiples of the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \left(n + \frac{5}{42}\right) \frac{1}{4096^n} = \frac{8}{21\pi} \tag{4.4}$$

which is also due to Ramanujan [14, Eq. (29)]. The series (4.3) and (4.4) correspond to the values $N = 3$ and $N = 7$ in [4, Table 6].

In a similar way, Conjectures (2.5), (2.6), and (2.9) in [19] are obtained by taking

$$(c, d, x) = \left(10, 1, \frac{-1}{16}\right), \quad \left(170, 37, \frac{1}{64}\right), \quad \left(1190, 163, -\frac{1}{64}\right)$$

respectively, in (4.2). The first set of parameter values gives a constant multiple of Ramanujan's series (4.3), while the other two sets of values both lead to multiples of (4.4).

The parameter values

$$(a, b, x) = \left(20, 7, \frac{-1}{16}\right) \quad \text{and} \quad (c, d, x) = \left(30, 11, \frac{1}{16}\right)$$

also lead to multiples of Ramanujan's series (4.3). However, the respective series on the left-hand sides of (4.1) and (4.2) are divergent, hence they are not listed among the conjectures in [19].

5 Conjectures (2.1)–(2.3): Level 6

Conjectures (2.1)–(2.3) of [19] are based on series of the form

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} x^k.$$

Numerical data suggest that $z = x/(1-4x)$ and this leads to:

Theorem 5.1. *The following identity holds in a neighborhood of $x = 0$:*

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{(1-4x)^{n+1/2}} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} x^k = \sum_{n=0}^{\infty} u(n) x^n \tag{5.1}$$

where

$$(n+1)^3 u(n+1) = (2n+1)(10n^2+10n+4)u(n) - 64n^3 u(n-1), \quad u(0) = 1,$$

or equivalently,

$$u(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}.$$

The numbers $u(n)$ are called *Domb numbers*. They are the sequence `A002895` in Sloane's database [18]. The series for $1/\pi$ that arise from the Domb numbers were first studied in [3]; see also the classification in [4, Table 9].

Conjectures (2.1), (2.2), and (2.3) in [19] involve the values $x = -1/8$, $x = -1/32$, and $x = 1/64$, respectively. However, the series on the right-hand side of (5.1) converges for $|x| < 1/16$, so Conjecture (2.1) cannot be handled by this formula. To obtain a formula that is convergent for all three conjectures, we recall the identity [16, Theorem 3.1] that holds in a neighborhood of $x = 0$:

$$\sum_{n=0}^{\infty} u(n) x^n = \sum_{n=0}^{\infty} \binom{3n}{n} \binom{2n}{n}^2 \frac{x^{2n}}{(1-4x)^{3n+1}}. \tag{5.2}$$

The identities (5.1) and (5.2) can be combined and used to produce the following:

Theorem 5.2. *The following identity holds in a neighborhood of $x = 0$:*

$$\sum_{n=0}^{\infty} \binom{2n}{n} (an+b) \frac{x^n}{(1-4x)^n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} x^k = \sum_{n=0}^{\infty} \binom{3n}{n} \binom{2n}{n}^2 (An+B) \frac{x^{2n}}{(1-4x)^{3n}}, \tag{5.3}$$

where A and B are given by

$$A = \frac{4a(1+2x)(1-x)}{3(1-4x+8x^2)\sqrt{1-4x}} \quad \text{and} \quad B = \frac{1}{\sqrt{1-4x}} \left(\frac{2ax(1-2x)}{1-4x+8x^2} + b \right).$$

The series on the right-hand side of (5.3) converges for $-1/2 < x < 1/16$. Conjectures (2.1), (2.2), and (2.3) correspond to the parameter values

$$(a,b,x) = \left(13, 4, \frac{-1}{8}\right), \quad \left(290, 61, \frac{-1}{32}\right) \quad \text{and} \quad \left(962, 137, \frac{1}{64}\right),$$

respectively. These values produce multiples of the series

$$\sum_{n=0}^{\infty} \binom{3n}{n}\binom{2n}{n}^2 \left(n+\frac{1}{6}\right)\frac{1}{6^{3n}} = \frac{\sqrt{3}}{2\pi}, \tag{5.4}$$

$$\sum_{n=0}^{\infty} \binom{3n}{n}\binom{2n}{n}^2 \left(n+\frac{2}{15}\right)\frac{1}{2^n \times 3^{6n}} = \frac{9}{20\pi} \tag{5.5}$$

and

$$\sum_{n=0}^{\infty} \binom{3n}{n}\binom{2n}{n}^2 \left(n+\frac{4}{33}\right)\frac{1}{15^{3n}} = \frac{5\sqrt{3}}{22\pi}. \tag{5.6}$$

The last two of these series are originally due to Ramanujan [14, Eqs. (31) and (32)] and the other series is due to J. M. Borwein and P. M. Borwein [2, p. 190]. These series correspond to the values $N = 2$, 4, and 5 in [4, Table 5]. This completes our discussion of Conjectures (2.1)–(2.3) in [19].

6 Conjectures (2.12)–(2.14), (2.18) and (2.20)–(2.22)

Conjectures (2.10)–(2.28) in [19] involve series of the form

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k} x^k.$$

We present two cases where the data allow z to be identified as a function of x.

6.1 Conjectures (2.13), (2.18)*, and* (2.22): *Level* 6

The data for these conjectures satisfy the relation $z = -x/(1-16x)$. This leads us to discover:

Theorem 6.1. *The following identity holds in a neighborhood of $x = 0$:*

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-x)^n}{(1-16x)^{n+1/2}} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k} x^k = \sum_{n=0}^{\infty} u(n) x^n, \tag{6.1}$$

where $\{u(n)\}$ are the Domb numbers introduced in Theorem 5.1.

Just as for Theorem 5.1, the radius of convergence of the series on the right-hand side of (6.1) is not large enough to handle all of the Conjectures (2.13), (2.18), and (2.22) in [19]. Therefore, we use (6.1) in conjunction with (5.2) to produce the identity

$$\sum_{n=0}^{\infty}\binom{2n}{n}(an+b)\frac{(-x)^n}{(1-16x)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}x^k = \sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2(An+B)\frac{x^{2n}}{(1-4x)^{3n}}, \tag{6.2}$$

where A and B are given by

$$A = \frac{4a(1-16x)^{3/2}(1+2x)}{3(1-4x)(1-8x)} \quad \text{and} \quad B = \frac{\sqrt{1-16x}}{1-4x}\left(b - \frac{4ax}{1-8x}\right).$$

The series on the right-hand side of (6.2) converges for $-1/2 < x < 1/16$. Conjectures (2.13), (2.18), and (2.22) in [19] correspond to the parameter values

$$(a,b,x) = \left(1, 0, \frac{-1}{8}\right), \quad \left(10, 1, \frac{-1}{32}\right) \quad \text{and} \quad \left(14, 3, \frac{1}{64}\right),$$

respectively. These values produce multiples of the series (5.4), (5.5), and (5.6), respectively. This completes our discussion of Conjectures (2.13), (2.18), and (2.22) in [19].

6.2 *Conjectures* (2.12), (2.14), (2.20)*, and* (2.21)*: Level* 6

The data for these conjectures satisfy the relation $z = x/(1+4x)^2$. Expanding in powers of x leads to:

Theorem 6.2. *The following identity holds in a neighborhood of $x = 0$:*

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+4x)^{2n+1}}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}x^k = \sum_{n=0}^{\infty} t(n)x^n, \tag{6.3}$$

where the sequence $\{t(n)\}$ is defined by the four-term recurrence relation

$$(n+1)^3 t(n+1) = -2n(n+1)(2n+1)t(n) + 16n(5n^2+1)t(n-1) - 96n(n-1)(2n-1)t(n-2) \tag{6.4}$$

and initial condition $t(0) = 1$. *The series on the left-hand side of* (6.3) *converges for* $-1/12 < x < 1/4$, *while the series on the right-hand side converges for* $|x| < 1/12$.

In order to gain access to properties of the sequence $\{t(n)\}$, we recall the following result of Chan et al. [3, Eq. (4.13)].

Lemma 6.3. *Let* z *and* y *be the level* 6 *modular forms defined by*

$$z = \prod_{j=1}^{\infty} \frac{(1-q^j)^4(1-q^{3j})^4}{(1-q^{2j})^2(1-q^{6j})^2} \quad \text{and} \quad y = q\prod_{j=1}^{\infty} \frac{(1-q^{2j})^6(1-q^{6j})^6}{(1-q^j)^6(1-q^{3j})^6}. \tag{6.5}$$

Let $\{u(n)\}$ *be the Domb numbers, which were defined in Theorem* 5.1. *Then in a neighborhood of* $y = 0$,

$$z = \sum_{n=0}^{\infty} (-1)^n u(n) y^n. \tag{6.6}$$

The next result gives a modular parameterization for the sequence $\{t(n)\}$. It also provides a connection with the Domb numbers.

Theorem 6.4. *Let* z *and* y *be the level* 6 *modular forms defined by* (6.5) *and let* $\{u(n)\}$ *be the Domb numbers, which were defined in Theorem* 5.1. *Let* Z *and* x *be defined by*

$$Z = (1+4y)z \quad \text{and} \quad x = \frac{y}{1+4y}. \tag{6.7}$$

Then in a neighborhood of $q = 0$,

$$\begin{aligned} Z &= \frac{1}{2}\left(3P(q^6) - P(q^2)\right) && (6.8)\\ &= \sum_{n=0}^{\infty} t(n)x^n && (6.9)\\ &= \sum_{n=0}^{\infty} (-1)^n u(n) \frac{x^n}{(1-4x)^{n+1}} && (6.10)\\ &= \sum_{n=0}^{\infty} \binom{3n}{n}\binom{2n}{n}^2 x^{2n}(1-4x)^n. && (6.11) \end{aligned}$$

Proof. By (6.5) and [10, Eqs. (32.66) and (33.2)], we have

$$\begin{aligned} Z &= (1+4y)z \\ &= \prod_{j=1}^{\infty} \frac{(1-q^j)^4(1-q^{3j})^4}{(1-q^{2j})^2(1-q^{6j})^2} + 4q\prod_{j=1}^{\infty} \frac{(1-q^{2j})^4(1-q^{6j})^4}{(1-q^j)^2(1-q^{3j})^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\left(12P(q^6) - 3P(q^3) - 4P(q^2) + P(q)\right) \\
&\quad + \frac{1}{6}\left(-3P(q^6) + 3P(q^3) + P(q^2) - P(q)\right) \\
&= \frac{1}{2}\left(3P(q^6) - P(q^2)\right).
\end{aligned}
$$

This proves (6.8).

Next, Chan et al. [3, Eq. (4.10)] showed that z satisfies a third-order differential equation with respect to y:

$$
y^2(1+4y)(1+16y)\frac{\mathrm{d}^3 z}{\mathrm{d}y^3} + 3y(1+30y+128y^2)\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} + (1+168y+448y^2)\frac{\mathrm{d}z}{\mathrm{d}y} + 4(1+16y)z = 0.
$$

On making the change of variables given by (6.7), we deduce that

$$
x^2(1-4x)^2(1+12x)\frac{\mathrm{d}^3 Z}{\mathrm{d}x^3} + 3x(1-4x)(1+10x-120x^2)\frac{\mathrm{d}^2 Z}{\mathrm{d}x^2} + (1+12x-576x^2+2304x^3)\frac{\mathrm{d}Z}{\mathrm{d}x} + 96x(6x-1)Z = 0.
$$

On expanding Z in powers of x and substituting into the differential equation, we obtain the recurrence relation (6.4). The proof of (6.9) may be completed by noting that $Z = 1$ and $x = 0$ when $q = 0$, therefore $t(0) = 1$.

To prove (6.10), use (6.6) and (6.7) to get

$$
Z = (1+4y)z = (1+4y)\sum_{n=0}^{\infty}(-1)^n u(n) y^n = \sum_{n=0}^{\infty}(-1)^n u(n)\frac{x^n}{(1-4x)^{n+1}}.
$$

Finally, (6.11) can be obtained by applying (5.2) to (6.10). □

Combining (6.3) with (6.9) and (6.11) gives the identity

$$
\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+4x)^{2n+1}}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}x^k = \sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2 x^{2n}(1-4x)^n. \quad (6.12)
$$

Equation (6.12) can be used to produce:

Theorem 6.5. *The following identity holds in a neighborhood of $x = 0$:*

$$\sum_{n=0}^{\infty}(an+b)\binom{2n}{n}\frac{x^n}{(1+4x)^{2n}}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}x^k$$
$$= \sum_{n=0}^{\infty}(An+B)\binom{3n}{n}\binom{2n}{n}^2 x^{2n}(1-4x)^n, \tag{6.13}$$

where

$$A = \frac{4a(1+4x)^2(1-6x)}{3(1-4x)^2} \quad \text{and} \quad B = (1+4x)\left(\frac{4ax}{1-4x}+b\right).$$

The series on the right-hand side of (6.13) *converges for* $-1/12 < x < 1/6$.

Conjectures (2.14), (2.20), and (2.21) in [19] correspond to the data

$$(a,b,x) = \left(6,-1,\frac{1}{12}\right), \quad \left(12,1,\frac{1}{36}\right) \quad \text{and} \quad \left(24,5,\frac{-1}{60}\right),$$

respectively. These values lead to multiples of the series (5.4), (5.5), and (5.6), respectively.

Conjecture (2.12) in [19] corresponds to the values $x = 1/6$, in which case the series on the right-hand side of (6.13) is divergent. Therefore, we proceed by a different method. By [12, Theorem 1], we have

$$\lim_{w\to 1^-}\sqrt{1-w}\sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2 n\left(\frac{w}{108}\right)^n = \frac{\sqrt{3}}{2\pi}.$$

Make the change of variables $w = 108x^2(1-4x)$ and observe that $w \to 1^-$ as $x \to (1/6)^-$. It follows that

$$\frac{\sqrt{3}}{2\pi} = \lim_{x\to(1/6)^-}\sqrt{(1-6x)^2(1+12x)}\sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2 n\,(x^2(1-4x))^n.$$

Now apply (6.13) with $a = 1$ and $b = -2$, and note that $4x/(1-4x) - 2$ vanishes at $x = 1/6$. This produces

$$\frac{\sqrt{3}}{2\pi} = \lim_{x\to(1/6)^-}\sqrt{1+12x}\cdot\frac{3(1-4x)^2}{4(1+4x)^2}\sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2$$
$$\times\left(\frac{4(1+4x)^2(1-6x)}{3(1-4x)^2}n + (1+4x)\left(\frac{4x}{1-4x}-2\right)\right)(x^2(1-4x))^n$$

$$= \lim_{x\to(1/6)^-} \sqrt{1+12x} \cdot \frac{3(1-4x)^2}{4(1+4x)^2}$$
$$\times \sum_{n=0}^{\infty}(n-2)\binom{2n}{n}\frac{x^n}{(1+4x)^{2n}}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}x^k$$
$$= \frac{3\sqrt{3}}{100}\sum_{n=0}^{\infty}(n-2)\binom{2n}{n}\left(\frac{3}{50}\right)^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}\left(\frac{1}{6}\right)^k.$$

This gives us a proof of Conjecture (2.12) in [19].

Equation (6.12) was deduced via a different path in [20, Theorem 12.4]: a Heun-type differential equation was obtained for the left-hand side, which was then explicitly solved and the solution transformed into the right-hand side. Conjecture (2.12) was also proved in [20], by first applying Clausen's theorem to convert the right-hand side of (6.12) into the square of a ${}_2F_1$, followed by evaluating the ${}_2F_1$'s with Gauss' second summation theorem and one of its contiguous versions.

There is also a companion result to Theorem 6.4:

Theorem 6.6. *Let z and y be the level 6 modular forms defined by (6.5). Let Z^* and x^* be defined by*

$$Z^* = (1+16y)z \quad \text{and} \quad x^* = \frac{y}{1+16y}.$$

Then in a neighborhood of $q=0$,

$$\begin{aligned} Z^* &= \frac{1}{2}\left(3P(q^3) - P(q)\right) \\ &= \sum_{n=0}^{\infty} v(n)(x^*)^n \\ &= \sum_{n=0}^{\infty}(-1)^n u(n)\frac{(x^*)^n}{(1-16x^*)^{n+1}} \\ &= \sum_{n=0}^{\infty}\binom{3n}{n}\binom{2n}{n}^2 (x^*)^n(1-16x^*)^{2n}, \end{aligned}$$

where the sequence $\{v(n)\}$ satisfies the recurrence relation

$$\begin{aligned}(n+1)^3 v(n+1) = (2n+1)(22n^2+22n+12)v(n) - 128n(5n^2+1)v(n-1) \\ + 1536n(n-1)(2n-1)v(n-2),\end{aligned}$$

and $\{u(n)\}$ are the Domb numbers.

It would be interesting to have an analog of Theorem 6.2 that involves the sequence $\{v(n)\}$.

7 Conjectures (6.3)–(6.13)

Conjectures (6.3)–(6.13) in [19] are based on the generating function

$$\sum_{n=0} \binom{2n}{n} z^n \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} x^k. \tag{7.1}$$

The numerical data for Conjectures (6.3)–(6.7) fit the relation

$$z = \frac{x}{(1-x)^2},$$

while for Conjectures (6.8)–(6.13), we have

$$z = -\frac{1}{2(1+4x)}.$$

We consider each case separately.

7.1 Conjectures (6.3)–(6.7): *Level* 14

Expanding in powers of x gives

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{(1-x)^{2n+1}} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} x^k = \sum_{n=0}^{\infty} a(n) x^n \tag{7.2}$$

where

$$\begin{aligned}(n+1)^3 a(n+1) &= (2n+1)(3n^2+3n+1)a(n) \\ &\quad + n(47n^2+4)a(n-1) + 14n(n-1)(2n-1)a(n-2)\end{aligned}$$

and $a(0) = 1$. The series expansion of a function in [24, Eq. (5)] involves the same coefficients, that is,

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{(1+x)^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \binom{2k}{k} x^k = \sum_{n=0}^{\infty} a(n) x^n. \tag{7.3}$$

The identities (7.2) and (7.3) can be used to establish the interesting result

$$\sum_{n=0}^{\infty}\left((1-x)^2n+(\lambda-1)\right)\binom{2n}{n}\frac{x^n}{(1+x)^{2n+1}}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\binom{2k}{k}x^k$$
$$=\sum_{n=0}^{\infty}\left((1+x)^2n+(\lambda+1)\right)\binom{2n}{n}\frac{x^n}{(1-x)^{2n+1}}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}x^k \quad (7.4)$$

which holds for any constant λ. This, in turn, can be used with the results in [24] to show that Conjectures (6.3)–(6.7) are equivalent to Conjectures (VII5), (VII1), (VII3), (VII4), and (VII6) in [19], respectively. To see the correspondence, compare the values of x in [24, Table 1] with the arguments of P_k in [19, Eqs. (6.3)–(6.7)]. Since Conjectures (VII1) and (VII3)–(VII6) have been proved in [24], the truth of Conjectures (6.3)–(6.7) follows from (7.4).

Before continuing to the next set of conjectures, we offer the following additional comments about the sequence $\{a(n)\}$. Equating coefficients of x^n in (7.2) and (7.3) leads to the following formulas for $a(n)$ as sums of binomial coefficients, respectively:

$$a(n)=\sum_{j,k}\binom{n+j}{2j+2k}\binom{2j+2k}{j+k}\binom{2k}{k}^2\binom{k}{j}$$
$$=\sum_{j,k}(-1)^{n-j}\binom{n+j}{2j+2k}\binom{2j+2k}{j+k}\binom{2k}{k}\binom{j+2k}{k}\binom{j+k}{k}. \quad (7.5)$$

It can be shown that

$$\sum_{n=0}^{\infty}a(n)\left(\frac{x}{1+5x+8x^2}\right)^{n+1}=\sum_{n=0}^{\infty}A(n)\left(\frac{x}{1+9x+8x^2}\right)^{n+1}, \quad (7.6)$$

where

$$(n+1)^3A(n+1)=(2n+1)(11n^2+11n+5)A(n)$$
$$-n(121n^2+20)A(n-1)+98n(n-1)(2n-1)A(n-2)$$

and $A(0)=1$. The sequence $\{A(n)\}$ was first studied in [12, Example 6]. It was shown in [9] that the sequence $\{A(n)\}$ can be parameterized by level 14 modular forms. The modular parameterization for $\{a(n)\}$ is inherited from this by (7.6). Both $\{a(n)\}$ and $\{A(n)\}$ possess many remarkable arithmetic properties that are beyond the scope of this work; we plan to discuss them in a forthcoming project in detail.

7.2 *Conjectures* (6.8)–(6.13): *Level* 2

The data for Conjectures (6.8)–(6.13) in [19] suggest that z and x in the generating function (7.1) are related by $z = -1/(2(1+4x))$. We replace x with $4x$ throughout, and consider the function

$$g(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{2^n(1+16x)^{n+1/2}} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (4x)^k.$$

The series g can be seen to converge in a neighborhood of $x = 0$ by noting that the nonzero terms in the inner sum occur only when $\lceil n/2 \rceil \le k \le n$, and so the series may be written in the form

$$g(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n (4x)^{\lceil n/2 \rceil}}{2^n(1+16x)^{n+1/2}} \sum_{k=\lceil n/2 \rceil}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (4x)^{k-\lceil n/2 \rceil}.$$

Expanding in powers of x gives

$$g(x) = \sum_{n=0}^{\infty} \binom{4n}{2n} \binom{2n}{n}^2 x^{2n}.$$

This can be used to produce the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} (an+b) \frac{(-1)^n}{2^n(1+16x)^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (4x)^k = \sum_{n=0}^{\infty} \binom{4n}{2n} \binom{2n}{n}^2 (An+B) x^{2n}, \tag{7.7}$$

where

$$A = \frac{4a(1+16x)^{3/2}}{1-48x} \quad \text{and} \quad B = (1+16x)^{1/2} \left(b + \frac{32ax}{1-48x} \right).$$

Conjectures (6.8)–(6.13) in [19] correspond to the data[2]

[2]Multiply the argument of P_k in each of the Conjectures (6.8)–(6.13) in [19] by 4, and then take the reciprocal to get the values of x in the data.

$$(a,b,x)=\left(130,41,\frac{-1}{784}\right),\ \left(46,13,\frac{1}{784}\right),\ \left(510,143,\frac{-1}{1584}\right),\ \left(42,11,\frac{1}{1584}\right)$$

and

$$\left(1848054,309217,\frac{-1}{396^2}\right),\ \left(171465,28643,\frac{1}{396^2}\right),$$

respectively. If either of the two data sets corresponding to $\pm 1/784$ are inserted in (7.7), the results are multiples of the series

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}^2\left(n+\frac{3}{40}\right)\frac{1}{28^{4n}}=\frac{49\sqrt{3}}{360\pi}.$$

Similarly, the data corresponding to $\pm 1/1584$ produce multiples of the series

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}^2\left(n+\frac{19}{280}\right)\frac{1}{1584^{2n}}=\frac{9\sqrt{11}}{140\pi},$$

while the data corresponding to $\pm 1/396^2$ lead to

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}^2\left(n+\frac{1103}{26390}\right)\frac{1}{396^{4n}}=\frac{9801\sqrt{2}}{105560\pi}.$$

These are Ramanujan's series [14, Eqs. (42)–(44)]. They correspond to the values $N=9, 11, 29$, and $q>0$ in [4, Table 4].

8 Further examples: the \$520 series

We mention one further set of examples for which the techniques of this paper can be used. The following identity holds in a neighborhood of $x=0$:

$$\sum_{n=0}^{\infty}\binom{2n}{n}(an+b)\frac{x^n}{(1+2x)^{2n}}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n-2k}{n-k}x^k =\sum_{n=0}^{\infty}(An+B)\left\{\sum_{k=0}^{\infty}\binom{n}{k}^4\right\}x^n, \quad (8.1)$$

Table 2 Specialization of the two-variable special series

Underlying series	Specialization	Reference	Level
$\sum_n \binom{2n}{n} z^n \sum_k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} x^k$	$z = \frac{x}{(1+4x)^2}$	Eq. (4.1)	4
	$z = \frac{-x}{1-8x}$	Eq. (4.2)	4
$\sum_n \binom{2n}{n} z^n \sum_k \binom{n}{k}^2 \binom{n+k}{k} x^k$	$z = \frac{x}{1-4x}$	Eq. (5.1)	6
	$x = \frac{1}{t+1},\ z = t^2$	[24]	3
$\sum_n \binom{2n}{n} z^n \sum_k \binom{2k}{k}^2 \binom{2n-2k}{n-k} x^k$	$z = \frac{x}{(1+4x)^2}$	Eq. (6.3)	6
	$z = \frac{-x}{1-16x}$	Eq. (6.1)	6
$\sum_n \binom{2n}{n} z^n \sum_k \binom{2k}{k}^2 \binom{k}{n-k} x^k$	$z = \frac{x}{(1-x)^2}$	Eq. (7.2)	14
	$z = \frac{-1}{2(1+4x)}$	Eq. (7.7)	2
$\sum_n \binom{2n}{n} z^n \sum_k \binom{n}{k}^2 \binom{2n-2k}{n-k} x^k$	$z = \frac{x}{(1+2x)^2}$	Eq. (8.1)	10
$\sum_n \binom{2n}{n} z^n \sum_k \binom{n}{k}^2 \binom{2n-2k}{n} x^k$	$x = t^2,\ z = \frac{t}{(1+3t)^2}$	Eq. (8.3)	14
	$z = \frac{x}{1+4x}$	Eq. (9.1)	10
$\sum_n \binom{2n}{n} z^n \sum_k \binom{n}{k}\binom{n+k}{n}\binom{2k}{k} x^k$	$z = \frac{x}{(1+x)^2}$	[24, Eq. (5)]	7

where

$$A = \frac{4a(1-x)(1+2x)^2}{5(1-4x)} \quad \text{and} \quad B = (1+2x)\left(b + \frac{6ax(2-x)}{5(1-4x)}\right).$$

Taking $a = 5440$, $b = 1201$, and $x = -1/64$ gives

$$\sum_{n=0}^{\infty} \binom{2n}{n}(5440n+1201)\left(\frac{-4}{31}\right)^{2n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-2k}{n-k}\left(\frac{-1}{64}\right)^k = \frac{62465}{16}\sum_{n=0}^{\infty}\left(n+\frac{1}{4}\right)\left\{\sum_{k=0}^{n}\binom{n}{k}^4\right\}\left(\frac{-1}{16}\right)^n. \quad (8.2)$$

The series on the left-hand side is [19, Eq. (3.24′)], which is equivalent to another identity [19, Eq. (3.24)] for which a \$520 prize was offered to the first correct solution. That solution was given by Rogers and Straub [17]. The value of the series on the right is given by [8, Theorem 5.3, $N = 9$]. Hence, we obtain another proof of the "\$520 challenge" series.

The identity (8.1) also provides alternative proofs of (3.28), (3.11′), (3.13′), (3.15′), (3.17′), and (3.25′) in [19], that were proved by Rogers and Straub [17]. We note here that proofs for (3.11′), (3.13′), and (3.15′), as well as (3.16′), (3.18′), and (3.19′), were given in [20].

The identities (3.12′), (3.14′), and (3.18′) are equivalent to (3.12), (3.14), and (3.18). They can be handled using

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+3x)^{2n+1}}\sum_{k=0}^{\infty}\binom{n}{k}^2\binom{2n-2k}{n}x^{2k}=\sum_{n=0}^{\infty}a(n)x^n \tag{8.3}$$

where $a(n)$ is the level 14 sequence that appears in (7.2) and (7.3). Equating coefficients of x^n gives yet another formula for $a(n)$ as a sum of binomial coefficients, to go along with (7.5), namely

$$a(n)=\sum_{j,k}\binom{n+j-k}{2j+2k}\binom{2j+2k}{j+k}\binom{j+k}{k}^2\binom{2j}{j+k}(-3)^{n-j-3k}.$$

9 Summary and afterthoughts

Table 2 summarizes the specializations of the two-variable special series used in this work: the resulting single-variable series are solutions of third-order linear differential equations, for which formulas for $1/\pi$ are already established in the literature. Most of the entries in Table 2 were originally guessed on the basis of Sun's conjectural identities in [19], but we also performed an independent computer investigation to search for other linear and quadratic specializations of the underlying series. The only additional series produced by the search is

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+4x)^{n+1/2}}\sum_{k=0}^{\infty}\binom{n}{k}^2\binom{2n-2k}{n}x^k=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}^4\right\}x^n. \tag{9.1}$$

One of the big surprises of our project is a solid presence, in modular parameterizations of third-order linear differential equations, of level 14 modular forms and functions (cf. Sections 7 and 8); at the same time, the similarly exotic level 15 [9] does not show up at all.

We note that the nonspecialized generating functions from [19] are expected to be representable as products of two power series, each satisfying a second-order equation; e.g., see the final Question in [24]. This expectation is shown to be true in many cases and it is the driving force behind the universal methods of establishing Sun's conjectures and similar identities in [6, 11, 17, 21, 22]. Two further examples of such factorizations follow from the two-variable identities

$$\sum_n \left(\frac{z}{y}\right)^n \sum_k \binom{n-k}{k}\binom{2k}{k}\binom{2n-2k}{n-k}^2 \left(\frac{y(y^2-1)}{4z}\right)^k = \sum_{m=0}^{\infty} \binom{2m}{m}^2 P_m\left(\frac{y^2+1}{2y}\right) z^m, \tag{9.2}$$

$$\sum_n \binom{2n}{n}\left(\frac{y^2-1}{4y^2}\right)^n \sum_k \binom{n}{k}^2 \binom{2k}{k}\left(\frac{4z}{y^2-1}\right)^k = y\sum_{m=0}^{\infty} \binom{2m}{m}^2 P_{2m}(y) z^m \tag{9.3}$$

and the corresponding factorizations [6, 22] of generating functions of Legendre polynomials

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

The former transformation (9.2) allows one to deal with [19, Conjecture (3.29)] (namely, by making it equivalent to [19, Eq. (I3)] established in [6]), while the latter one (9.3) paves the ground for proving the family of conjectures (3.N′) on Sun's list [19] in exactly the same way as in [17].

A drawback of using such two-variable factorizations in the proofs of the formulas for $1/\pi$ is the relatively cumbersome analysis: compare our proof of Sun's Conjecture (3.24′) from Section 8 with the proof of his (equivalent) Conjecture (3.24) given in [17]. An advantage is that transformations are also available for the two-variable series. One such example,

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}} y^k = \left(-\frac{16y}{1+y}\right)^n \sum_{k=0}^{n} \binom{k}{n-k}\binom{2k}{k}^2 \left(-\frac{(1+y)^2}{16y}\right)^k,$$

follows from the classical Whipple's quadratic transformation and reduces the verification of Sun's [19, Conjecture (6.14)] to one related to the generating function

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} x^k$$

considered in Section 7. Unfortunately, the corresponding values $x = -9/20$ and $z = -1/216$ or $z = -5/216$ (depending on whether $y = 1/5$ or $y = 5$) do not match the patterns we have discovered.

Acknowledgements We thank the referee for helpful comments and suggestions.

References

1. G.E. Andrews, R. Askey, R. Roy, *Special Functions* (Cambridge University Press, Cambridge, 1999)
2. J.M. Borwein, P.B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity* (Wiley, New York, 1987)
3. H.H. Chan, S.H. Chan, Z.-G. Liu, Domb's numbers and Ramanujan-Sato type series for $1/\pi$. Adv. Math. **186**, 396–410 (2004)
4. H.H. Chan, S. Cooper, Rational analogues of Ramanujan's series for $1/\pi$. Math. Proc. Camb. Phil. Soc. **153**, 361–383 (2012)
5. H.H. Chan, Y. Tanigawa, Y. Yang, W. Zudilin, New analogues of Clausen's identities arising from the theory of modular forms. Adv. Math. **228**, 1294–1314 (2011)
6. H.H. Chan, J.G. Wan, W. Zudilin, Legendre polynomials and Ramanujan-type series for $1/\pi$. Isr. J. Math. **194**, 183–207 (2013)
7. H.H. Chan, W. Zudilin, New representations for Apéry-like sequences. Mathematika **56**, 107–117 (2010)
8. S. Cooper, Level 10 analogues of Ramanujan's series for $1/\pi$. J. Ramanujan Math. Soc. **27**, 59–76 (2012)
9. S. Cooper, D. Ye, Level 14 and 15 analogues of Ramanujan's elliptic functions to alternative bases. Trans. Am. Math. Soc. **368**, 7883–7910 (2016)
10. N. Fine, *Basic Hypergeometric Series and Applications* (American Mathematical Society, Providence, RI, 1988)
11. J. Guillera, A family of Ramanujan–Orr formulas for $1/\pi$. Integral Transform. Spec. Funct. **26**, 531–538 (2015)
12. J. Guillera, W. Zudilin, Ramanujan-type formulae for $1/\pi$: the art of translation, in *The Legacy of Srinivasa Ramanujan,* eds. by B.C. Berndt, D. Prasad. Ramanujan Mathematical Society Lecture Notes Series, vol. 20 (2013), pp. 181–195
13. M. Petkovsek, H. Wilf, D. Zeilberger, $A = B$ (A.K. Peters, Wellesley, 1996)
14. S. Ramanujan, Modular equations and approximations to π. Quart. J. Math (Oxford) **45**, 350–372 (1914). Reprinted in [15, pp. 23–39] (2000)
15. S. Ramanujan, *Collected Papers, 3rd printing* (American Mathematical Society Chelsea, Providence, RI, 2000)
16. M. Rogers, New $_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$. Ramanujan J. **18**, 327–340 (2009)
17. M. Rogers, A. Straub, A solution of Sun's $520 challenge concerning $520/\pi$. Int. J. Number Theory **9**, 1273–1288 (2013)
18. N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences (2015), published electronically at https://oeis.org
19. Z.-W. Sun, List of conjectural series for powers of π and other constants (2014), 33 pp. http://arxiv.org/abs/1102.5649v47
20. J.G. Wan, *Random walks, elliptic integrals and related constants,* Ph.D. Thesis (University of Newcastle, NSW, Australia, 2013)
21. J.G. Wan, Series for $1/\pi$ using Legendre's relation. Integral Transform. Spec. Funct. **25**, 1–14 (2014)

22. J.G. Wan, W. Zudilin, Generating functions of Legendre polynomials: a tribute to Fred Brafman. J. Approx. Theory **164**, 488–503 (2012)
23. D. Ye, Private communication to S. Cooper (11 November, 2013)
24. W. Zudilin, A generating function of the squares of Legendre polynomials. Bull. Aust. Math. Soc. **89**, 125–131 (2014)

Integral Points on a Very Flat Convex Curve

Jean-Marc Deshouillers and Georges Grekos

To Krishnaswami Alladi, for his 60th birthday

Abstract The second named author studied in 1988 the possible relations between the length ℓ, the minimal radius of curvature r and the number of integral points N of a strictly convex *flat* curve in $\mathbb{R}^2$, stating that $N = O(\ell/r^{1/3})$ (*), a best possible bound even when imposing the tangent at one extremity of the curve; here *flat* means that one has $\ell = r^\alpha$ for some $\alpha \in [2/3, 1)$. He also proved that when $\alpha \leq 1/3$, the quantity N is bounded. In this paper, the authors prove that in general the bound (*) cannot be improved for *very flat* curves, i.e. those for which $\alpha \in (1/3, 2/3)$; however, if one imposes a 0 tangent at one extremity of the curve, then (*) is replaced by the sharper inequality $N \leq \ell^2/r + 1$.

Keywords Geometry of numbers · Integer points · Strictly convex curves

2010 Mathematics Subject Classification 11H06

Jean-Marc Deshouillers acknowledges the support of the binational research project MuDeRa, funded by the French and Austrian Science Funds ANR and FWF. The research of Georges Grekos has been supported by the French grant CAESAR ANR-12-BS01-0011.

J.-M. Deshouillers (✉)
Institut Mathématique de Bordeaux, UMR 5251 Université de Bordeaux,
CNRS, Bordeaux INP, 33405 TALENCE Cedex, France
e-mail: jean-marc.deshouillers@math.u-bordeaux.fr

G. Grekos
Institut Camille Jordan UMR 5208, Université de Saint-Etienne
(membre de l'Université de Lyon), CNRS, 23 rue du Dr Paul Michelon,
42023 SAINT-ÉTIENNE, France
e-mail: grekos@univ-st-etienne.fr

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_13

1 Introduction

In 1926, V. Jarník [2] started the study of integral points on general strictly convex curves in the euclidean plane, showing that the number of points of such a curve of length ℓ cannot be larger than $(C + o(1))\ell^{2/3}$ (where $C = 3(16\pi)^{-1/3}$) and that this bound is optimal. The second named author [1] considered in 1988 the case of *flat* curves, i.e. curves with a radius of curvature significantly larger than their length. Let us present the results of [1], starting with some definitions.

We let Γ be a $\mathscr{C}^2$ strictly convex curve in the euclidean plane $\mathbb{R}^2$. More precisely

$$\Gamma = \{M(t) = (x(t), y(t)) \in \mathbb{R}^2 : 0 \leq t \leq 1\}, \tag{1}$$

where x and y are two $\mathscr{C}^2$ functions on [0, 1], such that

$$\forall t \in [0, 1] : x'(t)y''(t) - x''(t)y'(t) \neq 0. \tag{2}$$

We denote the length of Γ by $\ell(\Gamma)$; to each point M on Γ, we associate the radius of curvature of Γ at M, denoted by $R(M)$, and define the *minimal radius of curvature* of Γ by the relation

$$r(\Gamma) = \min\{R(M) : M \in \Gamma\}; \tag{3}$$

notice that (2) implies $r(\Gamma) > 0$.

The number of “integral points” on Γ, i.e. points with coordinates in $\mathbb{Z}^2$, is denoted by $N(\Gamma)$.

Finally, the quantity α is defined by

$$\alpha = \frac{\log \ell(\Gamma)}{\log r(\Gamma)}. \tag{4}$$

The main results of [1] (Théorème 1 and Théorème 2) can be rephrased as Theorem 1.1 and Theorem 1.2 below.

Theorem 1.1. *If $r \geq 1$ and $\ell \geq r^{1/3}$, then, for any curve Γ with length ℓ and minimal radius of curvature r, we have*

$$N(\Gamma) \leq 2\ell r^{-1/3}. \tag{5}$$

The second result shows that this result is best possible, up to the numerical value of the constant, but moreover, that one can impose any value for the tangent at the origin of the curve Γ, determined by the vector

$$T_0(\Gamma) = \left(x'(0), y'(0)\right).$$

Theorem 1.2. *Let T be a nonzero vector in $\mathbb{R}^2$ and $\alpha \in (2/3, 1)$. For any $r \geq r_0(T, \alpha)$ there exists a curve Γ with $T_0(\Gamma) = T$, $\ell(\Gamma) = 10^5 r^\alpha$ such that*

$$N(\Gamma) \geq 10^{-6} \ell r^{-1/3}. \tag{6}$$

The first consequence of Theorem 1.1 is that for $r \geq 1$ and $\ell \leq r^{1/3}$ one has $N(\Gamma) \leq 2$: simply prolongate the curve with a suitable arc of a circle until its length is $r^{1/3}$. This implies that the case $\alpha \in [0, 1/3]$ is essentially trivial.

But Theorem 1.2 only deals with the case $\alpha \in (2/3, 1)$, hence a natural question arises: what happens when $\alpha \in (1/3, 2/3]$? It is conjectured in [1] that in this case one has

$$N(\Gamma) = O\left(\ell(\Gamma)^2 / r(\Gamma)\right). \tag{7}$$

Notice that $l^2/r < l/r^{-1/3}$ exactly when $\alpha < 2/3$.

The aim of this note is to show that this conjecture is partially correct, but partially false. More precisely, we have

Theorem 1.3. *Let Γ be a strictly convex curve such that $y'(0) = 0$ and $\ell(\Gamma) \leq r(\Gamma)$. We have*

$$N(\Gamma) \leq \left(\ell(\Gamma)^2 / r(\Gamma)\right) + 1. \tag{8}$$

On the other hand, if we do not fix T_0, Theorem 1.2 can be extended to any $\alpha > 1/3$; moreover, we may ask the radius of curvature to keep the same size, up to factor $(1 + o(1))$, all over the curve.

Theorem 1.4. *For any $\alpha \in (1/3, 2/3)$, $c_1 > 1$ and $c_2 < 2^{-1/3}$ there exists $r_0 = r_0(\alpha, c_1, c_2)$ such that for any $r \geq r_0$, there exists a strictly convex curve Γ such that*

$$\forall M \in \Gamma : r(\Gamma) = r \leq R(M) \leq c_1 r \tag{9}$$

$$\ell(\Gamma) = r^\alpha, \tag{10}$$

$$N(\Gamma) \geq c_2 \ell(\Gamma) / r(\Gamma)^{1/3}. \tag{11}$$

2 Proof of Theorem 1.3

We first introduce a technical tool. Lemma 2.1 is true without assuming the second part of (*iii*); however we do not see how to prove it without using some clumsy limiting process; in our case of interest, the case of the circle, this limiting process can be easily performed as it is done in Corollary 2.2.

Lemma 2.1. *Let $a < b$ be two real numbers and f and g be in $\mathscr{C}^2_{\mathbb{R}}[a,b]$ with the properties*

$$(i) \quad f(a) \le g(a) \ \ and \ \ f'(a) = g'(a),$$
$$(ii) \quad \forall x \in [a,b] \ : \ f''(x) > 0 \ and \ g''(x) > 0,$$
$$(iii) \quad \forall x \in [a,b] \ : \ r_f(x) \ge r_g(x) \ and \ \ r_f(a) > r_g(a),$$

where $r_f(x)$ denotes the radius of curvature of the graph of f at the point $(x, f(x))$. Then, for all $x \in [a,b] \ : \ f(x) \le g(x)$.

Proof 1. We first prove the following

$$\forall x \in (a,b) \ : \ f'(x) < g'(x). \tag{12}$$

We recall that we have

$$r_f(x) = \frac{\left(1 + f'(x)^2\right)^{3/2}}{|f''(x)|}. \tag{13}$$

Thanks to (i) and (iii) we have $f''(a) < g''(a)$ and there exists $c \in (a,b]$ such that $f'(x) < g'(x)$ for all $x \in (a,c)$; we choose c to be maximal and prove by contradiction that $c = b$, which proves (12). If $c < b$, we have $f'(c) = g'(c)$; by Rolle's theorem, there exists $d \in (a,c)$ with $f''(d) = g''(d)$; but $f'(d) < g'(d)$, which implies $r_f(d) < r_g(d)$, a contradiction.
From (12), we get for $x \in (a,b)$:

$$f(x) = f(a) + \int_a^x f'(t)dt \le g(a) + \int_a^x g'(t)dt = g(x).$$

□

We state a corollary of this lemma which will be more convenient in the sequel.

Corollary 2.2. *Let $R > 0$, $a < b \le a + R$ and $f \in \mathscr{C}^2_{\mathbb{R}}[a,b]$ be such that*

$$(i) \quad f'(a) = 0,$$
$$(ii) \quad \forall x \in [a,b) \ : \ f''(x) > 0,$$
$$(iii) \quad \forall x \in [a,b) \ : \ r_f(x) \ge R.$$

Then, for all $x \in [a,b] \ : \ f(a) \le f(x) \le f(a) + R - \sqrt{R^2 - (x-a)^2}$.

Proof 2. For the lower bound, notice that f is convex and thus above its tangent at the point 0. For the upper bound, apply Lemma 2.1 on the interval $[a, a + R - 2/N]$ with functions f and

$$g_N(x) = f(a) + R - 1/N - \sqrt{(R - 1/N)^2 - (x-a)^2}$$

for large N; notice that the graph of g_N is an arc of the circle with radius $R - 1/N$ and centre $(0, f(a) + R - 1/N)$. □

We now prove Theorem 1.3.

Proof 3. Let Γ be given by (1) with $y'(0) = 0$; by (2) we have $x'(0) \neq 0$ and $y''(0) \neq 0$. Without loss of generality, we may assume $x'(0) > 0$ and $y''(0) > 0$, since changing the sign of x or that of y or both correspond to symmetries which do not change either the length of the curve, or its radius of curvature, or the number of its integral points. Furthermore, when $\ell \leq r$, as we assumed, the curve Γ may be seen as the graph of a function f_Γ defined on $[x(0), x(1)]$. Brief explanation: if we denote by s the arc length on Γ and θ the angle between the abscissa axis and the tangent to Γ, we have the relation $s' = R\theta'$; since at each point of Γ the radius of curvature of Γ is at least $r = r(\Gamma)$, we have $s' \geq r\theta'$; integrating this relation, we get that at each point of the curve we have $\theta \leq \ell/r \leq 1 < \pi/2$. We now apply Corollary 2.2 to the function f_Γ, with $R = r(\Gamma)$, $a = x(0)$, $b = x(1)$; we have $b - a = x(1) - x(0) = \int_0^1 \sqrt{x'(t)^2} dt \leq \ell \leq R$; thus, for any $x \in [a, b]$ we have $f(x) \in [f(a), f(a) + R - \sqrt{R^2 - \ell^2}]$, an interval with length at most $R - R\sqrt{1 - (\ell/R)^2} \leq \ell^2/R$, which contains at most $\ell^2/R + 1$ integral points. The number of integral points on Γ is at most the number of points with integral ordinates, and since the function f_Γ is increasing, we have (8). □

3 Proof of Theorem 1.4

Proof 4. Let $\alpha \in (1/3, 2/3)$. For given $r \geq 1$ we let X and H be defined by

$$\frac{(1 + 4X^2)^{3/2}}{2} = r \quad \text{and} \quad \int_X^{X+H} (1 + 4x^2)^{1/2} dx = r^\alpha. \tag{14}$$

We easily check that we have

$$X \sim 4^{-1/3} r^{1/3} \quad \text{and} \quad H \sim 2^{-1/3} r^{\alpha - 1/3} = o(X), \quad \text{as } r \to \infty. \tag{15}$$

We consider for Γ the graph of the function f defined by

$$\forall x \in [X, X + H] : f(x) = x^2.$$

By (14) and (15) at any point $M = (x, f(x))$, we have

$$r = \frac{(1 + 4X^2)^{3/2}}{2} \leq \frac{(1 + 4x^2)^{3/2}}{2} = R(M) \leq \frac{(1 + 4(X + H)^2)^{3/2}}{2} \sim r,$$

which implies (9). We also have

$$\ell(\Gamma) = \int_X^{X+H} (1 + 4x^2)^{1/2} = r^\alpha,$$

which is (10). We finally have

$$N(\Gamma) = \mathrm{Card}([X, X + H] \cap \mathbb{Z}) \sim H \sim 2^{-1/3} r^{\alpha - 1/3},$$

which implies (11). □

References

1. G. Grekos, Sur le nombre de points entiers d'une courbe convexe. Bull. Sci. Math. **112**, 235–254 (1988)
2. V. Jarník, Über die Gitterpunkte auf konvexen Kurven. Math. Z. **24**, 500–518 (1926)

Unification, Refinements and Companions of Generalisations of Schur's Theorem

Jehanne Dousse

To Krishna Alladi in honour of his 60th birthday

Abstract We prove a general theorem on coloured overpartitions with difference conditions that unifies generalisations of Schur's theorem due to Alladi–Gordon, Andrews, Corteel–Lovejoy, Lovejoy and the author. This theorem also allows one to give companions and refinements of the generalisations of Andrews' theorems to overpartitions. The proof relies on a variant of the method of weighted words of Alladi and Gordon using q-difference equation techniques recently introduced by the author.

Keywords Integer partitions · Partition identities · Weighted words q-series · q-difference equations

2010 Mathematics Subject Classification 11P81 · 11P84 · 05A17

1 Introduction

A partition of n is a non-increasing sequence of natural numbers whose sum is n. In 1926, Schur [39] proved the following partition identity.

Theorem 1.1 (Schur). *For any integer n, let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3, and $B(n)$ the number of partitions*

J. Dousse (✉)
Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190,
8057 Zürich, Switzerland
e-mail: jehanne.dousse@math.uzh.ch

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_14

of n such that parts differ by at least 3 *and no two consecutive multiples of* 3 *appear. Then for all n,*

$$A(n) = B(n).$$

Schur's theorem became very influential and several proofs have been given using a variety of different techniques [2, 6, 8, 10, 13, 16]. For our purposes in this article, the most significant proofs are a proof of Alladi and Gordon [2] using the method of weighted words and two proofs of Andrews [6, 8] using recurrences and q-difference equations.

The idea of the method of weighted words of Alladi and Gordon is to give a combinatorial interpretation of the infinite product

$$\prod_{n\geq 1}(1+aq^n)(1+bq^n)$$

as the generating function for partitions whose parts appear in three colours a, b, ab.

More precisely, they consider the following ordering of colours

$$ab < a < b, \tag{1.1}$$

giving the following ordering on coloured positive integers

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < \cdots .$$

Denoting by $c(\lambda)$ the colour of λ, their refinement of Schur's theorem can be stated as follows.

Theorem 1.2 (Alladi–Gordon). *Let $A(u, v, n)$ be the number of partitions of n into u distinct parts coloured a and v distinct parts coloured b.*

Let $B(u, v, n)$ be the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n into distinct parts with no part 1_{ab}, such that the difference $\lambda_i - \lambda_{i+1} \geq 2$ if $c(\lambda_i) = ab$ or $c(\lambda_i) < c(\lambda_{i+1})$ in (1.1), having u parts a or ab and v parts b or ab.

Then

$$\sum_{u,v,n\geq 0} A(u, v, n)a^u b^v q^n = \sum_{u,v,n\geq 0} B(u, v, n)a^u b^v q^n = \prod_{n\geq 1}(1+aq^n)(1+bq^n).$$

Doing the transformations

$$q \to q^3, a \to aq^{-2}, b \to bq^{-1},$$

one obtains a refinement of Schur's theorem. For details, see [2].

On the other hand, using the ideas of his proofs with q-difference equations [6, 8], Andrews was able to generalise Schur's theorem in two different ways [7, 9]. Let us now recall some notation due to Andrews in order to state his generalisations.

Let $A = \{a(1), \dots, a(r)\}$ be a set of r distinct positive integers such that $\sum_{i=1}^{k-1} a(i) < a(k)$ for all $1 \leq k \leq r$. Note that the $2^r - 1$ possible sums of distinct elements of A are all distinct. We denote this set of sums by $A'=\{\alpha(1), \dots, \alpha(2^r - 1)\}$, where $\alpha(1) < \cdots < \alpha(2^r - 1)$. Let N be a positive integer with $N \geq \alpha(2^r - 1) = a(1) + \cdots + a(r)$. We further define $\alpha(2^r) = a(r+1) = N + a(1)$. Let A_N (resp. $-A_N$) denote the set of positive integers congruent to some $a(i) \mod N$ (resp. $-a(i) \mod N$), A'_N (resp. $-A'_N$) the set of positive integers congruent to some $\alpha(i) \mod N$ (resp. $-\alpha(i) \mod N$). Let $\beta_N(m)$ be the least positive residue of $m \mod N$. If $\alpha \in A'$, let $w_A(\alpha)$ be the number of terms appearing in the defining sum of α and $v_A(\alpha)$ (resp. $z_A(\alpha)$) the smallest (resp. the largest) $a(i)$ appearing in this sum.

The simplest example is the one where $a(k) = 2^{k-1}$ for $1 \leq k \leq r$ and $\alpha(k) = k$ for $1 \leq k \leq 2^r - 1$.

Theorem 1.3 **(Andrews).** *Let $D(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N. Let $E(A'_N; n)$ denote the number of partitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq N w_A(\beta_N(\lambda_{i+1})) + v_A(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $n \geq 0$,

$$D(A_N; n) = E(A'_N; n).$$

Theorem 1.4 **(Andrews).** *Let $F(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$. Let $G(-A'_N; n)$ denote the number of partitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq N w_A(\beta_N(-\lambda_i)) + v_A(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and

$$\lambda_s \geq N(w_A(\beta_N(-\lambda_s) - 1).$$

Then for all $n \geq 0$,

$$F(-A_N; n) = G(-A'_N; n).$$

Schur's theorem corresponds to the case $N = 3$, $r = 2$, $a(1) = 1$, $a(2) = 2$.

Andrews' identities led to a number of important developments in combinatorics [1, 21, 42], group representation theory [4] and quantum algebra [37].

A more general version of Theorem 1.3 (once reformulated), where the condition $\sum_{i=1}^{k-1} a(i) < a(k)$ is removed, has been proved by Andrews and Olsson in [4]. It was subsequently proved bijectively [13] and further generalised [14] by Bessenrodt.

In 2006, Corteel and Lovejoy [21] combined the ideas of Alladi-Gordon and Andrews to prove a general theorem on coloured partitions which unifies and refines Andrews' two hierarchies of partition identities. To state their refinement (slightly reformulated to fit our purposes), we need to introduce some more notation.

Let r be a positive integer. We define r primary colours $u_1, \dots, u_r$ and use them to define $2^r - 1$ colours $\tilde{u}_1, \dots, \tilde{u}_{2^r-1}$ as follows:

$$\tilde{u}_i := u_1^{\varepsilon_1(i)} \cdots u_r^{\varepsilon_r(i)},$$

where

$$\varepsilon_k(i) := \begin{cases} 1 \text{ if } 2^{k-1} \text{ appears in the binary expansion of } i \\ 0 \text{ otherwise.} \end{cases}$$

They are ordered in the natural ordering, namely

$$\tilde{u}_1 < \cdots < \tilde{u}_{2^r-1}.$$

Now for all $i \in \{1, \dots, 2^r - 1\}$, let $v(\tilde{u}_i)$ (resp. $z(\tilde{u}_i)$) be the smallest (resp. largest) primary colour appearing in the colour $\tilde{u}_i$ and $w(\tilde{u}_i)$ be the number of primary colours appearing in $\tilde{u}_i$. Finally, for $i, j \in \{1, \dots, 2^r - 1\}$, let

$$\delta(\tilde{u}_i, \tilde{u}_j) := \begin{cases} 1 \text{ if } z(\tilde{u}_i) < v(\tilde{u}_j) \\ 0 \text{ otherwise.} \end{cases}$$

In a slightly modified version, Corteel and Lovejoy's theorem may be stated as follows.

Theorem 1.5 (Corteel-Lovejoy). *Let $D(\ell_1, \dots, \ell_r; n)$ denote the number of partitions of n into distinct non-negative parts, each part being coloured in one of the primary colours $u_1, \dots, u_r$, having ℓ_i parts coloured u_i for all $i \in \{1, \dots, r\}$. Let $E(\ell_1, \dots, \ell_r; n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n into distinct non-negative parts, each part being coloured in one of the colours $\tilde{u}_1, \dots, \tilde{u}_{2^r-1}$, such that for all $i \in \{1, \dots, r\}$, ℓ_i parts have u_i as one of their primary colours, satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq w(c(\lambda_{i+1})) + \delta(c(\lambda_i), c(\lambda_{i+1})).$$

Then for all $\ell_1, \dots, \ell_r, n \geq 0$,

$$D(\ell_1, \dots, \ell_r; n) = E(\ell_1, \dots, \ell_r; n).$$

The proof of Theorem 1.5 relies on the iteration of a bijection originally discovered by Bressoud [15] and adapted by Alladi and Gordon to the context of weighted words [3].

Corteel and Lovejoy then noticed that the partitions counted by $D(\ell_1, \dots, \ell_r; n)$ and $E(\ell_1, \dots, \ell_r; n)$ have some symmetry properties and took advantage of them to prove an even more general theorem.

Let $\sigma \in S_r$ be a permutation. For every colour $\tilde{u}_i = u_1^{\varepsilon_1(i)} \cdots u_r^{\varepsilon_r(i)}$, we define the colour

$$\sigma(\tilde{u}_i) := u_{\sigma(1)}^{\varepsilon_1(i)} \cdots u_{\sigma(r)}^{\varepsilon_r(i)}.$$

Now for every partition λ counted by $E(\ell_1, \ldots, \ell_r; n)$, we define a new partition λ^σ obtained by setting $\lambda_i^\sigma = \lambda_i$ and $c(\lambda_i^\sigma) = \sigma(c(\lambda_i))$. This mapping is easily reversible by using the inverse permutation σ^{-1} on λ^σ. This transformation does not change $w(c(\lambda_{i+1}))$, so the difference condition we obtain on λ^σ is

$$\lambda_i^\sigma - \lambda_{i+1}^\sigma \geq w(c(\lambda_{i+1}^\sigma)) + \delta(\sigma^{-1}(c(\lambda_i^\sigma)), \sigma^{-1}(c(\lambda_{i+1}^\sigma))). \tag{1.2}$$

Thus $E(\ell_1, \ldots, \ell_r; n) = E^\sigma(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; n)$, where $E^\sigma(\ell_1, \ldots, \ell_r; n)$ denotes the number of partitions of n into distinct non-negative parts, each part being coloured in one of the colours $\tilde{u}_1, \ldots, \tilde{u}_{2^r-1}$, such that for all $i \in \{1, \ldots, r\}$, ℓ_i parts have u_i as one of their primary colours, satisfying the difference condition (1.2).

Moreover, by doing the same transformation on the partitions counted by $D(\ell_1, \ldots, \ell_r; n)$, one can see that

$$D(\ell_1, \ldots, \ell_r; n) = D(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; n).$$

Thus one has

Corollary 1.6 (Corteel-Lovejoy). *For every permutation $\sigma \in S_r$,*

$$D(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; n) = E^\sigma(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; n).$$

One obtains a refinement of Theorem 1.3 by using the permutation $\sigma = Id$ and doing the transformations

$$q \to q^N, u_1 \to u_1 q^{a(1)}, \ldots, u_r \to u_r q^{a(r)},$$

and a refinement of Theorem 1.4 by using the permutation $\sigma = (n, n-1, \ldots, 1)$ and doing the transformations

$$q \to q^N, u_1 \to u_1 q^{N-a(1)}, \ldots, u_r \to u_r q^{N-a(r)}.$$

More detail on how to recover Andrews' theorems is given in Section 2 in the case of overpartitions, which generalises the case of partitions.

Let us now mention the extensions of Schur's theorem and its generalisations to overpartitions. An overpartition of n is a partition of n in which the first occurrence of a number may be overlined. For example, there are 14 overpartitions of 4: $4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1$ and $\overline{1}+1+1+1$. Though they were not called overpartitions at the time, they were already used in 1967 by Andrews [5] to give combinatorial interpretations of the q-binomial theorem, Heine's transformation and Lebesgue's identity. Then, they were used in 1987 by Joichi and Stanton [29] in an algorithmic theory of bijective proofs of q-series identities. They also appear in bijective proofs of Ramanujan's ${}_1\psi_1$

summation and the q-Gauss summation [18, 19]. It was Corteel [18] who gave them their name in 2003, just before Corteel and Lovejoy [20] revealed their generality by giving combinatorial interpretations for several q-series identities. They went on to become a very interesting generalisation of partitions, and several partition identities have overpartition analogues or generalisations. For example, Lovejoy proved overpartition analogues of identities of Gordon [31], and Andrews-Santos and Gordon-Göllnitz [32]. Overpartitions also have interesting arithmetic properties [12, 17, 34, 41] and are related to the fields of Lie algebras [30], mathematical physics [22, 27, 28] and supersymmetric functions [22].

In 2005, Lovejoy [33] generalised Schur's theorem (in the weighted words version) to overpartitions by proving the following.

Theorem 1.7 (Lovejoy). *Let $\overline{A}(x_1, x_2; k, n)$ denote the number of overpartitions of n into x_1 parts congruent to 1 and x_2 parts congruent to 2 modulo 3, having k non-overlined parts. Let $\overline{B}(x_1, x_2; k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n, with x_1 parts congruent to 0 or 1 modulo 3 and x_2 parts congruent to 0 or 2 modulo 3, having k non-overlined parts and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 1, 2 \mod 3, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 0 \mod 3, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $x_1, x_2, k, n \geq 0$, $\overline{A}(x_1, x_2; k, n) = \overline{B}(x_1, x_2; k, n)$.

Schur's theorem (in the refined version of Alladi and Gordon) corresponds to the case $k = 0$ in Lovejoy's theorem.

Recently, the author generalised both of Andrews' theorems (Theorems 1.3 and 1.4) to overpartitions [24, 25] by proving the following (reusing the notation of Andrews' theorems).

Theorem 1.8 (Dousse). *Let $\overline{D}(A_N; k, n)$ denote the number of overpartitions of n into parts taken from A_N, having k non-overlined parts. Let $\overline{E}(A'_N; k, n)$ denote the number of overpartitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq N\left(w_A\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v_A(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}),$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k, n \geq 0$, $\overline{D}(A_N; k, n) = \overline{E}(A'_N; k, n)$.

Theorem 1.9 (Dousse). *Let $\overline{F}(-A_N; k, n)$ denote the number of overpartitions of n into parts taken from $-A_N$, having k non-overlined parts. Let $\overline{G}(-A'_N; k, n)$ denote the number of overpartitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq N\left(w_A\left(\beta_N(-\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v_A(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and

$$\lambda_s \geq N(w_A(\beta_N(-\lambda_s)) - 1).$$

Then for all $k, n \geq 0$, $\overline{F}(-A_N; k, n) = \overline{G}(-A'_N; k, n)$.

Lovejoy's theorem corresponds to $N = 3, r = 2, a(1) = 1, a(2) = 2$ in Theorems 1.8 and 1.9. The case $k = 0$ of Theorem 1.8 (resp. Theorem 1.9) gives Andrews' Theorem 1.3 (resp. Theorem 1.4).

While the statements of Theorems 1.8 and 1.9 resemble those of Andrews' theorems (Theorem 1.3 and 1.4), the proofs are more intricate. We used q-difference equations and recurrences as well, but in our case, we had equations of order r while those of Andrews' proofs were easily reducible to equations of order 1. Thus, we needed to prove the result by induction on r by going back and forth from q-difference equations on generating functions to recurrence equations on their coefficients.

The purpose of this paper is to generalise and refine Theorems 1.8 and 1.9 in the same way that Theorem 1.5 generalises Andrews' identities and to unify all the above-mentioned generalisations of Schur's theorem. We prove the following.

Theorem 1.10. *Let* $\overline{D}(\ell_1, \ldots, \ell_r; k, n)$ *denote the number of overpartitions of n into non-negative parts coloured* $u_1, \ldots, u_{r-1}$ *or* u_r*, having* ℓ_i *parts coloured* u_i *for all* $i \in \{1, \ldots, r\}$ *and k non-overlined parts. Let* $\overline{E}(\ell_1, \ldots, \ell_r; k, n)$ *denote the number of overpartitions* $\lambda_1 + \cdots + \lambda_s$ *of n into non-negative parts coloured* $\tilde{u}_1, \ldots, \tilde{u}_{2^r-2}$ *or* $\tilde{u}_{2^r-1}$*, such that for all* $i \in \{1, \ldots, r\}$*,* ℓ_i *parts have* u_i *as one of their primary colours, having k non-overlined parts and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq w(c(\lambda_{i+1})) + \chi(\overline{\lambda_{i+1}}) - 1 + \delta(c(\lambda_i), c(\lambda_{i+1})),$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ *if* λ_{i+1} *is overlined and* 0 *otherwise.*

Then for all $\ell_1, \ldots, \ell_r, k, n \geq 0$*,*

$$\overline{D}(\ell_1, \ldots, \ell_r; k, n) = \overline{E}(\ell_1, \ldots, \ell_r; k, n).$$

The proof of Theorem 1.10 relies on the combination of the method of weighted words of Alladi and Gordon [2] and the q-difference equations techniques introduced by the author in [24]. This idea of mixing the method of weighted words with q-difference equations was first introduced by the author in a recent paper [26] to prove a refinement and companion of Siladić's theorem [40], a partition identity that first arose in the study of Lie algebras.

As in the work of Corteel and Lovejoy, we can take advantage of the symmetries in Theorem 1.10. Let $\sigma \in S_r$ be a permutation. For every overpartition λ counted by $\overline{E}(\ell_1, \ldots, \ell_r; k, n)$, we define a new overpartition λ^σ obtained by setting $\lambda_i^\sigma = \lambda_i$ and $c(\lambda_i^\sigma) = \sigma(c(\lambda_i))$, and overlining λ_i^σ if and only if λ_i was overlined. This mapping is reversible and does not change $w(c(\lambda_{i+1}))$ or $\chi(\overline{\lambda_{i+1}})$, so the difference condition we obtain on λ^σ is

$$\lambda_i^\sigma - \lambda_{i+1}^\sigma \geq w(c(\lambda_{i+1}^\sigma)) + \chi(\overline{\lambda_{i+1}^\sigma}) - 1 + \delta(\sigma^{-1}(c(\lambda_i^\sigma)), \sigma^{-1}(c(\lambda_{i+1}^\sigma))). \quad (1.3)$$

Thus $\overline{E}(\ell_1, \ldots, \ell_r; k, n) = \overline{E}^{\sigma}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; k, n)$, where $\overline{E}^{\sigma}(\ell_1, \ldots, \ell_r; k, n)$ denotes the number of overpartitions of n into non-negative parts, each part being coloured in one of the colours $\tilde{u}_1, \ldots, \tilde{u}_{2^r-1}$, such that for all $i \in \{1, \ldots, r\}$, ℓ_i parts have u_i as one of their primary colours, satisfying the difference condition (1.3).

Moreover, by doing the same transformation on the overpartitions counted by $D(\ell_1, \ldots, \ell_r; n)$, one can see that

$$\overline{D}(\ell_1, \ldots, \ell_r; k, n) = \overline{D}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; k, n).$$

Thus one has

$$\overline{D}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; k, n) = \overline{E}^{\sigma}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(r)}; k, n),$$

and relabelling the colours gives

Corollary 1.11. *For every permutation $\sigma \in S_r$,*

$$\overline{D}(\ell_1, \ldots, \ell_r; k, n) = \overline{E}^{\sigma}(\ell_1, \ldots, \ell_r; k, n).$$

We introduce one more notation. For $\alpha, \beta \in A'$, let

$$\delta_A(\alpha, \beta) := \begin{cases} 1 \text{ if } z_A(\alpha) < v_A(\beta) \\ 0 \text{ otherwise.} \end{cases}$$

We also extend the permutations to every integer $\alpha = a(i_1) + \cdots + a(i_s) \in A'$ by setting

$$\sigma(\alpha) = a(\sigma(i_1)) + \cdots + a(\sigma(i_s)).$$

By doing the transformations

$$q \to q^N, u_1 \to u_1 q^{a(1)}, \ldots, u_r \to u_r q^{a(r)},$$

we obtain the following generalisation and refinement of Theorem 1.8. Details are given in Section 2.

Theorem 1.12. *Let $\overline{D}(A_N; \ell_1, \ldots, \ell_r; k, n)$ denote the number of overpartitions of n into parts taken from A_N, having k non-overlined parts, such that for all $i \in \{1, \ldots, r\}$, ℓ_i parts are congruent to $a(i)$ modulo N. Let $\overline{E}^{\sigma}(A'_N; \ell_1, \ldots, \ell_r; k, n)$ denote the number of overpartitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that for all $i \in \{1, \ldots, r\}$, ℓ_i is the number of parts λ_j such that $\beta_N(\lambda_j)$ uses $a(i)$ in its defining sum, and satisfying the difference conditions*

$$\begin{aligned}\lambda_i - \lambda_{i+1} \geq &N \left(w_A\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}}) + \delta_A(\sigma(\beta_N(\lambda_i)), \sigma(\beta_N(\lambda_{i+1})))\right) \\ &+ \beta_N(\lambda_i) - \beta_N(\lambda_{i+1}),\end{aligned}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ *if* λ_{i+1} *is overlined and* 0 *otherwise.*

Then for all $\ell_1, \ldots, \ell_r, k, n \geq 0$, $\sigma \in S_r$,

$$\overline{D}(A_N; \ell_1, \ldots, \ell_r; k, n) = \overline{E}^{\sigma}(A'_N; \ell_1, \ldots, \ell_r; k, n).$$

Similarly, by using the transformations

$$q \to q^N, u_1 \to u_1 q^{N-a(1)}, \ldots, u_r \to u_r q^{N-a(r)},$$

we obtain a refinement and generalisation of Theorem 1.9.

Theorem 1.13. *Let* $\overline{F}(-A_N; \ell_1, \ldots, \ell_r; k, n)$ *denote the number of overpartitions of* n *into parts taken from* $-A_N$*, having* k *non-overlined parts, such that for all* $i \in \{1, \ldots, r\}$, ℓ_i *parts are congruent to* $-a(i)$ *modulo* N. *Let* $\overline{G}^{\sigma}(-A'_N; \ell_1, \ldots, \ell_r; k, n)$ *denote the number of overpartitions of* n *into parts taken from* $-A'_N$ *of the form* $n = \lambda_1 + \cdots + \lambda_s$, *having* k *non-overlined parts, such that for all* $i \in \{1, \ldots, r\}$, ℓ_i *is the number of parts* λ_j *such that* $\beta_N(-\lambda_j)$ *uses* $a(i)$ *in its defining sum, and satisfying the difference conditions*

$$\begin{aligned}\lambda_i - \lambda_{i+1} \geq & N\left(w_A(\beta_N(-\lambda_i)) - 1 + \chi(\overline{\lambda_{i+1}}) + \delta_A(\sigma(\beta_N(-\lambda_i)), \sigma(\beta_N(-\lambda_{i+1})))\right) \\ & + \beta_N(-\lambda_{i+1}) - \beta_N(-\lambda_i),\end{aligned}$$

and

$$\lambda_s \geq N w_A(\beta_N(-\lambda_s)) - \beta_N(-\lambda_s).$$

Then for all $\ell_1, \ldots, \ell_r, k, n \geq 0$, $\sigma \in S_r$,

$$\overline{F}(-A_N; \ell_1, \ldots, \ell_r; k, n) = \overline{G}^{\sigma}(-A'_N; \ell_1, \ldots, \ell_r; k, n).$$

Setting $k = 0$ in Theorems 1.12 and 1.13 recovers two theorems of Corteel and Lovejoy [21].

Theorem 1.8 corresponds to the case $\sigma = Id$ in Theorem 1.12 and Theorem 1.9 to the case $\sigma = (n, n-1, \ldots, 1)$ in Theorem 1.13. Details on how to recover Theorems 1.12 and 1.13 are also given in Section 2.

Theorem 1.12 (resp. 1.13) gives $r! - 1$ new companions to Theorem 1.8 (resp. 1.9). For $r \geq 3$, the companions of Theorem 1.8 are different from those of Theorem 1.9. For $r = 2$, the two companions are the same when $a(1) = N - a(2)$. For example, when $r = 2$, $a(1) = 1$, $a(2) = 2$, setting $\sigma = (2, 1)$ in Theorem 1.12 gives the following theorem.

Corollary 1.14. *Let* $\overline{D}(N, \ell_1, \ell_2; k, n)$ *denote the number of overpartitions of* n *into parts* $\equiv 1, 2 \mod N$ *with* ℓ_1 *parts* $\equiv 1 \mod N$ *and* ℓ_2 *parts* $\equiv 2 \mod N$ *and having* k *non-overlined parts.*

Let $\overline{E}(N, \ell_1, \ell_2; k, n)$ *denote the number of overpartitions* $\lambda_1 + \cdots + \lambda_s$ *of* n *into parts* $\equiv 1, 2, 3 \mod N$ *with* ℓ_1 *parts* $\equiv 1, 3 \mod N$ *and* ℓ_2 *parts* $\equiv 2, 3 \mod N$, *having k non-overlined parts, such that the entry (x, y) in the matrix M_N gives the minimal difference between $\lambda_i \equiv x \mod N$ and $\lambda_{i+1} \equiv y \mod N$:*

$$M_N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{pmatrix} N\chi(\overline{\lambda_{i+1}}) & N\chi(\overline{\lambda_{i+1}}) - 1 & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) - 2 \\ N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) + 1 & N\chi(\overline{\lambda_{i+1}}) & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) - 1 \\ N\chi(\overline{\lambda_{i+1}}) + 2 & N\chi(\overline{\lambda_{i+1}}) + 1 & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) \end{pmatrix} \end{array}.$$

Then $\overline{D}(N, \ell_1, \ell_2; k, n) = \overline{E}(N, \ell_1, \ell_2; k, n)$.

On the other hand, setting $\sigma = Id$ in Theorem 1.13 gives the following.

Corollary 1.15. *Let $\overline{F}(N, \ell_1, \ell_2; k, n)$ denote the number of overpartitions of n into parts $\equiv -1, -2 \mod N$ with ℓ_1 parts $\equiv -1 \mod N$ and ℓ_2 parts $\equiv -2 \mod N$ and having k non-overlined parts.*

Let $\overline{G}(N, \ell_1, \ell_2; k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n into parts $\equiv -1, -2, -3 \mod N$ with ℓ_1 parts $\equiv -1, -3 \mod N$ and ℓ_2 parts $\equiv -2, -3 \mod N$, having k non-overlined parts, such that the entry (x, y) in the matrix M'_N gives the minimal difference between $\lambda_i \equiv x \mod N$ and $\lambda_{i+1} \equiv y \mod N$:

$$M'_N = \begin{array}{c} \\ -1 \\ -2 \\ -3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad -1 \quad & \quad -2 \quad & \quad -3 \quad \end{array} \\ \begin{pmatrix} N\chi(\overline{\lambda_{i+1}}) & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) + 1 & N\chi(\overline{\lambda_{i+1}}) + 2 \\ N\chi(\overline{\lambda_{i+1}}) - 1 & N\chi(\overline{\lambda_{i+1}}) & N\chi(\overline{\lambda_{i+1}}) + 1 \\ N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) - 2 & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) - 1 & N\left(\chi(\overline{\lambda_{i+1}}) + 1\right) \end{pmatrix} \end{array}.$$

Then $\overline{F}(N, \ell_1, \ell_2; k, n) = \overline{G}(N, \ell_1, \ell_2; k, n)$.

When $N = 3$, Corollaries 1.14 and 1.15 become the same companion to Lovejoy's theorem.

The generalisations of Schur's theorem stated above are summarised in Figure 1, where $A \longrightarrow B$ means that the theorem corresponding to the infinite product A is generalised by the theorem corresponding to the infinite product B. Here, we use the classical notation

$$(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j),$$

for $n \in \mathbb{N} \cup \{\infty\}$.

The rest of this paper organised as follows. In Section 2, we deduce Theorems 1.12 and 1.13 from Theorem 1.10 and explain how refinements of Andrews' theorems for overpartitions (Theorems 1.8 and 1.9) can be derived from them. In Section 3, we prove Theorem 1.10 using the method of weighted words, q-difference equations and an induction.

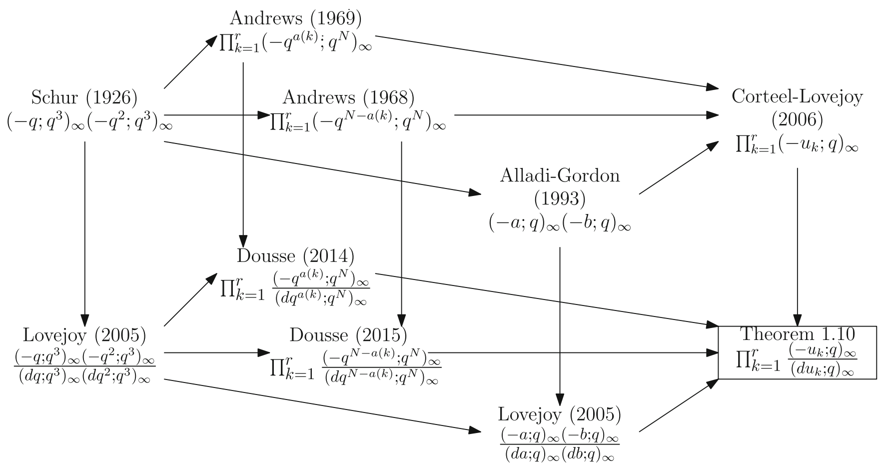


Fig. 1 Generalisations of Schur's theorem

2 Generalisations and refinements of Andrews' theorems for overpartitions

We start by showing how to deduce Theorems 1.12 and 1.13 from Theorem 1.10 and Corollary 1.11.

2.1 Proof of Theorem 1.12

Fix a permutation $\sigma \in S_r$. By Corollary 1.11, we have

$$\overline{D}(\ell_1,\dots,\ell_r;k,n) = \overline{E}^{\sigma^{-1}}(\ell_1,\dots,\ell_r;k,n).$$

Now transform the overpartitions counted by $\overline{D}(\ell_1,\dots,\ell_r;k,n)$ and $\overline{E}^{\sigma^{-1}}(\ell_1,\dots,\ell_r;k,n)$ by transforming each part λ_i of colour $\tilde{u}_j$ into a part

$$\lambda_i^{dil} = N\lambda_i + \alpha(j) = N\lambda_i + \varepsilon_1(j)a(1) + \cdots + \varepsilon_r(j)a(r).$$

This corresponds to doing the dilation $q \to q^N$ and the translations $u_i \to u_i q^{a(i)}$ for all $i \in \{1,\dots,r\}$ in the generating functions. The number k of non-overlined parts stays the same and the number n partitioned becomes

$$Nn + \ell_1 a(1) + \cdots + \ell_r a(r),$$

for both the overpartitions counted by $\overline{D}(\ell_1, \ldots, \ell_r; k, n)$ and by $\overline{E}^{\sigma^{-1}}(\ell_1, \ldots, \ell_r; k, n)$.

The parts before transformation were non-negative. After transformation, the parts of the overpartitions counted by $\overline{D}(\ell_1, \ldots, \ell_r; k, n)$ belong to A_N and those of the overpartitions counted by $\overline{E}^{\sigma^{-1}}(\ell_1, \ldots, \ell_r; k, n)$ belong to A'_N.

Let us now turn to the difference conditions. Before transformation, the overpartitions counted by $\overline{E}^{\sigma^{-1}}(\ell_1, \ldots, \ell_r; k, n)$ satisfied

$$\lambda_i - \lambda_{i+1} \geq w(c(\lambda_{i+1})) + \chi(\overline{\lambda_{i+1}}) - 1 + \delta(\sigma(c(\lambda_i)), \sigma(c(\lambda_{i+1}))).$$

After the transformations, it becomes

$$\lambda_i^{dil} - \lambda_{i+1}^{dil} \geq N\left(w_A\left(\alpha(c(\lambda_{i+1}))\right) - 1 + \chi(\overline{\lambda_{i+1}^{dil}}) + \delta_A(\sigma(\alpha(c(\lambda_i)), \sigma(\alpha(c(\lambda_{i+1}))))\right)$$
$$+ \alpha(c(\lambda_i)) - \alpha(c(\lambda_{i+1})),$$

By the definition of β_N and the transformations, we have the equality $\alpha(c(\lambda_i)) = \beta_N(\lambda_i^{dil})$. Thus the difference condition becomes

$$\lambda_i^{dil} - \lambda_{i+1}^{dil} \geq N\left(w_A\left(\beta_N(\lambda_{i+1}^{dil})\right) - 1 + \chi(\overline{\lambda_{i+1}^{dil}}) + \delta_A(\sigma(\beta_N(\lambda_i^{dil})), \sigma(\beta_N(\lambda_{i+1}^{dil})))\right)$$
$$+ \beta_N(\lambda_i^{dil}) - \beta_N(\lambda_{i+1}^{dil}),$$

This is exactly the difference condition from Theorem 1.12. This completes the proof.

2.2 *Proof of Theorem 1.13*

Let us now turn to the proof of Theorem 1.13. As before, we have

$$\overline{D}(\ell_1, \ldots, \ell_r; k, n) = \overline{E}^{\sigma^{-1}}(\ell_1, \ldots, \ell_r; k, n).$$

Now transform the overpartitions counted by $\overline{D}(\ell_1, \ldots, \ell_r; k, n)$ and $\overline{E}^{\sigma^{-1}}(\ell_1, \ldots, \ell_r; k, n)$ by transforming each part λ_i of colour $\tilde{u}_j$ into a part

$$\lambda_i^{dil'} = N\left(w(c(\lambda_i)) + \lambda_i\right) - \alpha(j) = N\left(w(c(\lambda_i)) + \lambda_i\right) - \varepsilon_1(j)a(1) - \cdots - \varepsilon_r(j)a(r).$$

This corresponds to doing the dilation $q \to q^N$ and the translations $u_i \to u_i q^{N-a(i)}$ for all $i \in \{1, \ldots, r\}$ in the generating functions. The number k of non-overlined parts stays the same and the number n partitioned becomes

$$N(n + \ell_1 + \cdots + \ell_r) - \ell_1 a(1) - \cdots - \ell_r a(r),$$

for both the overpartitions counted by $\overline{D}(\ell_1, \dots, \ell_r; k, n)$ and by $\overline{E}^{\sigma^{-1}}(\ell_1, \dots, \ell_r; k, n)$.

The parts before transformation were non-negative. After transformation, the parts of the overpartitions counted by $\overline{D}(\ell_1, \dots, \ell_r; k, n)$ belong to $-A_N$ and those of the overpartitions counted by $\overline{E}^{\sigma^{-1}}(\ell_1, \dots, \ell_r; k, n)$ belong to A'_N, with the additional condition that for all i,

$$\lambda_i^{dil'} \geq Nw(c(\lambda_i)) - \alpha(c(\lambda_i)) = Nw_A(\beta_N(-\lambda_i^{dil'})) - \beta_N(-\lambda_i^{dil'}). \tag{2.1}$$

Indeed by the definition of β_N and the transformations, we have now $\alpha(c(\lambda_i)) = \beta_N(-\lambda_i^{dil'})$.

The difference condition for the overpartitions counted by $\overline{E}^{\sigma^{-1}}(\ell_1, \dots, \ell_r; k, n)$ was

$$\lambda_i - \lambda_{i+1} \geq w(c(\lambda_{i+1})) + \chi(\overline{\lambda_{i+1}}) - 1 + \delta(\sigma(c(\lambda_i)), \sigma(c(\lambda_{i+1}))).$$

After the transformations, it becomes

$$\begin{aligned}\lambda_i^{dil'} - \lambda_{i+1}^{dil'} \geq & N\left(w_A\left(\alpha(c(\lambda_i))\right) - 1 + \chi(\overline{\lambda_{i+1}^{dil}}) + \delta_A(\sigma(\alpha(c(\lambda_i)), \sigma(\alpha(c(\lambda_{i+1}))))\right) \\ & - \alpha(c(\lambda_i)) + \alpha(c(\lambda_{i+1})),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\lambda_i^{dil'} - \lambda_{i+1}^{dil'} \geq & N\left(w_A\left(\beta_N(-\lambda_i^{dil})\right) - 1 + \chi(\overline{\lambda_{i+1}^{dil'}}) + \delta_A(\sigma(\beta_N(-\lambda_i^{dil})), \sigma(\beta_N(-\lambda_{i+1}^{dil})))\right) \\ & - \beta_N(-\lambda_i^{dil'}) + \beta_N(-\lambda_{i+1}^{dil'}),\end{aligned}$$

This is exactly the difference condition from Theorem 1.13. This completes the proof.

2.3 Refinement of Theorem 1.8

We now want to show that the case $\sigma = Id$ in Theorem 1.12 is actually a refinement of Theorem 1.8. To do so, let us reformulate Theorem 1.8. The minimal difference between two consecutive parts λ_i and λ_{i+1} is

$$\lambda_i - \lambda_{i+1} \geq N\left(w_A\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v_A(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

But by the definition of β_N, $\lambda_i - \lambda_{i+1}$ is always congruent to $\beta_N(\lambda_i) - \beta_N(\lambda_{i+1})$ modulo N. Therefore the difference condition is actually equivalent to having a minimal difference

$$N\left(w_A\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + \beta_N(\lambda_i) - \beta_N(\lambda_{i+1}),$$

if $v_A(\beta_N(\lambda_{i+1})) \leq \beta_N(\lambda_i)$, and

$$N\left(w_A\left(\beta_N(\lambda_{i+1})\right) + \chi(\overline{\lambda_{i+1}})\right) + \beta_N(\lambda_i) - \beta_N(\lambda_{i+1}),$$

if $v_A(\beta_N(\lambda_{i+1})) > \beta_N(\lambda_i)$.

We will be able to conclude using the following lemma.

Lemma 2.1. *For $\alpha, \beta \in A'$, we have $v_A(\alpha) > \beta$ if and only if $v_A(\alpha) > z_A(\beta)$.*

Proof. By the definition of z_A, $z_A(\beta) \leq \beta$. Thus if $v_A(\alpha) > \beta$, then $v_A(\alpha) > z_A(\beta)$.

Let us now show the other implication. Assume that $v_A(\alpha) > z_A(\beta)$. If we write $z_A(\beta) = a(k)$, then $v_A(\alpha) \geq a(k+1)$, but by the definition of A, we know that for all k,

$$\sum_{i=1}^{k} a(i) < a(k+1).$$

Thus

$$v_A(\alpha) \geq a(k+1) > \sum_{i=1}^{k} a(i) \geq z_A(\beta).$$

□

Hence by Lemma 2.1, the difference condition in Theorem 1.8 is actually equivalent to

$$\begin{aligned}\lambda_i - \lambda_{i+1} \geq & N\left(w_A\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}}) + \delta_A(\beta_N(\lambda_i), \beta_N(\lambda_{i+1}))\right) \\ & + \beta_N(\lambda_i) - \beta_N(\lambda_{i+1}),\end{aligned}$$

which is exactly the difference condition of Theorem 1.12 with $\sigma = Id$.

2.4 *Refinement of Theorem 1.9*

Finally, let us show that the case $\sigma = (n, n-1, \ldots, 1)$ in Theorem 1.13 is actually a refinement of Theorem 1.9. To do so, let us reformulate Theorem 1.9. The minimal difference between two consecutive parts λ_i and λ_{i+1} is

$$\lambda_i - \lambda_{i+1} \geq N\left(w_A\left(\beta_N(-\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v_A(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i).$$

But $\lambda_i - \lambda_{i+1}$ is always congruent to $-\beta_N(-\lambda_i) + \beta_N(-\lambda_{i+1})$ modulo N. Therefore, the difference condition is actually equivalent to having a minimal difference

$$N\left(w_A\left(\beta_N(-\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) - \beta_N(-\lambda_i) + \beta_N(-\lambda_{i+1}),$$

if $v_A(\beta_N(-\lambda_i)) \leq \beta_N(-\lambda_{i+1})$, and

$$N\left(w_A\left(\beta_N(-\lambda_i)\right) + \chi(\overline{\lambda_{i+1}})\right) - \beta_N(-\lambda_i) + \beta_N(-\lambda_{i+1}),$$

if $v_A(\beta_N(-\lambda_i)) > \beta_N(-\lambda_{i+1})$.

Again, by Lemma 2.1, this difference condition is equivalent to

$$\begin{aligned}\lambda_i - \lambda_{i+1} \geq & N\left(w_A\left(\beta_N(-\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}}) + \delta_A(\beta_N(-\lambda_{i+1}), \beta_N(-\lambda_i))\right) \\ & + \beta_N(-\lambda_{i+1}) - \beta_N(-\lambda_i).\end{aligned}$$

But when $\sigma = (n, n-1, \ldots, 1)$, then

$$\delta_A(\sigma(\beta_N(-\lambda_i)), \sigma(\beta_N(-\lambda_{i+1}))) = \delta_A(\beta_N(-\lambda_{i+1}), \beta_N(-\lambda_i)),$$

so we obtain exactly the same difference condition as in Theorem 1.9.

Finally, as λ_s is always congruent to $-\beta_N(-\lambda_s)$ modulo N, the condition $\lambda_s \geq N w_A(\beta_N(-\lambda_s)) - \beta_N(-\lambda_s)$ is equivalent to $\lambda_s \geq N(w_A(\beta_N(-\lambda_s)) - 1)$. This completes the proof.

3 Proof of Theorem 1.10

Let us now turn to the proof of Theorem 1.10.

It is clear that the generating function for the overpartitions with congruence conditions is

$$\sum_{\ell_1,\ldots,\ell_r,k,n\geq 0} \overline{D}(\ell_1,\ldots,\ell_r;k,n)u_1^{\ell_1}\cdots u_r^{\ell_r}d^k q^n = \prod_{k=1}^{r} \frac{(-u_k;q)_\infty}{(du_k;q)_\infty}.$$

The difficult task is to show that the generating function for overpartitions enumerated by $\overline{E}(\ell_1,\ldots,\ell_r;k,n)$ is the same. To do so, we adapt techniques introduced in [24] by taking colours into account. First, we establish the q-difference equation satisfied by the generating function with one added variable counting the number of parts, and then we prove by induction that a function satisfying this q-difference equation is equal to $\prod_{k=1}^{r} \frac{(-u_k;q)_\infty}{(du_k;q)_\infty}$ when the added variable is equal to 1.

3.1 The q-difference equation

Let us first establish the q-difference equation. Let $p_{i_{\tilde{u}_j}}(\ell_1, \dots, \ell_r; k, m, n)$ denote the number of overpartitions counted by $\overline{E}(\ell_1, \dots, \ell_r; k, n)$ having m parts such that the smallest part is at least $i_{\tilde{u}_j}$ (the non-negative integers are ordered according to their colours: $0_{\tilde{u}_1} < \cdots < 0_{\tilde{u}_{2^r-1}} < 1_{\tilde{u}_1} < \cdots$).

Let us define

$$\begin{aligned} f_{i_{\tilde{u}_j}}(x) &= f_{i_{\tilde{u}_j}}(u_1, \dots, u_r, d, x, q) \\ &:= \sum_{\ell_1,\dots,\ell_r,k,m,n \geq 0} p_{i_{\tilde{u}_j}}(\ell_1, \dots, \ell_r; k, m, n) u_1^{\ell_1} \cdots u_r^{\ell_r} d^k x^m q^n. \end{aligned} \tag{3.1}$$

We want to find an expression for $f_{0_{u_1}}(1)$, which is the generating function for all overpartitions counted by $\overline{E}(\ell_1, \dots, \ell_r; k, n)$.

We first prove the following lemma.

Lemma 3.1. *If* $1 \leq j \leq 2^r - 2$, *then*

$$\begin{aligned} f_{0_{\tilde{u}_j}}(x) - f_{0_{\tilde{u}_{j+1}}}(x) &= x u_1^{\varepsilon_1(j)} \cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(x q^{w(\tilde{u}_j)}) \\ &\quad + d x u_1^{\varepsilon_1(j)} \cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(x q^{w(\tilde{u}_j)-1}), \end{aligned} \tag{3.2}$$

$$f_{0_{\tilde{u}_{2^r-1}}}(x) - f_{1_{\tilde{u}_1}}(x) = x u_1 \cdots u_r f_{0_{u_1}}(x q^r) + d x u_1 \cdots u_r f_{0_{u_1}}(x q^{r-1}), \tag{3.3}$$

$$f_{1_{\tilde{u}_1}}(x) = f_{0_{\tilde{u}_1}}(xq). \tag{3.4}$$

Proof. We first prove the following recurrence equations for $1 \leq j \leq 2^r - 2$:

$$\begin{aligned} & p_{0_{\tilde{u}_j}}(\ell_1, \dots, \ell_r; k, m, n) - p_{0_{\tilde{u}_{j+1}}}(\ell_1, \dots, \ell_r; k, m, n) \\ &= p_{0_{v(\tilde{u}_j)}}(\ell_1 - \varepsilon_1(j), \dots, \ell_r - \varepsilon_r(j); k, m-1, n-(m-1)w(\tilde{u}_j)) \\ &+ p_{0_{v(\tilde{u}_j)}}(\ell_1 - \varepsilon_1(j), \dots, \ell_r - \varepsilon_r(j); k-1, m-1, n-(m-1)(w(\tilde{u}_j)-1)), \end{aligned} \tag{3.5}$$

$$\begin{aligned} & p_{0_{\tilde{u}_{2^r-1}}}(\ell_1, \dots, \ell_r; k, m, n) - p_{1_{\tilde{u}_1}}(\ell_1, \dots, \ell_r; k, m, n) \\ &= p_{0_{u_1}}(\ell_1 - 1, \dots, \ell_r - 1; k, m-1, n-(m-1)r) \\ &+ p_{0_{u_1}}(\ell_1 - 1, \dots, \ell_r - 1; k-1, m-1, n-(m-1)(r-1)), \end{aligned} \tag{3.6}$$

$$p_{1_{\tilde{u}_1}}(\ell_1, \dots, \ell_r; k, m, n) = p_{0_{\tilde{u}_1}}(\ell_1, \dots, \ell_r; k, m, n-m). \tag{3.7}$$

Let us first prove (3.5). The quantity

$$p_{0_{\tilde{u}_j}}(\ell_1, \dots, \ell_r; k, m, n) - p_{0_{\tilde{u}_{j+1}}}(\ell_1, \dots, \ell_r; k, m, n)$$

is the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ of n enumerated by $p_{0_{\tilde{u}_j}}(\ell_1, \ldots, \ell_r; k, m, n)$ such that the smallest part is equal to $0_{\tilde{u}_j}$.

If $\lambda_m = \overline{0_{\tilde{u}_j}}$ is overlined, then by the difference conditions in Theorem 1.10,

$$\lambda_{m-1} \geq 1 + w(\tilde{u}_j) + \delta(c(\lambda_{m-1}), v(\tilde{u}_j)).$$

This is equivalent to

$$\lambda_{m-1} \geq \begin{cases} w(\tilde{u}_j) \text{ if the colour of } \lambda_{m-1} \text{is at least } v(\tilde{u}_j), \\ 1 + w(\tilde{u}_j) \text{if the colour of } \lambda_{m-1} \text{is less than } v(\tilde{u}_j). \end{cases}$$

In other words,

$$\lambda_{m-1} \geq (w(\tilde{u}_j))_{v(\tilde{u}_j)}.$$

Then, we remove $\lambda_m = \overline{0_{\tilde{u}_j}}$ and subtract $w(\tilde{u}_j)$ from every other part. For all $i \in \{1, \ldots, r\}$, the number of parts using u_i as a primary colour decreases by 1 if and only if u_i appeared in $\tilde{u}_j$, i.e. if and only if $\varepsilon_i(j) = 1$. The number of parts is reduced to $m - 1$, the number of non-overlined parts is still k, and the number partitioned is now $n - (m - 1)w(\tilde{u}_j)$. Moreover, the smallest part is now at least $0_{v(\tilde{u}_j)}$. Therefore, we obtain an overpartition counted by

$$p_{0_{v(\tilde{u}_j)}}(\ell_1 - \varepsilon_1(j), \ldots, \ell_r - \varepsilon_r(j); k, m - 1, n - (m - 1)w(\tilde{u}_j)).$$

If $\lambda_m = 0_{\tilde{u}_j}$ is not overlined, then in the same way as before, by the difference conditions in Theorem 1.10,

$$\lambda_{m-1} \geq w(\tilde{u}_j) - 1 + \delta(c(\lambda_{m-1}), v(\tilde{u}_j)).$$

In other words,

$$\lambda_{m-1} \geq (w(\tilde{u}_j) - 1)_{v(\tilde{u}_j)}.$$

Then we remove $\lambda_m = 0_{\tilde{u}_j}$ and subtract $w(\tilde{u}_j) - 1$ from every other part. For all $i \in \{1, \ldots, r\}$, the number of parts using u_i as a primary colour decreases by 1 if and only if $\varepsilon_i(j) = 1$. The number of parts is reduced to $m - 1$, the number of non-overlined parts is reduced to $k - 1$, and the number partitioned is now $n - (m - 1)(w(\tilde{u}_j) - 1)$. Moreover, the smallest part is now at least $0_{v(\tilde{u}_j)}$. Therefore, we obtain an overpartition counted by

$$p_{0_{v(\tilde{u}_j)}}(\ell_1 - \varepsilon_1(j), \ldots, \ell_r - \varepsilon_r(j); k - 1, m - 1, n - (m - 1)(w(\tilde{u}_j) - 1)).$$

The proof of (3.6) is exactly the same with $j = 2^r - 1$.

Finally, to prove (3.7), we take a partition enumerated by $p_{1_{\tilde{u}_1}}(\ell_1, \ldots, \ell_r; k, m, n)$ and subtract 1 from each part. We obtain a partition enumerated by

$$p_{0_{\tilde{u}_1}}(\ell_1,\ldots,\ell_r;k,m,n-m).$$

The recurrences (3.5)-(3.7) can be translated as q-difference equations on the f_i's to complete the proof of Lemma 3.1. □

Let $2 \leq k \leq r$. Note that $\tilde{u}_{2^{k-1}} = u_k$. Adding equations (3.2) together for $1 \leq j \leq 2^{k-1}-1$ gives

$$\begin{aligned} f_{0_{u_1}}(x) - f_{0_{u_k}}(x) = \sum_{j=1}^{2^{k-1}-1} &\Big(xu_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)}) \\ &+dxu_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)-1})\Big). \end{aligned} \tag{3.8}$$

In the same way, adding equations (3.2) together for $2^{k-2} \leq j \leq 2^{k-1}-1$ gives

$$\begin{aligned} f_{0_{u_{k-1}}}(x) - f_{0_{u_k}}(x) = \sum_{j=2^{k-2}}^{2^{k-1}-1} &\Big(xu_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)}) \\ &+dxu_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)-1})\Big). \end{aligned} \tag{3.9}$$

For all $2^{k-2} \leq j \leq 2^{k-1}-1$, $\tilde{u}_j$ is of the form

$$\tilde{u}_j = u_1^{\varepsilon_1(j)}\cdots u_{k-2}^{\varepsilon_{k-2}(j)} u_{k-1}.$$

Thus (3.9) can be rewritten as

$$\begin{aligned} f_{0_{u_{k-1}}}(x) - f_{0_{u_k}}(x) &= xu_{k-1} f_{0_{u_{k-1}}}(xq) + dxu_{k-1} f_{0_{u_{k-1}}}(x) \\ &\quad + q^{-1}u_{k-1}\sum_{j=1}^{2^{k-2}-1} \Big(xqu_1^{\varepsilon_1(j)}\cdots u_{k-2}^{\varepsilon_{k-2}(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)+1}) \\ &\qquad\qquad +dxqu_1^{\varepsilon_1(j)}\cdots u_{k-2}^{\varepsilon_{k-2}(j)} f_{0_{v(\tilde{u}_j)}}(xq^{w(\tilde{u}_j)})\Big) \\ &= xu_{k-1} f_{0_{u_{k-1}}}(xq) + dxu_{k-1} f_{0_{u_{k-1}}}(x) \\ &\quad + q^{-1}u_{k-1}\Big(f_{0_{u_1}}(xq) - f_{0_{u_{k-1}}}(xq)\Big), \end{aligned}$$

where we used (3.8) with k replaced by $k-1$ and x replaced by xq to obtain the last equality.

Thus

$$f_{0_{u_k}}(x) = (1 - dxu_{k-1}) f_{0_{u_{k-1}}}(x) - q^{-1}u_{k-1} f_{0_{u_1}}(xq) \\ + q^{-1}u_{k-1}(1 - xq) f_{0_{u_{k-1}}}(xq). \tag{3.10}$$

In the same way, on can show that

$$f_{1_{u_1}}(x) = (1 - dxu_r) f_{0_{u_r}}(x) - q^{-1}u_r f_{0_{u_1}}(xq) \\ + q^{-1}u_r(1 - xq) f_{0_{u_r}}(xq). \tag{3.11}$$

We are almost ready to give the q-difference equation relating functions $f_{0_{u_1}}\left(xq^k\right)$ together for $k \geq 0$. To do so, recall that the q-binomial coefficients are defined as

$$\begin{bmatrix} m \\ r \end{bmatrix}_q := \begin{cases} \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} & \text{if } 0 \leq r \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

They are q-analogues of the binomial coefficients and satisfy q-analogues of the Pascal triangle identity [11, Equations (3.3.3) and (3.3.4)].

Proposition 3.2. *For all integers* $0 \leq r \leq m$*,*

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = q^r \begin{bmatrix} m-1 \\ r \end{bmatrix}_q + \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q, \tag{3.12}$$

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{bmatrix} m-1 \\ r \end{bmatrix}_q + q^{m-r} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q. \tag{3.13}$$

The following lemma will help us to obtain the desired q-difference equation.

Lemma 3.3. *For* $1 \leq k \leq r$*, we have*

$$\prod_{i=1}^{k-1}(1 - dxu_i)\, f_{0_{u_1}}(x) = f_{0_{u_k}}(x) \\ + \sum_{i=1}^{k-1}\left(\sum_{m=0}^{k-i-1} d^m \sum_{\substack{1 \leq j < 2^{k-1} \\ w(\tilde{u}_j)=i+m}} xu_1^{\varepsilon_1(j)} \cdots u_{k-1}^{\varepsilon_{k-1}(j)} \left((-x)^{m-1}\begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q \right.\right. \\ \left.\left. + (-x)^m \begin{bmatrix} i+m \\ m \end{bmatrix}_q \right)\right) \times \prod_{h=1}^{i-1}\left(1 - xq^h\right) f_{0_{u_1}}\left(xq^i\right). \tag{3.14}$$

Proof. The proof relies on an induction on k. For $k = 1$, (3.14) reduces to the trivial equation $f_{0_{u_1}}(x) = f_{0_{u_1}}(x)$. Now assume that (3.14) is true for some $1 \leq k \leq r-1$ and show it is also true for $k+1$. Let us define

$$s_k(x) := \sum_{i=1}^{k-1} \left(\sum_{m=0}^{k-i-1} d^m \sum_{\substack{1 \leq j < 2^{k-1} \\ w(\tilde{u}_j) = i+m}} x u_1^{\varepsilon_1(j)} \cdots u_{k-1}^{\varepsilon_{k-1}(j)} \left((-x)^{m-1} \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q \right.\right.$$
$$\left.\left. + (-x)^m \begin{bmatrix} i+m \\ m \end{bmatrix}_q \right) \right) \times \prod_{h=1}^{i-1} \left(1 - xq^h\right) f_{0_{u_1}}\left(xq^i\right).$$

We want to show that

$$\prod_{i=1}^{k} (1 - dxu_i)\, f_{0_{u_1}}(x) = f_{0_{u_{k+1}}}(x) + s_{k+1}(x).$$

One has

$$\begin{aligned}
&\prod_{i=1}^{k} (1 - dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_{k+1}}}(x) \\
&= (1 - dxu_k) \left(\prod_{i=1}^{k-1} (1 - dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_k}}(x) \right) \\
&\quad + (1 - dxu_k)\, f_{0_{u_k}}(x) - f_{0_{u_{k+1}}}(x) \\
&= (1 - dxu_k)\, s_k(x) + q^{-1} u_k f_{0_{u_1}}(xq) - q^{-1} u_k\, (1 - xq)\, f_{0_{u_k}}(xq),
\end{aligned}$$

where we used the induction hypothesis and (3.10) in the last equality. Then

$$\begin{aligned}
&\prod_{i=1}^{k} (1 - dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_{k+1}}}(x) \\
&= (1 - dxu_k)\, s_k(x) + q^{-1} u_k f_{0_{u_1}}(xq) \\
&\quad - q^{-1} u_k\, (1 - xq) \left(\prod_{i=1}^{k-1} (1 - dxqu_i)\, f_{0_{u_1}}(xq) - s_k\,(xq) \right) \\
&= (1 - dxu_k)\, s_k(x) + q^{-1} u_k\, (1 - xq)\, s_k\,(xq) \\
&\quad + q^{-1} u_k \left(1 - (1 - xq) \prod_{i=1}^{k-1} (1 - dxqu_i) \right) f_{0_{u_1}}(xq)
\end{aligned}$$

Expanding the last line gives

$$\begin{aligned}
&\prod_{i=1}^{k}(1-dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_{k+1}}}(x) \\
&= (1-dxu_k)\, s_k(x) + q^{-1}u_k\,(1-xq)\, s_k\,(xq) \\
&\quad + \left(xu_k + \sum_{m=1}^{k-1} d^m \sum_{\substack{2^{k-1}<j<2^k \\ w(\tilde{u}_j)=m+1}} xu_1^{\varepsilon_1(j)}\cdots u_{k-1}^{\varepsilon_{k-1}(j)} u_k \left((-xq)^{m-1} + (-xq)^m\right)\right) f_{0_{u_1}}(xq).
\end{aligned}$$

Now after replacing s_k by its definition and doing some calculations, we get

$$\begin{aligned}
&\prod_{i=1}^{k}(1-dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_{k+1}}}(x) \\
&= \sum_{i=1}^{k-1}\Bigg[\sum_{\substack{1\le j<2^k \\ w(\tilde{u}_j)=i}} xu_1^{\varepsilon_1(j)}\cdots u_{k-1}^{\varepsilon_{k-1}(j)} u_k^{\varepsilon_k(j)} \\
&\qquad + \sum_{m=1}^{k-i} d^m \Bigg(\sum_{\substack{1\le j<2^{k-1} \\ w(\tilde{u}_j)=i+m}} xu_1^{\varepsilon_1(j)}\cdots u_{k-1}^{\varepsilon_{k-1}(j)} \\
&\qquad\qquad \times \left((-x)^{m-1} \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q + (-x)^m \begin{bmatrix} i+m \\ m \end{bmatrix}_q \right) \\
&\qquad\qquad + \sum_{\substack{2^{k-1}<j<2^k \\ w(\tilde{u}_j)=i+m}} xu_1^{\varepsilon_1(j)}\cdots u_{k-1}^{\varepsilon_{k-1}(j)} u_k \\
&\qquad\qquad\quad \times \Bigg((-x)^{m-1} \left(\begin{bmatrix} i+m-2 \\ m-2 \end{bmatrix}_q + q^{(m-1)} \begin{bmatrix} i+m-2 \\ m-1 \end{bmatrix}_q \right) \\
&\qquad\qquad\qquad + (-x)^m \left(\begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} i+m-1 \\ m \end{bmatrix}_q \right) \Bigg) \Bigg) \Bigg] \\
&\quad \times \prod_{h=1}^{i-1}\left(1-xq^h\right) f_{0_{u_1}}(xq^i) \\
&\quad + xu_1\cdots u_k \prod_{h=1}^{k-1}\left(1-xq^h\right) f_{0_{u_1}}(xq^k).
\end{aligned}$$

Then, we use the first q-analogue of Pascal's triangle (3.12) in the last sum above and obtain

$$
\begin{aligned}
&\prod_{i=1}^{k}(1-dxu_i)\, f_{0_{u_1}}(x) - f_{0_{u_{k+1}}}(x) \\
&= \sum_{i=1}^{k} \left(\sum_{m=0}^{k-i} d^m \sum_{\substack{1 \le j < 2^k \\ w(\tilde{u}_j)=i+m}} x u_1^{\varepsilon_1(j)} \cdots u_{k-1}^{\varepsilon_{k-1}(j)} u_k^{\varepsilon_k(j)} \right. \\
&\qquad\qquad \left. \times \left((-x)^{m-1} \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q + (-x)^m \begin{bmatrix} i+m \\ m \end{bmatrix}_q \right) \right) \\
&\qquad \times \prod_{h=1}^{i-1} \left(1 - xq^h\right) f_{0_{u_1}}(xq^i) \\
&= s_{k+1}(x).
\end{aligned}
$$

This completes the proof. □

Starting from Equation (3.14) for $k = r$, using (3.11) and doing exactly the same computations as above, we obtain the following:

$$
\begin{aligned}
&\prod_{i=1}^{r}(1-dxu_i)\, f_{0_{u_1}}(x) = f_{1_{u_1}}(x) \\
&+ \sum_{i=1}^{r} \left(\sum_{m=0}^{r-i} d^m \sum_{\substack{1 \le j < 2^r \\ w(\tilde{u}_j)=i+m}} x u_1^{\varepsilon_1(j)} \cdots u_r^{\varepsilon_r(j)} \left((-x)^{m-1} \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q \right.\right. \\
&\qquad \left.\left. + (-x)^m \begin{bmatrix} i+m \\ m \end{bmatrix}_q \right) \right) \times \prod_{h=1}^{i-1} \left(1 - xq^h\right) f_{0_{u_1}}\left(xq^i\right).
\end{aligned}
\tag{3.15}
$$

Finally, using (3.4), we obtain the desired q-difference equation.

$$\prod_{i=1}^{r}(1-dxu_i)\, f_{0_{u_1}}(x) = f_{0_{u_1}}(xq)$$
$$+\sum_{i=1}^{r}\left(\sum_{m=0}^{r-i} d^m \sum_{\substack{1\leq j<2^r\\ w(\tilde{u}_j)=i+m}} xu_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)}\left((-x)^{m-1}\begin{bmatrix} i+m-1\\ m-1\end{bmatrix}_q\right.\right. \qquad (\mathrm{eq}_r)$$
$$\left.\left.+(-x)^m\begin{bmatrix} i+m\\ m\end{bmatrix}_q\right)\right)\times\prod_{h=1}^{i-1}\left(1-xq^h\right) f_{0_{u_1}}\left(xq^i\right).$$

3.2 The induction

Recall we want to find an expression for $f_{0_{u_1}}(1)$, which is the generating function for overpartitions counted by $\overline{E}(\ell_1,\ldots,\ell_r;k,n)$. We do so by proving the following theorem by induction on r.

Theorem 3.4. *Let r be a positive integer. Then for every function f satisfying the q-difference equation* (eq_r) *and the initial condition $f(0)=1$, we have*

$$f(1)=\prod_{k=1}^{r}\frac{(-u_k;q)_\infty}{(du_k;q)_\infty}.$$

As in [24], we start from a function satisfying (eq_r) and do some transformations to obtain a function satisfying (eq_{r-1}) and be able to use the induction hypothesis. More precisely, we make changes of unknown functions and switch between q-difference equations on a generating function and recurrences on its coefficients to lower the degree of the equation. Doing the transformations $q\to q^N, u_1\to q^{a(1)},\ldots,u_r\to q^{a(r)}$ (note that we do not keep the colours in the dilations) in the following proof of Theorem 3.4, we recover the one in [24]. The technical challenge here is to correctly keep track of all the colour variables $u_1,\ldots,u_r$ both in the changes of unknown functions and in the equations.

Lemma 3.5. *Let f and F be two functions such that*

$$F(x):=f(x)\prod_{n=0}^{\infty}\frac{1-dxu_rq^n}{1-xq^n}.$$

Then $f(0)=1$ and f satisfies (eq_r) *if and only if $F(0)=1$ and F satisfies the following q-difference equation*

$$\left(1+\sum_{i=1}^{r}(-x)^i\left(d^{i-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i-1}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}+d^i\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\right)F(x)$$
$$=F(xq)+\sum_{i=1}^{r}\sum_{\ell=1}^{r}\sum_{k=0}^{\min(i-1,\ell-1)}c_{k,i}b_{\ell-k,j}(-1)^{\ell-1}x^{\ell}F\left(xq^i\right), \qquad (\text{eq}'_r)$$

where

$$c_{k,i}:=d^ku_r^kq^{\frac{k(k+1)}{2}}\begin{bmatrix}i-1\\ k\end{bmatrix}_q,$$

and

$$b_{m,i}:=\left(d^{m-1}\sum_{\substack{1\leq j<2^{r}\\ w(\tilde{u}_j)=i+m-1}}u_1^{\varepsilon_1(j)}\cdots u_{r}^{\varepsilon_{r}(j)}+d^m\sum_{\substack{1\leq j<2^{r}\\ w(\tilde{u}_j)=i+m}}u_1^{\varepsilon_1(j)}\cdots u_{r}^{\varepsilon_{r}(j)}\right)\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}_q.$$

Proof. Starting from (eq_r) and writing f in terms of F, we obtain

$$(1-x)\prod_{i=1}^{r-1}(1-dxu_i)\,F(x)=F(xq)$$
$$+\sum_{i=1}^{r}\left(\sum_{m=0}^{r-i}d^m\sum_{\substack{1\leq j<2^{r}\\ w(\tilde{u}_j)=i+m}}xu_1^{\varepsilon_1(j)}\cdots u_{r}^{\varepsilon_{r}(j)}\left((-x)^{m-1}\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}_q\right.\right.$$
$$\left.\left.+(-x)^m\begin{bmatrix}i+m\\ m\end{bmatrix}_q\right)\right)\times\prod_{h=1}^{i-1}\left(1-dxu_rq^h\right)F\left(xq^i\right).$$

Using the conventions

$$\sum_{\substack{1\leq j<2^{r}\\ w(\tilde{u}_j)=n}}u_1^{\varepsilon_1(j)}\cdots u_{r}^{\varepsilon_{r}(j)}=0 \text{ for } n>r,$$

and

$$\sum_{\substack{1\leq j<2^{r}\\ w(\tilde{u}_j)=0}}xu_1^{\varepsilon_1(j)}\cdots u_{r}^{\varepsilon_{r}(j)}=1,$$

together with the q-binomial theorem (see for example [11, Equation (3.3.6)]), this can ban rewritten as

$$\left(1+\sum_{i=1}^{r}(-x)^i\left(d^{i-1}\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i-1}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right.\right.$$

$$\left.\left.+d^i\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\right)F(x)=F(xq)$$

$$+\sum_{i=1}^{r}\left(\sum_{m=1}^{r-i+1}\left(d^{m-1}\sum_{\substack{1\le j<2^{r}\\ w(\tilde{u}_j)=i+m-1}} u_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)}\right.\right.$$

$$\left.\left.+d^m\sum_{\substack{1\le j<2^{r}\\ w(\tilde{u}_j)=i+m}} u_1^{\varepsilon_1(j)}\cdots u_r^{\varepsilon_r(j)}\right)\begin{bmatrix} i+m-1\\ m-1\end{bmatrix}_q(-1)^{m-1}x^m\right)$$

$$\times\left(\sum_{k=0}^{i-1}d^k(-x)^k u_r^k q^{\frac{k(k-1)}{2}}\begin{bmatrix} i-1\\ k\end{bmatrix}_q\right)F\left(xq^i\right).$$

Expanding and noting that $b_{\ell-k,i}=0$ if $i+\ell-k-1\ge r$, we obtain (eq_r'). Moreover, $F(0)=f(0)=1$ and the lemma is proved. □

We can now transform (eq_r') into a recurrence equation on the coefficients of F as a power series in x.

Lemma 3.6. *Let F be a function and $(A_n)_{n\in\mathbb{N}}$ av sequence such that*

$$F(x)=:\sum_{n=0}^{\infty}A_nx^n.$$

Then F satisfies (eq_r') *and the initial condition $F(0)=1$ if and only if $A_0=1$ and $(A_n)_{n\in\mathbb{N}}$ satisfies the following recurrence equation*

$$\left(1-q^n\right)A_n = \sum_{m=1}^{r}\left(d^{m-1}\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=m-1}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=m}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)} + \sum_{i=1}^{r}\sum_{k=0}^{\min(i-1,m-1)} c_{k,i}b_{m-k,i}q^{i(n-m)}\right)(-1)^{m+1}A_{n-m}. \qquad (\mathrm{rec}_r)$$

For convenience, we now do transformations starting from (eq_{r-1}).

Lemma 3.7. *Let g and G be two functions such that*

$$G(x) := g(x)\prod_{n=0}^{\infty}\frac{1}{1-xq^n}.$$

Then g satisfies (eq_{r-1}) *and* $g(0)=1$ *if and only if* $G(0)=1$ *and G satisfies the following q-difference equation*

$$\left(1+\sum_{i=1}^{r}\left(d^{i-1}\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i-1}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^i \sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)(-x)^i\right)G(x)$$

$$= G(xq) + \sum_{i=1}^{r}\sum_{m=1}^{r-i}\left(d^{m-1}\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i+m-1}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i+m}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\begin{bmatrix} i+m-1 \\ m-1\end{bmatrix}_q (-1)^{m+1}x^m G\left(xq^i\right). \qquad (\mathrm{eq}''_{r-1})$$

Proof. By the definition of G and (eq_{r-1}), we have

$$
\begin{aligned}
(1-x)\prod_{i=1}^{r-1}(1-dxu_i)\,G(x) &= G(xq) \\
&+\sum_{i=1}^{r-1}\left(\sum_{m=0}^{r-i-1} d^m \sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i+m}} xu_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right. \\
&\qquad \left.\times\left((-x)^{m-1}\begin{bmatrix} i+m-1 \\ m-1\end{bmatrix}_q + (-x)^m \begin{bmatrix} i+m \\ m\end{bmatrix}_q\right)\right) G\left(xq^i\right).
\end{aligned}
$$

Then, using the q-binomial theorem as in the proof of Lemma 3.5, this can be reformulated as (eq''_{r-1}), and $G(0)=g(0)=1$. □

Again, we want to translate this into a recurrence equation on the coefficients of G written as a power series in the variable x.

Lemma 3.8. *Let G be a function and $(a_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$G(x) =: \sum_{n=0}^{\infty} a_n x^n.$$

Then G satisfies (eq''_{r-1}) *and $G(0)=1$ if and only if $a_0=1$ and $(a_n)_{n\in\mathbb{N}}$ satisfies the following recurrence equation*

$$
\begin{aligned}
\left(1-q^n\right)a_n &= \sum_{m=1}^{r}\sum_{i=0}^{r-1}\left(d^{m-1}\sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i+m-1}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right. \\
&\left.+d^m \sum_{\substack{1\le j<2^{r-1}\\ w(\tilde{u}_j)=i+m}} u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\begin{bmatrix} i+m-1 \\ m-1\end{bmatrix}_q q^{i(n-m)}(-1)^{m+1}a_{n-m}. \qquad (\mathrm{rec}'_{r-1})
\end{aligned}
$$

Proof. Plugging the definition of $(a_n)_{n\in\mathbb{N}}$ into (eq''_{r-1}) gives

$$\left(1-q^{n}\right)a_{n}=\sum_{m=1}^{r}\left(d^{m-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_{j})=m-1}}u_{1}^{\varepsilon_{1}(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right.$$

$$\left.+d^{m}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_{j})=m}}u_{1}^{\varepsilon_{1}(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)(-1)^{m+1}a_{n-m}$$

$$+\sum_{m=1}^{r-1}\sum_{i=1}^{r-1}\left(d^{m-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_{j})=i+m-1}}u_{1}^{\varepsilon_{1}(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right.$$

$$\left.+d^{m}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_{j})=i+m}}u_{1}^{\varepsilon_{1}(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\begin{bmatrix} i+m-1\\ m-1\end{bmatrix}_{q}q^{i(n-m)}(-1)^{m+1}a_{n-m}.$$

Gathering the sums and noting that $a_n = G(0) = 1$ completes the proof. □

We now do a final transformation and obtain the last recurrence equation.

Lemma 3.9. *Let $(a_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ be two sequences such that*

$$A'_n := a_n \prod_{k=0}^{n-1}\left(1+u_r q^k\right).$$

Then $(a_n)_{n\in\mathbb{N}}$ satisfies (rec''_{r-1}) *and the initial condition $a_0 = 1$ if and only if $A'_0 = 1$ and $(A'_n)_{n\in\mathbb{N}}$ satisfies the following recurrence equation*

$$\left(1-q^{n}\right)A'_{n}=\sum_{m=1}^{r}\left(\sum_{\nu=0}^{r-1}\sum_{\mu=0}^{\min(m-1,\nu)}f_{m,\mu}e_{m,\nu-\mu}q^{\nu(n-m)}\right.$$

$$\left.+\ u_r\sum_{\nu=1}^{r}\sum_{\mu=0}^{\min(m-1,\nu-1)}f_{m,\mu}e_{m,\nu-\mu-1}q^{\nu(n-m)}\right)(-1)^{m+1}A'_{n-m}, \qquad (\mathrm{rec}'_{r-1})$$

where

$$e_{m,i} := \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j) = i+m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right.$$
$$\left. + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j) = i+m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right) \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q ,$$

and

$$f_{m,k} := u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q .$$

Proof. Replacing the definition of $(A'_n)_{n \in \mathbb{N}}$ into (rec''_{r-1}), we have

$$\left(1 - q^n\right) A'_n = \sum_{m=1}^{r} \sum_{i=0}^{r-1} \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j) = i+m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right.$$
$$\left. + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j) = i+m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right)$$
$$\times \begin{bmatrix} i+m-1 \\ m-1 \end{bmatrix}_q q^{i(n-m)} (-1)^{m+1} \prod_{k=1}^{m} \left(1 + u_r q^{n-k}\right) A'_{n-m} .$$

Furthermore, by a change of variables and the q-binomial theorem, we obtain

$$\prod_{k=1}^{m} \left(1 + u_r q^{n-k}\right) = \left(1 + u_r q^{n-m}\right) \sum_{k=0}^{m-1} u_r^k q^{\frac{k(k+1)}{2} + k(n-m)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q .$$

Thus

$$\left(1-q^{n}\right)A'_{n}=$$

$$\sum_{m=1}^{r}\sum_{i=0}^{r-1}\left(d^{m-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m-1}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}+d^{m}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)$$

$$\times\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}_q q^{i(n-m)}\left(1+u_r q^{n-m}\right)\sum_{k=0}^{m-1}u_r^k q^{\frac{k(k+1)}{2}+k(n-m)}\begin{bmatrix}m-1\\ k\end{bmatrix}_q(-1)^{m+1}A'_{n-m}$$

$$=\sum_{m=1}^{r}\left[\sum_{i=0}^{r-1}\left(d^{m-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m-1}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}+d^{m}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)\right.$$

$$\times\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}_q q^{i(n-m)}\sum_{k=0}^{m-1}u_r^k q^{\frac{k(k+1)}{2}+k(n-m)}\begin{bmatrix}m-1\\ k\end{bmatrix}_q$$

$$+\sum_{i=0}^{r-1}\left(d^{m-1}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m-1}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}+d^{m}\sum_{\substack{1\leq j<2^{r-1}\\ w(\tilde{u}_j)=i+m}}u_1^{\varepsilon_1(j)}\cdots u_{r-1}^{\varepsilon_{r-1}(j)}\right)$$

$$\left.\times u_r\begin{bmatrix}i+m-1\\ m-1\end{bmatrix}_q q^{(i+1)(n-m)}\sum_{k=0}^{m-1}u_r^k q^{\frac{k(k+1)}{2}+k(n-m)}\begin{bmatrix}m-1\\ k\end{bmatrix}_q\right](-1)^{m+1}A'_{n-m}.$$

Therefore

$$\left(1-q^{n}\right)A'_{n}=\sum_{m=1}^{r}\left(\sum_{i=0}^{r-1}e_{m,i}q^{i(n-m)}\sum_{k=0}^{m-1}f_{m,k}q^{k(n-m)}\right.$$
$$\left.+\,u_r\sum_{i=1}^{r}e_{m,i-1}q^{i(n-m)}\sum_{k=0}^{m-1}f_{m,k}q^{k(n-m)}\right)(-1)^{m+1}A'_{n-m}.$$

Expanding gives (rec'_{r-1}), and $A'_0=a_0=1$. □

The key step is now to show that $(A_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ are equal.

Lemma 3.10. *Let $(A_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ be defined as in Lemmas 3.6 and 3.9. Then for every $n\in\mathbb{N}$, $A_n=A'_n$.*

Proof. To prove the equality, we show that for every $1 \leq m \leq r$, the coefficient of $(-1)^{m+1} A_{n-m}$ in (rec_r) and the coefficient of $(-1)^{m+1} A'_{n-m}$ in (rec'_{r-1}) are equal. Let $m \in \{1, ..., r\}$ and

$$\begin{aligned} S_m &:= \left[(-1)^{m+1} A_{n-m}\right] (\text{rec}_r) \\ &= d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j) = m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j) = m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \\ &+ \sum_{i=1}^{r} \sum_{k=0}^{\min(i-1, m-1)} c_{k,i} b_{m-k,i} q^{i(n-m)} \end{aligned}$$

and

$$\begin{aligned} S'_m &:= \left[(-1)^{m+1} A'_{n-m}\right] (\text{rec}'_{r-1}) \\ &= \sum_{\nu=0}^{r-1} \sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu} e_{m,\nu-\mu} q^{\nu(n-m)} + u_r \sum_{\nu=1}^{r} \sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu} e_{m,\nu-\mu-1} q^{\nu(n-m)} \\ &= f_{m,0} e_{m,0} + \sum_{\nu=1}^{r} \left(\sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu} e_{m,\nu-\mu} + u_r \sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu} e_{m,\nu-\mu-1} \right) q^{\nu(n-m)}, \end{aligned}$$

because $e_{m,r-\mu} = 0$ for all μ.

We start by noting that

$$f_{m,0} e_{m,0} = d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j) = m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j) = m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)}.$$

Now define

$$T_{m,i} := \sum_{k=0}^{\min(i-1,m-1)} c_{k,i} b_{m-k,i},$$

and

$$T'_{m,i} := \sum_{k=0}^{\min(m-1,i)} f_{m,k} e_{m,i-k} + u_r \sum_{k=0}^{\min(m-1,i-1)} f_{m,k} e_{m,i-k-1}.$$

So it only remains to show that for all $1 \leq i \leq r$,

$$T_{m,i} = T'_{m,i}.$$

We have

$$
\begin{aligned}
&c_{k,i}b_{m-k,i}\\
&= u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q\\
&\times \left(d^{m-1} \sum_{\substack{1\le j<2^r \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_r^{\varepsilon_r(j)} + d^m \sum_{\substack{1\le j<2^r \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_r^{\varepsilon_r(j)} \right)\\
&= u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q\\
&\times \left(d^{m-1} \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right)\\
&+ u_r^{k+1} q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q\\
&\times \left(d^{m-1} \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k-2}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right),
\end{aligned}
\tag{3.16}
$$

where the last equality follows from splitting the sum according to whether $\tilde{u}_j$ contains u_r as a primary colour or not.

On the other hand, one has

$$
\begin{aligned}
f_{m,k}e_{m,i-k} &= u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-1 \end{bmatrix}_q\\
&\times \left(d^{m-1} \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1\le j<2^{r-1} \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right),
\end{aligned}
\tag{3.17}
$$

and

$$u_r f_{m,k} e_{m,i-k-1} = u_r^{k+1} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-2 \\ m-1 \end{bmatrix}_q$$
$$\times \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-2}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right). \tag{3.18}$$

For all $j, k, m \in \mathbb{N}$, we have the following equality:

$$\begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} i+m-k-1 \\ m-1 \end{bmatrix}_{q^N} = \begin{bmatrix} i \\ k \end{bmatrix}_{q^N} \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N}. \tag{3.19}$$

Using (3.19), we obtain

$$T'_{m,i} = \chi(i \le m-1)\, u_r^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} m-1 \\ m-i-1 \end{bmatrix}_q$$
$$\times \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right)$$
$$+ \sum_{k=0}^{\min(m-1,i-1)} u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q$$
$$\times \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right)$$
$$+ \sum_{k=0}^{\min(m-1,i-1)} u_r^{k+1} q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-2 \\ m-k-1 \end{bmatrix}_q$$
$$\times \left(d^{m-1} \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-2}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \le j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right).$$

By the second q-analogue (3.13) of Pascal's triangle, we have

$$\begin{bmatrix} i \\ k \end{bmatrix}_q = \begin{bmatrix} i-1 \\ k \end{bmatrix}_q + q^{i-k} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix}_q,$$

$$\begin{bmatrix} i+m-k-2 \\ m-k-1 \end{bmatrix}_q = \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q - q^i \begin{bmatrix} i+m-k-2 \\ m-k-2 \end{bmatrix}_q.$$

Thus we can rewrite $T'_{m,i}$ as

$$\begin{aligned}
T'_{m,i} &= \chi(i \leq m-1)\, u_r^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} m-1 \\ m-i-1 \end{bmatrix}_q \\
&\times \left(d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=m-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=m}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right) \\
&+ \sum_{k=0}^{\min(m-1,i-1)} u_r^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q \\
&\times \left(d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right) \\
&+ \sum_{k=0}^{\min(m-1,i-1)} u_r^k q^{\frac{k(k-1)}{2}+j} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q \\
&\times \left(d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right) \\
&+ \sum_{k=0}^{\min(m-1,i-1)} u_r^{k+1} q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-1 \\ m-k-1 \end{bmatrix}_q \\
&\times \left(d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-2}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right) \\
&- \sum_{k=0}^{\min(m-2,i-1)} u_r^{k+1} q^{\frac{k(k+1)}{2}+j} \begin{bmatrix} i-1 \\ k \end{bmatrix}_q \begin{bmatrix} i+m-k-2 \\ m-k-2 \end{bmatrix}_q \\
&\times \left(d^{m-1} \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-2}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} + d^m \sum_{\substack{1 \leq j < 2^{r-1} \\ w(\tilde{u}_j)=i+m-k-1}} u_1^{\varepsilon_1(j)} \cdots u_{r-1}^{\varepsilon_{r-1}(j)} \right).
\end{aligned}$$

By (3.16), the sum of the second and fourth terms above is equal to $T_{m,j}$.

A simple computation shows that the sum of the first, third and fifth terms is zero. This completes the proof. □

We can finally use all the previous lemmas to prove Theorem 3.4.

Proof. (Proof of Theorem 3.4). Let us start by the initial case $r = 1$. Let f such that $f(0) = 1$ and

$$(1 - dxu_1)\, f(x) = f(xq) + xu_1 f(xq). \tag{eq$_1$}$$

Then

$$f(x) = \frac{1 + xu_1}{1 - dxu_1} f\,(xq)\,. \tag{3.20}$$

Iterating (3.20), we get

$$f(x) = \prod_{n=0}^{\infty} \frac{1 + xu_1 q^n}{1 - dxu_1 q^n} f(0).$$

Thus

$$f(1) = \frac{(-u_1; q)_\infty}{(du_1; q)_\infty}.$$

Now assume that Theorem 3.4 is true for some positive integer $r - 1$ and show that it is true for r too. Let f such that $f(0) = 1$ satisfying (eq$_r$). Let

$$F(x) := f(x) \prod_{n=0}^{\infty} \frac{1 - dxu_r q^n}{1 - xq^n}.$$

By Lemma 3.5, $F(0) = 1$ and F satisfies (eq$'_r$). Now let

$$F(x) =: \sum_{n=0}^{\infty} A_n x^n.$$

Then by Lemma 3.6 $A_0 = 1$ and $(A_n)_{n\in\mathbb{N}}$ satisfies (rec$_r$). But by Lemma 3.10, $(A_n)_{n\in\mathbb{N}}$ also satisfies (rec$'_{r-1}$). Now let

$$A_n =: a_n \prod_{k=0}^{n-1} \left(1 + u_r q^k\right).$$

By Lemma 3.9, $a_0 = 1$ and $(a_n)_{n\in\mathbb{N}}$ satisfies (rec$''_{r-1}$). Let

$$G(x) := \sum_{n=0}^{\infty} a_n x^n.$$

By Lemma 3.8, $G(0) = 1$ and G satisfies (eq$''_{r-1}$). Finally, let

$$g(x) := G(x) \prod_{n=0}^{\infty} \left(1 - xq^n\right).$$

By Lemma 3.7, $g(0) = 1$ and g satisfies (eq_{r-1}). By the induction hypothesis, we have

$$g(1) = \prod_{k=1}^{r-1} \frac{(-u_k; q)_\infty}{(du_k; q)_\infty}. \tag{3.21}$$

By Appell's comparison theorem [23],

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n \\
&= \lim_{x\to 1^-} (1-x) G(x) \\
&= \lim_{x\to 1^-} (1-x) \frac{g(x)}{\prod_{n=0}^{\infty} (1 - xq^n)} \\
&= \frac{g(1)}{\prod_{n=1}^{\infty} (1-q^n)}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} A_n = \prod_{k=0}^{\infty} \left(1 + u_r q^k\right) \frac{g(1)}{\prod_{n=1}^{\infty} \left(1 - q^{nN}\right)}.$$

Therefore, by Appell's lemma again,

$$\begin{aligned}
\lim_{x\to 1^-} (1-x) F(x) &= \lim_{n\to\infty} A_n \\
&= \prod_{k=0}^{\infty} \left(1 + u_r q^k\right) \frac{g(1)}{\prod_{n=1}^{\infty} (1-q^n)}.
\end{aligned} \tag{3.22}$$

Finally,

$$\begin{aligned}
f(1) &= \lim_{x\to 1^-} f(x) \\
&= \lim_{x\to 1^-} \prod_{n=0}^{\infty} \frac{1 - xq^n}{1 - dxu_r q^n} F(x) \\
&= \frac{\prod_{n=1}^{\infty} (1-q^n)}{\prod_{n=0}^{\infty} (1 - du_r q^n)} \prod_{k=0}^{\infty} \left(1 + u_r q^k\right) \frac{g(1)}{\prod_{n=1}^{\infty} (1-q^n)} \quad \text{by (3.22)} \\
&= \frac{(-u_r; q)_\infty}{(du_r; q)_\infty} g(1).
\end{aligned}$$

Then by (3.21),

$$f(1) = \prod_{k=1}^{r} \frac{(-u_k; q)_\infty}{(du_k; q)_\infty}.$$

This completes the proof. □

Now Theorem 1.10 is a simple corollary of Theorem 3.4.

Proof. (Proof of Theorem 1.10). By Lemma 3.3, $f_{1_{u_1}}$ satisfies (eq_r). Therefore

$$f_{0_{u_1}}(1) = \prod_{k=1}^{r} \frac{(-u_k; q)_\infty}{(du_k; q)_\infty}.$$

This infinite product is the generating function for the overpartitions counted by $\overline{D}(\ell_1, \ldots, \ell_r; k, n)$, and $f_{0_{u_1}}(1)$ the generating function for overpartitions counted by $\overline{E}(\ell_1, \ldots, \ell_r; k, n)$, thus

$$\overline{D}(\ell_1, \ldots, \ell_r; k, n) = \overline{E}(\ell_1, \ldots, \ell_r; k, n).$$

□

4 Conclusion

Our new version of the method of weighted words using q-difference equations has been successful in [26] and in the present paper to prove refinements of Rogers-Ramanujan type identities with intricate difference conditions which make the classical method difficult to apply. We are hopeful that this method can be used to refine a wide range of partition identities. For example, in an upcoming paper with Jeremy Lovejoy, we apply it to prove a conjectural partition identity of Primc [38] which arose from crystal base theory. It would be interesting to see whether it can also be applied to prove refined versions of partition identities arising from representation theory such as those of Meurman-Primc [35] or Nandi [36] for example.

Acknowledgements The author thanks Jeremy Lovejoy for his comments on an earlier version of this article. She also thanks the anonymous referee for carefully reading this paper in detail and giving very valuable advice to improve its readability. She would like to thank all involved.

References

1. K. Alladi, Refinements of Rogers–Ramanujan type identities, *Special Functions, q-Series and Related Topics*, vol. 14, Fields Institute Communications (1997), pp. 1–35
2. K. Alladi, B. Gordon, Generalizations of Schur's partition theorem. Manuscripta Math. **79**, 113–126 (1993)
3. K. Alladi, B. Gordon, Schur's partition theorem, companions, refinements and generalizations. Trans. Am. Math. Soc. **347**, 1591–1608 (1995)
4. G. Andrews, J. Olsson, Partition identities with an application to group representation theory. J. Reine Angew. Math. **413**, 198–212 (1991)
5. G.E. Andrews, Enumerative proofs of certain q-identities. Glasgow Math. J. **8**, 33–40 (1967)
6. G.E. Andrews, On Schur's second partition theorem. Glasgow Math. J. **8**, 127–132 (1967)
7. G.E. Andrews, A new generalization of Schur's second partition theorem. Acta Arith. **14**, 429–434 (1968)
8. G.E. Andrews, On partition functions related to Schur's second partition theorem. Proc. Am. Math. Soc. **19**, 441–444 (1968)
9. G.E. Andrews, A general theorem on partitions with difference conditions. Am. J. Math. **91**, 18–24 (1969)
10. G.E. Andrews, On a theorem of Schur and Gleissberg. Arch. Math. **22**, 165–167 (1971)
11. G.E. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, 1984)
12. G.E. Andrews, S.H. Chan, B. Kim, R. Osburn, The first positive rank and crank moments for overpartitions. Ann. Comb. **20**, 193–207 (2016)
13. C. Bessenrodt, A combinatorial proof of a refinement of the Andrews–Olsson partition identity. Eur. J. Combin. **12**, 271–276 (1991)
14. C. Bessenrodt, Generalisations of the Andrews–Olsson partition identity and applications. Discret. Math. **141**, 11–22 (1995)
15. D. Bressoud, On a partition theorem of Göllnitz. J. Reine Angew. Mat. **305**, 215–217 (1979)
16. D. Bressoud, A combinatorial proof of Schur's 1926 partition theorem. Proc. Am. Math. Soc. **79**, 338–340 (1980)
17. K. Bringmann, J. Lovejoy, Rank and congruences for overpartition pairs. Int. J. Number Theory **4**, 303–322 (2008)
18. S. Corteel, Particle seas and basic hypergeometric series. Adv. Appl. Math. **31**, 199–214 (2003)
19. S. Corteel, J. Lovejoy, Frobenius partitions and the combinatorics of Ramanujans ${}_1\psi_1$ summation. J. Combin. Theory Ser. A **97**, 177–183 (2002)
20. S. Corteel, J. Lovejoy, Overpartitions. Trans. Am. Math. Soc. **356**, 1623–1635 (2004)
21. S. Corteel, J. Lovejoy, An iterative-bijective approach to generalizations of Schur's theorem. Eur. J. Comb. **27**, 496–512 (2006)
22. P. Desrosiers, L. Lapointe, P. Mathieu, Jack superpolynomials, superpartition ordering and determinantal formulas. Commun. Math. Phys. **233**, 383–402 (2003)
23. P. Dienes, *The Taylor Series* (Dover, New York, 1957)
24. J. Dousse, A generalisation of a partition theorem of Andrews. Monatsh. Math. **179**, 227–251 (2016)
25. J. Dousse, A generalisation of a second partition theorem of Andrews to overpartitions. J. Combin. Theory Ser. A **145**, 101–128 (2017)
26. J. Dousse, Siladić's theorem: weighted words, refinement and companion. Proc. Am. Math. Soc. (to appear)
27. J.F. Fortin, P. Jacob, P. Mathieu, Generating function for K-restricted jagged partitions. Electron. J. Combin. **12** (2005)
28. J.F. Fortin, P. Jacob, P. Mathieu, Jagged partitions. Ramanujan J. **10**, 215–235 (2005)
29. J.T. Joichi, D. Stanton, Bijective proofs of basic hypergeometric series identities. Pacific J. Math. **127**, 103–120 (1987)
30. S.J. Kang, J.H. Kwon, Crystal bases of the Fock space representations and string functions. J. Algebra **280**, 313–349 (2004)

31. J. Lovejoy, Gordon's theorem for overpartitions. J. Combin. Theory Ser. A **103**, 393–401 (2003)
32. J. Lovejoy, Overpartition theorems of the Rogers–Ramanujan type. J. Lond. Math. Soc. **69**, 562–574 (2004)
33. J. Lovejoy, A theorem on seven-colored overpartitions and its applications. Int. J. Number Theory **1**, 215–224 (2005)
34. K. Mahlburg, The overpartition function modulo small powers of 2. Discret. Math. **286**, 263–267 (2004)
35. A. Meurman, M. Primc, Annihilating ideals of standard modules of $sl(2,\mathbb{C})^{\sim}$ and combinatorial identities. Adv. Math. **64**, 177–240 (1987)
36. D. Nandi, Partition identities arising from the standard $a_2^{(2)}$-modules of level 4. Ph.D. thesis, Rutgers University, 2014
37. S. Oh, The Andrews–Olsson identity and Bessenrodt insertion algorithm on Young walls. Eur. J. Comb. **43**, 8–31 (2015)
38. M. Primc, Some crystal Rogers–Ramanujan type identities. Glas. Math. **34**, 73–86 (1999)
39. I. Schur, Zur additiven zahlentheorie. Sitzungsberichte der Preussischen Akademie der Wissenschaften, pp. 488–495 (1926)
40. I. Siladić, Twisted $sl(3,\mathbb{C})^{\sim}$-modules and combinatorial identities (2002), arXiv:math/0204042v2
41. S. Treneer, Congruences for the coefficients of weakly holomorphic modular forms. Proc. Lond. Math. Soc. **93**, 304–324 (2006)
42. A. Yee, Alder's conjecture. J. Reine Angew. Math. **616**, 67–88 (2008)

Integrals Involving Rudin–Shapiro Polynomials and Sketch of a Proof of Saffari's Conjecture

Shalosh B. Ekhad and Doron Zeilberger

Dedicated to Krishnaswami "Krishna" Alladi, the tireless apostle of Srinivasa Ramanujan, yet a great mathematician in his own right

Abstract The Rudin–Shapiro polynomials $P_k(z)$, are defined by a certain linear functional recurrence equation and are of interest in signal processing due to their special autocorrelation properties. An algorithmic approach to computation of the moments of these polynomials is given. A proof sketch is given of Saffari's Conjecture for the asymptotic growth of these moments.

Keywords Rudin–Shapiro polynomials · Saffari's Conjecture

2010 Mathematics Subject Classification 11B83

Preface: Krishna Alladi

One of the greatest *disciples* of Srinivasa Ramanujan, who did so much to make him a household name in the mathematical community, and far beyond, is Krishnaswami "Krishna" Alladi. Among many other things, he founded and is still editor-in-chief, of the very successful *Ramanujan Journal* (very ably managed by managing editor Frank Garvan), and initiated the SASTRA Ramanujan prize given to promising young mathematicians.

But Krishna is not *just* a mathematical leader, he is also a great number-theorist with very broad interests, including analytic number theory and, inspired by Ramanujan, q-series and partitions. That's why it is not surprising that the conference to

S. B. Ekhad · D. Zeilberger (✉)
Department of Mathematics, Rutgers University (New Brunswick),
Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA
e-mail: zeilberg@math.rutgers.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_15

celebrate his 60th birthday, that took place last March, attracted attendees and speakers with very diverse interests, and enabled the participants to learn new things far afield from their own narrow specialty. That's how we found out, and got hooked on, *Rudin–Shapiro polynomials*.

Hugh Montgomery's Erdős's colloquium

One of the highlights of the conference was a fascinating talk by the eminent Michigan number theorist (and Krishna's former postdoc mentor) Hugh Montgomery, who talked about *Littlewood polynomials* of interest **both** in pure number theory and, surprisingly, in *signal processing*. These are polynomials whose coefficients are in $\{-1, 1\}$. Among these stand out the famous *Rudin–Shapiro* polynomials, introduced ([8, 9]) by Harold "Silent" Shapiro[1] and rediscovered by Walter Rudin ([7]).

The Rudin–Shapiro polynomials

The Rudin–Shapiro polynomials, $P_k(z)$, are best defined by the functional recurrence (see [10])

$$P_k(z) = P_{k-1}(z^2) + zP_{k-1}(-z^2), \qquad (DefiningRecurrence)$$

with the initial condition $P_0(z) = 1$.

As Hugh Montgomery described so well in his talk, these have amazing properties. Both number-theorists and signal-processors are very interested in the so-called *sequence of (even) moments*, whose definition usually involves the integral sign, but is better phrased entirely in terms of high-school algebra as follows.

$$M_n(k) := CT[P_k(z)^n P_k(z^{-1})^n],$$

where CT denotes the "constant term functional", that for any Laurent polynomial $f(z)$ of z, extracts the coefficient of z^0. For example $CT[4/z^2 + 11/z + 101 + 5z + 11z^{15}] = 101$.

Can we find closed-form expressions for $M_n(k)$, in k, for any given, specific, positive integer n? Failing this, can we find explicit expressions for the generating functions

[1]Harold S. Shapiro (S. originally stood for Seymour) was one of a brilliant cohort of students at City College, in the late 1940s, that included Leon Ehrenpreis, Donald Newman, Israel Aumann, and another Harold Shapiro, Harold N. Shapiro (N. originally stood for Nathaniel). But their friends, in order to distinguish between the two Harold Shapiros, called them "Silent" and "Noisy" respectively. It is ironic that Harold Silent Shapiro's son is the eminent, **but very loud**, MIT cosmologist, Max Tegmark.

$$R_n(t) := \sum_{k=0}^{\infty} M_n(k) t^k ?$$

The sequence $M_1(k)$ has a very nice closed-form, $M_1(k) = 2^k$. This is not very hard, even for humans. Indeed, using Eq. $(DefiningRecurrence)$, we get

$$P_k(z)P_k(z^{-1}) = \left(P_{k-1}(z^2) + zP_{k-1}(-z^2) \right) \cdot \left(P_{k-1}(z^{-2}) + z^{-1}P_{k-1}(-z^{-2}) \right)$$

$$= P_{k-1}(z^2)P_{k-1}(z^{-2}) \; + \; P_{k-1}(-z^2)P_{k-1}(-z^{-2}) \; + \; \left\{ zP_{k-1}(-z^2)P_{k-1}(z^{-2}) \right. \\ \left. + z^{-1}P_{k-1}(z^2)P_{k-1}(-z^{-2}) \right\}.$$

The quantity in the braces only has **odd** powers, so its constant term vanishes. Hence

$$M_1(k) = CT\,[\,P_k(z)P_k(z^{-1})\,] \;=\; CT\,[\,P_{k-1}(z^2)P_{k-1}(z^{-2})\,] \;+\; CT\,[\,P_{k-1}(-z^2)P_{k-1}(-z^{-2})\,].$$

Replacing z^2 by z in the first term on the right, and $-z^2$ by z in the second term, does not change the constant term, hence, we have the **linear recurrence equation with constant coefficients**

$$M_1(k) = 2M_1(k-1),$$

with the obvious initial condition $M_1(0) = 1$, that implies the explicit expression $M_1(k) = 2^k$. Equivalently, the generating function $R_1(t)$ is given by

$$R_1(t) = \frac{1}{1-2t}.$$

Let's move on to find an explicit formula for $M_2(k)$ and/or $R_2(t)$. That was already done by smart human John Littlewood ([5]) but let's do it again.

Once again, let's use the defining recurrence for the Rudin–Shapiro polynomials, but let's abbreviate

$$a(k)(z) = P_k(z) \quad , \quad b(k)(z) = P_k(-z) \quad , \quad A(k)(z) = P_k(z^{-1}) \quad , \quad B(k)(z) = P_k(-z^{-1}).$$

We have

$$P_k(z)^2 P_k(z^{-1})^2 = \left(P_{k-1}(z^2) + zP_{k-1}(-z^2) \right)^2 \cdot \left(P_{k-1}(z^{-2}) + z^{-1}P_{k-1}(-z^{-2}) \right)^2.$$

Expanding, discarding odd terms, replacing z^2 by z, and using trivial symmetries due to the fact that the functional CT is preserved under the dihedral group $\{z \to z, z \to -z, z \to z^{-1}, z \to -z^{-1}\}$, we get that

$$CT\,[a(k)^2A(k)^2] = 2CT[a(k-1)^2A(k-1)^2\,] - 2CT\,[za(k-1)^2B(k-1)^2] \\ + 4CT[a(k-1)A(k-1)b(k-1)B(k-1)\,].$$

The first term is an old friend, our quantity of interest with k replaced by $k-1$, but the other two are newcomers. So we do the same treatment to them. They in turn, may (and often do) introduce new quantities, but if all goes well, there would only be finitely many sequences, and we would get a **finite** system of first-order linear recurrences. This indeed happens, and one gets, for the generating functions of the encountered sequences, a system of six equations with six unknowns, and in particular, we get (in a split second, of course, we let Maple do it) that our desired object, the generating function of the sequence $CT\,[a(k)^2A(k)^2]$, alias, $R_2(t)$, is given by:

$$R_2(t) = \frac{4t+1}{(1+2t)(1-4t)} = \frac{4}{3}\frac{1}{1-4t} - \frac{1}{3}\frac{1}{1+2t}.$$

By extracting the coefficient of t^k, we even get a nice explicit expression for $M_2(k)$, already known to Littlewood

$$M_2(k) = \frac{4}{3}4^k - \frac{1}{3}(-2)^k.$$

This can be done for *any* monomial

$$z^{\alpha_0}a(k)^{\alpha_1}A(k)^{\alpha_2}b(k)^{\alpha_3}B(k)^{\alpha_4}.$$

Define the sequence

$$E[\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4](k) := CT\,[\,z^{\alpha_0}a(k)^{\alpha_1}A(k)^{\alpha_2}b(k)^{\alpha_3}B(k)^{\alpha_4}\,].$$

Replacing $a(k), A(k), b(k), B(k)$ by their expressions in terms of $z, a(k-1)$, $A(k-1), b(k-1), B(k-1)$, expanding, discarding odd terms, replacing z^2 by z, and replacing each monomial by its *canonical form*, implied by the above-mentioned action of the dihedral group that preserves CT, we can express, each such $E[.]$, in terms of other $E[.]$'s evaluated at $k-1$. It is (presumably, they may some issues about powers of z) possible to show (and it has been done by Doche and Habsieger [DH], using a different approach) that this process terminates and eventually we will not get any new sequences, leaving us with a finite system of linear equations for the corresponding generating functions, that can be automatically solved, and lead to an expression in terms of a rational function, since we get a first-order system

$$\mathbf{F}(t) = \mathbf{v} + t\mathbf{A}\mathbf{F}(t),$$

(where $\mathbf{F}(t)$ is the vector of generating functions whose first component is our desired one), for some matrix $\mathbf{A}$, of integers that the computer finds automatically, and our object of desire is the first component of $\mathbf{F}(t) = (\mathbf{I} - t\mathbf{A})^{-1}\mathbf{v}$.

While it is painful for a human to do this, a computer does not mind, and the Maple package
`HaroldSilentShapiro.txt`
accompanying this article does it for any desired monomial in z, $P_k(z)$, $P_k(-z)$, $P_k(z^{-1})$, $P_k(-z^{-1})$. See the output files accompanying this article, that may be viewed from the front of this article
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarim html/hss.html`.

Unlike the beautiful approach of Doche and Habsieger, that uses clever human pre-processing to establish an *algorithm*, that was then hard-programmed by hand, our approach is naive "dynamical programming", where we don't make any *a priori* human analysis, and let the computer introduce new quantities as needed. To guarantee that it *halts*, we input a parameter, that we call `K`, and if the size of the system exceeds K it returns `FAIL`, leaving us the option to forget about it, or try again with a larger `K`.

Higher moments and Saffari's Conjecture

Now that we have reduced, for any specific positive integer n, the computation of the generating function of the sequence of moments $M_n(k)$, that we call $R_n(t)$, to a routine calculation, we can ask our beloved computer to crank-out as many of them as it can output in a reasonable amount of time. According to the output file
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oHarold SilentShapiro1.txt`,

we get

$$R_1(t) = \frac{1}{1-2t} \quad ,$$

$$R_2(t) = \frac{1+4t}{(1+2t)(1-4t)} \quad ,$$

[both of which were already given above],

$$R_3(t) = \frac{1+16t}{(1+4t)(1-8t)} \quad ,$$

$$R_4(t) =$$

-(90194313216*t**11 - 15300820992*t**10 - 1979711488*t**9 - 292552704*t**8 - 22216704*t**7 + 10649600*t**6 - 1024*t**5 - 144384*t**4 + 7008*t**3 + 664*t**2 - 54*t - 1)/ ((8*t + 1)* (16*t - 1)*(1409286144*t**10 - 264241152*t**9 - 25690112*t**8 - 4128768*t**7 - 311296*t**6 + 170496*t**5 - 2624*t**4 - 2208*t**3 + 148*t**2 + 8*t - 1)),

$$R_5(t) =$$

- (369435906932736*t**11 - 32160715112448*t**10 - 2001454759936*t**9 - 145223581696*t**8 - 4454350848*t**7 + 1392508928*t**6 - 5865472*t**5 - 4599808*t**4 + 123648*t**3 + 4768*t**2 - 220*t - 1)/ ((1 + 16*t)*(32*t - 1)*(1443109011456*t**10 - 135291469824*t**9 - 6576668672*t**8 - 528482304* t **7 -19922944*t**6+5455872*t**5-41984*t**4-17664*t**3+592*t**2+16*t-1)).

To see $R_k(t)$ for $6 \leq k \leq 10$, look at the above-mentioned output file. Of course, one can easily go further. Note that these have already been computed in [3] (but their output is not easily accessible to the casual reader).

By looking at the smallest root of the denominator of $R_k(t)$ and computing the residue, one confirms for small (and not so small!) values of k (and one can easily go much further), the following conjecture of Bahman Saffari, as already done in [3] (for small k).

Saffari's Conjecture *For every positive integer n, as k* $\to \infty$*, the following asymptotic formula holds.*

$$M_n(k) \sim \frac{2^n}{n+1} \cdot (2^n)^k.$$

Saffari never published his conjecture, and it is mentioned as "private communication" in [3].

Sketch of a proof of Saffari's Conjecture

While for each *numeric* n, one can get an explicit expression, in *symbolic* t, for $R_n(t)$, these get more and more complicated as n gets larger, and there is (probably) no hope to get an explicit expression, in **symbolic** n, for $R_n(t)$, from which one can deduce that the smallest root (in absolute value) of the denominator is 2^{-n} and the residue is $\frac{2^n}{n+1}$.

But one can prove *rigorously* Saffari's conjectured asymptotic formula as follows.

Let n be a general (symbolic) positive integer. Recall that we are interested in the sequence

$$M_n(k) := CT[P_k(z)^n P_k(z^{-1})^n],$$

that we abbreviate

$$CT[\, a^n A^n \,],$$

under the convention

$$a = P_k(z), \quad b = P_k(-z), \quad A = P_k(z^{-1}), \quad B = P_k(-z^{-1}).$$

To get a scheme we use the *rewriting rules*, implied by the defining recurrence

$$a \to a + zb, \quad b \to a - zb, \quad A \to A + z^{-1}B, \quad B \to A - z^{-1}B,$$

where the discrete argument on the left is k and on the right $k - 1$, and the continuous argument on the left is z and on the right is z^2.

Using the binomial theorem, we have

$$a^n A^n \to (a + zb)^n (A + z^{-1}B)^n = \left(\sum_{i=0}^{n} \binom{n}{i} a^i (zb)^{n-i} \right) \left(\sum_{j=0}^{n} \binom{n}{j} A^j (z^{-1}B)^{n-j} \right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} a^i b^{n-i} A^j B^{n-j} z^{j-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i}^2 (aA)^i (bB)^{n-i} \quad + SmallChange,$$

where $SmallChange$ is a linear combination of unimportant monomials and we **define** an **important monomial** (in a, A, b, B, z) to be any member of the set of monomials

$$\{(aA)^m (bB)^{n-m} \,|\, 0 \le m \le n\}.$$

Let's try to find the "going down" evolution-step for the other important monomials.

We have

$$(aA)^m (bB)^{n-m} \to (a + zb)^m (A + z^{-1}B)^m (a - zb)^{n-m} (A - z^{-1}B)^{n-m}$$

$$= \left(\sum_{i_1=0}^{m} \binom{m}{i_1} a^{i_1} (zb)^{m-i_1} \right) \left(\sum_{i_2=0}^{m} \binom{m}{i_2} A^{i_2} (z^{-1}B)^{m-i_2} \right) \cdot$$

$$\left(\sum_{i_3=0}^{n-m} \binom{n-m}{i_3} a^{i_3} (-zb)^{n-m-i_3} \right) \left(\sum_{i_4=0}^{n-m} \binom{n-m}{i_4} A^{i_4} (-z^{-1}B)^{n-m-i_4} \right)$$

$$= \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{n-m} \sum_{i_4=0}^{n-m} \binom{m}{i_1} \binom{m}{i_2} \binom{n-m}{i_3} \binom{n-m}{i_4}$$
$$(-1)^{i_3+i_4} a^{i_1+i_3} A^{i_2+i_4} b^{n-i_1-i_3} B^{n-i_2-i_4} z^{i_2-i_1+i_4-i_3}.$$

The coefficient of a typical important monomial, $(aA)^r (bB)^{n-r}$ $(0 \le r \le n)$ in the above quadruple sum is

$$\sum_{i_1+i_3=r\,,\,i_2+i_4=r} (-1)^{i_3+i_4} \binom{m}{i_1}\binom{m}{i_2}\binom{n-m}{i_3}\binom{n-m}{i_4}$$

$$= \sum_{i_1=0}^{r}\sum_{i_2=0}^{r} (-1)^{i_1+i_2} \binom{m}{i_1}\binom{m}{i_2}\binom{n-m}{r-i_1}\binom{n-m}{r-i_2}$$

$$= \left(\sum_{i_1=0}^{r} (-1)^{i_1} \binom{m}{i_1}\binom{n-m}{r-i_1}\right)\left(\sum_{i_2=0}^{r} (-1)^{i_2} \binom{m}{i_2}\binom{n-m}{r-i_2}\right)$$

$$= \left(\sum_{i=0}^{r} (-1)^{i} \binom{m}{i}\binom{n-m}{r-i}\right)^2 .$$

This is an important quantity, so let's give it a name

$$K_n(m,r) := \left(\sum_{i=0}^{r} (-1)^{i} \binom{m}{i}\binom{n-m}{r-i}\right)^2 .$$

All the remaining monomials belong to *SmallChange*, and we have the general "evolution equation"

$$(aA)^m (bB)^{n-m} \to \sum_{r=0}^{n} K_n(m,r)(aA)^r (bB)^{n-r} + \quad SmallChange.$$

Assuming for now, that *SmallChange* is, asymptotically less than the "important monomials" (i.e. the rate of growth of a small change sequence divided by an "important monomial" sequence is $o(1)$), let α_n be the largest eigenvalue of the $n+1$ by $n+1$ matrix K_n (whose (m,r) entry is $K_n(m,r)$), then for $0 \le m \le n$

$$CT[(P_k(z)P_k(z^{-1}))^m (P_k(-z)P_k(-z^{-1}))^{n-m}] \sim c_m(\alpha_n)^k,$$

where $(c_0, \ldots, c_n)$ is an eigenvector corresponding to the largest eigenvalue, α_n.

We now need two elementary propositions that should be provable using the **Wilf–Zeilberger algorithmic proof theory**, (a suitable extension of [4, 12] for the first, [1, 11] for the second). They may be even provable by purely human means, but since we know *for sure* that they are both true, we do not bother.

Proposition 1. *The characteristic polynomial,* $\det(z\mathbf{I} - K_n)$*, of the* $(n+1) \times (n+1)$ *matrix* K_n *whose* (m,r) *entry is*

$$K_n(m,r) := \left(\sum_{i=0}^{r} (-1)^{i} \binom{m}{i}\binom{n-m}{r-i}\right)^2 ,$$

equals

$$\det(z\mathbf{I} - K_n) = z^{\lfloor (n+1)/2 \rfloor} \prod_{j=0}^{\lfloor n/4 \rfloor} \left(z - 2^{n-4j} \binom{4j}{2j} \right) \prod_{j=0}^{\lfloor (n-2)/4 \rfloor} \left(z + 2^{n-4j-2} \binom{4j+2}{2j+1} \right).$$

[To confirm this *shaloshable* determinant identity for $n \le N$, type, in the Maple package `HaroldSilentShapiro.txt`, `CheckCP(N);` . For example, `CheckCP(20);` returns `true` in one second, and `CheckCP(40);` returns `true` in 20 seconds.]

So the non-zero eigenvalues of the matrix K_n are

$$\{ 2^{n-4j} \binom{4j}{2j} \ ; \ 0 \le j \le \lfloor n/4 \rfloor \ \} \bigcup \{ -2^{n-4j-2} \binom{4j+2}{2j+1} \ ; \ 0 \le j \le \lfloor (n-2)/4 \rfloor \ \}.$$

In particular, the largest eigenvalue (in absolute value) is indeed 2^n. We also need the following *shaloshable* binomial coefficients identity.

Proposition 2. *The vector* $(c_0, \dots, c_n)$ *defined by* $c_r = \binom{n}{r}^{-1}$ $(0 \le r \le n)$ *is an eigenvector of the matrix* K_n *corresponding to its largest eigenvalue* 2^n *(with multiplicity* 1*). In other words, for* $0 \le m \le n$

$$\sum_{r=0}^{n} K_n(m,r) c_r = 2^n c_m.$$

[To confirm this *shaloshable* binomial coefficient identity for $n \le N$, type, in the Maple package `HaroldSilentShapiro.txt`, `CheckEV(N);` . For example, `CheckEV(50);` returns `true` in two seconds, and `CheckCP(100);` returns `true` in 30 seconds.]

But an eigenvector is only determined up to a constant multiple. Let's find it (modulo the Small Change hypothesis). We know that

$$CT \ [\ (P_k(z) P_k(z^{-1}))^m (P_k(-z) P_k(-z^{-1}))^{n-m} \ ] \sim \frac{C}{\binom{n}{m}} \cdot (2^n)^k,$$

for *some* constant C. To find it, we use the well-known, and easily proved identity (see [10])

$$P_k(z) P_k(z^{-1}) + P_k(-z) P_k(-z^{-1}) = 2^{k+1}.$$

Raising it to the n-th power, using the binomial theorem, and taking the constant term, we have

$$\sum_{m=0}^{n} \binom{n}{m} CT[(P_k(z)P_k(z^{-1}))^m (P_k(-z)P_k(-z^{-1}))^{n-m}] = 2^{(k+1)n}.$$

Hence

$$\sum_{m=0}^{n} \binom{n}{m} \frac{C}{\binom{n}{m}} \cdot (2^n)^k \quad = 2^{(k+1)n},$$

that implies that

$$C = \frac{2^n}{n+1}.$$

We just established

Proposition 3. *Modulo the Small Change Hypothesis, for* $0 \le m \le n < \infty$

$$CT[(P_k(z)P_k(z^{-1}))^m (P_k(-z)P_k(-z^{-1}))^{n-m}] \sim \frac{2^n}{(n+1)\binom{n}{m}} \cdot (2^n)^k.$$

In particular, taking $m = n$*, we get Saffari's conjecture (for even moments)*

$$M_n(k) = CT[P_k(z)^n P_k(z^{-1})^n] \sim \frac{2^n}{n+1} \cdot (2^n)^k.$$

Towards a Proof of the Small Change Hypothesis

It would have been great if the "children" of each unimportant monomial, in the evolution equation described above (implemented in procedure `GD` in our Maple package), would all be unimportant. Then we could have easily proved, by induction that, asymptotically, they are insignificant compared to the important monomials. It turns out that for *most* unimportant monomials, this is indeed the case, but there are a few, that we call *false pretenders* that do have important children.

It should not be hard to fully characterize these. In fact it turns out (empirically, for now) that for n even there are $(n/2)^2 - 1$ of them, and for n odd there are $(n^2 - 1)/4$. Then for those false pretenders one should be able to describe all their important children, and then prove that the leading terms of their contributions cancel out (using the inductive hypothesis, and Prop. 3).

This has been verified empirically up to $n \le 16$. See procedures `Medio` and `MedioP` in the Maple package `HaroldSilentShapiro.txt`.

Hugh Montgomery's Stronger Conjecture

In [6], Hugh Montgomery considered the more general sequences

$$M_{m,n}(k) := CT\,[\,P_k(z)^m P_k(z^{-1})^n\,].$$

He conjectured that, for $m \neq n$,

$$M_{m,n}(k) = o(\,2^{(m+n)k/2}\,).$$

Once again, the generating function, for each specific m and n, is always a rational function, and our Maple package (procedure `RS(m,n,t,K)`) computes them, and procedure `MamarH(N,K,t)` prints out an article confirming Hugh Montgomery's conjecture, as well as giving the generating functions for $1 \leq m < n \leq N$. (`K` is a parameter that should be made large enough, say 1000).

To see the output for $1 \leq m < n \leq 7$, go to: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oHaroldSilentShapiro2.txt`, that contains the explicit expressions for all these cases, and confirms Montgomery's conjecture with a vengeance. Unlike the $m = n$ case, the smallest root (alias the reciprocal of the largest eigenvalue) is not "nice", and there are usually several roots with smallest absolute value, hence the sequences often oscillate. Nevertheless, Montgomery's conjecture is true for all $1 \leq m < n \leq 7$, and one could go much further.

Let's Generalize!

The same approach works for *any* sequence of Laurent polynomials defined by a recurrence of the form

$$P_k(z) = C_1(z)P_{k-1}(z^r) + C_2(z)P_{k-1}(-z^r) + C_3(z)P_{k-1}(z^{-r}) + C_4(z)P_{k-1}(-z^{-r}),$$

with the initial condition $P_0(z) = 1$, where $C_1(z)$, $C_2(z)$, $C_3(z)$, $C_4(z)$ are Laurent polynomials of degree less than r and low-degree larger than $-r$, for *any* positive integer r larger than 1.

One always gets a finite scheme (disclaimer: we don't have a rigorous proof, but we believe that such a proof exists, at any rate, it is true in all the cases that we tried out) and hence a rational generating function for the sequence

$$S[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4](k) := CT\left[z^{\alpha_0} P_k(z)^{\alpha_1} P_k(z^{-1})^{\alpha_2} P_k(-z)^{\alpha_3} P_k(-z^{-1})^{\alpha_4}\right],$$

for any non-negative $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$. This is implemented in the Maple package `ShapiroGeneral.txt` also available from the webpage of this article, or directly from `http://www.math.rutgers.edu/~zeilberg/tokhniot/ShapiroGeneral.txt`.

Let's (not!) Generalize Even More!

The set $\{1, -1\}$ is a multiplicative subgroup of the field of complex numbers. For any finite multiplicative subgroup G of the field of complex numbers, and any positive integer r larger than 1, the same approach should be able to handle sequences of polynomials given by a recurrence

$$P_k(z) = \sum_{g \in G} \alpha_g(z) P_{k-1}(gz^r) + \sum_{g \in G} \beta_g(z) P_{k-1}(gz^{-r}), \quad P_0(z) = 1,$$

where $\alpha_g(z)$, $\beta_g(z)$ are $2|G|$ given Laurent polynomials in z of degree $< r$ and low-degree $> -r$.

This includes the case treated in [2], where G is a cyclotomic group.

We could go even further, with *higher order* recurrences (as opposed to only first order), several continuous variables (as opposed to only z), and, presumably, even several discrete variables (as opposed to only k), but *enough is enough!*

Added May 27, 2016: Brad Rodgers, independently, and simultaneously, found a (complete) proof of Saffari's conjecture, that he is writing up now and will soon post in the arxiv. Meanwhile, you can read his proof in a letter posted out in `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/BradleyRodgersLetter.pdf`.

Added June 7, 2016: Brad Rodgers' beautiful paper, that also proves the more general Montgomery conjecture, mentioned above, is now available here: `http://arxiv.org/abs/1606.01637`.

References

1. M. Apagodu, D. Zeilberger, Multi-variable Zeilberger and Almkvist–Zeilberger algorithms and the sharpening of Wilf–Zeilberger theory. Adv. Appl. Math. **37**, 139–152 (2006), http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multiZ.html
2. C. Doche, Even moments of generalized Rudin–Shapiro polynomials. Math. Comput. **74**, 1923–1935 (2005), http://www.ams.org/mcom/2005-74-252/S0025-5718-05-01736-9/S0025-5718-05-01736-9.pdf. Accessed 6 May 2016
3. C. Doche, L. Habsieger, Moments of the Rudin–Shapiro polynomials. J. Fourier Anal. Appl. **10**, 497–505 (2004), http://web.science.mq.edu.au/~doche/049.pdf. Accessed 6 May 2016
4. C. Koutschan, M. Kauers, D. Zeilberger, A proof of George Andrews' and David Robbins' q-TSPP conjecture. Proc. Nat. Acad. Sci. **108**(6), 2196–2199 (2011), http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/qtsppRig.html
5. J.E. Littlewood, *Some Problems in Real and Complex Analysis* (D.C. Heath and Co. Raytheon Education Co, Lexington, Mass, 1968)

6. H. Montgomery, Problem, in *Problem Session*, p. 3029, *Mathematisches Forschungsinstitut Oberwolfach, Report No. 51/2013*, https://doi.org/10.4171/OWR/2013/51; Analytic Number Theory, 20 October–26 October 2013, pp. 3029–3030, https://www.mfo.de/document/1343/OWR_2013_51.pdf. Accessed 6 May 2016
7. W. Rudin, Some theorems on fourier coefficients. Proc. Am. Math. Soc. **10**, 855–859 (1959)
8. H.S. Shapiro, Extremal problems for polynomial and power series, M.Sc. thesis, MIT 1951, http://dspace.mit.edu/bitstream/handle/1721.1/12198/30502786-MIT.pdf?sequence=2. Accessed 11 May 2016
9. H.S. Shapiro, Extremal problems for polynomial and power series, Ph. D. thesis, MIT 1952, http://dspace.mit.edu/bitstream/handle/1721.1/12247/30752144-MIT.pdf?sequence=2. Accessed 11 May 2016
10. The Wikipedia Foundation, Shapiro Polynomials. https://en.wikipedia.org/wiki/Shapiro_polynomials. Accessed 13 May 2016
11. H.S. Wilf, D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. **108**, 575–633 (1992), http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multiwz.html
12. D. Zeilberger, The Holonomic ansatz II: automatic discovery(!) and proof(!!) of holonomic determinant evaluations. Ann. Comb. **11**, 241–247 (2007), http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ansatzII.html

From Ramanujan to Groups of Rationals: A Personal History of Abstract Multiplicative Functions

P. D. T. A. Elliott

In celebration of the sixtieth birthday of Krishnaswami Alladi

Keywords Multiplicative function · Mean value · Automorphic form

2010 Mathematics Subject Classification 11N37 · 11F03 · 11F30 · 11K65 11L99 · 11M99 · 11N60 · 11N64

Introduction

The following furnishes details to a lecture that I gave at the March 17–21, 2016, meeting held in Gainsville, Florida, to celebrate the sixtieth birthday of Krishnaswami Alladi.

A complex-valued function defined on the positive integers is *arithmetic*. An arithmetic function is *additive* if on mutually prime integers a, b it satisfies $f(ab) = f(a) + f(b)$, *multiplicative* if under the same circumstances it satisfies $g(ab) = g(a)g(b)$. It is *completely additive* respectively *completely multiplicative* if the coprimality condition may be omitted.

I begin by considering multiplicative functions in as wide a generality as possible and end by viewing them as characters on the multiplicative group of positive rationals.

P. D. T. A. Elliott (✉)
Department of Mathematics, University of Colorado Boulder,
Boulder, CO 80309-0395, USA
e-mail: pdtae@euclid.colorado.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_16

Starting from two results of Ramanujan, one established with Hardy and one without, there are two passes over the twentieth century. The first pass largely concentrates on the application of abstract multiplicative functions to the study of Fourier coefficients of automorphic forms. The second pass is a commentary on the first pass, carried out within the aesthetic of Probabilistic Number Theory.

FIRST PASS

1

1916. In section 16 of an extensive paper 'On Certain Arithmetical Functions' [63] Ramanujan introduces the function τ defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\left(\prod_{j=1}^{\infty}(1-x^j)\right)^{24}$$

and derives an identity for the number of ways a positive integer n may be written as the sum of 24 squares:

$$r_{24}(n) = \tfrac{16}{691}\sum_{d|n} d^{11} + \tfrac{128}{691}\cdot 259\tau(n)$$

if n is odd,

$$r_{24}(n) = \tfrac{16}{691}\sum_{d|n}(-1)^d d^{11} - \tfrac{128}{691}\left(259\tau(n) + 512\tau(\tfrac{1}{2}n)\right)$$

if n is even.

Having shown that $\tau(n)$ does not exceed a constant multiple of n^7 in size, in §18 of that same paper Ramanujan introduces two conjectures connecting the behaviour of τ on the integers to its behaviour on the primes: that

$$\sum_{n=1}^{\infty} \tau(n)n^{-t} = \prod_p \left(1 - \tau(p)p^{-t} + p^{11-2t}\right)^{-1},$$

and that $\left(\frac{1}{2}\tau(p)\right)^2 \le p^{11}$.

That $\tau(n)$ should satisfy the first of these conjectures and, in particular, that it should be multiplicative, comes out of the blue.

There is a further conjecture concerning $\tau(n)$ in §25 of Ramanujan's paper.

2

The following year Mordell [61], anticipating the notion of a Hecke operator and introducing the analytic theory of complex variables, established the validity of Ramanujan's first conjecture.

Replacing x in the definition of τ by $e^{2\pi iz}$ with z in the complex upper half plane H:

$$e^{2\pi iz}\left(\prod_{j=1}^{\infty}(1-e^{2\pi ijz})\right)^{24}=\sum_{n=1}^{\infty}\tau(n)e^{2\pi inz},$$

the left hand side becomes an holomorphic cusp form of weight 12 for the action of $SL(2,\mathbb{Z})$ on H, the right hand side its Fourier expansion at infinity.

The collection of such cusp forms may be viewed as a linear space over the complex numbers. From this distance we might say that Mordell constructs a second cusp form in the space to deliver what Ramanujan requires and, since the space has dimension 1, an identity is secured.

Mordell argued with a ratio of forms, an extension of Liouville's theorem and as was his wont, pioneered.

3

1939. We move to a paper of Rankin on the function τ and similar arithmetical functions, the second of three [66], [67], [68] and in which he establishes the analytic continuation of $2(4\pi)^{-s}\Gamma(s)\Gamma(s-11)\zeta(2s-22)\sum_{n=1}^{\infty}\tau(n)^2n^{-s}$ to the whole complex plane save for simple poles at $s=11$ and $s=12$.

A key step in Rankin's argument is the reduction of a representation for that function by an integral over the product of $\mathbb{R}/\mathbb{Z}$ and the positive reals to a representation by an integral over a fundamental domain for the action of $SL(2,\mathbb{Z})$ upon H, i.e. over the coset space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

It followed from a theorem of Landau that for a positive constant A,

$$\sum_{n\le x}\tau(n)^2=Ax^{12}+O(x^{12-2/5}),\qquad x\ge 2,$$

hence $\tau(n)\ll n^{6-1/5}$.

Present remarks apart, accounts of the foregoing results with additional references may all be found in Hardy's 1940 Cambridge volume *Ramanujan. Twelve lectures on Subjects suggested by his Life and Work*, [45]. It is interesting to read in Chapter IX §9.4 of that account Hardy's impressed appraisal of Ramanujan's 1916 paper and its companion paper on trigonometrical sums [64], 1918.

4

1943. Lehmer, [59], asks whether $\tau(n)$ for positive integers n ever vanishes? That it does not is now known as Lehmer's conjecture.

5

1974. As a consequence of proving the Riemann hypothesis for zeta functions of algebraic varieties over finite fields, Deligne, [6], settles in the affirmative the second of Ramanujan's conjectures and its Petersson extension to holomorphic new forms.

It has taken fifty years and an elaborate development in algebraic geometry to show that in Ramanujan's representation

$$n^{-11/2}\tau(n) = \prod_{p^\alpha \| n} \frac{\sin((1+\alpha)\theta_p)}{\sin\theta_p}, \qquad 2\cos\theta_p = p^{-11/2}\tau(p),$$

the otherwise complex θ_p may be chosen real, with Ramanujan's subsequent conclusions valid, [65] pp. 153–154.

This allows us to better appreciate the corresponding example of the Sato-Tate conjecture: that

$$\pi(x)^{-1} \sum_{\substack{p\le x \\ \alpha<\theta_p\le\beta}} 1 \to \frac{2}{\pi}\int_\alpha^\beta (\sin\phi)^2\, d\phi, \qquad x\to\infty,$$

for each pair α, β, $0 \le \alpha \le \beta \le \pi$.

A version of the Sato-Tate conjecture attached to a study of elliptic curves, with its connection to conjectural analytic properties of symmetric power L-series as considered by Tate [78], 1965, may be found in an example of Serre [73], pp. I-25 to I-26.

6

In 1980, whilst visiting Imperial College, London on a Guggenheim and studying the distribution of differences $f(an+b) - f(An+B)$ of an additive function, $a > 0$, $A > 0$, $aB - Ab \ne 0$, a topic in the Probabilistic Theory of Numbers to which I shall return, I asked myself what might be the minimal requirement of an abstract multiplicative arithmetic function in order that one might improve the bound on $\sum_{n\le x} |\tau(n)|n^{-11/2}$, that follows from the work of Hardy and Rankin, from $O(x)$ to $o(x)$, as $x\to\infty$?

It seemed unlikely that an analytic continuation of $\sum_{n=1}^{\infty} |\tau(n)| n^{-11/2-s}$ was in the offing, and anyway there would likely be an essential singularity at $s = 1$. Importing argument from Probabilistic Number Theory I could establish

Theorem 1. *If g is a non-negative real-valued multiplicative function for which*

$$A_1 = \lim_{x\to\infty} x^{-1} \sum_{n\le x} g(n)$$

exists, and $0 < \delta < 1$, *then*

$$A_\delta = \lim_{x\to\infty} x^{-1} \sum_{n\le x} g(n)^\delta$$

exists. Moreover, every A_δ with $0 < \delta < 1$ *is zero unless the series*

$$\sum_p p^{-1} \left(g(p)^{1/2} - 1\right)^2$$

converges.

In fact the final assertion of this theorem holds with $g(p)^{1/2}$ replaced by $g(p)^\theta$ for each fixed θ, $0 < \theta < 1$.

Since with $g(p) = \tau(p)^2 p^{-11}$ the final assertion of Theorem 1 would not be consistent with the Sato-Tate conjecture, I conjectured that every A_δ with $0 < \delta < 1$ be zero.

A proof of Theorem 1 together with this conjecture and the conjecture that τ satisfies an analogue of the Central Limit Theorem in the Theory of Probability:

$$[x]^{-1} \sum_{\substack{n\le x \\ |\tau(n)|n^{-11/2} \le (\log x)^{-1/2} \exp(z\mu(\log\log x)^{1/2})}} 1 \to (2\pi)^{-1/2} \int_{-\infty}^{z} e^{-u^2/2}\, du, \qquad x \to \infty,$$

for a certain positive constant μ and all real z, may be found in Elliott, [20]. At the time I had only an integral representation for the constant μ, which I later evaluated to be $(\frac{\pi^2}{12} + \frac{1}{2})^{1/2}$, c.f. Elliott, [33].

7

Four years later, in a triple paper Moreno, Shahidi and I, [39] established

Theorem 2. *For each δ,* $0 < \delta < 1$,

$$w(\delta) = \inf_{-1\le y\le 3} y^{-2}(1 + \delta y - (1 + y)^\delta)$$

is positive. In particular, $w(1/2) = 1/18$.

Moreover,

$$\sum_{n\le x}(|\tau(n)|n^{-11/2})^{2\delta} \ll x(\log x)^{-w(\delta)}, \qquad x \ge 2,$$

the implied constant depending upon δ.

The argument employs the analytic continuation of $\sum_{n=1}^{\infty}\tau(n)^4 n^{-22-s}$ over a double pole at $s = 1$. To effect the explicit upper bound the function $y \to |y|$ on a suitable real interval is approximated by a quadratic polynomial in y.

8

This result was generalised to a class of holomorphic cusp forms by Rankin, c.f. [69] 1983, §6, p. 235, [70] 1985, [71] 1986.

Note that in many applications of Lemma 1 in the second of these papers it is essential to check that $H_2(s)$ does not vanish at $s = 1$.

Note also that at the time the Euler products of the L-functions considered by Rankin in the third of these papers and employed in all results similar to Theorem 2 were restricted to non-archimedean non-ramified factors and, when suitably renormalised, analytic in an open set containing the half-plane $\mathrm{Re}\,(s) \ge 1$, but otherwise known only to be meromorphic.

Moreover, their corresponding representations of the group $GL_2(A_{\mathbb{Q}})$ of invertible 2×2 matrices with rational adèle entries were to have trivial central character, c.f. Shahidi [76] 1994, in particular §7, a constraint only removed by Shahidi in that same survey paper.

In what follows an *holomorphic form* attached to the action of a congruence group $\Gamma_0(N)$ upon the complex upper half-plane, of weight at least two, arbitrary level and character, and eigenfunction of the appropriate Hecke operators, will be deemed *classical*.

9

1981. Combining an ℓ-adic representation of the Galois group of the algebraic closure of the rationals, due to Deligne, with the Chebotarev density theorem, Serre, [74] §7.4, establishes that the integers on which the Fourier coefficients of a classical non-CM holomorphic form do not vanish have positive asymptotic density. Indeed, for any $c < 3/2$ the Fourier coefficients can vanish on at most $O(x(\log x)^{-c})$ primes not exceeding x.

Note that for cusp forms attached to elliptic curves over the rationals a result of Elkies, [8] 1987, guarantees the density to be less than 1.

The Fourier coefficients of the corresponding forms with complex multiplication vanish on the inertial primes of a quadratic extension of the rationals and do not vanish on $(1 + o(1))Ax(\log x)^{-1/2}$, $A > 0$, of the integers not exceeding x, Serre loc. cit. §7.5.

There is apparently no current analogue of these results for Maass forms.

10

2011. (*Holomorphic Sato-Tate*). After Deligne, the Fourier coefficients a_n of a classical new cusp form of weight k, nebentypus (character) $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$, have a representation $a_p = 2p^{(k-1)/2}\rho \cos \gamma_p$, $0 \le \gamma_p \le \pi$, with ρ^2 a fixed value of ψ.

In Barnet-Lamb, Geraghty, Harris, Taylor, [2], the authors show that, for an arbitrary Dirichlet character χ (mod M), each corresponding symmetric product function $L(\text{sym}^m \times \chi, s)$, $m = 1, 2, \ldots$, defined in a right-hand half-plane by an Euler product restricted to primes that do not divide MN, has a meromorphic continuation to the whole complex s-plane, analytic in an open set containing the half-plane $\text{Re}\,(s) \ge 1 + m(k-1)/2$.

As a consequence,

$$\pi_\rho(x)^{-1} \sum_{\substack{p \le x, \psi(p) = \rho^2 \\ \alpha < \gamma_p \le \beta}} 1 \qquad \to \frac{2}{\pi} \int_\alpha^\beta (\sin\theta)^2 \, d\theta, \qquad x \to \infty,$$

where $0 \le \alpha \le \beta \le \pi$ and $\pi_\rho(x)$ denotes the number of primes p, not exceeding x, for which $\psi(p) = \rho^2$.

In particular, with $k = 12$, $N = 1$, ψ trivial, $\rho = 1$, this holds for Ramanujan's function $\tau(n)$.

11

Conjectures of Sato-Tate type concern the distribution of Hecke (operator) eigenvalues on the primes. Conjectures of Central Limit type, as in the author's Australian Mathematical Society paper [20], concern their distribution on the integers.

Classical cusp forms of holomorphic or Maass type with trivial characters may be realised by elements of the Hilbert space $L^2(A_\mathbb{Q}^\times GL_2(\mathbb{Q})\backslash GL_2(A_\mathbb{Q}))$ attached to a cuspidal representation of the group $GL_2(A_\mathbb{Q})$. It is then natural to consider the corresponding Fourier or Fourier–Whittaker coefficients in mean square. As an example:

Theorem 3. *For each real w,*

$$(\alpha x)^{-1} \sum_{\substack{n \le x \\ |\tau(n)|n^{-11/2} \le \exp(A(x) + \frac{w}{2}((\frac{\pi^2}{3} - \frac{5}{2}) \log\log x)^{1/2})}} \tau(n)^2 n^{-11} \to (2\pi)^{-1/2} \int_{-\infty}^{w} e^{-u^2/2}\, du, \qquad x \to \infty$$

with

$$A(x) = \sum_{p \le x} \tau(p)^2 p^{-12} \log(|\tau(p)| p^{-11/2}) = \left(\tfrac{1}{4} + o(1)\right) \log\log x,$$

the sum over primes for which τ *does not vanish, and where*

$$\alpha = \lim_{x \to \infty} x^{-1} \sum_{n \le x} \tau(n)^2 n^{-11}.$$

The Fourier coefficients of cusp forms that are simultaneous eigenfunctions of the appropriate Hecke operators are multiplicative, and the following further theorem for abstract multiplicative functions, Elliott [30] 2012, may be applied.

2012. (*Abstract Central Limit Theorem for Eigenforms*).

Theorem 4. *Let g be a real, non-negative multiplicative arithmetic function that satisfies*

$$\sum_{n \le x} g(n) = Bx + O(x^{1-\delta}) \tag{1}$$

for positive constants B, δ *and all* $x \ge 1$. *Furthermore, let*

$$\sum_{x^{1/2} < p \le x} p^{-k} g(p^k) \to 0, \qquad x \to \infty, \tag{2}$$

for each integer k, $2 \le k < \delta^{-1}$, *with summation over primes, p.*

Define

$$E(x) = \sum_{q \le x} q^{-1} g(q) \log g(q),$$

$$F(x) = \left(\sum_{q \le x} q^{-1} g(q) (\log g(q))^2 \right)^{1/2} \ge 0,$$

the summation over prime-powers, q, for which $g(q)$ *is positive.*

Assume that $F(x) \to \infty$ *and that for each positive y,* $F(x^y)/F(x) \to 1$ *as* $x \to \infty$.

Then the frequencies

$$\left(\sum_{m\le x} g(m)\right)^{-1} \sum_{\substack{n\le x\\ g(n)\le \exp(E(x)+zF(x))}} g(n), \qquad z\in\mathbb{R}, \tag{3}$$

approach a limit law as $x\to\infty$ *if and only if the frequencies*

$$F(x)^{-2}\sum_{0<g(q)\le \exp(uF(x))} q^{-1}g(q)(\log g(q))^2$$

converge weakly to a function $K(u)$ *on the real line. The characteristic function of the limit law is then*

$$\exp\left(\int_{-\infty}^{\infty}(e^{itu}-1-itu)u^{-2}\,dK(u)\right), \qquad t\in\mathbb{R},$$

where the integrand is $-t^2/2$ *at* $u=0$.

Assuming only (1) *and* (2) *to hold, a sufficient condition for the frequencies* (3) *to approach the normal law with mean zero and variance 1 is that* $F(x)\to\infty$ *and for each* $\varepsilon>0$, *the normalized sums*

$$F(x)^{-2}{\sum_{q\le x}}' q^{-1}g(q)(\log g(q))^2, \tag{4}$$

taken over those prime-powers q *for which* $0<g(q)\le\exp(-\varepsilon F(x))$ *or* $g(q)>\exp(\varepsilon F(x))$ *holds, converge to zero as* $x\to\infty$.

If

$$\sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \qquad a_1=1,$$

is the Fourier expansion at infinity of a (classical) holomorphic new cusp form of weight $k\ge 2$, setting $g(n)=|a_n|^2 n^{-(k-1)}$ and appealing to the analytic continuation of corresponding second and fourth symmetric product restricted renormalised L-functions to a neighbourhood of $s=1$, c.f. Shahidi [76], Barnet-Lamb et al., loc. cit., we may obtain the analogue of Theorem 3 with summation condition

$$|a_n|n^{-(k-1)/2}\le \exp(A(x)+(w/2)B(x)),$$

where

$$A(x)=\sum_{p\le x}|a_p|^2p^{-k}\log(|a_p|p^{-(k-1)/2}),$$

$$B(x)=\left(\sum_{p\le x}|a_p|^2p^{-k}\left(\log(|a_p|p^{-(k-1)/2})\right)^2\right)^{1/2}\ge 0,$$

the summation over primes. This holds whether the cusp form is of complex multiplication type or not. However, if it is not then the holomorphic Sato-Tate conjecture, now a theorem, allows the replacement of each function $B(x)$ by $((\pi^2/3 - 5/2)\log\log x)^{1/2}$, and an estimate $A(x) = (1/4 + o(1))\log\log x$, $x \to \infty$.

Moreover, as an analogue for Maass forms, let

$$f = \sum_{n \neq 0} a_n (2\pi y)^{1/2} K_w(2\pi |n| y) e^{2\pi i n x}, \qquad a_1 = 1,$$

be a non-zero Maass form for the action of $SL(2, \mathbb{Z})$ upon the complex upper half-plane, i.e. a solution to the Laplace equation

$$-y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = \left(\frac{1}{4} - w^2 \right) f$$

that belongs to the Hilbert space $L^2(SL(2, \mathbb{Z}) \backslash H)$ with the Petersson inner product, eigenfunction of the appropriate Hecke operators. Then

$$(\gamma x)^{-1} \sum_{\substack{n \leq x \\ |a_n| \leq \exp(A(x) + \lambda B(x))}} |a_n|^2 \qquad \to (2\pi)^{-1/2} \int_{-\infty}^{\lambda} e^{-u^2/2} \, du, \qquad x \to \infty,$$

where

$$\gamma = \lim_{x \to \infty} x^{-1} \sum_{n \leq x} |a_n|^2$$

and $A(x)$, $B(x)$ are defined as for holomorphic forms with k replaced by 1.

In this case, absent a valid analogue of either the Ramanujan-Petersson or Sato-Tate conjecture, adequate control of exceptionally large Fourier coefficients may be derived from the partial analytic continuation of a corresponding sixth symmetric square L-function, this last a result of Kim and Shahidi [56], 2000/2002.

As an example in the control of large values of Fourier–Whittaker coefficients in the lack of an adequate universal upper bound, I sketch an argument for an analogue of Theorem 2 for the coefficients a_n of a Maass form of the above type. For ease of presentation I consider only the case $\delta = 1/2$.

Since $(1 - y^2)^2 = (1 + (y^2 - 1)/2 - y)2(1 + y)^2$,

$$\rho_w = \inf_{0 \leq y \leq w} (1 - y^2)^{-1} (1 + (y^2 - 1)/2 - y) = (1 + w)^{-2}/2.$$

Setting $y = |a_p|$ and dividing the range of primes according to whether $|a_p| \leq w$ or not, we see that

$$\sum_{p \leq x} p^{-1} |a_p| \leq \sum_{p \leq x} p^{-1} (1 + (a_p^2 - 1)/2 - \rho_w (a_p^2 - 1)^2) + \sum_{p \leq x, |a_p| > w} p^{-1} \, |a_p|.$$

For $w > 0$, a typical summand of the second bounding sum does not exceed $w^{-3}a_p^4$. From the analytic properties of the corresponding second and fourth symmetric product L-functions alone, the upper bound is at most $(1 - \rho_w + 2w^{-3} + o(1)) \log\log x$ as $x \to \infty$, and for w sufficiently large, e.g. $w = 5$, $2w^{-3} - (1 + w)^{-2}/2$ is negative.

We may now appeal to a standard bound for multiplicative functions, c.f. Elliott [26], Lemma 2.2, together with Elliott [30], Lemma 6.

A self-standing version of Theorem 3 with appeal to symmetric product L-function results of Shahidi [75], Moreno and Shahidi [62], without appeal to the holomorphic Sato-Tate conjecture, is given in the author's paper [31], 2012.

A complete proof of the abstract central limit theorem for eigenforms is in Elliott [30], 2012.

A detailed account of central limit theorems for the Fourier coefficients of classical cusp forms, including the foregoing results and with attention to variants impelled by the vanishing of coefficients, may be found in the author's paper [33], 2015. Note that the occasional appropriate renormalisation, as in the penultimate line of the statement of Theorem 1, is inadvertently omitted.

I note that the author's 1981 conjecture concerning the unweighted frequency distribution of Ramanujan's tau function, with appropriate modification if Lehmer's conjecture fails, would be accessible to a quantitative version of the Barnet-Lamb et al. Sato-Tate theorem whose asymptotic estimate for the value distribution of $\tau(n)$ on the primes in the interval $[1, x]$ held within an error of $o(x(\log x)^{-3})$ as $x \to \infty$.

12

(*Sign changes in Fourier coefficients*). The Fourier coefficients, of a classical non CM holomorphic new cusp form of level N and nebentypus $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ are real on the integers prime to N if and only if ψ is principal. It is natural to ask whether the non-zero coefficients are then uniformly distributed between the positive and negative. Indeed, they are.

More generally, we may consider the same question of Maass cusp form coefficients. Once again the non-availability of the Ramanujan-Petersson conjecture or a Sato-Tate conjecture is a constraint. Moreover, the analogue of Serre's 1981 result on the non-vanishing of Fourier coefficients is also not available. To offset this we appeal to deeper results in the abstract theory of multiplicative functions.

As an example, consider the Fourier–Whittaker coefficients a_n of a Maass cusp form for the action of $SL(2, \mathbb{Z})$ on H, as in section 11 on Central Limit Theorems.

Theorem 5. *Let $S(x)$ denote the number of integers n, not exceeding x, for which a_n does not vanish. Then $S(x) \gg x(\log x)^{-1/2}$ and there are positive constants a, b, c for which*

$$a \le S(x)\left(\frac{x}{\log x}\prod_{\substack{p\le x\\ a_p\ne 0}}\left(1+\frac{1}{p}\right)\right)^{-1} \le b,$$

$$S(x)^{-1}\sum_{\substack{n\le x\\ a_n<0}} 1 = 1/2 + O((\log x)^{-c}), \qquad x\ge 2.$$

A proof of Theorem 5 with $24000c = 1$, which also accomplishes the analogous result for holomorphic cusp form coefficients, is given in Elliott and Kish, [38].

An asymptotic estimate for $S(x)$ is currently unavailable. The upper and lower bounds on $S(x)$, comparable in size, are sensitive to the vanishing of coefficients. The demand that their form places upon the abstract theory of multiplicative functions during the proof of Theorem 5 is met with the following result, established in the same paper, that estimates the mean-value of a multiplicative function that may have a seriously reduced support on the primes.

Theorem 6. *Let $3/2 \le Y \le x$. Let g be a multiplicative function that for positive constants β, c, c_1 satisfies $|g(p)| \le \beta$,*

$$\sum_{w<p\le x}(|g(p)|-c)p^{-1} \ge -c_1, \qquad Y\le w\le x,$$

on the primes. Suppose, further, that the series

$$\sum_q |g(q)|q^{-1}(\log q)^{\gamma}, \qquad \gamma = 1 + c\beta(c+\beta)^{-1},$$

taken over the prime-powers $q = p^k$ with $k\ge 2$, converges.

Then with

$$\lambda = \min_{|t|\le T}\sum_{Y<p\le x}(|g(p)| - \operatorname{Re} g(p)p^{it})p^{-1},$$

$$\sum_{n\le x} g(n) \ll x(\log x)^{-1}\prod_{p\le x}(1+|g(p)|p^{-1})(\exp(-\lambda c(c+\beta)^{-1}) + T^{-1/2})$$

uniformly for Y, x, $T>0$, the implied constant depending at most upon β, c, c_1 and a bound for the sum of the series over higher prime-powers.

It is shown in Elliott [9], Theorem 6, that raising to the power $c/(3c+1)$ the final factor in the upper bound for the mean value of g on the integers allows the factor $(\log q)^{\gamma}$ in the second boundary constraint to be removed.

Uniform sign changes in the coefficients of second and third order symmetric product L-functions attached to Maass cusp forms may also be found in Elliott [9], where it is demonstrated that in such investigations appeal to bounds on Satake parameters and application of the Wiener-Ikehara theorem may be replaced by appeal to results of Heilbronn and Landau [47], [48], that have considerably less demanding hypotheses.

The abstract theory of multiplicative functions is now sufficiently developed that in applications to questions involving a uniformity or the distribution of values of a particular multiplicative function, g, requirements are often effectively reduced to that of a weak lower bound on sums $\sum_{p\le x}|g(p)|p^{-1}\log p$ or, even less, upon sums $\sum_{p\le x}|g(p)|p^{-1}$. As a consequence, if the study involves L-functions, appeal to analytic continuation over a critical line is reduced to analytic continuation into a neighbourhood of a critical point. In some sense, the background requirement becomes the continuity of a parametrised structure at a single critical point.

13

As a final example in this first pass exhibiting the efficacy of abstract multiplicative functions, I consider their distribution over residue classes to varying moduli, a topic that, when applied to the Möbius function, directly concerns the uniform distribution of rational primes.

Theorem 7. *Let a_n be the Fourier coefficient function of a Maass cusp form attached to the action of $SL(2,\mathbb{Z})$ on the complex upper half-plane, eigenfunction of the appropriate Hecke operators. Then for each pair of mutually prime positive integers b, D,*

$$\sum_{\substack{n\le x, a_n\ne 0\\ n\equiv b \ (\mathrm{mod}\ D)}} 1 = \frac{1}{\phi(D)}\left(1+O\left(\left(\frac{\log D}{\log x}\right)^{1/49}\right)\right)\sum_{\substack{n\le x, a_n\ne 0\\ (n,D)=1}} 1$$

uniformly for $(b, D) = 1$, $1 \le D \le x$.

Theorem 7 may be found as Example 4 in Elliott and Kish, [37].

A comprehensive study of multiplicative functions and their Dirichlet character braided sums, with emphasis on uniformities and assumption only that the functions take values in the complex unit disc, is carried out in Elliott and Kish, [37].

With standard exceptions, all the foregoing results that hold for Maass cusp forms attached to the action of $SL(2,\mathbb{Z})$ upon H, also hold for Maass cusp forms attached to the action of $\Gamma_0(N)$ upon H.

SECOND PASS

14

In 1916, Ramanujan formulated two conjectures that immediately catalysed developments in the theory of modular functions.

Within a year, Ramanujan publishes with Hardy a paper, [46], that will catalyse a new discipline: Probabilistic Number Theory.

1917. Hardy and Ramanujan define an arithmetical function f to have a normal order $\phi(n)$ if ϕ is elementary and increasing and, for each positive ε, the integers n for which $|f(n) - \phi(n)| > \varepsilon\phi(n)$ have asymptotic density zero.

Hardy and Ramanujan prove by induction that the interval $[1, x]$, $x \geq 2$, contains at most

$$\frac{c_0 x}{\log x} \frac{(\log\log x + c_1)^{k-1}}{(k-1)!}, \qquad c_0 > 0, c_1 > 0,$$

integers with exactly k distinct prime divisors, $k = 1, 2, \ldots$. As a consequence, for any unbounded non-decreasing function $\psi(n)$, $n = 1, 2, \ldots$, the function $\omega(n)$, that counts the number of distinct prime divisors of n, satisfies $|\omega(n) - \log\log n| \leq \psi(n)(\log\log n)^{1/2}$ save possibly on a set of integers of density zero. In particular, $\omega(n)$ has normal order $\log\log n$.

They prove like results for the function $\Omega(n)$ that counts the prime divisors of n with multiplicity and ask, c.f. [46] §V, whether similar results hold for other standard arithmetical functions, such as the divisor function, $d(n)$?

The results of Hardy and Ramanujan in this paper are redolent of a Poisson distribution with an attendant weak law of large numbers – but not least among Hardy's attributes is his partisanship for mathematical rigour, and the Kolmogorov axioms for a probability space will not appear until 1933.

For seventeen years nothing happens. Then in

15

1934. Turán [80], gives, in Hardy's own words, [45] §3.7, a 'very simple and elegant' proof for the Hardy and Ramanujan quantitative version of the normalcy of $\omega(n)$.

Moreover, in 1936 [81], Turán employs the same argument to show that for any additive function $\sum_{p|n} f(p)$, with the $f(p)$ non-negative and uniformly bounded, $A(N) = \sum_{p \leq N} p^{-1} f(p)$ unbounded, the inequality

$$\sum_{n \leq N} (f(n) - A(N))^2 \ll N A(N)^2$$

holds, so that $f(n)$ has normal order $A(N)$.

The argument: expand the square and cancel the leading terms of the resulting asymptotic estimates, is becoming abstract. In form it reminds of Chebyshev's inequality, save that at that time, as Erdős told me, neither he nor Turán knew anything of Probability.

16

1939. Erdős and Kac [41] prove that a real-valued additive function $\sum_{p|n} f(p)$, $|f(p)| \le 1$, with $A(N) = \sum_{p\le N} p^{-1} f(p)$, $B(N) = (\sum_{p\le N} p^{-1}|f(p)|^2)^{1/2}$ unbounded, satisfies

$$\frac{1}{N}\sum_{\substack{n\le N\\ f(n)-A(N)\le\lambda B(N)}} 1 \quad \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}\,du, \qquad N\to\infty, \lambda\in\mathbb{R}.$$

In particular, the frequency of integers in the interval $[1, N]$ for which $\omega(n) - \log\log N \le \lambda(\log\log N)^{1/2}$ converges weakly to the standard normal distribution, mean zero, variance 1.

The tour-de-force argument, apparently written up by Kac, combines Kac' notion that divisibility of an integer by distinct primes represents independent events, with Erdős' expertise in the method of Brun's sieve.

17

The Laplace operator $-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ has a self-adjoint extension to the space $L^2(SL(2,\mathbb{Z})\backslash H)$ with Petersson inner-product $\langle\ ,\ \rangle$, hence a spectral resolution. An explicit realisation was established by Selberg, c.f. [72]

Let η_j, $j = 1, 2, \dots$, be an orthonormal basis of Maass cusp forms, eigenfunctions of the appropriate Hecke operators, η_0 the constant function $(3/\pi)^{1/2}$. Then for any f in $L^2(SL(2,\mathbb{Z})\backslash H)$

$$f(z) = \sum_{j=0}^{\infty}\langle f, \eta_j\rangle\eta_j(z) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle f, E(*, s)\rangle E(z, s)\,ds,$$

where $E(z, s)$ is the Eisenstein series function

$$\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)=1}}\frac{y^s}{|cz+d|^{2s}}, \qquad \mathrm{Re}\,(s) > 1, z\in H,$$

with a Fourier (-Whittaker) expansion

$$y^s + y^{1-s}\phi(s) + \frac{2\pi^s y^{1/2}}{\Gamma(s)\zeta(2s)}\sum_{n\ne 0}\sigma_{1-2s}(n)|n|^{s-1/2}K_{s-1/2}(2\pi|n|y)e^{2\pi i n x},$$

$$\phi(s) = \frac{\pi^{1/2}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}, \qquad \sigma_w(n) = \sum_{d|n} d^w.$$

The arithmetic coefficient $\sigma_{1-2s}(n)|n|^{s-1/2}$ coincides with the divisor function $d(n)$ when $s = 1/2$. Since $E(1/2, z) = 0$ identically in z, $d(n)$ effectively appears in the Fourier expansion of $dE(s, z)/ds$, at $s = 1/2$.

From this point of view, the Erdős-Kac theorem is an extremal version of the central limit theorem for cuspidular eigenforms, corresponding to the continuous rather than the point spectrum of the Laplacian.

The inequality $2^{\omega(n)} \le d(n) \le 2^{\Omega(n)}$, that already appears in the 1916 paper of Hardy and Ramanujan, has the consequence, noted by Kac in [51], 1941, that for each real λ

$$x^{-1} \sum_{\substack{n \le x \\ d(n) \le 2^{\log\log x + \lambda\sqrt{\log\log x}}}} 1 \quad \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-u^2/2}\, du, \qquad x \to \infty.$$

A closer accord with the results for holomorphic or Maass cusp forms is had from the case $\alpha = 2$ of the following result, c.f. Elliott [34] 2014, which, not being susceptible to the method of Erdős-Kac, relies upon the analytic continuation of complex powers of $\zeta(s)$.

Theorem 8. *Let $d(n)$ denote the number of divisors of the positive integer n. Let α be a positive number, $\beta = 2^\alpha$. Denote by $N(x)$ the sum $\sum_{n \le x} d(n)^\alpha$.*

Then for each real λ,

$$N(x)^{-1} \sum_{\substack{n \le x \\ \omega(n) - \beta \log\log x \le \lambda(\beta \log\log x)^{1/2}}} d(n)^\alpha \qquad \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-u^2/2}\, du, \qquad x \to \infty.$$

Moreover, the summation condition on $\omega(n)$ may be replaced by

$$d(n) \le (\log x)^{\beta \log 2} \exp(\lambda \log 2 (\beta \log\log x)^{1/2})$$

or by a similar inequality with n in place of x.

Note that $N(x) = (1 + o(1))c_\alpha x(\log x)^{\beta-1}$, $c_\alpha > 0$, as $x \to \infty$.

I note that in the foregoing commentary the Maass forms have weight zero, and there are no holomorphic cusp forms of weight zero for the action of $SL(2, \mathbb{Z})$ on H. Holomorphic cusp forms for such an action appear first with Ramanujan's function $\tau(n)$, corresponding to an eigenfunction attached to the bottom of the spectrum of the operator $-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + 12iy\frac{\partial}{\partial x}$, c.f. Goldfeld and Hundley [43], Chapter 3.

18

1956. Kubilius [57] 1956, [58] 1964, sets out to axiomatize the Erdős-Kac argument by constructing for a given interval $[1, N]$, an explicit finite probability space on which to accurately approximate the value distribution of a given additive function in terms of a sum of explicit independent random variables.

In practice, in order to avoid the implicit possibility of giving a positive measure to an empty set, it is preferable to increase the abstraction or to allow the sets in the underlying sigma-algebra of the model to be infinite; c.f. the remarks in the introduction to the second of the author's two volume Grundlehren work on Probabilistic Number Theory, [19] 1980.

The upshot amounts to the introduction of mutually approximating measures on a (necessarily finite) sigma-algebra subset of $\mathbb{Z}/D\mathbb{Z}$ for a modulus D as large as the application of an appropriate sieve method will allow.

Truncation of the random variables concomitant with the application of a sieve is offset by appeal to what is now the *Turán-Kubilius inequality* for a complex-valued additive function f:

$$\sum_{n\le N}\left|f(n)-\sum_{p\le N}p^{-1}f(p)\right|^2 \ll N\sum_{q\le N}q^{-1}|f(q)|^2, \qquad N\ge 1,$$

q traversing the prime-powers, the implied constant absolute.

At all events, I found it valuable to view each model as parametrised by two variables that might be vibrated in tandem to yield local structural information.

19

A generalised modified Kubilius model underlies the author's central limit theorems for classical cusp forms, considered earlier.

There is a corresponding Generalised Turán-Kubilius Inequality:

Theorem 9. *Let g be a real, non-negative, multiplicative, arithmetic function that satisfies*

$$\sum_{n\le x}g(n)=Hx+O(x^{1-\delta})$$

for positive constants H, δ and all $x\ge 1$.

Then for any complex-valued additive function f:

$$\sum_{n\le N}g(n)\left|f(n)-\sum_{q\le x}q^{-1}g(q)f(q)\right|^2 \ll x\sum_{q\le N}q^{-1}g(q)|f(q)|^2,$$

the implied constant depending at most upon g.

In particular, one may choose $g(n) = \tau(n)^2 n^{-11}$.

Analogues of Theorem 9 under weaker constraints, via an argument different from that of Turán and Kubilius, are established in the author's paper [23].

20

Here I may draw attention to a long Crelle [1], 1982 paper that Krishna Alladi wrote on the Turán-Kubilius inequality, the additive function restricted to integers without large prime factors; before saying 'farewell' and moving to q-series.

Krishna's paper pioneers what is now a parallel mathematical universe in which probabilistic number theory is studied on what are often called *nombres friables*; c.f. the Leitfaden in Elliott [32], and a recent summary in de la Bretèche and Tenenbaum [4].

21

Although allowing limit laws other than the Normal, as applied in his 1964 monograph [58], a Kubilius model prescribed a single possible unbounded renormalisation for the additive function under consideration, in particular excluded consideration of the additive function $n \to A \log n$, $A \neq 0$, the restriction of a canonical continuous homomorphism from the multiplicative positive reals to the additive reals.

To advance I replaced the interval $[1, N]$ with integer N by an interval $[1, x]$ with x arbitrary positive real, and considered frequencies

$$[x]^{-1} \sum_{\substack{n \le x \\ f(n)-\alpha(x) \le \lambda\beta(x)}} 1 \qquad , \qquad \lambda \in \mathbb{R}, x \to \infty,$$

the otherwise arbitrary renormalising functions $\alpha(x)$, $\beta(x)$ to be classified under the action of the group of transformations $x \to x^y$, $y > 0$.

I combined the application of a perturbed Kubilius model with an extensive Fourier analysis, operating through the agency of the corresponding characteristic functions

$$[x]^{-1} \exp(-it\alpha(x)\beta(x)^{-1}) \sum_{n \le x} g(n)$$

with $g(n) = \exp(it\beta(x)^{-1} f(n))$ multiplicative; c.f. Delange [5] 1961, Halász [44] 1968, Elliott [12] 1975, [13] 1976.

Having reinterpreted Linnik's Large Sieve in functional analytic terms [10], I adopted the philosophy *If operator corresponds to sufficiency then dual operator corresponds to necessity*, dualised the classical Turán-Kubilius inequality and showed the existence of a limit law to guarantee that suitably translated the renormalising function $\alpha(x)$ locally became an in-measure homomorphism on the product of two copies of the multiplicative positive reals, the associated Haar measure scaled by a parameter attached to an appropriate Kubilius model.

I anticipated that once a suitable obstruction had been removed, an additive function would again behave like a sum of independent random variables; and so it proved.

An illustrative example is the author's solution, for additive functions, of the 1917 Hardy and Ramanujan question whether arithmetic functions other than $\omega(n)$ or $\Omega(n)$ might possess a normal order; c.f. [14], [15], 1976. Birch [3] 1967, had already shown that the only real-valued multiplicative functions with a non-decreasing normal order are the fixed power maps $n \to n^{\alpha}$, $\alpha > 0$.

That a normal order should be elementary, researchers have dropped.

Theorem 10. *In order that the non-zero real additive function $f(n)$ possess a normal order it is necessary and sufficient that there exist a real function $g(x)$, for sufficiently large values of x positive and non-decreasing, and which satisfies the following conditions:*

(i) There is a decomposition $g(x) = u(x) + v(x)$ where, for each $y > 0$, the asymptotic relations $u(x^y) = yu(x) + o(g(x))$, $v(x^y) = v(x) + o(g(x))$ hold.

(ii) For each $\varepsilon > 0$ the function $h(p, x) = f(p) - u(x)\log p/\log x$ satisfies

$$\sum_{\substack{p \le x \\ |h(p,x)| > \varepsilon g(x)}} p^{-1} \to 0, \qquad g(x)^{-2} \sum_{\substack{p \le x \\ |h(p,x)| \le \varepsilon g(x)}} p^{-1} h(p, x)^2 \to 0.$$

(iii) Moreover,

$$g(x) = u(x) + \sum_{\substack{p \le x \\ |h(p,x)| \le g(x)}} p^{-1} h(p, x) + o(g(x)),$$

the relations to hold as $x \to \infty$.

When these relations are satisfied $f(n)$ has the normal order $g(n)$.

22

1968. Kátai [52], defines a sequence of positive integers a_n, $n = 1, 2, \ldots$, to be a *set of uniqueness* if every real completely additive function that vanishes on the a_n vanishes identically. In particular, he proves [53], that adjoining finitely many integers to the shifted primes $p + 1$ provides such a set, and conjectures that the shifted primes alone are a set of uniqueness.

In 1969 and 1970, Kátai [54], [55] further proposes a number of questions of which the following is typical: Characterise the real-valued additive functions f that satisfy

$$f(an+b) - f(An+B) \to c, \qquad n \to \infty,$$

where the integers $a > 0$, $A > 0$, b, B are constrained by $\Delta = aB - Ab \neq 0$.

These questions seek to widen several studies in the characterisation of the logarithm as an additive arithmetic function, the earliest being those of Erdős [40], 1946, a particularly impressive example that of Wirsing [82], 1970, with its application, after Turán, to the characterisation of the Riemann zeta function amongst Dirichlet series with Euler products.

1974 [11]. With an argument completely different from that of Kátai, I proved that the shifted primes are indeed a set of uniqueness.

1978. Arguing via vector spaces over the field of rational numbers, Wolke [83], Dress and Volkmann [7], prove that the sequence a_n, $n = 1, 2, \ldots$, is a set of uniqueness if and only if every positive integer r has a multiplicative representation

$$r^v = \prod a_{j_i}^{\varepsilon_i}$$

with $\varepsilon_i = \pm 1$ and the exponent v possibly varying with r.

Further, Meyer [60] 1980, proves that one may take $v = 1$ in such representations if and only if every completely multiplicative complex-valued function that assumes the value 1 on the a_n is identically 1; his argument resting upon the freedom of the multiplicative group of positive rationals when viewed as a $\mathbb{Z}$-module.

Detailed discussion of these results may be found in Elliott [21], and Chapter 15 of Elliott [22].

23

Moving from the consideration of one additive function to the simultaneous consideration of arbitrarily many [16], [18], I could reduce the characterisation of additive functions f_h, $h = 1, \ldots, k$, for which

$$n \to \sum_{h=1}^{k} f_h(a_h n + b_h), \qquad \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \neq 0, 1 \leq i < j \leq k,$$

is uniformly bounded, to the requirement that for some p_0 the sums

$$(x(\log x)^2)^{-1} \sum_{p_o < p \leq x} f_h(p)^2 p^{-1}, \qquad h = 1, \ldots, k,$$

are bounded uniformly in $x \geq 2$. Clearly necessary, an achievement of this bound seems to be of independent difficulty.

Concentrating on the case $k = 2$, I developed a localised L^2-theory of the differences $f(an + b) - f(An + B)$, having settled Kátai's question along the way: f has the form $c \log$ on the integers prime to $aA\Delta$; I gave necessary and sufficient conditions for the weak convergence of suitably renormalised frequencies

$$[x]^{-1} \sum_{f(an+b)-f(An+B)\leq\lambda\beta(x)} 1 \qquad , \qquad \lambda \in \mathbb{R}, x \to \infty,$$

with concomitant convergence of mean and variance; introduced the quotient group $G = \mathbb{Q}^*/\Gamma$ of the multiplicative positive rationals by its subgroup generated by a sequence of positive rationals a_n, $n = 1, 2, \ldots$, considering its homomorphims into $(\mathbb{R}, +)$ and $\mathbb{C}^\times$ to be characters and, along with many related results, wrote straight into a third Springer Grundlehren volume [22] 1985, organised as a first systematic application of abstract arithmetic functions to problems in algebra.

Determination of a general group G amounts to the explicit realisation of an abelian group with denumerably many generators and denumerably many relations, a problem for which a (finite recursive) decision procedure is known not to exist. Application of harmonic analysis offers an alternative.

Note that from this point of view a combination of the arguments of Kátai [53] and Elliott [11], shows that Γ generated by the shifted primes the group G is finite, although that was not realised at the time.

In particular, I proved that with Γ generated by the ratios $(an + b)/(An + B)$, $a > 0$, $A > 0$, $\Delta \neq 0$, n exceeding an arbitrary given value, the group G is finitely generated. I gave an explicit set of generators for its free component and determined the membership of its torsion group, the former via estimates for Kloosterman sums, the latter via homomorphisms into the additive reals.

Thus, every positive rational r has infinitely many representations

$$r^v = \prod_j \left(\frac{3n_j + 1}{5n_j + 2}\right)^{\varepsilon_j}, \qquad \varepsilon_j = \pm 1, n_j > k,$$

with $v = |G|$. The best value of v for each r would require a closer study of the multiplicative characters on G.

I conjectured that in the general case the group G would be *arithmic*, i.e. the homomorphic image of the direct sum of finitely many reduced residue class groups derived from $\mathbb{Z}$, c.f. [22], Chapter 22, and Chapter 23, Problem 12.

24

Meanwhile, Probabilistic Number Theory continued apace, including important studies by Hildebrand, on multiplicative functions on short intervals [49], 1987, and on the differences of additive functions $f(n+1) - f(n)$, [50] 1988.

25

By 1994, in an AMS memoir on the correlation of multiplicative and the sum of additive arithmetic functions, I could establish the following version of a central limit theorem for a pair of additive functions, [25] Theorem 10.1.

Theorem 11. *For $j = 1, 2$, let $a_j > 0$, b_j be integers, $a_1 b_2 \neq a_2 b_1$, $\beta_j(x)$ positive real measurable functions, defined for $x \geq 2$, which satisfy $\beta_j(x) \to \infty$ and $\beta_j(x^y)/\beta_j(x) \to 1$ for each fixed $y > 0$, as $x \to \infty$.*

In order that for suitably chosen real $\alpha_j(x)$ the frequencies $[x]^{-1} \sum'_{n \leq x} 1$, taken over the integers n for which

$$\sum_{j=1}^{2} \beta_j(x)^{-1}(f_j(a_j n + b_j) - \alpha_j(x)) \leq z, \qquad z \in \mathbb{R},$$

should converge weakly to a distribution function as $x \to \infty$, it is both necessary and sufficient that there exist constants λ_j, $j = 1, 2$, and a real $\gamma(x)$ so that if the independent random variables Y_p, one for each prime p not exceeding x, are distributed according to

$$Y_p = \begin{cases} \beta_1(x)^{-1}(f_1(p) - \lambda_1 \log p) & \text{with probability } \dfrac{1}{p}, \\ \beta_2(x)^{-1}(f_2(p) - \lambda_2 \log p) & \text{with probability } \dfrac{1}{p}, \\ 0 & \text{with probability } 1 - \dfrac{2}{p}, \end{cases}$$

then the

$$P\left(\sum_{p \leq x} Y_p - \gamma(x) \leq z\right)$$

converge weakly to the same distribution function.

The random variables Y_p are infinitesimal. Necessary and sufficient conditions for the weak convergence of the final distribution functions may be read off from a classical theorem of Gnedenko in the Theory of Probability, c.f. Gnedenko and Kolmogorov [42], Chapter 4, §2.5.

The somewhat elaborate argument for Theorem 11 employs the associated characteristic functions

$$t \to [x]^{-1} \exp\left(-it \sum_{j=1}^{2} \beta_j(x)^{-1}\alpha_j(x)\right) \sum_{n \le x} g_1(a_1 n + b_1) g_2(a_2 n + b_2),$$

correlations with multiplicative functions $g_j(n) = \exp(it\beta_j(x)^{-1} f_j(n))$ that depend individually upon the local parameter x.

Abstract multiplicative functions and their correlations are considered in the author's paper [24], 1988, applications to sums of additive functions including to the translations of a sum $f_1(N-n) + f_2(n)$, with $f_j(0) = 0$, over an interval of integers $1 \le n \le N$. Although I had worked out the details in that case, a complete analogue of Theorem 11 I published only in 2010, [29]. The following is a simple example.

Theorem 12. *Let $\beta(x) \to \infty$ and for each $y > 0$ satisfy $\beta(x^y)/\beta(x) \to 1$ as $x \to \infty$. In order that for some $\theta(N)$ an additive function f should satisfy*

$$\frac{1}{N} \sum_{\substack{n \le N \\ f(n)+f(N-n)-\theta(N) \le z\beta(N)}} 1 \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du, \qquad N \to \infty,$$

it is necessary and sufficient that for a real λ and each $u > 0$,

$$\sum_{\substack{p \le N \\ |f(p)-\lambda \log p| > u\beta(N)}} \frac{1}{p} \to 0, \qquad \sum_{\substack{p \le N \\ |f(p)-\lambda \log p| \le u\beta(N)}} \frac{(f(p)-\lambda \log p)^2}{p\beta(N)^2} \to \frac{1}{2},$$

$$\sum_{\substack{p | N \\ |f(p)-\lambda \log p| \le u\beta(n)}} \frac{(f(p)-\lambda \log p)^2}{p\beta(N)^2} \to 0,$$

as $N \to \infty$.

Of particular interest is the final condition on the prime divisors of the integer N, which has both an arithmetic and a probabilistic aspect.

The same 1988 paper contains two conjectures concerning the correlation

$$S(x) = [x]^{-1} \sum_{n \le x} \prod_{j=1}^{k} g_j(a_j n + b_j), \qquad \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \ne 0, 1 \le i < j \le k,$$

the second of which asserts that if $\limsup_{x\to\infty} S(x) > 0$ then for each multiplicative function g_j, with values in the complex unit disc, there is a Dirichlet character χ_j and real τ_j for which the series $\sum p^{-1}(1 - \operatorname{Re} g_j(p)\chi_j(p)p^{i\tau_j})$ converges, a conjecture slightly modified according to the background aesthetic of a particular application, c.f. [25] Conjecture III, p. 65, which last, modified according to a remark in the author's 1997 Cambridge Tract [26], Chapter 34. p. 315 becomes, for some $c > 0$, that

$$x^{-1}\sum_{n\le x} g(n)h(n+1) \ll \left(T^{-1} + \exp\left(-\min_{\substack{\chi \pmod D\\ D\le T}} \min_{|\tau|\le T} \sum_{p\le x} p^{-1}(1-\mathrm{Re}\, g(p)\chi(p)p^{i\tau})\right)\right)^{c},$$

uniformly for multiplicative functions g, h with values in the complex unit disc, $T \ge 1$, $x \ge 2$.

26

In 2015, Tao [77] establishes the validity of a logarithmically weighted variant:

Let the integers $a > 0$, $b > 0$, c, d satisfy $ad - bc \ne 0$. Let $\varepsilon > 0$ and suppose that A_0 is sufficiently large depending upon ε, a, b, c, d. Let $x \ge w \ge A_0$ and let g_1, g_2 be multiplicative functions, with values in the complex unit disc, for which

$$\sum_{p\le x} p^{-1}(1-\mathrm{Re}\, g_1(p)\overline{\chi}(p)p^{-it}) \ge A_0$$

for all Dirichlet characters of period at most A_0, and all real numbers t with $|t| \le A_0 x$.

Then

$$\left|\sum_{x/w<n\le x} n^{-1} g_1(an+b)g_2(cn+d)\right| \le \varepsilon \log w.$$

27

Harmonic analysis on the positive rationals sufficiently developed, the following result of Elliott and Kish [36], settles in the affirmative the thirty year old conjecture of the first author's Grundlehren volume [22], Chapter 23, Unsolved problems 11, 12, mentioned in § 23

Theorem 13. *Let integers $a > 0$, $A > 0$, b, B satisfy $\Delta = aB - Ab \ne 0$, and let k be a further positive integer. Let $\mathbb{Q}^*$ be the multiplicative group of positive rationals, Γ its subgroup generated by the rationals $(an+b)/(An+B)$, $n > k$.*

Then the factor group $\mathbb{Q}^*/\Gamma$ may be explicitly determined through its dual. In particular, its torsion group is a homomorphic image of the reduced residue class group $\pmod{6(aA\Delta)^3}$.

Three examples of the outcome may here suffice.

The group $\mathbb{Q}^*/\Gamma_k$ generated by the ratios $(3n+1)/(5n+2)$, $n > k$, is trivial. Each positive rational r has infinitely many representations

$$r = \prod_j \left(\frac{3n_j+1}{5n_j+2}\right)^{\varepsilon_j}, \qquad \varepsilon_j = \pm 1.$$

The group $\mathbb{Q}^*/\Gamma_k$ generated by the ratios $(5n+1)/(5n-1)$, $n > k$, has the single free generator 5, and a torsion group of order 2 determined by its dual through the quadratic Dirichlet character $\pmod 5$. There are infinitely many representations

$$57^2 = \prod_j \left(\frac{5n_j+1}{5n_j-1}\right)^{\varepsilon_j}, \qquad \varepsilon_j = \pm 1,$$

but no such representation is available to 57 itself.

The argument of Theorem 13 extends immediately to embrace negative rationals. If $\mathbb{Q}_5^\times$ denotes the multiplicative group of all rationals that in reduced terms are not divisible by 5, Γ_k its subgroup generated by the ratios $(5n+1)/(-5n+1)$, $n > k$, then $\mathbb{Q}_5^\times/\Gamma_k$ has order 4, its dual group generated by a quartic character $\pmod 5$, unitary characters on $\mathbb{Q}_5^*$ extended to $\mathbb{Q}_5^\times$ by being given a value ± 1 on the rational -1. There are infinitely many representations

$$57^4 = \prod_j \left(\frac{5n_j+1}{-5n_j+1}\right)^{\varepsilon_j}, \qquad \varepsilon_j = \pm 1,$$

but no similar representation for 57^2.

Concluding Remarks

Probabilistic Number Theory up to 1964 is covered in the American Mathematical Society translation volume, Kubilius, [58].

A comprehensive survey of Probabilistic Number Theory up to 1978 may be found in the author's two-volume Grundlehren work [17] 1979, [19] 1980.

More recent results may be found in the author's AMS memoir [25], 1994 and in Tenenbaum's volume on Analytic and Probabilistic Number Theory [79].

The present author's appreciation of the mathematicians Paul Erdős, Paul Turán and Jonas Kubilius and of their contributions to Probabilistic Number Theory may be found in [27], [32], [35], respectively. Each of these accounts is, in an individual way, comprehensive. Mark Kac may be found in the second Grundlehren volume, [19].

Foundations for a systematic study of multiplicative functions as characters attached to harmonic analysis on the positive rationals may be found in Elliott and Kish [37], [38].

Further details on the rôle played by general characters $n \to \chi(n)n^{i\tau}$ in approximating characters on the group $\mathbb{Q}^*$ may be found in the author's closing remarks to the paper Elliott and Kish [36].

An extensive overview of groups generated by products of rationals with attendant $\mathbb{Q}^*$-character sums is given in the author's paper [28], 2002.

The present paper employs Harmonic Analysis on the Positive Rationals to effect a chronological account connecting the apparently disparate disciplines of Probabilistic Number Theory, Automorphic Forms and the Theory of Denumerably Infinite Abelian Groups.

Detailed accounts of improvements in individual results await another occasion.

I thank George Andrews and Frank Garvan, organisers of the Gainesville meeting, for their kind invitation to speak and for their financial support.

Happy Birthday Krishna!

References

1. K. Alladi, The Turán-Kubilius inequality for integers without large prime factors. J. Reine Angew. Math. **335**, 180–196 (1982)
2. T. Barnet-Lamb, D. Geraghty, M. Harris, R. Taylor, A family of Calabi-Yau varieties and potential automorphy II. Publ. Res. Inst. Math. Sci. **47**(1), 29–98 (2011)
3. B.J. Birch, Multiplicative functions with non-decreasing normal order. J. Lond. Math. Soc. **42**, 149–151 (1967)
4. R. de la Bretèche, G. Tenenbaum, On the friable Turán-Kubilius inequality, in *Analytic and Probabilistic Methods in Number Theory*, *Proceedings of the Fifth International Conference in Honour of J. Kubilius, Palanga, Lithuania, 4–10 September, 2011* ed. by F. Schweiger, E. Manstavičius. New trends in probability and statistics (TEV, Vilnius, 2012), pp. 111–117
5. H. Delange, Sur les fonctions arithmétiques multiplicatives. Ann. Sci. École Norm. Sup. (3) **78**(1), 273–304 (1961)
6. P. Deligne, La conjecture de Weil. I, Inst. Hautes Études Sci. Publ. Math. **43**, 273–307 (1974)
7. F. Dress, B. Volkmann, Ensembles d'unicité pour les fonctions arithmétiques additives ou multiplicatives. C. R. Acad. Sci. Paris Sér. A-B **287**(2), A43–A46 (1978)
8. N.D. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over $\mathbb{Q}$. Invent. Math. **89**(3), 561–567 (1987)
9. P.D.T.A. Elliott, Multiplicative functions and the sign of Maass form Fourier coefficients, in *From Arithmetic to Zeta functions, a volume in memory of Wolfgang Schwarz,* ed. by J. Sander, J. Steuding, R. Steuding (Springer, 2016) pp. 109–120
10. P.D.T.A. Elliott, On inequalities of large sieve type. Acta Arith. **18**(1), 405–422 (1971) (Davenport Memorial Volume)
11. P.D.T.A. Elliott, A conjecture of Kátai. Acta Arith. **26**(1), 11–20 (1974/75)
12. P.D.T.A. Elliott, The law of large numbers for additive arithmetic functions. Math. Proc. Camb. Philos. Soc. **78**(1), 33–71 (1975)
13. P.D.T.A. Elliott, General asymptotic distributions for additive arithmetic functions. Math. Proc. Camb. Philos. Soc. **79**(1), 43–54 (1976)
14. P.D.T.A. Elliott, On a conjecture of Narkiewicz about functions with non-decreasing normal order. Colloq. Math. **36**(2), 289–294 (1976)
15. P.D.T.A. Elliott, On a problem of Hardy and Ramanujan. Mathematika **23**(1), 10–17 (1976)
16. P.D.T.A. Elliott, On the differences of additive arithmetic functions. Mathematika **24**(2), 153–165 (1977)

17. P.D.T.A. Elliott, *Probabilistic Number Theory I: Mean-Value Theorems*, Grundlehren der mathematischen Wissenschaften, vol. 239 (Springer, New York, 1979)
18. P.D.T.A. Elliott, Sums and differences of additive arithmetic functions in mean square. J. Reine Angew. Math. **309**, 21–54 (1979)
19. P.D.T.A. Elliott, *Probabilistic Number Theory II: Central Limit Theorems*, Grundlehren der mathematischen Wissenschaften, vol. 240 (Springer, New York, 1980)
20. P.D.T.A. Elliott, Multiplicative functions and Ramanujan's τ-function. J. Aust. Math. Soc. Ser. A **30**(4), 461–468 (1981)
21. P.D.T.A. Elliott, On representing integers as products of integers of a prescribed type. J. Aust. Math. Soc. Ser. A **35**(2), 143–161 (1983)
22. P.D.T.A. Elliott, *Arithmetic Functions and Integer Products*, Grundlehren der mathematischen Wissenschaften, vol. 272 (Springer, New York, 1985)
23. P.D.T.A. Elliott, The norms of compositions of arithmetic operators. Bull. Lond. Math. Soc. **19**(6), 522–530 (1987)
24. P.D.T.A. Elliott, Multiplicative functions $|g| \leq 1$ and their convolutions: an overview, Séminaire de Théorie des Nombres, Paris 1987–88, ed. by C. Goldstein. Progress in Mathematics, vol. 81 (Birkhäuser, Boston, 1990), pp. 63–73
25. P.D.T.A. Elliott, On the correlation of multiplicative and the sum of additive arithmetic functions. Mem. Am. Math. Soc. **112**(538), viii+88 (1994)
26. P.D.T.A. Elliott, *Duality in Analytic Number Theory*, Cambridge Tracts in Mathematics, vol. 122 (Cambridge University Press, Cambridge, 1997)
27. P.D.T.A. Elliott, *Cast thy bread upon the waters… A personal view of the mathematician Paul Erdős*, Paul Erdős and his mathematics. I, ed. by G. Halász, L. Lovász, M. Simonovits, V.T. Sós, Bolyai Society Mathematical Studies, vol. 11, János Bolyai Mathematical Society (Budapest, 2002), pp. 175–218
28. P.D.T.A. Elliott, Product representations by rationals, in *Number Theoretic Methods: Future Trends*, *Proceedings of the Second China-Japan Seminar, Iizuka, Japan, March 12–16, 2001*, ed. by S. Kanemitsu, C. Jia. Devolopment in Mathematics, vol. 8 (Kluwer Academic Publications, Dordrecht, 2002), pp. 119–150
29. P.D.T.A. Elliott, The value distribution of additive arithmetic functions on a line. J. Reine Angew. Math. **642**, 57–108 (2010)
30. P.D.T.A. Elliott, An abstract central limit theorem for eigenforms, in *Analytic and Probabilistic Methods in Number Theory, Proceedings of the Fifth International Conference in Honour of J. Kubilius, Palanga, Lithuania, 4–10 September, 2011*, ed. by E. Manstavičius, A. Laurinčikas, G. Stepanauskas (TEV, Vilnius, 2012), pp. 131–142
31. P.D.T.A. Elliott, A central limit theorem for Ramanujan's tau function. Ramanujan J. **29**(1–3), 145–161 (2012) (Ramanujan's 125th anniversary special volume)
32. P.D.T.A. Elliott, Paul Turán and probabilistic number theory, a Personal View, in *Number Theory, Analysis, and Combinatorics, Proceedings of the Paul Turán Memorial Conference held August 22–26, 2011 in Budapest*, ed. by J. Pintz, A. Biró, K. Győry, G. Harcos, M. Simonovits, J. Szabados, Proceedings in Mathematics (De Gruyter, Berlin/Boston, 2014), pp. 41–62
33. P.D.T.A. Elliott, Central limit theorems for classical cusp forms. Ramanujan J. **36**(1–2), 81–98 (2015). Basil Gordon memorial volume
34. P.D.T.A. Elliott, Corrigendum: central limit theorems for classical cusp forms. Ramanujan J. **36**(1–2), 99–102 (2015) (Basil Gordon memorial volume; published online Oct. 2014)
35. P.D.T.A. Elliott, Jonas Kubilius and probabilistic number theory. some personal reflections. Lith. Math. J. **55**(1), 2–24 (2015)
36. P.D.T.A. Elliott, J. Kish, Harmonic analysis on the positive rationals. Determination of the group generated by the ratios $(an + b)/(An + B)$ (2016), Mathematik (2017), arXiv:1602.03263
37. P.D.T.A. Elliott, J. Kish, Harmonic analysis on the positive rationals I: basic results. J. Fac. Sci. Univ. Tokyo **23**(3) (2016), arXiv:1405.7130
38. P.D.T.A. Elliott, J. Kish, Harmonic analysis on the positive rationals II: multiplicative functions and maass forms. J. Math. Sci. Univ. Tokyo **23**(3) (2016), arXiv:1405.7132

39. P.D.T.A. Elliott, C.J. Moreno, F. Shahidi, On the absolute value of Ramanujan's τ-function. Math. Ann. **266**(4), 507–511 (1984)
40. P. Erdős, On the distribution function of additive functions. Ann. Math. **47**(2), 1–20 (1946)
41. P. Erdős, M. Kac, The Gaussian law of errors in the theory of additive number theoretic functions. Am. J. Math. **62**, 738–742 (1940)
42. B.V. Gnedenko, A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley Publishing Co., Reading, 1968) (Translated from the 1949 Russian edition and annotated by K.L. Chung)
43. D. Goldfeld, J. Hundley, *Automorphic Representations and L Functions for the General Linear Group. Volume I*, Cambridge Studies in Advanced Mathematics, vol. 129 (Cambridge University Press, New York, 2011)
44. G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen. Acta Math. Acad. Sci. Hungar. **19**, 365–403 (1968)
45. G.H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work* (Cambridge University Press, Cambridge, 1940) (Reprinted, with additional corrections and a commentary by Bruce C. Berndt, AMS Chelsea, Providence, Rhode Island, 1999)
46. G.H. Hardy, S. Ramanujan, The normal number of prime factors of a number n. Q. J. Math. **48**, 76–92 (1917)
47. H. Heilbronn, E. Landau, Anwendungen der N. Wienerschen Methode. Math. Z. **37**(1), 18–21 (1933) (c.f. *The Collected Papers of Hans Arnold Heilbronn*, ed. by E.J. Kari, R.A. Smith. Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, 1988)
48. H. Heilbronn, E. Landau, Bemerkungen zur vorstehenden Arbeit von Herrn Bochner. Math. Z. **37**(1), 10–16 (1933) (c.f. *The Collected Papers of Hans Arnold Heilbronn*, ed. by E.J. Kari, R.A. Smith. Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, 1988)
49. A. Hildebrand, Multiplicative functions in short intervals. Can. J. Math. **39**(3), 646–672 (1987)
50. A. Hildebrand, An Erdős-Wintner theorem for differences of additive functions. Trans. Am. Math. Soc. **310**(1), 257–276 (1988)
51. M. Kac, Note on the distribution of values of the arithmetic function $d(m)$. Bull. Am. Math. Soc. **47**, 815–817 (1941)
52. I. Kátai, On sets characterizing number-theoretical functions. Acta Arith. **13**(3), 315–320 (1968)
53. I. Kátai, On sets characterizing number-theoretical functions (II) (The set of "prime plus one" 's is a set of quasi-uniqueness). Acta Arith. **16**(1), 1–4 (1969)
54. I. Kátai, Some results and problems in the theory of additive functions. Acta Sci. Math. (Szeged) **30**(3–4), 305–311 (1969)
55. I. Kátai, On number-theoretical functions, in *Number Theory, Proceedings of the Number-Theoretic Colloquium, Debrecen, Hungary, April 4–8, 1968*, ed. by P. Turán. Colloquia Mathematica Societatis János Bolyai, vol. 2 (North-Holland, Amsterdam, 1970), pp. 133–137
56. H.H. Kim, F. Shahidi, Cuspidality of symmetric powers with applications. Duke Math. J. **112**(1), 177–197 (2002)
57. J. Kubilius, Probabilistic methods in the theory of numbers (Russian). Uspekhi Mat. Nauk **11**(2(68)), 31–66 (1956) (see Am. Math. Soc. Trans. **19**, 1962, 47–85)
58. J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, 2nd edn. (Vilnius, 1962) (corrected and revised by J. Kubilius, translated from the Russian by G. Burgie and S. Schuur, American Mathematical Society. Translations of Mathematical Monographs, vol. 11, Providence, RI, 1964)
59. D.H. Lehmer, Ramanujan's function $\tau(n)$. Duke Math. J. **10**, 483–492 (1943)
60. J. Meyer, Ensembles d'unicité pour les fonctions additives. Étude analogue dans le cas des fonctions multiplicatives, in *Proceedings of the Journées de Théorie Analytique et Elémentaire des Nombres, Université de Paris-Sud, Orsay, France, June 2–3, 1980*, vol. 81 (Publ. Math. Orsay, no. 1, Université de Paris-Sud, Orsay, 1981), pp. 19–29
61. L.J. Mordell, On Mr. Ramanujan's empirical expansions of modular functions. Proc. Camb. Philos. Soc. **19**, 117–124 (1917)

62. C.J. Moreno, F. Shahidi, The fourth moment of Ramanujan τ-function. Math. Ann. **266**(2), 233–239 (1983)
63. S. Ramanujan, On certain arithmetical functions. Trans. Camb. Philos. Soc. **22**(9), 159–184 (1916)
64. S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers. Trans. Camb. Philos. Soc. **22**(13), 259–276 (1918)
65. S. Ramanujan, in *Collected Papers of Srinivasa Ramanujan*, ed. by G.H. Hardy, P.V. Seshu Aiyar, B.M. Wilson (Cambridge University Press, Cambridge, 1927) (Reprinted, with a new preface and commentary by Bruce C. Berndt, AMS Chelsea, Providence, Rhode Island, 2000)
66. R.A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions I. The zeros of the function $\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$ on the line $Re\, s = \frac{13}{2}$. Math. Proc. Camb. Philos. Soc. **35**(3), 351–356 (1939)
67. R.A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of the Fourier coefficients of integral modular forms. Math. Proc. Camb. Philos. Soc. **35**(3), 357–372 (1939)
68. R.A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions III. A note on the sum function of the Fourier coefficients of integral modular forms. Math. Proc. Camb. Philos. Soc. **36**(2), 150–151 (1940)
69. R.A. Rankin, Sums of powers of cusp form coefficients. Math. Ann. **263**(2), 227–236 (1983)
70. R.A. Rankin, Sums of powers of cusp form coefficients. II. Math. Ann. **272**(4), 593–600 (1985)
71. R.A. Rankin, Fourier coefficients of cusp forms. Math. Proc. Camb. Philos. Soc. **100**(1), 5–29 (1986)
72. A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc. (N.S.) **20**, 47–87 (1956)
73. J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, (W.A. Benjamin Inc., New York 1968) (republished as *Research Notes in Mathematics*, vol. 7, A K Peters Ltd., Wellesley, MA, 1998)
74. J.-P. Serre, Quelques applications du théorème de densité de Chebotarev. Inst. Hautes Études Sci. Publ. Math. **54**, 123–201 (1981)
75. F. Shahidi, On certain L-functions. Am. J. Math. **103**(2), 297–355 (1981)
76. F. Shahidi, in *Symmetric power L-functions for $GL(2)$*, Elliptic curves and related topics (H. Kisilevsky and M. R. Murty, eds.), Centre de Recherches Mathématiques, Université de Montréal, CRM Proceedings and Lecture Notes, vol. 4 (AMS, Providence, RI, 1994), pp. 159–182
77. T. Tao, The logarithmic averaged Chowla and Elliott conjectures for two-point correlations. Forum. Math. Pi. **4**(e8), 36 (2016), arXiv:1509.05422v2
78. J. Tate, Algebraic cycles and poles of zeta functions, in *Arithmetical Algebraic Geometry: Proceedings of a Conference Held at Purdue University December 5–7, 1963*, ed. by O.F.G. Schilling. Harper's Series in Modern Mathematics (Harper and Row, New York, 1965), pp. 93–110
79. G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, vol. 13 (Institut Elie Cartan, Université de Nancy I, 1990), English translation *Introduction to Analytic and Probabilistic Number Theory* (Cambridge University Press, Cambridge, 1994) (Second edition of French edition published by SMF, 1996)
80. P. Turán, On a theorem of Hardy and Ramanujan. J. Lond. Math. Soc. **9**(4), 274–276 (1934)
81. P. Turán, Uber einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan. J. Lond. Math. Soc. **11**(2), 125–133 (1936)
82. E. Wirsing, A characterization of $\log n$ as an additive arithmetic function, in *Symposia Mathematica, Proceedings of the Rome Conference on Number Theory, Istituto Nazionale di Alta Matematica Roma, December 9–12, 1968 (London)*, vol. 4 (Academic Press, London, 1970), pp. 45–57
83. D. Wolke, Bemerkungen über Eindeutigkeitsmengen additiver Funktionen. Elem. Math. **33**(1), 14–16 (1978)

On an Additive Prime Divisor Function of Alladi and Erdős

Dorian Goldfeld

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract This paper discusses the additive prime divisor function $A(n) := \sum_{p^\alpha || n} \alpha\, p$ which was introduced by Alladi and Erdős in 1977. It is shown that $A(n)$ is uniformly distributed (mod q) for any fixed integer $q > 1$ with an explicit bound for the error.

1 Introduction

Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the unique prime decomposition of a positive integer n. In 1977, Alladi and Erdős [1] introduced the additive function

$$A(n) := \sum_{i=1}^{r} a_i \cdot p_i.$$

Among several other things they proved that $A(n)$ is uniformly distributed modulo 2. This was obtained from the identity

This research was partially supported by NSA grant H98230-15-1-0035.

D. Goldfeld (✉)
Department of Mathematics, Columbia University, New York, NY 10027, USA
e-mail: goldfeld@columbia.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_17

$$\sum_{n=1}^{\infty} \frac{(-1)^{A(n)}}{n^s} = \frac{2^s+1}{2^s-1} \cdot \frac{\zeta(2s)}{\zeta(s)} \tag{1}$$

together with the known zero-free region for the Riemann zeta function. As a consequence they proved that there exists a constant $c > 0$ such that

$$\sum_{n \le x} (-1)^{A(n)} = \mathscr{O}\left(x\, e^{-c\sqrt{\log x \log\log x}}\right),$$

for $x \to \infty$.

In 1969 Delange [3] gave a necessary and sufficient condition for uniform distribution in progressions for integral valued additive functions which easily implies that $A(n)$ is uniformly distributed (mod q) for all $q \ge 2$ (although without a bound for the error in the asymptotic formula). The main goal of this paper is to show that $A(n)$ is uniformly distributed modulo q for any integer $q \ge 2$ with an explicit bound for the error.

Unfortunately, it is not possible to obtain such a simple identity as in (1) for the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i \frac{hA(n)}{q}}}{n^s}$$

when $q > 2$ and h, q are coprime. Instead we require a representation involving a product of rational powers of Dirichlet L-functions which will have branch points at the zeros of the L-functions.

The uniform distribution of $A(n)$ is a consequence of the following theorem (1.1) which is proved in §3. To state the theorem we require some standard notation. Let μ denote the Mobius function and let ϕ denote Euler's function. For any Dirichlet character $\chi \pmod q$ (with $q > 1$) let $\tau(\chi) = \sum\limits_{\ell \pmod q} \chi(\ell) e^{\frac{2\pi i \ell}{q}}$ denote the associated Gauss sum and let $L(s, \chi)$ denote the Dirichlet L-function associated to χ.

Theorem 1.1. *Let h, q be fixed coprime integers with $q > 2$. Then for $x \to \infty$ we have the asymptotic formula*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \begin{cases} C_{h,q} \cdot x\,(\log x)^{-1+\frac{\mu(q)}{\phi(q)}} \left(1 + \mathscr{O}\left((\log x)^{-1}\right)\right) & \text{if } \mu(q) \ne 0, \\ \mathscr{O}\left(x\, e^{-c_0\sqrt{\log x}}\right) & \text{if } \mu(q) = 0, \end{cases}$$

where $c_0 > 0$ is a constant depending at most on h, q,

$$C_{h,q} = \frac{V_{h,q} \cdot \sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi}\, \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \prod_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}},$$

and

$$V_{h,q} := \exp\left[-\frac{\mu(q)}{\phi(q)} \sum_{p|q} \sum_{k=1}^{\infty} \frac{1}{kp^k} + \sum_{p\,|\,q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^k} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\,p^k} \right].$$

Theorem 1.1 has the following easily proved corollary.

Corollary 1.2. *Let $q > 1$ and let h be an arbitrary integer. Then*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \mathcal{O}\left(\frac{x}{\sqrt{\log x}} \right).$$

The above corollary can then be used to obtain the desired uniform distribution theorem.

Theorem 1.3. *Let h, q be fixed integers with $q > 2$. Then for $x \to \infty$, we have*

$$\sum_{\substack{n \le x \\ A(n) \equiv h \pmod q}} 1 = \frac{x}{q} + \mathcal{O}\left(\frac{x}{\sqrt{\log x}} \right).$$

We remark that the error term in theorem 1.3 can be replaced by a second order asymptotic term which is not uniformly distributed (mod q).

The proof of theorem (1.1) relies on explicitly constructing an L-function with coefficients of the form $e^{2\pi i \frac{hA(n)}{q}}$. It will turn out that this L-function will be a product of Dirichlet L-functions raised to complex powers. The techniques for obtaining asymptotic formulae and dealing with branch singularities arising from complex powers of ordinary L-series were first introduced by Selberg [7], and see also Tenenbaum [8] for a very nice exposition with different applications. In [4–6], one finds a larger class of additive functions where these methods can also be applied yielding similar results but with different constants.

2 On the function $L(s, \psi_{h/q})$

Let h, q be coprime integers with $q > 1$. In this paper we shall investigate the completely multiplicative function

$$\psi_{h/q}(n) := e^{\frac{2\pi i h A(n)}{q}}.$$

Then the L-function associated to $\psi_{h/q}$ is defined by the absolutely convergent series

$$L(s, \psi_{h/q}) := \sum_{n=1}^{\infty} \psi_{h/q}(n) n^{-s}, \tag{2}$$

in the region $\Re(s) > 1$, and has an Euler product representation (product over rational primes) of the form

$$L(s, \psi_{h/q}) := \prod_{p} \left(1 - \frac{e^{\frac{2\pi i h p}{q}}}{p^s}\right)^{-1}. \tag{3}$$

The Euler product (3) converges absolutely to a non-vanishing function for $\Re(s) > 1$. We would like to show it has analytic continuation to a larger region.

Lemma 2.1. *Let* $\Re(s) > 1$. *Then*

$$\log\left(L(s, \psi_{h/q})\right) = \sum_{p} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + T_{h,q}(s)$$

where, for any $\varepsilon > 0$, *the function*

$$T_{h,q}(s) := \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}$$

is holomorphic for $\Re(s) > \frac{1}{2} + \varepsilon$ *and satisfies* $|T_{h,q}(s)| = \mathcal{O}_{\varepsilon}(1)$ *where the* $\mathcal{O}_{\varepsilon}$-*constant is independent of* q *and depends at most on* ε.

Proof. Taking log's, we obtain

$$\begin{aligned} \log\left(L(s, \psi_{h/q})\right) &= \sum_{p} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}}}{k\, p^{sk}} \\ &= \sum_{p} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}. \end{aligned}$$

Hence, we may take

$$T_{h,q}(s) = \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}},$$

which is easily seen to converge absolutely for $\Re(s) > \frac{1}{2}$. □

For $q > 2$, let χ denote a Dirichlet character $\pmod q$ with associated Gauss sum $\tau(\chi)$. We also let χ_0 be the trivial character $\pmod q$.

We require the following lemma.

Lemma 2.2. *Let $h, q \in \mathbf{Z}$ with $q > 2$ and $(h, q) = 1$. Then*

$$e^{\frac{2\pi i h}{q}} = \left(\frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi) \cdot \overline{\chi(h)} \right) + \frac{\mu(q)}{\phi(q)}.$$

Proof. Since $(h, q) = 1$, it follows that for $\chi \pmod q$ with $\chi \neq \chi_0$,

$$\tau(\chi)\,\overline{\chi(h)} = \sum_{\ell=1}^{q} \chi(\ell) e^{\frac{2\pi i \ell h}{q}}.$$

This implies that

$$\begin{aligned}\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi)\,\overline{\chi(h)} &= (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} + \sum_{\substack{\ell=2 \\ (\ell,q)=1}}^{q} \left(\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \chi(\ell) \right) e^{\frac{2\pi i \ell h}{q}} \\ &= (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} - \sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} + e^{\frac{2\pi i h}{q}}.\end{aligned}$$

The proof is completed upon noting that the Ramanujan sum on the right side above can be evaluated as

$$\sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} = \sum_{d \mid (q,h)} \mu\left(\frac{q}{d}\right) d \; = \; \mu(q).$$ □

Theorem 2.3. *Let $s \in \mathbf{C}$ with $\Re(s) > 1$. Then we have the representation*

$$L(s, \psi_{h/q}) = \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(s, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot \zeta(s)^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(s)},$$

where

$$U_{h,q}(s) := -\frac{\mu(q)}{\phi(q)} \sum_{p \mid q} \sum_{k=1}^{\infty} \frac{1}{k p^{sk}} + \sum_{p \mid q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}.$$

Proof. If we combine lemmas (2.1) and (2.2) it follows that for $\Re(s) > 1$,

$$\log\big(L(s,\psi_{h/q})\big) = \sum_p \sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} \;+\; T_{h,q}(s)$$

$$= \sum_{p\nmid q}\sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} \;+\; \sum_{p\mid q}\sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \;T_{h,q}(s)$$

$$= \sum_{p\nmid q}\sum_{k=1}^\infty \frac{\left(\frac{1}{\phi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\chi)\cdot\overline{\chi(h\,p^k)} \;+\; \frac{\mu(q)}{\phi(q)}\right)}{k\, p^{sk}} \;+\; \sum_{p\mid q}\sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} + \;T_{h,q}(s).$$

Hence

$$\log\big(L(s,\psi_{h/q})\big) = \frac{1}{\phi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\chi)\overline{\chi(h)}\,\log(L(s,\overline{\chi}) \;+\; \frac{\mu(q)}{\phi(q)}\log\big(\zeta(s)\big)$$

$$-\frac{\mu(q)}{\phi(q)}\sum_{p|q}\sum_{k=1}^\infty \frac{1}{kp^{sk}} \;+\; \sum_{p\mid q}\sum_{k=1}^\infty \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} \;+\; T_{h,q}(s).$$

The theorem immediately follows after taking exponentials. □

The representation of $L(s,\psi_{h/q})$ given in theorem 2.3 allows one to analytically continue the function $L(s,\psi_{h/q})$ to a larger region which lies to the left of the line $\Re(s) = 1+\varepsilon$ $(\varepsilon > 0)$. This is a region which does not include the branch points of $L(s,\psi_{h/q})$ at the zeros and poles of $L(s,\chi)$, $\zeta(s)$.

Assume that $q > 1$ and $\chi \pmod q$. It is well known (see [2]) that the Dirichlet L-functions $L(\sigma+it,\chi))$ do not vanish in the region

$$\sigma \ge \begin{cases} 1-\frac{c_1}{\log q|t|} & \text{if } |t|\ge 1,\\ 1-\frac{c_2}{\log q} & \text{if } |t|\le 1,\end{cases} \qquad \text{(for absolute constants } c_1,c_2>0), \tag{4}$$

unless χ is the exceptional real character which has a simple real zero (Siegel zero) near $s=1$.

Similarly, $\zeta(\sigma+it)$ does not vanish for

$$\sigma \ge 1-\frac{c_3}{\log(|t|+2)}, \qquad \text{(for an absolute constant } c_3>0). \tag{5}$$

Assume $q>1$ and that there is no exceptional real character (mod q). It follows from (4) and (5) that $L(s,\psi_{h/q})$ is holomorphic in the region to the right of the contour $\mathscr{C}_q$ displayed in Figure 1.

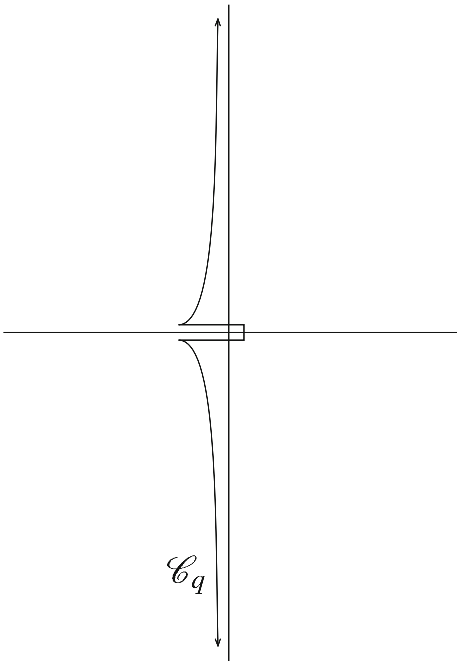


Fig. 1 The contour $\mathscr{C}_q$

To construct the contour $\mathscr{C}_q$ first take a slit along the real axis from $1 - \frac{c_2}{\log q}$ to 1 and construct a line just above and just below the slit. Then take two asymptotes to the line $\Re(s) = 1$ with the property that if $\sigma + it$ is on the asymptote and $|t| \geq 1$, then σ satisfies (4). If $q = 1$, we do a similar construction using (5).

3 Proof of theorem 1.1

The proof of theorem 1.1 is based on the following theorem.

Theorem 3.1. *Let h, q be fixed coprime integers with $q > 2$ and $\mu(q) \neq 0$. Then for $x \to \infty$ there exist absolute constants $c, c' > 0$ such that*

$$\sum_{n\le x} e^{2\pi i \frac{hA(n)}{q}}$$

$$= \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{H_{h,q}(\sigma)} \, \frac{x^\sigma}{\sigma} \, d\sigma$$

$$+ \mathscr{O}\left(xe^{-c'\sqrt{\log x}}\right).$$

On the other hand if $\mu(q) = 0$, *then* $\sum_{n\le x} e^{2\pi i \frac{hA(n)}{q}} = \mathscr{O}\left(xe^{-c'\sqrt{\log x}}\right)$.

Proof. The proof of theorem 3.1 relies on the following lemma taken from [2].

Lemma 3.2. *Let*

$$\delta(x) := \begin{cases} 0, & \text{if } 0 < x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 1, & \text{if } x > 1, \end{cases}$$

then for $x, T > 0$, *we have*

$$\left| \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} \frac{x^s}{s} \, ds \; - \; \delta(x) \right| \; < \; \begin{cases} x^c \cdot \min\left(1, \frac{1}{T|\log x|}\right), & \text{if } x \neq 1, \\ cT^{-1}, & \text{if } x = 1. \end{cases}$$

It follows from lemma 3.2, for $x, T \gg 1$ and $c = 1 + \frac{1}{\log x}$, that

$$\frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds = \sum_{n\le x} \psi_{h/q}(n) \; + \; \mathscr{O}\left(\frac{x\log x}{T}\right). \tag{6}$$

Fix large constants $c_1, c_2 > 0$. Next, shift the integral in (6) to the left and deform the line of integration to a contour

$$L^+ \; + \; \mathscr{C}_{T,x} \; + \; L^-$$

as in figure 2 below which contains two short horizontal lines:

$$L^\pm = \left\{ \sigma \pm iT \;\middle|\; 1 - \frac{c_1}{\log qT} \;\le\; \sigma \;\le\; 1 + \frac{1}{\log x} \right\},$$

together with the contour $C_{T,x}$ which is similar to C_q except that the two curves asymptotic to the line $\Re(s) = 1$ go from $1 - \frac{c_1}{\sqrt{\log qT}} + iT$ to $1 - \frac{c_2}{\sqrt{\log x}} + i\varepsilon$ and $1 - \frac{c_2}{\sqrt{\log x}} - i\varepsilon$ to $1 - \frac{c_1}{\sqrt{\log qT}} - iT$, respectively, for $0 < \varepsilon \to 0$.

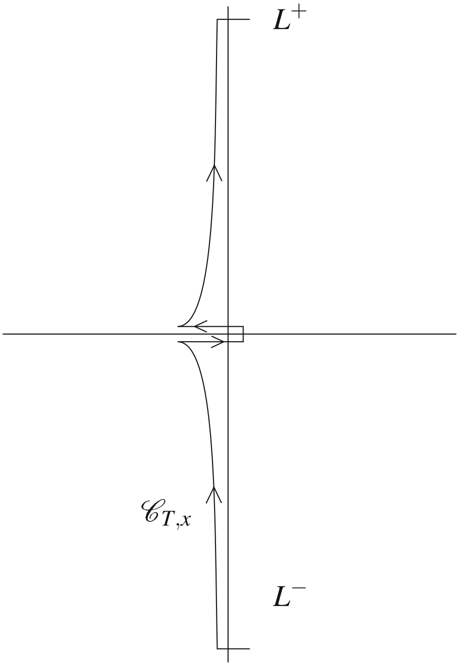


Fig. 2 The contour $\mathscr{C}_{T,x}$

Now, by the zero-free regions (4), (5), the region to the right of the contour $L^+ + \mathscr{C}_{T,x} + L^-$ does not contain any branch points or poles of the L-functions $L(s, \chi)$ for any $\chi \pmod q$. It follows that

$$\frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds \;=\; \frac{1}{2\pi i} \left(\int_{L^+} + \int_{\mathscr{C}_\varepsilon} + \int_{L^-} \right) L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds. \quad (7)$$

The main contribution for the integral along $L^+ + \mathscr{C}_{T,x} + L^-$ in (7) comes from the integrals along the straight lines above and below the slit on the real axis $\left[1 - \frac{c_2}{\sqrt{\log x}},\ 1\right]$. These integrals cancel if the function $L\big(s, \psi_{h/q}\big)$ has no branch points or poles on the slit. It follows from theorem 2.3 that this will be the case if $\mu(q) = 0$. The remaining integrals in (7) can then be estimated as in the proof of the prime number theorem for arithmetic progressions (see [2]), yielding an error term of the form $\mathscr{O}\left(x e^{-c' \sqrt{\log x}}\right)$. This proves the second part of theorem 3.1.

Next, assume $\mu(q) \neq 0$. In this case $L(s, \psi_{h/q})$ has a branch point at $s = 1$ coming from the Riemann zeta function, it is necessary to keep track of the change in argument. Let 0^+i denote the upper part of the slit and let 0^-i denote the lower part of the slit. Then we have $\log[\zeta(\sigma + 0^+ i) = \log|\zeta(\sigma)| - i\pi$ and $\log[\zeta(\sigma + 0^- i) = \log|\zeta(\sigma)| + i\pi$.

By the standard proof of the prime number theorem for arithmetic progressions it follows that (with an error $\mathscr{O}\big(e^{-c'\sqrt{\log x}}\big)$) the right hand side of (7) is asymptotic to

$$\mathscr{I}_{\text{slit}} := \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \Big[\exp\Big(\log\big(L\left(\sigma + 0^+ i,\ \psi_{h/q}\right)\big)\Big) - \exp\Big(\log\big(L\left(\sigma - 0^- i,\ \psi_{h/q}\right)\big)\Big)\Big] \frac{x^\sigma}{\sigma}\, d\sigma. \tag{8}$$

We may evaluate $\mathscr{I}_{\text{slit}}$ using theorem 2.3. This gives

$$\begin{aligned}
\mathscr{I}_{\text{slit}} &= \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot e^{U_{h,q}(\sigma)} \\
&\quad \cdot \left[\exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| - i\pi\Big)\right) - \exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| + i\pi\Big)\right)\right] \frac{x^\sigma}{\sigma}\, d\sigma \\
&= \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\, \frac{x^\sigma}{\sigma}\, d\sigma.
\end{aligned}$$

As in the previous case when $\mu(q) = 0$, the remaining integrals in (7) can then be estimated as in the proof of the prime number theorem for arithmetic progressions, yielding an error term of the form $\mathscr{O}\left(xe^{-c'\sqrt{\log x}}\right)$. This completes the proof of theorem 3.1. □

The proof of theorem 1.1 follows from theorem 3.1 if we can obtain an asymptotic formula for the integral

$$\mathscr{I}_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma,\ \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\, \frac{x^\sigma}{\sigma}\, d\sigma. \tag{9}$$

Since we have assumed q is fixed, it immediately follows that for arbitrarily large $c \gg 1$ and $x \to \infty$, we have

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \int_{1-\frac{c\log\log x}{\log x}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)} \frac{x^\sigma}{\sigma}\, d\sigma + \mathcal{O}\left(\frac{x}{(\log x)^c}\right).$$

Now, in the region $1 - \frac{c\log\log x}{\log x} \leq \sigma \leq 1$,

$$\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \cdot \frac{e^{H_{h,q}(\sigma)}}{\sigma} = \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} + \mathcal{O}\left(\frac{\log\log x}{\log x}\right).$$

Consequently,

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} \int_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}} x^\sigma\, d\sigma$$
$$+ \mathcal{O}\left(\frac{\log\log x}{\log x} \left| \int_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}} x^\sigma\, d\sigma \right| \right). \tag{10}$$

It remains to compute the integral of $|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}}$ occurring in (10). For σ very close to 1, we have

$$|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma-1|} + \mathcal{O}(1)\right)^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma-1|}\right)^{\frac{\mu(q)}{\phi(q)}} + \mathcal{O}\left(\left(\frac{1}{|\sigma-1|}\right)^{\frac{\mu(q)}{\phi(q)}-1}\right).$$

It follows that

$$\int_{1-\frac{c\log\log x}{\log x}}^{1} |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} x^\sigma\, d\sigma = \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} + \mathcal{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right). \tag{11}$$

Combining equations (10) and (11) we obtain

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\pi}{\phi(q)}\right)}{\pi} \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} e^{U_{h,q}(1)} \frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} + \mathscr{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right).$$

Remark: As pointed out to me by Gérald Tenenbaum, it is also possible to deduce Corollary 1.2 directly from theorem 2.3 by using theorem II.5.2 of [8]. In this manner one can obtain an explicit asymptotic expansion which, furthermore, is valid for values of q tending to infinity with x.

4 Examples of equidistribution (mod 3) and (mod 9)

Equidistribution (mod 3): Theorem (1.1) says that for $h = 1,\ q = 3$:

$$\sum_{n \le x} e^{\frac{2\pi i A(n)}{3}} = \frac{-V_{1,3}}{\pi} \Gamma\left(\frac{3}{2}\right) \prod_{\substack{\chi \pmod 3 \\ \chi \neq \chi_0}} L(1, \chi)^{\frac{G(\overline{\chi})}{2}} \frac{x}{(\log x)^{\frac{3}{2}}} \left(1 + \mathscr{O}\left(\frac{1}{\log x}\right)\right)$$

$$\approx (-0.503073 + 0.24042\, i) \frac{x}{(\log x)^{\frac{3}{2}}}.$$

We computed the above sum for $x = 10^7$ and obtained

$$\sum_{n \le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -98,423.00 + 55,650.79\, i.$$

Our theorem predicts that

$$\sum_{n \le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -88,870.8 + 42,471.7\, i.$$

Since $\log\left(10^7\right) \approx 16.1$ is small, this explains the discrepancy between the actual and predicted results.

As $x \to \infty$, we have

$$\sum_{\substack{n \le x \\ A(n) \equiv a \pmod 3}} = \frac{1}{3} \sum_{h=0}^{2} \sum_{n \le x} e^{\frac{2\pi i A(n) h}{3}} e^{-\frac{2\pi i h a}{3}} = \frac{x}{3} + c_a \frac{x}{(\log x)^{\frac{3}{2}}} + \mathscr{O}\left(\frac{x}{(\log x)^{\frac{5}{2}}}\right)$$

where

$$c_0 = -0.335382, \qquad c_1 \approx 0.306498, \qquad c_2 \approx 0.0288842.$$

Equidistribution (mod 9):

Our theorem says that for $h \neq 3, 6$ $(1 \leq h < 9)$ and $q = 9$:

$$\sum_{n \leq x} e^{\frac{2\pi i h A(n)}{9}} = \mathcal{O}\left(x\, e^{-c_0\sqrt{\log x}}\right).$$

Surprisingly!! there is a huge amount of cancellation when $x = 10^7$:

$$\sum_{n \leq 10^7} e^{\frac{2\pi i h A(n)}{9}} \approx \begin{cases} -315.2 - 140.4\, i & \text{if } h = 1, \\ 282.2 - 543.4\, i & \text{if } h = 2, \\ 94.5 + 321.9\, i & \text{if } h = 4, \\ 94.5 - 321.9\, i & \text{if } h = 5, \\ 282.2 + 543.4\, i & \text{if } h = 7, \\ -315.2 + 140.4\, i & \text{if } h = 8. \end{cases}$$

Acknowledgements The author would like to thank Ada Goldfeld for creating the figures in this paper and would also like to thank Wladyslaw Narkiewiz for pointing out the reference [3].

References

1. K. Alladi, P. Erdős, On an additive arithmetic function. Pac. J. Math. **71**(2), 275–294 (1977)
2. H. Davenport, *Multiplicative Number Theory*, vol. 74, 2nd edn., Graduate Texts in Mathematics (Springer, Berlin, 1967). (revised by Hugh Montgomery)
3. H. Delange, On integral valued additive functions. J. Number Theory **1**, 419–430 (1969)
4. A. Ivić, *On Certain Large Additive Functions, (English summary) Paul Erdős and His Mathematics I*, vol. 11, Bolyai Society Mathematical Studies (János Bolyai Mathematical Society, Budapest, 1999,2002), pp. 319–331
5. A. Ivić, P. Erdős, Estimates for sums involving the largest prime factor of an integer and certain related additive functions. Studia Sci. Math. Hungar. **15**(1–3), 183–199 (1980)
6. A, Ivić, P. Erdős, The distribution of quotients of small and large additive functions, II, in *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, (University of Salerno, Salerno, 1992), pp. 83–93
7. A. Selberg, Note on a paper by L.G. Sathe. J. Indian Math. Soc. (N.S.) **18**, 83–87 (1954)
8. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, vol. 163, 3rd edn., Graduate Studies in Mathematics (American Mathematical Society, Providence, 2015). Translated from the 2008 French edition by Patrick D.F. Ion

Ramanujan's Tau Function

Michael D. Hirschhorn

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Ramanujan's tau function is defined by

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1 - q^n)$. It is known that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right),$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n. We give proofs of this relation for $p = 2,\ 3,\ 5,\ 7$ and 13. which rely on nothing more than Jacobi's triple product identity. I believe that the case $p = 11$ is intrinsically more difficult, and I do not attempt it here.

Keywords Ramanujan's tau function · Jacobi's triple product identity · Theta series

2010 Mathematics Subject Classification 11B65 · 11F11 · 11F20 · 11F25

M. D. Hirschhorn (✉)
School of Mathematics and Statistics, UNSW, Sydney, NSW 2052, Australia
e-mail: m.hirschhorn@unsw.edu.au

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_18

1 Introduction

Ramanujan's tau function is defined by

$$\sum_{n\geq 1}\tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1-q^n)$.

The tau function has many fascinating properties.

One of these is that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right), \tag{1.1}$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n.

It follows easily from (1.1) that tau is multiplicative,

$$\tau(mn) = \tau(m)\tau(n) \tag{1.2}$$

provided m and n have no common divisor other than 1,
and that, at least formally,

$$\sum_{n\geq 1}\frac{\tau(n)}{n^s} = \prod_{p \text{ prime}}\left(1 - \frac{\tau(p)}{p^s} + \frac{1}{p^{2s-11}}\right)^{-1}. \tag{1.3}$$

I have found proofs of (1.1) for $p = 2,\ 3,\ 5,\ 7$ and 13 which require nothing more than Jacobi's triple product identity,

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty}a^nq^{n^2}. \tag{1.4}$$

Completely different elementary proofs of (1.1) for $p = 2$ and 3 have recently been given by Kenneth S. Williams [6].

A modern proof of (1.1) may be found in [2].

2 $p = 2$

Let

$$\phi(q) = \sum_{-\infty}^{\infty}q^{n^2}, \quad \psi(q) = \sum_{n\geq 0}q^{(n^2+n)/2} = \sum_{-\infty}^{\infty}q^{2n^2+n}.$$

It can be shown with, or even without, (1.4), [4, chap. 1] that

$$\phi(q)\phi(-q) = \phi(-q^2)^2 \text{ and } \phi(q)\psi(q^2) = \psi(q)^2.$$

Also, by (1.4),

$$\phi(-q) = \frac{E(q)^2}{E(q^2)} \text{ and } \psi(q) = \frac{E(q^2)^2}{E(q)}.$$

It is easy to see that

$$\phi(q) = \phi(q^4) + 2q\psi(q^8) \tag{2.1}$$

Put $-q$ for q in (2.1).

$$\phi(-q) = \phi(q^4) - 2q\psi(q^8). \tag{2.2}$$

Multiply (2.1) by (2.2).

$$\phi(-q^2)^2 = \phi(q^4)^2 - 4q^2\psi(q^8)^2. \tag{2.3}$$

Put q for q^2 in (2.3).

$$\phi(-q)^2 = \phi(q^2)^2 - 4q\psi(q^4)^2. \tag{2.4}$$

Put $-q$ for q in (2.4).

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{2.5}$$

Multiply (2.4) by (2.5).

$$\phi(-q^2)^4 = \phi(q^2)^4 - 16q^2\psi(q^4)^4. \tag{2.6}$$

Put q for q^2 in (2.6) and rearrange.

$$\phi(q)^4 - \phi(-q)^4 = 16q\psi(q^2)^4. \tag{2.7}$$

We have

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24} \tag{2.8}$$

$$= qE(q^2)^{12}\left(\frac{E(q)^2}{E(q^2)}\right)^{12}$$

$$= qE(q^2)^{12}\phi(-q)^{12}$$

$$= qE(q^2)^{12}\left(\phi(-q)^2\right)^6$$
$$= qE(q^2)^{12}\left(\phi(q^2)^2 - 4q\psi(q^4)^2\right)^6$$

$$= qE(q^2)^{12}\left(\phi(q^2)^{12} - 24q\phi(q^2)^{10}\psi(q^4)^2 + 240q^2\phi(q^2)^8\psi(q^4)^4 \right.$$
$$-1280q^3\phi(q^2)^6\psi(q^4)^6 + 3840q^4\phi(q^2)^4\psi(q^4)^8 - 6144q^5\phi(q^2)^2\psi(q^4)^4$$
$$\left. + 4096q^6\psi(q^4)^6\right).$$

If we extract the even powers and replace q^2 by q, we obtain

$$\sum_{n\geq 1}\tau(2n)q^n = -8qE(q)^{12}\phi(q)^2\psi(q^2)^2\left(3\phi(q)^8 + 160q\phi(q)^4\psi(q^2)^4 + 768q^2\psi(q^2)^8\right) \tag{2.9}$$

$$= -8qE(q)^{12}\left(\phi(q)\psi(q^2)\right)^2\left(3\left(\phi(q)^4 - 16q\psi(q^2)^4\right)^2 + 256q\phi(q)^4\psi(q^2)^4\right)$$

$$= -8qE(q)^{12}\psi(q)^4\left(3\left(\phi(-q)^4\right)^2 + 256q\psi(q)^8\right)$$

$$= -8qE(q)^{12}\left(\frac{E(q^2)^2}{E(q)}\right)^4\left(3\left(\frac{E(q)^2}{E(q^2)}\right)^8 + 256q\left(\frac{E(q^2)^2}{E(q)}\right)^8\right)$$
$$= -24qE(q)^{24} - 2^{11}q^2E(q^2)^{24}$$

$$= -24\sum_{n\geq 1}\tau(n)q^n - 2^{11}\sum_{n\geq 1}\tau(n)q^{2n}.$$

The term $n = 1$ in (2.9) gives

$$\tau(2) = -24\tau(1) = -24,$$

so (2.9) becomes

$$\sum_{n\geq 1}\tau(2n)q^n = \tau(2)\sum_{n\geq 1}\tau(n)q^n - 2^{11}\sum_{n\geq 1}\tau(n)q^{2n},$$

as claimed.

Aside: (2.7) can be written

$$\left(\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\right)^4 - \left(\prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n}\right)^4 \tag{2.10}$$
$$= 16q\left(\prod_{n\geq 1}\frac{(1-q^{4n})^2}{(1-q^{2n})}\right)^4.$$

If we divide (2.10) by $\prod_{n\geq 1}(1-q^{2n})^4$, we find

$$\prod_{n\geq 1}(1+q^{2n-1})^8 - \prod_{n\geq 1}(1-q^{2n-1})^8 = 16q\prod_{n\geq 1}(1+q^{2n})^8. \tag{2.11}$$

Jacobi described (2.11) as *"aequatio identica satis abstrusa"*. ("A fairly obscure identity".)

(2.11) can perhaps most strikingly be written [4, chap. 19]

$$\mathbf{O}\Big(\prod_{\substack{n\geq 1 \\ n\not\equiv 0 \pmod 4}}(1-q^n)^8\Big) = -8q.$$

Observe that

$$\prod_{\substack{n\geq 1 \\ n\not\equiv 0 \pmod 4}}(1-q^n)^8 = 1-8q+20q^2-62q^4+216q^6-641q^8$$
$$+1636q^{10}-3778q^{12}+8248q^{14}+\cdots.$$

3 $p = 3$

Using Jacobi's formula for the cube of Euler's product, which follows from (1.4), namely

$$E(q)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}, \tag{3.1}$$

we have the 3-dissection

$$\begin{aligned} E(q)^3 &= \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \\ &= 1-3q+5q^3-7q^6+9q^{10}-11q^{15}+\cdots \\ &= A(q^3)-3qE(q^9)^3. \end{aligned} \tag{3.2}$$

So,

$$\begin{aligned} \sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24} \\ &= q\left(E(q)^3\right)^8 \\ &= q\left(A(q^3)-3qE(q^9)^3\right)^8 \end{aligned} \tag{3.3}$$

$$= q\Big(A(q^3)^8 - 24qA(q^3)^7E(q^9)^3 + 252q^2A(q^3)^6E(q^9)^6 - 1512q^3A(q^3)^5E(q^9)^9 + 5670q^4A(q^3)^4E(q^9)^{12} - 13608q^5A(q^3)^3E(q^9)^{15} + 20412q^6A(q^3)^2E(q^9)^{18} - 17496q^7A(q^3)E(q^9)^{21} + 6561q^8E(q^9)^{24}\Big).$$

If we extract those terms in which the power of q is a multiple of 3, and replace q^3 by q, we obtain

$$\sum_{n\geq 1}\tau(3n)q^n = 252qA(q)^6E(q^3)^6 - 13608q^2A(q)^3E(q^3)^{15} + 6561q^3E(q^3)^{24}. \tag{3.4}$$

If we put q, ωq, $\omega^2 q$ for q in (3.2) and multiply the three results, we find

$$E(q)^3E(\omega q)^3E(\omega^2 q)^3 = A(q^3)^3 - 27q^3E(q^9)^9, \tag{3.5}$$

or,

$$\left(\frac{E(q^3)^4}{E(q^9)}\right)^3 = A(q^3)^3 - 27q^3E(q^9)^9. \tag{3.6}$$

If in (3.6) we replace q^3 by q and rearrange, we obtain

$$A(q)^3 = \frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9. \tag{3.7}$$

If we substitute (3.7) into (3.4), we obtain

$$\sum_{n\geq 1}\tau(3n)q^n = 252qE(q^3)^6\left(\frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9\right)^2 - 13608q^2E(q^3)^{15}\left(\frac{E(q)^{12}}{E(q^3)^3} + 27qE(q^3)^9\right) + 6561q^3E(q^3)^{24} \tag{3.8}$$

$$= 252qE(q)^{24} - 3^{11}q^3E(q^3)^{24}$$

$$= 252\sum_{n\geq 1}\tau(n)q^n - 3^{11}\sum_{n\geq 1}\tau(n)q^{3n}.$$

The term $n = 1$ in (3.8) gives

$$\tau(3) = 252\tau(1) = 252,$$

so (3.8) becomes

$$\sum_{n\geq 0} \tau(3n)q^n = \tau(3)\sum_{n\geq 1} \tau(n)q^n - 3^{11}\sum_{n\geq 1} \tau(n)q^{3n},$$

as claimed.

Aside: It can be shown [4, chap. 21] that

$$\begin{aligned} A(q) &= E(q)\left(1 + 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \\ &= E(q)\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \end{aligned}$$

4 $p = 5$

We have Euler's pentagonal numbers theorem (this follows from (1.4))

$$\begin{aligned} E(q) &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - - + + \cdots \\ &= E_0 + E_1 + E_2 \end{aligned} \tag{4.1}$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 5. ($i = 0,\ 1,\ 2.$)

It is easy to prove that

$$E_1 = -qE(q^{25}) \tag{4.2}$$

and, using Jacobi's formula for the cube of Euler's product (3.1),

$$(E_0 + E_1 + E_2)^3 = \prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \tag{4.3}$$

that

$$E_0E_2 = -E_1^2. \tag{4.4}$$

If we write

$$\alpha = -\frac{E_0}{E_1} \text{ and } \beta = -\frac{E_2}{E_1} \tag{4.5}$$

then $\alpha\beta = -1$ and

$$E(q) = qE(q^{25})\,(\alpha - 1 + \beta)\,. \tag{4.6}$$

We have

$$\sum_{n\geq 1}\tau(n)q^n = qE(q)^{24} \tag{4.7}$$

$$\begin{aligned}
&= q^{25}E(q^{25})^{24}(\alpha-1+\beta)^{24}\\
&= q^{25}E(q^{25})^{24}\big(\alpha^{24}-24\alpha^{23}+252\alpha^{22}-1472\alpha^{21}+4830\alpha^{20}-6072\alpha^{19}\\
&\quad -16192\alpha^{18}+78936\alpha^{17}-82731\alpha^{16}-212520\alpha^{15}+649704\alpha^{14}\\
&\quad -73416\alpha^{13}-1977862\alpha^{12}+2034672\alpha^{11}+3487260\alpha^{10}\\
&\quad -7072408\alpha^{9}-3432198\alpha^{8}+15343944\alpha^{7}+134596\alpha^{6}\\
&\quad -25077360\alpha^{5}+6067446\alpha^{4}+33474936\alpha^{3}-12286968\alpha^{2}\\
&\quad -38228232\alpha+14903725-38228232\beta-12286968\beta^{2}+33474936\beta^{3}\\
&\quad +6067446\beta^{4}-25077360\beta^{5}+134596\beta^{6}+15343944\beta^{7}-3432198\beta^{8}\\
&\quad -7072408\beta^{9}+3487260\beta^{10}+2034672\beta^{11}-1977862\beta^{12}-73416\beta^{13}\\
&\quad +649704\beta^{14}-212520\beta^{15}-82731\beta^{16}+78936\beta^{17}-16192\beta^{18}\\
&\quad -6072\beta^{19}+4830\beta^{20}-1472\beta^{21}+252\beta^{22}-24\beta^{23}+\beta^{24}\big).
\end{aligned}$$

If we extract those terms in which the power of q is a multiple of 5, we obtain

$$\begin{aligned}\sum_{n\geq 1}\tau(5n)q^{5n} &= q^{25}E(q^{25})^{24}\Big(4830\alpha^{20}-212520\alpha^{15}+3487260\alpha^{10}-25077360\alpha^{5}\\
&\quad +14903725-25077360\beta^{5}+3487260\beta^{10}-212520\beta^{15}+4830\beta^{20}\Big).\end{aligned} \tag{4.8}$$

Miraculously, this can be written

$$\sum_{n\geq 1}\tau(5n)q^{5n} = q^{25}E(q^{25})^{24}\left(4830(\alpha^5-11+\beta^5)^4-5^{11}\right). \tag{4.9}$$

If in (4.6) we replace q by q, ηq, $\eta^2 q$, $\eta^3 q$ and $\eta^4 q$ where η is a fifth root of unity other than 1, and multiply the five results, we obtain

$$E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) = q^5E(q^{25})^5(\alpha^5-11+\beta^5). \tag{4.10}$$

or,

$$\alpha^5-11+\beta^5 = \frac{E(q^5)^6}{q^5E(q^{25})^6}. \tag{4.11}$$

If we substitute (4.11) into (4.9) we find

$$\begin{aligned}\sum_{n\geq 1}\tau(5n)q^{5n} &= q^{25}E(q^{25})^{24}\left(4830\left(\frac{E(q^5)^6}{q^5E(q^{25})^6}\right)^4-5^{11}\right)\\
&= 4830q^5E(q^5)^{24}-5^{11}q^{25}E(q^{25})^{24}.\end{aligned} \tag{4.12}$$

If in (4.12) we replace q^5 by q, we obtain

$$\sum_{n\geq 1} \tau(5n)q^n = 4830q E(q)^{24} - 5^{11} q^5 E(q^5)^{24} \tag{4.13}$$

$$= 4830 \sum_{n\geq 1} \tau(n)q^n - 5^{11} \sum_{n\geq 1} \tau(n)q^{5n}.$$

The term $n = 1$ in (4.13) gives

$$\tau(5) = 4830\tau(1) = 4830,$$

so (4.13) becomes

$$\sum_{n\geq 1} \tau(5n)q^n = \tau(5) \sum_{n\geq 1} \tau(n)q^n - 5^{11} \sum_{n\geq 1} \tau(n)q^{5n},$$

as claimed.

Aside: It can be shown [4, chap. 8] that

$$\alpha = r(q^5)^{-1}, \quad \beta = -r(q^5),$$

where

$$r(q) = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}}$$

$$= q^{\frac{1}{5}} \prod_{n\geq 0} \frac{(1 - q^{5n+1})(1 - q^{5n+4})}{(1 - q^{5n+2})(1 - q^{5n+3})}.$$

5 $p = 7$

We can write

$$E(q) = E_0 + E_1 + E_2 + E_5 \tag{5.1}$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 7. ($i = 0,\ 1,\ 2,\ 5.$)

It is easy to show that

$$E_2 = -q^2 E(q^{49}). \tag{5.2}$$

If we write

$$\alpha = -\frac{E_0}{E_2}, \quad \beta = -\frac{E_1}{E_2}, \quad \gamma = -\frac{E_5}{E_2}, \tag{5.3}$$

then

$$E(q) = q^2 E(q^{49})(\alpha + \beta - 1 + \gamma). \tag{5.4}$$

Jacobi's identity (3.1) yields

$$\alpha\beta\gamma = -1, \tag{5.5}$$

$$-\alpha^2 + \alpha\beta^2 + \gamma = 0, \tag{5.6}$$

$$\alpha - \beta^2 + \beta\gamma^2 = 0 \tag{5.7}$$

and

$$\alpha^2\gamma + \beta - \gamma^2 = 0. \tag{5.8}$$

We have

$$\begin{aligned} \sum_{n\geq 1} \tau(n)q^n &= qE(q)^{24} \\ &= q^{49}E(q^{49})^{24}(\alpha + \beta - 1 + \gamma)^{24}. \end{aligned} \tag{5.9}$$

We can expand the right side of (5.9) and extract those terms in which the power of q is a multiple of 7. Thus, if H is the Huffing operator modulo 7, given by

$$H\left(\sum_n a(n)q^n\right) = \sum_n a(7n)q^{7n}, \tag{5.10}$$

and if we apply H to (5.9), we find

$$\sum_{n\geq 1} \tau(7n)q^{7n} = q^{49}E(q^{49})^{24} H\left((\alpha + \beta - 1 + \gamma)^{24}\right). \tag{5.11}$$

Let

$$\zeta = \alpha + \beta - 1 + \gamma = \frac{E(q)}{q^2E(q^{49})}. \tag{5.12}$$

Then (5.11) becomes

$$\sum_{n\geq 1} \tau(7n)q^{7n} = q^{49}E(q^{49})^{24}H(\zeta^{24}). \tag{5.13}$$

Now,

$$H(\zeta^0) = H(1) = 1, \tag{5.14}$$

$$H(\zeta) = H(\alpha + \beta - 1 + \gamma) = -1, \tag{5.15}$$

$$H(\zeta^2) = H(\alpha^2 + 2\alpha\beta + (\beta^2 - 2\alpha) - 2\beta + 1 + 2\alpha\gamma + 2\beta\gamma + 2\gamma + \gamma^2) = 1 \tag{5.16}$$

and from Jacobi's identity (3.1)

$$\begin{aligned} E(q)^3 &= \sum_{n \geq 0} (-1)^n (2n+1) q^{(n^2+n)/2} \\ &= A(q^7) - 3qB(q^7) + 5q^3 C(q^7) - 7q^6 E(q^{49})^3 \end{aligned} \tag{5.17}$$

it follows that

$$H(\zeta^3) = -7. \tag{5.18}$$

It can also be shown (see [3] or [4, chap. 7]) that

$$H(\zeta^4) = -4T - 7, \tag{5.19}$$

$$H(\zeta^5) = 10T + 49 \tag{5.20}$$

and

$$H(\zeta^6) = 49 \tag{5.21}$$

where

$$T = \frac{E(q^7)^4}{q^7 E(q^{49})^4}. \tag{5.22}$$

It can then be shown that ζ satisfies the so-called modular equation

$$\zeta^7 + 7\zeta^6 + 21\zeta^5 + 49\zeta^4 + (7T + 147)\zeta^3 + (35T + 343)\zeta^2 + (49T + 343)\zeta - T^2 = 0. \tag{5.23}$$

It follows that for $i \geq 0$,

$$\begin{aligned} H(\zeta^{i+7}) + 7H(\zeta^{i+6}) + 21H(\zeta^{i+5}) + 49\zeta^{i+4}) + (7T + 147)H(\zeta^{i+3}) \\ +(35T + 343)H(\zeta^{i+2}) + (49T + 343)H(\zeta^{i+1}) - T^2 H(\zeta^i) = 0. \end{aligned} \tag{5.24}$$

Now write

$$u_i = H(\zeta^i). \tag{5.25}$$

Then

$$u_0 = 1, \quad u_1 = -1, \quad u_2 = 1, \quad u_3 = 7, \quad u_4 = -4T - 7, \quad u_5 = 10T + 49, \quad u_6 = 49 \tag{5.26}$$

and for $i \geq 0$,

$$u_{i+7} + 7u_{i+6} + 21u_{i+5} + 49u_{i+4} + (7T + 147)u_{i+3} \\ +(35T + 343)u_{i+2} + (49T + 343)u_{i+1} - T^2 u_i = 0. \tag{5.27}$$

If we write

$$U = \sum_{i \geq 0} u_i z^i \tag{5.28}$$

it follows from (5.26) and (5.27) that

$$(1 + 7z + 21z^2 + 49z^3 + (7T + 147)z^4 + (35T + 343)z^5 + (49T + 343)z^6 - T^2 z^7)U \\ = 1 + 6z + 15z^2 + 28z^3 + (3T + 63)z^4 + (10T + 98)z^5 + (7T + 49)z^6 \tag{5.29}$$

and so

$$U = \frac{1 + 6z + 15z^2 + 28z^3 + (3T + 63)z^4 + (10T + 98)z^5 + (7T + 49)z^6}{1 + 7z + 21z^2 + 49z^3 + (7T + 147)z^4 + (35T + 343T)z^5 + (49T + 343T)z^6 - T^2 z^7}. \tag{5.30}$$

If we expand the right side of (5.30) as a series, we find

$$u_{24} = -16744T^6 - 7^{11}. \tag{5.31}$$

That is,

$$H(\zeta^{24}) = -16744 \left(\frac{E(q^7)^4}{q^7 E(q^{49})^4} \right)^6 - 7^{11}. \tag{5.32}$$

If we substitute (5.32) into (5.13), we find

$$\sum_{n \geq 1} \tau(7n) q^{7n} = q^{49} E(q^{49})^{24} \left(-16744 \left(\frac{E(q^7)^4}{q^7 E(q^{49})^4} \right)^6 - 7^{11} \right) \\ = -16744 q^7 E(q^7)^{24} - 7^{11} q^{49} E(q^{49})^{24}. \tag{5.33}$$

If in (5.33) we replace q^7 by q, we obtain

$$\sum_{n \geq 1} \tau(7n) q^n = -16744 q E(q)^{24} - 7^{11} q^7 E(q^7)^{24} \tag{5.34}$$

$$= -16744 \sum_{n \geq 1} \tau(n) q^n - 7^{11} \sum_{n \geq 1} \tau(n) q^{7n}$$

The term $n = 1$ in (5.34) gives

$$\tau(7) = -16744\tau(1) = -16744,$$

so (5.33) becomes

$$\sum_{n\geq 1} \tau(7n)q^n = \tau(7)\sum_{n\geq 1} \tau(n)q^n - 7^{11}\sum_{n\geq 1} \tau(n)q^{7n},$$

as claimed.

Aside: It can be shown [4, chap. 10], using the quintuple product identity, that

$$\alpha = q^{-2}\prod_{n\geq 0} \frac{(1-q^{49n+14})(1-q^{49n+35})}{(1-q^{49n+7})(1-q^{49n+42})},$$
$$\beta = -q^{-1}\prod_{n\geq 0} \frac{(1-q^{49n+21})(1-q^{49n+28})}{(1-q^{49n+14})(1-q^{49n+35})},$$
$$\gamma = q^{3}\prod_{n\geq 0} \frac{(1-q^{49n+7})(1-q^{49n+42})}{(1-q^{49n+21})(1-q^{49n+28})}.$$

6 $p = 13$

Define

$$\zeta = \frac{E(q)}{q^7 E(q^{169})}, \quad T = \frac{E(q^{13})^2}{q^{13}E(q^{169})^2}. \tag{6.1}$$

The following results may be proved in a fashion similar to the proofs of (5.15)–(5.21), using the work of O'Brien [5], Part 1, Sections 1–3 and Bilgici and Ekin [1]. We omit the details.

$$H(\zeta) = 1, \tag{6.2}$$
$$H(\zeta^2) = -2T - 1, \tag{6.3}$$
$$H(\zeta^3) = 13, \tag{6.4}$$
$$H(\zeta^4) = 2T^2 - 13, \tag{6.5}$$
$$H(\zeta^5) = -20T^2 - 10\times 13T - 13^2, \tag{6.6}$$
$$H(\zeta^6) = 10T^3 - 13^2, \tag{6.7}$$
$$H(\zeta^7) = 98T^3 + 28\times 13T^2 - 13^3, \tag{6.8}$$
$$H(\zeta^8) = -70T^4 - 13^3, \tag{6.9}$$
$$H(\zeta^9) = -162T^4 + 108\times 13T^3 + 72\times 13^2T^2 + 18\times 13^3T + 13^4, \tag{6.10}$$
$$H(\zeta^{10}) = 238T^5 - 13^4, \tag{6.11}$$
$$H(\zeta^{11}) = -902T^5 - 1672\times 13T^4 - 792\times 13^2T^3 - 198\times 13^3T^2 - 22\times 13^4T - 13^5, \tag{6.12}$$
$$H(\zeta^{12}) = -418T^6 - 13^5. \tag{6.13}$$

For $0 \leq i \leq 12$ let

$$\zeta_i = \zeta(\eta^i q) \tag{6.14}$$

where η is a 13th root of unity other than 1.

Then

$$\sum_i \zeta_i = 13, \tag{6.15}$$

$$\sum_i \zeta_i^2 = -2 \times 13T - 13, \tag{6.16}$$

$$\sum_i \zeta_i^3 = 13^2, \tag{6.17}$$

$$\sum_i \zeta_i^4 = 2 \times 13T^2 - 13^2, \tag{6.18}$$

$$\sum_i \zeta_i^5 = -20 \times 13T^2 - 10 \times 13^2 T - 13^3, \tag{6.19}$$

$$\sum_i \zeta_i^6 = 10 \times 13T^3 - 13^3, \tag{6.20}$$

$$\sum_i \zeta_i^7 = 98 \times 13T^3 + 28 \times 13^2 T^2 - 13^4, \tag{6.21}$$

$$\sum_i \zeta_i^8 = -70 \times 13T^4 - 13^4, \tag{6.22}$$

$$\sum_i \zeta_i^9 = -162 \times 13T^4 + 108 \times 13^2 T^3 + 72 \times 13^3 T^2 + 18 \times 13^4 T + 13^5, \tag{6.23}$$

$$\sum_i \zeta_i^{10} = 238 \times 13T^5 - 13^5, \tag{6.24}$$

$$\sum_i \zeta_i^{11} = -902 \times 13T^5 - 1672 \times 13^2 T^4 - 792 \times 13^3 T^3 - 198 \times 13^4 T^2 - 22 \times 13^5 T - 13^6, \tag{6.25}$$

$$\sum_i \zeta_i^{12} = -418 \times 13T^6 - 13^6. \tag{6.26}$$

From these we obtain the symmetric functions,

$$\sigma_1 = \sum_i \zeta_i = 13, \tag{6.27}$$

$$\sigma_2 = \sum_{i<j} \zeta_i \zeta_j = 13T + 7 \times 13, \tag{6.28}$$

$$\sigma_3 = \sum_{i<j<k} \zeta_i \zeta_j \zeta_k = 13^2 T + 3 \times 13^2, \tag{6.29}$$

$$\sigma_4 = 6 \times 13T^2 + 7 \times 13^2 T + 15 \times 13^2, \tag{6.30}$$

$$\sigma_5 = 74 \times 13T^2 + 37 \times 13^2 T + 5 \times 13^3, \tag{6.31}$$

$$\sigma_6 = 20 \times 13T^3 + 38 \times 13^2 T^2 + 13^4 T + 19 \times 13^3, \tag{6.32}$$

$$\sigma_7 = 222 \times 13T^3 + 184 \times 13^2 T^2 + 51 \times 13^3 T + 5 \times 13^4, \tag{6.33}$$

$$\sigma_8 = 38 \times 13T^4 + 102 \times 13^2 T^3 + 56 \times 13^3 T^2 + 13^5 T + 15 \times 13^4, \tag{6.34}$$

$$\sigma_9 = 346 \times 13T^4 + 422 \times 13^2 T^3 + 184 \times 13^3 T^2 + 37 \times 13^4 T + 3 \times 13^5, \tag{6.35}$$

$$\sigma_{10} = 36 \times 13T^5 + 126 \times 13^2 T^4 + 102 \times 13^3 T^3 + 38 \times 13^4 T^2 + 7 \times 13^5 T + 7 \times 13^5, \tag{6.36}$$

$$\sigma_{11} = 204 \times 13T^5 + 346 \times 13^2 T^4 + 222 \times 13^3 T^3 + 74 \times 13^4 T^2 + 13^6 T + 13^6 \tag{6.37}$$

$$\sigma_{12} = 11 \times 13T^6 + 36 \times 13^2 T^5 + 38 \times 13^3 T^4 + 20 \times 13^4 T^3 + 6 \times 13^5 T^2 + 13^6 T + 13^6 \tag{6.38}$$

and

$$\sigma_{13} = \prod_i \zeta_i = \frac{E(q^{13})^{14}}{q^{91} E(q^{169})^{14}} = T^7. \tag{6.39}$$

It follows that the modular equation is

$$\begin{aligned}
&\zeta^{13} - 13\zeta^{12} + (13T + 7 \times 13)\zeta^{11} - (13^2 T + 3 \times 13^2)\zeta^{10} + (6 \times 13T^2 + 7 \times 13^2 T + 15 \times 13^2)\zeta^9 \\
&-(74 \times 13T^2 + 37 \times 13^2 T + 5 \times 13^3)\zeta^8 + (20 \times 13T^3 + 38 \times 13^2 T^2 + 13^4 T + 19 \times 13^3)\zeta^7 \\
&-(222 \times 13T^3 + 184 \times 13^2 T^2 + 51 \times 13^3 T + 5 \times 13^4)\zeta^6 \\
&+(38 \times 13T^4 + 102 \times 13^2 T^3 + 56 \times 13^3 T^2 + 13^5 T + 15 \times 13^4)\zeta^5 \\
&-(346 \times 13T^4 + 422 \times 13^2 T^3 + 184 \times 13^3 T^2 + 37 \times 13^4 T + 3 \times 13^5)\zeta^4 \\
&+(36 \times 13T^5 + 126 \times 13^2 T^4 + 102 \times 13^3 T^3 + 38 \times 13^4 T^2 + 7 \times 13^5 T + 7 \times 13^5)\zeta^3 \\
&-(204 \times 13T^5 + 346 \times 13^2 T^4 + 222 \times 13^3 T^3 + 74 \times 13^4 T^2 + 13^6 T + 13^6)\zeta^2 \\
&+(11 \times 13T^6 + 36 \times 13^2 T^5 + 38 \times 13^3 T^4 + 20 \times 13^4 T^3 + 6 \times 13^5 T^2 + 13^6 T + 13^6)\zeta - T^7 = 0.
\end{aligned} \tag{6.40}$$

If, as before, we let $u_i = H(\zeta^i)$ and $U = \sum_{i \geq 0} u_i z^i$, then

$$U = \frac{N}{D} \tag{6.41}$$

where

$$D = 1 - 13z + (13T + 7 \times 13)z^2 - (13^2T + 3 \times 13^2)z^3 + (6 \times 13T^2 + 7 \times 13^2T + 15 \times 13^2)z^4$$
$$-(74 \times 13T^2 + 37 \times 13^2T + 5 \times 13^3)z^5 + (20 \times 13T^3 + 38 \times 13^2T^2 + 13^4T + 19 \times 13^3)z^6$$
$$-(222 \times 13T^3 + 184 \times 13^2T^2 + 51 \times 13^3T + 5 \times 13^4)z^7$$
$$+(38 \times 13T^4 + 102 \times 13^2T^3 + 56 \times 13^3T^2 + 13^5T + 15 \times 13^4)z^8$$
$$-(346 \times 13T^4 + 422 \times 13^2T^3 + 184 \times 13^3T^2 + 37 \times 13^4T + 3 \times 13^5)z^9$$
$$+(36 \times 13T^5 + 126 \times 13^2T^4 + 102 \times 13^3T^3 + 38 \times 13^4T^2 + 7 \times 13^5T + 7 \times 13^5)z^{10}$$
$$-(204 \times 13T^5 + 346 \times 13^2T^4 + 222 \times 13^3T^3 + 74 \times 13^4T^2 + 13^6T + 13^6)z^{11}$$
$$+(11 \times 13T^6 + 36 \times 13^2T^5 + 38 \times 13^3T^4 + 20 \times 13^4T^3 + 6 \times 13^5T^2 + 13^6T + 13^6)z^{12} - T^7z^{13} \tag{6.42}$$

and

$$N = 1 - 12z + (11T + 77)z^2 - (10 \times 13T + 30 \times 13)z^3 + (54T^2 + 63 \times 13T + 135 \times 13)z^4$$
$$-(592T^2 + 296 \times 13T + 40 \times 13^2)z^5 + (140T^3 + 266 \times 13T^2 + 7 \times 13^3T + 133 \times 13^2)z^6$$
$$-(1332T^3 + 1104 \times 13T^2 + 306 \times 13^2T + 30 \times 13^3)z^7$$
$$+(190T^4 + 510 \times 13T^3 + 280 \times 13^2T^2 + 5 \times 13^4T + 75 \times 13^3)z^8$$
$$-(1384T^4 + 1688 \times 13T^3 + 736 \times 13^2T^2 + 148 \times 13^3T + 12 \times 13^4)z^9$$
$$+(108T^5 + 378 \times 13T^4 + 306 \times 13^2T^3 + 114 \times 13^3T^2 + 21 \times 13^4T + 21 \times 13^4)z^{10}$$
$$-(408T^5 + 692 \times 13T^4 + 444 \times 13^2T^3 + 148 \times 13^3T^2 + 2 \times 13^5T + 2 \times 13^5)z^{11}$$
$$+(11T^6 + 36 \times 13T^5 + 38 \times 13^2T^4 + 20 \times 13^3T^3 + 6 \times 13^4T^2 + 13^5T + 13^5)z^{12}. \tag{6.43}$$

If we expand U to the power 24, we find that

$$H(\zeta^{24}) = u_{24} = -577738T^{12} - 13^{11}. \tag{6.44}$$

We then have

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24} \tag{6.45}$$
$$= q^{169}E(q^{169})^{24}\left(\frac{E(q)}{q^7E(q^{169})}\right)^{24}$$
$$= q^{169}E(q^{169})^{24}\zeta^{24},$$

$$\sum_{n\geq 1} \tau(13n)q^{13n} = q^{169}E(q^{169})^{24}H(\zeta^{24}) \tag{6.46}$$
$$= q^{169}E(q^{169})^{24}\left(-577738T^{12} - 13^{11}\right)$$
$$= q^{169}E(q^{169})^{24}\left(-577738\left(\frac{E(q^{13})^2}{q^{13}E(q^{169})^2}\right)^{12} - 13^{11}\right)$$

$$= -577738q^{13}E(q^{13})^{24} - 13^{11}q^{169}E(q^{169})^{24}$$

and

$$\sum_{n\geq 1}\tau(13n)q^n = -577738qE(q)^{24} - 13^{11}q^{13}E(q^{13})^{24} \tag{6.47}$$
$$= -577738\sum_{n\geq 1}\tau(n)q^n - 13^{11}\sum_{n\geq 1}\tau(n)q^{13n}.$$

The term $n = 1$ gives

$$\tau(13) = -577738\tau(1) = -577738,$$

so (6.47) becomes

$$\sum_{n\geq 1}\tau(13n)q^n = \tau(13)\sum_{n\geq 1}\tau(n)q^n - 13^{11}\sum_{n\geq 1}\tau(n)q^{13n}, \tag{6.48}$$

as claimed.

Aside: It can be shown, using the quintuple product identity, that

$$\zeta = \alpha + \beta + \gamma + \delta + 1 + \varepsilon + \theta$$

where

$$\alpha = q^{-7}\prod_{n\geq 0}\frac{(1-q^{169n+52})(1-q^{169n+117})}{(1-q^{169n+26})(1-q^{169n+143})}$$
$$\beta = -q^{-6}\prod_{n\geq 0}\frac{(1-q^{169n+78})(1-q^{169n+91})}{(1-q^{169n+39})(1-q^{169n+130})}$$
$$\gamma = -q^{-5}\prod_{n\geq 0}\frac{(1-q^{169n+26})(1-q^{169n+143})}{(1-q^{169n+13})(1-q^{169n+156})}$$
$$\delta = q^{-2}\prod_{n\geq 0}\frac{(1-q^{169n+65})(1-q^{169n+104})}{(1-q^{169n+52})(1-q^{169n+117})}$$
$$\varepsilon = q^{5}\prod_{n\geq 0}\frac{(1-q^{169n+39})(1-q^{169n+130})}{(1-q^{169n+65})(1-q^{169n+104})}$$
$$\theta = q^{15}\prod_{n\geq 0}\frac{(1-q^{169n+13})(1-q^{169n+156})}{(1-q^{169n+78})(1-q^{169n+91})}$$

References

1. G. Bilgici, A.B. Ekin, Some congruences for modulus 13 related to partition generating function. Ramanujan J. **33**, 197–218 (2014)
2. H.M. Farkas, I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group* (American Mathematical Society, Providence, 2001)
3. F.G. Garvan, A simple proof of Watson's partition congruences for powers of 7. J. Aust. Math. Soc. Ser. A **36**(3), 316–334 (1984)
4. M.D. Hirschhorn, *The Power of q*. Developments in Mathematics Vol. **49** (Springer, Berlin, 2017)
5. J. N. O'Brien, Some properties of partitions with special reference to primes other than 5, 7, and 11, Ph.D. thesis, University of Durham, England, 1966
6. K.S. Williams, Historical remark on Ramanujan's tau function. Am. Math. Mon. **122**(1), 30–35 (2015)

Construction of Cusp Forms Using Rankin–Cohen Brackets

Abhash Kumar Jha and Arvind Kumar

Dedicated to Professor Krishnaswami Alladi on the occasion of his 60th birthday

Abstract For a fixed modular form we consider a family of linear maps constructed using Rankin–Cohen brackets. We explicitly compute the adjoint of these maps with respect to the Petersson scalar product. The Fourier coefficients of the image of a cusp form under the adjoint maps are, up to a constant, a special value of a certain shifted Rankin–Selberg convolution attached to them. This is a generalization of the work due to Kohnen (Math. Z. 207 (1991), 657–660) and Herrero (Ramanujan J. 36 (2014), no. 3, 529–536) in the case of integral weight modular forms to half-integral weight modular forms. As a consequence we get non-vanishing and asymptotic bound for the special values of a certain shifted Rankin–Selberg convolution of modular forms.

Keywords Modular forms · Rankin–Cohen brackets · Adjoint map
Dirichlet series

2010 Mathematics Subject Classification Primary 11F37 · Secondary 11F25
11F66

A. K. Jha (✉)
Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
e-mail: abhashkumarjha@gmail.com; abhashjha@iisc.ac.in

A. Kumar
Harish-Chandra Research Institute, HBNI, Chhatnag Road,
Jhunsi, Allahabad 211 019, India
e-mail: kumararvind@hri.res.in

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_19

1 Introduction

W. Kohnen [9] constructed certain cusp forms whose Fourier coefficients are special values of certain shifted Dirichlet series. More precisely, he computed the adjoint of the product map by a fixed cusp form with respect to the Petersson scalar product. This result has been generalized by several authors to other automorphic forms (see the list [1, 10, 11, 13, 15]).

Rankin [12] described all polynomials in the derivatives of modular forms with values again in modular forms. Cohen [2], constructed certain bilinear operators and obtained elliptic modular forms. Zagier [16] investigated algebraic properties of these operators and called them Rankin–Cohen brackets. The Rankin–Cohen bracket is a generalization of the usual product (the 0-th Rankin–Cohen bracket being the product). Recently, the work of Kohnen [9] has been generalized by Herrero [3], where the author constructed cusp forms by computing the adjoint of the map constructed using Rankin–Cohen brackets by a fixed cusp form instead of the product. Recently, the first author and B. Sahu extended this result to the case of Jacobi forms [6] and Siegel modular forms of genus two [7]. The aim of this article is to extend the work of Herrero [3] for elliptic modular forms (both integer and half integer weight) with character and for any congruence subgroup to construct cusp forms. We apply this result to get non-vanishing of special value of certain shifted Rankin–Selberg convolution of modular forms. An asymptotic bound of the special values of these shifted Rankin–Selberg convolution has been also obtained.

2 Preliminaries

In this section, we briefly recall some basic definitions and Rankin–Cohen brackets on modular forms.

2.1 Modular forms

The full modular group $SL_2(\mathbb{Z})$ and congruence subgroup $\Gamma_0(N),\ N \in \mathbb{N}$ is defined as follows;

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \mod N \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on the complex upper half plane $\mathscr{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by fractional linear transformation as follows;

$$\gamma \cdot z := \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } z \in \mathscr{H}.$$

Let $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then for a holomorphic function $f : \mathscr{H} \to \mathbb{C}$ define the slash operator as follows;

$$f \mid_k \gamma(z) := (cz+d)^{-k} f(\gamma \cdot z). \tag{1}$$

Unless otherwise stated we assume that $\Gamma = \Gamma_0(N)$, for a fixed positive integer N.

Definition 2.1 *A modular form of weight $k \in \mathbb{Z}$ on Γ for a character χ modulo N is a complex-valued holomorphic function f on $\mathscr{H}$ satisfying;*

$$f \mid_k \gamma(z) = \chi(d) f(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and holomorphic at the cusps of Γ. Further if f also vanishes at the cusps of Γ, then f is called a cusp form.

Let $M_k(\Gamma, \chi)$ (respectively $S_k(\Gamma, \chi)$) denote the space of modular forms (respectively cusp forms) of integral weight k on Γ for character χ.
We define the Petersson scalar product on $S_k(\Gamma, \chi)$ as follows;

$$\langle f, g \rangle = \int\limits_{\Gamma \backslash \mathscr{H}} f(z)\overline{g(z)}\, (Im(z))^k \, d^*z, \tag{2}$$

where $\Gamma \setminus \mathscr{H}$ is a fundamental domain and $d^*z = \frac{dxdy}{y^2}, (z = x + iy)$ is an invariant measure under the action on Γ on $\mathscr{H}$. For more details on the theory of modular forms, we refer to [8].

2.2 *Modular Forms of half-integral Weight*

Let $k \in \mathbb{Z}$, $\Gamma = \Gamma_0(N)$ where $N \in 4\mathbb{N}$. For $\gamma = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in \Gamma$ and a holomorphic function $f : \mathscr{H} \to \mathbb{C}$ define the slash operator as follows;

$$f \tilde{|}_{k+\frac{1}{2}} \gamma(z) := \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+\frac{1}{2}} (cz+d)^{-k-\frac{1}{2}} f(\gamma \cdot z), \tag{3}$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol.

Let $M_{k+\frac{1}{2}}(\Gamma, \chi)$ denote the space of modular forms of weight $k+\frac{1}{2}$ and character χ modulo N for Γ, that is the space of all complex-valued holomorphic functions f satisfying;

$$f\tilde{|}_{k+\frac{1}{2}}\gamma(z) = \chi(d)f(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and holomorphic at the cusps of Γ. Further, if f vanishes at cusps of Γ, then f is called a cusp form and we denote the space of all cusp forms of weight $k+\frac{1}{2}$ and character χ for Γ by $S_{k+\frac{1}{2}}(\Gamma, \chi)$. If χ is the trivial character then we denote the spaces by $M_k(\Gamma)$ and $S_k(\Gamma)$.

The Petersson scalar product on $S_{k+\frac{1}{2}}(\Gamma, \chi)$ can be defined in similar way as in (2). For more details on the theory of modular forms of half-integral weight, we refer to [8] and [14].

Definition 2.2 *Let n be a be a positive integer. The n-th Poincaré series of weight k, where $k \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ is defined by*

$$P_{k,n}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z} \,||_k\, \gamma, \tag{4}$$

where $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Z} \right\}$ and $||_k$ is the slash operator defined by (1) and (3) for $k \in \mathbb{Z}$ and $k \in \mathbb{Z}+\frac{1}{2}$, respectively. It is well known that $P_{k,n} \in S_k(\Gamma)$ for $k > 2$.

The Poincaré series $P_{k,n}(z)$ is characterized by the following property:

Lemma 2.3 *Let $k \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ and $f \in S_k(\Gamma)$ with Fourier expansion $f(z) = \sum_{m=1}^{\infty} a(m)q^m$. Then*

$$\langle f,\ P_{k,n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\, a(n). \tag{5}$$

The following lemmas tell about the growth of the Fourier coefficients of a modular form.

Lemma 2.4 *[5] If $f \in M_k(\Gamma, \chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll |n|^{k-1+\varepsilon},$$

and moreover, if f is a cusp form, then

$$a(n) \ll |n|^{\frac{k}{2}-\frac{1}{4}+\varepsilon}.$$

Lemma 2.5 *[4] If $f \in M_{k+\frac{1}{2}}(\Gamma, \chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll |n|^{k-\frac{1}{2}+\varepsilon},$$

and moreover, if $f \in S_{k+\frac{1}{2}}(\Gamma, \chi)$ *is a cusp form, then*

$$a(n) \ll |n|^{\frac{k}{2}+\varepsilon}.$$

2.3 Rankin–Cohen Brackets

Let k and l be real numbers and $\nu \geq 0$ be an integer. Let f and g be two complex-valued holomorphic functions on $\mathscr{H}$. Define the ν-th Rankin–Cohen bracket of f and g by

$$[f, g]_\nu := \sum_{r=0}^{\nu} C_r(k, l; \nu) D^r f D^{\nu-r} g, \tag{6}$$

where $D^r f = \frac{1}{(2\pi i)^r} \frac{d^r f}{dz^r}$ and $C_r(k, l; \nu) = (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)}$ and $\Gamma(x)$ is the usual Gamma function.

Remark 2.6 *[2] It is easy to verify that*

$$[f|_k\gamma, g|_l\gamma]_\nu = [f, g]|_{k+l+2\nu}\gamma, \quad \forall \gamma \in \Gamma. \tag{7}$$

Theorem 2.7 *[2] Let* $\nu \geq 0$ *be an integer and* $f \in M_k(\Gamma, \chi_1)$ *and* $g \in M_l(\Gamma, \chi_2)$. *Then* $[f, g]_\nu \in M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$,

$$where \chi = \begin{cases} 1, & if\ both\ k, l \in \mathbb{Z}, \\ \psi^k, & if\ k \in \mathbb{Z}\ and\ l \in \mathbb{Z} + \frac{1}{2}, \\ \psi^l, & if\ k \in \mathbb{Z} + \frac{1}{2}\ and\ l \in \mathbb{Z}, \\ \psi^{k+l} & if\ both\ k, l \in \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{8}$$

Moreover, if $\nu > 0$, *then* $[f, g]_\nu \in S_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$. *In fact,* $[\ ,\ ]_\nu$ *is a bilinear map from* $M_k(\Gamma, \chi_1) \times M_l(\Gamma, \chi_2)$ *to* $M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$. *Here* ψ *is the character defined by* $\psi(x) = (\frac{-4}{x})$.

3 Statement of the Theorem

Let $k, l \in \frac{1}{2}\mathbb{Z}$ and $\nu \geq 0$ be an integer. Also assume that $\Gamma = \Gamma_0(N)$, $N \in 4\mathbb{N}$ if either of k or l is non-integer. For a fixed $g \in M_l(\Gamma, \chi_2)$, we consider the map

$$T_{g,\nu} : S_k(\Gamma) \to S_{k+l+2\nu}(\Gamma, \chi_2),$$

defined by $T_{g,\nu}(f) = [f, g]_\nu$. $T_{g,\nu}$ is a $\mathbb{C}$-linear map of finite dimensional Hilbert spaces and therefore has an adjoint map $T^*_{g,\nu} : S_{k+l+2\nu}(\Gamma, \chi_2) \to S_k(\Gamma)$ such that

$$\langle f, T_{g,\nu}(h)\rangle = \langle T^*_{g,\nu}(f), h\rangle, \quad \forall\, f \in S_{k+l+2\nu}(\Gamma, \chi_2) \ \text{ and } \ h \in S_k(\Gamma).$$

Remark 3.1 *In [3], Herrero computed the adjoint map for the case when $k, l \in \mathbb{Z}$, $\Gamma = SL_2(\mathbb{Z})$ and χ_2 is the trivial character. One can prove the similar result for the case when Γ is a congruence subgroup of level N and χ_2 is any character mod N using the techniques used in proof of Theorem 3.2.*

In view of remark 3.1, we are left with the case where at least one of k and l is non-integer. From now on, k, l and ν denote non-negative integers. Consider the following maps:

1. $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$, with $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$,
2. $T_{g,\nu} : S_k(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$, with $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$,
3. $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$, with $g \in M_l(\Gamma, \chi_2)$.

We exhibit explicitly the Fourier coefficients of $T^*_{g,\nu}(f)$ for $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ in case (1) and by using the same method, we can find the analogous maps in case (2) and (3) (see the remark 3.3). These involve special values of certain shifted Dirichlet series of Rankin–Selberg type associated to f and g. We now state the main theorem.

Theorem 3.2 *Let $g(z) = \sum_{m=0}^{\infty} b(m)q^m \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$. Suppose that either (a) g is a cusp form and $k > 2$ or (b) g is not cusp form and $l < k - \frac{3}{2}$.*
Then the image of any cusp form $f(z) = \sum_{m=1}^{\infty} a(m)q^m \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ under $T^*_{g,\nu}$ is given by

$$T^*_{g,\nu}(f)(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{k+\frac{1}{2}}(\Gamma),$$

where

$$c(n) = \beta(k, l, \nu; n) L_{f,g,\nu,n}(\gamma), \tag{9}$$

$$\gamma = k + l + 2\nu, \ \ \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu)\, n^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2})(4\pi)^{l+2\nu+\frac{1}{2}}},$$

and $L_{f,g,\nu,n}$ is the shifted, twisted Rankin–Selberg convolution of f and g, defined by

$$L_{f,g,\nu,n}(s) = \sum_{m=1}^{\infty} \frac{a(n+m)\overline{b(m)}\ \ \alpha(k, l, \nu, m, n)}{(n+m)^s}, \qquad s \in \mathbb{C} \tag{10}$$

with

$$\alpha(k,l,\nu,n,m) \;=\; \sum_{r=0}^{\nu}(-1)^{\nu-r}\binom{\nu}{r}\frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)}n^r m^{\nu-r}.$$

Remark 3.3 *We have the similar results for the map in case (2) with*

$$\gamma = k+l+2\nu-\frac{1}{2}, \text{ and } \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\; n^{k-1}}{\Gamma(k-1)\;(4\pi)^{l+2\nu+\frac{1}{2}}},$$

and for the map in case (3) with

$$\gamma = k+l+2\nu-\frac{1}{2}, \text{ and } \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\; n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})\;(4\pi)^{l+2\nu}},$$

with the assumption that either (a) g is a cusp form and $k>3$ or (b) g is not cusp form and $l<k-2$.

4 Applications

4.1 An asymptotic bound for twisted and shifted Rankin–Selberg convolution

The Dirichlet series $L_{f,g,\nu,n}(s)$ defined in (10) is absolutely convergent for $\mathrm{Re}(s) > \frac{k}{2}+l+\nu-\frac{1}{4}$ by Lemma 2.4 and Lemma 2.5 and has meromorphic continuation to $\mathbb{C}$. For $\nu=0$, this is similar to the Dirichlet series which appeared in [9].

Let $g \in M_{l+\frac{1}{2}}(\Gamma,\chi_2)$ and $f(z) \in S_{k+l+2\nu+\frac{1}{2}}(\Gamma,\chi_2\chi)$ and consider the case (2). By Theorem 3.2 and remark 3.3, we have

$$L_{f,g,\nu,n}(k+l+2\nu-\frac{1}{2}) = \frac{c(n)}{\beta(k,l,\nu;n)},$$

where $c(n)$ is the n-th Fourier coefficient of a cusp form of weight k. By using Lemma 2.4, a straightforward calculation gives,

$$L_{f,g,\nu,n}(k+l+2\nu-\frac{1}{2}) \ll n^{-\frac{k}{2}+\frac{3}{4}}.$$

Note that the asymptotic bound of special values of $L-$functions in other cases can be obtained in the similar way.

4.2 Non-vanishing of special values of certain Dirichlet series

Consider the linear map $T^*_{g,\nu} \circ T_{g,\nu}$ on $S_k(\Gamma)$ with $g(z) \in M_l(\Gamma, \chi_2)$. If λ is a eigenvalue of $T^*_{g,\nu} \circ T_{g,\nu}$, then $\lambda \geqslant 0$. Suppose that $S_k(\Gamma)$ is one dimensional space generated by $f(z) = \sum_m a(n)q^n$. Then $T^*_{g,\nu} \circ T_{g,\nu}(h) = \lambda f$, for all $h \in S_k(\Gamma)$. In particular, $T^*_{g,\nu} \circ T_{g,\nu}(f) = \lambda f$ with $\lambda \geqslant 0$ and if we write $T^*_{g,\nu} \circ T_{g,\nu}(f) = \sum_n c(n)q^n$ then

$$c(n) = \frac{\Gamma(k+l+2\nu-1)}{\Gamma(k-1)} \frac{n^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(n+m)\overline{b(m)}\ \alpha(k,l,\nu,n,m)}{(n+m)^{k+l+2\nu-1}},$$

where $a_{T_{g,\nu}(f)}(n)$ is the n-th Fourier coefficient of $T_{g,\nu}(f) = [f, g]_\nu$. If $a(m_0)$ is the first non-zero Fourier coefficient of f then by comparing the Fourier coefficients in $T^*_{g,\nu} \circ T_{g,\nu}(f) = \lambda f$, we have

$$\lambda = \frac{\Gamma(k+l+2\nu-1)}{a(m_0)\Gamma(k-1)} \frac{m_0^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(m_0+m)\overline{b(m)}\ \alpha(k,l,\nu,m_0,m)}{(m_0+m)^{k+l+2\nu-1}} \geqslant 0.$$

In particular, if we take $l = 0,\ k = 6$ and $\nu = 0$ with $g(z) = \theta(z) = \sum\limits_n q^{n^2}$ and the unique newform $\Delta_{4,6}(z) = \sum_n \tau_{4,6}(n)q^n \in S_6(\Gamma_0(4))$, in case (2) then $m_0 = 1,\ \alpha(k,l,\nu,m_0,m) = 1$, and

$$\lambda = \frac{\Gamma(\frac{11}{2})}{\Gamma(5)2\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

or equivalently

$$\sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0. \tag{11}$$

Now $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ is the $(m+1)$-th Fourier coefficient of $\theta(z)\Delta_{4,6}(z)$ and equals to $\sum\limits_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)$. Putting the value of $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ in (11), we have

$$\sum_{m=1}^{\infty} \frac{\left(\sum\limits_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)\right)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

or equivalently

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r \le \sqrt{m^2+1}} \tau_{4,6}(m^2+1-r^2) \right)}{(m^2+1)^{\frac{11}{2}}} > 0.$$

5 Proof of the Theorem 3.2

We need the following lemma to proof the main theorem.

Lemma 5.1 *Using the same notation in Theorem 3.2, we have*

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash \mathscr{H}} \mid f(z) \, \overline{[e^{2\pi i n z} \mid_k \gamma, g]_\nu} \, (Im(z))^{k+l+2\nu+1} \mid \, d^* z$$

converges.

Proof. The proof is similar to Lemma 1 in [3].

Now we give a proof of Theorem 3.2. Put

$$T^*_{g,\nu}(f)(z) = \sum_{n=1}^{\infty} c(n) q^n.$$

We consider the n-th Poincaré series of weight $k + \frac{1}{2}$ as given in (4). Then using the Lemma 2.3, we have

$$\langle T^*_{g,\nu} f, P_{k+\frac{1}{2},n} \rangle = \frac{\Gamma(k-\frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}} \, c(n).$$

On the other hand, by definition of the adjoint map we have

$$\langle T^*_{g,\nu} f, P_{k+\frac{1}{2},n} \rangle = \langle f, T_{g,\nu}(P_{k+\frac{1}{2},n}) \rangle = \langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle.$$

Hence we get

$$c(n) = \frac{(4\pi n)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \, \langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle. \tag{12}$$

By definition,

$$\begin{aligned}\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle &= \int_{\Gamma\backslash\mathscr{H}} f(z)\, \overline{[P_{k+\frac{1}{2},n}, g]_\nu(z)}\, (Im(z))^{k+l+2\nu+1}\, d^*z\\ &= \int_{\Gamma\backslash\mathscr{H}} f(z)\, \overline{[\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e^{2\pi inz}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(z)}\, (Im(z))^{k+l+2\nu+1}\, d^*z\\ &= \int_{\Gamma\backslash\mathscr{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(z)\, \overline{[e^{2\pi inz}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(z)}\, (Im(z))^{k+l+2\nu+1}\, d^*z.\end{aligned}$$

By Lemma 5.1, we can interchange the sum and integration in last expression. Hence we get,

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathscr{H}} f(z)\, \overline{[e^{2\pi inz}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(z)}\, (Im(z))^{k+l+2\nu+1}\, d^*z.$$

Since $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$, $g\tilde{|}_{l+\frac{1}{2}}\gamma = \chi_2(d)g(z)$, for every $\gamma = \begin{pmatrix} a\, b\\ c\, d\end{pmatrix} \in \Gamma$. Therefore

$$\begin{aligned}&\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle\\ &= \sum_{\gamma=\begin{pmatrix} a\, b\\ c\, d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathscr{H}} f(z)\, \overline{[e^{2\pi inz}\tilde{|}_{k+\frac{1}{2}}\gamma, \frac{1}{\chi_2(d)}\, g\tilde{|}_{l+\frac{1}{2}}\gamma]_\nu(z)}\, (Im(z))^{k+l+2\nu+1}\, d^*z\\ &= \sum_{\gamma=\begin{pmatrix} a\, b\\ c\, d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int_{\Gamma\backslash\mathscr{H}} f(z)\, \overline{[e^{2\pi inz}\, |_{k+\frac{1}{2}}\, \gamma, g\, |_{l+\frac{1}{2}}\, \gamma]_\nu(z)} (Im(z))^{k+l+2\nu+1}\, d^*z.\end{aligned}$$

Using the change of variable z to $\gamma^{-1}\cdot z$ in each integral, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals

$$\begin{aligned}&\sum_{\gamma=\begin{pmatrix} a\, b\\ c\, d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int_{\Gamma\backslash\mathscr{H}} f(\gamma^{-1}\cdot z)\, \overline{[e^{2\pi inz}\, |_{k+\frac{1}{2}}\, \gamma, g\, |_{l+\frac{1}{2}}\, \gamma]_\nu(\gamma^{-1}\cdot z)}\\ &\times (Im(\gamma^{-1}\cdot z))^{k+l+2\nu+1} d^*(\gamma^{-1}\cdot z).\end{aligned}$$

Since $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$, $f(\gamma^{-1}\cdot z) = \chi_2(a)\chi(a)(-cz+a)^{k+l+2\nu+1} f(z)$, for every $\gamma = \begin{pmatrix} a\, b\\ c\, d\end{pmatrix} \in \Gamma$, and hence

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle = \sum_{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int_{\Gamma\backslash\mathscr{H}} \chi_2(a)\chi(a)(-cz+a)^{k+l+2\nu+1} f(z) \times$$

$$\overline{(-cz+a)^{k+l+2\nu+1}([e^{2\pi inz}\mid_{k+\frac{1}{2}}\gamma, g\mid_{l+\frac{1}{2}}\gamma]_\nu \mid_{k+l+2\nu+1}\gamma^{-1})(z)}\left(\frac{Im(z)}{|-cz+a|^2}\right)^{k+l+2\nu+1} d^*z.$$

Now using (7), we get

$$\begin{aligned}
&\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle \\
&= \sum_{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)}\chi_2(a)\chi(a) \int_{\gamma\cdot\Gamma\backslash\mathscr{H}} f(z)\,\overline{[e^{2\pi inz}, g]_\nu}\,(Im(z))^{k+l+2\nu+1}\,d^*z \\
&= \sum_{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)}\chi_2(a)\left(\frac{-4}{a}\right)^{k+l+1} \int_{\gamma\cdot\Gamma\backslash\mathscr{H}} f(z)\overline{[e^{2\pi inz}, g]_\nu}\,(Im(z))^{k+l+2\nu+1} d^*z.
\end{aligned}$$

The quantity appearing before integral is equals to 1, for all $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_\infty \setminus \Gamma$, hence we get

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma\cdot\Gamma\backslash\mathscr{H}} f(z)\,\overline{[e^{2\pi inz}, g]_\nu}\,(Im(z))^{k+l+2\nu+1}\,d^*z.$$

Now using Rankin unfolding argument, we have

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle &= \int_{\Gamma_\infty\backslash\mathscr{H}} f(z)\,\overline{[e^{2\pi inz}, g]_\nu}\,(Im(z))^{k+l+2\nu+1}\,d^*z \qquad (13) \\
&= \int_{\Gamma_\infty\backslash\mathscr{H}} f(z)\sum_{r=0}^{\nu} C_r(k,l;\nu)\,\overline{D^r(e^{2\pi inz})D^{\nu-r}(g)}\,(Im(z))^{k+l+2\nu+1}\,d^*z
\end{aligned}$$

Now replacing f and g by their Fourier series expansions in (13), $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals

$$\begin{aligned}
&\sum_{r=0}^{\nu} C_r(k,l;\nu) \int_{\Gamma_\infty\backslash\mathscr{H}} \left(\sum_s a(s)e^{2\pi isz}\right) n^r\,\overline{e^{2\pi inz}} m^{\nu-r}\left(\sum_m \overline{b(m)}\,\overline{e^{2\pi imz}}\right)(Im(z))^{k+l+2\nu+1} d^*z \\
&= \int_{\Gamma_\infty\backslash\mathscr{H}} \sum_s\sum_m \alpha(k,l,\nu,n,m)a(s)\overline{b(m)}\,e^{2\pi isz}\,\overline{e^{2\pi inz}}\,\overline{e^{2\pi imz}}\,(Im(z))^{k+l+2\nu+1}\,d^*z
\end{aligned}$$

$$= \sum_s \sum_m \alpha(k,l,\nu,n,m) a(s) \overline{b(m)} \int_{\Gamma_\infty \backslash \mathscr{H}} e^{2\pi i s z}\, \overline{e^{2\pi i n z}}\, \overline{e^{2\pi i m z}}\, (Im(z))^{k+l+2\nu+1}\, d^* z.$$

A fundamental domain for the action of Γ_∞ on $\mathscr{H}$ is given by $[0, 1] \times [0, \infty)$. Integrating on this region after substituting $z = x + iy$,

$$\begin{aligned}
&\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle \\
&= \sum_s \sum_m \alpha(k,l,\nu,n,m) a(s) \overline{b(m)} \int_0^1 \int_0^\infty e^{2\pi i (s-n-m)x} e^{-2\pi(s+n+m)y} y^{k+l+2\nu-1} dx dy \\
&= \sum_m \alpha(k,l,\nu,n,m) a(n+m) \overline{b(m)} \int_0^\infty e^{-4\pi(n+m)y} y^{k+l+2\nu-1} dy \\
&= \frac{\Gamma(k+l+2\nu)}{(4\pi)^{k+l+2\nu}} \sum_m \frac{a(n+m)\overline{b(m)} \alpha(k,l,\nu,n,m)}{(n+m)^{k+l+2\nu}}.
\end{aligned}$$

Substituting the above value of $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$ in (12), we get the required expression for $c(n)$ given in Theorem 3.2.

Acknowledgements We gratefully acknowledge the many helpful suggestions of B. Ramakrishnan and B. Sahu during the preparation of the paper. We would like to thank the referee for useful comments on an earlier version of the paper. The authors would also like to thank Council of Scientific and Industrial Research (CSIR), India for financial support. The first author thanks the organizers of the conference "ALLADI 60" for partial support and Science and Engineering Research Board (SERB), India for providing travel support through "International Travel Support Scheme" to attend the conference.

References

1. Y. Choie, H. Kim, M. Knopp, Construction of Jacobi forms. Math. Z. **219**(1), 71–76 (1995)
2. H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters. Math. Ann. **217**(3), 271–285 (1975)
3. S.D. Herrero, The adjoint of some linear maps constructed with the Rankin-Cohen brackets. Ramanujan J. **36**(3), 529–536 (2015)
4. H. Iwaniec, Fourier coefficients of modular forms of half-integral weight. Invent. Math. **87**(2), 385–401 (1987)
5. H. Iwaniec, *Topics in Classical Automorphic Forms*, vol. 17, Graduate Studies in Mathematics (American Mathematical Society, Providence, 1977)
6. A.K. Jha, B. Sahu, Rankin-Cohen brackets on Jacobi forms and the adjoint of some linear maps. Ramanujan J. **39**(3), 533–544 (2016)
7. A.K. Jha, B. Sahu, Rankin-Cohen brackets on Siegel modular forms and special values of certain Dirichlet series. Ramanujan J. **44**(1), 63–73 (2017)
8. N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, vol. 97, 2nd edn., Graduate Texts in Mathematics (Springer, Berlin, 1993)
9. W. Kohnen, Cusp forms and special value of certain Dirichlet series. Math. Z. **207**(4), 657–660 (1991)
10. M.H. Lee, Siegel cusp forms and special values of Dirichlet series of Rankin type. Complex Var. Theory Appl. **31**(2), 97–103 (1996)

11. M.H. Lee, D.Y. Suh, Fourier coefficients of cusp forms associated to mixed cusp forms. Panam. Math. J. **8**(1), 31–38 (1998)
12. R.A. Rankin, The construction of automorphic forms from the derivatives of a given form. J. Indian Math. Soc. **20**, 103–116 (1956)
13. H. Sakata, Construction of Jacobi cusp forms. Proc. Japan. Acad. Ser. A Math. Sci. **74**(7), 117–119 (1998)
14. G. Shimura, On modular forms of half integral weight. Ann. of Math. **97**(2), 440–481 (1973)
15. X. Wang, P. Ding Yi, Hilbert modular forms and special values of some Dirichlet series. Acta. Math. Sin. **38**(3), 336–343 (1995)
16. D. Zagier, Modular forms and differential operators. Proc. Indian Acad. Sci. Math. Sci. **104**(1), 57–75 (1994)

An Open Problem of Corteel, Lovejoy, and Mallet

Shashank Kanade, Kağan Kurşungöz and Matthew Russell

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Corteel, Lovejoy, and Mallet concluded their paper "An extension to overpartitions of the Rogers–Ramanujan identities for even moduli" with an open question of investigating the combinatorial properties of a q-series with two additional parameters. We settle their question, unfortunately in the negative, by showing that the series yields only the known results in overpartitions. However, when one annihilates one of the parameters, the resulting series have nice integer partitions interpretations. Those series appeared in another publication as well. In particular, Corteel, Lovejoy, and Mallet's series involve an index d. This index unifies two classes of overpartition identities for $d = 1$ and $d = 2$, but does not give additional overpartition identities for $d \geq 3$. Upon setting one of the parameters zero, one does get regular partition identities for all d. The proofs are conventional, formal verifications for brevity, but we show how to make the proofs constructive.

S. K. is supported by a postdoctoral fellowship awarded by the Pacific Institute for Mathematical Sciences (PIMS).

S. Kanade (✉)
University of Alberta, Edmonton, AB, Canada
e-mail: kanade@alberta.ca

K. Kurşungöz
Sabancı University, İstanbul, Turkey
e-mail: kursungoz@sabanciuniv.edu

M. Russell
Rutgers University, Piscataway, NJ, USA
e-mail: russell2@math.rutgers.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_20

Keywords Partition identities · Rogers–Ramanujan generalization · Partition generating functions

2010 Mathematics Subject Classification Primary 11P84 · 05A17 · 05A15 Secondary 05A19

1 Introduction

Andrews' H and J functions [3] are not only a framework for many results hitherto known, but also a source of inspiration for a wave of results after it.

$$H_{k,a}(y;x;q) = \sum_{n\geq 0} \frac{x^{kn}q^{kn^2+n-an}y^n(1-x^aq^{2na})(yxq^{n+1})_\infty(1/y)_n}{(q)_n(xq^n)_\infty} \tag{1.1}$$

$$J_{k,a}(y;x;q) = H_{k,a}(y;xq;q) - yxqH_{k,a-1}(y;xq;q) \tag{1.2}$$

Above and elsewhere, we employ the standard q-series notation [10]

$$(a)_n = (a;q)_n = \prod_{j=1}^{n}(1-aq^{j-1}),$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n,$$

$$(a_1,a_2,\ldots,a_s)_n = (a_1,a_2,\ldots,a_s;q)_n = \prod_{j=1}^{s}(a_j)_n.$$

$a, a_1, \ldots, a_s$ are called parameters, and q is called the base. If the base is not specified, it is understood to be q. $|q| < 1$ is enough for absolute convergence of the infinite products. $|x| < 1/|q|$ in addition is required for absolute convergence of (1.1) and (1.2). In this note, however, one does not need to worry about convergence.

The series (1.1) and (1.2) satisfy a number of functional equations. They admit application of Jacobi's triple product identity [10, eq. (1.6.1)] under various substitutions. This yields integer partition identities. Some examples are stated below.

Definition 1.1. A partition of a nonnegative integer n is a nonincreasing sum of positive integers

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$. The number of parts m is also known as the length of the partition.

Alternatively, one can write

$$n = 1f_1 + 2f_2 + 3f_3 + \cdots$$

for the same partition, where f_i denotes the number of occurrences, or the frequency, of i among $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Obviously, only finitely many of the f_i can be nonzero. For example, the nonincreasing sum

$$5 + 5 + 3 + 3 + 2 + 1 + 1$$

is a partition of 20, where

$$f_1 = 2, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 0, \quad f_5 = 2, \quad \text{and } f_i = 0 \text{ for } i \geq 6.$$

Theorem 1.1 (Rogers–Ramanujan identities [12, 16, 17]). *Given a nonnegative integer n, the number of partitions of n where* $f_i + f_{i+1} < 2$ *equals the number of partitions of n where* $f_{5j} = f_{5j\pm 2} = 0$.

The number of partitions of n where $f_i + f_{i+1} < 2$ *and* $f_1 = 0$ *equals the number of partitions of n where* $f_{5j} = f_{5j\pm 1} = 0$.

The conventional way to express the first Rogers–Ramanujan identity is that the number of partitions of n into distinct and nonconsecutive parts equals the number of partitions of n into parts that are 1 or 4 modulo 5. Notice that the condition $f_i + f_{i+1} < 2$ stipulates that the parts cannot repeat, and i and $i + 1$ cannot appear together. Also, the condition $f_{5j} = f_{5j\pm 2} = 0$ amounts to disallowing parts that are 0, 2, or 3 modulo 5, so that only those 1 or 4 modulo 5 can be used.

We stick to the frequency notation so that all results in this paper can be written in a unified and succinct manner [3, Ch. 7].

The specialization here is

$$J_{2,2}(0; x; q) \text{ and } J_{2,1}(0; x; q),$$

respectively. These series appeared in [18] previously.

Theorem 1.2 (Rogers–Ramanujan–Gordon identities [11]). *Given a nonnegative integer n, and integers k and a such that* $k \geq 2$, $1 \leq a \leq k$, *the number of partitions of n where* $f_i + f_{i+1} < k$ *and* $f_1 < a$ *equals the number of partitions of n where* $f_{(2k+1)j} = f_{(2k+1)j\pm a} = 0$.

Andrews' proof of Gordon's theorem [1] uses

$$J_{k,a}(0, x; q).$$

Of course, an immediate open problem after Gordon's theorem was the possibility of an even moduli extension (instead of $(2k + 1)$). This problem is partially solved by Andrews [2], who found a modulo $(4k + 2)$ analog. A full solution was given by Bressoud [5].

Theorem 1.3 (Bressoud's theorem). *Suppose n is a nonnegative integer, k and a are integers such that $k \geq 2$, $1 \leq a < k$. Let $A(n)$ be the number of partitions of n where $f_{2k} = f_{2k\pm a} = 0$, let $B(n)$ be the number of partitions of n where $f_i + f_{i+1} < k$, $f_1 < a$, and if $f_i + f_{i+1} = k - 1$, then $if_i + (i+1)f_{i+1} \equiv a - 1 \pmod 2$. Then $A(n) = B(n)$.*

The proof of Bressoud's theorem [5] uses

$$(-xq)_\infty J_{\frac{k-1}{2},\frac{a}{2}}(0; x^2; q^2).$$

Later, Corteel and Lovejoy introduced overpartitions [8].

Definition 1.2. An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of each part may be overlined. One can write

$$n = 1f_1 + 1f_{\overline{1}} + 2f_2 + 2f_{\overline{2}} + 3f_3 + 3f_{\overline{3}} + \cdots$$

where f_i denotes the number of occurrences, or the frequency, of i (nonoverlined), and $f_{\overline{i}}$ denotes that of $\overline{i}$ (overlined).

Again, only finitely many of f_i or $f_{\overline{i}}$s can be nonzero. In addition, $f_{\overline{i}}$'s may be 0 or 1 only. For example,

$$8 + 8 + 7 + 7 + 5 + 5 + \overline{4} + 3 + 3 + \overline{2} + \overline{1} + 1 \tag{1.3}$$

is an overpartition of 54 where

$$\begin{gathered} f_1 = 1,\ f_{\overline{1}} = 1,\ f_2 = 0,\ f_{\overline{2}} = 1,\ f_3 = 2,\ f_{\overline{3}} = 0,\ f_4 = 0,\ f_{\overline{4}} = 1, \\ f_5 = 2,\ f_{\overline{5}} = 0,\ f_6 = 0,\ f_{\overline{6}} = 0,\ f_7 = 2,\ f_{\overline{7}} = 0,\ f_8 = 2,\ f_{\overline{8}} = 0, \\ \text{and } f_i = f_{\overline{i}} = 0 \text{ for } i \geq 9. \end{gathered}$$

Lovejoy gave an overpartition analog of Gordon's theorem for overpartitions in two cases [14]. His proof used

$$J_{k,k}(-1; x; q) \text{ and } J_{k,1}(-1/q; x; q).$$

In fact, Lovejoy considered $J_{k,a}(y; x; q)$ but only the two cases above admitted (single) infinite product representations. Chen, Sang, and Shi obtained the general theorem [6].

Theorem 1.4 (Gordon's theorem for overpartitions). *Suppose n is a nonnegative integer, and k and a are integers such that $k \geq 2$, and $1 \leq a \leq k$.*

Let $D_{k,a}(n)$ be the number of overpartitions of n where $f_i + f_{\overline{i}} + f_{i+1} < k$ and $f_1 < a$.

For $1 \leq a < k$, let $C_{k,i}(n)$ be the number of overpartitions of n where $f_{2kj} = f_{2kj\pm a} = 0$.

Let $C_{k,k}(n)$ be the number of overpartitions of n where $f_{kj} = f_{\overline{kj}} = 0$. Then, $C_{k,a}(n) = D_{k,a}(n)$.

It should be noted that the $a = k$ case in Theorem 1.4 is different from Lovejoy's $a = k$ case [14]. Chen, Sang and Shi's proof used

$$H_{k,a}(-1/q; q; q),$$

instead of the specializations of J-function.

Corteel, Lovejoy, and Mallet extended Bressoud's theorem to overpartitions in one case [9]. They utilized the following statistic.

Definition 1.3. Given an overpartition and an arbitrary positive integer i that need not occur in the overpartition,

$$V(i) = \sum_{j=1}^{i} f_{\overline{j}}.$$

In other words, $V(i)$ is the number of overlined parts that are less than or equal to i.

For instance, the overpartition (1.3) has

$$V(1) = 1, \; V(2) = 2, \; V(3) = 2, \quad \text{and } V(i) = 3, \; \text{for } i \geq 4.$$

Again, Chen, Sang, and Shi proved the remaining cases [7].

Theorem 1.5 (Bressoud's theorem for overpartitions). *Suppose n is a nonnegative integer, and k and a are integers such that $k \geq 2$, and $1 \leq a \leq k$.*

Let $D_{k,a}(n)$ be the number of overpartitions of n where $f_i + f_{\overline{i}} + f_{i+1} < k$, $f_1 < a$, and if $f_i + f_{\overline{i}} + f_{i+1} = k - 1$, then $if_i + if_{\overline{i}} + (i+1)f_{i+1} \equiv V(i) + a - 1 \pmod 2$.

Let $C_{k,a}(n)$ be the number of overpartitions of n where $f_{(2k-1)j} = f_{(2k-1)j\pm a} = 0$. Then, $C_{k,a}(n) = D_{k,a}(n)$.

Corteel, Lovejoy, and Mallet introduced the following variant of the H and J functions.

$$\widetilde{H}_{k,a}(y; x; q) = \sum_{n\geq 0} \frac{(-y)^n q^{kn^2-n(n-1)/2+n-an} x^{(k-1)n} (1 - x^a q^{2na})(-x, -1/y)_n (-yxq^{n+1})_\infty}{(q^2; q^2)_n (xq^n)_\infty}$$

$$\widetilde{J}_{k,a}(y; x, q) = \widetilde{H}_{k,a}(y; xq; q) + yxq\widetilde{H}_{k,a-1}(y; xq; q)$$

Corteel, Lovejoy, and Mallet's result uses

$$\widetilde{J}_{k,1}(1/q; x; q),$$

whereas Chen, Sang, and Shi use

$$\widetilde{H}_{k,a}(1/q; xq; q).$$

in the proofs. For $y = 0$ instead of $y = 1/q$, Theorem 1.5 reduces to Theorem 1.3.

Corteel, Lovejoy, and Mallet concluded their paper with the open question of the combinatorial merit of the $J_{k,a,d}(y; x, q)$ series defined below.

$$\begin{aligned} H_{k,a,d}(y; x; q) \\ &= \sum_{n\geq 0} (-y)^n q^{kn^2+n-an-(d-1)n(n-1)/2} x^{k-d+1}(1 - x^a q^{2na}) \\ &\quad \times \frac{(-1/y)_n(-yxq^{n+1})_\infty (x^d; q^d)_n}{(q^d; q^d)_n (x)_\infty}, \end{aligned} \tag{1.4}$$

$$J_{k,a,d}(y; x; q) = H_{k,a,d}(y; xq; q) + yxq H_{k,a-1,d}(y; xq; q). \tag{1.5}$$

All of the above-listed results have the same *formal verification* method in their proofs. One starts with the multiplicity conditions imposed on the partitions. The functional equations along with the initial conditions their generating functions satisfy are found. Of course, these functional equations and initial conditions must uniquely determine the generating functions, hence the partition or overpartition enumerants.

Then, one verifies that a particular specialization or twist of (1.1) or (1.2) satisfies the same functional equations and the same initial conditions. Therefore, one argues, the series at hand must be the generating function. Finally, one renders all variables but q ineffective (by substituting a power of q, 0, or 1) and applies Jacobi's triple product identity. This yields the congruence condition on the partitions or overpartitions, and hence completes the proof.

Now we turn to the discussion of the case $d \geq 3$ for (1.5). For many well-known classes of (over)partitions, it is easy to derive the recursions satisfied by their generating functions. Conversely, given a set of generating functions along with the recursions they satisfy, one can conceive of naïvely reversing this procedure to guess the (over)partitions counted by the generating functions. Such a process is carried out implicitly in many problems, for instance, in some well-known proofs of Rogers–Ramanujan, Gordon–Andrews, Göllnitz-Gordon–Andrews, Andrews–Bressoud identities, etc. We explore such a reverse engineering procedure to guess what partitions might be counted by the series $J_{k,a,d}(y; x, q)$. Unfortunately, our naïve explorations for the case $k = 5$, $d = 3$ suggest that these series may not have easily deducible combinatorial interpretations. We discuss our findings in Section 4 below. We first explain our procedure by applying it to the well-known example of Rogers–Ramanujan recursion, then we build a formal framework to go beyond. Our results for the specific case $k = 5$, $d = 3$ are explained in Section 4.4.

The paper is organized as follows. In Section 2, we state and prove the main result, and indicate some implications. Although our proofs are formal verifications as well, we demonstrate how to *construct* those series in section 3. Thus, the proofs may be made into constructive proofs. We also display the constructed generating functions when we alter the characterization of classes of overpartitions slightly. In Section 4, we turn to a concrete exploration done in the case $k = 5$, $d = 3$. We conclude with some further exploration topics in section 5.

2 Main results

In this section, we indicate that the answer to Corteel, Lovejoy, and Mallet's question is most likely negative for overpartitions. The answer is affirmative for regular partitions, when the parameter y is set to zero [13]. We need another overpartition statistic [13] before we proceed.

Definition 2.1. Given an overpartition and an arbitrary positive integer i that need not occur in the overpartition,

$$\rho(i) = \sum_{j=1}^{i} (-1)^j f_{\overline{j}}.$$

In other words, $\rho(i)$ is the signed sum of number of occurrences of overlined parts that are less than or equal to i.

For instance, the overpartition (1.3) has

$$\rho(1) = -1, \;\; \rho(2) = 0, \;\; \rho(3) = 0, \quad \text{and } \rho(i) = 1, \;\; \text{for } i \geq 4.$$

We will place the series defined by (1.4) in the following class of series.

$$\begin{aligned} H^s_{k,a,d}(y; x; q) = & \frac{(x^d; q^d)_\infty}{(x; q)_\infty} \\ & \times \sum_{n \geq 0} (-1)^n x^{n(k+1-d)} q^{(2k+1-d)n(n-1)/2+(k+1)n-an} \\ & \quad \times \frac{y^n (-1/y; q)_n (-yxq^{n+1}; q)_\infty}{(q^d; q^d)_n (x^d q^{nd}; q^d)_\infty} \\ & \quad \times q^{-sn} \left[q^{dn} \frac{x^{d-s} - x^d}{1 - x^d} + \frac{1 - x^{d-s}}{1 - x^d} \right] \\ & \quad - (-1)^n x^{n(k+1-d)+a} q^{(2k+1-d)n(n-1)/2+(k+1)n+a(n+1)} \end{aligned}$$

$$\times \frac{y^n(-1/y;q)_n(-yxq^{n+1};q)_\infty}{(q^d;q^d)_n(x^dq^{nd};q^d)_\infty}$$
$$\times q^{sn}\left[q^{-dn}\frac{1-x^s}{1-x^d}+\frac{x^s-x^d}{1-x^d}\right]$$

$H^d_{k,a,d}(y;x;q) = H^0_{k,a,d}(y;x;q)$ is (1.4).

Theorem 2.1. *Suppose m, n, and r are nonnegative integers, and k, a, d, and s are integers such that*

$$k \geq 2, \quad 1 \leq a \leq k, \quad 1 \leq d \leq k, \quad 0 \leq s \leq d-1.$$

Let ${}_d\overline{b}^s_{k,a}(m,n,r)$ *be the number of overpartitions of n into m parts, r of which are overlined, such that*

$f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 < a,$
$f_{\overline{1}} = 0$, *i.e., 1 cannot be overlined,*
if $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ *for* $\delta = 1, 2 \ldots, d-1$,
then $a + s - 1 - f_{odd(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$.

Here, $odd(i) = i$ *if* i *is odd,* $odd(i) = i+1$ *if* i *is even; and* $\chi_e(\overline{i+1})$ *is 1 if* $(i+1)$ *is even and* $f_{\overline{i+1}} = 1$, $\chi_e(\overline{i+1})$ *is 0 otherwise. Then,*

$$H^s_{k,a,d}(y;xq;q) = \sum_{m,n,r\geq 0} {}_d\overline{b}^s_{k,a}(m,n,r)x^mq^ny^r$$

for $d = 1$ *or* $d = 2$.

This theorem is a one-parameter extension of Corollary 12 in [13] up to a substitution. The formal verification proofs are more or less the same, but we include the proof here for the sake of completeness.

Proof. Let
$$ {}_d\overline{\mathscr{B}}^s_{k,a}(y;x;q) = \sum_{m,n,r\geq 0} {}_d\overline{b}^s_{k,a}(m,n,r)x^mq^ny^r.$$

First, we argue that

$$\begin{aligned} &{}_d\overline{\mathscr{B}}^s_{k,a}(y;x;q) - {}_d\overline{\mathscr{B}}^{s+1}_{k,a-1}(y;x;q) \\ &= x^{a-1}q^{a-1}{}_d\overline{\mathscr{B}}^0_{k,k-s-a+1}(y;xq;q) + yx^aq^{a+1}{}_d\overline{\mathscr{B}}^0_{k,k-s-a}(y;xq;q), \qquad (2.1) \\ &{}_d\overline{\mathscr{B}}^s_{k,0}(y;x;q) = 0, \qquad (2.2) \\ &{}_d\overline{\mathscr{B}}^s_{k,a}(y;0;q) = 1. \qquad (2.3) \end{aligned}$$

Notice that the functional equation (2.1) and the initial conditions (2.2) and (2.3) are equivalent to the following recurrence and initial values.

$$
\begin{aligned}
{}_d\overline{b}^{s}_{k,a}(m,n,r) &= {}_d\overline{b}^{s+1}_{k,a-1}(m,n,r)\\
&+ {}_d\overline{b}^{0}_{k,k-s-a+1}(m-a+1,n-a+1,r)\\
&+ {}_d\overline{b}^{0}_{k,k-s-a}(m-a,n-a-1,r-1), \qquad (2.4)\\
{}_d\overline{b}^{s}_{k,0}(m,n,r) &= 0, \qquad (2.5)\\
{}_d\overline{b}^{s}_{k,a}(0,n,r) &= \begin{cases} 1 & \text{if } n=r=0,\\ 0 & \text{otherwise.} \end{cases} \qquad (2.6)
\end{aligned}
$$

It is easy to see that (2.4)–(2.6) uniquely determine ${}_d\overline{b}^{s}_{k,a}(m,n,r)$ because each application of (2.4) decreases one or more parameters. (2.5) and (2.6) are a complete collection of initial conditions. Therefore (2.1), (2.2), and (2.3), uniquely determine ${}_d\overline{\mathscr{B}}^{s}_{k,a}(y;x;q)$.

It appears at first that we also need

$$ {}_d\overline{b}^{s}_{k,a}(m,n,r) = 0 \text{ if } m,n, \text{ or } r < 0. $$

We will momentarily show that the indices cannot go negative. So, the last condition is not essential although it is clearly true.

The initial condition (2.5) is for the fact that there are no overpartitions with $f_1 < 0$. No part can appear a negative number of times. The initial condition (2.6) captures the empty overpartition of zero, which is the only overpartition with no parts.

For the recurrence (2.4), we consider the following collections of overpartitions.

$\mathscr{T}$ = the collection of overpartitions enumerated by ${}_d\overline{b}^{s}_{k,a}(m,n,r)$, i.e., the overpartition of n into m parts, r of which are overlined, such that $f_1 < a$, $f_{\overline{1}} = 0$, $f_i + f_{\overline{i+1}} + f_{i+1} < k$,
and if $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for $\delta = 1, 2, \ldots, d-1$,
then $a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$.

$\mathscr{U}$ = the overpartitions in $\mathscr{T}$ in which $f_1 < a - 1$,

$\mathscr{V}$ = the overpartitions in $\mathscr{T}$ in which $f_1 = a - 1$ and $f_{\overline{2}} = 0$,

$\mathscr{W}$ = the overpartitions in $\mathscr{T}$ in which $f_1 = a - 1$ and $f_{\overline{2}} = 1$.

It is immediate that $\mathscr{T}$ is the disjoint union of $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$. Because the conditions $f_1 < a - 1$; $f_1 = a - 1$ and $f_{\overline{2}} = 0$; and $f_1 = a - 1$ and $f_{\overline{2}} = 1$ are mutually exclusive and complementary.

$\mathscr{U}$ is enumerated by ${}_d\overline{b}^{s+1}_{k,a-1}(m,n,r)$, because $f_1 < a - 1$ and if $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for $\delta = 1, 2, \ldots, d-1$, then $(a-1) + (s+1) - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$, which is equivalent to $a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$.

Next, we will show that the overpartitions in $\mathscr{V}$ are in one-to-one correspondence with the overpartitions counted by ${}_d\overline{b}^0_{k,k-s-a+1}(m-a+1, n-a+1, r)$. In the course, we will argue that the indices cannot go negative.

If we delete the $(a-1)$ 1's and subtract 1 from the remaining parts, nonoverlined and overlined alike, then the parts change parity. The number of parts decreases by $(a-1)$, and becomes $(m-a+1)$. At the beginning, necessarily, $m \geq a-1$ and $n \geq a-1$, so that $m-a+1 \geq 0$ and $n-a+1 \geq 0$ after the transformation.

We know that $f_1 = a-1$ and $f_{\overline{2}} = 0$, so $a-1+f_2 < k$ and $\rho(2) = 0$. Moreover, if $a-1+f_2 = k-\delta$ for $\delta = 1, 2, \ldots, d-1$, then $a+s-1-(a-1)+0 \equiv 0, 1, \ldots, \delta-1 \pmod{d}$, or simply $s \equiv 0, 1, \ldots, \delta-1 \pmod{d}$. Taking $0 \leq s \leq d$ and $1 \leq \delta \leq d$ into consideration, the last congruence asserts $s < \delta$. This, in turn, implies $k-s > k-\delta$, that is $a-1+f_2 = k-\delta < k-s$, so $f_2 < k-s-a+1$. Because $f_2 \geq 0$, $k-s-a+1 > 0$.

Therefore, after the subtraction $f_{\overline{1}} = 0$ and $f_1 < k-s-a+1$.

Suppose λ is a specific overpartition in $\mathscr{V}$. Call the resulting overpartition $\tilde{\lambda}$ after the removal of $(a-1)$ 1's and subtraction of 1's from the other parts. For arbitrary but fixed $i \geq 1$, when $\rho(i) = A$ in λ, then $\tilde{\rho}(i-1) = -A$ in $\tilde{\lambda}$, since no overlined part is deleted and all parts changed parity. Here, $\tilde{\rho}$ denotes the ρ–statistic in $\tilde{\lambda}$. Only one condition remains to verify $\tilde{\lambda}$ is counted by ${}_d\overline{b}^0_{k,k-s-a+1}(m-a+1, n-a+1, r)$. Namely, for $i \geq 2$, if $\tilde{f}_{i-1} + \tilde{f}_{\overline{i}} + \tilde{f}_i = k-\delta$ for $\delta = 1, 2, \ldots, d-1$, then

$$(k-s-a+1)+0-1-\tilde{f}_{\mathrm{odd}(i-1)} - \chi_e(\overline{i}) + \tilde{\rho}(i) \overset{?}{\equiv} 0, 1, \ldots, \delta-1 \pmod{d}, \tag{2.7}$$

where $\tilde{f}$ denotes the frequencies in $\tilde{\lambda}$.

We know that $\tilde{f}_i = f_{i+1}$ and $\chi_e(\overline{i}) + \chi_e(\overline{i+1}) = \tilde{f}_{\overline{i}} = f_{\overline{i+1}}$, so that $\tilde{f}_{\mathrm{odd}(i-1)} + \chi_e(\overline{i}) + \chi_e(\overline{i+1}) + f_{\mathrm{odd}(i)} = k-\delta$. We saw that $\tilde{\rho}(i) = -\rho(i+1)$ as well. Thus, (2.7) is equivalent to

$$k-s-a+1+0-1-(k-\delta-f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1})) - \rho(i+1) \overset{?}{\equiv} 0, 1, \ldots, \delta-1 \pmod{d},$$

or, after some rearrangement, to

$$-a-s+1+f_{\mathrm{odd}(i)} + \chi_e(\overline{i+1}) - \rho(i+1) \overset{?}{\equiv} 0, -1, \ldots, -\delta+1 \pmod{d},$$

that is, after negating both sides,

$$a+s-1-f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \overset{?}{\equiv} 0, 1, \ldots, \delta-1 \pmod{d}.$$

The last condition is satisfied by λ. Therefore (2.7) is satisfied by $\tilde{\lambda}$. It follows that the number of overpartitions in $\mathscr{V}$ is equal to ${}_d\overline{b}^0_{k,k-s-a+1}(m-a+1, n-a+1, r)$.

The correspondence between overpartitions in $\mathscr{W}$ and overpartitions counted by ${}_d\overline{b}^0_{k,k-s-a}(m-a, n-a-1, r-1)$ is constructed likewise. The difference is that there is a $\overline{2}$ in overpartitions in $\mathscr{W}$, so we delete it alongside the $(a-1)$ 1's. A particular overpartition λ in $\mathscr{W}$ after the deletions and subtraction of 1 from the remaining parts becomes $\tilde{\lambda}$. $\tilde{\lambda}$ has $m-a$ parts, $r-1$ of which are overlined, and it yields an overpartition of $n-a-1$. If $\rho(i+1)=A$ in λ, then $\tilde{\rho}(i)=-(A-1)$ because of the deleted $\overline{2}$.

The above arguments establish (2.4), and consequently (2.1), (2.2), and (2.3).

Next, we investigate when $H^s_{k,a,d}(y;xq;q)$ satisfies (2.1), (2.2), and (2.3). For convenience, set

$$\overline{c}_n(y;xq;q)$$
$$=\frac{(-1)^n x^{n(k+1-d)} q^{(2k+1-d)n(n+1)/2+n} y^n (-1/y;q)_n (-yxq^{n+2};q)_\infty}{(q^d;q^d)_n (x^d q^{(n+1)d};q^d)_\infty},$$

so that

$$H^s_{k,a,d}(y;xq;q)=\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}$$
$$\times\sum_{n\geq 0}\overline{c}_n(y;xq;q)q^{-an}q^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]$$
$$-\overline{c}_n(y;xq;q)x^a q^{a(n+1)}q^{sn}\left[q^{-dn}\frac{1-(xq)^s}{1-(xq)^d}+\frac{(xq)^s-(xq)^d}{1-(xq)^d}\right].$$

Observe that $\overline{c}_n(y;xq;q)$ depends on k and d, but not on a or s. The series in (1.4) are

$$H^d_{k,a,d}(y;xq;q)=H^0_{k,a,d}(y;xq;q)$$
$$=\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}\sum_{n\geq 0}\overline{c}_n(y;xq;q)q^{-an}-\overline{c}_n(y;xq;q)x^a q^{a(n+1)}.$$

It is clear that

$$\overline{c}_n(y;0;q)=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

So,

$$H^s_{k,a,d}(y;0;q)=\left[\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]_{x=0}=1. \tag{2.8}$$

Then, we examine

$$H^s_{k,0,d}(y;xq;q) = \frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty} \sum_{n\geq 0} \frac{\overline{c}_n(y;xq;q)}{1-(xq)^d}$$
$$\times \left[q^{(d-s)n}\left((xq)^{d-s}-(xq)^d\right) + q^{-sn}\left(1-(xq)^{d-s}\right)\right.$$
$$\left. -q^{(s-d)n}\left(1-(xq)^s\right) - q^{sn}\left((xq)^s-(xq)^d\right)\right].$$

The expression inside brackets in the last two lines vanishes for $s=0$ or $2s=d$, i.e., $2s \equiv 0 \pmod d$. Empirical evidence suggests that $H^s_{k,0,d}(y;xq;q)$ is nonzero in all other cases, but we do not have a proof of this. So we have to be content with saying

$$H^s_{k,0,d}(y;xq;q) = 0 \quad \text{if} \quad 2s \equiv 0 \pmod d. \tag{2.9}$$

It is worth noting that there are no missing cases for $d=2$.

We finally argue that

$$H^s_{k,a,d}(y;xq;q) - H^{s+1}_{k,a-1,d}(y;xq;q) = (xq)^{a-1} H^0_{k,k-s-a+1,d}(y;xq^2;q)$$
$$+ yx^a q^{a+1} H^0_{k,k-s-a,d}(y;xq^2;q). \tag{2.10}$$

(2.10) is implied by the following relations.

$$\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}\overline{c}_n(y;xq;q)\left[q^{-an}q^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\right.$$
$$\left.-q^{-(a-1)n}q^{-(s+1)n}\left[q^{dn}\frac{(xq)^{d-(s+1)}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-(s+1)}}{1-(xq)^d}\right]\right]$$
$$=\frac{\left((xq^2)^d;q^d\right)_\infty}{(xq^2;q)_\infty}\overline{c}_{n-1}(y;xq^2;q)$$
$$\times\left(-(xq)^{a-1}(xq^{n+1})^{k-s-a+1}+yx^a q^{a+1}(xq^{n+1})^{k-s-a}\right),$$

$$\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}\overline{c}_n(y;xq;q)$$
$$\times\left[-(xq^{n+1})^a q^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\right.$$
$$\left.+(xq^{n+1})^{a-1}q^{-(s+1)n}\left[q^{dn}\frac{(xq)^{d-(s+1)}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-(s+1)}}{1-(xq)^d}\right]\right]$$
$$=\frac{\left((xq^2)^d;q^d\right)_\infty}{(xq^2;q)_\infty}\overline{c}_n(y;xq^2;q)$$
$$\times\left((xq)^{a-1}(q^{-n})^{k-s-a+1}-yx^a q^{a+1}(q^{-n})^{k-s-a}\right).$$

These are straightforward verifications.

The left-hand side of (2.10) suggests that the quantity $(a+s)$ is an invariant. On the other hand, (2.9) imposes

$$2(a+s) \equiv 0 \pmod{d}.$$

Applying this to the right-hand side of (2.10), we have

$$2(k-s-a+1) \equiv 0 \pmod{d}, \quad \text{and} \quad 2(k-s-a) \equiv 0 \pmod{d},$$

which forces $2 \equiv 0 \pmod{d}$. In other words, $d=1$ or $d=2$.

Since (2.1), (2.2), and (2.3) uniquely determine ${}_d\overline{\mathscr{B}}^s_{k,a}(y;x;q)$, we conclude that

$$H^s_{k,a,d}(y;xq;q) = {}_d\overline{\mathscr{B}}^s_{k,a}(y;x;q) \quad \text{for } d=1 \text{ or } d=2,$$

by (2.8), (2.9), (2.10), and the above congruences. That is,

$$H^s_{k,a,d}(y;xq;q) = \sum_{m,n,r\geq 0} {}_d\overline{b}^s_{k,a}(m,n,r)x^m q^n y^r$$

for $d=1$ or $d=2$.

Corollary 2.1. *Let ${}_d\overline{\eta}^s_{k,a}(m,n,r)$ be the number of overpartitions of n into m parts, r of which are overlined, such that*

$$\begin{aligned}
&f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 + f_{\overline{1}} < a,\\
&\textit{if } f_i + f_{\overline{i+1}} + f_{i+1} = k-\delta \textit{ for } \delta = 1,2\ldots,d-1,\\
&\textit{then } a+s-1-f_{odd(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0,1,\ldots,\delta-1 \pmod{d}.
\end{aligned}$$

Then,

$$\begin{aligned}
J^s_{k,a,d}(y;x;q) &:= H^s_{k,a,d}(y;xq;q) + xyqH^s_{k,a-1,d}(y;xq;q)\\
&= \sum_{m,n,r\geq 0} {}_d\overline{\eta}^s_{k,a}(m,n,r)x^m q^n y^r
\end{aligned}$$

for $d=1$ or $d=2$.

Proof. Let λ be an overpartition enumerated by ${}_d\overline{\eta}^s_{k,a}(m,n,r)$. If λ has no $\overline{1}$, then it is also counted by ${}_d\overline{b}^s_{k,a}(m,n,r)$.

If λ has an $\overline{1}$, then erase it to obtain $\tilde{\lambda}$. $\tilde{\lambda}$ is an overpartition of $n-1$ into $m-1$ parts, $r-1$ of which are overlined, because of the deleted $\overline{1}$. $\tilde{\rho}(i) = \rho(i)+1$ for the same reason, where $\tilde{\rho}$ is the ρ-statistic in $\tilde{\lambda}$. Also, $f_1 < a-1$ in $\tilde{\lambda}$, because $f_1 + f_{\overline{1}} < a$ and $f_{\overline{1}} = 1$ in λ.

Now, λ satisfies

$$a + s - 1 - f_{\text{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$$

when $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for some $\delta = 1, 2, \ldots, d$ and $i \in \mathbb{Z}^+$. So,

$$(a-1) + s - 1 - f_{\text{odd}(i)} - \chi_e(\overline{i+1}) + \tilde{\rho}(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$$

for $\tilde{\lambda}$. Therefore, $\tilde{\lambda}$ is enumerated by ${}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$. Conversely, we can append an $\overline{1}$ to any overpartition counted by ${}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$, and obtain one counted by ${}_d\overline{\eta}^s_{k,a}(m, n, r)$.

Having or lacking $\overline{1}$ are mutually exclusive and complementary cases for λ. We have shown that

$${}_d\overline{\eta}^s_{k,a}(m, n, r) = {}_d\overline{b}^s_{k,a}(m, n, r) + {}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$$

which implies the corollary.

In [14], Lovejoy states that $J^0_{k,a,1}(1; 1; q)$ is not an infinite product, but a combination of two infinite products. One still has a partition identity in this case, because we can interpret both infinite products as partition enumerants, and obtain an identity in the form of

$$A(n) - A(n-*) = B(n) - B(n-*) + C(n) - C(n-*),$$

which admittedly is not as elegant as the classical partition identities. Above, $*$ stands for various fixed positive integers, not necessarily the same in each occurence.

Lovejoy, however, observes also that $J^0_{k,a,1}(1/q; 1; q)$ is a combination of two infinite products. In this case, as in the proof of the above Corollary, one subtracts another 1 from all overlined parts, and a possibility of an overlined zero arises. Then, an overpartition identity cannot be obtained since some partitions will be counted twice. One needs further work to eliminate the occurrence of overlined zero.

It is interesting to substitute $y = 0$ in the series $H^s_{k,a,d}(y; xq; q)$ and see what one obtains:

$$\begin{aligned} &J^s_{k,a,d}(0; x; q) = H^s_{k,a,d}(0; xq; q) \\ =&\frac{((xq)^d; q^d)_\infty}{(xq; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n x^{n(k+1-d)} q^{(2k+2-d)n(n+1)/2 - an}}{(q^d; q^d)_n (x^d q^{nd}; q^d)_\infty} \\ &\times q^{-sn} \left[q^{dn} \frac{(xq)^{d-s} - (xq)^d}{1 - (xq)^d} + \frac{1 - (xq)^{d-s}}{1 - (xq)^d} \right] \end{aligned}$$

$$- \frac{(-1)^n x^{n(k+1-d)+a} q^{(2k+2-d)n(n+1)/2+a(n+1)}}{(q^d;q^d)_n (x^d q^{nd};q^d)_\infty} \times q^{sn} \left[q^{-dn} \frac{1-(xq)^s}{1-(xq)^d} + \frac{(xq)^s - (xq)^d}{1-(xq)^d} \right].$$

This the exact same series as [13, Lemma 11]. There, the series was constructed from scratch.

3 Constructions

In this section, we will show that all results stated above can be proven linearly and constructively. In other words, there is no need for formal verifications that a proposed series is indeed the generating function sought for. Given the description of partition classes, we will *construct* their generating function as a series. We will carry out computations for one example in detail.

Let's recall the partition enumerant in Corollary 2.1. Let ${}_d\overline{\eta}^s_{k,a}(m,n,r)$ be the number of overpartitions of n into m parts, r of which are overlined, such that

$$\begin{aligned}
&f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 + f_{\overline{1}} < a, \\
&\text{if } f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta \text{ for } \delta = 1, 2 \ldots, d-1, \\
&\text{then } a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}.
\end{aligned}$$

It is possible (as in the proof of Theorem 2.1) to justify the following recurrences and initial conditions.

$$\begin{aligned}
{}_d\overline{\eta}^s_{k,a}(m,n,r) = {}&{}_d\overline{\eta}^{s+1}_{k,a-1}(m,n,r) \\
&+ {}_d\overline{\eta}^0_{k,k-a-s+1}(m-a+1, n-a+1, r) \\
&+ {}_d\overline{\eta}^0_{k,k-a-s+2}(m-a+1, n-a+1, r-1),
\end{aligned} \tag{3.1}$$

$${}_d\overline{\eta}^s_{k,a}(0,n,r) = \begin{cases} 1 & \text{if } n = r = 0, \\ 0 & \text{otherwise }, \end{cases} \tag{3.2}$$

$${}_d\overline{\eta}^s_{k,1}(m,n,r) = {}_d\overline{\eta}^0_{k,k-s}(m,n,r). \tag{3.3}$$

It is fairly clear that equations (3.1) for $a = 2, 3, \ldots, k$, (3.2), and (3.3) uniquely determine ${}_d\overline{\eta}^s_{k,a}(m,n,r)$. The reason we did not use ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$ is that the equation (3.1) already needs a reinterpretation for $a = 1$, and the reinterpretation (3.3) implies that ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$. Another reason for not explicitly stating ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$ is that the computations will not yield it explicitly. Still we will be able to construct the series.

Set

$$Q_a^s(x) := Q_{k,a,d}^s(y;x,q) = \sum_{m,n,r\geq 0} {}_d\overline{\eta}_{k,a}^s(m,n,r)x^m y^r q^n.$$

We suppress writing d, k, y, and q because they are unchanged throughout the computations.

The conditions (3.1)–(3.3) are translated as the following.

$$Q_a^s(x) - Q_{a-1}^{s+1} = (xq)^{a-1} Q_{k-a-s+1}^0(xq) + y(xq)^{a-1} Q_{k-a-s+2}^0(xq) \tag{3.4}$$

for $a = 2, 3, \ldots, k$,

$$Q_a^s(0) = 1, \tag{3.5}$$

$$Q_1^s(x) = Q_{k-s}^0(xq), \tag{3.6}$$

and these functional equations and the initial conditions uniquely determine $Q_a^s(x)$.

The next step is taking Andrews's analytic proof of Gordon's theorem [1] as a black box, and assuming that $Q_a^s(x)$ is of the form

$$Q_a^s(x) = \sum_{n\geq 0} \alpha_n^s(x) q^{-na} + \beta_n^s(x)(xq^{n+1})^a. \tag{3.7}$$

$\alpha_n^s(x)$ and $\beta_n^s(x)$ depend on d, k, y, and q; but not on a. Again, we imitate the mechanism in [1] to assert that

$$\begin{aligned}\alpha_n^s(x)q^{-na} - {}&\alpha_n^{s+1}(x)q^{-na+n} \\ &= (xq)^{a-1}\beta_{n-1}^0(xq)(xq^{n+1})^{k-a-s+1} \\ &+ y(xq)^{a-1}\beta_{n-1}^0(xq)(xq^{n+1})^{k-a-s+2},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\beta_n^s(x)(xq^{n+1})^a - \beta_n^{s+1}(x)(xq^{n-1})^{a-1} \\ &= (xq)^{a-1}\alpha_n^0(xq)(q^{-n})^{k-a-s+1} + y(xq)^{a-1}\alpha_n^0(xq)(q^{-n})^{k-a-s+2}.\end{aligned} \tag{3.9}$$

It is useful to keep in mind that although $s = 0, 1, \ldots, d-1$, it is interpreted as a residue class $\pmod d$. So, $\alpha_n^d(x) = \alpha_n^0(x)$, and $\beta_n^d(x) = \beta_n^0(x)$.

The recurrences (3.8) and (3.9) imply (3.4). The idea is to discover α's and β's first, then imposing the initial conditions (3.5) and (3.6).

The reader can check that if one tries to make α's and β's independent of s as well, one either encounters inconsistent equations, or has to adjust the format of (3.7). In the latter case, the adjustment is more difficult to come up with, and the resulting equations are much harder to solve. Empirical evidence shows that this is a convenient way to proceed.

The equations (3.8) and (3.9) can be simplified as

$$\alpha_n^s(x) - q^n\alpha_n^{s+1}(x) = (xq^{n+1})^k q^n (xq^{n+1})^{-s}(1+yxq^{n+1})\beta_{n-1}^0(xq),$$

and

$$xq^{n+1}\beta_n^s(x) - \beta_n^{s+1}(x) = (q^{-n})kq^{ns}yq^{-n}(1+q^n/y)\alpha_n^0(xq).$$

We can collect equations for various s's and write them in matrix form.

$$\begin{bmatrix} 1 & -q^n & & \\ & 1 & -q^n & \\ & & \ddots & \\ -q^n & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_n^0(x) \\ \alpha_n^1(x) \\ \vdots \\ \alpha_n^{d-1}(x) \end{bmatrix}$$
$$= (xq^{n+1})^k q^n (1+yxq^{n+1})\beta_{n-1}^0(xq) \begin{bmatrix} 1 \\ (xq^{n+1})^{-1} \\ \vdots \\ (xq^{n+1})^{-d+1} \end{bmatrix},$$

and

$$\begin{bmatrix} xq^{n+1} & -1 & & \\ & xq^{n+1} & -1 & \\ & & \ddots & \\ -1 & & & xq^{n+1} \end{bmatrix} \begin{bmatrix} \beta_n^0(x) \\ \beta_n^1(x) \\ \vdots \\ \beta_n^{d-1}(x) \end{bmatrix}$$
$$= (q^{-n})^k q^{-n} y(1+q^n/y)\alpha_n^0(xq) \begin{bmatrix} 1 \\ q^n \\ \vdots \\ (q^n)^{d-1} \end{bmatrix}.$$

The displayed matrices have the following respective inverses.

$$\frac{1}{(1-q^{dn})} \begin{bmatrix} 1 & q^n & q^{2n} & \cdots & q^{dn-n} \\ q^{dn-n} & 1 & q^n & \cdots & q^{dn-2n} \\ & & \vdots & & \\ q^n & q^{2n} & q^{3n} & \cdots & 1 \end{bmatrix},$$

and

$$\frac{(-1)}{(1-(xq^{n+1})^d)}\begin{bmatrix}(xq^{n+1})^{d-1} & (xq^{n+1})^{d-2} & \cdots & 1\\ 1 & (xq^{n+1})^{d-1} & \cdots & (xq^{n+1})\\ & \vdots & & \\ (xq^{n+1})^{d-2} & (xq^{n+1})^{d-3} & \cdots & (xq^{n+1})^{d-1}\end{bmatrix}.$$

Multiplying by the corresponding inverse matrix on both sides, and performing the matrix-vector multiplication, we obtain

$$\begin{aligned}\alpha_n^s(x) &= \frac{(xq^{n+1})^k q^n (xq)^{1-d}(1+yxq^{n+1})(1-(xq)^d)}{(1-q^{dn})(1-xq)}\beta_{n-1}^0(xq)\\ &\times\left[q^{(d-s)n}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+q^{-sn}\frac{1-(xq)^{d-s}}{1-(xq)^d}\right],\end{aligned}$$

and

$$\begin{aligned}\beta_n^s(x) &= \frac{(-1)(q^{-n})^k q^{-n}(q^{-n})^{1-d}y(1+q^n/y)(1-(xq)^d)}{(1-(xq^{n+1})^d)(1-xq)}\alpha_n^0(xq)\\ &\times\left[q^{(s-d)n}\frac{1-(xq)^s}{1-(xq)^d}+q^{sn}\frac{(xq)^s-(xq)^d}{1-(xq)^d}\right].\end{aligned}$$

Please notice that the only part involving s in both recurrences is inside the brackets on the right-hand sides, and both brackets evaluate to 1 for $s=0$ or $s=d$.

Unfolding the last two equations, we first find

$$\begin{aligned}\alpha_n^0(x) &= (-1)(xq^2)^{k+1-d}q^{n(d-1)+1}y\\ &\times\frac{(1+q^{n-1}/y)(1+yxq^{n+1})(1-(xq)^d)(1-(xq^2)^d)}{(1-q^{dn})(1-(xq^{n+1})^d)(1-xq)(1-xq^2)}\alpha_{n-1}^0(xq^2),\end{aligned}$$

and then

$$\begin{aligned}\alpha_n^0(x) &= (-1)^n x^{(k+1-d)n}q^{(2k+1-d)\binom{n+1}{2}+n}y^n\\ &\times\frac{(-1/y;q)_n(-yxq^{n+1};q)_n((xq)^d;q^d)_{2n}}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_n(xq;q)_{2n}}\alpha_0^0(xq^{2n}).\end{aligned}$$

Defining

$$\widetilde{\alpha}_0^0(x) = \frac{((xq)^d;q^d)_\infty(xq;q)_\infty}{(-yxq;q)_\infty((xq)^d;q^d)_\infty}\alpha_0^0(x),$$

we finally get

$$\alpha_n^0(x) = (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+n} y^n \\ \times \frac{(-1/y;q)_n(-yxq^{n+1};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty}\widetilde{\alpha}_0^0(xq^{2n}).$$

Then, in the order given below, we find

$$\beta_n^0(x) = -(-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^{n+1} \\ \times \frac{(-1/y;q)_{n+1}(-yxq^{n+2};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty}\widetilde{\alpha}_0^0(xq^{2n+1}),$$

$$\alpha_n^s(x) = (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+n} y^n \\ \times \frac{(-1/y;q)_n(-yxq^{n+1};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty} \\ \times \left[q^{(d-s)n}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d} + q^{-sn}\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\widetilde{\alpha}_0^0(xq^{2n}),$$

and

$$\beta_n^s(x) = -(-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^{n+1} \\ \times \frac{(-1/y;q)_{n+1}(-yxq^{n+2};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty} \\ \times \left[q^{(s-d)n}\frac{1-(xq)^s}{1-(xq)^d} + q^{sn}\frac{(xq)^s-(xq)^d}{1-(xq)^d}\right]\widetilde{\alpha}_0^0(xq^{2n+1}).$$

The first initial condition (3.5) is easily seen to hold as long as $\widetilde{\alpha}_0^0(0) = 1$. The subsequent computations will show that there is no harm in taking $\widetilde{\alpha}_0^0(x) = 1$.

There are two options for the other initial condition. The more obvious

$$Q_0^s(x) = 0$$

does not seem to hold, unfortunately. Therefore, one has to resort to (3.6), which is

$$Q_1^s(x) = Q_{k-s}^0(xq).$$

$Q_1^s(x)$ will be used as is. $Q_{k-s}^0(xq)$ needs a small transformation.

$$
\begin{aligned}
Q^0_{k-s}(xq) = \sum_{n\geq 0} & (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(s+2-d)n} y^n \\
& \times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
& - (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(k+1-d)n} y^{n+1} \\
& \times \frac{(-1/y;q)_{n+1}(-yxq^{n+3};q)_\infty((xq^2)^d;q^d)_\infty(xq^{n+2})^{k-s}}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
= \sum_{n\geq 0} & (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(s+2-d)n} y^n \\
& \times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
& + (-1)^n x^{(k+1-d)n+d-1-s} q^{(2k+1-d)\binom{n+1}{2}-sn+d-s-1} y^n \\
& \times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_{n-1}((xq^{n+1})^d;q^d)_\infty(xq^2;q)_\infty}
\end{aligned}
$$

In particular, we shifted the index $n \leftarrow (n-1)$ in the second term. The introduction of the $n = -1$ term in the sum is no problem since $1/(q^d;q^d)_{-1} = 0$. Noticing the common factor

$$
\begin{aligned}
\mathscr{C}_n :=& (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^n \\
& \times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty},
\end{aligned}
$$

the series at hand become

$$
\begin{aligned}
Q^s_1(x) =& \sum_{n\geq 0} \mathscr{C}_n \left\{(1+yxq^{n+1})\right. \\
& \times \left[q^{(d-s)n}\left((xq)^{d-s}-(xq)^d\right) + q^{-sn}\left(1-(xq)^d\right)\right] \\
& - xq^{n+1}y(1+q^n/y) \\
& \times \left.\left[q^{(s-d)n}\left(1-(xq)^s\right) + q^{sn}\left((xq)^s-(xq)^d\right)\right]\right\}, \\
Q^0_{k-s}(xq) =& \sum_{n\geq 0} \mathscr{C}_n \left\{q^{(s+2-d)n}\left(1-(xq^{n+1})^d\right)(1-xq)\right. \\
& \left. + x^{d-1-s} q^{-s(n+1)+d-1}(1-q^{dn})(1-xq)\right\}.
\end{aligned}
$$

Now one can use a computer algebra system to examine the difference of the expressions in curly braces. Their difference is zero for $d = 1, 2$ and all corresponding s. It is empirically nonzero for $d \geq 3$ and various s. This ends the construction along with the proof of Corollary 2.1.

It should be possible to examine the aforementioned differences, and prove that the condition $d = 1, 2$ in the results is not only sufficient but also necessary.

4 The Case of $d \geq 3$

In this section, we report on an exploration that shows that a naïve approach to finding interpretations of the series (1.5) yields complicated (or fascinating, as per one's taste) results. At the end, for concreteness, we work with $d = 3, k = 5$.

The main idea of this exploration is to "reverse engineer" the process of deducing recurrences satisfied by generating functions of a certain class of partitions. Such a process is an important step in the motivated proof of Rogers–Ramanujan identities as given by Andrews and Baxter [4].

4.1 *Motivating example*

As an example of what we mean, let us explain this reverse engineering process applied to the familiar Rogers–Ramanujan identities. Suppose that one is presented with a formal series

$$F(x,q) = \sum_{m,n\geq 0} f_{m,n} x^m q^n$$

with integral coefficients with the following conditions:

$$f_{0,0} = 1 \tag{4.1}$$

$$f_{m,n} = 0 \quad \text{if } m > n \tag{4.2}$$

$$F(x,q) = F(xq,q) + xqF(xq^2,q). \tag{4.3}$$

These conditions tell us that the coefficients of F are nonnegative. Now, the first two conditions hint at the fact that perhaps $f_{m,n}$ counts certain partitions of n with m parts. With this ansatz, we can now make additional guesses. The transformation $x \mapsto xq^j$ corresponds to adding j to every part of the partition, and multiplication by xq^j corresponds to inserting the part j in the partition. Now we can start "building" the partitions possibly counted by F using the recurrence (4.3).

Let us call the class of partitions of n with exactly m parts counted in F by $\pi_{m,n}$. Let us denote the null partition by $\mathbf{0}$. Thus, $a_{0,0} = 1$ counts this null partition. Note that by definition, we let $\pi_{m,n} = \{\}$ if $(m,n) \notin \{(x,y) \,|\, x \geq 1, y \geq 1, x \leq y\} \cup \{(0,0)\}$. Then, (4.3) written with partitions in mind reads as follows:

To find $\pi_{m,n}$, take the union of the following two sets:

1. For each partition appearing in $\pi_{m,n-m}$, add 1 to every part (corresponds to the term $F(xq,q)$).
2. For each partition appearing in $\pi_{m-1,n-1-2(m-1)}$, add 2 to every part and then adjoin the part 1 to each of the resulting partitions. (corresponds to the term $xqF(xq^2,q)$).

Doing this process, we arrive a "partition generating function" $\Pi(x, q)$ of the sets of partitions $\pi_{n,m}$ as follows:

$$\begin{aligned}
&\Pi(x, q)\\
&= \{\mathbf{0}\}x^0q^0\\
&+ \{(1)\}xq + \{(2)\}xq^2 + \{(3)\}xq^3 + \{(4)\}xq^4 + \{(5)\}xq^3 + \{(6)\}xq^4 + \cdots\\
&+ \{(1, 3)\}x^2q^4 + \{(1, 4)\}x^2q^5 + \{(1, 5), (2, 4)\}x^2q^6 + \cdots\\
&+ \{(1, 3, 5)\}x^3q^9 + \{(1, 3, 6)\}x^3q^{10} + \{(1, 3, 7), (1, 4, 6)\}x^3q^{11} + \cdots\\
&+ \cdots .
\end{aligned}$$

Doing this for sufficiently high powers $x^i q^j$, one can see a pattern emerging:

> $f_{m,n}$ counts the number of partitions of n with exactly m parts in which adjacent parts differ by at least 2.

What we have done is a naïve enrichment of F to a "partition generating function" and a naïve enrichment of (4.3) to a recurrence of "partition generating functions." However, for an arbitrary recurrence, the following problems could arise:

1. The coefficients $f_{m,n}$ may not be (manifestly) nonnegative.
2. The sets of partitions that arise from various summands may not be disjoint.

These necessitate that we instead look at "partition generating functions" with integral weights attached to the partitions. We, therefore, formalize the reverse engineering process given above in an algebraic language as given in the next subsection.

4.2 An algebraic formalism

One can avoid such an algebraic language altogether, however, it facilitates a succinct exposition of our ideas.

Definition 4.1. Let $\mathscr{P}$ denote the set of all partitions and $\mathscr{P}_{m,n}$ denote the set of all partitions of n with exactly m parts. By convention, $\mathscr{P}_{0,0} = \{\mathbf{0}\}$.

Definition 4.2. Let $\mathbf{P}$ denote the free $\mathbb{Z}$-module generated by $\mathscr{P}$. Similarly, let $\mathbf{P}_{m,n}$ denote the free $\mathbb{Z}$-module generated by $\mathscr{P}_{m,n}$.

Definition 4.3. Let $\mathbf{P}[[x, q]]$ denote the space of two variable generating functions with coefficients in $\mathbf{P}$. We say that $f \in \mathbf{P}[[x, q]]$ is a partition generating function if the coefficient of $x^m q^n$ of f lies in $\mathbf{P}_{m,n}$. We denote the space of partition generating functions by $\mathbf{F}$. It is clear that $\mathbf{F}$ is a $\mathbb{Z}$-submodule of $\mathbf{P}[[x, q]]$.

Henceforth, we shall employ the following convention.

Convention 4.1 *We shall write the partitions as tuples of positive integers in a nondecreasing order. For instance,* $\pi = (1, 1, 2, 3, 4, 15) \in \mathscr{P}_{6,26}$. *We shall also think of* $\mathscr{P}_{m,n}$ *as a subset of* $\mathbf{P}_{m,n}$. *Given* $f \in \mathbf{F}$, *we will denote the coefficient of* $x^m q^n$ *by* $f_{m,n}$.

We define the following $\mathbb{Z}$-linear maps:

Definition 4.4 Let $\sigma : \mathbf{P}_{m,n} x^m q^n \to \mathbf{P}_{m,n+m} x^m q^{n+m}$ be the unique map such that $\sigma(\pi x^m q^n) = \tilde{\pi} x^m q^{n+m}$ where $\pi \in \mathscr{P}_{m,n}$ and $\tilde{\pi} \in \mathscr{P}_{m,n+n}$ is obtained by adding 1 to every part of π. Note that the null partition $\mathbf{0}$ does not have any parts, and hence $\sigma(\mathbf{0}) = \mathbf{0}$.

Definition 4.5 Let $\alpha : \mathbf{P}_{m,n} x^m q^n \to \mathbf{P}_{m+1,n+1} x^{m+1} q^{n+1}$ be the unique map such that $\sigma(\pi x^m q^n) = \tilde{\pi} x^{m+1} q^{n+1}$ where $\pi \in \mathscr{P}_{m,n}$ and $\tilde{\pi} \in \mathscr{P}_{m,n+n}$ is obtained by adjoining 1 to π.

Definition 4.6. Let $\chi : \mathbf{P}_{m,n} x^m q^m \to \mathbb{Z} x^m q^n$ be the unique map such that $\chi(\pi x^m q^n) = x^m q^n$ where $\pi \in \mathscr{P}_{m,n}$.

We may and do extend the maps σ, α, and χ to the space $\mathbf{F}$ of partition generating functions.

Proposition 4.1. *Let* $f \in \mathbf{F}$. *Then, the following hold.*

$$(\chi(\sigma(f)))(x, q) = (\chi(f))(xq, q)$$
$$(\chi(\alpha(f)))(x, q) = xq \cdot (\chi(f))(x, q).$$

Now we can lift the recurrence (4.3) as a recurrence of partition generating function as follows:

$$\Pi(x, q) = (\sigma \Pi)(x, q) + (\alpha\sigma^2 \Pi)(x, q).$$

With the help of computers, it is a trivial matter to generate enough data for such a generating function.

4.3 Recurrence for $J_{k,a,d}$

In this subsection, we derive the recurrences followed by $J_{k,a,d}$. The following statements are easy generalizations of the results in [9]. First, recall from [9], with $[d]_x = (1 + x + \cdots + x^{d-1})$:

$$J_{k,a,d}(y, xq, q) = H_{k,a,d}(y, xq, q) + yxq H_{k,a-1,d}(y, xq, q) \tag{4.4}$$
$$H_{k,0,d}(y, x, q) = 0 \tag{4.5}$$
$$H_{k,-a,d}(y, x, q) = -x^{-a} H_{k,a,d}(y, x, q) \tag{4.6}$$
$$H_{k,a,d}(y, x, q) - H_{k,a-d,d}(y, x, q) = x^{a-d} [d]_x J_{k,k-a+1,d}(y, x, q) \tag{4.7}$$

Invoking (4.7) with $a = t$ and $a = d - t$ and dropping the implicit arguments y, x, q, we have:

$$H_{k,t,d} - H_{k,t-d,d} = x^{t-d}[d]_x J_{k,k-t+1,d}$$
$$H_{k,d-t,d} - H_{k,-t,d} = x^{-t}[d]_x J_{k,k-d+t+1,d}.$$

Using equation (4.6), rearranging:

$$H_{k,t,d} + x^{t-d} H_{k,d-t,d} = x^{t-d}[d]_x J_{k,k-t+1,d}$$
$$x^{-t} H_{k,t,d} + H_{k,d-t,d} = x^{-t}[d]_x J_{k,k-d+t+1,d}.$$

Solving, we get:

$$(1 - x^{-d}) H_{k,d-t,d} = [d]_x (x^{-t} J_{k,k+t+1-d,d} - x^{-d} J_{k,k-t+1,d})$$

The equation one gets for $H_{k,t,d}$ is just $t \mapsto d - t$. Simplifying,

$$(x^d - 1) H_{k,d-t,d} = [d]_x (x^{d-t} J_{k,k+t+1-d,d} - J_{k,k-t+1,d})$$

Hence,

$$H_{k,d-t,d} = \frac{x^{d-t} J_{k,k+t+1-d,d} - J_{k,k-t+1,d}}{x - 1}$$

Letting $t \mapsto d - t$:

$$H_{k,t,d} = \frac{x^t J_{k,k-t+1,d} - J_{k,k+t-d+1,d}}{x - 1}$$

We can now deduce the following:

1. For $a = 1$, we have that:

$$\begin{aligned} J_{k,1,d}(y, x, q) &\\ &= H_{k,1,d}(y, xq, q) + yxq H_{k,0,d}(y, xq, q) = H_{k,1,d}(y, xq, q) \\ &= \frac{1}{xq - 1} \left\{ xq J_{k,k,d}(y, xq, q) - J_{k,k+2-d,d}(y, xq, q) \right\} \end{aligned} \tag{4.8}$$

   We get the correct (2.4) from [9] with $d = 2$.
2. Let $1 < a < d$:

$$\begin{aligned} J_{k,a,d}(y, x, q) &\\ &= H_{k,a,d}(y, xq, q) + yxq H_{k,a-1,d}(y, xq, q) \end{aligned}$$

$$= \frac{1}{xq-1}\left\{(xq)^a J_{k,k-a+1,d}(y,xq,q) - J_{k,k+a-d+1,d}(y,xq,q) \right.$$
$$\left. + y(xq)^a J_{k,k-a+2,d}(y,xq,q) - yxq J_{k,k+a-d,d}(y,xq,q)\right\}. \tag{4.9}$$

3. Letting $a = d$ and then using the expression for H in terms of J:

$$\begin{aligned} &J_{k,d,d}(y,x,q) \\ &= H_{k,d,d}(y,xq,q) + yxqH_{k,d-1,d}(y,xq,q) \\ &= H_{k,d,d}(y,xq,q) - H_{k,0,d}(y,xq,q) + axqH_{k,d-1,d}(y,xq,q) \\ &= [d]_{xq}J_{k,k-d+1,d}(y,xq,q) + yxqH_{k,d-1,d}(d,xq,q) \\ &= [d]_{xq}J_{k,k-d+1,d}(y,xq,q) + yxq\frac{(xq)^{d-1}J_{k,k+2-d,d}(y,xq,q) - J_{k,k,d}(y,xq,q)}{xq-1} \end{aligned} \tag{4.10}$$

Note that when $d = 2$, this specializes to (2.5) of [9].

4. For $d + 1 \le a \le k$,

$$\begin{aligned} &J_{k,a,d}(y,x,q) - J_{k,a-d,d}(y,x,q) \\ &= H_{k,a,d}(y,xq,q) + yxqH_{k,a-1,d}(y,xq,q) \\ &\quad - H_{k,a-d,d}(y,xq,q) - yxqH_{k,a-d-1,d}(y,xq,q) \\ &= (xq)^{a-d}[d]_{xq}\left(J_{k,k-a+1,d}(y,xq,q) + yJ_{k,k-a+2,d}(y,xq,q)\right) \end{aligned} \tag{4.11}$$

Note that for $d = 2$, we correctly get (2.6) of [9].

4.4 A concrete exploration

For our explorations, we let $d = 3$, $k = 5$, and $y \mapsto 0$, to begin with. We have the following recurrences. For convenience, we shall abbreviate $J_{5,a,3}(0, x, q)$ by $J_a(x, q)$.

$$(xq-1)J_1(x,q) = xqJ_5(xq,q) - J_4(xq,q) \tag{4.12}$$
$$(xq-1)J_2(x,q) = (xq)^2J_4(xq,q) - J_5(xq,q) \tag{4.13}$$
$$J_3(x,q) = X_d(xq)J_3(xq,q) \tag{4.14}$$
$$J_4(x,q) - J_1(x,q) = xqX_d(xq)J_2(xq,q) \tag{4.15}$$
$$J_5(x,q) - J_2(x,q) = (xq)^2X_d(xq)J_1(xq,q). \tag{4.16}$$

We deduce the following functional equations for the "partition generating functions" (we drop the implicit arguments x, q):

$$F_1 = \alpha F_1 - \alpha\sigma F_5 + \sigma F_4 \tag{4.17}$$

$$F_2 = \alpha F_2 - \alpha^2\sigma F_4 + \sigma F_5 \tag{4.18}$$

$$F_3 = (1 + \alpha^1 + \alpha^2)\sigma F_3 \tag{4.19}$$

$$F_4 = F_1 + (\alpha + \alpha^2 + \alpha^3)\sigma F_2 \tag{4.20}$$

$$F_5 = F_2 + (\alpha^2 + \alpha^3 + \alpha^4)\sigma F_1. \tag{4.21}$$

The partition generating functions hold some nice patterns for small partitions, but one quickly gets complicated coefficients. A computer search reveals the following coefficients in the expansion of F_1:

$$\begin{aligned}
&2\cdot(1,2,2,2,2,3,3,4)x^8q^{19}\\
&3\cdot(1,2,2,2,2,3,3,4,4,5)x^{10}q^{28}\\
&2\cdot(1,2,2,2,2,3,3,4,4,5,5,6)x^{12}q^{39}\\
&-2\cdot(1,2,2,2,2,3,3,4,4,5,5,6,6,7)x^{14}q^{52}\\
&-8\cdot(1,2,2,2,2,3,3,4,4,5,5,6,6,7,7,8)x^{16}q^{67}\\
&-12\cdot(1,2,2,2,2,3,3,4,4,5,5,6,6,7,7,8,8,9)x^{18}q^{84}.
\end{aligned}$$

Of course, for each $x^m q^n$ appearing above, there are also other partitions of n with length m besides the ones mentioned above that yield nonzero coefficients.

One may very easily do explorations with y not specialized to 0. In this case, we assume that we are working with two-colored partition where parts may appear overlined, with y counting overlined parts and x counting total number of parts.

Looking at equations (4.8)–(4.11), observe the following: Whenever a new overlined part is introduced, that is, whenever we have a factor of y on the right-hand sides, the newly introduced overlined part is always a $\overline{1}$ (a term of the sort $y(xq)^t$ corresponds to introducing one $\overline{1}$ and $t-1$ nonoverlined 1s). Moreover, the term with y on the right-hand sides of equations (4.8)–(4.11) is always multiplied with a shifted, (that is, $x \mapsto xq$) generating function. This implies that we get nonzero coefficients in the corresponding partition generating functions only if the overlined parts do not repeat.

However, further computer search reveals interesting patterns; we have the following terms as a sample:

$$\begin{aligned}
&2\cdot(1,2,2,3,3,\overline{4})\,ax^6q^{15}\\
&2\cdot(1,1,2,2,3,3,\overline{4})\,ax^7q^{16}\\
&2\cdot(1,1,1,2,2,3,3,\overline{4})\,ax^8q^{17},
\end{aligned}$$

and

$$-2\cdot(1, 2, 3, 3, 3, \overline{3})\, ax^6 q^{15}$$
$$-2\cdot(1, 1, 2, 3, 3, 3, \overline{3})\, ax^7 q^{16}$$
$$-2\cdot(1, 1, 1, 2, 3, 3, 3, \overline{3})\, ax^8 q^{17},$$

etc.

5 Further research

We suggest the following directions for further research:

1. Carry out the explorations in Subsection 4.2 with other values of d and k.
2. First lesson to be learnt from Section 2 is that maybe it is too much to hope that partition generating functions like F_i count something meaningful. Instead, it will be worthwhile to explore if linear combinations of F_i hold interesting information.
3. Second lesson to be learnt is that may be only certain combinations of values (k, a, d) yield interesting results. However, which values of k, a to choose when $d \geq 3$ is not clear yet.
4. Write a computer algebra program to automate the construction in §3. This is a partial converse to the theory developed in [15] in the context of Rogers–Ramanujan generalizations. The $WZ-$theory constructs recurrences given series. In contrast, §3 constructs q-series given functional equations.

Acknowledgements The authors would like to thank the referee for helpful suggestions.

References

1. G.E. Andrews, An analytical proof of the Rogers-Ramanujan-Gordon identities. Am. J. Math. **88**, 844–846 (1966). https://doi.org/10.2307/2373082
2. G.E. Andrews, Some new partition theorems. J. Comb. Theory **2**(4), 431–436 (1967). https://doi.org/10.1016/S0021-9800(67)80054-1
3. G.E. Andrews, *The Theory of Partitions* (Cambridge University Press, 1984)
4. George E. Andrews, Rodney J. Baxter, A motivated proof of the Rogers-Ramanujan identities. Am. Math. Monthly **96**(5), 401–409 (1989)
5. D.M. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli. JCT A **27**, 64–68 (1979). https://doi.org/10.1016/0097-3165(79)90008-6
6. W.Y.C. Chen, D.D.M. Sang, D.Y.H. Shi, The Rogers-Ramanujan-Gordon theorem for overpartitions. Proc. Lond. Math. Soc. (2013). https://doi.org/10.1112/plms/pds056
7. W.Y.C. Chen, D.D.M. Sang, D.Y.H. Shi, An overpartition analogue of Bressoud's theorem of Rogers-Ramanujan-Gordon type. Ramanujan J. (2013). https://doi.org/10.1007/s11139-012-9458-7
8. S. Corteel, J. Lovejoy, Overpartitions. Trans. AMS **356**(4), 1623–1635 (2003). https://doi.org/10.1090/S0002-9947-03-03328-2

9. S. Corteel, J. Lovejoy, O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli. Journal of Number Theory **128**, 1602–1621 (2008). https://doi.org/10.1016/j.jnt.2007.03.016
10. G. Gasper, M. Rahman, *Encyclopedia of Mathematics and its Applications*, 2nd edn., Basic Hypergeometric Series (Cambridge University Press, 2004)
11. B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities. Am. J. Math **83**, 393–399 (1961). https://doi.org/10.2307/2372962
12. G.H. Hardy, *Ramanujan, Cambridge Univ* (Press, London and New York; reprinted by Chelsea, New York, 1940)
13. K. Kurşungöz, Bressoud style partition identities for regular partitions and overpartitions. J. Number Theory **168**, 45–63 (2016). https://doi.org/10.1016/j.jnt.2016.04.001
14. J. Lovejoy, Gordon's theorem for overpartitions. JCT A **103**, 393–401 (2003). https://doi.org/10.1016/S0097-3165(03)00116-X
15. M. Petkovšek, H.S. Wilf, D. Zeilberger, *A=B* (Taylor & Francis, 1996)
16. L.J. Rogers, Second memoir on the expansion of certain infinite products. Proc. Lond. Math. Soc. **25**, 318–343 (1894)
17. I. Schur, *Ein Beitrag zur Additiven Zahlentheorie und zur Theorie der Kettenbrche* (Akad. Wiss. Berlin, Sitzungsber, 1917), pp. 302–321
18. A. Selberg, Uber einige arithmetische Identitäten. Avhandlinger Norske Akademie **8**, (1936)

On the Asymptotics of Partial Theta Functions

Susie Kimport

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Partial theta functions are sums whose terms resemble those of modular theta functions, save that the sums are taken over an incomplete lattice. In one of his notebooks, Ramanujan wrote down an asymptotic expansion for one particular partial theta functions as $q \to 1^-$. In 2011, Berndt and Kim generalized this type of asymptotic expansion to a related family, also for $q \to 1^-$. In this article, we extend the asymptotic results of Berndt and Kim to the case of $q \to e^{2\pi ih/k}$, any root of unity, and present new asymptotic expansions of another family of partial theta functions. These results were part of the author's 2015 Ph.D. thesis, where the partial theta functions discussed are related to a certain infinite family of quantum modular forms.

Keywords Theta functions · Partial theta functions · Asymptotic expansion Euler numbers · Hermite polynomials · Ramanujan's notebooks

2010 Mathematics Subject Classification Primary 11F27 · 33D15 · Secondary 11B68

S. Kimport (✉)
Stanford University, Stanford, California 94305, USA
e-mail: skimport@stanford.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_21

1 Introduction

A partial theta function is a function that is not modular but whose q-series expansion resembles that of an ordinary theta function, save that the sum is taken over an incomplete lattice. In one of his notebooks [3, page 324], Ramanujan wrote down an asymptotic expansion for the partial theta function

$$2\sum_{n\geq 0}(-1)^n q^{n^2+n} = 2\sum_{n\geq 0}(-1)^n \left(\frac{1-t}{1+t}\right)^{n^2+n} \sim 1+t+t^2+2t^3+5t^4+\ldots,$$

where $q = \frac{1-t}{1+t} \to 1^-$ as $t \to 0^+$. Galway [11] showed that all the coefficients of t^n in this expansion are positive integers. Further, Stanley [17] provided a combinatorial interpretation of these coefficients as the number of fixed-point-free alternating involutions in the symmetric group S_{2n}.

For any real numbers a and b, with $a > 0$, Berndt and Kim [6] generalized this type of asymptotic expansion to the partial theta functions

$$\begin{aligned}\Upsilon_{1/2}\left(-1, a, b; \frac{1-t}{1+t}\right) &:= 2\sum_{n\geq 0}(-1)^n \left(\frac{1-t}{1+t}\right)^{an^2+bn}, \\ \Upsilon_{1/2}\left(1, a, b; \frac{1-t}{1+t}\right) &:= 2\sum_{n\geq 0} \left(\frac{1-t}{1+t}\right)^{an^2+bn},\end{aligned} \tag{1.1}$$

as $t \to 0^+$, that is, as $q = \left(\frac{1-t}{1+t}\right) \to 1^-$. We note that by using the asymptotic of Berndt and Kim for $\Upsilon_{1/2}\left(-1, 1, 1; \frac{1-t}{1+}\right)$, one recovers Ramanujan's asymptotic.

Berndt and Kim also considered the coefficients of t^n arising in their expansion of $\Upsilon_{1/2}\left(-1, a, b; \frac{1-t}{1+t}\right)$ and showed that, if b is a positive integer, then each coefficient of t^n is an integer. They conjectured that for sufficiently large n, these coefficients have the same sign. Bringmann and Folsom [7] proved this conjecture and provided a complete classification based on b for the exact sign of the coefficients.

Partial theta functions have appeared in various areas of mathematics, including the theory of partitions [1, 2], quantum invariants of 3-manifolds [13, 15], and Vassiliev knot invariants [18], among others. Despite these applications and their similarities with the q-series of modular theta functions, their analytic theory has not been well understood. Recent results, due to Bringmann and Rolen [8], and Folsom, Ono, and Rhoades [9, 10] have shed new light on the role played by partial theta functions in the generalized theory of modular forms and their connection with quantum modular forms. See also [14].

In this paper, we generalize the work of Berndt and Kim (summarized in Section 2) and compute the asymptotic expansions for the partial theta functions in (1.1), hereafter referred to as *"weight 1/2" partial theta functions*, for any $a, b \in \mathbb{Q}$, $a > 0$, as q approaches other roots of unity. In particular, we will let $q = \zeta_k^h\left(\frac{1-t}{1+t}\right)$, where

$\zeta_k^h := e^{2\pi i h/k}$ (h, k coprime) and take $t \to 0^+$. These results are given in Theorems 3.2 and 3.3.

Further, we generalize their results to consider *"weight* $3/2$*" partial theta functions*

$$\Upsilon_{3/2}(-1, a, b; q) := 2\sum_{n\geq 0}(-1)^n n q^{an^2+bn},$$
$$\Upsilon_{3/2}(1, a, b; q) := 2\sum_{n\geq 0} n q^{an^2+bn}. \tag{1.2}$$

The results for $q = \frac{1-t}{1+t} \to 1^-$ are given in Theorems 4.2 and 4.4. These results are then generalized to the case where $q = \zeta_k^h\left(\frac{1-t}{1+t}\right) \to \zeta_k^h$ in Theorems 4.5 and 4.6. Finally, in Section 5, we explicitly write down the low order terms for each of the asymptotic expansions of the partial theta functions in (1.1) and (1.2).

Remark 1.1 The "weight" associated with these partial theta functions has two sources. First, by summing over all $n \in \mathbb{Z}$ (instead of just $n \geq 0$), each $\Upsilon_{j/2}$ becomes an ordinary theta function of weight $j/2$ for $j = 1, 3$. Second, in [14], the author constructs an infinite family of quantum modular forms of weight $j/2$ for $j = 1, 3$. These quantum modular forms, defined on a dense subset of the rational numbers, extend to functions of $\tau \in \mathbb{H}$ that can be expressed in terms of $\Upsilon_{j/2}(\pm 1, a, b; e^{2\pi i\tau})$ for specific a and b. See [14] for full details.

2 Berndt and Kim's results

Before we can state the asymptotic expansions of Berndt and Kim, we must recall some notation. The *Euler numbers* E_n, $n \in \mathbb{N}_0$, are a sequence of integers given by the generating function [12, formula 9.630]

$$\frac{1}{\cosh t} = \sum_{n\geq 0} E_n \frac{t^n}{n!}, \tag{2.1}$$

for $0 < |t| < \frac{\pi}{2}$. The *Bernoulli numbers* B_n, $n \in \mathbb{N}_0$, are a sequence of rational numbers given by the generating function [12, formula 9.610]

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!},$$

for $0 < |t| < 2\pi$. Finally, we recall the *Hermite polynomials* $H_n(x)$, $n \in \mathbb{N}_0$, given by [12, formula 8.950 and 8.951]

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (x + it)^n e^{-t^2}\, dt.$$

In particular, for $n \in \mathbb{N}_0$, we have (see [16, page 250] for example)

$$H_{2n}(x) = (2n)! \sum_{m=0}^{n} \frac{(-1)^m}{m!(2n-2m)!} (2x)^{2n-2m}, \tag{2.2}$$

and

$$H_{2n+1}(x) = 2x(2n+1)! \sum_{m=0}^{n} \frac{(-1)^m}{m!(2n+1-2m)!} (2x)^{2n-2m}. \tag{2.3}$$

We also have the following interesting relationships between the Hermite polynomials and integrals of certain exponential and trigonometric functions.

Lemma 2.1 **([12, formula 3.952, no. 9 and 10]).** *If n is a nonnegative integer, $|\arg \beta| < \frac{\pi}{4}$, and $c > 0$, then*

$$\int_0^\infty x^{2n+1} e^{-\beta^2 x^2} \sin(cx)\, dx = (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} e^{-c^2/(4\beta^2)} H_{2n+1}\left(\frac{c}{2\beta}\right),$$

and

$$\int_0^\infty x^{2n} e^{-\beta^2 x^2} \cos(cx)\, dx = (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} e^{-c^2/(4\beta^2)} H_{2n}\left(\frac{c}{2\beta}\right).$$

Lemma 2.2. *Let $c > 0$, $b \in \mathbb{R}$, and $\theta > 0$. Then we have that*

$$\frac{1}{2i\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} z^{2n+1} e^{biz} e^{-z^2/\theta}\, dz = \frac{(-1)^n \sqrt{\theta}^{2n+1}}{2^{2n+2}} e^{-b^2\theta/4} H_{2n+1}\left(\frac{b\sqrt{\theta}}{2}\right).$$

Lemma 2.3. *Let $c > 0$, $b \in \mathbb{R}$, and $\theta > 0$. Then we have that*

$$\frac{1}{2\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} z^{2n} e^{biz} e^{-z^2/\theta}\, dz = \frac{(-1)^n \theta^n}{2^{2n+1}} e^{-b^2\theta/4} H_{2n}\left(\frac{b\sqrt{\theta}}{2}\right).$$

Lemmas 2.2 and 2.3 are constant multiples of [6, Lemma 3.2] and [6, Lemma 2.2], respectively, and we refer the reader to [6] for their proofs.

We are now ready to state the asymptotic expansions of Berndt and Kim for the partial theta functions $\Upsilon_{1/2}(\pm 1, a, b; q)$ in (1.1).

Theorem 2.4 **([6, Theorem 1.1]).** *Let $a > 0$ and $b \in \mathbb{R}$. Then, as $t \to 0^+$, we have that*

$$2\sum_{n\geq 0} (-1)^n \left(\frac{1-t}{1+t}\right)^{an^2+bn} \sim \left(\frac{1-t}{1+t}\right)^{\frac{a-2b}{4}} \sum_{n\geq 0} T_{1/2}\left(a, b, n; \log\left(\frac{1+t}{1-t}\right)\right), \tag{2.4}$$

where

$$T_{1/2}(a,b,n;\theta) := \frac{E_{2n}}{(2n)!2^{2n}}(a\theta)^n H_{2n}\left(\frac{(b-a)\sqrt{a\theta}}{2a}\right). \tag{2.5}$$

Theorem 2.5 ([6, Theorem 3.4]). *Let $a > 0$ and $b \in \mathbb{R}$. Then, as $t \to 0^+$, we have that*

$$2\sum_{n\geq 0}\left(\frac{1-t}{1+t}\right)^{an^2+bn} \sim \left(\frac{1-t}{1+t}\right)^{-b^2/4a}\left[C_{1/2}\left(a,b;\log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$\left. - \left(\frac{1-t}{1+t}\right)^{\frac{(b-a)^2}{4a}}\sum_{n\geq 0} S_{1/2}\left(a,b,n;\log\left(\frac{1+t}{1-t}\right)\right)\right],$$

where

$$C_{1/2}(a,b;\theta) := \sqrt{\frac{\pi}{a\theta}} - \frac{1}{\sqrt{a\theta}}\left(e^{-\frac{\theta(b-a)^2}{4a}}\sinh\left(\frac{(b-a)\sqrt{a\theta}}{a}\right)\right) \tag{2.6}$$

and

$$S_{1/2}(a,b,n;\theta) := \frac{(2^{2n+1}-1)(a\theta)^{n+\frac{1}{2}}B_{2n+2}}{2^{2n}(2n+2)!}H_{2n+1}\left(\frac{(b-a)\sqrt{a\theta}}{2a}\right). \tag{2.7}$$

Remark 2.6 We note the following:

(1) In fact, Berndt and Kim only state Theorems 2.4 and 2.5 for $a = 1$. The statements here immediately follow from letting $b \to \frac{b}{a}$ and $q \to q^a$ in their theorems.
(2) Theorem 2.4 corrects a typo in Berndt and Kim's theorem as to the exponent of $\frac{1-t}{1+t}$ on the right-hand side of (2.4), which is correctly stated in [6, (2.9)].
(3) Berndt and Kim did not use the notation of $T_{1/2}$, $C_{1/2}$, and $S_{1/2}$. We have defined this notation here to allow for ease of expression in what follows.

3 Asymptotic expansions for "weight 1/2" partial theta functions

We will now use the results of Berndt and Kim to build new asymptotic expansions of the partial theta functions in (1.1) as q approaches other roots of unity, $\zeta_k^h := e^{2\pi i h/k}$, where $q = \zeta_k^h\left(\frac{1-t}{1+t}\right)$ and $t \to 0^+$.

In order to study the behavior as we move towards other roots of unity, we restrict to $a, b \in \mathbb{Q}$ and relabel $a = \frac{u}{v}$ and $b = \frac{s}{w}$, where $u, v, s, w \in \mathbb{Z}$ and $u, v, w > 0$. We can then describe the asymptotic behavior of

$$\Upsilon_{1/2}\left(\pm 1, \frac{u}{v}, \frac{s}{w}; q\right) = 2\sum_{n\geq 0}(\pm 1)^n q^{\frac{u}{v}n^2+\frac{s}{w}n}$$

as $q \to \zeta_k^h$ by considering the behavior, as $t \to 0^+$, of

$$\Upsilon_{1/2}\left(\pm 1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(\pm 1)^n \left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}. \quad (3.1)$$

Remark 3.1. We note that we do not require $\gcd(u, v) = 1$ or $\gcd(s, w) = 1$. In fact, one can easily check that the asymptotic estimates obtained in this section and the next also hold for $\gcd(u, v) > 1$ and/or $\gcd(s, w) > 1$. The reason for introducing this notation is to easily refer to the numerators and denominators of $a = \frac{u}{v}$ and $b = \frac{s}{w}$.

Theorem 3.2. *Let $h, k, u, v, s, w \in \mathbb{Z}$ with $k, u, v, w > 0$ and $\gcd(h, k) = 1$. Further, we let $\ell := vwk/\gcd(vwk, huw, svh)$. Then, as $t \to 0^+$, the following are true:*

(i) If ℓ is odd, we have that

$$\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(-1)^n \left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}$$
$$\sim \sum_{r=0}^{\ell-1}(-1)^r \left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}r^2+\frac{s}{w}r+\frac{u\ell^2}{4v}-\frac{ur\ell}{v}-\frac{s\ell}{2w}}$$
$$\times \sum_{n\geq 0} T_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right).$$

(ii) If ℓ is even, we have that

$$\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(-1)^n \left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}$$
$$\sim \left(\frac{1-t}{1+t}\right)^{-\frac{s^2v}{4uw^2}}\sum_{r=0}^{\ell-1}(-1)^r \left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left[C_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$\left.-\left(\frac{1-t}{1+t}\right)^{\frac{(sv-uw\ell+2ruw)^2}{4uvw^2}}\sum_{n\geq 0} S_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right].$$

Proof. We start with $\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right)$ as in (3.1) and note that

$$\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n} = e^{2\pi i(huwn^2+svhn)/vwk} \quad (3.2)$$

is a periodic function. To determine the period, we note that the exponent can be simplified if there are any common factors between huw, svh, and vwk. So we define $d := \gcd(vwk, huw, svh)$ and see that, for $0 \le r < \ell = vwk/d$,

$$\left(\zeta_k^h\right)^{\frac{u}{v}(r+\ell)^2+\frac{s}{w}(r+\ell)} = \left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}.$$

Thus, ℓ is the period of (3.2) and we can split the sum in (3.1) across residue classes modulo ℓ by letting $n \to r + n\ell$. This yields

$$\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = \sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}r^2+\frac{s}{w}r} \times 2\sum_{n\ge 0}(-1)^{n\ell}\left(\frac{1-t}{1+t}\right)^{\frac{u\ell^2}{v}n^2+\left(\frac{2ur\ell}{v}+\frac{s}{w}\right)n}. \tag{3.3}$$

We now see the need to differentiate between the case when ℓ is odd and when ℓ is even. In the case when ℓ is odd, we see that $(-1)^{n\ell} = (-1)^n$ and thus the innermost sum in (3.3) is exactly $\Upsilon_{1/2}\left(-1, \frac{u\ell^2}{v}, \frac{2ur\ell}{v} + \frac{s}{w}; \left(\frac{1-t}{1+t}\right)\right)$. Statement (i) then follows from a straightforward application of Theorem 2.4.

However, if ℓ is even, we have that $(-1)^{n\ell} = 1$ for all n. Then the innermost sum in (3.3) is $\Upsilon_{1/2}\left(1, \frac{u\ell^2}{v}, \frac{2ur\ell}{v} + \frac{s}{w}; \left(\frac{1-t}{1+t}\right)\right)$ and we must apply Theorem 2.5 to obtain statement (ii). □

We now turn our attention to the second family in (1.1). For this family of partial theta functions, we can follow the same argument as in Theorem 3.2, however, we no longer require the cases based on the parity of ℓ. We, therefore, have the following asymptotic expansion.

Theorem 3.3. *Let $h, k, u, v, s, w \in \mathbb{Z}$ with $k, u, v, w > 0$ and $\gcd(h, k) = 1$. Further, we let $\ell := vwk/\gcd(vwk, huw, svh)$. Then, as $t \to 0^+$, we have,*

$$\Upsilon_{1/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\ge 0}\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}$$
$$\sim \left(\frac{1-t}{1+t}\right)^{-\frac{s^2v}{4uw^2}}\sum_{r=0}^{\ell-1}\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left[C_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$\left. - \left(\frac{1-t}{1+t}\right)^{\frac{(sv-uw\ell+2ruw)^2}{4uvw^2}}\sum_{n\ge 0}S_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right]$$

Proof. Exactly as in the proof of Theorem 3.2, we can rewrite (3.1) as

$$\Upsilon_{1/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = \sum_{r=0}^{\ell-1}(\zeta_k^h)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}r^2+\frac{s}{w}r}$$
$$\times 2\sum_{n\geq 0}\left(\frac{1-t}{1+t}\right)^{\frac{u\ell^2}{v}n^2+(\frac{2ur\ell}{v}+\frac{s}{w})n}. \qquad (3.4)$$

From here, we see that the innermost sum in (3.4) can be expanded via Theorem 2.5 to obtain the desired result. □

4 Asymptotic expansions of "weight 3/2" partial theta functions

We now move to considering the "weight 3/2" partial theta functions by generalizing the methods employed by Berndt and Kim to this case. In Sections 4.1 and 4.2, we obtain the asymptotic expansions of the families of partial theta functions in (1.2) as $q = \frac{1-t}{1+t} \to 1^-$. In Section 4.3, we generalize these results to obtain asymptotic expansions as q approaches other roots of unity, $\zeta_k^h := e^{2\pi i h/k}$, by taking $q = \zeta_k^h\left(\frac{1-t}{1+t}\right)$ and $t \to 0^+$.

4.1 Asymptotic expansion for $\Upsilon_{3/2}(-1, a, b; q)$

We start by considering the first family in (1.2), namely $\Upsilon_{3/2}(-1, a, b; q)$. We will need Lemmas 2.1 and 2.2, as well as the following result of Berndt and Kim [6].

Lemma 4.1 ([6, Lemma 2.4]). *If $c, \theta > 0$, $b \in \mathbb{R}$ and $n \in \mathbb{N}_0$, then we have that*

$$\frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(2n+b)iz}\,dz = e^{-(n+b/2)^2\theta}.$$

We can now state and prove the following theorem.

Theorem 4.2. *For $a > 0$, $b \in \mathbb{R}$, as $t \to 0^+$, we have that*

$$\Upsilon_{3/2}\left(-1, a, b; \frac{1-t}{1+t}\right) = 2\sum_{n\geq 0}(-1)^n n\left(\frac{1-t}{1+t}\right)^{an^2+bn}$$
$$\sim -\left(\frac{1-t}{1+t}\right)^{\frac{a-2b}{4}}\sum_{n\geq 0}T_{3/2}\left(a, b, n; \log\left(\frac{1+t}{1-t}\right)\right) + \frac{1}{2}T_{1/2}\left(a, b, n; \log\left(\frac{1+t}{1-t}\right)\right),$$

where $T_{1/2}$ is as defined in (2.5) and

$$T_{3/2}(a,b,n;\theta) := \frac{E_{2n+2}}{(2n+1)!2^{2n+2}}(a\theta)^{n+\frac{1}{2}}H_{2n+1}\left(\frac{(b-a)\sqrt{a\theta}}{2a}\right). \tag{4.1}$$

Proof. For the proof, we consider the case when $a = 1$. The general result for $a > 0$ then follows by letting $q \mapsto q^a$ and $b \mapsto b/a$. Following the method of Berndt and Kim [6], we let $q = e^{-\theta}$ where $\theta := \log\left(\frac{1+t}{1-t}\right)$ and rewrite our theta function as

$$2\sum_{n\geq 0}(-1)^n nq^{n^2+bn} = e^{b^2\theta/4}2\sum_{n\geq 0}(-1)^n ne^{-(n+b/2)^2\theta} =: e^{b^2\theta/4}G_1(\theta). \tag{4.2}$$

By Lemma 4.1, we see that, for $c > 0$,

$$\begin{aligned} G_1(\theta) &= \frac{2}{\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta}\sum_{n\geq 0}(-1)^n ne^{(2n+b)iz}\,dz \\ &= \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz}\left(2\sum_{n\geq 0}(-1)^n ne^{(2n+1)iz}\right)dz. \end{aligned} \tag{4.3}$$

It is not hard to show that

$$\frac{1}{\cos z} = 2\sum_{n\geq 0}(-1)^n e^{(2n+1)iz}$$

for all $z \in \mathbb{H}$, meaning

$$\frac{d}{dz}\frac{1}{\cos z} = \sec(z)\tan(z) = 2\sum_{n\geq 0}(-1)^n(2n+1)ie^{(2n+1)iz}.$$

Thus, applying absolute and uniform convergence of the integral in (4.3) (which follows from the fact that $c > 0$), we can rewrite (4.3) as

$$\begin{aligned} G_1(\theta) = \frac{1}{2i\sqrt{\pi\theta}}&\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz}\,\sec(z)\tan(z)\,dz \\ &- \frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz}\,\frac{dz}{\cos(z)}. \end{aligned} \tag{4.4}$$

For the first integral in (4.4), we start with a consequence of (2.1): for $0 < |x| < \frac{\pi}{2}$,

$$\sec(x) = \sum_{n\geq 0}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n},$$

where we have used that $\cos(x) = \cosh(-ix)$ and that the Euler numbers of odd index are zero. Differentiating both sides gives us that, for $0 < |x| < \frac{\pi}{2}$,

$$\sec(x)\tan(x) = \sum_{n\geq 0} \frac{(-1)^{n+1}E_{2n+2}}{(2n+1)!}x^{2n+1}.$$

Thus, for any $N \in \mathbb{N}_0$, we can write the first term in (4.4) as

$$\frac{1}{2i\sqrt{\pi\theta}} \sum_{n=0}^{N} \frac{(-1)^{n+1}E_{2n+2}}{(2n+1)!} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} z^{2n+1}\, dz + R_N,$$

where R_N is given by

$$R_N := \frac{1}{2i\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} \left(\sec(z)\tan(z) - \sum_{n=0}^{N} \frac{(-1)^{n+1}E_{2n+2}}{(2n+1)!} z^{2n+1} \right) dz.$$

We bound this "remainder" R_N using a similar argument to that in [3, pages 546-547], and consider $|R_N|$. Since $\sec(z)\tan(z)$ is analytic for $|z| < \frac{\pi}{2}$, we have that

$$\left| \sec(z)\tan(z) - \sum_{n=0}^{N} \frac{(-1)^{n+1}E_{2n+2}}{(2n+1)!} z^{2n+1} \right| \leq C_1|z|^{2N+3}$$

for $|z| \leq \frac{\pi}{4}$, where C_1 depends on N but not on z.

Further, for $0 < c \leq 1$ and $|z| \geq \frac{\pi}{4}$ with z on the contour $(-\infty + ci, \infty + ci)$, is not hard to see that

$$\frac{1}{z^{2N+3}} \quad \text{and} \quad \frac{1}{z^{2N+3}} \sum_{n=0}^{N} \frac{(-1)^{n+1}E_{2n+2}}{(2n+1)!} z^{2n+1}$$

are bounded by constants independent of c and z. It remains to bound $\sec(z)\tan(z)$ for z on this contour. This is not necessarily possible as we let $c \to 0^+$, but we are able to bound $c^2 \sec(z)\tan(z)$. Indeed, for $z = x + ci$, $x \in \mathbb{R}$, we one can show that

$$\left|\cos(x + ci)\right| \geq \sinh(c) \quad \text{and} \quad \left|\sin(x + ci)\right| \leq \cosh(c). \tag{4.5}$$

Equality in both expressions in (4.5) is obtained when $x = (2n-1)\pi/2$ for $n \in \mathbb{Z}$. Thus, since $\sinh(c) > 0$ for $c > 0$, we only need to address $z = (2n-1)\pi/2 + ci$ as $c \to 0^+$. It is straightforward to show that

$$\lim_{c\to 0^+} \left| c^2 \sec\left(\frac{(2n-1)\pi}{2} + ci\right) \tan\left(\frac{(2n-1)\pi}{2} + ci\right) \right| = \lim_{c\to 0^+} \frac{c^2\cosh(c)}{\sinh^2(c)} = 1,$$

which shows that we are able to bound $c^2 \sec(z)\tan(z)$ for $z \in (-\infty + ci, \infty + ci)$ as $c \to 0^+$ and that this bound is independent of z and c. Therefore, for all points z on the contour $(-\infty + ci, \infty + ci)$, we have that,

$$\left| \frac{c^2}{z^{2N+3}} \left(\sec(z)\tan(z) - \sum_{n=0}^{N} \frac{(-1)^{n+1} E_{2n+2}}{(2n+1)!} z^{2n+1} \right) \right| \le C_2,$$

where C_2 is some positive constant independent of both z and c. This means,

$$\begin{aligned}
|R_N| &\le \frac{C_2}{2ic^2\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} \left| e^{-z^2/\theta+(b-1)iz} z^{2N+3} \right| dz \\
&= \frac{C_2}{2ic^2\sqrt{\pi\theta}} \int_{-\infty}^{\infty} \left| e^{-(x+ci)^2/\theta+(b-1)i(x+ci)} (x+ci)^{2N+3} \right| dx \\
&= \frac{C_2}{2ic^2\sqrt{\pi\theta}} \int_{-\infty}^{\infty} e^{(c^2-x^2)/\theta-(b-1)c} (x^2+c^2)^{N+3/2}\, dx \\
&< \frac{C_2 e^{-(b-1)c}}{2ic^2\sqrt{\pi\theta}} \int_{-\infty}^{\infty} e^{(c^2-x^2)/\theta} \left((2x^2)^{N+3/2} + (2c^2)^{N+3/2} \right) dx \\
&= \frac{2^{N+1/2} C_2 e^{c^2/\theta-(b-1)c}}{ic^2\sqrt{\pi}} \left(\Gamma(N+2)\,\theta^{N+3/2} + \Gamma\left(\frac{1}{2}\right) c^{2N+3} \right)
\end{aligned}$$

where we have followed the calculation as in [3] and $\Gamma(t)$ is the well-known function given by

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx.$$

If we now take $c = \sqrt{\theta}$ and use the Taylor expansion for e^x, we see that

$$R_N = O(\theta^{N+1/2}) \quad \text{as} \quad N \to \infty,\ \theta \to 0^+.$$

Finally, from Lemma 2.2, we have that

$$\begin{aligned}
&\frac{1}{2i\sqrt{\pi\theta}} \sum_{n=0}^{N} \frac{(-1)^{n+1} E_{2n+2}}{(2n+1)!} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} z^{2n+1}\, dz \\
&\qquad = -e^{-(b-1)^2\theta/4} \sum_{n=0}^{N} \frac{E_{2n+2}}{(2n+1)!} \frac{\sqrt{\theta}^{2n+1}}{2^{2n+2}} H_{2n+1}\left(\frac{(b-1)\sqrt{\theta}}{2} \right).
\end{aligned}$$

This gives us the asymptotic for the first integral in (4.4). For the second integral in (4.4), we turn to Berndt and Kim's proof of Theorem 2.4 [6, proof of Theorem 1.1]:

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} \frac{dz}{\cos(z)} \\
&\qquad = e^{-(b-1)^2\theta/4} \sum_{n=0}^{N} \frac{\theta^n E_{2n}}{2^{2n+1}(2n)!} H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2} \right) + O(\theta^{N+1/2}).
\end{aligned}$$

Putting this all together, we have that

$$G_1(\theta) = -e^{-(b-1)^2\theta/4}\sum_{n=0}^{N}\frac{\theta^{n+\frac{1}{2}}E_{2n+2}}{2^{2n+2}(2n+1)!}H_{2n+1}\left(\frac{(b-1)\sqrt{\theta}}{2}\right)$$
$$- e^{-(b-1)^2\theta/4}\sum_{n=0}^{N}\frac{\theta^{n}E_{2n}}{2^{2n+1}(2n)!}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(\theta^{N+1/2}).$$

The result in the case $a = 1$ now follows as $N \to \infty$, $t \to 0^+$, given that our desired partial theta function is $e^{b^2\theta/4}G_1(\theta)$ (see (4.2)) and that $\theta := \log\left(\frac{1+t}{1-t}\right)$. Then, by letting $\theta \mapsto a\theta$ and $b \mapsto b/a$, we obtain the general result as stated in the theorem. □

4.2 Asymptotic expansion for $\Upsilon_{3/2}(1, a, b; q)$

We now turn our attention to the second family of "weight 3/2" partial theta functions in (1.2). Without loss of generality, we will take $a = 1$ and, as before, define $G_2(\theta)$ as

$$2\sum_{n\geq 0} nq^{n^2+bn} = e^{b^2\theta/4}2\sum_{n\geq 0} ne^{-(n+b/2)^2\theta} =: e^{b^2\theta/4}G_2(\theta), \tag{4.6}$$

and consider the asymptotic expansion of $G_2(\theta)$ when $\theta := \log\left(\frac{1+t}{1-t}\right)$. Our proof will rely on Lemmas 2.1 and 2.3, as well as the following result.

Lemma 4.3. *Let $c, \theta > 0$ and $b \in \mathbb{R}$. Then we have that*

$$-\frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci}\frac{1}{z^2}e^{-z^2/\theta+biz}\,dz = -\frac{b}{2}\sqrt{\frac{\pi}{\theta}} + \frac{b}{2\sqrt{\theta}}e^{-b^2\theta/4}\sinh(b\sqrt{\theta}) + \frac{e^{-b^2\theta/4}}{\theta}.$$

Proof. We start by letting $u = z/\sqrt{\theta}$. Then

$$-\frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci}\frac{1}{z^2}e^{biz-z^2/\theta}\,dz = -\frac{1}{2\theta\sqrt{\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}\frac{1}{u^2}e^{bi\sqrt{\theta}u-u^2}\,du. \tag{4.7}$$

This integrand is holomorphic for $u \in \mathbb{C}$ except for a double pole at 0, which will make it difficult to evaluate this integral using Cauchy's theorem. Instead, we apply integration by parts to obtain

$$
-\frac{1}{2\theta\sqrt{\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}\frac{1}{u^2}e^{bi\sqrt{\theta}u-u^2}\,du
$$

$$
=\frac{1}{2\theta\sqrt{\pi}}\frac{1}{u}e^{bi\sqrt{\theta}u-u^2}\Big|_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}-\frac{1}{2\theta\sqrt{\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}\frac{1}{u}(bi\sqrt{\theta}-2u)e^{bi\sqrt{\theta}u-u^2}\,du. \tag{4.8}
$$

Applying L'Hopital's rule, it is straightforward to see that the first term in (4.8) is 0. Therefore, we see that (4.7) is equal to

$$
-\frac{1}{2\theta\sqrt{\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}\frac{1}{u}(bi\sqrt{\theta}-2u)e^{bi\sqrt{\theta}u-u^2}\,du
$$

$$
=-\frac{bi}{2\sqrt{\theta\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}\frac{1}{u}e^{bi\sqrt{\theta}u-u^2}\,du+\frac{1}{\theta\sqrt{\pi}}\int_{-\infty+\frac{c}{\sqrt{\theta}}i}^{\infty+\frac{c}{\sqrt{\theta}}i}e^{bi\sqrt{\theta}u-u^2}\,du \tag{4.9}
$$

$$
=-\frac{b}{2}\sqrt{\frac{\pi}{\theta}}+\frac{b}{2\sqrt{\theta}}e^{-b^2\theta/4}\sinh(b\sqrt{\theta})+\frac{e^{-b^2\theta/4}}{\theta},
$$

where we have used [6, Lemma 3.3] to evaluate the first integral in (4.9) and Cauchy's theorem to evaluate the second integral in (4.9). □

We are now prepared to state and prove the asymptotic expansion for this family of "weight 3/2" partial theta functions.

Theorem 4.4. *For $a > 0$, $b \in \mathbb{R}$, as $t \to 0^+$, we have that*

$$
\Upsilon_{3/2}\left(1,a,b;\frac{1-t}{1+t}\right)=2\sum_{n\geq 0}n\left(\frac{1-t}{1+t}\right)^{an^2+bn}
$$

$$
\sim\left(\frac{1-t}{1+t}\right)^{-b^2/(4a)}\left[C_{3/2}\left(a,b;\log\left(\frac{1+t}{1-t}\right)\right)\right.
$$

$$
\left.+\left(\frac{1-t}{1+t}\right)^{\frac{(b-a)^2}{4a}}\sum_{n\geq 0}S_{3/2}\left(a,b,n;\log\left(\frac{1+t}{1-t}\right)\right)+\frac{1}{2}S_{1/2}\left(a,b,n;\log\left(\frac{1+t}{1-t}\right)\right)\right],
$$

where $S_{1/2}$ is defined in (2.7),

$$
C_{3/2}(a,b;\theta):=-\frac{b}{2a}C_{1/2}(a,b;\theta)+\frac{1}{a\theta}e^{-\frac{\theta(b-a)^2}{4a}} \tag{4.10}
$$

with $C_{1/2}$ as defined in (2.6), and

$$
S_{3/2}(a,b,n;\theta):=\frac{(2^{2n+1}-1)(2n+1)(a\theta)^n B_{2n+2}}{2^{2n}(2n+2)!}H_{2n}\left(\frac{(b-a)\sqrt{a\theta}}{2a}\right). \tag{4.11}
$$

Proof. Once again, we prove the case of $a = 1$ and note that the general theorem can be obtained by letting $q \mapsto q^a$ and $b \mapsto b/a$. As stated in (4.6), we have that

$$2\sum_{n\geq 0} nq^{n^2+bn} = e^{b^2\theta/4} G_2(\theta),$$

where we have let $q = e^{-\theta}$ and $\theta := \log\left(\frac{1+t}{1-t}\right)$. We proceed as in the proof of Theorem 4.2 to see that

$$G_2(\theta) = \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} \left(2\sum_{n\geq 0} ne^{(2n+1)iz}\right) dz. \tag{4.12}$$

Now, it is easy to see that

$$\frac{i}{\sin z} = 2\sum_{n\geq 0} e^{(2n+1)iz},$$

for all $z \in \mathbb{H}$, meaning we can recognize the sum in (4.12) as

$$2\sum_{n\geq 0} ne^{(2n+1)iz} = \frac{1}{2i}\frac{d}{dz}\frac{i}{\sin z} - \frac{1}{2}\frac{i}{\sin z}.$$

Therefore, we can rewrite (4.12) as

$$\begin{aligned} G_2(\theta) = -\frac{1}{2\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} \frac{\cos z}{\sin^2 z}\, dz \\ - \frac{i}{2\sqrt{\pi\theta}} \int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz} \frac{dz}{\sin z}. \end{aligned} \tag{4.13}$$

We start with the first term in (4.13). We use the fact [12, formula 1.411, no. 11] that, for $|z| < \pi$,

$$\csc z = \frac{1}{z} + 2\sum_{n\geq 0} \frac{(-1)^n(2^{2n+1}-1)B_{2n+2}}{(2n+2)!} z^{2n+1}$$

to obtain the Taylor series for the derivative for $|z| < \pi$,

$$-\frac{\cos z}{\sin^2 z} = -\frac{1}{z^2} + 2\sum_{n\geq 0} \frac{(-1)^n(2^{2n+1}-1)(2n+1)B_{2n+2}}{(2n+2)!} z^{2n}.$$

Thus, the first term in (4.13) is

$$
-\frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz}\frac{\cos z}{\sin^2 z}\,dz
$$
$$
= -\frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci}\frac{1}{z^2}e^{-z^2/\theta+(b-1)iz}\,dz \tag{4.14}
$$
$$
+\frac{1}{\sqrt{\pi\theta}}\sum_{n=0}^{N}\frac{(-1)^n(2^{2n+1}-1)(2n+1)B_{2n+2}}{(2n+2)!}\int_{-\infty+ci}^{\infty+ci} z^{2n}e^{-z^2/\theta+(b-1)iz}\,dz+R'_N,
$$

where here we define

$$
R'_N := \frac{1}{2\sqrt{\pi\theta}}\int_{-\infty+ci}^{\infty+ci} e^{-z^2/\theta+(b-1)iz}
\times\left(-\frac{\cos z}{\sin^2 z}+\frac{1}{z^2}-2\sum_{n=0}^{N}\frac{(-1)^n(2^{2n+1}-1)(2n+1)B_{2n+2}}{(2n+2)!}z^{2n}\right)dz.
$$

Lemmas 2.3 and 4.3 allow us to evaluate the integrals in (4.14). Next, we will show that we can bound R'_N. This follows from a very similar argument to that used in the proof of Theorem 4.2, obtaining that

$$
R'_N = O(\theta^N) \quad \text{as} \quad N\to\infty,\ \theta\to 0^+.
$$

Thus, along with Berndt and Kim's expansion [6, Theorem 3.4] of the second integral in (4.13), we have obtained the desired asymptotic for $a=1$ as $N\to\infty$, $t\to 0^+$. □

4.3 *Asymptotic expansions as q approaches other roots of unity*

We now want to consider the "weight 3/2" partial theta functions in (1.2) as we approach any root of unity. To achieve this, we once again restrict to $a=\frac{u}{v}\in\mathbb{Q}$ and $b=\frac{s}{w}\in\mathbb{Q}$ with $u,v,s,w\in\mathbb{Z}$, $u,v,w>0$ and consider

$$
\Upsilon_{3/2}\left(\pm 1,\frac{u}{v},\frac{s}{w};\zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(\pm 1)^n n\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}. \tag{4.15}
$$

As noted in Remark 3.1, we do not require $\gcd(u,v)=1$ and $\gcd(s,w)=1$.

The method for obtaining asymptotic expansions for each of these partial theta functions is exactly the same as that used in the proofs of Theorem 3.2 and 3.3: we use the periodicity of the root of unity to split the infinite sum in (4.15) into a finite sum of partial theta functions as in (3.3) and (3.4). We note that, in doing this, we will see that the partial theta functions in the finite sum are themselves sums of a "weight 3/2" and a "weight 1/2" partial theta function.

Theorem 4.5. *Let $h, k, u, v, s, w \in \mathbb{Z}$ with $k, u, v, w > 0$ and $\gcd(h, k) = 1$. Further, we let $\ell := vwk/\gcd(vwk, huw, svh)$. Then, as $t \to 0^+$, the following are true:*

(i) If ℓ is odd, we have that

$$\Upsilon_{3/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(-1)^n n\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}$$
$$\sim -\sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}r^2+\frac{s}{w}r+\frac{u\ell^2}{4v}-\frac{ur\ell}{v}-\frac{s\ell}{2w}}$$
$$\times\left[\ell\sum_{n\geq 0}T_{3/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$\left.+\left(\frac{1}{2}-\frac{r}{\ell}\right)T_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right]$$

(ii) If ℓ is even, we have that

$$\Upsilon_{3/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right) = 2\sum_{n\geq 0}(-1)^n n\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n}$$
$$\sim\left(\frac{1-t}{1+t}\right)^{-\frac{s^2v}{4uw^2}}\sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}$$
$$\times\left[\ell C_{3/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$+rC_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)$$
$$-\ell\left(\frac{1-t}{1+t}\right)^{\frac{uw\ell^2+4\ell ruw+2s\ell v}{4vw}}$$
$$\times\sum_{n\geq 0}S_{3/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)$$
$$\left.+\left(\frac{1}{2}+\frac{r}{\ell}\right)S_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right]$$

Theorem 4.6. *Let $h, k, u, v, s, w \in \mathbb{Z}$ with $k, u, v, w > 0$ and $\gcd(h, k) = 1$. Further, we let $\ell := vwk/\gcd(vwk, huw, svh)$. Then, as $t \to 0^+$, we have that*

$$\Upsilon_{3/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h\left(\frac{1-t}{1+t}\right)\right)$$
$$= 2\sum_{n\geq 0} n\left(\zeta_k^h\right)^{\frac{u}{v}n^2+\frac{s}{w}n}\left(\frac{1-t}{1+t}\right)^{\frac{u}{v}n^2+\frac{s}{w}n} \sim \left(\frac{1-t}{1+t}\right)^{-\frac{s^2v}{4uw^2}}\sum_{r=0}^{\ell-1}\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}$$
$$\times\left[\ell C_{3/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)\right.$$
$$+rC_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}; \log\left(\frac{1+t}{1-t}\right)\right)$$
$$-\ell\left(\frac{1-t}{1+t}\right)^{\frac{uw\ell^2+4\ell ruw+2s\ell v}{4vw}}$$
$$\times\sum_{n\geq 0} S_{3/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)$$
$$\left.+\left(\frac{1}{2}+\frac{r}{\ell}\right)S_{1/2}\left(\frac{u\ell^2}{v}, \frac{2ur\ell}{v}+\frac{s\ell}{w}, n; \log\left(\frac{1+t}{1-t}\right)\right)\right]$$

The proofs of Theorems 4.5 and 4.6 follow the arguments given in the proofs of Theorems 3.2 and 3.3. We omit them here for brevity.

5 Low order terms

The theorems in Sections 3 and 4 give a complete characterization of the asymptotic properties of $\Upsilon_{1/2}\left(\pm 1, \frac{u}{v}, \frac{s}{w}; q\right)$ and $\Upsilon_{3/2}\left(\pm 1, \frac{u}{v}, \frac{s}{w}; q\right)$ as q approaches any root of unity, ζ_k^h, along the path $q = \zeta_k^h\left(\frac{1-t}{1+t}\right)$, $t \to 0^+$. However, applications of these asymptotic estimates as $t \to 0^+$ are often most concerned with the low order terms of the expansion. In this section, we rewrite the results of Sections 3 and 4 to highlight those terms of degree less than $1/2$.

We begin by considering the low order terms of the building blocks of these asymptotic expansions, namely $T_{j/2}$, $C_{j/2}$, and $S_{j/2}$ for $j = 1$ or 3, which are themselves built from well-known series. In particular, for $t, x, \alpha \in \mathbb{R}$ with $t \geq 0$ and $n \in \mathbb{N}_0$, we have that

$$\left.\begin{aligned}\theta^\alpha &:= \log^\alpha\left(\frac{1+t}{1-t}\right) = (2t)^\alpha\left(1+\frac{\alpha}{3}t^2+O(t^4)\right),\\ \left(\frac{1-t}{1+t}\right)^\alpha &= 1-2\alpha t+2\alpha^2t^2+O(t^3),\end{aligned}\right\}\quad (5.1)$$

$$\left.\begin{aligned} H_{2n}(x) &= \frac{(-1)^n(2n)!}{n!} - \frac{2(-1)^n(2n)!}{(n-1)!}x^2 + O(x^4), \\ H_{2n+1}(x) &= \frac{2(-1)^n(2n+1)!}{n!}x - \frac{4(-1)^n(2n+1)!}{3(n-1)!}x^3 + O(x^5), \\ \frac{\sinh(\alpha x)}{x} &= \alpha + \frac{\alpha^3}{6}x^2 + O(x^4), \end{aligned}\right\} \tag{5.2}$$

where the expansions of $H_{2n}(x)$ and $H_{2n+1}(x)$ follow from (2.2) and (2.3), respectively.

5.1 "Weight 1/2" low order terms

Using those series in (5.1) and (5.2), we can obtain the following series for the building blocks of our asymptotic expansions in Theorems 3.2 and 3.3. In particular, for $a > 0$, $b \in \mathbb{R}$, $n \in \mathbb{Z}$ and $\theta := \log\left(\frac{1+t}{1-t}\right)$, we have that $T_{1/2}(a, b, n; \theta)$ as defined in (2.5) has an expansion of the form

$$\begin{aligned} T_{1/2}(a, b, n; \theta) &= \frac{a^n E_{2n}}{(2n)!2^{2n}}(2t)^n \left(1 + \frac{n}{3}t^2 + O(t^4)\right) \\ &\qquad \times (-1)^n(2n)! \left(\frac{1}{n!} - \frac{(b-a)^2}{a(n-1)!}t + O(t^2)\right) \\ &= \frac{(-a)^n E_{2n}}{2^n} t^n \left(\frac{1}{n!} - \frac{(b-a)^2}{a(n-1)!}t + O(t^2)\right). \end{aligned} \tag{5.3}$$

Similarly, we see that $C_{1/2}(a, b; \theta)$ as defined in (2.6) has an expansion of the form

$$\begin{aligned} C_{1/2}(a, b; \theta) &= \sqrt{\frac{\pi}{2at}} - \frac{1}{6}\sqrt{\frac{\pi}{2a}}t^{3/2} \\ &\qquad - \left(1 - \frac{(b-a)^2}{2a}t + O(t^2)\right)\left(\frac{b-a}{a} + \frac{(b-a)^3}{3a^2}t + O(t^2)\right) \\ &= \sqrt{\frac{\pi}{2at}} - \frac{b-a}{a} + \frac{(b-a)^3}{6a^2}t - \frac{1}{6}\sqrt{\frac{\pi}{2a}}t^{3/2} + O(t^2), \end{aligned} \tag{5.4}$$

and $S_{1/2}(a, b, n; \theta)$ as defined in (2.7) has an expansion of the form

$$\begin{aligned} S_{1/2}(a, b, n; \theta) &= \frac{(2^{2n+1}-1)a^{n+\frac{1}{2}}B_{2n+2}}{2^{2n}(2n+2)!}(2t)^{n+\frac{1}{2}}\left(1 + \frac{(2n+1)}{6}t^2 + O(t^4)\right) \\ &\quad \times \frac{(-1)^n(b-a)(2n+1)!\sqrt{2}}{\sqrt{a}}\left(\frac{1}{n!}t^{1/2} - \frac{(b-a)^3}{3a(n-1)!}t^{3/2} + O(t^{5/2})\right) \end{aligned}$$

$$= \frac{(-a)^n(2^{2n+1}-1)(b-a)B_{2n+2}}{2^{n-1}(2n+2)} t^{n+1} \left(\frac{1}{n!} - \frac{(b-a)^3}{3a(n-1)!} t + O(t^2) \right). \quad (5.5)$$

Now that we understand the building blocks, we can state the low order terms of the expansions appearing in Theorems 3.2 and 3.3.

Corollary 5.1. *Let* $h, k, u, v, s, w \in \mathbb{Z}$ *with* $k, u, v, w > 0$ *and* $\gcd(h, k) = 1$. *Further, we let* $\ell := vwk/\gcd(vwk, huw, svh)$ *and* $\theta := \log\left(\frac{1+t}{1-t}\right)$. *Then, as* $t \to 0^+$, *the following are true:*

(i) If ℓ *is odd, we have that*

$$\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = \sum_{r=0}^{\ell-1} (-1)^r \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r} + O(t). \quad (5.6)$$

(ii) If ℓ *is even, we have that*

$$\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = \left(\sqrt{\frac{\pi v}{2u\ell^2 t}} - \frac{sv}{uw\ell} + 1\right) \sum_{r=0}^{\ell-1} (-1)^r \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r}$$
$$+ \frac{2}{\ell} \sum_{r=0}^{\ell-1} (-1)^r r \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r} + O(t^{1/2}). \quad (5.7)$$

(iii) For any ℓ, *we have that*

$$\Upsilon_{1/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = \left(\sqrt{\frac{\pi v}{2u\ell^2 t}} - \frac{sv}{uw\ell} + 1\right) \sum_{r=0}^{\ell-1} \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r} \quad (5.8)$$
$$+ \frac{2}{\ell} \sum_{r=0}^{\ell-1} r \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r} + O(t^{1/2}).$$

Proof. Working from the statement of Theorem 3.2, we first notice that (5.3) implies that, for all $a > 0$ and $b \in \mathbb{R}$, $T_{1/2}(a, b, 0; \theta) = 1 + O(t)$ and $T_{1/2}(a, b, n; \theta) = O(t)$ for all $n \geq 1$. Therefore, (5.6) follows immediately from (5.1).

Next, we turn our attention to proving (5.7) and (5.8), which starts with recalling the statements of Theorem 3.2 and Theorem 3.3, respectively. We notice that (5.5) implies that $S_{1/2}(a, b, n; \theta) = O(t)$ for all $n \in \mathbb{N}_0$, $a > 0$, and $b \in \mathbb{R}$. Thus, both (5.7) and (5.8) follow from (5.1) and (5.4). □

Remark 5.2. The sums appearing in the constant terms of (5.6), (5.7), and (5.8), namely

$$\sum_{r=0}^{\ell-1} (\pm 1)^r \left(\zeta_k^h\right)^{\frac{u}{v}r^2 + \frac{s}{w}r}, \quad (5.9)$$

are related to *generalized Gauss sums*. In particular, a generalized Gauss sum is a sum of the form

$$\sum_{n=0}^{c-1} e^{2\pi i\left(\frac{an^2+bn}{c}\right)}$$

for some $a, b, c \in \mathbb{N}_0$, $c \neq 0$.

Understanding when the sum in (5.9) is zero allows one to determine at which roots of unity the partial theta function $\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$ vanishes in the case that ℓ is odd. Similarly, understanding the generalized Gauss sum in asymptotic expansions related to $\Upsilon_{1/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$ when ℓ is even and $\Upsilon_{1/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$ allow us to determine which roots of unity are poles of the corresponding partial theta function. We refer the reader to [4, 5] for a more detailed discussion of Guass sums.

5.2 "Weight 3/2" low order terms

We now turn our attention to the asymptotic expansions in Theorems 4.5 and 4.6. As in Section 5.1, we begin with the building blocks of these expansions, namely $T_{3/2}$, $C_{3/2}$ and $S_{3/2}$. For $a > 0$, $b \in \mathbb{R}$, $n \in \mathbb{Z}$ and $\theta := \log\left(\frac{1+t}{1-t}\right)$, we have that $T_{3/2}(a, b, n; \theta)$ as defined in (4.1) has an expansion of the form

$$T_{3/2}(a, b, n; \theta) = \frac{(-a)^n (b-a) E_{2n+2}}{2^{n+1} n!} t^{n+1} \left(1 + \frac{n+1}{3} t^2 + O(t^4)\right). \tag{5.10}$$

Similarly, we see that $C_{3/2}(a, b; \theta)$ as defined in (4.10) has an expansion of the form

$$\begin{aligned} C_{3/2}(a, b; \theta) =& \frac{1}{2at} - \frac{b}{2a}\sqrt{\frac{\pi}{2at}} + \frac{b^2 - a^2}{4a^2} \\ &+ \frac{(-b^4 + 6a^2b^2 - 8a^3b + 3a^4 - 8a^2)}{48a^3} t + \frac{b}{12a}\sqrt{\frac{\pi}{2a}} t^{3/2} + O(t^2), \end{aligned} \tag{5.11}$$

and $S_{3/2}(a, b, n; \theta)$ as defined in (4.11) has an expansion of the form

$$S_{3/2}(a, b, n; \theta) = \frac{(-a)^n (2^{2n+1} - 1) B_{2n+2}}{2^n (2n+2)} t^n \left(\frac{1}{n!} - \frac{(b-a)^2}{a(n-1)!} t + O(t^2)\right). \tag{5.12}$$

We are now prepared to state the low order terms of the expansions appearing in Theorems 4.5 and 4.6.

Corollary 5.3. *Let $h, k, u, v, s, w \in \mathbb{Z}$ with $k, u, v, w > 0$ and $\gcd(h, k) = 1$. Further, we let $\ell := vwk/\gcd(vwk, huw, svh)$ and $\theta := \log\left(\frac{1+t}{1-t}\right)$. Then, as $t \to 0^+$, the following are true:*

(i) If ℓ is odd, we have that

$$\Upsilon_{3/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = -\sum_{r=0}^{\ell-1}(-1)^r\left(\frac{\ell}{2}-r\right)\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r} + O(t). \quad (5.13)$$

(ii) If ℓ is even, we have that

$$\begin{aligned}\Upsilon_{3/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = &\frac{v}{2u\ell t}\sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r} \qquad (5.14)\\ &-\frac{sv}{2uw}\sqrt{\frac{\pi v}{2u\ell^2 t}}\sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\\ &+\sum_{r=0}^{\ell-1}(-1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(-\frac{\ell}{3}+r-\frac{r^2}{\ell}+\frac{s^2v^2}{2\ell u^2w^2}\right)\\ &+O(t^{1/2}).\end{aligned}$$

(iii) For all ℓ, we have that

$$\begin{aligned}\Upsilon_{3/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right) = &\frac{v}{2u\ell t}\sum_{r=0}^{\ell-1}\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r} - \frac{sv}{2uw}\sqrt{\frac{\pi v}{2u\ell^2 t}}\sum_{r=0}^{\ell-1}\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r} \qquad (5.15)\\ &+\sum_{r=0}^{\ell-1}\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}\left(-\frac{\ell}{3}+r-\frac{r^2}{\ell}+\frac{s^2v^2}{2\ell u^2w^2}\right)+O(t^{1/2}).\end{aligned}$$

Proof. To prove (5.13), we note that, by (5.10), $T_{3/2}(a, b, n; \theta) = O(t)$ for all $n \in \mathbb{N}_0$, $a > 0$, and $b \in \mathbb{R}$. Then, recalling the statement of Theorem 4.5, (5.13) follows from (5.1) and (5.3).

To prove (5.14) and (5.15), we notice (5.12) implies that for all $a > 0$ and $b \in \mathbb{R}$, $S_{3/2}(a, b, 0; \theta) = 1 + O(t)$ and $S_{3/2}(a, b, n; \theta) = O(t)$ for all $n \geq 1$. Then, recalling Theorems 4.5 and 4.6, (5.14) and (5.15) follow from (5.1), (5.4), (5.11), and (5.12) (since, as noted in the proof of Corollary 5.1, $S_{1/2}(a, b, n; \theta) = O(t)$ for all $n \in \mathbb{N}_0$). □

Remark 5.4. As in Corollary 5.1, we again see the same generalized Gauss sums

$$\sum_{r=0}^{\ell-1}(\pm 1)^r\left(\zeta_k^h\right)^{\frac{u}{v}r^2+\frac{s}{w}r}$$

appearing in the low order terms of $\Upsilon_{3/2}\left(\pm 1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$. Understanding when these sums vanish allows us to determine at which roots of unity the partial theta functions $\Upsilon_{3/2}\left(-1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$ with ℓ even and $\Upsilon_{3/2}\left(1, \frac{u}{v}, \frac{s}{w}; \zeta_k^h e^{-\theta}\right)$ have poles.

References

1. K. Alladi, A new combinatorial study of the Rogers-Fine identity and a related partial theta series. Int. J. Number Theory **5**(7), 1311–1320 (2009). URL https://doi.org/10.1142/S1793042109002675.
2. G.E. Andrews, *Partitions: Yesterday and Today* (New Zealand Mathematical Society, Wellington, 1979). With a foreword by J. C. Turner
3. B.C. Berndt, *Ramanujan's Notebooks. Part V*. (Springer, New York, 1998). URL https://doi.org/10.1007/978-1-4612-1624-7.
4. B.C. Berndt, R.J. Evans, The determination of Gauss sums. Bull. Am. Math. Soc. (N.S.) **5**(2), 107–129 (1981). URL http://dx.doi.org/10.1090/S0273-0979-1981-14930-2
5. B.C. Berndt, R.J. Evans, K.S. Williams, *Gauss and Jacobi Sums*. Canadian Mathematical Society Series of Monographs and Advanced Texts. (John Wiley and Sons, Inc., New York ,1998). A Wiley-Interscience Publication
6. B.C. Berndt, B. Kim, Asymptotic expansions of certain partial theta functions. Proc. Am. Math. Soc. **139**(11), 3779–3788 (2011). URL http://dx.doi.org/10.1090/S0002-9939-2011-11062-1
7. K. Bringmann, A. Folsom, On a conjecture of B. Berndt and B. Kim. Ramanujan J. **32**(1), 1–4 (2013). URL http://dx.doi.org/10.1007/s11139-012-9378-6
8. K. Bringmann, L. Rolen, Half-integral weight Eichler integrals and quantum modular forms. J. Number Theory **161**, 240–254 (2016). URL http://dx.doi.org/10.1016/j.jnt.2015.03.001
9. A. Folsom, K. Ono, R.C. Rhoades, Mock theta functions and quantum modular forms. Forum Math. Pi **1**(e2), 27 (2013)
10. A. Folsom, K. Ono, R.C. Rhoades, Ramanujans radial limits. Ramanujan **125**(627), 91 (2014)
11. W.F. Galway, An asymptotic expansion of Ramanujan. Number theory (Ottawa, ON, 1996), CRM Proc **19**, 107–110 (1999). Lecture Notes
12. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th edn. (Elsevier, Academic Press, Amsterdam, 2007), pp. 00–00. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX)
13. K. Hikami, Quantum invariant for torus link and modular forms. Commun. Math. Phys. **246**(2), 403–426 (2004). URL http://dx.doi.org/10.1007/s00220-004-1046-2
14. S. Kimport, Quantum modular forms, mock modular forms, and partial theta functions. Ph.D. thesis, Yale University, 2015
15. R. Lawrence, D. Zagier, Modular forms and quantum invariants of 3-manifolds. Asian J. Math. **3**(1), 93–107 (1999). Sir Michael Atiyah: a great mathematician of the twentieth century
16. W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and theorems for the special functions of mathematical physics. 3rd enlarged edn. Die Grundlehren der mathematischen Wissenschaften, Band 52 (Springer-Verlag New York, Inc., New York 1966)
17. R.P. Stanley, A survey of alternating permutations. in *Combinatorics and graphs*, Contemp. Math. **531**, 165–196 (2010). URL http://dx.doi.org/10.1090/conm/531/10466
18. D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function. Topology **40**(5), 945–960 (2001). URL http://dx.doi.org/10.1016/S0040-9383(00)00005-7

Overpartitions and Truncated Partition Identities

Louis W. Kolitsch

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Recently, Andrews and Merca, and Kolitsch and Burnette gave partition interpretations of the Truncated Pentagonal Number Theorem. In this paper, another partition interpretation for the Truncated Pentagonal Number Theorem involving overpartitions will be presented. Overpartitions will also be used to explain some specific truncations of Jacobi's Triple Product Identity.

Keywords Overpartitions · Truncated Identities

2010 Mathematics Subject Classification 11P81

1 Introduction

Euler's Pentagonal Number Theorem yields the recurrence

$$p(n) = \sum_{j=1}^{\infty}(-1)^{j-1}\big(p(n-(3j^2-j)/2) + p(n-(3j^2+j)/2)\big), \qquad (1.1)$$

This research was supported by a UTMartin Reagan Leave.

L. W. Kolitsch (✉)
Department of Mathematics and Statistics, The University of Tennessee at Martin, Martin, TN 38238, USA
e-mail: lkolitsc@utm.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_22

where $p(m) = 0$ if $m < 0$ for the ordinary partition function. The Truncated Pentagonal Number Theorem addresses the fact that,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\big(p(n-(3j^2+j)/2)-p(n-(3j^2+5j+2)/2)\big)\geq 0. \quad (1.2)$$

Andrews and Merca [1] proved that this nonnegative quantity is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$. Kolitsch and Burnette [4] proved that this nonnegative quantity is the number of partition pairs (S, U) where S is a partition with parts greater than k, U is a partition with $k-1$ distinct parts all of which are greater than the smallest part in S, and the sum of the parts in $S \cup U$ is n. Kolitsch and Burnette [4] also proved that

$$(-1)^k\left(p(n)+\sum_{j=1}^{k}(-1)^j\big(p(n-(3j^2-j)/2)-p(n-(3j^2+j)/2)\big)\right) \quad (1.3)$$

is nonnegative and counts the number of partition pairs (S, T) where S is a partition with parts greater than k, T is a partition with k distinct parts all of which are greater than the smallest part in S, and the sum of the parts in $S \cup T$ is n. In this paper, overpartitions will be used to give a new interpretation of these nonnegative quantities. Some truncated identities associated with Jacobi's Triple Product Identity will also be explored.

2 The Main Theorem

Letting $p_e(n)$ (respectively, $p_o(n)$) be the number of partitions of n into an even (odd) number of distinct parts, the combinatorial proof of Euler's Pentagonal Number Theorem shows that $p_e(n) - p_o(n) = 0$ unless $n = \frac{3j^2-j}{2}$ or $\frac{3j^2+j}{2}$. In these exceptional cases, $p_e(n) - p_o(n) = (-1)^j$. The new interpretation of the Truncated Pentagonal Number Theorem presented here will look at the difference in the number of overpartitions in two categories separated by parity. In order to state the main theorem, we need to introduce the polynomials $F_r(q)$ with $r \geq 2$ defined by $1+\sum_{j=1}^{(r-1)/2}(-1)^j(q^{(3j^2-j)/2}+q^{(3j^2+j)/2})$ for r odd and $\sum_{j=0}^{(r-2)/2}(-1)^j(q^{(3j^2+j)/2}-q^{(3j^2+5j+2)/2})$ for r even.

Theorem 2.1. *The coefficient of q^n in the generating function $\frac{F_r(q)}{(q;q)_\infty}$ is the number of overpartitions of n with an even number of overlined parts minus the number of overpartitions of n with an odd number of overlined parts where all parts are $\geq \lfloor\frac{r+2}{2}\rfloor$ and at most $\lfloor\frac{r-1}{2}\rfloor$ parts are overlined.*

To prove this theorem, we need the generating functions for the overpartitions described in the theorem. The generating function for unrestricted overpartitions is $\frac{(-q;q)_\infty}{(q;q)_\infty}$ [3]. For our theorem, we will need to keep track of the size of the parts and the number of overlined parts. To do this, we will insert parameters b and c into the generating function to get $\frac{(-bq^c;q)_\infty}{(q^c;q)_\infty}$. This function will now generate the number of overpartitions, where the parts are $\geq c$ and the parameter b will keep track of the number of overlined parts in the partition. Using Corollary 2.2 in [2], we can rewrite

$$\frac{(-bq^c;q)_\infty}{(q^c;q)_\infty} = \frac{1}{(q^c;q)_\infty}\sum_{j=0}^{\infty}\frac{b^j q^{(j^2-j)/2+cj}}{(q;q)_j}. \tag{2.1}$$

Thus, the generating function for overpartitions, where all parts are $\geq c$ and exactly j parts are overlined, is $\frac{1}{(q^c;q)_\infty}\left(\frac{q^{(j^2-j)/2+cj}}{(q;q)_j}\right)$. Hence, the generating function for the number of overpartitions of n with an even number of overlined parts minus the number of overpartitions of n with an odd number of overlined parts, where all parts are $\geq \lfloor\frac{r+2}{2}\rfloor$ and at most $\lfloor\frac{r-1}{2}\rfloor$ parts are overlined, is

$$\frac{1}{(q^{k+1};q)_\infty}\sum_{j=0}^{k}\left(\frac{(-1)^j q^{(j^2-j)/2+(k+1)j}}{(q;q)_j}\right) \tag{2.2}$$

when $r = 2k+1$ and is

$$\frac{1}{(q^{k+2};q)_\infty}\sum_{j=0}^{k}\left(\frac{(-1)^j q^{(j^2-j)/2+(k+2)j}}{(q;q)_j}\right) \tag{2.3}$$

when $r = 2k+2$. These generating functions appear in equations (2.5) and (2.9) in [5] and are equal to $\frac{F_{2k+1}(q)}{(q;q)_\infty}$ and $\frac{F_{2k+2}(q)}{(q;q)_\infty}$, respectively, which completes the proof of Theorem 2.1.

3 The Non-negativity Results

Let π_c be the set of ordinary partitions of n with parts $\geq c$ and let λ denote an element of this set. Let $\bar{p}_c(n)$ be the number of overpartitions of n with parts $\geq c$. Also, let $\beta(\lambda)$ be the number of different parts in λ. For example, the partition $\lambda = 1+1+2+3+3+3+5$ of 18 in π_1 has $\beta(\lambda) = 4$ since the four different parts in λ are 1, 2, 3, and 5. With these definitions, the following lemma can easily be observed since the parts that are overlined have to be selected from the different parts in λ.

Lemma 3.1.

$$\bar{p}_c(n) = \sum_{\lambda \in \pi_c} \sum_{i=0}^{\beta(\lambda)} \binom{\beta(\lambda)}{i}. \tag{3.1}$$

The following theorem is a consequence of this lemma and a property of alternating sums and differences of binomial coefficients.

Theorem 3.2. *The number of overpartitions of n with an even number of overlined parts minus the number of overpartitions of n with an odd number of overlined parts, where all parts are $\geq c$ and at most k parts are overlined, is given by*

$$\sum_{\lambda \in \pi_c} \sum_{i=0}^{k} (-1)^i \binom{\beta(\lambda)}{i} = (-1)^k \sum_{\lambda \in \pi_c} \binom{\beta(\lambda) - 1}{k}. \tag{3.2}$$

The nonnegativity results associated with the Truncated Pentagonal Number Theorem follow immediately by taking $c = \lfloor \frac{r+2}{2} \rfloor$ and $k = \lfloor \frac{r-1}{2} \rfloor$.

4 Some Results for Jacobi's Triple Product

For $r \geq 2$ and $s \leq \lfloor \frac{r}{2} \rfloor$, Jacobi's Triple Product Identity can be stated as

$$(q^s; q^r)_\infty (q^{r-s}; q^r)_\infty (q^r; q^r)_\infty = \sum_{i=0}^{\infty} (-1)^i q^{(ri^2 + (r-2s)i)/2} (1 - q^{s(2i+1)}). \tag{4.1}$$

We will define $a_{r,s}(n)$ by

$$\frac{1}{(q^s; q^r)_\infty (q^{r-s}; q^r)_\infty (q^r; q^r)_\infty} = \sum_{n=0}^{\infty} a_{r,s}(n) q^n. \tag{4.2}$$

The results in [6] show that, for $r \geq 2$ and $s \leq \lfloor \frac{r}{2} \rfloor$,

$$(-1)^k \sum_{i=0}^{k} (-1)^i \left(a_{r,s}\big(n - \frac{ri^2 + (r-2s)i}{2}\big) - a_{r,s}\big(n - \frac{ri^2 + (r+2s)i + 2s}{2}\big) \right) \tag{4.3}$$

is nonnegative. In this section, Yee's result for $k = 1$ will be interpreted in terms of overpartitions.

Theorem 4.1. *For $r \geq 3$, $s < \lfloor \frac{r}{2} \rfloor$, and $n \geq 1$,*

$$a_{r,s}(n) - a_{r,s}(n-s) - a_{r,s}(n-(r-s)) + a_{r,s}(n-(r+2s)) \tag{4.4}$$

is the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part $\geq r - s$ *and congruent to* $\pm s$ *(mod r) and the non-overlined parts are congruent to* $0, \pm s$ *(mod r) and s and r do not occur as parts.*

A generating function that can be used to generate the overpartitions described in the theorem is

$$\frac{(-bq^{r+s};q^r)_\infty(-bq^{r-s};q^r)_\infty}{(q^{r+s};q^r)_\infty(q^{r-s};q^r)_\infty(q^{2r};q^r)_\infty}$$
$$= \frac{1}{(q^{r+s};q^r)_\infty(q^{r-s};q^r)_\infty(q^{2r};q^r)_\infty}\sum_{i,j\geq 0}\frac{b^{i+j}q^{r((i^2-i)/2+(j^2-j)/2)+(r+s)i+(r-s)j}}{(q^r;q^r)_i(q^r;q^r)_j}. \tag{4.5}$$

The coefficient of b^0 in this generating function yields the generating function for the overpartitions of n described in the theorem that have no overlined parts and the coefficient of b^1 in this function yields the generating function for overpartitions of n described in the theorem that have one overlined part. Respectively, these two generating functions are $\frac{1}{(q^{r+s};q^r)_\infty(q^{r-s};q^r)_\infty(q^{2r};q^r)_\infty}$ and $\frac{1}{(q^{r+s};q^r)_\infty(q^{r-s};q^r)_\infty(q^{2r};q^r)_\infty}\left(\frac{q^{r+s}+q^{r-s}}{(q^r;q^r)_1}\right)$. Hence, the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part $\geq r$ and congruent to $\pm s(\text{mod } r)$ and the non-overlined parts are congruent to $0, \pm s(\text{mod } r)$, and not equal to s or r is the coefficient of q^n in

$$\frac{1}{(q^{r+s};q^r)_\infty(q^{r-s};q^r)_\infty(q^{2r};q^r)_\infty}\left(1-\frac{q^{r+s}+q^{r-s}}{1-q^r}\right)$$
$$= \frac{1}{(q^r;q^r)_\infty(q^s;q^r)_\infty(q^{r-s};q^r)_\infty}\left(1-q^s-q^{r-s}+q^{r+2s}\right), \tag{4.6}$$

which gives the desired result.

We can also use overpartitions to explain what the quantity $a_{r,s}(n) - a_{r,s}(n - s) - a_{r,s}(n - (r - s))$ enumerates.

Theorem 4.2. *For* $r \geq 3, s < \lfloor \frac{r}{2} \rfloor$, *and* $n \geq 1$, $a_{r,s}(n) - a_{r,s}(n - s) - a_{r,s}(n - (r - s))$ *is the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part where the overlined parts are* $\geq r - s$ *and in the sequence* $r + si$ *for* $i \geq -1$ *and the non-overlined parts are* $\geq r - s$ *and congruent to* $0, \pm s$ *(mod r).*

A generating function that can be used to generate the overpartitions described in the theorem is

$$\frac{(-bq^{r-s};q^s)_\infty}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}$$
$$= \frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}\sum_{j=0}^{\infty}\frac{b^j q^{s(j^2-j)/2+(r-s)j}}{(q^s;q^s)_j}. \tag{4.7}$$

The coefficient of b^0 in this generating function yields the generating function for the overpartitions of n described in the theorem that have no overlined parts and the coefficient of b^1 in this function yields the generating function for overpartitions of n described in the theorem that have one overlined part. Respectively, these two generating functions are $\frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}$ and $\frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}\left(\frac{q^{(r-s)}}{(q^s;q^s)_1}\right)$. Hence, the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part, where the overlined parts are $\geq r-s$ and in the sequence $r+si$ for $i \geq -1$ and the non-overlined parts are $\geq r-s$ and congruent to 0, $\pm s (\bmod r)$ is the coefficient of q^n in

$$\begin{aligned} &\frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty} - \frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}\left(\frac{q^{(r-s)}}{(q^s;q^s)_1}\right) \\ &\quad = \frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^{r+s};q^r)_\infty}\left(1-\frac{q^{r-s}}{1-q^s}\right) \\ &\quad = \frac{1}{(q^{r-s};q^r)_\infty(q^r;q^r)_\infty(q^s;q^r)_\infty}\left(1-q^s-q^{r-s}\right), \end{aligned} \tag{4.8}$$

which gives the desired result.

Unfortunately, the next truncation of Jacobi's Triple Product, the case $k=2$ from Yee's paper, does not follow in a similar manner. This case looks at the quantity

$$\begin{aligned} &a_{r,s}(n) - a_{r,s}(n-s) - a_{r,s}(n-(r-s)) + a_{r,s}(n-(r+2s)) \\ &+ a_{r,s}(n-(3r-2s)) - a_{r,s}(n-(3r+3s)). \end{aligned} \tag{4.9}$$

Our method yields a result for

$$\begin{aligned} &a_{r,s}(n) - a_{r,s}(n-s) - a_{r,s}(n-(r-s)) + a_{r,s}(n-(r+2s)) \\ &+ a_{r,s}(n-(3r-2s)) - a_{r,s}(n-3r) \end{aligned} \tag{4.10}$$

instead of the desired truncation.

Theorem 4.3. *For $r \geq 3$, $s < \lfloor \frac{r}{2} \rfloor$, and $n \geq 1$,*

$$\begin{aligned} &a_{r,s}(n) - a_{r,s}(n-s) - a_{r,s}(n-(r-s)) + a_{r,s}(n-(r+2s)) \\ &+ a_{r,s}(n-(3r-2s)) - a_{r,s}(n-3r) \end{aligned} \tag{4.11}$$

is the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part where the parts are $\geq r+s$ and congruent to 0, $\pm s$ (mod r).

A generating function that can be used to generate the overpartitions described in the theorem is

$$\frac{(-bq^{r+s};q^r)_\infty(-bq^{2r-s};q^r)_\infty(-bq^{2r};q^r)_\infty}{(q^{r+s};q^r)_\infty(q^{2r-s};q^r)_\infty(q^{2r};q^r)_\infty} = \frac{1}{(q^{r+s};q^r)_\infty(q^{2r-s};q^r)_\infty(q^{2r};q^r)_\infty} \tag{4.12}$$

$$\times \sum_{i,j,k\geq 0} \frac{b^{i+j+k}q^{r((i^2-i)/2+(j^2-j)/2+(k^2-k)/2)+(r+s)i+(2r-s)j+2rk}}{(q^r;q^r)_i(q^r;q^r)_j(q^r;q^r)_k}.$$

The coefficient of b^0 in this generating function yields the generating function for the overpartitions of n described in the theorem that have no overlined parts and the coefficient of b^1 in this function yields the generating function for overpartitions of n described in the theorem that have one overlined part. Respectively, these two generating functions are $\frac{1}{(q^{r+s};q^r)_\infty(q^{2r-s};q^r)_\infty(q^{2r};q^r)_\infty}$ and $\frac{1}{(q^{r+s};q^r)_\infty(q^{2r-s};q^r)_\infty(q^{2r};q^r)_\infty}$ $\left(\frac{q^{r+s}+q^{2r-s}+q^{2r}}{(q^r;q^r)_1}\right)$. Hence, the number of overpartitions of n with no overlined parts minus the number of overpartitions of n with one overlined part, where the parts are $\geq r+s$ and congruent to 0, $\pm s$ (mod r) is the coefficient of q^n in

$$\frac{1}{(q^{r+s};q^r)_\infty(q^{2r-s};q^r)_\infty(q^{2r};q^r)_\infty}\left(1-\frac{q^{r+s}+q^{2r-s}+q^{2r}}{1-q^r}\right)$$

$$= \frac{1}{(q^r;q^r)_\infty(q^s;q^r)_\infty(q^{r-s};q^r)_\infty}\left(1-q^s-q^{r-s}+q^{r+2s}+q^{3r-2s}-q^{3r}\right), \tag{4.13}$$

which gives the desired result.

5 Concluding Remarks

Since the $r=3, s=1$ case of Jacobi's Triple Product is the Pentagonal Number Theorem, it would be nice to find a comprehensive result that easily explains each truncation of Jacobi's Triple Product as a generalization of an interpretation of the Truncated Pentagonal Number Theorem.

Acknowledgements The author would like to thank the organizers for a wonderful conference.

References

1. G.E. Andrews, M. Merca, The truncated pentagonal number theorem. J. Comb. Theory Ser. A **119**(8), 1639–1643 (2012)
2. G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications (Addison-Wesley, Reading, 1976)
3. S. Corteel, J. Lovejoy, Overpartitions. Trans. Am. Math. Soc. **356**(4), 1623–1635 (2004)
4. L. Kolitsch, M. Burnette, Interpreting the truncated pentagonal number theorem using partition pairs. Electron. J. Comb. **22**(2), (2015). Paper 2.55

5. L. Kolitsch, Another approach to the truncated pentagonal number theorem. Int. J. Number Theory **11**(5), 1563–1569 (2015)
6. A.J. Yee, A truncated Jacobi triple product theorem. J. Comb. Theory Ser. A **130**, 1–14 (2015)

Congruences Modulo Powers of 2 for the Number of Unique Path Partitions

C. Krattenthaler

Dedicated to Krishnaswami Alladi on the occasion of his 60th birthday

Abstract We compute the congruence class modulo 16 of the number of unique path partitions of n (as defined by Olsson), thus generalising previous results by Bessenrodt, Olsson and Sellers [*Ann. Combin.* **13** (2013), 591–602].

Keywords Unique path partitions · Congruences · q-series

2010 Mathematics Subject Classification Primary 05A15 · Secondary 05A17 11A07 · 11P83

1 Introduction

Unique path partitions were introduced by Olsson in [3]. Their study is motivated from the Murnaghan–Nakayama rule for the calculation of the value of characters of the symmetric group. They were completely characterised by Bessenrodt, Olsson and Sellers in [1]. They used this characterisation to derive a formula for the generating

Research partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S50-N15, the latter in the framework of the Special Research Program "Algorithmic and Enumerative Combinatorics".

C. Krattenthaler (✉)
Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1,
A-1090 Vienna, Austria
e-mail: Christian.Krattenthaler@univie.ac.at
URL: http://www.mat.univie.ac.at/∼kratt

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_23

function for the number $u(n)$ of all unique path partitions of n. This formula reads (cf. [1, Remark 3.6])

$$\sum_{n\ge1} u(n)q^n = 2\sum_{i\ge1} q^{2^i-1}(1+q^{2^{i-1}})\prod_{j=0}^{i-2}\frac{1}{1-q^{2^j}}$$
$$= 2\left(q(1+q)+\sum_{i\ge2}\frac{q^{-1}+1}{1-q^2}\cdot\frac{q^{2^i}(1+q^{2^{i-1}})}{\prod_{j=1}^{i-2}(1-q^{2^j})}\right). \tag{1.1}$$

The final part in [1] concerns congruences modulo 8 for $u(n)$. The corresponding main result [1, Theorem 4.6] provides a complete description of the behaviour of $u(n)$ modulo 8 (in terms of the related sequence of numbers $w(n)$; see the next section for the definition of $w(n)$). The arguments to arrive at this result are mainly of a recursive nature.

The purpose of this note is to show that a more convenient and more powerful method to derive congruences (modulo powers of 2) is by an analysis of the generating function (1.1). Not only are we able to recover the result from [1], but in addition we succeed in determining the congruence class of $u(n)$ modulo 16, see (2.1) and Theorem 7, thus solving the problem left open in the last paragraph of [1]. We point out that the approach presented here is very much inspired by calculations in [2, Appendix], where expressions similar to the one on the right-hand side of (1.1) appear, with the role of the prime number 2 replaced by 3, though.

2 An equivalent expression for the generating function

We start with the observation (already made in [1]) that, first, all numbers $u(n)$ are divisible by 2, and, second, we have $u(2n) = u(2n-1)$ for all n. This is easy to see from the right-hand side of (1.1) since it has the form $2(1+q)f(q^2)$, where $f(t)$ is a formal power series in t. We therefore divide the right-hand side of (1.1) by $2(1+q^{-1})$, subsequently replace q by $q^{1/2}$, and consider the "reduced" generating function

$$\sum_{n\ge2} w(n)q^n = \sum_{i\ge2} q^{2^{i-1}}(1+q^{2^{i-2}})\frac{1}{(1-q)\prod_{j=0}^{i-3}(1-q^{2^j})}.$$

In other words, we have

$$2w(n) = u(2n) = u(2n-1) \tag{2.1}$$

for all n.

Using the convention

$$\sum_{k=M}^{N-1} \operatorname{Expr}(k) = \begin{cases} \sum_{k=M}^{N-1} \operatorname{Expr}(k), & N > M, \\ 0, & N = M, \\ -\sum_{k=N}^{M-1} \operatorname{Expr}(k), & N < M. \end{cases} \tag{2.2}$$

for sums, we rewrite the above equation in the following way:

$$\begin{aligned}
\sum_{n\ge 2} w(n)q^n &= \sum_{i\ge 2} q^{2^{i-1}} \frac{1+q^{2^{i-2}}}{(1-2q+q^2)\prod_{j=1}^{i-3}(1-q^{2^j})} \\
&= \sum_{i\ge 2} q^{2^{i-1}} \frac{1+q^{2^{i-2}}}{(1-\frac{2q}{1+q^2})(1+q^2)\prod_{j=1}^{i-3}(1-q^{2^j})} \\
&= \sum_{i\ge 2} q^{2^{i-1}} \frac{1+q^{2^{i-2}}}{(1-\frac{2q}{1+q^2})(1-q^4)\prod_{j=2}^{i-3}(1-q^{2^j})} \\
&= \sum_{i\ge 2} q^{2^{i-1}} \frac{1+q^{2^{i-2}}}{(1-\frac{2q}{1+q^2})(1-q^4)^2\prod_{j=3}^{i-3}(1-q^{2^j})} \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \sum_{i\ge 1} q^{2^{2i-1}} \frac{1+q^{2^{2i-2}}}{(1-\frac{2q}{1+q^2})(1-\frac{2q^4}{1+q^8})\cdots(1-\frac{2q^{2^{2i-4}}}{1+q^{2^{2i-3}}})(1-q^{2^{2i-2}})} \\
&\qquad + \sum_{i\ge 1} q^{2^{2i}} \frac{1}{(1-\frac{2q}{1+q^2})(1-\frac{2q^4}{1+q^8})\cdots(1-\frac{2q^{2^{2i-2}}}{1+q^{2^{2i-1}}})} \\
&= \sum_{i\ge 1} q^{2^{2i-1}} \frac{1+\frac{2q^{2^{2i-2}}}{1-q^{2^{2i-2}}}}{\prod_{j=0}^{i-2}(1-\frac{2q^{2^{2j}}}{1+q^{2^{2j+1}}})} + \sum_{i\ge 1} q^{2^{2i}} \frac{1}{\prod_{j=0}^{i-1}(1-\frac{2q^{2^{2j}}}{1+q^{2^{2j+1}}})}.
\end{aligned} \tag{2.3}$$

From the last expression, it is immediately obvious that $w(n)$ is odd if and only if n is a power of 2, thus recovering the first assertion of [1, Cor. 4.3]. The above-mentioned mod-4 result [1, Theorem 4.6] for $w(n)$—which, by (2.1), translates into a mod-8 result for $u(n)$—can also be derived within a few lines from the above expression.

In the next section, we show how to obtain congruences modulo 8 for $w(n)$, which, by (2.1), translate into congruences modulo 16 for the unique path partition numbers $u(n)$.

3 Congruences modulo powers of 2

In what follows, we write

$$f(q) = g(q) \text{ modulo } 2^\gamma$$

to mean that the coefficients of q^i in $f(q)$ and $g(q)$ agree modulo 2^γ for all i. We apply geometric series expansion in (2.3), and at the same time, we neglect terms which are divisible by 8. For example, we expand

$$\frac{1}{1-\frac{2q}{1+q^2}} = 1 + \frac{2q}{1+q^2} + \frac{4q^2}{(1+q^2)^2} \quad \text{modulo } 8.$$

In this manner, we obtain the congruence

$$\sum_{n\ge2} w(n)q^n = \sum_{i\ge1} q^{2^{2i-1}} \Bigg(1 + \frac{2q^{2^{2i-2}}}{1-q^{2^{2i-2}}} + 2\sum_{j=0}^{i-2} \frac{q^{2^{2j}}}{1+q^{2^{2j+1}}} \\ + 4\frac{q^{2^{2i-2}}}{1-q^{2^{2i-2}}} \sum_{j=0}^{i-2} \frac{q^{2^{2j}}}{1+q^{2^{2j+1}}} + 4\sum_{0\le s\le t\le i-2} \frac{q^{2^{2s}+2^{2t}}}{(1+q^{2^{2s+1}})(1+q^{2^{2t+1}})}\Bigg) \\ + \sum_{i\ge1} q^{2^{2i}} \Bigg(1 + 2\sum_{j=0}^{i-1} \frac{q^{2^{2j}}}{1+q^{2^{2j+1}}} + 4\sum_{0\le s\le t\le i-1} \frac{q^{2^{2s}+2^{2t}}}{(1+q^{2^{2s+1}})(1+q^{2^{2t+1}})}\Bigg) \\ \text{modulo } 8.$$

After rearrangement, this becomes

$$\sum_{n\ge2} w(n)q^n = \sum_{i\ge1} q^{2^i} + \frac{2q^3}{1-q} + 2\sum_{j\ge1} \frac{1}{1-q^{2^{2j}}} \Bigg(q^{2^{2j}+2^{2j+1}} + q^{2^{2j-2}}(1-q^{2^{2j-1}}) \sum_{\ell\ge2j} q^{2^\ell}\Bigg) \\ + 4\sum_{1\le s<t} \frac{q^{2^{2s-2}+2^{2t-2}}}{(1-q^{2^{2s-1}})(1-q^{2^{2t-1}})} \Bigg(q^{2^{2t-1}}(1+q^{2^{2t-2}}) + \sum_{\ell\ge2t} q^{2^\ell}\Bigg) \\ + 4\sum_{s\ge1} \frac{q^{2^{2s-1}}}{(1-q^{2^{2s}})} \sum_{\ell\ge2s} q^{2^\ell} \qquad \text{modulo } 8. \tag{3.1}$$

We must now analyse the individual sums in (3.1).

Lemma 1. *Let $n \ge 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$, with $0 \le n_i \le 1$ for all i and $n_a \ne 0 \ne n_e$. Then, the coefficient of q^n in*

$$\sum_{j\ge1} \frac{q^{2^{2j}+2^{2j+1}}}{1-q^{2^{2j}}} \tag{3.2}$$

is equal to $\lfloor a/2 \rfloor$ *if* n *is not a power of* 2*, and it is equal to* $\max\{\lfloor a/2 \rfloor - 1, 0\}$ *otherwise.*

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.2) is equal to the number of possibilities to write $n = (k+3)2^{2j}$ for some $j \geq 1$ and $k \geq 0$. For fixed j, we can find a suitable k if and only if $n \geq 3 \cdot 2^{2j}$. If n is not a power of 2, this is equivalent to the condition that $2j \leq a$. The claim follows immediately. □

Lemma 2. *Let* $n \geq 2$*, and write* $n = \sum_{i=a}^{e} n_i \cdot 2^i$ *as in Lemma 1. Then, the coefficient of* q^n *in*

$$\sum_{j \geq 1} \frac{q^{2^{2j-2}}}{1 - q^{2^{2j}}} \sum_{\ell \geq 2j} q^{2^\ell} \tag{3.3}$$

is equal to $e - 2j + 1$ *if* $a = 2j - 2$*,* $n_{a+1} = n_{2j-1} = 0$*, and* n *is not a power of* 2*, and it is equal to* 0 *otherwise.*

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.3) is equal to the number of possibilities to write $n = 2^{2j-2} + k \cdot 2^{2j} + 2^\ell$ for some $j \geq 1$, $\ell \geq 2j$, and $k \geq 0$. The claim follows immediately. □

Lemma 3. *Let* $n \geq 2$*, and write* $n = \sum_{i=a}^{e} n_i \cdot 2^i$ *as in Lemma 1. Then, the coefficient of* q^n *in*

$$\sum_{j \geq 1} \frac{q^{2^{2j-2}+2^{2j-1}}}{1 - q^{2^{2j}}} \sum_{\ell \geq 2j} q^{2^\ell} \tag{3.4}$$

is equal to $e - 2j + 1$ *if* $a = 2j - 2$*,* $n_{a+1} = n_{2j-1} = 1$*, and it is equal to* 0 *otherwise.*

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.4) is equal to the number of possibilities to write $n = 2^{2j-2} + 2^{2j-1} + k \cdot 2^{2j} + 2^\ell$ for some $j \geq 1$, $\ell \geq 2j$, and $k \geq 0$. The claim follows immediately. □

Lemma 4. *Let* $n \geq 2$*, and write* $n = \sum_{i=a}^{e} n_i \cdot 2^i$ *as in Lemma 1. Then the coefficient of* q^n *in*

$$\sum_{s \geq 1} \frac{q^{2^{2s-1}}}{1 - q^{2^{2s}}} \sum_{\ell \geq 2s} q^{2^\ell} \tag{3.5}$$

is equal to $e - 2s + 1$ *if* $a = 2s - 1$ *and* n *is not a power of* 2*, and it is equal to* 0 *otherwise.*

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.5) is equal to the number of possibilities to write $n = 2^{2s-1} + k \cdot 2^{2s} + 2^\ell$ for some $s \geq 1$, $\ell \geq 2s$, and $k \geq 0$. The claim follows immediately. □

Lemma 5. *Let* $n \geq 2$*, and write* $n = \sum_{i=a}^{e} n_i \cdot 2^i$ *as in Lemma 1. Then, the coefficient of* q^n *in*

$$\sum_{1\le s<t} \frac{q^{2^{2s-2}+2^{2t-2}}}{(1-q^{2^{2s-1}})(1-q^{2^{2t-1}})} \sum_{\ell\ge 2t-1} q^{2^\ell} \tag{3.6}$$

is congruent to

$$e \sum_{i=a+2}^{e-\chi(e \text{ even})} n_i - a\cdot n_{a+2} + \left\lfloor \tfrac{1}{2}(e-a-1)\right\rfloor \pmod 2, \tag{3.7}$$

where $\chi(\mathscr{S})=1$ *if* $\mathscr{S}$ *is true and* $\chi(\mathscr{S})=0$ *otherwise.*

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.6) is equal to the number of possibilities to write

$$n = (2k_1+1)2^{2s-2} + (2k_2+1)2^{2t-2} + 2^{2t-1+k_3} \tag{3.8}$$

for some s and t with $1\le s<t$ and $k_1,k_2,k_3\ge 0$. Clearly, we need a to be even in order that the number of these possibilities be non-zero. Given that $a=2s-2$, we just have to count the number of possible triples (t,k_2,k_3) in (3.8), since the appropriate k_1 can certainly be found. If we fix t and k_3, the number of possible k_2's is

$$\left\lfloor \frac{1}{2}\cdot\frac{n-2^{2t-1+k_3}}{2^{2t-2}} + \frac{1}{2}\right\rfloor = \left\lfloor \frac{n}{2^{2t-1}} + \frac{1}{2}\right\rfloor - 2^{k_3}.$$

This needs to be summed over all t and k_3 with $\frac{1}{2}(a+2)=s<t\le\frac{1}{2}(e+1)$ and $0\le k_3\le e-2t+1$. We obtain

$$\begin{aligned}
&\sum_{t=s+1}^{\lfloor\frac{1}{2}(e+1)\rfloor}\sum_{k_3=0}^{e-2t+1}\left(\left\lfloor\frac{n}{2^{2t-1}}+\frac{1}{2}\right\rfloor - 2^{k_3}\right)\\
&\quad\equiv \sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor}\sum_{k_3=0}^{e-2t+1}\lfloor n_a\cdot 2^{a-2t+1}+\dots+(n_{2t-2}+1)\cdot 2^{-1}\\
&\qquad\qquad + n_{2t-1}+n_{2t}\cdot 2+\dots+n_e\cdot 2^{e-2t+1}\rfloor - \left\lfloor\tfrac{1}{2}(e-a-1)\right\rfloor\\
&\quad\equiv \sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor}(e-2t+2)(n_{2t-2}+n_{2t-1}) + \left\lfloor\tfrac{1}{2}(e-a-1)\right\rfloor\\
&\quad\equiv e\sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor} n_{2t-1} + e\sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e-1)\rfloor} n_{2t} + (e-a)n_{a+2} + \left\lfloor\tfrac{1}{2}(e-a-1)\right\rfloor \pmod 2.
\end{aligned}$$

□

Lemma 6. *Let* $n\ge 2$*, and write* $n=\sum_{i=a}^e n_i\cdot 2^i$ *as in Lemma 1. Then, the coefficient of* q^n *in*

$$\sum_{1\le s<t} \frac{q^{2^{2s-2}+2^{2t}}}{(1-q^{2^{2s-1}})(1-q^{2^{2t-1}})} \tag{3.9}$$

is congruent to

$$\sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor} n_{2t-1} + \left\lfloor \tfrac{1}{2}(e-a-1)\right\rfloor. \quad \pmod 2. \tag{3.10}$$

Proof. By geometric series expansion, we see that the coefficient of q^n in (3.9) is equal to the number of possibilities to write

$$n = (2k_1+1)2^{2s-2} + (k_2+2)2^{2t-1} \tag{3.11}$$

for some s and t with $1 \le s < t$ and $k_1, k_2 \ge 0$. Clearly again, we need a to be even in order that the number of these possibilities be non-zero. Given that $a = 2s-2$, we just have to count the number of possible pairs (t, k_2) in (3.11), since the appropriate k_1 can certainly be found. If we fix t, the number of possible k_2's is

$$\left\lfloor \frac{n-2^{2t}}{2^{2t-1}} + 1 \right\rfloor = \left\lfloor \frac{n}{2^{2t-1}} \right\rfloor - 1.$$

This needs to be summed over all t with $\frac{1}{2}(a+2) = s < t \le \frac{1}{2}(e+1)$. We obtain

$$\begin{aligned}\sum_{t=s+1}^{\lfloor\frac{1}{2}(e+1)\rfloor} \left(\left\lfloor \frac{n}{2^{2t-1}}\right\rfloor - 1\right) &\equiv \sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor} \Big\lfloor n_a \cdot 2^{a-2t+1} + \cdots + n_{2t-2}\cdot 2^{-1} \\ &\qquad + (n_{2t-1}-1) + n_{2t}\cdot 2 + \cdots + n_e \cdot 2^{e-2t+1} \Big\rfloor \\ &\equiv \sum_{t=\frac{1}{2}(a+4)}^{\lfloor\frac{1}{2}(e+1)\rfloor} n_{2t-1} - \left\lfloor \tfrac{1}{2}(e-a-1)\right\rfloor \pmod 2.\end{aligned}$$

□

We are finally in the position to state and prove our main result. It expresses the congruence class of $w(n)$ modulo 8—and thus, by (2.1), the congruence class of the unique path partition number $u(n)$ modulo 16—in terms of the binary digits of n. We point out that the assertion (3.12) already appeared in [1, Prop. 4.5].

Theorem 7. *Let $n \ge 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$ as in Lemma 1. Then, if $a = e$ (i.e. if n is a power of 2), the number $w(n)$ is congruent to*

$$2\lfloor a/2 \rfloor + 1 \pmod 8, \tag{3.12}$$

while it is congruent to

$$2 + 2\lfloor a/2 \rfloor + 2\chi(a \text{ even})(1 - 2n_{a+1})(e - a - 1) + 4\chi(a \text{ odd})(e - a)$$
$$+ 4\chi(a \text{ even})\left(e \sum_{i=a+2}^{e-\chi(e \text{ even})} n_i + a \cdot n_{a+2} + \sum_{t=\frac{1}{2}(a+4)}^{\lfloor \frac{1}{2}(e+1) \rfloor} n_{2t-1}\right) \pmod 8 \qquad (3.13)$$

otherwise.

Proof. Let first $n = 2^a$. We must then read the coefficient of q^n on the right-hand side of (3.1) and reduce the result modulo 8. Non-zero contributions come from the very first sum, from the series $2q^3/(1 - q^2)$, and from the series which is discussed in Lemma 1. Altogether, we obtain

$$1 + 2\chi(a \geq 2) + 2 \max\{\lfloor a/2 \rfloor - 1, 0\},$$

which can be simplified to (3.12).

Now let n be different from a power of 2. The non-zero contributions when reading the coefficient of q^n on the right-hand side of (3.1) come again from the series $2q^3/(1 - q^2)$, and from the series discussed in Lemmas 1–6. These contributions add up to

$$2\chi(n \geq 3) + 2\lfloor a/2 \rfloor + 2\chi(a \text{ even}, n_{a+1} = 0)(e - a - 1)$$
$$+ 2\chi(a \text{ even}, n_{a+1} = 1)(e - a - 1) + 4\chi(a \text{ odd})(e - a)$$
$$+ 4\chi(a \text{ even})\left(e \sum_{i=a+2}^{e-\chi(e \text{ even})} n_i - a \cdot n_{a+2} + \left\lfloor \tfrac{1}{2}(e - a - 1) \right\rfloor + \sum_{t=\frac{1}{2}(a+4)}^{\lfloor \frac{1}{2}(e+1) \rfloor} n_{2t-1} + \left\lfloor \tfrac{1}{2}(e - a - 1) \right\rfloor\right).$$

This expression can be simplified to result in (3.13). □

It is clear that, in the same way, one could also derive a result for $w(n)$ modulo 16, 32, …, albeit at the cost of considerably more work.

References

1. C. Bessenrodt, J. Olsson, J.A. Sellers, Unique path partitions: characterization and congruences. Ann. Comb. **13**, 591–602 (2013)
2. C. Krattenthaler, T.W. Müller, A method for determining the mod-3^k behaviour of recursive sequences, preprint, 83pp, arxiv:1308.2856
3. J. Olsson, Sign conjugacy classes in symmetric groups. J. Algebr. **322**, 2793–2800 (2009)

Complex Form of Classical and Quantum Electrodynamics

Sergey I. Kryuchkov, Nathan A. Lanfear and Sergei K. Suslov

Physical laws should have mathematical beauty.

P.A.M. Dirac

Dedicated to Krishna Alladi on the occasion of his 60th birthday and to the memory of his late father Professor Alladi Ramakrishnan

Abstract We consider a complex covariant form of the macroscopic Maxwell equations, in a moving medium or at rest, following the original ideas of Minkowski. A compact, Lorentz invariant, derivation of the energy-momentum tensor and the corresponding differential balance equations are given. Conservation laws and quantization of the electromagnetic field will be discussed in this covariant approach elsewhere.

Keywords Macroscopic Maxwell's equations · Complex electromagnetic fields Energy-momentum balance equations · Cherenkov radiation

2010 Mathematics Subject Classification 35Q60 · 35Q61

S. I. Kryuchkov (✉) · N. A. Lanfear · S. K. Suslov
School of Mathematical and Statistical Sciences, Arizona State University,
Tempe, AZ 85287–1804, USA
e-mail: sergeykryuchkov@yahoo.com

N. A. Lanfear
e-mail: nlanfear@asu.edu

S. K. Suslov
e-mail: sks@asu.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_24

1 Introduction

Although a systematic study of electromagnetic phenomena in media is not possible without methods of quantum mechanics, statistical physics and kinetics, in practice a standard mathematical model based on phenomenological Maxwell's equations provides a good approximation to many important problems. As is well known, one should be able to obtain the electromagnetic laws for continuous media from those for the interaction of fields and point particles [18], [34], [42], [51], [57], [66], [91]. As a result of the hard work of several generations of researchers and engineers, the classical electrodynamics, especially in its current complex covariant form, undoubtedly satisfies Dirac's criteria of mathematical beauty[1], being a state of the art mathematical description of nature.

In macroscopic electrodynamics, the volume (mechanical or ponderomotive) forces, acting on a medium, and the corresponding energy density and energy flux are introduced with the help of the energy-momentum tensors and differential balance relations [24], [31], [51], [72], [86], [91]. These forces occur in the equations of motion for a medium or individual charges and, in principle, they can be experimentally tested [32], [69], [74], [92] (see also the references therein). But interpretation of the results should depend on the accepted model of the interaction between the matter and radiation.

In this methodological note, we discuss a complex version of Minkowski's phenomenological electrodynamics (at rest or in a moving medium) without assuming any particular form of material equations as far as possible. Lorentz invariance of the corresponding differential balance equations is emphasized in view of long-standing uncertainties about the electromagnetic stresses and momentum density, the so-called Abraham–Minkowski controversy (see, for example, [5], [15], [19], [22], [24], [30], [31], [32], [34], [36], [51], [62], [63], [67], [68], [69], [72], [73], [74], [78], [80], [85], [89], [92], [93], [94], [95] and the references therein).

The paper is organized as follows. In sections 2 to 4, we describe the $3D$-complex version of Maxwell's equations and derive the corresponding differential balance density laws for the electromagnetic fields. Their covariant versions are given in sections 5 to 9. The case of a uniformly moving medium is discussed in section 10 and complex Lagrangians are introduced in section 11. Some useful tools are collected in appendices A to C for the reader's benefit.

2 Maxwell's Equations in $3D$-Complex Form

Traditionally, the macroscopic Maxwell equations in a fixed frame of reference are given by

[1]During a seminar at Moscow State University on October 3, 1956, when asked to summarize his philosophy of physics, Dirac wrote the above-cited sentence on the blackboard in capital letters [25], [38], [88].

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \text{ (Faraday)}, \qquad \text{div } \mathbf{B} = 0 \text{ (no magnetic charge)} \tag{2.1}$$

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_{\text{free}} \text{ (Biot\&Savart)}, \quad \text{div } \mathbf{D} = 4\pi\rho_{\text{free}} \text{ (Coulomb)}. \tag{2.2}$$

Here, $\mathbf{E}$ is the electric field,[2] $\mathbf{D}$ is the displacement field; $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the induction field. These equations, which are obtained by averaging of microscopic Maxwell's equations in the vacuum, provide a good mathematical description of electromagnetic phenomena in various media, when complemented by the corresponding material equations. In the simplest case of an isotropic medium at rest, one usually has

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{j} = \sigma\mathbf{E}, \tag{2.3}$$

where ε is the dielectric constant, μ is the magnetic permeability, and σ describes the conductivity of the medium (see, for example, [1], [6], [7], [15], [16], [18], [21], [23], [28], [34], [37], [51], [57], [70], [72], [82], [86], [88], [90], [91] for fundamentals of classical electrodynamics).

Introduction of two complex fields

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}, \qquad \mathbf{G} = \mathbf{D} + i\mathbf{B} \tag{2.4}$$

allows one to rewrite the phenomenological Maxwell equations in the following compact form

$$\frac{i}{c}\left(\frac{\partial \mathbf{G}}{\partial t} + 4\pi\mathbf{j}\right) = \text{curl } \mathbf{F}, \qquad \mathbf{j} = \mathbf{j}^*, \tag{2.5}$$

$$\text{div } \mathbf{G} = 4\pi\rho, \qquad \rho = \rho^*, \tag{2.6}$$

where the asterisk stands for complex conjugation (see also [6], [47] and [79]). As we shall demonstrate, different complex forms of Maxwell's equations are particularly convenient for study of the corresponding "energy-momentum" balance equations for the electromagnetic fields in the presence of the "free" charges and currents in a medium.

3 Hertz Symmetric Stress Tensor

We begin from a complex $3D$-interpretation of the traditional symmetric energy-momentum tensor [72]. By definition,

---

[2] From this point, we shall write $\rho_{\text{free}} = \rho$ and $\mathbf{j}_{\text{free}} = \mathbf{j}$. A detailed analysis of electromagnetic laws for continuous media from those for point particles is given in [34] (statistical description of material media).

$$T_{pq} = \frac{1}{16\pi}\left[F_p G_q^* + F_p^* G_q + F_q G_p^* + F_q^* G_p \right. \\ \left. - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^* + \mathbf{F}^*\cdot\mathbf{G}\right)\right] = T_{qp} \qquad (p, q = 1, 2, 3) \tag{3.1}$$

and the corresponding "momentum" balance equation,

$$\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p + \frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B}\right)\right]_p \\ = \frac{\partial T_{pq}}{\partial x_q} + \frac{1}{16\pi}\left[\text{curl}\left(\mathbf{F}\times\mathbf{G}^* + \mathbf{F}^*\times\mathbf{G}\right)\right]_p \\ + \frac{1}{16\pi}\left(F_q\frac{\partial G_q^*}{\partial x_p} - G_q\frac{\partial F_q^*}{\partial x_p} + F_q^*\frac{\partial G_q}{\partial x_p} - G_q^*\frac{\partial F_q}{\partial x_p}\right), \tag{3.2}$$

can be obtained from Maxwell's equations (2.5)–(2.6) as a result of elementary but rather tedious vector calculus calculations usually omitted in textbooks. (We use Einstein summation convention over any two repeated indices unless otherwise stated. In this paper, Greek indices run from 0 to 3, while Latin indices may have values from 1 to 3 inclusive.)

Proof. Indeed, in a $3D$-complex form,

$$\frac{\partial}{\partial x_q}\left(F_p G_q^* + F_q G_p^* - \delta_{pq}\mathbf{F}\cdot\mathbf{G}^*\right) \\ = \frac{\partial F_p}{\partial x_q}G_q^* + F_p\frac{\partial G_q^*}{\partial x_q} + \frac{\partial F_q}{\partial x_q}G_p^* + F_q\frac{\partial G_p^*}{\partial x_q} - \frac{\partial}{\partial x_p}\left(F_q G_q^*\right) \\ = F_q\left(\frac{\partial G_p^*}{\partial x_q} - \frac{\partial G_q^*}{\partial x_p}\right) + \left(\frac{\partial F_p}{\partial x_q} - \frac{\partial F_q}{\partial x_p}\right)G_q^* \\ + F_p \,\text{div}\, \mathbf{G}^* + G_p^* \,\text{div}\, \mathbf{F} \\ = F_p \,\text{div}\, \mathbf{G}^* - \left(\mathbf{F}\times\text{curl}\,\mathbf{G}^*\right)_p + G_p^* \,\text{div}\, \mathbf{F} - \left(\mathbf{G}^*\times\text{curl}\,\mathbf{F}\right)_p \tag{3.3}$$

due to an identity [86]:

$$(\mathbf{A}\times\text{curl}\,\mathbf{B})_p = A_q\left(\frac{\partial B_q}{\partial x_p} - \frac{\partial B_p}{\partial x_q}\right). \tag{3.4}$$

Taking into account the complex conjugate, we derive

$$\frac{1}{2}\frac{\partial}{\partial x_q}\left[F_p G_q^* + F_p^* G_q + F_q G_p^* + F_q^* G_p - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^* + \mathbf{F}^*\cdot\mathbf{G}\right)\right] \\ = \frac{1}{2}\left(\mathbf{F}\,\text{div}\,\mathbf{G}^* - \mathbf{G}^*\times\text{curl}\,\mathbf{F} + \mathbf{F}^*\,\text{div}\,\mathbf{G} - \mathbf{G}\times\text{curl}\,\mathbf{F}^*\right)_p \\ + \frac{1}{2}\left(\mathbf{G}\,\text{div}\,\mathbf{F}^* - \mathbf{F}^*\times\text{curl}\,\mathbf{G} + \mathbf{G}^*\,\text{div}\,\mathbf{F} - \mathbf{F}\times\text{curl}\,\mathbf{G}^*\right)_p \tag{3.5}$$

as our first important fact.

On the other hand, in view of Maxwell's equations (2.5)–(2.6), one gets

$$\begin{aligned}&\mathbf{F}\ \mathrm{div}\ \mathbf{G}^* - \mathbf{G}^* \times \mathrm{curl}\ \mathbf{F} \\ &\quad = 4\pi\rho\mathbf{F} + \frac{i}{c}\left(\frac{\partial \mathbf{G}}{\partial t} \times \mathbf{G}^* + 4\pi \mathbf{j} \times \mathbf{G}^*\right)\end{aligned} \tag{3.6}$$

and, with the help of its complex conjugate,

$$\begin{aligned}&\mathbf{F}\ \mathrm{div}\ \mathbf{G}^* - \mathbf{G}^* \times \mathrm{curl}\ \mathbf{F} + \mathbf{F}^*\ \mathrm{div}\ \mathbf{G} - \mathbf{G} \times \mathrm{curl}\ \mathbf{F}^* \\ &\quad = 4\pi\rho\left(\mathbf{F} + \mathbf{F}^*\right) + \frac{i}{c}\frac{\partial}{\partial t}\left(\mathbf{G} \times \mathbf{G}^*\right) + \frac{4\pi i}{c}\mathbf{j} \times \left(\mathbf{G}^* - \mathbf{G}\right),\end{aligned} \tag{3.7}$$

or

$$\begin{aligned}&\frac{1}{2}\left(\mathbf{F}\ \mathrm{div}\ \mathbf{G}^* - \mathbf{G}^* \times \mathrm{curl}\ \mathbf{F} + \mathbf{F}^*\ \mathrm{div}\ \mathbf{G} - \mathbf{G} \times \mathrm{curl}\ \mathbf{F}^*\right) \\ &\quad = 4\pi\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B}\right),\end{aligned} \tag{3.8}$$

providing the second important fact. (Up to the constant, the first term on the right-hand side represents the density of Lorentz's force acting on the "free" charges and currents in the medium under consideration [85], [86].)

In view of (3.8) and (3.5), we can write

$$\begin{aligned}&4\pi\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}\right)_p + \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B}\right)_p \\ &= \frac{1}{2}\frac{\partial}{\partial x_q}\left[F_p G_q^* + F_p^* G_q + F_q G_p^* + F_q^* G_p - \delta_{pq}\left(\mathbf{F} \cdot \mathbf{G}^* + \mathbf{F}^* \cdot \mathbf{G}\right)\right] \\ &\quad - \frac{1}{2}\left(\mathbf{G}\ \mathrm{div}\ \mathbf{F}^* - \mathbf{F}^* \times \mathrm{curl}\ \mathbf{G} + \mathbf{G}^*\ \mathrm{div}\ \mathbf{F} - \mathbf{F} \times \mathrm{curl}\ \mathbf{G}^*\right)_p \\ &= \frac{1}{4}\frac{\partial}{\partial x_q}\left[F_p G_q^* + F_p^* G_q + F_q G_p^* + F_q^* G_p - \delta_{pq}\left(\mathbf{F} \cdot \mathbf{G}^* + \mathbf{F}^* \cdot \mathbf{G}\right)\right] \\ &\quad - \frac{1}{4}\left(\mathbf{G}\ \mathrm{div}\ \mathbf{F}^* - \mathbf{F}^* \times \mathrm{curl}\ \mathbf{G} + \mathbf{G}^*\ \mathrm{div}\ \mathbf{F} - \mathbf{F} \times \mathrm{curl}\ \mathbf{G}^*\right)_p \\ &\quad + \frac{1}{4}\left(\mathbf{F}\ \mathrm{div}\ \mathbf{G}^* - \mathbf{G}^* \times \mathrm{curl}\ \mathbf{F} + \mathbf{F}^*\ \mathrm{div}\ \mathbf{G} - \mathbf{G} \times \mathrm{curl}\ \mathbf{F}^*\right)_p \\ &= 4\pi\frac{\partial T_{pq}}{\partial x_q} + \frac{1}{4}\left(\mathbf{F}\ \mathrm{div}\ \mathbf{G}^* - \mathbf{G}^*\ \mathrm{div}\ \mathbf{F} + \mathbf{F}^*\ \mathrm{div}\ \mathbf{G} - \mathbf{G}\ \mathrm{div}\ \mathbf{F}^*\right)_p \\ &\quad + \frac{1}{4}\left(\mathbf{F} \times \mathrm{curl}\ \mathbf{G}^* - \mathbf{G}^* \times \mathrm{curl}\ \mathbf{F} + \mathbf{F}^* \times \mathrm{curl}\ \mathbf{G} - \mathbf{G} \times \mathrm{curl}\ \mathbf{F}^*\right)_p.\end{aligned} \tag{3.9}$$

Finally, in the last two lines, one can utilize the following differential vector calculus identity,

$$[\mathbf{A}\ \text{div}\ \mathbf{B} - \mathbf{B}\ \text{div}\ \mathbf{A} + \mathbf{A} \times \text{curl}\ \mathbf{B} - \mathbf{B} \times \text{curl}\ \mathbf{A} - \text{curl}\ (\mathbf{A} \times \mathbf{B})]_p \\ = A_q \frac{\partial B_q}{\partial x_p} - B_q \frac{\partial A_q}{\partial x_p}, \tag{3.10}$$

see (A.5), with $\mathbf{A} = \mathbf{F}$, $\mathbf{B} = \mathbf{G}^*$ and its complex conjugates, in order to obtain (3.2) and/or (3.16), which completes the proof. (An independent proof will be given in section 7.)

Derivation of the corresponding differential "energy" balance equation is much simpler. By (2.5),

$$\mathbf{F} \cdot \frac{\partial \mathbf{G}^*}{\partial t} + \mathbf{F}^* \cdot \frac{\partial \mathbf{G}}{\partial t} + 4\pi \mathbf{j} \cdot \left(\mathbf{F} + \mathbf{F}^*\right) = \frac{c}{i}\ \text{div}\ \left(\mathbf{F} \times \mathbf{F}^*\right) \tag{3.11}$$

due to a familiar vector calculus identity (A.1):

$$\text{div}\ (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl}\ \mathbf{A} - \mathbf{A} \cdot \text{curl}\ \mathbf{B}. \tag{3.12}$$

In a traditional form,

$$\frac{1}{4\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right) + \mathbf{j} \cdot \mathbf{E} + \text{div}\ \left(\frac{c}{4\pi}\mathbf{E} \times \mathbf{H}\right) = 0 \tag{3.13}$$

(see, for example, [18], [86]), where one can substitute

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}\right) \\ + \frac{1}{2}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t}\right). \tag{3.14}$$

As a result, $3D$-differential "energy-momentum" balance equations are given by

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}}{8\pi}\right) + \text{div}\ \left(\frac{c}{4\pi}\mathbf{E} \times \mathbf{H}\right) + \mathbf{j} \cdot \mathbf{E} \\ + \frac{1}{8\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t}\right) = 0 \tag{3.15}$$

and

$$
-\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B}\right)\right]_p+\frac{\partial T_{pq}}{\partial x_q}-\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p \tag{3.16}
$$
$$
+\frac{1}{8\pi}\left[\text{curl }\left(\mathbf{E}\times\mathbf{D}+\mathbf{H}\times\mathbf{B}\right)\right]_p
$$
$$
+\frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x_p}-\mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x_p}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x_p}-\mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x_p}\right)=0,
$$

respectively (see also [32], [62]). The real form of the symmetric stress tensor (3.1), namely,

$$
T_{pq}=\frac{1}{8\pi}\left[E_pD_q+E_qD_p+H_pB_q+H_qB_p -\delta_{pq}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)\right] \qquad (p,q=1,2,3)\,, \tag{3.17}
$$

is due to Hertz [72].

Equations (3.15)–(3.16) are related to a fundamental concept of conservation of mechanical and electromagnetic energy and momentum. Here, these balance conditions are presented in differential forms in terms of the corresponding local field densities. They can be integrated over a given volume in $\mathbb{R}^3$ in order to obtain, in a traditional way, the corresponding conservation laws of the electromagnetic fields (see, for example, [50], [51], [88], [90], [91]). These laws made it necessary to ascribe a definite linear momentum and energy to the field of an electromagnetic wave, which can be observed, for example, as light pressure.

Note. At this point, the Lorentz invariance of these differential balance equations is not obvious in our $3D$-analysis. But one can introduce the four-vector $x^\mu=(ct,\mathbf{r})$ and try to match (3.15)–(3.16) with the expression,

$$
\frac{\partial}{\partial x^\nu}T_\mu{}^\nu=\frac{\partial T_\mu{}^0}{\partial x_0}+\frac{\partial T_\mu{}^q}{\partial x_q} \qquad (\mu,\nu=0,1,2,3;\quad p,q=1,2,3)\,, \tag{3.18}
$$

as an initial step, in order to guess the corresponding four-tensor form. An independent covariant derivation will be given in section 7.

Note. In an isotropic non-homogeneous variable medium (without dispersion and/or compression), when $\mathbf{D}=\varepsilon\left(\mathbf{r},t\right)\mathbf{E}$ and $\mathbf{B}=\mu\left(\mathbf{r},t\right)\mathbf{H}$, the "ponderomotive forces" in (3.15) and (3.16) take the form [86]:

$$
\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x^\nu}-\mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x^\nu}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x^\nu}-\mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x^\nu} \tag{3.19}
$$
$$
=\frac{\partial\varepsilon}{\partial x^\nu}\mathbf{E}^2+\frac{\partial\mu}{\partial x^\nu}\mathbf{H}^2=\begin{pmatrix}\dfrac{1}{c}\left(\dfrac{\partial\varepsilon}{\partial t}\mathbf{E}^2+\dfrac{\partial\mu}{\partial t}\mathbf{H}^2\right)\\[2ex] \mathbf{E}^2\nabla\varepsilon+\mathbf{H}^2\nabla\mu\end{pmatrix},
$$

which may be interpreted as a four-vector "energy-force" acting from an inhomogeneous and time-variable medium. Its covariance is analyzed in section 7.

4 "Angular Momentum" Balance

The $3D$-"linear momentum" differential balance equation (3.16), can be rewritten in a more compact form,

$$\frac{\partial T_{pq}}{\partial x_q} = \mathscr{F}_p + \frac{\partial \mathscr{G}_p}{\partial t}, \qquad \overrightarrow{\mathscr{G}} = \frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B}\right), \tag{4.1}$$

with the help of the Hertz symmetric stress tensor $T_{pq} = T_{qp}$ defined by (3.17). A "net force" is given by

$$\begin{aligned}\mathscr{F}_p = {} & \left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p - \frac{1}{8\pi}\left[\text{curl }\left(\mathbf{E}\times\mathbf{D} + \mathbf{H}\times\mathbf{B}\right)\right]_p \\ & - \frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x_p} - \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x_p} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x_p} - \mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x_p}\right).\end{aligned} \tag{4.2}$$

In this notation, we state the $3D$-"angular momentum" differential balance equation as follows

$$\frac{\partial M_{pq}}{\partial x_q} = \mathscr{T}_p + \frac{\partial \mathscr{L}_p}{\partial t}, \qquad \overrightarrow{\mathscr{L}} = \mathbf{r}\times\overrightarrow{\mathscr{G}}, \quad \overrightarrow{\mathscr{T}} = \mathbf{r}\times\overrightarrow{\mathscr{F}}, \tag{4.3}$$

where the "field angular momentum density" is defined by

$$\overrightarrow{\mathscr{L}} = \frac{1}{4\pi c}\mathbf{r}\times\left(\mathbf{D}\times\mathbf{B}\right) \tag{4.4}$$

and the "flux of angular momentum" is described by the following tensor [37]:

$$M_{pq} = e_{prs}x_r T_{sq}. \tag{4.5}$$

(Here, e_{pqr} is the totally anti-symmetric Levi-Civita symbol with $e_{123} = +1$). An elementary example of conservation of the total angular momentum is discussed in [86].

Proof. Indeed, in view of (4.1), one can write

$$\begin{aligned}\frac{\partial M_{pq}}{\partial x_q} &= e_{prs}T_{sr} + e_{prs}x_r\frac{\partial T_{sq}}{\partial x_q} \\ &= e_{pqr}x_q\mathscr{F}_r + \frac{\partial}{\partial t}\left(e_{pqr}x_q\mathscr{G}_r\right),\end{aligned} \tag{4.6}$$

which completes the proof.

Note. Once again, in $3D$-form, the Lorentz invariance of this differential balance equation for the local densities is not obvious. An independent covariant derivation will be given in section 8.

5 Complex Covariant Form of Macroscopic Maxwell's Equations

With the help of complex fields $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and $\mathbf{G} = \mathbf{D} + i\mathbf{B}$, we introduce the following anti-symmetric four-tensor,

$$Q^{\mu\nu} = -Q^{\nu\mu} = \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} \tag{5.1}$$

and use the standard four-vectors, $x^\mu = (ct, \mathbf{r})$ and $j^\mu = (c\rho, \mathbf{j})$ for contravariant coordinates and current, respectively.

Maxwell's equations then take the covariant form [47], [54]:

$$\frac{\partial}{\partial x^\nu} Q^{\mu\nu} = -\frac{\partial}{\partial x^\nu} Q^{\nu\mu} = -\frac{4\pi}{c} j^\mu \tag{5.2}$$

with summation over two repeated indices. Indeed, in block form, we have

$$\frac{\partial Q^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \begin{pmatrix} 0 & -G_q \\ G_p & ie_{pqr}F_r \end{pmatrix} = \begin{pmatrix} -\operatorname{div} \mathbf{G} = -4\pi\rho \\ \dfrac{1}{c}\dfrac{\partial \mathbf{G}}{\partial t} + i \operatorname{curl} \mathbf{F} = -\dfrac{4\pi}{c}\mathbf{j} \end{pmatrix}, \tag{5.3}$$

which verifies this fact. The continuity equation,

$$0 \equiv \frac{\partial^2 Q^{\mu\nu}}{\partial x^\mu \partial x^\nu} = -\frac{4\pi}{c}\frac{\partial j^\mu}{\partial x^\mu}, \tag{5.4}$$

or in the $3D$-form,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \tag{5.5}$$

describes conservation of the electrical charge. The latter equation can also be derived in the complex $3D$-form from (2.5)–(2.6).

Note. In vacuum, when $\mathbf{G} = \mathbf{F}$ and $\rho = 0$, $\mathbf{j} = 0$, one can write due to (B.5)–(B.6):

$$Q^{\mu\nu} = F^{\mu\nu} - \frac{i}{2} e^{\mu\nu\sigma\tau} F_{\sigma\tau}, \qquad F^{\mu\nu} = g^{\mu\sigma} g^{\nu\tau} F_{\sigma\tau}, \qquad g_{\mu\sigma} g_{\nu\tau} Q^{\sigma\tau} = Q_{\mu\nu}. \quad (5.6)$$

As a result, the following self-duality property holds

$$e_{\mu\nu\sigma\tau} Q^{\sigma\tau} = 2i Q_{\mu\nu}, \qquad 2i Q^{\mu\nu} = e^{\mu\nu\sigma\tau} Q_{\sigma\tau} \quad (5.7)$$

(see, for example, [8], [48] and appendix B). Two covariant forms of Maxwell's equations are given by

$$\partial_\nu Q^{\mu\nu} = 0, \qquad \partial^\nu Q_{\mu\nu} = 0, \quad (5.8)$$

where $\partial^\nu = g^{\nu\mu} \partial_\mu$, $\partial_\mu = \partial / \partial x^\mu$ and $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The last equation can be derived from a more general equation, involving a rank three tensor,

$$g^{\alpha\alpha} e_{\alpha\mu\nu\tau} \partial^\nu Q^{\tau\beta} - g^{\beta\beta} e_{\beta\mu\nu\tau} \partial^\nu Q^{\tau\alpha} = -i \partial_\mu Q^{\alpha\beta} \quad (5.9)$$

($\alpha, \beta = 0, 1, 2, 3$ are fixed; no summation is assumed over these two indices), which is related to the Pauli–Lubański vector from the representation theory of the Poincaré group [47]. Different spinor forms of Maxwell's equations are analyzed in [48] (see also the references therein).

6 Dual Electromagnetic Field Tensors

Two dual anti-symmetric field tensors of complex fields, $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and $\mathbf{G} = \mathbf{D} + i\mathbf{B}$, are given by

$$Q^{\mu\nu} = \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} = R^{\mu\nu} + iS^{\mu\nu} \quad (6.1)$$
$$= \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & -H_3 & H_2 \\ D_2 & H_3 & 0 & -H_1 \\ D_3 & -H_2 & H_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

and

$$P_{\mu\nu} = \begin{pmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & iG_3 & -iG_2 \\ -F_2 & -iG_3 & 0 & iG_1 \\ -F_3 & iG_2 & -iG_1 & 0 \end{pmatrix} = F_{\mu\nu} + iG_{\mu\nu} \quad (6.2)$$

$$= \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & D_3 & -D_2 \\ -H_2 & -D_3 & 0 & D_1 \\ -H_3 & D_2 & -D_1 & 0 \end{pmatrix}.$$

The real part of the latter represents the standard electromagnetic field tensor in a medium [6], [72], [91]. As for the imaginary part of (6.1), which, ironically, Pauli called an "artificiality" in view of its non-standard behavior under spatial inversion [72], the use of complex conjugation restores this important symmetry for our complex field tensors.

The dual tensor identities are given by

$$e_{\mu\nu\sigma\tau} Q^{\sigma\tau} = 2i P_{\mu\nu}, \qquad 2i Q^{\mu\nu} = e^{\mu\nu\sigma\tau} P_{\sigma\tau}. \tag{6.3}$$

Here $e^{\mu\nu\sigma\tau} = -e_{\mu\nu\sigma\tau}$ and $e_{0123} = +1$ is the Levi-Civita four-symbol [27]. Then

$$6i \frac{\partial Q^{\mu\nu}}{\partial x^\nu} = e^{\mu\nu\lambda\sigma} \left(\frac{\partial P_{\lambda\sigma}}{\partial x^\nu} + \frac{\partial P_{\nu\lambda}}{\partial x^\sigma} + \frac{\partial P_{\sigma\nu}}{\partial x^\lambda} \right) \tag{6.4}$$

and both pairs of Maxwell's equations can also be presented in the form [47]

$$\frac{\partial P_{\mu\nu}}{\partial x^\lambda} + \frac{\partial P_{\nu\lambda}}{\partial x^\mu} + \frac{\partial P_{\lambda\mu}}{\partial x^\nu} = -\frac{4\pi i}{c} e_{\mu\nu\lambda\sigma} j^\sigma \tag{6.5}$$

in addition to the one given above

$$\frac{\partial Q^{\mu\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\mu. \tag{6.6}$$

The real part of the first equation traditionally represents the first (homogeneous) pair of Maxwell's equation and the real part of the second one gives the remaining pair. In our approach, both pairs of Maxwell's equations appear together (see also [6], [8], [9], [54], and [87] for the case in vacuum). Moreover, a generalization to complex-valued four-current may naturally represent magnetic charge and magnetic current not yet observed in nature [79].

An important cofactor matrix identity,

$$P_{\mu\nu} Q^{\nu\lambda} = (\mathbf{F} \cdot \mathbf{G})\, \delta_\mu^\lambda = \frac{1}{4} (P_{\sigma\tau} Q^{\tau\sigma})\, \delta_\mu^\lambda, \tag{6.7}$$

was originally established, in a general form, by Minkowski [65]. Once again, the dual tensors are given by

$$P_{\mu\nu} = \begin{pmatrix} 0 & F_q \\ -F_p & ie_{pqr}G_r \end{pmatrix}, \quad Q^{\mu\nu} = \begin{pmatrix} 0 & -G_q \\ G_p & ie_{pqr}F_r \end{pmatrix}, \tag{6.8}$$

in block form. A complete list of relevant tensor and matrix identities is given in appendix B.

7 Covariant Derivation of Energy-Momentum Balance Equations

7.1 Preliminaries

As has been announced in [47] (see also [48]), the covariant form of the differential balance equations can be presented as follows[3]

$$\frac{\partial}{\partial x^{\nu}} \left[\frac{1}{16\pi} \left(P^{*}_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda} \overset{*}{Q}^{\lambda\nu} \right) \right] \tag{7.1}$$
$$+ \frac{1}{32\pi} \left(P^{*}_{\sigma\tau} \frac{\partial Q^{\tau\sigma}}{\partial x^{\mu}} + P_{\sigma\tau} \frac{\partial \overset{*}{Q}^{\tau\sigma}}{\partial x^{\mu}} \right) = -\frac{1}{c} F_{\mu\lambda} j^{\lambda} = \begin{pmatrix} -\mathbf{j}\cdot\mathbf{E}/c \\ \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}/c \end{pmatrix}.$$

In our complex form, when $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and $\mathbf{G} = \mathbf{D} + i\mathbf{B}$, the energy-momentum tensor is given by

$$16\pi T_{\mu}{}^{\nu} = P^{*}_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda} \overset{*}{Q}^{\lambda\nu} \tag{7.2}$$
$$= \begin{pmatrix} \mathbf{F}\cdot\mathbf{G}^{*} + \mathbf{F}^{*}\cdot\mathbf{G} & 2i\left(\mathbf{F}\times\mathbf{F}^{*}\right)_q \\ -2i\left(\mathbf{G}\times\mathbf{G}^{*}\right)_p & 2\left(F_p G^{*}_q + F^{*}_p G_q\right) - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^{*} + \mathbf{F}^{*}\cdot\mathbf{G}\right) \end{pmatrix}.$$

Here, we point out for the reader's convenience that

$$i\left(\mathbf{F}\times\mathbf{F}^{*}\right) = 2\left(\mathbf{E}\times\mathbf{H}\right), \qquad i\left(\mathbf{G}\times\mathbf{G}^{*}\right) = 2\left(\mathbf{D}\times\mathbf{B}\right), \tag{7.3}$$
$$\mathbf{F}\cdot\mathbf{G}^{*} + \mathbf{F}^{*}\cdot\mathbf{G} = 2\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$$

[3] From now on we abbreviate $\left(Q^{\lambda\nu}\right)^{*} = \overset{*}{Q}^{\lambda\nu}$.

and, in real form,

$$4\pi T_{\mu}{}^{\nu} = \begin{pmatrix} (\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})/2 & (\mathbf{E}\times\mathbf{H})_q \\ -(\mathbf{D}\times\mathbf{B})_p & E_p D_q + H_p B_q - \delta_{pq}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})/2 \end{pmatrix}. \tag{7.4}$$

The covariant form of the differential balance equation allows one to clarify the physical meaning of different energy-momentum tensors. For instance, it is worth noting that the non-symmetric Maxwell and Heaviside form of the $3D$-stress tensor [72],

$$\widetilde{T}_{pq} = \frac{1}{4\pi}\left(E_p D_q + H_p B_q\right) - \frac{1}{8\pi}\delta_{pq}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right), \tag{7.5}$$

appears here in the corresponding "momentum" balance equation [86]:

$$\begin{aligned} &-\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}(\mathbf{D}\times\mathbf{B})\right]_p + \frac{\partial \widetilde{T}_{pq}}{\partial x_q} - \left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p \\ &+ \frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x_p} - \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x_p} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x_p} - \mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x_p}\right) = 0. \end{aligned} \tag{7.6}$$

At the same time, in view of (3.16), use of the form (7.5) differs from Hertz's symmetric tensors in (3.1) and (3.17) only in the case of anisotropic media (crystals) [72], [85].

Indeed,

$$8\pi\frac{\partial}{\partial x_q}\left(\widetilde{T}_{pq} - T_{pq}\right) = [\operatorname{curl}\,(\mathbf{E}\times\mathbf{D}+\mathbf{H}\times\mathbf{B})]_p\,. \tag{7.7}$$

Moreover, with the help of elementary identities,

$$[\operatorname{curl}\,(\mathbf{A}\times\mathbf{B})]_p = \frac{\partial}{\partial x_q}\left(A_p B_q - A_q B_p\right) \tag{7.8}$$

and

$$2\frac{\partial}{\partial x_q}\left(A_p B_q\right) = \frac{\partial}{\partial x_q}\left(A_p B_q + A_q B_p\right) + [\operatorname{curl}\,(\mathbf{A}\times\mathbf{B})]_p\,, \tag{7.9}$$

one can transform the latter balance equation into its "symmetric" form, which provides an independent proof of (3.16).

(For further discussion of symmetric and non-symmetric forms of the energy-momentum and stress tensors, the interested reader is referred to the classical accounts [32], [62], [72], [85]; see also the references therein.)

7.2 Proof

The fact that Maxwell's equations can be united with the help of a complex second rank (anti-symmetric) tensor allows us to utilize the standard Sturm–Liouville type argument in order to establish the energy-momentum differential balance equations in covariant form. Indeed, by adding matrix equation

$$P^*_{\mu\lambda}\left(\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda\right) \tag{7.10}$$

and its complex conjugate

$$P_{\mu\lambda}\left(\frac{\partial \overset{*}{Q}{}^{\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda\right) \tag{7.11}$$

one gets

$$P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} + P_{\mu\lambda}\frac{\partial \overset{*}{Q}{}^{\lambda\nu}}{\partial x^\nu} = -\frac{8\pi}{c} F_{\mu\lambda} j^\lambda. \tag{7.12}$$

A simple decomposition,

$$f\frac{\partial g}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(fg) + \frac{1}{2}\left(f\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} g\right) \tag{7.13}$$

with $f = P^*_{\mu\lambda}$ and $g = Q^{\lambda\nu}$ (and their complex conjugates), results in

$$\begin{aligned} &\frac{\partial}{\partial x^\nu}\left[\frac{1}{16\pi}\left(P^*_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu}\right)\right] \\ &+ \frac{1}{16\pi}\left[\left(P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} - \frac{\partial P_{\mu\lambda}}{\partial x^\nu}\overset{*}{Q}{}^{\lambda\nu}\right) + (\text{c.c.})\right] = -\frac{1}{c} F_{\mu\lambda} j^\lambda. \end{aligned} \tag{7.14}$$

By a direct substitution, one can verify that

$$\begin{aligned} Z_\mu = P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} - \frac{\partial P_{\mu\lambda}}{\partial x^\nu}\overset{*}{Q}{}^{\lambda\nu} &= \frac{1}{2} P^*_{\sigma\tau}\frac{\partial Q^{\tau\sigma}}{\partial x^\mu} \\ = -\frac{1}{2}\overset{*}{Q}{}^{\sigma\tau}\frac{\partial P_{\tau\sigma}}{\partial x^\mu} &= \mathbf{F}^* \cdot \frac{\partial \mathbf{G}}{\partial x^\mu} - \mathbf{G}^* \cdot \frac{\partial \mathbf{F}}{\partial x^\mu}. \end{aligned} \tag{7.15}$$

(An independent covariant proof of these identities is given in appendix C.) Finally, introducing

$$16\pi X_\mu = Z_\mu + Z^*_\mu, \tag{7.16}$$

we obtain (7.1) with the explicitly covariant expression for the ponderomotive force (3.19), which completes the proof.

As a result, the covariant energy-momentum balance equation is given by

$$\frac{\partial}{\partial x^{\nu}} T_{\mu}{}^{\nu} + X_{\mu} = -\frac{1}{c} F_{\mu\lambda} j^{\lambda}, \tag{7.17}$$

in a compact form. If these differential balance equations are written for a stationary medium, then the corresponding equations for moving bodies are uniquely determined, since the components of a tensor in any inertial coordinate system can be derived by a proper Lorentz transformation [72].

8 Covariant Derivation of Angular Momentum Balance

By definition, $x_{\mu} = g_{\mu\nu} x^{\nu} = (ct, -\mathbf{r})$ and $T_{\mu\lambda} = T_{\mu}{}^{\nu} g_{\nu\lambda}$, where $g_{\mu\nu} = \partial x_{\mu}/\partial x^{\nu} = \operatorname{diag}(1, -1, -1, -1)$. In view of (7.17), we derive

$$\begin{aligned} \frac{\partial}{\partial x^{\nu}} \left(x_{\lambda} T_{\mu}{}^{\nu} - x_{\mu} T_{\lambda}{}^{\nu} \right) = \left(T_{\mu\lambda} - T_{\lambda\mu} \right) \\ - \left(x_{\lambda} X_{\mu} - x_{\mu} X_{\lambda} \right) - \frac{1}{c} \left(x_{\lambda} F_{\mu\nu} - x_{\mu} F_{\lambda\nu} \right) j^{\nu} \end{aligned} \tag{8.1}$$

as a required differential balance equation.

With the help of familiar dual relations (B.4), one can get another covariant form of the angular momentum balance equation:

$$\begin{aligned} \frac{\partial}{\partial x^{\nu}} \left(e^{\mu\lambda\sigma\tau} x_{\sigma} T_{\tau}{}^{\nu} \right) + e^{\mu\lambda\sigma\tau} T_{\sigma\tau} \\ + e^{\mu\lambda\sigma\tau} x_{\sigma} X_{\tau} + \frac{1}{c} e^{\mu\lambda\sigma\tau} x_{\sigma} F_{\tau\nu} j^{\nu} = 0^{\mu\lambda}. \end{aligned} \tag{8.2}$$

In $3D$-form, the latter relation can be reduced to (4.3)–(4.5).

Indeed, when $\mu = 0$ and $\lambda = p = 1, 2, 3,$ one gets

$$\begin{aligned} -\frac{1}{4\pi c} \frac{\partial}{\partial t} \left[e_{pqr} x_q \left(\mathbf{D} \times \mathbf{B} \right)_r \right] + \frac{\partial}{\partial x_s} \left(e_{pqr} x_q \widetilde{T}_{rs} \right) \\ + e_{pqr} \widetilde{T}_{qr} + e_{pqr} x_q \left(X_r + Y_r \right) = 0, \end{aligned} \tag{8.3}$$

where $-\mathbf{Y} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B}/c$ is the familiar Lorentz force. Substitution, $\widetilde{T}_{rs} = T_{rs} + \left(\widetilde{T}_{rs} - T_{rs} \right)$, results in (4.3) in view of identity (7.7). The remaining cases, when $\mu, \nu = p, q = 1, 2, 3,$ can be analyzed in a similar fashion. In $3D$-form, the corresponding equations are equivalent to (3.15) and (7.6). Details are left to the reader.

Thus the angular momentum law has the form of a local balance equation, not a conservation law, since in general, the energy-momentum tensor will not be symmetric [34]. A torque, for instance, may occur, which cannot be compensated for by a change in the electromagnetic angular momentum, though not in contradiction with experiment [72].

9 Transformation Laws of Complex Electromagnetic Fields

Let **v** be a constant real-valued velocity vector representing uniform motion of one frame of reference with respect to another one. Let us consider the following orthogonal decompositions,

$$\mathbf{F} = \mathbf{F}_{\parallel} + \mathbf{F}_{\perp}, \qquad \mathbf{G} = \mathbf{G}_{\parallel} + \mathbf{G}_{\perp}, \tag{9.1}$$

such that our complex vectors $\left\{\mathbf{F}_{\parallel}, \mathbf{G}_{\parallel}\right\}$ are collinear with the velocity vector **v** and $\{\mathbf{F}_{\perp}, \mathbf{G}_{\perp}\}$ are perpendicular to it (Figure 1). The Lorentz transformation of electric and magnetic fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ take the following complex form

$$\mathbf{F}'_{\parallel} = \mathbf{F}_{\parallel}, \qquad \mathbf{G}'_{\parallel} = \mathbf{G}_{\parallel} \tag{9.2}$$

and

$$\mathbf{F}'_{\perp} = \frac{\mathbf{F}_{\perp} - \dfrac{i}{c}\left(\mathbf{v} \times \mathbf{G}\right)}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{G}'_{\perp} = \frac{\mathbf{G}_{\perp} - \dfrac{i}{c}\left(\mathbf{v} \times \mathbf{F}\right)}{\sqrt{1 - v^2/c^2}}. \tag{9.3}$$

(Although this transformation was found by Lorentz, it was Minkowski who realized that this is the law of transformation of the second rank anti-symmetric four-tensors [58], [65] ; a brief historical overview is given in [72].) This complex $3D$-form of the Lorentz transformation of electric and magnetic fields was known to Minkowski (1908), but apparently only in vacuum, when $\mathbf{G} = \mathbf{F}$ (see also [88]). Moreover,

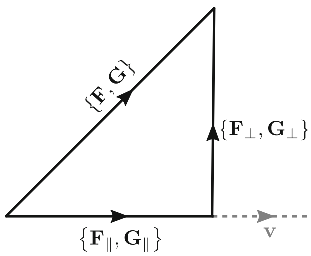


Fig. 1 Complex electromagnetic fields decomposition.

$$\mathbf{r}'_{\parallel} = \frac{\mathbf{r}_{\parallel} - \mathbf{v}t}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \qquad t' = \frac{t - (\mathbf{v} \cdot \mathbf{r})/c^2}{\sqrt{1 - v^2/c^2}}, \tag{9.4}$$

in the same notation [72].

The latter equations can be rewritten as follows

$$\mathbf{r}' = \mathbf{r} + \left[(\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{r}}{v^2} - \gamma t \right] \mathbf{v}, \qquad t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c^2} \right), \tag{9.5}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. In a similar fashion, one gets

$$\mathbf{F}' = \gamma \left(\mathbf{F} - \frac{i}{c} \mathbf{v} \times \mathbf{G} \right) - (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{F}}{v^2} \mathbf{v}, \tag{9.6}$$

$$\mathbf{G}' = \gamma \left(\mathbf{G} - \frac{i}{c} \mathbf{v} \times \mathbf{F} \right) - (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{G}}{v^2} \mathbf{v}, \tag{9.7}$$

as a compact $3D$-version of the Lorentz transformation for the complex electromagnetic fields.

In complex four-tensor form,

$$Q'^{\,\mu\nu}\left(x'\right) = \Lambda^{\mu}_{\ \sigma} \Lambda^{\nu}_{\ \tau} Q^{\sigma\tau}(x), \qquad x' = \Lambda x. \tag{9.8}$$

Although Minkowski considered the transformation of electric and magnetic fields in a complex $3D$-vector form, see Eqs. (8)–(9) and (15) in [65] (or Eqs. (25.5)–(25.6) in [50]), he seems never to have combined the corresponding four-tensors into the complex forms (6.1)–(6.2). In the second article [66], Max Born, who used Minkowski's notes, didn't mention the complex fields. As a result, the complex field tensor seems only to have appeared, for the first time, in [54] (see also [87]). The complex identity, $\mathbf{F} \cdot \mathbf{G} =$ invariant under a similarity transformation, follows from Minkowski's determinant relations (B.23)–(B.25).

Example. Let $\{\mathbf{e}_k\}_{k=1}^3$ be an orthonormal basis in $\mathbb{R}^3$. We choose $\mathbf{v} = v\mathbf{e}_1$ and write $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ with

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \tag{9.9}$$

for the corresponding Lorentz boost (Figure 2).

In view of (9.8), by matrix multiplication one gets

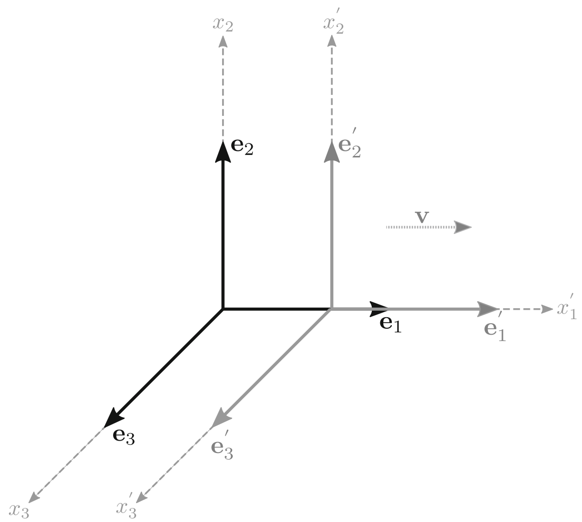


Fig. 2 Example of a moving frame.

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -G_1 & -\gamma G_2 - i\beta\gamma F_3 & -\gamma G_3 + i\beta\gamma F_2 \\ G_1 & 0 & \beta\gamma G_2 + i\gamma F_3 & \beta\gamma G_3 - i\gamma F_2 \\ \gamma G_2 + i\beta\gamma F_3 & -\beta\gamma G_2 - i\gamma F_3 & 0 & iF_1 \\ \gamma G_3 - i\beta\gamma F_2 & -\beta\gamma G_3 + i\gamma F_2 & -iF_1 & 0 \end{pmatrix}. \tag{9.10}$$

Thus $G_1' = G_1$ and

$$G_2' = \gamma G_2 + i\beta\gamma F_3 = \frac{G_2 + i\,(v/c)\,F_3}{\sqrt{1 - v^2/c^2}} = \frac{G_2 - \dfrac{i}{c}\,(\mathbf{v} \times \mathbf{F})_2}{\sqrt{1 - v^2/c^2}}, \tag{9.11}$$

$$G_3' = \gamma G_3 - i\beta\gamma F_2 = \frac{G_3 - i\,(v/c)\,F_2}{\sqrt{1 - v^2/c^2}} = \frac{G_3 - \dfrac{i}{c}\,(\mathbf{v} \times \mathbf{F})_3}{\sqrt{1 - v^2/c^2}}.$$

In a similar fashion, $F_1' = F_1$ and

$$F_2' = \gamma F_2 + i\beta\gamma G_3 = \frac{F_2 - \dfrac{i}{c}(\mathbf{v}\times\mathbf{G})_2}{\sqrt{1-v^2/c^2}}, \tag{9.12}$$
$$F_3' = \gamma F_3 - i\beta\gamma G_2 = \frac{F_3 - \dfrac{i}{c}(\mathbf{v}\times\mathbf{G})_3}{\sqrt{1-v^2/c^2}}.$$

The reader can easily verify that the latter relations are in agreement with the complex field transformations (9.2)–(9.3).

In block form, one gets

$$\begin{pmatrix} F_1' \\ F_2' \\ G_3' \\ G_2' \\ F_3' \\ G_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(i\theta) & \sin(i\theta) & 0 & 0 & 0 \\ 0 & -\sin(i\theta) & \cos(i\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(i\theta) & \sin(i\theta) & 0 \\ 0 & 0 & 0 & -\sin(i\theta) & \cos(i\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ G_3 \\ G_2 \\ F_3 \\ G_1 \end{pmatrix}, \tag{9.13}$$

where, by definition,

$$\cos(i\theta) = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \sin(i\theta) = i\beta\gamma = \frac{i\beta}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}. \tag{9.14}$$

As a result, the transformation law of the complex electromagnetic fields $\{\mathbf{F}, \mathbf{G}\}$ under the Lorentz boost can be thought of as a complex rotation in $\mathbb{C}^6$, corresponding to a reducible representation of the one-parameter subgroup of $SO(3, \mathbb{C})$. (Cyclic permutation of the spatial indices cover the two remaining cases; see also [88].)

10 Material Equations, Potentials, and Energy-Momentum Tensor for Moving Isotropic Media

Electromagnetic phenomena in moving media are important in relativistic astrophysics, the study of accelerated plasma clusters and high-energy electron beams [15], [16], [26], [91].

10.1 Material equations

Minkowski's field- and connecting-equations [65], [66] were derived from the corresponding laws for the bodies at rest by means of a Lorentz transformation (see [15], [18], [34], [51], [67], [72],[91]). Explicitly covariant forms, which are applicable both in the rest frame and for moving media, are analyzed in [15], [16], [34], [39], [40], [67], [71], [72], [75], [77], [88], [91] (see also the references therein). In standard notation,

$$\beta = v/c, \qquad \gamma = \left(1 - \beta^2\right)^{-1/2}, \qquad v = |\mathbf{v}|, \qquad \kappa = \varepsilon\mu - 1, \tag{10.1}$$

one can write [15], [16], [18], [91]:

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} + \frac{\kappa\gamma^2}{\mu}\left[\beta^2 \mathbf{E} - \frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\mathbf{E}) + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right], \\ \mathbf{H} &= \frac{1}{\mu}\mathbf{B} + \frac{\kappa\gamma^2}{\mu}\left[-\beta^2 \mathbf{B} + \frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\mathbf{B}) + \frac{1}{c}(\mathbf{v}\times\mathbf{E})\right]. \end{aligned} \tag{10.2}$$

In covariant form, these relations are given by

$$\begin{aligned} R^{\lambda\nu} &= \varepsilon^{\lambda\nu\sigma\tau} F_{\sigma\tau} = \frac{1}{2}\left(\varepsilon^{\lambda\nu\sigma\tau} - \varepsilon^{\lambda\nu\tau\sigma}\right) F_{\sigma\tau} \\ &= \frac{1}{4}\left(\varepsilon^{\lambda\nu\sigma\tau} - \varepsilon^{\lambda\nu\tau\sigma} + \varepsilon^{\nu\lambda\tau\sigma} - \varepsilon^{\nu\lambda\sigma\tau}\right) F_{\sigma\tau} \end{aligned} \tag{10.3}$$

(see [14], [15], [16], [39], [40], [75], [77], [91] and the references therein). Here,

$$\varepsilon^{\lambda\nu\sigma\tau} = \frac{1}{\mu}\left(g^{\lambda\sigma} + \kappa u^\lambda u^\sigma\right)\left(g^{\nu\tau} + \kappa u^\nu u^\tau\right) = \varepsilon^{\nu\lambda\tau\sigma} \tag{10.4}$$

is the four-tensor of electric and magnetic permeabilities[4] and

$$u^\lambda = (\gamma, \gamma\mathbf{v}/c), \qquad u^\lambda u_\lambda = 1 \tag{10.5}$$

is the four-velocity of the medium (a computer algebra verification of these relations is given in [52]). In a complex covariant form,

$$\left(Q^{\mu\nu} + \overset{*}{Q}{}^{\mu\nu}\right) = \varepsilon^{\mu\nu\sigma\tau}\left(P_{\sigma\tau} + \overset{*}{P}_{\sigma\tau}\right). \tag{10.6}$$

In view of (10.3) and (B.5)–(B.6), we get

$$Q^{\mu\nu} = \left(\varepsilon^{\mu\nu\sigma\tau} - \frac{i}{2} e^{\mu\nu\sigma\tau}\right) F_{\sigma\tau}, \qquad P_{\mu\nu} = \left(\delta_\mu^\lambda \delta_\nu^\rho - \frac{i}{2} e_{\mu\nu\sigma\tau}\varepsilon^{\sigma\tau\lambda\rho}\right) F_{\lambda\rho}, \tag{10.7}$$

in terms of the real-valued electromagnetic field tensor.

[4]Originally introduced by Tamm in a general case of the moving anisotropic medium [83], [84]; expression (10.4) for an isotropic medium is due to Ryazanov [75], [77].

10.2 Potentials

In practice, one can choose

$$F_{\sigma\tau}=\frac{\partial A_{\tau}}{\partial x^{\sigma}}-\frac{\partial A_{\sigma}}{\partial x^{\tau}}, \tag{10.8}$$

for the real-valued four-vector potential $A_{\lambda}(x)$. Then

$$\begin{aligned}\partial_{\nu}Q^{\lambda\nu}&=\varepsilon^{\lambda\nu\sigma\tau}\partial_{\nu}\left(\partial_{\sigma}A_{\tau}-\partial_{\tau}A_{\sigma}\right)-\frac{i}{2}e^{\lambda\nu\sigma\tau}\partial_{\nu}\left(\partial_{\sigma}A_{\tau}-\partial_{\tau}A_{\sigma}\right)\\&=\frac{1}{\mu}\left(g^{\lambda\sigma}+\kappa u^{\lambda}u^{\sigma}\right)\left(g^{\nu\tau}+\kappa u^{\nu}u^{\tau}\right)\partial_{\nu}\left(\partial_{\sigma}A_{\tau}-\partial_{\tau}A_{\sigma}\right)\end{aligned}$$

by (10.4). Substitution into Maxwell's equations (6.6) or (6.5) results in

$$\begin{aligned}&\left(g^{\lambda\sigma}+\kappa u^{\lambda}u^{\sigma}\right)\left\{-\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]A_{\sigma}\right.\\&\left.+\partial_{\sigma}\left(\partial^{\tau}A_{\tau}+\kappa u^{\nu}u^{\tau}\partial_{\nu}A_{\tau}\right)\right\}=-\frac{4\pi\mu}{c}j^{\lambda},\end{aligned} \tag{10.9}$$

where $-\partial^{\tau}\partial_{\tau}=-g^{\sigma\tau}\partial_{\sigma}\partial_{\tau}=\Delta-(\partial/c\partial t)^{2}$ is the d'Alembert operator. In view of an inverse matrix identity,

$$\left(g_{\lambda\rho}-\frac{\kappa}{1+\kappa}u_{\lambda}u_{\rho}\right)\left(g^{\lambda\sigma}+\kappa u^{\lambda}u^{\sigma}\right)=\delta_{\rho}^{\sigma}, \tag{10.10}$$

the latter equations take the form[5]

$$\begin{aligned}&\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]A_{\sigma}-\partial_{\sigma}\left(\partial^{\tau}A_{\tau}+\kappa u^{\nu}u^{\tau}\partial_{\nu}A_{\tau}\right)\\&=\frac{4\pi\mu}{c}\left(g_{\sigma\lambda}-\frac{\kappa}{1+\kappa}u_{\sigma}u_{\lambda}\right)j^{\lambda}.\end{aligned} \tag{10.11}$$

Subject to the subsidiary condition,

$$\partial^{\tau}A_{\tau}+\kappa u^{\nu}u^{\tau}\partial_{\nu}A_{\tau}=\left(g^{\nu\tau}+\kappa u^{\nu}u^{\tau}\right)\partial_{\nu}A_{\tau}=0, \tag{10.12}$$

these equations were studied in detail for the sake of development of the phenomenological classical and quantum electrodynamics in a moving medium (see [13], [14], [15], [16], [71],[75], [76], [77], [91] and the references therein). In particular, Green's function of the photon in a moving medium was studied in [39], [75], [76] (with applications to quantum electrodynamics).

[5]Equations (10.8) and (10.11), together with the gauge condition (10.12), may be considered as the fundamentals of the theory [39]. Our complex fields are given by (10.7).

10.3 Hertz's tensor and vectors

We follow [15], [16], [91] with somewhat different details. The substitution,

$$A^{\mu}(x) = \left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda} - \delta^{\mu}_{\lambda}\right)\partial_{\sigma}Z^{\lambda\sigma}(x) \tag{10.13}$$

(a generalization of Hertz's potentials for a moving medium [15], [91]), into the gauge condition (10.12) results in $Z^{\lambda\sigma} = -Z^{\sigma\lambda}$, in view of

$$\begin{aligned}&\left(g_{\nu\mu} + \kappa u_{\nu}u_{\mu}\right)\partial^{\nu}A^{\mu}\\ &\quad= \left(g_{\nu\mu} + \kappa u_{\nu}u_{\mu}\right)\left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda} - \delta^{\mu}_{\lambda}\right)\partial^{\nu}\partial_{\sigma}Z^{\lambda\sigma}\\ &\quad= -g_{\nu\lambda}\partial^{\nu}\partial_{\sigma}Z^{\lambda\sigma} = -\partial_{\lambda}\partial_{\sigma}Z^{\lambda\sigma} \equiv 0.\end{aligned}$$

Then, equations (10.11) take the form

$$\left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]\partial_{\sigma}Z^{\lambda\sigma} = -\frac{4\pi\mu}{c}j^{\lambda}. \tag{10.14}$$

Indeed, the left-hand side of (10.11) is given by

$$\begin{aligned}&\left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]A_{\sigma} = \left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]g_{\sigma\mu}A^{\mu}\\ &\quad= \left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]g_{\sigma\mu}\left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda} - \delta^{\mu}_{\lambda}\right)\partial_{\rho}Z^{\lambda\rho}\\ &\quad= \left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]\left(\frac{\kappa}{1+\kappa}u_{\sigma}u_{\lambda} - g_{\sigma\lambda}\right)\partial_{\rho}Z^{\lambda\rho}\\ &\quad= \frac{4\pi\mu}{c}\left(g_{\sigma\lambda} - \frac{\kappa}{1+\kappa}u_{\sigma}u_{\lambda}\right)j^{\lambda},\end{aligned}$$

from which the result follows due to (10.10).

Finally, with the help of the standard substitution,

$$j^{\lambda} = c\partial_{\sigma}p^{\lambda\sigma}, \qquad p^{\lambda\sigma} = -p^{\sigma\lambda} \tag{10.15}$$

(in view of $\partial_{\lambda}j^{\lambda} = c\partial_{\lambda}\partial_{\sigma}p^{\lambda\sigma} \equiv 0$), we arrive at

$$\partial_{\sigma}\left\{\left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]Z^{\lambda\sigma} + 4\pi\mu p^{\lambda\sigma}\right\} = 0. \tag{10.16}$$

Therefore, one can choose

$$\left[\partial^{\tau}\partial_{\tau} + \kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]Z^{\lambda\nu} = -4\pi\mu p^{\lambda\nu}. \tag{10.17}$$

Here, by definition,

$$p^{\lambda\nu} = \begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 \\ p_1 & 0 & m_3 & -m_2 \\ p_2 & -m_3 & 0 & m_1 \\ p_3 & m_2 & -m_1 & 0 \end{pmatrix} \tag{10.18}$$

is an anti-symmetric four-tensor [15], [16], [91]. The "electric" and "magnetic" Hertz vectors, $\mathbf{Z}^{(e)}$ and $\mathbf{Z}^{(m)}$, respectively, are also introduced in terms of a single four-tensor,

$$Z^{\lambda\nu} = \begin{pmatrix} 0 & Z_1^{(e)} & Z_2^{(e)} & Z_3^{(e)} \\ -Z_1^{(e)} & 0 & -Z_3^{(m)} & Z_2^{(m)} \\ -Z_2^{(e)} & Z_3^{(m)} & 0 & -Z_1^{(m)} \\ -Z_3^{(e)} & -Z_2^{(m)} & Z_1^{(m)} & 0 \end{pmatrix}. \tag{10.19}$$

In view of (10.13), for the four-vector potential, $A^\lambda = (\varphi, \mathbf{A})$, we obtain

$$\varphi = -\left(1 - \frac{\kappa\gamma^2}{1+\kappa}\right) \operatorname{div} \mathbf{Z}^{(e)} + \frac{\kappa\gamma^2}{(1+\kappa)\,c} \mathbf{v} \cdot \left(\frac{\partial \mathbf{Z}^{(e)}}{c\partial t} + \operatorname{curl} \mathbf{Z}^{(m)}\right) \tag{10.20}$$

and

$$\begin{aligned} \mathbf{A} &= \frac{\partial \mathbf{Z}^{(e)}}{c\partial t} + \operatorname{curl} \mathbf{Z}^{(m)} \\ &+ \frac{\kappa\gamma^2 \mathbf{v}}{(1+\kappa)\,c^2} \left[c \operatorname{div} \mathbf{Z}^{(e)} + \frac{\partial}{c\partial t}\left(\mathbf{v}\cdot\mathbf{Z}^{(e)}\right) + \mathbf{v} \cdot \operatorname{curl} \mathbf{Z}^{(m)}\right]. \end{aligned} \tag{10.21}$$

Then, equations (10.17) take the form

$$\left[\partial^\tau \partial_\tau + \kappa\,(u^\tau \partial_\tau)^2\right] \mathbf{Z}^{(e)} = 4\pi\mu\mathbf{p}, \qquad \left[\partial^\tau \partial_\tau + \kappa\,(u^\tau \partial_\tau)^2\right] \mathbf{Z}^{(m)} = 4\pi\mu\mathbf{m} \tag{10.22}$$

and, for the four-current, $j^\lambda = (c\rho, \mathbf{j})$, one gets

$$\rho = -\operatorname{div} \mathbf{p}, \qquad \mathbf{j} = \frac{\partial \mathbf{p}}{\partial t} + c \operatorname{curl} \mathbf{m} \tag{10.23}$$

in terms of the corresponding electric $\mathbf{p}$ and magnetic $\mathbf{m}$ moments, respectively (see [15], [16], [91] for more details).

Moreover, in $3D$-complex form,

$$\left[\partial^\tau \partial_\tau + \kappa\,(u^\tau \partial_\tau)^2\right] \mathbf{W} = 4\pi\mu\boldsymbol{\zeta}, \tag{10.24}$$

where $\mathbf{W} = \mathbf{Z}^{(e)} + i\mathbf{Z}^{(m)}$ and $\boldsymbol{\zeta} = \mathbf{p} + i\mathbf{m}$, by definition. In a similar fashion,

$$\left[\partial^\tau \partial_\tau + \kappa\,(u^\tau \partial_\tau)^2\right] W^{\lambda\nu} = -4\pi\mu\zeta^{\lambda\nu}, \tag{10.25}$$

for the corresponding (self-dual) four-tensors:

$$W^{\lambda\nu} = Z^{\lambda\nu} + \frac{i}{2} e^{\lambda\nu\sigma\tau} Z_{\sigma\tau}, \qquad \zeta^{\lambda\nu} = p^{\lambda\nu} + \frac{i}{2} e^{\lambda\nu\sigma\tau} p_{\sigma\tau}. \tag{10.26}$$

The Hertz vector and tensor potentials, for a moving medium and at rest, were utilized in [15], [16], [28], [41], [86], [91], [96] (see also the references therein). Many classical problems of radiation and propagation can be consistently solved by using these potentials.

10.4 Energy-momentum tensor

In the case of the covariant version of the energy-momentum tensor given by (7.2), the differential balance equations under consideration are independent of the particular choice of the frame of reference. Therefore, our relations (10.7) are useful for derivation of the expressions for the energy-momentum tensor and the ponderomotive force for moving bodies from those for bodies at rest which were extensively studied in the literature. For example, one gets

$$4\pi T_{\mu}^{\ \nu} = F_{\mu\lambda} \varepsilon^{\lambda\nu\sigma\tau} F_{\sigma\tau} + \frac{1}{4} \delta_{\mu}^{\nu} F_{\sigma\tau} \varepsilon^{\sigma\tau\lambda\rho} F_{\lambda\rho} \tag{10.27}$$

with the help of (10.3)–(10.4) and (B.14) (see also [84]).

10.5 Tamm's problem and Cherenkov radiation

Let a stationary point charge q be located at the origin of laboratory frame in a moving dispersionless medium with the velocity $\mathbf{v}$. The time-independent potentials can be written in terms of piecewise defined functions as follows [15], [16], [91]:

$$\varphi(\mathbf{r}) = \frac{qf}{\varepsilon} \frac{\alpha\gamma^2}{\left(r_{\parallel}^2 + \alpha\gamma^2 r_{\perp}^2\right)^{1/2}}, \qquad \mathbf{A}(\mathbf{r}) = -\frac{qf}{\varepsilon} \frac{\kappa\gamma^2}{\left(r_{\parallel}^2 + \alpha\gamma^2 r_{\perp}^2\right)^{1/2}} \frac{\mathbf{v}}{c}, \tag{10.28}$$

where $\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$ and $\alpha = 1 - \varepsilon\mu\beta^2$. Here, in the "slower-than-light" case, when $\alpha > 0$, one gets $f(\mathbf{r}) = 1$; while in the "faster-than-light" case, $\alpha < 0$, we should substitute:

$$f(\mathbf{r}) = \begin{cases} 2, & \text{when } \mathbf{r}_{\parallel} \text{ is parallel to } \mathbf{v} \text{ and } r_{\parallel}^2 > |\alpha|\,\gamma^2 r_{\perp}^2; \\ 0, & \text{otherwise; if } \mathbf{r}_{\parallel} \text{ is anti-parallel to } \mathbf{v}, \text{ or } r_{\parallel}^2 < |\alpha|\,\gamma^2 r_{\perp}^2 \end{cases} \tag{10.29}$$

(see [15], [16] for more details and [52] for a direct Mathematica verification). The corresponding (static) electric and magnetic fields are given by

$$\mathbf{E}(\mathbf{r}) = qf \frac{\alpha\gamma^2}{\left(r_\parallel^2 + \alpha\gamma^2 r_\perp^2\right)^{3/2}} \left(\mathbf{r}_\parallel + \alpha\gamma^2 \mathbf{r}_\perp\right), \quad \mathbf{H}(\mathbf{r}) = \mathbf{0}, \tag{10.30}$$

$$\mathbf{D}(\mathbf{r}) = \varepsilon qf \frac{\alpha\gamma^2}{\left(r_\parallel^2 + \alpha\gamma^2 r_\perp^2\right)^{3/2}} \mathbf{r}, \qquad \mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp,$$

$$\mathbf{B}(\mathbf{r}) = \frac{\kappa}{\alpha}\left(\mathbf{E} \times \frac{\mathbf{v}}{c}\right) = qf \frac{\alpha\kappa\gamma^4}{\left(r_\parallel^2 + \alpha\gamma^2 r_\perp^2\right)^{3/2}} \left(\mathbf{r} \times \frac{\mathbf{v}}{c}\right).$$

On the other hand, if a charge q is moving with constant velocity $\mathbf{v}$ and the medium is at rest, by a Lorentz transformation, one gets

$$\varphi(\mathbf{r}, t) = \frac{qf'(\mathbf{r}, t)}{\varepsilon\left(\left(\mathbf{r}_\parallel - \mathbf{v}t\right)^2 + \alpha\mathbf{r}_\perp^2\right)^{1/2}}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu qf'(\mathbf{r}, t)}{\left(\left(\mathbf{r}_\parallel - \mathbf{v}t\right)^2 + \alpha\mathbf{r}_\perp^2\right)^{1/2}} \frac{\mathbf{v}}{c}, \tag{10.31}$$

provided

$$\operatorname{div} \mathbf{A} + \frac{\varepsilon\mu}{c}\frac{\partial\varphi}{\partial t} = 0. \tag{10.32}$$

Here, $f'(\mathbf{r}, t) = 1$, if $\alpha > 0$ and

$$f'(\mathbf{r}, t) = \begin{cases} 2, & \text{when } r_\parallel < vt - r_\perp |\alpha|^{1/2}; \\ 0, & \text{otherwise}, \end{cases} \tag{10.33}$$

if $\alpha < 0$ (see [70], [91] for the vacuum case). Properties of the Cherenkov radiation, when $\alpha < 0$ (the charge velocity is greater than the speed of light in the medium under consideration), are discussed in detail in [3], [15], [16], [91] following the original article [85]. (At every given moment of time, the field is confined to the cone with a vertex angle defined by $\sin\theta = c/v\sqrt{\varepsilon\mu}$.)

11 Real versus Complex Lagrangians

In modern presentations of the classical and quantum field theories, the Lagrangian approach is usually utilized.

11.1 Complex forms

We introduce two quadratic "Lagrangian" densities

$$\begin{aligned}\mathscr{L}_0 = \mathscr{L}_0^* &= \frac{1}{2}\left(P_{\sigma\tau}Q^{\tau\sigma} + P_{\sigma\tau}^{*}\overset{*}{Q}{}^{\tau\sigma}\right) \\ &= \frac{i}{4}e^{\sigma\tau\kappa\rho}\left(P_{\sigma\tau}P_{\kappa\rho} - P_{\sigma\tau}^{*}P_{\kappa\rho}^{*}\right) \\ &= F_{\sigma\tau}R^{\tau\sigma} - G_{\sigma\tau}S^{\tau\sigma} = 2F_{\sigma\tau}R^{\tau\sigma} \\ &= 4\left(\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}\right)\end{aligned} \tag{11.1}$$

and

$$\begin{aligned}\mathscr{L}_1 = -\mathscr{L}_1^* = P_{\sigma\tau}^{*}Q^{\tau\sigma} &= \frac{1}{2}\left(P_{\sigma\tau}^{*}Q^{\tau\sigma} - P_{\sigma\tau}\overset{*}{Q}{}^{\tau\sigma}\right) \\ = \frac{i}{2}e^{\sigma\tau\kappa\rho}P_{\sigma\tau}P_{\kappa\rho}^{*} &= 4i\left(\mathbf{E}\cdot\mathbf{B} - \mathbf{H}\cdot\mathbf{D}\right).\end{aligned} \tag{11.2}$$

Then, by formal differentiation,

$$\frac{\partial\mathscr{L}_0}{\partial P_{\alpha\beta}} = Q^{\beta\alpha}, \qquad \frac{\partial\mathscr{L}_0}{\partial P_{\alpha\beta}^{*}} = \overset{*}{Q}{}^{\beta\alpha} \tag{11.3}$$

and

$$\frac{\partial\mathscr{L}_1}{\partial P_{\alpha\beta}^{*}} = Q^{\beta\alpha}, \qquad \frac{\partial\mathscr{L}_1^{*}}{\partial P_{\alpha\beta}} = \overset{*}{Q}{}^{\beta\alpha} \tag{11.4}$$

in view of (B.7).

The complex covariant Maxwell equations (5.2) take the forms

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial\mathscr{L}_0}{\partial P_{\nu\mu}}\right) = -\frac{4\pi}{c}j^{\mu}, \qquad \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial\mathscr{L}_1}{\partial P_{\nu\mu}}\right) = \frac{4\pi}{c}j^{\mu} \tag{11.5}$$

and the covariant energy-momentum balance relations (7.1) are given by

$$\begin{aligned}&\frac{\partial}{\partial x^{\nu}}\left[\frac{1}{16\pi}\left(P_{\mu\lambda}^{*}\frac{\partial\mathscr{L}_0}{\partial P_{\nu\lambda}} + P_{\mu\lambda}\frac{\partial\mathscr{L}_0}{\partial P_{\nu\lambda}^{*}}\right)\right] \\ &\quad + \frac{1}{32\pi}\left[P_{\sigma\tau}^{*}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial\mathscr{L}_0}{\partial P_{\sigma\tau}}\right) + P_{\sigma\tau}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial\mathscr{L}_0}{\partial P_{\sigma\tau}^{*}}\right)\right] = -\frac{1}{c}F_{\mu\lambda}j^{\lambda}\end{aligned} \tag{11.6}$$

and

$$\frac{\partial}{\partial x^{\nu}}\left[\frac{1}{16\pi}\left(P_{\mu\lambda}\frac{\partial \mathscr{L}_1}{\partial P_{\nu\lambda}}+P^{*}_{\mu\lambda}\frac{\partial \mathscr{L}^{*}_1}{\partial P_{\nu\lambda}}\right)\right] + \frac{1}{32\pi}\left[P_{\sigma\tau}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathscr{L}_1}{\partial P_{\sigma\tau}}\right)+P^{*}_{\sigma\tau}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathscr{L}^{*}_1}{\partial P^{*}_{\sigma\tau}}\right)\right]=\frac{1}{c}F_{\mu\lambda}j^{\lambda} \tag{11.7}$$

in terms of the complex Lagrangians under consideration, respectively.

Finally, with the help of the following densities,

$$L_0=\mathscr{L}_0-\frac{4\pi}{c}j^{\nu}A_{\nu}, \qquad L_1=\mathscr{L}_1+\frac{4\pi}{c}j^{\nu}A_{\nu}, \tag{11.8}$$

one can derive analogs of the Euler–Lagrange equations for electromagnetic fields in media:

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L_{0,1}}{\partial P_{\nu\mu}}\right)-\frac{\partial L_{0,1}}{\partial A_{\mu}}=0. \tag{11.9}$$

In the case of a moving isotropic medium, a relation between $P_{\nu\mu}$ and A_{μ} is given by our equations (10.7)–(10.8).

11.2 Real form

Taking the real and imaginary parts, Maxwell's equations (6.6) can be written as follows

$$\partial_{\nu}R^{\mu\nu}=-\frac{4\pi}{c}j^{\mu}, \qquad \partial_{\nu}S^{\mu\nu}=0. \tag{11.10}$$

Here,

$$-6\partial_{\nu}S^{\mu\nu}=e^{\mu\nu\lambda\sigma}\left(\partial_{\nu}F_{\lambda\sigma}+\partial_{\sigma}F_{\nu\lambda}+\partial_{\lambda}F_{\sigma\nu}\right)\equiv 0,$$

with the help of (6.4) and (10.8). Thus the second set of equations is automatically satisfied when we introduce the four-vector potential. For the inhomogeneous pair of Maxwell's equations, the Lagrangian density is given by

$$\begin{aligned} L &= \frac{1}{4}F_{\sigma\tau}R^{\tau\sigma}-\frac{4\pi}{c}j^{\sigma}A_{\sigma} \\ &= \frac{1}{4}F_{\sigma\tau}\varepsilon^{\tau\sigma\lambda\rho}F_{\lambda\rho}-\frac{4\pi}{c}j^{\sigma}A_{\sigma}, \end{aligned} \tag{11.11}$$

in view of (10.3). Then, for "conjugate momenta" to the four-potential field A_{μ}, one gets

$$\frac{\partial L}{\partial\left(\partial_{\nu}A_{\mu}\right)}=\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial F_{\sigma\tau}}{\partial\left(\partial_{\nu}A_{\mu}\right)}=R^{\mu\nu} \tag{11.12}$$

and the corresponding Euler–Lagrange equations take a familiar form

$$\partial_\nu \left(\frac{\partial L}{\partial \left(\partial_\nu A_\mu \right)} \right) - \frac{\partial L}{\partial A_\mu} = 0. \tag{11.13}$$

The latter equation can also be derived with the help of the least action principle [72], [88], [90]. The corresponding Hamiltonian and quantization are discussed in [35], [39], [75] among other classical accounts.

In conclusion, it is worth noting the role of complex fields in quantum electrodynamics, quadratic invariants and quantization (see, for instance, [2], [8], [9], [20], [39], [40], [44], [46], [53], [55], [56], [75], [76], [77], [90], [97]). The classical and quantum theory of Cherenkov radiation is reviewed in [3], [11], [13], [29], [31], [81], [85]. For paraxial approximation in optics, see [28], [43], [45], [60], [61] and the references therein. Maxwell's equations in the gravitational field are discussed in [17], [27]. One may hope that our detailed mathematical consideration of several aspects of macroscopic electrodynamics will be useful for future investigations and pedagogy.

Acknowledgements We dedicate this article to the memory of Professor Alladi Ramakrishnan who made significant contributions to probability and statistics, elementary particle physics, cosmic rays and astrophysics, matrix theory, and the special theory of relativity [4].

We are grateful to Krishna Alladi, Albert Boggess, Mark P. Faifman, John R. Klauder, Vladimir I. Man'ko, and Igor N. Toptygin for valuable comments and help. The authors are indebted to Kamal Barley for graphics enhancement. The useful suggestions from the referee are very much appreciated.

Appendix A: Formulas from Vector Calculus

Among useful differential relations are

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) . \tag{A.1}$$

$$\nabla \cdot (f \mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f (\nabla \cdot \mathbf{A}) . \tag{A.2}$$

$$\nabla \times (f \mathbf{A}) = (\nabla f) \times \mathbf{A} + f (\nabla \times \mathbf{A}) . \tag{A.3}$$

$$\begin{aligned} &\mathbf{A} \cdot (\nabla \times (f \nabla \times \mathbf{B})) - \mathbf{B} \cdot (\nabla \times (f \nabla \times \mathbf{A})) \\ &\quad = \nabla \cdot (f (\mathbf{B} \times (\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{B}))) . \end{aligned} \tag{A.4}$$

$$\mathbf{A}\,(\nabla\cdot\mathbf{B})-\mathbf{B}\,(\nabla\cdot\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{B})-\mathbf{B}\times(\nabla\times\mathbf{A}) \tag{A.5}$$
$$-\nabla\times(\mathbf{A}\times\mathbf{B})=\sum_{\alpha=1}^{3}A_\alpha^2\nabla\left(\frac{B_\alpha}{A_\alpha}\right)=-\sum_{\alpha=1}^{3}B_\alpha^2\nabla\left(\frac{A_\alpha}{B_\alpha}\right).$$

(See also [1], [79] and [90].) Here, div $\mathbf{A}=\nabla\cdot\mathbf{A}$ and curl $\mathbf{A}=\nabla\times\mathbf{A}$.

Appendix B: Dual Tensor Identities

In this article, $e^{\mu\nu\sigma\tau}=-e_{\mu\nu\sigma\tau}$ and $e_{0123}=+1$ is the Levi-Civita four-symbol [27] with familiar contractions:

$$e^{\mu\nu\sigma\tau}e_{\mu\kappa\lambda\rho}=-\begin{vmatrix}\delta_\kappa^\nu & \delta_\lambda^\nu & \delta_\rho^\nu\\ \delta_\kappa^\sigma & \delta_\lambda^\sigma & \delta_\rho^\sigma\\ \delta_\kappa^\tau & \delta_\lambda^\tau & \delta_\rho^\tau\end{vmatrix}, \tag{B.1}$$

$$e^{\mu\nu\sigma\tau}e_{\mu\nu\lambda\rho}=-2\begin{vmatrix}\delta_\lambda^\sigma & \delta_\rho^\sigma\\ \delta_\lambda^\tau & \delta_\rho^\tau\end{vmatrix}=-2\left(\delta_\lambda^\sigma\delta_\rho^\tau-\delta_\rho^\sigma\delta_\lambda^\tau\right), \tag{B.2}$$

$$e^{\mu\nu\sigma\tau}e_{\mu\nu\sigma\rho}=-6\delta_\rho^\tau,\qquad e^{\mu\nu\sigma\tau}e_{\mu\nu\sigma\rho}=-24. \tag{B.3}$$

Dual second rank four-tensor identities are given by [27]:

$$e^{\mu\nu\sigma\tau}A_{\sigma\tau}=2B^{\mu\nu},\qquad e_{\mu\nu\sigma\tau}B^{\sigma\tau}=A_{\nu\mu}-A_{\mu\nu}. \tag{B.4}$$

In particular,

$$Q^{\mu\nu}=R^{\mu\nu}+iS^{\mu\nu}=R^{\mu\nu}-\frac{i}{2}e^{\mu\nu\sigma\tau}F_{\sigma\tau}, \tag{B.5}$$
$$P_{\mu\nu}=F_{\mu\nu}+iG_{\mu\nu}=F_{\mu\nu}-\frac{i}{2}e_{\mu\nu\sigma\tau}R^{\sigma\tau}. \tag{B.6}$$

$$e_{\mu\nu\sigma\tau}Q^{\sigma\tau}=2iP_{\mu\nu},\qquad 2iQ^{\mu\nu}=e^{\mu\nu\sigma\tau}P_{\sigma\tau}. \tag{B.7}$$

$$2R^{\mu\nu}=e^{\mu\nu\sigma\tau}G_{\sigma\tau},\qquad -2S^{\mu\nu}=e^{\mu\nu\sigma\tau}F_{\sigma\tau}. \tag{B.8}$$

$$2G_{\mu\nu}=-e_{\mu\nu\sigma\tau}R^{\sigma\tau},\qquad 2F_{\mu\nu}=e_{\mu\nu\sigma\tau}S^{\sigma\tau}. \tag{B.9}$$

$$P_{\mu\nu}Q^{\mu\nu}=2F_{\mu\nu}R^{\mu\nu}-\frac{i}{2}\left(e^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}+e_{\mu\nu\sigma\tau}R^{\mu\nu}R^{\sigma\tau}\right). \tag{B.10}$$

By direct calculation,

$$F_{\mu\nu}R^{\mu\nu}=2\,(\mathbf{H}\cdot\mathbf{B}-\mathbf{E}\cdot\mathbf{D}), \tag{B.11}$$

$$e^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} = 8\mathbf{E} \cdot \mathbf{B}, \qquad e_{\mu\nu\sigma\tau} R^{\mu\nu} R^{\sigma\tau} = 8\mathbf{H} \cdot \mathbf{D}. \tag{B.12}$$

As a result,

$$\frac{1}{4} P_{\mu\nu} Q^{\mu\nu} = \mathbf{H} \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D} - i\left(\mathbf{E} \cdot \mathbf{B} + \mathbf{H} \cdot \mathbf{D}\right). \tag{B.13}$$

An important decomposition,

$$\begin{aligned} P^{*}_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda} \overset{*}{Q}{}^{\lambda\nu} &= 2\left(F_{\mu\lambda} R^{\lambda\nu} + G_{\mu\lambda} S^{\lambda\nu}\right) \\ &= 4F_{\mu\lambda} R^{\lambda\nu} + \delta^{\nu}_{\mu} F_{\sigma\tau} R^{\sigma\tau} \\ &= 4F_{\mu\lambda} R^{\lambda\nu} - 2\delta^{\nu}_{\mu}\left(\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B}\right), \end{aligned} \tag{B.14}$$

is complemented by an identity,

$$\begin{aligned} P_{\mu\lambda} Q^{\lambda\nu} + P^{*}_{\mu\lambda} \overset{*}{Q}{}^{\lambda\nu} &= \frac{1}{4}\left(P_{\sigma\tau} Q^{\tau\sigma} + P^{*}_{\sigma\tau} \overset{*}{Q}{}^{\tau\sigma}\right)\delta^{\nu}_{\mu} \\ &= \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B}\right)\delta^{\nu}_{\mu}. \end{aligned} \tag{B.15}$$

In matrix form,

$$PQ = (F + iG)(R + iS) = (FR - GS) + i(FS + GR), \tag{B.16}$$

$$P^{*}Q = (F - iG)(R + iS) = (FR + GS) + i(FS - GR). \tag{B.17}$$

Here,

$$FS = \frac{1}{4}\operatorname{Tr}(FS)\, I = (\mathbf{E} \cdot \mathbf{B})\, I, \tag{B.18}$$

$$GR = \frac{1}{4}\operatorname{Tr}(GR)\, I = (\mathbf{H} \cdot \mathbf{D})\, I. \tag{B.19}$$

$$FR - GS = \frac{1}{2}\operatorname{Tr}(FR)\, I = (\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B})\, I, \tag{B.20}$$

$$\begin{aligned} FR + GS &= 2FR - \frac{1}{2}\operatorname{Tr}(FR)\, I \\ &= 2FR - (\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B})\, I. \end{aligned} \tag{B.21}$$

$$\operatorname{Tr}(FR + GS) = 0, \tag{B.22}$$

where $I = \operatorname{diag}(1, 1, 1, 1)$ is the identity matrix.

Also,

$$PQ = QP = (\mathbf{F}\cdot\mathbf{G})\, I, \tag{B.23}$$

$$\det P = \det Q = -\,(\mathbf{F}\cdot\mathbf{G})^2 \tag{B.24}$$

and

$$\begin{aligned}\mathbf{F}\cdot\mathbf{G} &= (\mathbf{E}+i\mathbf{H})\cdot(\mathbf{D}+i\mathbf{B}) \\ &= (\mathbf{E}\cdot\mathbf{D}-\mathbf{H}\cdot\mathbf{B}) + i\,(\mathbf{E}\cdot\mathbf{B}+\mathbf{H}\cdot\mathbf{D})\,.\end{aligned} \tag{B.25}$$

Other useful dual four-tensor identities are given by [27]:

$$e^{\mu\nu\sigma\tau} A_{\nu\sigma\tau} = 6B^{\mu}, \qquad A_{\mu\nu\lambda} = e_{\mu\nu\lambda\sigma} B^{\sigma}. \tag{B.26}$$

In particular,

$$6i\,\frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = e^{\mu\nu\lambda\sigma}\left(\frac{\partial P_{\lambda\sigma}}{\partial x^{\nu}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial P_{\sigma\nu}}{\partial x^{\lambda}}\right), \tag{B.27}$$

and

$$\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = i e_{\mu\nu\lambda\sigma}\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}} \tag{B.28}$$

(see also [47]).

Appendix C: Proof of Identities

In view of (6.3), or (B.7), and (B.28), we can write

$$\left(\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = i e_{\mu\nu\lambda\sigma}\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}}\right) \overset{*}{Q}{}^{\lambda\nu}, \tag{C.1}$$

or

$$\begin{aligned} 2\,\overset{*}{Q}{}^{\lambda\nu}\,\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \overset{*}{Q}{}^{\lambda\nu}\,\frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} \\ = i\left(e_{\mu\nu\lambda\sigma}\,\overset{*}{Q}{}^{\lambda\nu}\right)\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}} = -2P^{*}_{\mu\sigma}\,\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}}\end{aligned}$$

by (B.7). Therefore,

$$P^{*}_{\mu\lambda}\,\frac{\partial Q^{\lambda\nu}}{\partial x^{\nu}} - \frac{\partial P_{\mu\lambda}}{\partial x^{\nu}}\,\overset{*}{Q}{}^{\lambda\nu} = -\frac{1}{2}\,\overset{*}{Q}{}^{\sigma\tau}\,\frac{\partial P_{\tau\sigma}}{\partial x^{\mu}}. \tag{C.2}$$

In addition, with the help of (B.7) one gets

$$2i\left(P^{*}_{\sigma\tau}\frac{\partial Q^{\tau\sigma}}{\partial x^{\mu}}\right)=P^{*}_{\sigma\tau}e^{\tau\sigma\lambda\nu}\frac{\partial P_{\lambda\nu}}{\partial x^{\mu}}$$
$$=e^{\sigma\tau\nu\lambda}P^{*}_{\sigma\tau}\frac{\partial P_{\lambda\nu}}{\partial x^{\mu}}=-2i\left(\overset{*}{Q}{}^{\sigma\tau}\frac{\partial P_{\tau\sigma}}{\partial x^{\mu}}\right),$$

which completes the proof.

References

1. M. Abraham, *The Classical Theory of Electricity and Magnetism* (Blackie and Son, Glasgow, 1948). Translation of the Eighth German Edition
2. P.B. Acosta-Humánez, S.I. Kryuchkov, E. Suazo, S.K. Suslov, Degenerate parametric amplification of squeezed photons: explicit solutions, statistics, means and variances. J. Nonlinear Opt. Phys. Mater. **24**(2), 1550021 (27pp) (2015)
3. G.N. Afanasiev, *Vavilov–Cherenkov and Synchrotron Radiation: Foundations and Applications* (Kluwer, Dordrecht, 2004)
4. K. Alladi, J.R. Klauder, C.R. Rao (eds.), *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences* (Springer, New York, 2010)
5. S.M. Barnett, Resolution of the Abraham–Minkowski dilemma. Phys. Rev. Lett. **104**(7), 070401 (4pp) (2010)
6. A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980)
7. R. Becker, *Electromagnetic Fields and Interactions* (Blaisdell Publishing Co., New York, 1964)
8. I. Białynicki-Birula, Z. Białynicki-Birula, *Quantum Electrodynamics* (Pergamon and PWN–Polish Scientific Publishers, Oxford, 1975)
9. I. Białynicki-Birula, Z. Białynicki-Birula, The role of Riemann-Silberstein vector in classical and quantum theories of electromagnetism. J. Phys. B At. Mol. Opt. Phys. **46**, 053001 (32pp) (2013)
10. N.N. Bogolubov, A.A. Logunov, A.I. Oksak, I.T. Todorov, *General Principles of Quantum Field Theory*, 7th edn. (Kluwer, Dordrecht, 1990)
11. B.M. Bolotovskii, Theory of Vavilov–Cherenkov effect (I–II). Usp. Fiz. Nauk **62**(3), 201–246 (1957). [in Russian]
12. B.M. Bolotovskii, Theory of Cherenkov radiation (III). Sov. Phys.–Usp. **4**(5), 781–811 (1962)
13. B.M. Bolotovskii, Vavilov–Cherenkov radiation: its discovery and application. Phys.–Usp. **52**(11), 1099–1110 (2009)
14. B.M. Bolotovskii, A.A. Rukhadze, Field of a charged particle in a moving medium. Sov. Phys. JETP **10**(5), 958–961 (1960)
15. B.M. Bolotovskii, S.N. Stolyarov, Current status of the electrodynamics of moving media (infinite media). Sov. Phys.–Usp. **17**(6), 875–895 (1975). See also a revised version with the annotated bibliography in Einstein's Collection 1974, pp. 179–275, Nauka, Moscow, 1976 [in Russian]
16. B.M. Bolotovskii, S.N. Stolyarov, *The Fields of Emitting Sources in Moving Media*, Einstein's Collection 1978–1979 (Nauka, Moscow, 1983), pp. 173–277. [in Russian]
17. M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (World Scientific, New Jersey, 2001)
18. V.I. Denisov, Introduction to Electrodynamics of Material Media, Izd. MGU, Moscow State University, Moscow (1989). [in Russian]
19. I.Y. Dodin, N.J. Fisch, Axiomatic geometrical optics, Abraham–Minkowski controversy, and photon properties derived classically. Phys. Rev. A **86**(5), 053834 (16pp) (2012)

20. V.V. Dodonov, V.I. Man'ko, Invariants and Correlated States of Nonstationary Quantum Systems, *Invariants and the Evolution of Nonstationary Quantum Systems*, vol. 183, Proceedings of Lebedev Physics Institute (Nauka, Moscow, 1987), pp. 71–181. [in Russian]; English translation published by Nova Science, Commack, New York, 1989, pp. 103–261
21. A. Einstein, Zur Electrodynamik der bewegter Körper. Ann. Phys. **17**, 891–921 (1905)
22. A. Einstein, Eine neue formale Deutung der Maxwellschen Feildgleichungen der Electrodynamik. Sitzungsber. preuss Akad. Wiss. **1**, 184–188 (1916)
23. A. Einstein, J. Laub, Über die elektromagnetischen Grundgleichungen für bewegte Körper. Ann. Phys. **26**, 532–540 (1908); Bemerkungen zu unserer Arbeit < < Über die elektromagnetischen Grundgleichungen für bewegte Körper > >. Ann. Phys. **28**, 445–447 (1908)
24. A. Einstein, J. Laub, Über die im elektromagnetischen Felde auf ruhende Körper ausgeubten pondemotorischen Kräfte. Ann. Phys. Wiss. **26**, 541–550 (1908)
25. G. Farmelo, *The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom* (Faber and Faber, London, 2009)
26. G.D. Fleishman, I.N. Toptygin, *Cosmic Electrodynamics: Electrodynamics and Magnetic Hydrodynamics of Cosmic Plasmas* (Springer, New York, 2013)
27. V.A. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press, New York, 1964)
28. V.A. Fock, *Electromagnetic Diffraction and Propagation Problems* (Pergamon Press, London, 1965)
29. V.L. Ginzburg, Quantum theory of electromagnetic radiation of an electron uniformly moving in a medium. Sov. Phys. JETP **10**(6), 589–600 (1940). [in Russian]
30. V.L. Ginzburg, The laws of conservation of energy and momentum in emission of electromagnetic waves (photons) in a medium and the energy-momentum tensor in macroscopic electrodynamics. Sov. Phys.–Usp. **16**(3), 434–439 (1973)
31. V.L. Ginzburg, *Applications of Electrodynamics in Theoretical Physics and Astrophysics*, 2nd edn. (Gordon and Breach, New York, 1989)
32. V.L. Ginsburg, V.A. Ugarov, Remarks on forces and the energy-momentum tensor in macroscopic electrodynamics. Sov. Phys.–Usp. **19**(1), 94–101 (1976)
33. M.V. Gorkunov, A.V. Kondratov, Microscopic view of light pressure on a continuous medium. Phys. Rev. A **88**(1), 011804(R) (5pp) (2013)
34. S.R. de Groot, L.G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972)
35. W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago Press, Chicago, 1930); Dover, New York (1949)
36. W. Israel, J.M. Steward, Progress in relativistic thermodynamics and electrodynamics of continous media, in *General Relativity and Gravitation (One hundred years after the birth of Albert Einstein)*, vol. 2, ed. by A. Held (Plenum Press, New York, 1980), pp. 491–525
37. J.D. Jackson, *Classical Electrodynamics*, 2nd edn. (Wiley, New York, 1962)
38. M. Jacob, Antimatter, https://cds.cern.ch/record/294366/files/open-96-005.pdf
39. J.M. Jauch, K.M. Watson, Phenomenological quantum-electrodynamics. Phys. Rev. **74**(8), 950–957 (1948)
40. J.M. Jauch, K.M. Watson, Phenomenological quantum electrodynamics. Part II. Interaction of the field with charges. Phys. Rev. **74**(10), 1485–1493 (1948)
41. L. Kannenberg, A note on the Hertz potentials in electromagnetism. Am. J. Phys. **18**(4), 370–372 (1987)
42. A.N. Kaufman, Maxwell equations in nonuniformly moving media. Ann. Phys. **18**(2), 264–273 (1962)
43. A.P. Kiselev, A.B. Plachenov, Laplace-Gauss and Helmholtz–Gauss paraxial modes in media with quadratic refraction index. J. Opt. Soc. Am. A **33**(4), 663–666 (2016)
44. J.R. Klauder, E.C.G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin Inc., New York, 1968)
45. C. Koutschan, E. Suazo, S.K. Suslov, Fundamental laser modes in paraxial optics: from computer algebra and simulations to experimental observation. Appl. Phys. B **121**(3), 315–336 (2015)

46. C. Krattenthaler, S.I. Kryuchkov, A. Mahalov, S.K. Suslov, On the problem of electromagnetic-field quantization. Int. J. Theor. Phys. **52**(12), 4445–4460 (2013)
47. S.I. Kryuchkov, N.A. Lanfear, S.K. Suslov, The Pauli–Lubański vector, complex electrodynamics, and photon helicity. Phys. Scr. **90**, 074065 (8pp) (2015)
48. S.I. Kryuchkov, N.A. Lanfear, S.K. Suslov, The role of the Pauli–Lubański vector for the Dirac, Weyl, Proca, Maxwell and Fierz–Pauli equations. Phys. Scr. **91**, 035301 (15pp) (2016)
49. S.I. Kryuchkov, S.K. Suslov, J.M. Vega-Guzmán, The minimum-uncertainty squeezed states for atoms and photons in a cavity. J. Phys. B At. Mol. Opt. Phys. **46**, 104007 (15pp) (2013). IOP Select and Highlight of 2013
50. L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Butterworth-Heinenann, Oxford, 1975)
51. L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd edn. (Pergamon, Oxford, 1989)
52. N. Lanfear, Mathematica notebook: ComplexElectrodynamics.nb
53. N. Lanfear, R.M. López, S.K. Suslov, Exact wave functions for generalized harmonic oscillators. J. Russ. Laser Res. **32**(4), 352–361 (2011)
54. O. Laporte, G.E. Uhlenbeck, Applications of spinor analysis to the Maxwell and Dirac equations. Phys. Rev. **37**, 1380–1397 (1931)
55. R.M. López, S.K. Suslov, J.M. Vega-Guzmán, Reconstracting the Schrödinger groups. Phys. Scr. **87**(3), 038118 (6pp) (2013)
56. R.M. López, S.K. Suslov, J.M. Vega-Guzmán, On a hidden symmetry of quantum harmonic oscillators. J. Differ. Eqs. Appl. **19**(4), 543–554 (2013)
57. H.A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat*, 2nd edn. (Teubner, Leipzig, 1916)
58. H.A. Lorentz, A. Einstein, H. Minkowski, *Das Relativitätsprinzip, eine Sammlung von Abhandlungen* (Teubner, Berlin, 1913); English translation in: A. Einstein et al., The Principle of Relativity, A Collection of Original Papers on the Special and General Theory of Relativity, 4th edn. (Dover, New York, 1932)
59. A. Mahalov, E. Suazo, S.K. Suslov, Spiral laser beams in inhomogeneous media. Opt. Lett. **38**(15), 2763–2766 (2013)
60. A. Mahalov, S.K. Suslov, An “Airy gun”: self-accelerating solutions of the time-dependent Schrödinger equation in vacuum. Phys. Lett. A **377**, 33–38 (2012)
61. A. Mahalov, S.K. Suslov, Solution of paraxial wave equation for inhomogeneous media in linear and quadratic approximation. Proc. Am. Math. Soc. **143**(2), 595–610 (2015)
62. V.P. Makarov, A.A. Rukhadze, Force acting on a substance in an electromagnetic field. Phys.–Usp. **52**(9), 937–943 (2009)
63. V.P. Makarov, A.A. Rukhadze, Negative group velocity electromagnetic waves and the energy-momentum tensor. Phys.–Usp. **54**(12), 1285–1296 (2011)
64. H. Minkowski, Raum und Zeit. Jahresbericht der Deutschen Mathematiker-Vereinigung 75–88 (1909)
65. H. Minkowski, Die Grundlagen für die elektromagnetischen Vorgänge in bewegten Körpern. Nachr. König. Ges. Wiss. Göttingen, math.-phys. Kl. 53–111 (1908); The fundamental equations for electromagnetic processes in moving bodies, English translation in *The Principle of Relativity*, ed. by M. Saha (University Press, Calcutta, 1920), pp. 1–69
66. H. Minkowski, Eine Ableitung der Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern vom Standpukt der Electronentheorie. Math. Ann. **68**, 526–556 (1908)
67. V.V. Nesterenko, A.V. Nesterenko, Symmetric energy-momentum tensor: the Abraham form and the explicit covariant formula. J. Math. Phys. **57**(3), 032901 (14pp) (2016)
68. Y.N. Obukhov, Electromagnetic energy and momentum in moving media. Ann. Phys. (Berlin) **17**(9–10), 830–851 (2008)
69. Y.N. Obukhov, F.W. Hehl, Electromagnetic energy-momentum and forces in matter. Phys. Lett. A **311**, 277–284 (2003)
70. W.K.H. Panofsky, M. Phillips, *Classical Electricity and Magnetism*, 2nd edn. (Addison-Wesley, Reading, 1962)

71. C.H. Papas, *Theory of Electromagnetic Wave Propagation* (Dover, New York, 2011)
72. W. Pauli, *Theory of Relativity* (Pergamon Press, Oxford, 1958)
73. V.I. Pavlov, On discussions concerning the problem of ponderomotive forces. Sov. Phys.-Usp. **21**(2), 171–173 (1978)
74. R.N.C. Pfeifer, T.A. Nieminen, N.R. Heckenberg, H. Rubinsztein-Dunlop, Momentum of an electromagnetic wave in dielectric media. Rev. Mod. Phys. **79**, 1197–1216 (2007)
75. M.I. Riazanov, Phenomenological study of the effect of nonconducting medium in quantum electrodynamics. Sov. Phys. JETP **5**(5), 1013–1015 (1957)
76. M.I. Riazanov, Radiative corrections to Compton scattering taking into account polarization of the surrounding medium. Sov. Phys. JETP **7**(5), 869–875 (1958)
77. M.I. Ryazanov, Covariant formulation of phenomenological quantum electrodynamics, *Topics in Theoretical Physics* (Atomizdat, Moscow, 1958), pp. 75–86. [in Russian]
78. G.L.J.A. Rikken, B.A. van Tiggelen, Observation of the intrinsic Abraham force in time-varying magnetic and electric fields. Phys. Rev. Lett. **108**(23), 230402 (4pp) (2012)
79. J. Schwinger, L.L. DeRoad Jr., K.A. Milton, Wu-y Tsai, *Classical Electrodynamics* (Perseus Books, Reading, 1998)
80. D.V. Skobel'tsyn, The momentum-energy tensor of the electromagnetic field. Sov. Phys.–Usp. **16**(3), 381–401 (1973)
81. A. Sokolov, Quantum theory of radiation of elementary particles. Doklady Phys. **28**(5), 415–417 (1940). [in Russian]
82. J.A. Stratton, *Electromagnetic Theory* (McGraw Hill, New York, 1941)
83. I.E. Tamm, Electrodynamics of anisotropic medium in special theory of relativity. J. Russ. Phys. Chem. Soc. **56**(2–3), 248–262 (1924). [in Russian]
84. I.E. Tamm, Crystal optics of the relativity theory in connection with geometry of a biquadratic form. J. Russ. Phys. Chem. Soc. **57**(3–4), 209–214 (1925). [in Russian]
85. I.E. Tamm, Radiation emitted by uniformly moving electrons. J. Phys. USSR **1**(5–6), 439–454 (1939)
86. I.E. Tamm, *Fundamentals of the Theory of Electricity* (Mir, Moscow, 1979)
87. N.W. Taylor, A simplified form of the relativistic electromagnetic equations. Aust. J. Sci. Res. Ser. A Phys. Sci. **5**, 423–429 (1952)
88. YaP Terletskii, YuP Rybakov, *Electrodynamics*, 2nd edn. (Vysshaya Skola, Moscow, 1990). [in Russian]
89. M. Testa, The momentum of an electromagnetic wave inside a dielectric. Ann. Phys. **336**(1), 1–11 (2013)
90. I.N. Toptygin, *Foundations of Classical and Quantum Electrodynamics* (Wiley-VCH, Weinheim, 2014)
91. I.N. Toptygin, *Electromagnetic Phenomena in Matter: Statistical and Quantum Approaches* (Wiley-VCH, Weinheim, 2015)
92. I.N. Toptygin, K. Levina, Energy-momentum tensor of the electromagnetic field in dispersive media. Phys.–Usp. **59**(2), 141–152 (2016)
93. V.G. Veselago, Energy, linear momentum, and mass transfer by an electromagnetic wave in a negative-refraction medium. Phys.–Usp. **52**(6), 649–654 (2009)
94. V.G. Veselago, Waves in metamaterials: their role in physics. Phys.–Usp. **54**(11), 1161–1165 (2011)
95. V.G. Veselago, V.V. Shchavlev, Force acting on a substance in an electromagnetic field. Phys.–Usp. **53**(3), 317–318 (2010)
96. M.B. Vinogradova, O.V. Rudenko, A.P. Sukhorukov, *Theory of Waves*, 2nd edn. (Nauka, Moscow, 1990). [in Russian]
97. K.M. Watson, J.M. Jauch, Phenomenological quantum electrodynamics. Part III. Dispersion. Phys. Rev. **75**(8), 1249–1261 (1949)

On a System of q-Partial Differential Equations with Applications to q-Series

Zhi-Guo Liu

This paper is dedicated to Professor Krishnaswami Alladi on the occasion of his 60th birthday

Abstract Using the theory of functions of several variables and q-calculus, we prove an expansion theorem for the analytic function in several variables which satisfies a system of q-partial differential equations. Some curious applications of this expansion theorem to q-series are discussed. In particular, an extension of Andrews' transformation formula for the q-Lauricella function is given.

Keywords q-Partial derivative · q-Partial differential equations · q-Integrals Analytic function

2010 Mathematics Subject Classification 05A30 · 33D05 · 33D15 · 32A05 32A10

1 Introduction and preliminaries

The q-analog of the partial differential equation $f_x = f_y$ was studied by us in [6, 8]. This investigation led us to develop a systematic method of deriving q-formulas, and

---

This work was supported by the National Natural Science Foundation of China (Grant No. 11571114) and Science and Technology Commission of Shanghai Municipality (Grant No. 13dz2260400).

Z.-G. Liu (✉)
Department of Mathematics, East China Normal University,
Shanghai 200241, P.R. China
e-mail: liuzg@hotmail.com

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_25

many interesting results in q-series are proved with this method [6–8, 10–12]. In particular, in [8, Proposition 10.2] we establish a general q-transformation formula which includes Watson's q-analog of Whipple's theorem as a special case.

In [9] we further investigated the q-analog of the partial differential equation $f_x = \alpha f_y$, where $\alpha \neq 0$.

In this paper, we continue our investigation to discuss the q-extension of the partial differential equation $\beta f_x = \alpha f_y$, where α, β are two nonzero complex numbers.

The motivation of this paper is to find the q-extension of $f(\alpha x + \beta y)$ through a q-partial differential equation.

As usual, we use $\mathbb{C}$ to denote the set of all complex numbers and $\mathbb{C}^k = \mathbb{C} \times \cdots \times \mathbb{C}$ the set of all k-dimensional complex numbers.

For $a \in \mathbb{C}$ and any positive integer n, we define the q-shifted factorial as

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

and it is understood that $(a; q)_0 = 1$. For $|q| < 1$, we further define the q-shifted factorial $(a; q)_\infty$ by

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n = \prod_{k=0}^{\infty} (1 - aq^k),$$

since this infinite product converges when $|q| < 1$.

If m is a positive integer, we define the multiple q-shifted factorial as follows

$$(a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n ... (a_m; q)_n,$$

where n is an integer or ∞.

Unless otherwise stated, we suppose throughout that $|q| < 1$, which ensures that all the sums and products appear in the paper converge.

The basic hypergeometric series or the q-hypergeometric series ${}_r\phi_s(\cdot)$ are defined as

$${}_r\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n.$$

We now introduce some basic concepts in the q-calculus. The q-derivative was introduced by Leopold Schendel [14] in 1877 and Frank Hilton Jackson [5] in 1908, which is the q-analog of the ordinary derivative.

Definition 1.1 *If q is a complex number, then, for any function $f(x)$ of one variable, the q-derivative of $f(x)$ with respect to x, is defined as*

$$D_q\{f(x)\} = \frac{f(x) - f(qx)}{x},$$

and we further define $D_q^0\{f\} = f$, *and for* $n \geq 1$, $D_q^n\{f\} = D_q\{D_q^{n-1}\{f\}\}$.

Now we give the definitions of the q-partial derivative and the q-partial differential equations.

Definition 1.2 *A q-partial derivative of a function of several variables is its q-derivative with respect to one of those variables, regarding other variables as constants. The q-partial derivative of a function f with respect to the variable x is denoted by* $\partial_{q,x}\{f\}$.

Definition 1.3 *A q-partial differential equation is an equation that contains unknown multivariable functions and their q-partial derivatives.*

The Gaussian binomial coefficients also called the q-binomial coefficients are the q-analogs of the binomial coefficients, which are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

We now introduce the homogeneous polynomials $\Phi_n^{(\alpha,\beta)}(x, y|q)$ in the following definition.

Definition 1.4 *The homogeneous polynomials* $\Phi_n^{(\alpha,\beta)}(x, y|q)$ *are defined by*

$$\Phi_n^{(\alpha,\beta)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\alpha;q)_k (\beta;q)_{n-k} x^k y^{n-k}.$$

When $\alpha = \beta = 0$, the homogeneous polynomials $\Phi_n^{(\alpha,\beta)}(x, y|q)$ reduce to the homogeneous Rogers–Szegő polynomials (see, for example [8])

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

If we set $\beta = 0$ in $\Phi_n^{(\alpha,\beta)}(x, y|q)$, we can obtain the homogeneous Hahn polynomials (see, for example [9]) which are given by

$$\Phi_n^{(\alpha)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\alpha;q)_k x^k y^{n-k}.$$

If we set $\alpha = \beta$ in $\Phi_n^{(\alpha,\beta)}(x, y|q)$, we can obtain the homogeneous continuous q-ultraspherical polynomials

$$C_n(x, y; \beta|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\beta; q)_k (\beta; q)_{n-k} x^k y^{n-k}.$$

The polynomials $\Phi_n^{(\alpha,\beta)}(x, 1|q)$ have been studied by Verma and Jain [16] among others. It is obvious that $\Phi_n^{(\alpha,\beta)}(x, y|q) = y^n \Phi_n^{(\alpha,\beta)}(x/y, 1|q)$, but an important difference between $\Phi_n^{(\alpha,\beta)}(x, 1|q)$ and $\Phi_n^{(\alpha,\beta)}(x, y|q)$ is that the latter satisfies the following q-partial differential equation, which does not appear in the literature before.

Proposition 1.5 *The homogeneous polynomials $\Phi_n^{(\alpha,\beta)}(x, y|q)$ satisfy the q-partial differential equation*

$$\begin{aligned} &\partial_{q,x} \left\{ \Phi_n^{(\alpha,\beta)}(x, y|q) - \beta \Phi_n^{(\alpha,\beta)}(x, qy|q) \right\} \\ &= \partial_{q,y} \left\{ \Phi_n^{(\alpha,\beta)}(x, y|q) - \alpha \Phi_n^{(\alpha,\beta)}(qx, y|q) \right\}. \end{aligned}$$

Proof. Using the q-identity, $\partial_{q,x}\{x^k\} = (1 - q^k)x^{k-1}$, we easily deduce that

$$\begin{aligned} &\partial_{q,x} \left\{ \Phi_n^{(\alpha,\beta)}(x, y|q) - \beta \Phi_n^{(\alpha,\beta)}(x, qy|q) \right\} \\ &= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\alpha; q)_k (1 - q^k)(\beta; q)_{n+1-k} x^{k-1} y^{n-k}. \end{aligned}$$

In the same way, using the identity, $\partial_{q,y}\{y^{n-k}\} = (1 - q^{n-k})y^{n-k-1}$, we conclude that

$$\begin{aligned} &\partial_{q,y} \left\{ \Phi_n^{(\alpha,\beta)}(x, y|q) - \alpha \Phi_n^{(\alpha,\beta)}(qx, y|q) \right\} \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q (\alpha; q)_{k+1} (\beta; q)_{n-k} (1 - q^{n-k}) x^k y^{n-k-1}. \end{aligned}$$

If we make the variable change $k + 1 \to k$ in the right-hand side of the above equation, we can find that

$$\begin{aligned} &\partial_{q,y} \left\{ \Phi_n^{(\alpha,\beta)}(x, y|q) - \alpha \Phi_n^{(\alpha,\beta)}(qx, y|q) \right\} \\ &= \sum_{k=1}^{n} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q (\alpha; q)_k (\beta; q)_{n+1-k} (1 - q^{n-k+1}) x^{k-1} y^{n-k}. \end{aligned}$$

From the definition of the q-binomial coefficients, it is easy to verify that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q (1 - q^k) = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q (1 - q^{n-k+1}).$$

Combining the above three equations, we complete the proof of Proposition 1.5. □

Definition 1.6 *If $f(x_1,\ldots,x_k)$ is a k-variable function, then, for $j=1,2,\ldots,k$, the q-shift operator η_{x_j} on the variable x_j is defined as*

$$\eta_{x_j}\{f(x_1,\ldots,x_k)\}=f(x_1,\ldots,x_{j-1},qx_j,x_{j+1},\ldots,x_k)$$

The principal result of this paper is the following expansion theorem for the analytic functions in several variables.

Theorem 1.7 *If $f(x_1,y_1,\ldots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,\cdots,0)\in\mathbb{C}^{2k}$, then, f can be expanded in terms of*

$$\Phi_{n_1}^{(a_1,b_1)}(x_1,y_1|q)\cdots\Phi_{n_k}^{(a_k,b_k)}(x_k,y_k|q)$$

if and only if f satisfies the following system of q-partial differential equations:

$$\partial_{q,x_j}(1-b_j\eta_{y_j})\{f\}=\partial_{q,y_j}(1-a_j\eta_{x_j})\{f\},\quad j\in\{1,2,\ldots,k\}.$$

Theorem 1.7 is a very powerful tool for proving q-formulas, which allows us to derive some deep q-formulas. The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.7. In Section 3, we will illustrate our approach by using Theorem 1.7 to prove two q-formulas. Andrews [2] proved the following transformation formula for the q-Lauricella function.

Proposition 1.8 *For* $\max\{|a|,|c|,|y_1|,\ldots,|y_k|\}<1$, *we have*

$$\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{(a;q)_{n_1+\cdots+n_k}(\beta_1;q)_{n_1}\cdots(\beta_k;q)_{n_k}y_1^{n_1}\cdots y_k^{n_k}}{(c;q)_{n_1+\cdots+n_k}(q;q)_{n_1}\cdots(q;q)_{n_k}}$$
$$=\frac{(a,\beta_1y_1,\ldots,\beta_ky_k;q)_\infty}{(c,y_1,\ldots,y_k;q)_\infty}{}_{k+1}\phi_k\left(\begin{matrix}c/a,y_1,\ldots,y_k\\ \beta_1y_1,\ldots,\beta_ky_k\end{matrix};q,a\right).$$

In Section 4, we extend the Andrews formula to the following q-formula by using Theorem 1.7.

Theorem 1.9 *If* $\max\{|a|,|c|,|x_1|,|y_1|,\ldots,|x_k|,|y_k|\}<1$, *then, we have*

$$\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{(a;q)_{n_1+\cdots+n_k}\Phi_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)\cdots\Phi_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q)}{(c;q)_{n_1+\cdots+n_k}(q;q)_{n_1}\cdots(q;q)_{n_k}}$$
$$=\frac{(a,\alpha_1x_1,\beta_1y_1,\ldots\alpha_kx_k,\beta_ky_k;q)_\infty}{(c,x_1,y_1,\ldots,x_k,y_k;q)_\infty}$$
$$\times{}_{2k+1}\phi_{2k}\left(\begin{matrix}c/a,x_1,y_1,\ldots,x_k,y_k\\ \alpha_1x_1,\beta_1y_1,\ldots,\alpha_kx_k,\beta_ky_k\end{matrix};q,a\right).$$

When $x_1=x_2=\ldots=x_k=0$, Theorem 1.9 immediately reduces to Andrews' formula in Proposition 1.8, and when $\beta_1=\beta_2=\ldots=\beta_k=0$, Theorem 1.9 becomes

[9, Theorem 6.1]. Theorem 1.9 may be regarded as a multilinear generating function for $\Phi_n^{(a,b)}(x, y|q)$. In Section 5, we prove another multilinear generating function for $\Phi_n^{(a,b)}(x, y|q)$. In Section 6, we will use Theorem 1.7 to derive a q-integral formula.

2 The proof of Theorem 1.7

To prove Theorem 1.7, we need the following fundamental property of several complex variables (see, for example [13, p. 5, Proposition 1]).

Proposition 2.1 *If $f(x_1, x_2, \ldots, x_k)$ is analytic at the origin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely convergent power series,*

$$f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k=0}^{\infty} \lambda_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Now we begin to prove Theorem 1.7 with the help of Proposition 2.1.

Proof. This theorem can be proved by induction. We first prove the theorem for the case $k = 1$. Since f is analytic at $(0, 0)$, we know that f can be expanded in an absolutely convergent power series in a neighborhood of $(0, 0)$. Thus there exists a sequence $\lambda_{k,l}$ independent of x_1 and y_1 such that

$$f(x_1, y_1) = \sum_{k,l=0}^{\infty} \lambda_{k,l}(a_1; q)_k (b_1; q)_l \begin{bmatrix} k+l \\ k \end{bmatrix}_q x_1^k y_1^l. \tag{2.1}$$

Substituting this into the q-partial differential equation, $\partial_{q,x_1}\{f(x_1, y_1) - b_1 f(x_1, qy_1)\} = \partial_{q,y_1}\{f(x_1, y_1) - a_1 f(qx_1, y_1)\}$, using the identities, $\partial_{q,x_1}\{x_1^k\} = (1-q^k)$ x_1^{k-1} and $\partial_{q,y_1}\{y_1^l\} = (1-q^l)y_1^{l-1}$, we find that

$$\begin{aligned} &\sum_{k,l=0}^{\infty} \lambda_{k,l}(a_1; q)_k (b_1; q)_{l+1} (1-q^k) \begin{bmatrix} k+l \\ k \end{bmatrix}_q x_1^{k-1} y_1^l \\ &= \sum_{k,l=0}^{\infty} \lambda_{k,l}(a_1; q)_{k+1} (b_1; q)_l (1-q^l) \begin{bmatrix} k+l \\ k \end{bmatrix}_q x_1^k y_1^{l-1}. \end{aligned}$$

Equating the coefficients of $x^{k-1}y^l$ on both sides of the above equation, we deduce that

$$\lambda_{k,l}(a_1; q)_k (b_1; q)_{l+1} \frac{(q; q)_{k+l}}{(q; q)_{k-1}(q; q)_l} = \lambda_{k-1,l+1}(a_1; q)_k (b_1; q)_{l+1} \frac{(q; q)_{k+l}}{(q; q)_{k-1}(q; q)_l}.$$

It follows that $\lambda_{k,l} = \lambda_{k-1,l+1}$. Iterating this relation $(k-1)$ times, we deduce that $\lambda_{k,l} = \lambda_{0,l+k}$. Substituting this into (2.1), we arrive at

$$f(x_1,y_1)=\sum_{k,l=0}^{\infty}\lambda_{0,k+l}(a_1;q)_k(b_1;q)_l\begin{bmatrix}k+l\\ k\end{bmatrix}_q x_1^k y_1^l.$$

Making the variable change $n=k+l$ and interchanging the order of summation, we find that

$$f(x_1,y_1)=\sum_{n=0}^{\infty}\lambda_{0,n}\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q (a;q)_k(b;q)_{n-k}x^k y^{n-k}.$$

Conversely, if $f(x_1,y_1)$ can be expanded in terms of $\Phi_n^{(a,b)}(x_1,y_1|q)$, then using Proposition 1.5, we find that

$$\partial_{q,x_1}\{f(x_1,y_1)-b_1 f(x_1,qy_1)\}=\partial_{q,y_1}\{f(x_1,y_1)-a_1 f(qx_1,y_1)\}.$$

We conclude that Theorem 1.7 holds when $k=1$.

Now, we assume that the theorem is true for the case $k-1$ and consider the case k. If we regard $f(x_1,y_1,\ldots,x_k,y_k)$ as a function of x_1 and y_1, then f is analytic at $(0,0)$ and satisfies

$$\partial_{q,x_1}\{f(x_1,y_1)-b_1 f(x_1,qy_1)\}=\partial_{q,y_1}\{f(x_1,y_1)-a_1 f(qx_1,y_1)\}.$$

Thus, there exists a sequence $\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$ independent of x_1 and y_1 such that

$$f(x_1,y_1,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\Phi_{n_1}^{(a_1,b_1)}(x_1,y_1|q). \tag{2.2}$$

Setting $x_1=0$ in the above equation and using $\Phi_{n_1}^{(a_1,b_1)}(0,y_1|q)=(b_1;q)_{n_1}y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}(b_1;q)_{n_1}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)y_1^{n_1}.$$

Using the Maclaurin expansion theorem, we immediately deduce that

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\left.\frac{\partial^{n_1}f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{(b_1;q)_{n_1}n_1!\partial {y_1}^{n_1}}\right|_{y_1=0}.$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k}$, from the above equation, we know that $c_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k-2}$. Substituting (2.2) into the q-partial differential equations in Theorem 1.7, we find that for $j=2,\ldots,k$,

$$\sum_{n_1=0}^{\infty} \partial_{q,x_j}(1-b_j\eta_{y_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\Phi_{n_1}^{(a_1,b_1)}(x_1,y_1|q)$$
$$=\sum_{n_1=0}^{\infty} \partial_{q,y_j}(1-a_j\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\Phi_{n_1}^{(a_1,b_1)}(x_1,y_1|q).$$

By equating the coefficients of $\Phi_{n_1}^{(a_1,b_1)}(x_1,y_1|q)$ in the above equation, we find that for $j=2,\ldots,k$,

$$\partial_{q,x_j}(1-b_j\eta_{y_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$$
$$=\partial_{q,y_j}(1-a_j\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}.$$

Thus by the inductive hypothesis, there exists a sequence $\lambda_{n_1,n_2,\ldots,n_k}$ independent of $x_2,y_2,\ldots,x_k,y_k$ (of course independent of x_1 and y_1) such that

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\sum_{n_2,\ldots,n_k=0}^{\infty}\lambda_{n_1,n_2,\ldots,n_k}$$
$$\times\Phi_{n_2}^{(a_2,b_2)}(x_2,y_2|q)\ldots\Phi_{n_k}^{(a_k,b_k)}(x_k,y_k|q).$$

Substituting this equation into (2.2), we complete the proof of the theorem. □

To determine if a given function is an analytic function in several complex variables, one can use the following theorem (see, for example, [15, p. 28]).

Theorem 2.2 (Hartogs' theorem). *If a complex valued function $f(z_1,z_2,\ldots,z_n)$ is holomorphic (analytic) in each variable separately in a domain $U\in\mathbb{C}^n$, then, it is holomorphic (analytic) in U.*

3 Some generating functions for $\Phi_n^{(a,b)}(x,y|q)$

The following proposition is equivalent to the formula [16, Equation (2.1)]. Verma and Jain proved this formula by using many known results in q-series and the technique of interchanging the order of summation. In this section, we will use Theorem 1.7 to prove this proposition and the proof is quite different from that of Verma and Jain.

Theorem 3.1 *If $m\ge 0$ is an integer and* $\max\{|xt|,|yt|\}<1$, *then, we have*

$$\sum_{n=0}^{\infty}\Phi_{n+m}^{(a,b)}(x,y|q)\frac{t^{n+m}}{(q;q)_n}=\frac{(axt,byt;q)_\infty}{(xt,yt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q^{-m},xt,yt\\ axt,byt\end{matrix};q,q\right).$$

Proof. Denote the right-hand side of the above equation by $f(x, y)$. It is easily seen that $f(x, y)$ is an analytic function of x and y, for $\max\{|xt|, |yt|\} < 1$. Thus, $f(x, y)$ is analytic at $(0, 0) \in \mathbb{C}^2$. A direct computation shows that

$$\begin{aligned}
&\partial_{q,x}\{f(x, y) - bf(x, qy)\} = \partial_{q,y}\{f(x, y) - af(qx, y)\} \\
&= (1-a)(1-b)t\frac{(axtq, bytq; q)_\infty}{(xt, yt; q)_\infty} {}_3\phi_2\left(\begin{matrix} q^{-m}, xt, yt \\ axtq, bytq \end{matrix}; q, q^2\right).
\end{aligned}$$

Thus, by Theorem 1.7, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{(axt, byt; q)_\infty}{(xt, yt; q)_\infty} {}_3\phi_2\left(\begin{matrix} q^{-m}, xt, yt \\ axt, byt \end{matrix}; q, q\right) = \sum_{n=0}^{\infty} \lambda_n \Phi_n^{(a,b)}(x, y|q).$$

Putting $x = 0$ in this equation, using $\Phi_n^{(a,b)}(0, y|q) = (b; q)_n y^n$, we find that

$$\frac{(byt; q)_\infty}{(yt; q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-m}, yt \\ byt \end{matrix}; q, q\right) = \sum_{n=0}^{\infty} \lambda_n (b; q)_n y^n.$$

Equating the coefficients of y^n and using the q-binomial theorem, we find that

$$\lambda_n = \frac{t^n}{(q; q)_n} \sum_{k=0}^{m} \frac{(q^{-m}; q)_k}{(q; q)_k} q^{k(1+n)} = \frac{t^n}{(q; q)_{n-m}}.$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda_n \Phi_n^{(a,b)}(x, y|q) &= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_{n-m}} \Phi_n^{(a,b)}(x, y|q) \\
&= \sum_{n=m}^{\infty} \frac{t^n}{(q; q)_{n-m}} \Phi_n^{(a,b)}(x, y|q).
\end{aligned}$$

Making the variable change $n - m$ to n, we complete the proof of the theorem. □

When $m = 0$, Theorem 3.1 reduces to the following proposition, which can be obtained easily by multiplying two copies of the q-binomial theorem together.

Proposition 3.2 *If* $\max\{|xt|, |yt|\} < 1$, *then, we have the formula*

$$\sum_{n=0}^{\infty} \Phi_n^{(a,b)}(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(axt, byt; q)_\infty}{(xt, yt; q)_\infty}.$$

Next, we will use Theorem 1.7 to prove the following theorem.

Theorem 3.3 *If* $\max\{|c|, |ab|, |tx|, |ty|\} < 1$ *and* $cd = ab$, *then, we have*

$$\sum_{n=0}^{\infty} \frac{(c;q)_n \Phi_n^{(a,b)}(x,y|q)t^n}{(q,ab;q)_n} = \frac{(c,axt,byt;q)_\infty}{(ab,tx,ty;q)_\infty} {}_3\phi_2\left(\begin{matrix} d, xt, yt \\ axt, byt \end{matrix}; q, c\right).$$

Proof. If we use $f(x, y)$ to denote the right-hand side of the above equation, then, using the ratio test, we find that $f(x, y)$ is an analytic function of x and y, for $\max\{|c|, |ab|, |xt|, |yt|\} < 1$. Thus, $f(x, y)$ is analytic at $(0, 0) \in \mathbb{C}^2$. A direct computation shows that

$$\begin{aligned} &\partial_{q,x}\{f(x,y) - bf(x,qy)\} = \partial_{q,y}\{f(x,y) - af(qx,y)\} \\ &= (1-a)(1-b)t\frac{(c,axtq,bytq;q)_\infty}{(ab,tx,ty;q)_\infty} {}_3\phi_2\left(\begin{matrix} d, xt, yt \\ qaxt, qbyt \end{matrix}; q, qc\right). \end{aligned}$$

Thus, by Theorem 1.7, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{(c,axt,byt;q)_\infty}{(ab,tx,ty;q)_\infty} {}_3\phi_2\left(\begin{matrix} d, xt, yt \\ axt, byt \end{matrix}; q, c\right) = \sum_{n=0}^{\infty} \lambda_n \Phi_n^{(a,b)}(x,y|q).$$

Setting $y = 0$ in this equation, using $\Phi_n^{(a,b)}(x, 0|q) = (a;q)_n x^n$, we find that

$$\frac{(c,axt;q)_\infty}{(ab,tx;q)_\infty} {}_2\phi_1\left(\begin{matrix} d, xt \\ axt \end{matrix}; q, c\right) = \sum_{n=0}^{\infty} \lambda_n (a;q)_n x^n.$$

Using the Heine transformation formula and noting that $ab = cd$, we have

$$\frac{(c,axt;q)_\infty}{(ab,tx;q)_\infty} {}_2\phi_1\left(\begin{matrix} d, xt \\ axt \end{matrix}; q, c\right) = {}_2\phi_1\left(\begin{matrix} a, c \\ ab \end{matrix}; q, xt\right).$$

Comparing the above two equations, we find that $(q, ab; q)_n \lambda_n = (c; q)_n t^n$. This completes the proof of Theorem 3.3. □

4 The proof of Theorem 1.9

Proof. If we use $f(x_1, y_1, \ldots, x_k, y_k)$ to denote the right-hand side of the equation in Theorem 1.9, then, using the ratio test, we find that f is an analytic function of $x_1, y_1, \ldots, x_k, y_k$ for $\max\{|a|, |c|, |x_1|, |y_1|, \ldots, |x_k|, |y_k|\} < 1$. By a direct computation, we deduce that for $j = 1, 2 \ldots, k$,

$$\begin{aligned}
&\partial_{q,x_j}(1-\beta_j\eta_{y_j})\{f\} = \partial_{q,y_j}(1-\alpha_j\eta_{x_j})\{f\}\\
&= \frac{(1-\alpha_j)(1-\beta_j)(a, \alpha_1 x_1, \beta_1 y_1, \ldots, q\alpha_j x_j, q\beta_j y_j, \ldots \alpha_k x_k, \beta_k y_k; q)_\infty}{(c, x_1, y_1, \ldots, x_j, y_j, \ldots, x_k, y_k; q)_\infty}\\
&\quad \times {}_{2k+1}\phi_{2k}\left(\begin{matrix} c/a, x_1, y_1, \ldots, x_j, y_j, \ldots, x_k, y_k \\ \alpha_1 x_1, \beta_1 y_1, \ldots, q\alpha_j x_j, q\beta_j y_j, \ldots, \alpha_k x_k, \beta_k y_k \end{matrix}; q, qa\right).
\end{aligned}$$

Thus, by Theorem 1.7, there exists a sequence $\{\lambda_{n_1,\ldots,n_k}\}$ independent of $x_1, y_1, \ldots, x_k, y_k$ such that

$$\begin{aligned}
&f(x_1, y_1, \ldots, x_k, y_k)\\
&= \sum_{n_1,\ldots,n_k=0}^{\infty} \lambda_{n_1,\ldots,n_k} \Phi_{n_1}^{(\alpha_1,\beta_1)}(x_1, y_1|q)) \cdots \Phi_{n_k}^{(\alpha_k,\beta_k)}(x_k, y_k|q).
\end{aligned} \tag{4.1}$$

Setting $x_1 = x_2 \cdots = x_k = 0$ in this equation, we immediately deduce that

$$\begin{aligned}
&\sum_{n_1,\ldots,n_k=0}^{\infty} \lambda_{n_1,\ldots,n_k} (\beta_1; q)_{n_1} \cdots (\beta_k; q)_{n_k} y_1^{n_1} \cdots y_k^{n_k}\\
&= \frac{(a, \beta_1 y_1, \ldots, \beta_k y_k; q)_\infty}{(c, y_1, \ldots, y_k; q)_\infty} {}_{k+1}\phi_k\left(\begin{matrix} c/a, y_1, \ldots, y_k \\ \beta_1 y_1, \ldots, \beta_k y_k \end{matrix}; q, a\right).
\end{aligned} \tag{4.2}$$

Applying the Andrews identity in Proposition 1.8 to the right-hand side of the above equation, we obtain

$$\begin{aligned}
&\sum_{n_1,\ldots,n_k=0}^{\infty} \lambda_{n_1,\ldots,n_k} (\beta_1; q)_{n_1} \cdots (\beta_k; q)_{n_k} y_1^{n_1} \cdots y_k^{n_k}\\
&= \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(a; q)_{n_1+\cdots+n_k} (\beta_1; q)_{n_1} \cdots (\beta_k; q)_{n_k} y_1^{n_1} \cdots y_k^{n_k}}{(c; q)_{n_1+\cdots+n_k} (q; q)_{n_1} \cdots (q; q)_{n_k}}.
\end{aligned} \tag{4.3}$$

Equating the coefficients of $y_1^{n_1} \cdots y_k^{n_k}$ on both sides of the equation, we find that

$$\lambda_{n_1,\ldots,n_k} = \frac{(a; q)_{n_1+n_2+\cdots+n_k}}{(c; q)_{n_1+\cdots+n_k} (q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_k}}.$$

Substituting this into (4.1), we complete the proof of Theorem 1.9. □

5 Another multilinear generating function

In this section, we will discuss some applications of Theorem 1.7 to q-beta integrals. The Jackson q-integral of the function $f(x)$ from a to b is defined as

$$\int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty}[bf(bq^n) - af(aq^n)]q^n.$$

Using the q-integral notation, one can write some q-formulas in more compact forms. For example, Sears' nonterminating extension of the q-Saalschütz summation can be rewritten in the following beautiful form, which was first noticed by Al-Salam and Verma [1].

Proposition 5.1 *If there are no zero factors in the denominator of the integral and* $\max\{|a|, |b|, |cx|, |cy|, |ax/y|, |by/x|\} < 1$*, then, we have*

$$\int_x^y \frac{(qz/x, qz/y, abcz; q)_\infty}{(az/y, bz/x, cz; q)_\infty} d_qz = \frac{(1-q)y(q, x/y, qy/x, ab, acx, bcy; q)_\infty}{(ax/y, by/x, a, b, cx, cy; q)_\infty}.$$

Using this proposition we can find the following q-integral representation for $\Phi_k^{(a,b)}(x, y|q)$.

Proposition 5.2 *If there are no zero factors in the denominator of the integral, then, we have*

$$\Phi_k^{(a,b)}(x, y|q) = \frac{(ab; q)_k(a, b, by/x, ax/y; q)_\infty}{(1-q)y(q, ab, x/y, qy/x; q)_\infty} \int_x^y \frac{(qz/x, qz/y; q)_\infty z^k}{(bz/x, az/y; q)_\infty} d_qz.$$

Proof. Using the generating function for $\Phi_n^{(a,b)}(x, y|q)$ in Proposition 3.2, we find that

$$\frac{(acx, bcy; q)_\infty}{(cx, cy; q)_\infty} = \sum_{n=0}^{\infty} \frac{\Phi_n^{(a,b)}(x, y|q)c^n}{(q; q)_n}.$$

It follows that

$$\int_x^y \frac{(qz/x, qz/y, abcz; q)_\infty}{(az/y, bz/x, cz; q)_\infty} d_qz = \frac{(1-q)y(q, x/y, qy/x, ab; q)_\infty}{(ax/y, by/x, a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{\Phi_n^{(a,b)}(x, y|q)c^n}{(q; q)_n}.$$

Applying the q-partial derivative operator $\partial_{q,c}^n$ to act both sides of the equation, we deduce that

$$\begin{aligned}&(ab;q)_k\int_x^y \frac{(qz/x,qz/y,q^kabcz;q)_\infty}{(az/y,bz/x,cz;q)_\infty}d_qz\\&=\frac{(1-q)y(q,x/y,qy/x,ab;q)_\infty}{(ax/y,by/x,a,b;q)_\infty}\sum_{n=k}^\infty \frac{\Phi_n^{(a,b)}(x,y|q)c^{n-k}}{(q;q)_{n-k}}.\end{aligned}$$

Setting $c=0$ in this equation, we complete the proof of Proposition 5.2. □

Putting $a=b=0$ in Proposition 5.2, we immediately find the following q-integral representation for the homogeneous Rogers–Szegő polynomials.

Proposition 5.3 *We have*

$$h_k(x,y|q)=\frac{1}{(1-q)y(q,x/y,qy/x;q)_\infty}\int_x^y (qz/x,qz/y;q)_\infty z^k d_qz.$$

Using Proposition 5.2 and Theorem 1.7, we can prove the following theorem.

Theorem 5.4 *For* $\max_{j\in\{1,\ldots,k\}}\{|xt|,|yt|,|xu_j|,|xv_j|,|yu_j|,|yv_j|\}<1$, *we have*

$$\begin{aligned}&\sum_{n_1,\ldots,n_k=0}^\infty \frac{\Phi_{n_1}^{(\alpha_1,\beta_1)}(u_1,v_1|q)\cdots\Phi_{n_k}^{(\alpha_k,\beta_k)}(u_k,v_k|q)}{(q;q)_{n_1}\cdots(q;q)_{n_k}}\\&\quad\times\int_x^y z^{n_1+\cdots+n_k}\frac{(qz/x,qz/y,\gamma zt;q)_\infty}{(bz/x,az/y,zt;q)_\infty}d_qz\\&=\int_x^y \frac{(qz/x,qz/y,\gamma zt,\alpha_1u_1z,\beta_1v_1z,\ldots,\alpha_ku_kz,\beta_kv_kz;q)_\infty}{(bz/x,az/y,zt,zu_1,zv_1,\ldots,zu_k,zv_k;q)_\infty}d_qz.\end{aligned}$$

Proof. If we use $f(u_1,v_1,\ldots,u_k,v_k)$ to denote the right-hand side of the equation in Theorem 5.4, then, using the ratio test, we find that f is analytic function at $(0,\ldots,0)\in\mathbb{C}^{2k}$. For simplicity, we use $I(\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k;z)$ to denote the integrand function.

By a direct computation, we easily deduce that for $j=1,2\ldots,k$,

$$\begin{aligned}&\partial_{q,u_j}(1-\beta_j\eta_{v_j})\{f\}=\partial_{q,v_j}(1-\alpha_j\eta_{u_j})\{f\}\\&=(1-\alpha_j)(1-\beta_j)\int_x^y zI(\alpha_1,\beta_1,\ldots,q\alpha_j,q\beta_j,\ldots\alpha_k,\beta_k;z)d_qz.\end{aligned}$$

Thus, by Theorem 1.7, there exists a sequence $\{c_{n_1,\ldots,n_k}\}$ independent of $u_1,v_1,\ldots,u_k,v_k$ such that

$$\begin{aligned}&f(u_1,v_1,\ldots,u_k,v_k)\\&=\sum_{n_1,\ldots,n_k=0}^\infty c_{n_1,\ldots,n_k}\Phi_{n_1}^{(\alpha_1,\beta_1)}(u_1,v_1|q))\cdots\Phi_{n_k}^{(\alpha_k,\beta_k)}(u_k,v_k|q).\end{aligned}\tag{5.1}$$

Putting $v_1 = \cdots = v_k = 0$ in this equation and noting that $\Phi_{n_j}^{(\alpha_j,\beta_j)}(u_j, 0|q) = (\alpha_j; q)_{n_j} u_j^{n_j}$ for $j = 1, 2 \ldots, k$, we find that

$$\sum_{n_1,\ldots,n_k=0}^{\infty} c_{n_1,\ldots,n_k} (\alpha_1; q)_{n_1} \cdots (\alpha_k; q)_{n_k} u_1^{n_1} \cdots u_k^{n_k}$$
$$= \int_x^y \frac{(qz/x, qz/y, \gamma zt, \alpha_1 u_1 z, \ldots, \alpha_k u_k z; q)_\infty}{(bz/x, az/y, zt, zu_1, \ldots, zu_k; q)_\infty} d_q z.$$

Applying the operator $\partial_{q,u_1}^{n_1} \cdots \partial_{q,u_k}^{n_k}$ to act both sides of the equation and then setting $u_1 = \cdots = u_k = 0$, we deduce that

$$c_{n_1,\ldots,n_k} (\alpha_1; q)_{n_1} \cdots (\alpha_k; q)_{n_k} (q; q)_{n_1} \cdots (q; q)_{n_k}$$
$$= (\alpha_1; q)_{n_1} \cdots (\alpha_k; q)_{n_k} \int_x^y z^{n_1+\cdots+n_k} \frac{(qz/x, qz/y, \gamma zt; q)_\infty}{(bz/x, az/y, zt; q)_\infty} d_q z.$$

It follows that

$$c_{n_1,\ldots,n_k} (q; q)_{n_1} \cdots (q; q)_{n_k}$$
$$= \int_x^y z^{n_1+\cdots+n_k} \frac{(qz/x, qz/y, \gamma zt; q)_\infty}{(bz/x, az/y, zt; q)_\infty} d_q z.$$

Substituting this equation into (5.1), we complete the proof of Theorem 5.4. □

Setting $a = b = 0$ in Theorem 5.4 and then equating the coefficients of t^m on both sides of the resulting equation, we conclude the following proposition, which is equivalent to [9, Theorem 7.3].

Proposition 5.5 *For* $\max_{j\in\{1,\ldots,k\}}\{|xt|, |yt|, |xu_j|, |xv_j|, |yu_j|, |yv_j|\} < 1$, *we have*

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{\Phi_{n_1}^{(\alpha_1,\beta_1)}(u_1, v_1|q) \cdots \Phi_{n_k}^{(\alpha_k,\beta_k)}(u_k, v_k|q)}{(q; q)_{n_1} \cdots (q; q)_{n_k}}$$
$$\times \int_x^y z^{m+n_1+\cdots+n_k} (qz/x, qz/y; q)_\infty d_q z$$
$$= \int_x^y \frac{z^m (qz/x, qz/y, \alpha_1 u_1 z, \beta_1 v_1 z, \ldots, \alpha_k u_k z, \beta_k v_k z; q)_\infty}{(zu_1, zv_1, \ldots, zu_k, zv_k; q)_\infty} d_q z.$$

6 A q-integral formula

The Andrews–Askey integral formula [3, Theorem 1] is stated in the following proposition.

Proposition 6.1 *If there are no zero factors in the denominator of the integral, then, we have*

$$\int_u^v \frac{(qx/u, qx/v; q)_\infty}{(cx, dx; q)_\infty} d_q x = \frac{(1-q)v(q, u/v, qv/u, cduv; q)_\infty}{(cu, cv, du, dv; q)_\infty}. \tag{6.1}$$

Applying $\partial_{q,c}^n$ to act both sides of (6.1) and then using the q-Leibniz rule, one can easily find the following proposition [17].

Proposition 6.2 *If there are no zero factors in the denominator of the integral, then, we have*

$$\int_u^v \frac{x^n (qx/u, qx/v; q)_\infty}{(cx, dx; q)_\infty} d_q x = \frac{(1-q)v(q, u/v, qv/u, cduv; q)_\infty}{(cu, cv, du, dv; q)_\infty} \times \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(cv, dv; q)_j}{(cduv; q)_j} u^j v^{n-j}.$$

The main result of this section is the following q-integral formula.

Theorem 6.3 *If there are no zero factors in the denominator of the integral, then, we have*

$$\begin{aligned} &\int_u^v \frac{(qx/u, qx/v, \alpha ax, \beta bx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} d_q x \\ &= \frac{(1-q)v(q, u/v, qv/u, cduv; q)_\infty}{(cu, cv, du, dv; q)_\infty} \\ &\quad \times \sum_{n=0}^\infty \frac{\Phi_n^{(\alpha,\beta)}(a, b|q)}{(q;q)_n} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(cv, dv; q)_j}{(cduv; q)_j} u^j v^{n-j}. \end{aligned}$$

Proof. If we use $I(a, b)$ to denote the q-integral in the above theorem, then it is easy to show that $I(a, b)$ is analytic near $(0, 0) \in \mathbb{C}^2$. A straightforward evaluation shows that

$$\begin{aligned} &\partial_{q,a}\{I(a, b) - \beta I(a, bq)\} = \partial_{q,b}\{I(a, b) - \alpha I(aq, b)\} \\ &= (1-\alpha)(1-\beta) \int_u^v \frac{x(qx/u, qx/v, \alpha qax, \beta qbx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} d_q x. \end{aligned}$$

Thus, by Theorem 1.7, there exists a sequence $\{\lambda_n\}$ independent of a and b such that

$$I(a,b)=\int_u^v \frac{(qx/u,qx/v,\alpha ax,\beta bx;q)_\infty}{(ax,bx,cx,dx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\lambda_n \Phi_n^{(\alpha,\beta)}(a,b|q). \tag{6.2}$$

Setting $b=0$ in the above equation and using $\Phi_n^{(\alpha,\beta)}(a,0|q)=(\alpha;q)_n a^n$, we immediately deduce that

$$\int_u^v \frac{(qx/u,qx/v,\alpha ax;q)_\infty}{(ax,cx,dx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\lambda_n(\alpha;q)_n a^n.$$

Applying $\partial_{q,a}^n$ to act both sides of the equation, setting $a=0$, and using Proposition 6.2, we obtain

$$\begin{aligned}\lambda_n &= \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty(q;q)_n}\\ &\quad\times\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q \frac{(cv,dv;q)_j}{(cduv;q)_j}u^j v^{n-j}.\end{aligned}$$

Substituting the above equation into (6.2), we complete the proof Theorem 6.3. □

Acknowledgements I am grateful to the anonymous referee for careful reading of the manuscript and many invaluable suggestions and comments.

References

1. W.A. Al-Salam, A. Verma, Some remarks on q-beta integral. Proc. Am. Math. Soc. **85**, 360–362 (1982)
2. G.E. Andrews, Summations and transformations for basic Appell series. J. Lond. Math. Soc. **2**(4), 618–622 (1972)
3. G.E. Andrews, R. Askey, Another q-extension of the beta function. Proc. Am. Math. Soc. **81**, 97–100 (1981)
4. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edn. (Cambridge University Press, Cambridge, 2004)
5. F.H. Jackson, On q-functions and a certain difference operator. Trans. R. Soc. Edin. **46**, 253–281 (1908)
6. Z.-G. Liu, Two q-difference equations and q-operator identities. J. Differ. Equ. Appl. **16**, 1293–1307 (2010)
7. Z.-G. Liu, An extension of the non-terminating ${}_6\phi_5$ summation and the Askey-Wilson polynomials. J. Differ. Equ. Appl. **17**, 1401–1411 (2011)
8. Z.-G Liu, *On the q-Partial Differential Equations and q-Series*, *the Legacy of Srinivasa Ramanujan*, Ramanujan Mathematical Society Lecture Notes Series (Ramanujan Mathematical Society, Mysore, 2013), pp. 213–250
9. Z.-G. Liu, A q-extension of a partial differential equation and the Hahn polynomials. Ramanujan J. **38**, 481–501 (2015)

10. Z.-G. Liu, J. Zeng, Two expansion formulas involving the Rogers-Szegő polynomials with applications. Int. J. Number Theory **11**, 507–525 (2015)
11. Z.-G. Liu, On a Reduction Formula for a Kind of Double q-Integrals. Symmetry **8**, 44 (2016)
12. Z.-G. Liu, Extensions of Ramanujan's reciprocity theorem and the Andrews-Askey integral. J. Math. Anal. Appl. **443**, 1110–1129 (2016)
13. B. Malgrange, *Lectures on Functions of Several Complex Variables* (Springer, Berlin, 1984)
14. L. Schendel, Zur theorie der functionen. J. Reine Angew. Math. **84**, 80–84 (1877)
15. J. Taylor, *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups*, Graduate Studies in Mathematics (American Mathematical Society, Providence, 2002)
16. A. Verma, V.K. Jain, Poisson kernel and multilinear generating functions of some orthogonal polynomials. J. Math. Anal. Appl. **146**, 333–352 (1990)
17. M. Wang, q-integral representation of the Al-Salam-Carlitz polynomials. Appl. Math. Lett. **22**, 943–945 (2009)

Asymmetric Generalizations of Schur's Theorem

Jeremy Lovejoy

For Krishna Alladi on his 60th birthday

Abstract We extend a theorem of Alladi and Gordon asymmetrically to overpartitions. As special cases, we find asymmetric generalizations of Schur's theorem and partition identities closely related to Capparelli's identity and the Alladi–Andrews dual of Göllnitz' theorem.

Keywords Partitions · Overpartitions · Schur's theorem · Capparelli's theorem
Weighted words

2010 Mathematics Subject Classification 11P81 · 11P84

1 Introduction and Statement of Results

1.1 Introduction

Recall that a partition λ of n is a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose sum is n. Like many papers on partition identities, this one begins with an influential theorem of Schur [15].

Theorem 1 (Schur). *Let $S(n)$ denote the number of partitions of n such that*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 6, & \text{if } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3, \\ 3, & \text{otherwise}. \end{cases} \tag{1.1}$$

J. Lovejoy (✉)
CNRS, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, France
e-mail: lovejoy@math.cnrs.fr

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_26

Then $S(n)$ is equal to the number of partitions of n into parts congruent to 1 *or* 5 *modulo* 6*.*

In terms of generating functions, Schur's theorem may be written

$$\sum_{n\geq 0} S(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}, \tag{1.2}$$

where we use the usual q-hypergeometric notation,

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \tag{1.3}$$

valid for $n \in \mathbb{N} \cup \infty$. Given the simple fact that

$$\frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty} = (-q;q^3)_\infty (-q^2;q^3)_\infty, \tag{1.4}$$

the $S(n)$ in Schur's theorem is also equal to the number of partitions of n into distinct parts not divisible by 3. From this perspective, Alladi and Gordon [7, 8] gave a generalization and refinement of Schur's theorem, which we now describe.

Consider the positive integers in the three colors a, b, and ab, with the order

$$ab < a < b, \tag{1.5}$$

so that the integers are ordered

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < \cdots . \tag{1.6}$$

Let $S(u, v, n)$ denote the number of three-colored partitions of n with no part 1_{ab}, u parts colored a or ab, v parts colored b or ab, and satisfying the difference conditions in the matrix

$$A = \begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \end{array}. \tag{1.7}$$

By this we mean that the entry (x, y) gives the minimal difference between λ_i of color x and λ_{i+1} of color y. Alladi and Gordon [7, 8] established the following elegant generating function for $S(u, v, n)$.

Theorem 2 (Alladi–Gordon). *We have*

$$\sum_{u,v,n\geq 0} S(u, v, n)a^u b^v q^n = (-aq;q)_\infty (-bq;q)_\infty. \tag{1.8}$$

Setting $q = q^3$, $a = aq^{-2}$ and $b = bq^{-1}$, the three-colored positive integers become the ordinary positive integers, with parts congruent to 0, 1, or 2 modulo 3 labeled ab, a, or b, respectively. The matrix of difference conditions in (1.7) becomes

$$\begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 3 & 5 & 4 \\ 4 & 3 & 5 \\ 5 & 4 & 6 \end{pmatrix} \end{array}, \tag{1.9}$$

which is equivalent to (1.1), and we recover Schur's theorem. In fact, we have a refinement of Schur's theorem, thanks to the extra parameters a and b.

Alladi and Gordon's treatment of Schur's theorem marked the beginning of the so-called *method of weighted words*, which would subsequently be used to find refinements and generalizations of partition identities such as those of Göllnitz [5], Capparelli [6], and Siladić [12], as well as to discover a number of new identities. For more on this, see [1, 2, 4].

It turns out that Theorem 2 is a special case of an identity for overpartitions. Recall that an overpartition is a partition in which the first occurrence of a given integer may be overlined. We consider overpartitions with the same three colors and the same ordering as in (1.5) and (1.6), allowing the first occurrence of a given colored integer to be overlined. We append the label d to the color of a non-overlined part. Let $\overline{S}(u, v, m, n)$ denote the number of overpartitions of n having m non-overlined parts, no part 1_{abd} or $\overline{1}_{ab}$, u parts having a in their color, v parts having b in their color, and satisfying the difference conditions in the matrix

$$\overline{A} = \begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix} \end{array}. \tag{1.10}$$

These overpartitions are also generated by a simple infinite product.

Theorem 3 (See [16]). *We have*

$$\sum_{u,v,m,n \geq 0} \overline{S}(u, v, m, n) a^u b^v d^m q^n = \frac{(-aq; q)_\infty (-bq; q)_\infty}{(adq; q)_\infty (bdq; q)_\infty}. \tag{1.11}$$

When $d = 0$ we recover Theorem 2. With the same substitutions as before, $q = q^3$, $a = aq^{-2}$ and $b = bq^{-1}$, the matrix $\overline{A}$ becomes

$$\begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\ \begin{pmatrix} 3 & 5 & 4 & 0 & 2 & 1 \\ 4 & 3 & 5 & 1 & 0 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 3 & 5 & 4 & 0 & 2 & 1 \\ 4 & 3 & 5 & 1 & 0 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix} \end{array}, \tag{1.12}$$

and we have a result known as *Schur's theorem for overpartitions*.

Theorem 4 (See [16]). *Let $\overline{S}(m,n)$ denote the number of overpartitions of n with m non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3, & \text{if } \lambda_{i+1} \text{ is overlined or if } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3, \\ 6, & \text{if } \lambda_{i+1} \text{ is overlined and } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3). \end{cases} \tag{1.13}$$

Then $\overline{S}(m,n)$ is equal to the number of overpartitions of n into parts not divisible by 3, m of which are non-overlined.

Note that when there are no non-overlined parts (i.e., $m = 0$) we recover Schur's theorem. See [11] for a proof of Theorem 4 using q-difference equations and [14] for a bijective proof.

1.2 Statement of Results

In this paper, we prove two asymmetric extensions of Theorem 2 to overpartitions. The word *asymmetric* refers to the fact that one of the terms in the denominator of (1.11) is missing from each of (1.14) and (1.15) below.

Theorem 5. *The following are true.*

(i) Let $\overline{S}_1(u,v,m,n)$ denote the number of overpartitions counted by $\overline{S}(u,v,m,n)$ where, in addition, the s smallest parts must be overlined, where s is the number of parts of color b or bd. Then

$$\sum_{u,v,m,n\geq 0} \overline{S}_1(u,v,m,n)a^u b^v d^m q^n = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(adq;q)_\infty}, \tag{1.14}$$

(ii) Let $\overline{S}_2(u,v,m,n)$ denote the number of overpartitions counted by $\overline{S}(u,v,m,n)$ where, in addition, the r smallest parts must be overlined, where r is the number of parts of color a or ad. Then

$$\sum_{u,v,m,n\geq 0} \overline{S}_2(u,v,m,n)a^u b^v d^m q^n = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(bdq;q)_\infty}. \tag{1.15}$$

Note that if $m = 0$ in either (1.14) or (1.15), we recover the Alladi–Gordon result in Theorem 2. Also note that although we recover the overpartitions in Theorem 3 if either of the extra conditions is omitted, Theorem 5 is not a special case of Theorem 3.

With the usual substitutions $q = q^3$, $a = aq^{-2}$, and $b = bq^{-1}$, we obtain a pair of results which may be compared with Schur's theorem and Schur's theorem for overpartitions.

Corollary 1. *For $j = 1$ or 2, let $\overline{S}_j(m, n)$ denote the number of overpartitions counted by $\overline{S}(m, n)$ in Theorem 4 with the extra condition that the smallest s parts are overlined, where s is the number of parts congruent to $3 - j$ modulo 3. Then $\overline{S}_j(m, n)$ is equal to the number of overpartitions of n into overlined parts not divisible by 3 and m non-overlined parts congruent to j modulo 3.*

We highlight two other special cases of Theorem 5, where the overpartitions become ordinary partitions.

Corollary 2. *Let $C(n)$ denote the number of partitions of n satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 5, & \text{if } \lambda_{i+1} \text{ is even or if } \lambda_{i+1} \equiv 5 \pmod 6 \text{ and } \lambda_i \equiv 0, 5 \pmod 6, \\ 11, & \text{if } \lambda_{i+1} \equiv 0 \pmod 6 \text{ and } \lambda_i \equiv 0, 5 \pmod 6, \end{cases} \tag{1.16}$$

and, in addition, having the s smallest parts even, where s is the number of parts congruent to 1 or 2 modulo 6. Then $C(n)$ is equal to the number of partitions of n into distinct parts not congruent to ± 1 modulo 6.

Corollary 3. *Let $G(n)$ denote the number of partitions of n satisfying the difference conditions in (1.16), and, in addition, having the r smallest parts even, where r is the number of parts congruent to 3 or 4 modulo 6. Then $G(n)$ is equal to the number of partitions of n into parts congruent 1, 2, or 4 modulo 6, where only parts congruent to 1 modulo 6 may repeat.*

Note that the partitions into distinct parts not congruent to ± 1 modulo 6 in Corollary 2 are precisely those in Capparelli's partition identity [6], while the partitions into parts 1, 2, or 4 modulo 6 in Corollary 3 are nearly those in the dual Göllnitz theorem due to Alladi and Andrews [3]. For other partitions related to Capparelli's identity, see [10].

The remainder of the paper is organized as follows. In the next section, we prove Theorem 5 using a q-series identity, reviewing the work on Schur's theorem and Schur's theorem for overpartitions along the way. The colored partitions are similar in all three cases, but while Schur's theorem uses a staircase and the overpartition version uses a generalized staircase, the asymmetric version uses what we call a *partial staircase*. In Section 3 we give a bijective proof of Theorem 5. In Section 4 we deduce Corollaries 2 and 3 from Theorem 5. In Section 5 Corollaries 2 and 3 are illustrated with examples. We close in Section 6 with some final remarks.

2 Weighted words and the proof of Theorem 5

2.1 Schur's theorem

Recall that we have been considering the positive integers in the three colors a, b, and ab, with the order $ab < a < b$. Take one ordinary partition with parts colored a, another ordinary partition with parts colored b, and one partition into distinct parts ≥ 2 colored ab. If we then order the three-colored integers accordingly, we obtain a three-colored partition with no 1_{ab} and the matrix of difference conditions,

$$A' = \begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}. \tag{2.1}$$

Let $S'(u, v, n)$ denote the number of such three-colored partitions of n, where u is the number of parts with a in their color and v is the number of parts with b in their color. Then it is quite clear that

$$\sum_{u,v,n\geq 0} S'(u, v, n)a^u b^v q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t}, \tag{2.2}$$

the first two terms corresponding to the ordinary partitions colored a and b and the third term to the partition into distinct parts ≥ 2 colored ab.

Next, let us add a "staircase" to the three-colored partition. That is, we add 0 to the smallest part, 1 to the next smallest part, and so on. This augments each minimal difference by one, giving us a partition with no part 1_{ab} and the difference conditions in (1.7). The quantities u and v do not change, and so we have the partitions counted by our $S(u, v, n)$ defined in the introduction.

Now, to compute the generating function for $S(u, v, n)$, we observe that adding a staircase simply corresponds to multiplying the summand on the right-hand side of (2.2) by $q^{\binom{r+s+t}{2}}$, and we have the generating function

$$\sum S(u, v, n)a^u b^v q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} q^{\binom{r+s+t}{2}}. \tag{2.3}$$

To simplify this sum (and some later ones), we recall several basic q-series facts (see [13]). First, we have

$$(a)_{n-k} = \frac{(a)_n}{(q^{1-n}/a)_k}(-q/a)^k q^{\binom{k}{2}-nk}, \tag{2.4}$$

so that

$$(q)_{n-k} = \frac{(q)_n}{(q^{-n})_k}(-1)^k q^{\binom{k}{2}-nk} \tag{2.5}$$

and

$$\frac{(aq^{-n})_n}{(bq^{-n})_n} = \frac{(q/a)_n}{(q/b)_n}(a/b)^n. \tag{2.6}$$

We also recall the q-Chu-Vandermonde summation,

$$\sum_{k=0}^{n} \frac{(a)_k (q^{-n})_k q^k}{(q)_k (c)_k} = \frac{(c/a)_n a^n}{(c)_n}, \tag{2.7}$$

and the q-binomial identity,

$$\sum_{n\geq 0} \frac{z^n(-a)_n}{(q)_n} = \frac{(-az)_\infty}{(z)_\infty}, \tag{2.8}$$

noting the special cases

$$\sum_{k=0}^{n} \frac{(q^{-n})_k (q^{-m})_k q^k}{(q)_k} = q^{-mn} \tag{2.9}$$

and

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}} z^n}{(q)_n} = (-zq)_\infty. \tag{2.10}$$

We now evaluate (2.3) as follows:

$$\begin{aligned}
\sum S(u,v,n)a^u b^v q^n &= \sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+t}{2}+r+s+t+\binom{t+1}{2}} a^{r+t} b^{s+t}}{(q)_r (q)_s (q)_t} \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+s-t}{2}+r+s+\binom{t}{2}} a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s)=(r-t,s-t)) \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}+rs} a^r b^s (q^{-r})_t (q^{-s})_t q^t}{(q)_r (q)_s (q)_t} \qquad \text{(from (2.5))} \\
&= \sum_{r,s\geq 0} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}} a^r b^s}{(q)_r (q)_s} \qquad \text{(by (2.9))} \\
&= (-aq)_\infty (-bq)_\infty \qquad \text{(by (2.10))}.
\end{aligned}$$

This is Theorem 2.

2.2 Schur's theorem for overpartitions

Now let us go back to the three-colored partitions counted by $S'(u, v, n)$. Instead of adding a staircase to such a partition, we will add a *generalized staircase*. This corresponds the term $d^{r+s+t}(-1/d)_{r+s+t}$, as follows. For each part k between 0 and $r+s+t-1$ in the partition into distinct parts generated by $(-1/d)_{r+s+t}$, we add 1 to each of the k largest parts and then overline the $k+1$st part. Notice that the exponent of d counts the number of non-overlined parts, and when $d = 0$ we just have the staircase $q^{\binom{r+s+t}{2}}$.

Thus we obtain a three-colored overpartition λ where the minimal difference between λ_i and λ_{i+1} is as in (1.7) if λ_{i+1} is overlined, but as in (2.1) if λ_{i+1} is non-overlined; that is, as in (1.10). There is no part 1_{ab} or 1_{abd}, and u and v count the same quantities as before. With m counting the number of non-overlined parts, then, we have the overpartitions counted by $\overline{S}(u, v, m, n)$. So,

$$\sum \overline{S}(u, v, m, n) a^u b^v d^m q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} (-1/d)_{r+s+t} d^{r+s+t}. \tag{2.11}$$

We emphasize that the only difference with the generating function for $S(u, v, n)$ in (2.3) is that the generalized staircase $(-1/d)_{r+s+t} d^{r+s+t}$ replaces the staircase $q^{\binom{r+s+t}{2}}$.

We now evaluate (2.11) as follows:

$$\begin{aligned}
\sum \overline{S}(u, v, m, n) a^u b^v d^m q^n &= \sum_{r,s,t\geq 0} (-1/d)_{r+s+t} d^{r+s+t} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} \\
&= \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{(-1/d)_{r+s-t} d^{r+s-t} q^{r+s+\binom{t}{2}} a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s) = (r-t, s-t)) \\
&= \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{(-1/d)_{r+s} d^{r+s} a^r b^s q^{r+s} (q^{-r})_t (q^{-s})_t q^t}{(q)_r (q)_s (q)_t (-dq^{1-r-s})_t} \qquad \text{(by (2.5) and (2.4))} \\
&= \sum_{r,s\geq 0} \frac{(-1/d)_{r+s} d^{r+s} a^r b^s q^{r+s}}{(q)_r (q)_s} \frac{(-dq^{1-s})_s}{(-dq^{1-r-s})_s} q^{-rs} \qquad \text{(by (2.7))} \\
&= \sum_{r,s\geq 0} \frac{(-1/d)_{r+s} d^{r+s} a^r b^s q^{r+s}}{(q)_r (q)_s} \frac{(-1/d)_s}{(-q^r/d)_s} \qquad \text{(by (2.6))} \\
&= \sum_{r,s\geq 0} \frac{(-1/d)_{r+s} d^{r+s} a^r b^s q^{r+s}}{(q)_r (q)_s} \frac{(-1/d)_s (-1/d)_r}{(-1/d)_{r+s}} \\
&= \sum_{r,s\geq 0} \frac{(-1/d)_r (-1/d)_s d^{r+s} a^r b^s q^{r+s}}{(q)_r (q)_s} \\
&= \frac{(-aq)_\infty (-bq)_\infty}{(adq)_\infty (bdq)_\infty} \qquad \text{(by (2.8))}.
\end{aligned}$$

This is Theorem 3.

2.3 The asymmetric Schur's theorem for overpartitions

Finally, we turn to the asymmetric case. Instead of a staircase or generalized staircase, we use a *partial staircase*, which is a kind of generalized staircase which is an actual staircase at the top. If we have $r+s+t$ parts, we require that the s largest steps in the staircase occur, namely $r+s+t-1, r+s+t-2, \ldots, r+t$. Then we allow a generalized staircase from $r+t-1$ down to 0. The result is the partial staircase corresponding to the term

$$q^{\binom{r+s+t}{2}-\binom{r+t}{2}}(-1/d)_{r+t}d^{r+t} = q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}. \tag{2.12}$$

Adding such a partial staircase to a three-colored partition counted by $S'(u, v, n)$ gives an overpartition counted by $\overline{S}_1(u, v, m, n)$, where as usual m denotes the number of non-overlined parts.

In terms of generating functions, we have

$$\sum \overline{S}_1(u, v, m, n) r^u s^v d^m q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}, \tag{2.13}$$

which may compared with (2.11) and (2.3). This triple sum may be evaluated as follows:

$$\begin{aligned}
&\sum \overline{S}_1(u, v, m, n) a^u b^v d^m q^n \\
&= \sum_{r,s,t\geq 0} \frac{q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}q^{r+s+t+\binom{t+1}{2}}a^{r+t}b^{s+t}}{(q)_r(q)_s(q)_t} \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{s-t}{2}+(r-t)(s-t)+(s-t)t}(-1/d)_r d^r q^{r+s+\binom{t}{2}}a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s)=(r-t,s-t)) \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{s+1}{2}+rs+r+t}(-1/d)_r d^r a^r b^s (q^{-r})_t (q^{-s})_t}{(q)_r(q)_s(q)_t} \qquad \text{(by (2.5))} \\
&= \sum_{r,s\geq 0} \frac{q^{\binom{s+1}{2}+r}(-1/d)_r d^r a^r b^s}{(q)_r(q)_s} \qquad \text{(by (2.9))} \\
&= \frac{(-aq)_\infty(-bq)_\infty}{(adq)_\infty} \qquad \text{(by (2.10) and (2.8)).}
\end{aligned}$$

This is the first part of Theorem 5. Note that by symmetry we can exchange the roles of r and s in the partial staircase (2.12) and the same argument would give the product

$$\frac{(-aq)_\infty(-bq)_\infty}{(bdq)_\infty}, \tag{2.14}$$

corresponding to the overpartitions counted by $\overline{S}_2(u, v, m, n)$. This completes the proof. $\square$

3 A bijective proof

Here we give a bijective proof of Theorem 5. We give details only for the first part. We start with the product side, namely a partition λ corresponding to $(-bq)_\infty$ and an overpartition μ corresponding to $(-aq)_\infty/(adq)_\infty$. To illustrate the steps in the bijection, we follow the example

$$\lambda = (23_b, 22_b, 19_b, 15_b, 14_b, 11_b, 7_b, 4_b, 3_b, 1_b)$$

and

$$\mu = (15_a, \overline{13}_a, 13_a, \overline{10}_a, \overline{9}_a, 8_a, 8_a, 8_a, \overline{5}_a, 5_a, 5_a, 4_a, 3_a, \overline{1}_a).$$

(We omit the label d from the colors of the non-overlined parts.) Let r be the number of parts in μ. Then, for each part x_b of λ which is $\leq r$, we add 1 to the x largest parts of μ and change the color of the xth part to ab. This gives us λ' and μ'. In our example, we have

$$\lambda' = (23_b, 22_b, 19_b, 15_b)$$

and

$$\mu' = (21_{ab}, \overline{18}_a, 18_{ab}, \overline{14}_{ab}, \overline{12}_a, 11_a, 11_{ab}, 10_a, \overline{7}_a, 7_a, 7_{ab}, 5_a, 4_a, \overline{2}_{ab}).$$

Next, we remove a generalized staircase from μ' (reversing the process described at the beginning of Sect. 2.2) and then remove r from the smallest part of λ', $r+1$ from the next smallest part, and so on. The result is λ'', μ'', and the removed parts in ν. In our example, we have

$$\lambda'' = (6_b, 6_b, 4_b, 1_b),$$

$$\mu'' = (16_{ab}, 14_a, 14_{ab}, 11_{ab}, 10_a, 9_a, 9_{ab}, 8_a, 6_a, 6_a, 6_{ab}, 4_a, 3_a, 2_{ab}),$$

and the partial staircase

$$\nu = (17, 16, 15, 14, 13, 8, 4, 3, 1).$$

Since there were 4 parts in λ', the 4 largest parts of ν form a staircase. Now we recall the order $ab < a < b$ and put the parts of λ'' into μ'' in the proper place. Continuing our example, we have a partition

$$\mu''' = (16_{ab}, 14_a, 14_{ab}, 11_{ab}, 10_a, 9_a, 9_{ab}, 8_a, 6_b, 6_b, 6_a, 6_a, 6_{ab}, 4_b, 4_a, 3_a, 2_{ab}, 1_b).$$

Finally, we add the partial staircase ν back on to μ'''. In our case, we have

$$(25_{ab}, \overline{22}_a, 22_{ab}, \overline{18}_{ab}, \overline{16}_a, 15_a, 15_{ab}, 14_a, \overline{11}_b, 11_b, 11_a, 11_a, 11_{ab}, \overline{8}_b, \overline{7}_a, \overline{5}_a, \overline{3}_{ab}, \overline{1}_b).$$

Notice that because we are adding the partial staircase in the manner described in Section 2.2, and since the s largest possible parts of ν occur (where s is the number of b-parts), the s smallest parts of the final overpartition will be overlined. (In our example, $s = 4$.) A little thought reveals that the difference conditions between parts match what is claimed in (1.10) and that the operation is reversible.

4 Proofs of Corollaries 2 and 3

We begin by treating Corollary 2. For this, we use (1.15) with the substitutions $q = q^6$, $a = q^{-4}$, $b = q^{-2}$, and $d = q^{-1}$. The product side is then

$$\frac{(-q^2; q^6)_\infty(-q^4; q^6)_\infty}{(q^3; q^6)_\infty} = (-q^2; q^6)_\infty(-q^4; q^6)_\infty(-q^3; q^3)_\infty, \tag{4.1}$$

which is the generating function for partitions into distinct parts not congruent to ± 1 modulo 6. On the other hand, in the colored partitions counted by $\overline{S}_2(u, v, m, n)$, a part x of color a, b, ab, ad, bd, or abd becomes the integer $6x - 4$, $6x - 2$, $6x - 6$, $6x - 5$, $6x - 3$, or $6x - 7$, respectively. (Recall that the label d corresponds to a non-overlined part.) Since there was no part 1_{ab} or 1_{abd}, this is the full set of positive integers. The matrix of difference conditions in (1.10) becomes

$$\begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\ \begin{pmatrix} 6 & 10 & 8 & 1 & 5 & 3 \\ 8 & 6 & 10 & 3 & 1 & 5 \\ 10 & 8 & 12 & 5 & 3 & 7 \\ 5 & 9 & 7 & 0 & 4 & 2 \\ 7 & 5 & 9 & 2 & 0 & 4 \\ 9 & 7 & 11 & 4 & 2 & 6 \end{pmatrix} \end{array}, \tag{4.2}$$

which is succinctly summarized by the difference conditions in (1.16) To finish, we note that the parts colored a or ad in overpartitions counted by $\overline{S}_2(u, v, m, n)$ become parts of the form $6x - 4$ or $6x - 5$. This gives Corollary 2

Corollary 3 is similar. We use the same substitutions $q = q^6$, $a = q^{-4}$, $b = q^{-2}$, and $d = q^{-1}$, but this time in (1.14). On the product side we have

$$\frac{(-q^2; q^6)_\infty(-q^4; q^6)_\infty}{(q; q^6)_\infty}, \tag{4.3}$$

which is the generating function for the number of partitions into parts 1, 2, or 4 modulo 6, where only parts congruent to 1 modulo 6 may be repeated. From $\overline{S}_1(u, v, m, n)$ we have the same difference conditions as in (4.2) (and hence (1.16)). Finally, the parts colored b or bd correspond to parts of the form $6x - 3$ and $6x - 2$.

5 Examples

5.1 Generalizations of Schur's theorem

Here we illustrate Theorem 4 and Corollary 1 for $n = 6$. To begin, there are 24 overpartitions of 6 satisfying the difference conditions in (1.13),

$$(\overline{6}), (6), (\overline{5}, \overline{1}), (\overline{5}, 1), (5, \overline{1}), (5, 1), (\overline{4}, 2), (4, 2), (\overline{4}, \overline{1}, 1), (\overline{4}, 1, 1), (4, \overline{1}, 1), (4, 1, 1),$$
$$(\overline{3}, 2, 1), (3, 2, 1), (\overline{3}, 1, 1, 1), (3, 1, 1, 1), (\overline{2}, 2, 2), (2, 2, 2), (\overline{2}, 2, 1, 1,), (2, 2, 1, 1),$$
$$(\overline{2}, 1, 1, 1, 1), (2, 1, 1, 1, 1), (\overline{1}, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1), \tag{5.1}$$

as well as 24 overpartitions of 6 into parts not divisible by 3,

$$(\overline{5}, \overline{1}), (\overline{5}, 1), (5, \overline{1}), (5, 1), (\overline{4}, \overline{2}), (\overline{4}, 2), (4, \overline{2}), (4, 2), (\overline{4}, \overline{1}, 1), (\overline{4}, 1, 1), (4, \overline{1}, 1), (4, 1, 1),$$
$$(\overline{2}, 2, 2), (2, 2, 2), (\overline{2}, 2, \overline{1}, 1), (\overline{2}, 2, 1, 1), (2, 2, \overline{1}, 1), (2, 2, 1, 1),$$
$$(\overline{2}, \overline{1}, 1, 1, 1), (\overline{2}, 1, 1, 1, 1), (2, \overline{1}, 1, 1, 1), (2, 1, 1, 1, 1), (\overline{1}, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1), \tag{5.2}$$

confirming Theorem 4 for $n = 6$ (and $0 \leq m \leq 6$).

Of the overpartitions in (5.1), 12 of them have their s smallest parts overlined, where s is the number of parts congruent to 2 modulo 3. These are

$$(\overline{6}), (6), (\overline{5}, \overline{1}), (5, \overline{1}), (\overline{4}, \overline{1}, 1), (\overline{4}, 1, 1), (4, \overline{1}, 1), (4, 1, 1),$$
$$(\overline{3}, 1, 1, 1), (3, 1, 1, 1), (\overline{1}, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1).$$

And, as predicted by Theorem 1 for $j = 1$, there are 12 overpartitions in (5.2) whose non-overlined parts are all congruent to 1 modulo 3,

$$(\overline{5}, \overline{1}), (\overline{5}, 1), (\overline{4}, \overline{2}), (4, \overline{2}), (\overline{4}, \overline{1}, 1), (\overline{4}, 1, 1), (4, \overline{1}, 1), (4, 1, 1),$$
$$(\overline{2}, \overline{1}, 1, 1, 1), (\overline{2}, 1, 1, 1, 1), (\overline{1}, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1).$$

Similarly, of the overpartitions in (5.1), there are 6 which have their s smallest parts overlined, where s is the number of parts congruent to 1 modulo 3,

$$(\overline{6}), (6), (\overline{5}, \overline{1}), (5, \overline{1}), (\overline{2}, 2, 2), (2, 2, 2),$$

and there are 6 overparititons in (5.2) whose non-overlined parts are all congruent to 2 modulo 3,

$$(\overline{5}, \overline{1}), (5, \overline{1}), (\overline{4}, \overline{2}), (\overline{4}, 2), (\overline{2}, 2, 2), (2, 2, 2).$$

5.2 Corollaries 2 and 3

Next, we illustrate Corollaries 2 and 3 for $n = 10$. There are 19 partitions of 10 which satisfy the difference conditions in (1.16). They are

$$\begin{aligned}&(10), (9, 1), (8, 2), (8, 1, 1), (7, 3), (7, 2, 1), (7, 1, 1, 1), (6, 3, 1), (6, 1, 1, 1, 1), (5, 3, 1, 1),\\ &(5, 1, 1, 1, 1, 1), (4, 3, 3), (4, 3, 1, 1, 1), (4, 1, 1, 1, 1, 1, 1), (3, 3, 3, 1), (3, 3, 1, 1, 1, 1),\\ &(3, 1, 1, 1, 1, 1, 1, 1), (2, 1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1).\end{aligned} \tag{5.3}$$

Of these, the ones that have their s smallest parts even, where s is the number of parts congruent to 1 or 2 modulo 6, are

$$(10), (8, 2), (4, 3, 3).$$

Thus $C(10) = 3$, and as predicted, there are 3 partitions of 10 into distinct parts not congruent to ± 1 modulo 6,

$$(10), (8, 2), (6, 4).$$

On the other hand, nine of the partitions in (5.3) have their r smallest parts even, where r is the number of parts congruent to 3 or 4 modulo 6. These are

$$\begin{aligned}&(10), (8, 2), (8, 1, 1), (7, 2, 1), (7, 1, 1, 1), (6, 1, 1, 1, 1), (5, 1, 1, 1, 1, 1),\\ &(2, 1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1).\end{aligned}$$

Thus $G(10) = 9$, and the nine partitions of 10 into parts congruent 1, 2, or 4 modulo 6 with only parts congruent to 1 modulo 6 allowed to repeat are

$$\begin{aligned}&(10), (8, 2), (8, 1, 1), (7, 2, 1), (7, 1, 1, 1), (4, 2, 1, 1, 1, 1), (4, 1, 1, 1, 1, 1, 1),\\ &(2, 1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1).\end{aligned}$$

6 Conclusion

In establishing Theorem 5, we have shown that using a partial staircase in the context of weighted words can lead to elegant infinite product generating functions, just as with staircases and generalized staircases. We have limited ourselves to the framework of Schur's theorem, but partial staircases can be used to asymmetrically extend other partition identities, such as Göllnitz's theorem [5] or the Alladi-Andrews-Berkovich identity [4]. We leave the details to the motivated reader.

We have also seen that partial staircases work well with bijective arguments. It remains to be seen, however, whether proofs of Schur's theorem [9] and Schur's theorem for overpartitions [12] using q-difference equations can be adapted to the asymmetric case.

References

1. K. Alladi, The Method of Weighted Words and Applications to Partitions, *Number Theory (Paris, 1992–1993)*, vol. 215, London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, 1995), pp. 1–36
2. K. Alladi, Refinements of Rogers–Ramanujan type identities, Special functions, q-series and related topics (Toronto, ON, 1995), in Fields Institute Communication, vol. 14 (Amererican Mathematical Society, Providence, 1997), pp. 1–35
3. K. Alladi, G.E. Andrews, The dual of Göllnitz's (big) partition theorem. Ramanujan J. **36**, 171–201 (2015)
4. K. Alladi, G.E. Andrews, A. Berkovich, A new four parameter q-series identity and its partition implications. Invent. Math. **153**, 231–260 (2003)
5. K. Alladi, G.E. Andrews, B. Gordon, Generalizations and refinements of a partition theorem of Göllnitz. J. Rein. Angew. Math. **460**, 165–188 (1995)
6. K. Alladi, G.E. Andrews, B. Gordon, Refinements and generalizations of Capparelli's conjecture on partitions. J. Algebra **174**, 636–658 (1995)
7. K. Alladi, B. Gordon, Generalizations of Schur's partition theorem. Manus. Math. **79**, 113–126 (1993)
8. K. Alladi, B. Gordon, Schur's partition theorem, companions, refinements and generalizations. Trans. Am. Math. Soc. **347**, 1591–1608 (1995)
9. G.E. Andrews, On Schur's second partition theorem. Glasg. Math. J. **8**, 127–132 (1967)
10. A. Berkovich, A.K. Uncu, A new companion to Capparelli's identities. Adv. Appl. Math. **71**, 125–137 (2015)
11. J. Dousse, On generalizations of partition theorems of Schur and Andrews to overpartitions. Ramanujan J. **35**, 339–360 (2014)
12. J. Dousse, Sliadič's theorem: weighted words, refinement, and companion, preprint
13. G. Gasper, M. Rahman, *Basic Hypergeometric Series* (Cambridge University Press, Cambridge, 2004)
14. R. Raghavendra, Padmavathamma, Bressoud's generalization of Schur's theorem extension to overpartitions. N. Z. J. Math. **39**, 25–32 (2009)
15. I. Schur, Zur additiven Zahlentheorie, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. (1926), pp. 488–495
16. J. Lovejoy, A theorem on seven-colored partitions and its applications. Int. J. Number Theory **1**, 215–224 (2005)

Local Behavior of the Composition of the Aliquot and Co-Totient Functions

Florian Luca and Carl Pomerance

For Krishna Alladi on his 60th birthday

Abstract We study the local behavior of the composition of the aliquot function $s(n) = \sigma(n) - n$ and the co-totient function $s_\varphi(n) = n - \varphi(n)$, where σ is the sum-of-divisors function and φ is the Euler function. In particular, we show that $s \circ s_\varphi$ and $s_\varphi \circ s$ are independent in the sense of Erdős, Győry, and Papp.

Keywords sum-of-divisors function · Euler function · applications of sieve methods

2010 Mathematics Subject Classification 11N25 · 11N36

1 Introduction

In [5], two arithmetic functions $f(n)$ and $g(n)$ are called independent if for all $k \geq 2$ and permutations $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ of $\{1, 2, \ldots, k\}$, there exist infinitely many n such that

---

F. Luca (✉)
School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa
e-mail: florian.luca@wits.ac.za

F. Luca
Centro de Ciencias Matemáticas, UNAM, Morelia, Mexico

C. Pomerance
Department of Mathematics, Dartmouth College, HB 6188, Hanover, NH 03755, USA
e-mail: carl.pomerance@dartmouth.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_27

$$f(n+i_1) < f(n+i_2) < \cdots < f(n+i_k),$$
$$g(n+j_1) < g(n+j_2) < \cdots < g(n+j_k). \tag{1.1}$$

In [5], it was shown that the number-of-prime-divisors function, denoted ω, and the number-of-divisors function, denoted τ, are independent. They also showed that σ, the sum-of-divisors function, and φ, Euler's function, are *not* independent (when $k \geq 5$). In [2], it was shown that φ and the Carmichael function λ are independent. In [8], it was shown that the compositions $\sigma \circ \varphi$ and $\varphi \circ \sigma$ are independent.

Here, we put

$$s(n) = \sigma(n) - n \quad \text{and} \quad s_\varphi(n) = n - \varphi(n).$$

These functions are well-known in the literature, and the first has an ancient history, dating to Pythagoras. It is not known if the sets of values of these functions has an asymptotic density, though recent progress was made in [7]. Due to the result in [5] that σ and φ are *not* independent, it seems likely that s and s_φ are also not independent. Our principal result is the following theorem.

Theorem 1 *The functions $s \circ s_\varphi$ and $s_\varphi \circ s$ are independent.*

We also show the following result, by somewhat different methods.

Theorem 2 *The closure of the set of rationals*

$$\left\{ \frac{s(n)}{s_\varphi(n)} : n > 1 \right\}$$

is the interval $[1, \infty)$. *The closure of the set of rationals*

$$\left\{ \frac{(s \circ s_\varphi)(n)}{(s_\varphi \circ s)(n)} : n \text{ composite} \right\}$$

is $[0, \infty)$. *The same is true for the rationals $(s \circ s)(n)/(s_\varphi \circ s_\varphi)(n)$ with n composite.*

Note that $s(1) = s_\varphi(1) = 0$ and $(s \circ s_\varphi)(p) = (s_\varphi \circ s)(p) = 0$ for p prime, and this is why there are certain values of n excluded in the sets in Theorem 2.

One can also ask about typical behavior; we find it to be markedly different.

Theorem 3 *There is a set $\mathscr{A}$ of asymptotic density 1 such that*

$$\frac{(s \circ s_\varphi)(n)}{n} \sim \frac{s(n)s_\varphi(n)}{n^2} \sim \frac{(s_\varphi \circ s)(n)}{n}$$

as $n \to \infty$, $n \in \mathscr{A}$. In particular,

$$\lim_{n \to \infty,\, n \in \mathscr{A}} \frac{(s \circ s_\varphi)(n)}{(s_\varphi \circ s)(n)} = 1.$$

In addition, as $n \to \infty$, $n \in \mathscr{A}$,

$$\frac{(s \circ s)(n)}{n} \sim \left(\frac{s(n)}{n}\right)^2, \quad \frac{(s_\varphi \circ s_\varphi)(n)}{n} \sim \left(\frac{s_\varphi(n)}{n}\right)^2.$$

That $s(s(n))/n$ is normally asymptotic to $(s(n)/n)^2$ is essentially [4, Theorem 5.1 and (5.1)]. Probably $s(n)/s_\varphi(n)$ has a continuous and strictly increasing distribution function on $[1, \infty)$, but we haven't been able to show this. The existence of a distribution function may follow from the methods of [13, Section 3] and [11, Section 3].

Throughout this paper, we use the Landau symbols O, o and the Vinogradov symbols $\gg$, $\ll$ with their usual meaning. The constants implied by them might depend on the fixed parameter k. For a set $\mathscr{A}$ of integers and a real number $t \geq 1$, let $\mathscr{A}(t) = \mathscr{A} \cap [1, t]$. The letters p, q run over primes. Our conjecture about $s(n)/s_\varphi(n)$ at the end of Sect. 1 was recently proved by [15].

2 The proof of Theorem 1

Let $i \geq 1$ be an integer and

$$f_i(t) = \left(\frac{\sigma(i)}{i}t - 1\right)\left(1 - \frac{\varphi(i)}{it}\right) \quad \text{for real} \quad t \geq 1.$$

Clearly $f_i(t) \to \infty$ *as* $t \to \infty$ and

$$f_i'(t) = \frac{\sigma(i)}{i} - \frac{\varphi(i)}{it^2} > 0 \quad \text{for} \quad t > 1,$$

so $f_i(t)$ is increasing for $t \geq 1$. Let $k \geq 2$, $C_k := \max\{2, f_i(2) : 1 \leq i \leq k\}$ and consider two permutations $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ of $\{1, \ldots, k\}$. Choose real numbers

$$C_k < \alpha_{j_1} < \cdots < \alpha_{j_k},$$

and solve the equations

$$f_i(u_i) = \alpha_i \quad \text{for} \quad u_i > 2 \quad \text{and} \quad i \in \{1, \ldots, k\}.$$

This is possible because $\alpha_i > C_k \geq f_i(2)$. Now choose real numbers

$$0 < \beta_{i_1} < \beta_{i_2} < \cdots < \beta_{i_k} \leq 1$$

and put

$$v_i = \frac{\varphi(i)/i}{1 - \beta_i/\alpha_i + (\beta_i/\alpha_i)(\varphi(i)/(iu_i))}.$$

Since $\beta_i/\alpha_i \le \frac{1}{2}$ and $0 < \varphi(i)/(iu_i) < 1$, we have

$$0 < \frac{\varphi(i)}{i} < v_i < \frac{\varphi(i)/i}{1 - \beta_i/\alpha_i} \le 2\frac{\varphi(i)}{i} \le 2,$$

so that for $i = 1, \ldots, k$, we have $0 < v_i < u_i$. Further, the way we have chosen v_i and u_i gives

$$\left(\frac{\sigma(i)}{i}u_i - 1\right)\left(1 - \frac{\varphi(i)}{iu_i}\right) = \alpha_i, \tag{2.1}$$

$$\left(\frac{\sigma(i)}{i}u_i - 1\right)\left(1 - \frac{\varphi(i)}{iv_i}\right) = \beta_i, \tag{2.2}$$

for all $i \in \{1, \ldots, k\}$.

Let $\mathcal{Q}$ be the set of odd primes $3 = q_1 < q_2 < \cdots$ such that for all $\ell \ge 2$, q_ℓ is the smallest odd prime with $q_\ell \not\equiv 1 \pmod{q_j}$ for any $j = 1, \ldots, \ell - 1$. The first elements of $\mathcal{Q}$ are 3, 5, 17, ... Erdős [3] showed that

$$\#\mathcal{Q}(t) = (1 + o(1))\frac{t}{\log t \log\log t} \quad \text{for} \quad t \to \infty. \tag{2.3}$$

In particular, by Abel summation,

$$\sum_{\substack{a<q<b \\ q\in\mathcal{Q}}} \frac{1}{q} = \log\log\log b - \log\log\log a + o(1) \tag{2.4}$$

uniformly in $b > a$ and $a \to \infty$. We now let x be large, and put

$$y := (\log\log x)^4, \quad z := e^{(\log\log x)^{1/2}}, \quad \varepsilon := (\log\log x)^{-1}.$$

By (2.4), it follows easily that

$$\prod_{\substack{y<q<z \\ q\in\mathcal{Q}}} \left(1 + \frac{1}{q}\right) = \left(\frac{1}{2} + o(1)\right)\frac{\log\log\log x}{\log\log\log\log x} \quad \text{for} \quad x \to \infty. \tag{2.5}$$

We choose pairwise disjoint sets of primes

$$\mathcal{Q}_i \subset \mathcal{Q} \cap (y, z) \quad \text{for all} \quad i \in \{1, \ldots, k\}, \tag{2.6}$$

such that

$$\prod_{q\in\mathscr{Q}_i}\left(1+\frac{1}{q}\right)\in(u_i-\varepsilon,u_i+\varepsilon)\quad\text{for all}\quad i\in\{1,\ldots,k\}. \tag{2.7}$$

We select subsets $\mathscr{R}_i\subseteq\mathscr{Q}_i$ such that

$$\prod_{q\in\mathscr{R}_i}\left(1+\frac{1}{q}\right)\in(v_i-\varepsilon,v_i+\varepsilon)\quad\text{for all}\quad i\in\{1,\ldots,k\}. \tag{2.8}$$

All this is possible because of (2.5) and because $v_i<u_i$ for all $i\in\{1,\ldots,k\}$. Put

$$Q_i=\prod_{q\in\mathscr{Q}_i}q\quad\text{and}\quad R_i=\prod_{q\in\mathscr{R}_i}q\quad\text{for all}\quad i\in\{1,\ldots,k\}.$$

We now choose for each $i\in\{1,\ldots,k\}$, U_i to be the smallest prime such that

$$\begin{aligned}
&U_i\equiv -1+R_i+2R_i^2 \pmod{R_i^3}\\
&U_i\equiv 2+2(Q_i/R_i)^2 \pmod{(Q_i/R_i)^3}\\
&U_i\equiv 2+2Q_j^2 \pmod{Q_j^3}\quad\text{for all}\quad j\in\{1,\ldots,k\}\backslash\{i\}.
\end{aligned} \tag{2.9}$$

Note that $U_1,\ldots,U_k$ are distinct and $\{U_1,\ldots,U_k\}$ is disjoint from $\bigcup_{i=1}^k\mathscr{Q}_i$ because any prime among the U's is larger than any member of $\bigcup_{i=1}^k\mathscr{Q}_i$. The congruences (2.9) put U_i in a certain arithmetic progression modulo $(\prod_{i=1}^k Q_i)^3$. By a result of Xylouris [14], we have

$$U_i\ll\left(\prod_{i=1}^k Q_i^3\right)^5<\left(\prod_{p<z}p\right)^{15}. \tag{2.10}$$

Finally, let

$$P=\prod_{\substack{2k<p<3z^2\\ p\notin\bigcup_{i=1}^k\mathscr{Q}_i\bigcup\{U_1,\ldots,U_k\}}}p.$$

Consider the Chinese Remainder Theorem system of congruences:

$$\begin{aligned}
&n\equiv 0 \pmod{(2k)!P}\\
&n\equiv -i+Q_i^2U_i \pmod{Q_i^3U_i^2}\quad\text{for all}\quad i=1,\ldots,k.
\end{aligned} \tag{2.11}$$

Congruences (2.11) put n into an arithmetic progression of modulus

$$M:=(2k)!P\prod_{i=1}^k(Q_i^3U_i^2). \tag{2.12}$$

Recalling that implied constants depend on the choice of k, note that from (2.10),

$$\begin{aligned} M &\ll P\left(\prod_{j=1}^{k} Q_i\right)^3\left(\prod_{i=1}^{k} U_i\right)^2 \ll P\left(\prod_{i=1}^{k} Q_i\right)^{3+30k} \\ &< \left(\prod_{p<3z^2} p\right)\left(\prod_{p<z} p\right)^{3+30k} \ll e^{4z^2}, \end{aligned} \tag{2.13}$$

so that $M \le e^{(\log x)^{o(1)}}$ as $x \to \infty$. Write

$$n = M\lambda + N_0,$$

where $0 < N_0 < M$ is the smallest positive integer in the progression. Then

$$\begin{aligned} n+i \ &= \ M\lambda + (i+N_0) = i Q_i^2 U_i (M_i\lambda + N_i), \\ \text{where } \ M_i &:= \frac{M}{i Q_i^2 U_i} \\ \text{and } \ N_i &:= \frac{i+N_0}{i Q_i^2 U_i} \quad \text{for all} \quad i \in \{1, \ldots, k\}. \end{aligned} \tag{2.14}$$

Fix $i \in \{1, \ldots, k\}$. We start by noting that M_i is divisible by all primes $p \le 3z^2$, while the numbers N_i are coprime to all primes $p \le 3z^2$ for $i \in \{1, \ldots, k\}$. To justify this claim, first of all let us note that for large x, we have $y > 2k$, so all primes in $\mathcal{Q}_i$, all primes dividing P, and all the primes $U_1, \ldots, U_k$ are larger than $2k$. Then:

(i) Since $i \le k$, we get $i^2 \mid (k!)^2 \mid (2k)! \mid M$, so $i \mid M_i$.
(ii) If $p \in (k, 2k)$, then $p \mid (2k)!$ and $p \nmid i$ for any $\in \{1, \ldots, k\}$, so $p \mid M_i$.
(iii) Since, $Q_i^3 \mid M$, we have $Q_i \mid M_i$.
(iv) If $U_i \le 3z^2$ for some $i \in \{1, \ldots, k\}$, then $U_i^2 \mid M$, so $U_i \mid M_i$.
(v) If $p \in (2k, 3z^2]$ and $p \notin \bigcup_{i=1}^{k} \mathcal{Q}_i \cup \{U_1, \ldots, U_k\}$, then $p \mid P \mid M_i$.

From (i)–(v) above, we get that $p \mid M_i$ for all $p \le 3z^2$.

Similar observations show that N_i is not a multiple of any prime $p \le 3z^2$. Indeed, if $p \mid i$, then $p^2 \mid i^2 \mid (2k)! \mid N_0$, so p does not divide $(i + N_0)/i$. If $p \le k$ does not divide i, then p divides N_0 but not $N_0 + i$, so $p \nmid N_i$. If $p \in (k, 2k)$, then $p \mid N_0$ and $p \nmid i$, so p does not divide $i + N_0$, and in particular $p \nmid N_i$. If $p \in \mathcal{Q}_i$, then $p^2 \parallel i + N_0$, so $p \nmid N_i$. Similar arguments show that p does not divide N_i if p is either in $\{U_1, \ldots, U_k\}$, or if it divides P.

So, N_i is coprime to all primes $p \le 3z^2$ as well as with the primes in $\{U_1, \ldots, U_k\}$. In particular,

$$\gcd(M_i, N_i) = 1.$$

Further, $M_i\lambda + N_i$ is neither a multiple of any prime $p \le 3z^2$ nor a multiple of any of the primes in $\{U_1, \ldots, U_k\}$. Then we have for $i = 1, \ldots, k$,

$$\begin{aligned} s_\varphi(n+i) &= (n+i) - \varphi(n+i) \\ &= iQ_i^2 U_i(M_i\lambda + N_i) - \varphi(i)Q_i(U_i - 1)\varphi(M_i\lambda + N_i)\prod_{q\in\mathcal{Q}_i}(q-1) \\ &= iQ_iT_i, \end{aligned} \tag{2.15}$$

where

$$T_i := Q_iU_i(M_i\lambda + N_i) - \varphi(i)(U_i - 1)(\varphi(M_i\lambda + N_i)/i)\prod_{q\in\mathcal{Q}_i}(q-1).$$

Also,

$$\begin{aligned} s(n+i) &= \sigma(n+i) - (n+i) \\ &= \sigma(i)(U_i + 1)\sigma(M_i\lambda + N_i)\prod_{q\in\mathcal{Q}_i}(q^2 + q + 1) - iQ_i^2U_i(M_i\lambda + N_i) \\ &= iR_iS_i, \end{aligned} \tag{2.16}$$

where

$$S_i = \sigma(i)((U_i + 1)/R_i)(\sigma(M_i\lambda + N_i)/i)\prod_{q\in\mathcal{Q}_i}(q^2 + q + 1) - Q_i(Q_i/R_i)U_i(M_i\lambda + N_i).$$

Now we start sieving. Note that if $\lambda \le x/M$, then $M_i\lambda + N_i < n < 2x$. Let us throw away some values of $\lambda \le x/M$ in such a way that at each step we only throw an amount of λ of order of magnitude

$$o\left(\frac{x}{M}\right) \quad \text{as} \quad x \to \infty.$$

There exists an absolute constant c_0 such that for every $i \in \{1, \ldots, k\}$, the set Λ_1 of $\lambda \le x/M$ such that $\varphi(M_i\lambda + N_i)$ or $\sigma(M_i\lambda + N_i)$ is not divisible by all numbers $m < c_0 \log\log x/\log\log\log x$ is of cardinality $\ll x/(M \log\log x)$. For φ and without the arithmetic progression, this follows from Lemma 2 in [6]. The proof of that lemma can be adapted in a straightforward way to yield the current result. So, we ignore $\lambda \in \Lambda_1$, and assume from now on that

$$\varphi(M_i\lambda + N_i), \sigma(M_i\lambda + N_i) \text{ are multiples of all numbers } m \le c_0\frac{\log\log x}{\log\log\log x}.$$

In particular, this implies that T_i, S_i are integers.

We eliminate $\lambda \in \Lambda_2$, where this set is such that for some $i \in \{1, \ldots, k\}$ we have that $M_i\lambda + N_i$ is a multiple of a prime $p > x/M$. Assume $\lambda \in \Lambda_2$. Then for some

$i \in \{1, \ldots, k\}$ we have that $M_i\lambda + N_i = pm$, where $m < 2M$. Fixing m, this puts $\lambda \le x/M$ into a certain progression modulo m, such that $M_i\lambda + N_i \equiv 0 \pmod{m}$ and $(M_i\lambda + N_i)/m = p$ is prime. Thus, the number of $\lambda \le x/M$ satisfying these conditions is

$$\ll \frac{M_i m}{\varphi(M_i m)} \frac{x/Mm}{\log(x/Mm)} \ll \frac{x \log\log x}{Mm \log x},$$

using the minimal order of φ and (2.13). Summing on $m < 2M$ and on $i = 1, \ldots, k$ and again using (2.13), we have

$$\#\Lambda_2 \ll \frac{x \log M \log\log x}{M \log x} \ll \frac{xz^2 \log\log x}{M \log x} \ll \frac{x}{M\sqrt{\log x}} = o\left(\frac{x}{M}\right)$$

as $x \to \infty$.

We eliminate $\lambda \in \Lambda_3$ such that $M_i\lambda + N_i$ is not squarefree. So, assume that $i \in \{1, \ldots, k\}$ and $M_i\lambda + N_i$ is not squarefree. Thus, some $p^2 \mid M_i\lambda + N_i$. Assume first that $p^2 < x/M$. Then the number of such $\lambda \le x/M$ is $\ll x/(Mp^2)$. Summing over all $i \in \{1, \ldots, k\}$ and $p > 3z^2$, we get a bound of

$$\ll \sum_{p > 3z^2} \frac{x}{Mp^2} \ll \frac{x}{M} \sum_{p > 3z^2} \frac{1}{p^2} \ll \frac{x}{Mz^2} = o\left(\frac{x}{M}\right) \quad \text{as} \quad x \to \infty. \tag{2.17}$$

Assume now that $p^2 > x/M$. Since $M_i\lambda + N_i < 2x$, we have $p < \sqrt{2x}$. Moreover, each such p gives rise to at most one value of $\lambda \le x/M$. Thus, the number of choices of λ in this case is at most $\pi(\sqrt{2x}) < \sqrt{x} = o(x/M)$ as $x \to \infty$, by (2.13). We deduce from (2.17) that

$$\#\Lambda_3 = o\left(\frac{x}{M}\right) \quad \text{as} \quad x \to \infty.$$

We eliminate $\lambda \in \Lambda_4$ such that for some $i \in \{1, \ldots, k\}$,

$$\omega(M_i\lambda + N_i) \ge 10 \log\log x.$$

For this, write $M_i\lambda + N_i = m'm$ where the least prime factor of m' exceeds the greatest prime factor of m and m is maximal with $m \le x/M$. Since $\sqrt{M_i\lambda + N_i} < \sqrt{2x} < x/M$ and since $M_i\lambda + N_i$ is squarefree (using $\lambda \notin \Lambda_3$), it follows that $\omega(m) \ge \frac{1}{2}\omega(M_i\lambda + N_i)$. Thus, summing over $i = 1, \ldots, k$,

$$\#\Lambda_4 \ll \sum_{\substack{m \le x/M \\ m \text{ squarefree} \\ \omega(m) \ge 5 \log\log x}} \frac{x}{Mm} \le \frac{x}{M} \sum_{j \ge 5 \log\log x} \frac{1}{j!} \left(\sum_{p \le x/M} \frac{1}{p}\right)^j \ll \frac{x}{M(\log x)^3},$$

since the inner sum on p is $\log\log x + O(1)$. Thus, $\#\Lambda_4 = o(x/M)$ as $x \to \infty$.

We now eliminate those $\lambda \in \Lambda_5$ where for some $i = 1, \dots, k$, we have a prime $p \mid M_i\lambda + N_i$ with $\omega(p-1) > 5 \log\log x$. Since $\lambda \notin \Lambda_2$, we may assume that $p \le x/M$. For a given p with $\omega(p-1) > 5 \log\log x$ and a given $i \in \{1, \dots, k\}$, the number of $\lambda \le x/M$ with $M_i\lambda + N_i$ divisible by p is $\ll x/Mp < x/M(p-1)$. Writing $p - 1 = m$ and ignoring that p is prime, we have

$$\#\Lambda_5 \ll \frac{x}{M} \sum_{\substack{m \le x/M \\ \omega(m) > 5 \log\log x}} \frac{1}{m} \le \frac{x}{M} \sum_{j > 5 \log\log x} \frac{1}{j!} \left(\sum_{q^a \le x/M} \frac{1}{q^a} \right)^j,$$

where q^a runs over prime powers. Since the inner sum is $\log\log x + O(1)$, we have, as with the calculation for Λ_4, that $\#\Lambda_5 = o(x/M)$ as $x \to \infty$.

We eliminate $\lambda \in \Lambda_6$ such that for some $i \in \{1, \dots, k\}$, we have that

$$\gcd((M_i\lambda + N_i)U_i, \varphi(M_i\lambda + N_i)) > 1.$$

Since $\lambda \notin \Lambda_3$, we have that $M_i\lambda + N_i$ is squarefree. Thus, if for some $i \in \{1, \dots, k\}$, the number $M_i\lambda + N_i$ and its Euler function are not coprime, then there are primes p and q with $pq \mid M_i\lambda + N_i$ and $p \mid q - 1$. There are two cases here to consider. If $pq \le x/M$, then we fix i, p, q and we get that the number of such $\lambda \le x/M$ is $\ll x/Mpq$. Summing up this inequality over all $q \equiv 1 \pmod{p}$ with $q \le x/M$ (using [12, Theorem 1, Remark 1]), then over all $p \in (3z^2, x/M)$, then over all $i \in \{1, \dots, k\}$, we get a bound of

$$k \frac{x \log\log x}{Mz^2} = o\left(\frac{x}{M}\right) \quad \text{for} \quad x \to \infty.$$

The other case to consider is when $pq > x/M$. We then write $M_i\lambda + N_i = pqm$, where $m < 2M$ and fix m. Since $q > p$, we get that $p < 2x^{1/2}$. So, $pm < 4x^{1/2}M < x/M$ for large x. Fixing also p, we get that $\lambda \le x/M$ is in a certain arithmetic progression modulo pm such that $pm \mid M_i\lambda + N_i$ and $(M_i\lambda + N_i)/pm$ is prime. The number of such λ is, using the minimal order of φ,

$$\ll \frac{M_i m}{\varphi(M_i m)} \frac{x}{Mpm \log x} \ll \frac{x \log\log x}{Mpm \log x}.$$

Summing over $p \le 2x^{1/2}$, $m < 2M$, $i \le k$, we get an estimate that is

$$\ll \frac{x (\log\log x)^2 \log M}{M \log x} = o\left(\frac{x}{M}\right),$$

as $x \to \infty$, using (2.13). Finally, consider the case that $U_i \mid \varphi(M_i\lambda + N_i)$. Then there is a prime $q \equiv 1 \pmod{U_i}$ with $q \mid M_i\lambda + N_i$. Again using [12], $\sum 1/q \ll$

$(\log\log x)/U_i$, so the number of such $\lambda \le x/M$ is $o(x/M)$ as $x \to \infty$. Thus, $\#\Lambda_6 = o(x/M)$ as $x \to \infty$.

We eliminate $\lambda \in \Lambda_7$ such that for some $i \in \{1, \dots, k\}$ we have

$$\gcd(M_i\lambda + N_i, U_i - 1) > 1.$$

Assume that $i \in \{1, \dots, k\}$ and that there is a prime $p \mid \gcd(M_i\lambda + N_i, U_i - 1)$. Then $p > 3z^2$. Fixing p, we have $p \le x/M$ because $\lambda \notin \Lambda_2$, therefore, the number of such $\lambda \le x/M$ is $\le 1 + x/Mp \le 2x/Mp$. Summing this over all the prime divisors $p > 3z^2$ of $U_i - 1$, we get a bound of

$$\ll \frac{x\omega(U_i - 1)}{Mz^2} \ll \frac{x \log U_i}{Mz^2 \log\log U_i} \ll \frac{x \log M}{Mz^2 \log\log M} \ll \frac{x}{M \log\log M}$$

where we used the maximal order of $\omega(m)$ together with (2.13). Summing this up over $i \in \{1, \dots, k\}$, we get

$$\#\Lambda_7 \ll \frac{x}{M \log\log M} = o\left(\frac{x}{M}\right) \quad \text{as} \quad x \to \infty.$$

Now let us look at

$$T_i = Q_i U_i (M_i\lambda + N_i) - \varphi(i)(U_i - 1)(\varphi(M_i\lambda + N_i)/i)\varphi(Q_i).$$

Note that

$$\gcd(Q_i U_i (M_i\lambda + N_i), \varphi(i)(U_i - 1)\varphi(Q_i)) = 1;$$
$$\gcd(U_i(M_i\lambda + N_i), \varphi(M_i\lambda + N_i)) = 1.$$

Indeed, Q_i is coprime to $\varphi(Q_i)$ due to the definition of the set $\mathcal{Q}$. The other relations follow from the sizes of the primes involved, the definition of U_i, and because $\lambda \notin \Lambda_6 \cup \Lambda_7$. So, it follows that

$$\gcd(Q_i U_i (M_i\lambda + N_i), \varphi(i)(U_i - 1)(\varphi(M_i\lambda + N_i)/i)\varphi(Q_i)) = W_i,$$

where

$$W_i := \gcd(Q_i, \varphi(M_i\lambda + N_i)).$$

We may write

$$T_i = W_i' T_i',$$

where W_i' is a multiple of W_i and is the largest divisor of T_i supported on the prime factors of W_i, and where T_i' is coprime to W_i and its least prime factor exceeds $c_0 \log\log x / \log\log\log x$ (because $\lambda \notin \Lambda_1$).

We continue with the sieving. We put

$$Y := \exp\left(\frac{\log x \log\log\log x}{\log\log x}\right),$$

and eliminate $\lambda \in \Lambda_8$ such that for some $i \in \{1, \dots, k\}$ we have that the largest prime factor P of $M_i\lambda + N_i$ satisfies $P \le Y$. Since the conditions (1.10) for the main theorem in [1] are fulfilled, we get that

$$\#\Lambda_8 \ll \frac{x}{M \exp(u \log u + u \log\log u)} = o\left(\frac{x}{M}\right) \quad \text{as} \quad x \to \infty,$$

where $u = \log(xM_i/M)/\log Y = (1 + o(1)) \log\log x / \log\log\log x$ as $x \to \infty$.

We eliminate $\lambda \in \Lambda_9$ such that $\omega(T_i') > 100 \log\log x$. Fix $i \in \{1, \dots, k\}$ and assume that $\omega(T_i') > 100 \log\log x$. Write $M_i\lambda + N_i = Pm$, where $Y < P \le x/M$ and $m < 2x/Y$. This is possible because $\lambda \notin \Lambda_2 \cup \Lambda_8$. Further, P and m are coprime, and m is squarefree because $\lambda \notin \Lambda_3$. Substituting $M_i\lambda + N_i = Pm$ into

$$iT_i = iQ_iU_i(M_i\lambda + N_i) - \varphi(i)(U_i - 1)\varphi(M_i\lambda + N_i)\varphi(Q_i) = iW_i'T_i',$$

we get

$$A_iP + B_i = iW_i'T_i',$$

where

$$A_i = iQ_iU_im - \varphi(i)(U_i - 1)\varphi(m)\varphi(Q_i), \quad B_i = \varphi(i)(U_i - 1)\varphi(m)\varphi(Q_i).$$

It follows from the definition of T_i' that T_i' is coprime to $\gcd(A_i, B_i)$, and so Q_i, m, T_i' are pairwise coprime. We consider the prime factors of T_i' in three ranges:

(i) At least $x^{1/(20 \log\log x)}$ (the number of such is at most $20 \log\log x$);
(ii) At most $x^{1/(\log\log x)^2}$;
(iii) In the interval $I = [x^{1/(\log\log x)^2}, x^{1/(20 \log\log x)}]$.

Suppose that T_i' has at least $5 \log\log x$ prime divisors in the range (ii) above and let τ be the product of $\lceil 5 \log\log x \rceil$ of them. Then $\tau < x^{6/\log\log x} < \sqrt{Y}$. The relation $\tau \mid A_iP + B_i$ with τ coprime to $\gcd(A_i, B_i)$ puts P in a certain arithmetic progression modulo τ, which for a fixed value of m, puts λ in a particular arithmetic progression modulo $m\tau$. Ignoring that P is prime and since $m\tau < 2x/\sqrt{Y} < x/M$, the number of such $\lambda \le x/M$ is

$$\ll \frac{x}{Mm\tau}.$$

We now sum over all possible $m < 2x/Y$ and all possible τ, a number which is at most x/M but has $\lceil 5\log\log x\rceil$ distinct prime factors. We get

$$\frac{x}{M}\sum_{m\le x}\frac{1}{m}\sum_{\substack{\tau\le x/M\\ \omega(\tau)\ge 5\log\log x}}\frac{1}{\tau}\ll\frac{x\log x}{M}\sum_{\substack{\tau\le x/M\\ \omega(\tau)\ge 5\log\log x}}\frac{1}{\tau}\ll\frac{x}{M(\log x)^2}=o\left(\frac{x}{M}\right),$$

as $x\to\infty$, where the last bound follows from the argument above for Λ_4 or Λ_5.

Assume now that T_i' has at least $K := \lfloor 5\log\log\log x\rfloor$ prime divisors in I and let τ' be a product of K of them. Then

$$\tau' < x^{5\log\log\log x/20\log\log x} < \sqrt{Y}.$$

Thus, for fixed τ', m, we have the number of $\lambda \le x/M$ with λ in the particular class mod $m\tau'$ and $(M_i\lambda + N_i)/m = p$ prime, is at most the number of primes $p \le 2M_i x/(Mm)$ which are in a fixed congruence class modulo $M_i\tau'$, and this is

$$\ll\frac{M_i\tau'}{\varphi(M_i\tau')}\frac{x}{Mm\tau'\log((2x)/(Mm\tau'))}\ll\frac{x\log\log x}{Mm\tau'\log Y},$$

using $m\tau' \ll x/\sqrt{Y}$ and (2.13). (We have also used the minimal order of φ.) Thus, by an argument similar to the preceding one we get that the number of such numbers $\lambda \le x/M$ is

$$\ll\frac{x(\log\log x)^2}{M\log x}\sum_{m\le x}\frac{1}{m}\sum_{\substack{\tau' \text{ squarefree}\\ \omega(\tau')=K\\ p\mid\tau'\Rightarrow p\in I}}\frac{1}{\tau'}\ll\frac{x(\log\log x)^2}{M}\frac{1}{K!}\left(\sum_{p\in I}\frac{1}{p}\right)^K.$$

The inner sum is $\log\log\log x + O(1)$, and since $5(\log 5 - 1) > 3$, the estimate is $o(x/M)$ as $x\to\infty$. Thus, we get that the number of λ for which there are at least K prime factors of T_i' in I is $o(x/M)$ as $x\to\infty$. To summarize, except for a set of λ of cardinality $o(x/M)$, the number T_i' has at most $25\log\log x + 5\log\log\log x$ prime factors. Thus, Λ_9 has cardinality $o(x/M)$ as $x\to\infty$ and we may assume, therefore, that $\omega(T_i') < 100\log\log x$.

Now we have

$$\begin{aligned}(s\circ s_\varphi)(n+i) &= \sigma(s_\varphi(n+i)) - s_\varphi(n+i)\\ &= \sigma(iQ_iW_i'T_i') - iQ_iW_i'T_i'\\ &= s_\varphi(n+i)\left(\frac{\sigma(i)}{i}\frac{\sigma(Q_iW_i')}{Q_iW_i'}\frac{\sigma(T_i')}{T_i'}-1\right).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{(s\circ s_{\varphi})(n+i)}{n+i} &= \frac{(s\circ s_{\varphi})(n+i)}{s_{\varphi}(n+i)}\frac{s_{\varphi}(n+i)}{n+i}\\ &= \left(\frac{\sigma(i)}{i}\frac{\sigma(Q_iW_i')}{Q_iW_i'}\frac{\sigma(T_i')}{T_i'}-1\right)\\ &\quad\times\left(1-\frac{\varphi(i)}{i}\left(1-\frac{1}{U_i}\right)\frac{\varphi(Q_i)}{Q_i}\frac{\varphi(M_i\lambda+N_i)}{M_i\lambda+N_i}\right).\end{aligned}$$

So, let us see what we have. Since all primes dividing W_i' are in $\mathcal{Q}_i$, it follows that

$$\frac{\sigma(Q_i)}{Q_i}\le\frac{\sigma(Q_iW_i')}{Q_iW_i'}\le\frac{\sigma(Q_i)}{Q_i}\exp\left(O\left(\sum_{q>y}\frac{1}{q^2}\right)\right)=(1+o(1))\frac{\sigma(Q_i)}{Q_i},$$

for $x\to\infty$. Further, since primes dividing T_i' exceed $c_0\log\log x/\log\log\log x$ and the number of them is $<100\log\log x$, we get that for large x,

$$\frac{\sigma(T_i')}{T_i'}=1+o(1)\quad\text{as}\quad x\to\infty.$$

Since $\lambda\notin\Lambda_4$, $M_i\lambda+N_i$ has $O(\log\log x)$ prime factors all larger than $3z^2>y=(\log\log x)^4$, so

$$\frac{\varphi(M_i\lambda+N_i)}{M_i\lambda+N_i}=1+o(1)\quad\text{as}\quad x\to\infty.$$

Finally,

$$\frac{\sigma(Q_i)\varphi(Q_i)}{Q_i^2}=\prod_{q\in\mathcal{Q}_i}\left(1-\frac{1}{q^2}\right)=\exp\left(O\left(\sum_{q>y}\frac{1}{q^2}\right)\right)=\exp\left(O\left(\frac{1}{y}\right)\right),$$

which is $1+o(1)$ as $x\to\infty$. Summarizing all these observations, we get that

$$\begin{aligned}\frac{(s\circ s_{\varphi})(n+i)}{n+i} &= (1+o(1))\left(\frac{\sigma(i)}{i}\frac{\sigma(Q_i)}{Q_i}-1\right)\left(1-\frac{\varphi(i)}{i}\frac{Q_i}{\sigma(Q_i)}\right)\\ &= (1+o(1))f_i(u_i)=(1+o(1))\alpha_i,\end{aligned}$$

as $x\to\infty$.

Now we deal with $s(n+i)$ given by formula (2.16). Here, much like in the case of $s_{\varphi}(n+i)$, we have

$$\begin{aligned}&\gcd(Q_iU_i(M_i\lambda+N_i),\sigma(i)((U_i+1)/R_i)\sigma(Q_i^2))=1;\\ &\gcd(U_i(M_i\lambda+N_i),\sigma(M_i\lambda+N_i))=1.\end{aligned}$$

Indeed, because the primes in $Q_i U_i(M_i\lambda + N_i)$ are all large, they do not divide $\sigma(i)$. Similarly, $M_i\lambda + N_i$ is coprime to $\sigma(Q_i^2)$. We have U_i coprime to $\sigma(Q_i^2)$ by (2.9). The fact that all prime factors in $\mathcal{Q}_i$ are congruent to 2 modulo 3, implies they cannot divide $\sigma(q^2) = q^2 + q + 1$ for any prime q, so Q_i is coprime to $\sigma(Q_i^2)$. Also, the fact that $\gcd(M_i\lambda + N_i, U_i + 1) = 1$ can be achieved by removing a set of values of λ similar to Λ_7 whose cardinality is $o(x/M)$ for $x \to \infty$. As for the second line above, this is certainly true if we exclude a set of λ similar to Λ_6, namely the set of $\lambda \in \Lambda_{10}$ such that for some $i \in \{1, \ldots, k\}$ we have $\gcd((M_i\lambda + N_i)U_i, \sigma(M_i\lambda + N_i)) > 1$, a set which by the arguments used to deal with Λ_6 can be proved to have cardinality $o(x/M)$ as $x \to \infty$. It then follows that

$$\gcd(\sigma(i)((U_i + 1)/R_i)(\sigma(M_i\lambda + N_i)/i)\sigma(Q_i^2), Q_i U_i(Q_i/R_i)(M_i\lambda + N_i))$$
$$= \gcd(Q_i^2/R_i, \sigma(M_i\lambda + N_i)) =: Z_i,$$

say. Writing

$$S_i = Z_i' S_i',$$

where $Z_i \mid Z_i'$, Z_i' is the largest divisor of S_i supported on the primes from Z_i, we have that S_i' is coprime to Z_i' and all its prime factors exceed $c_0 \log\log x/\log\log\log x$. This last condition holds since $\lambda \notin \Lambda_1$. Since $M_i\lambda + N_i$ is squarefree (using $\lambda \notin \Lambda_3$), we have $\omega(Z_i') \le \sum_{p \mid M_i\lambda + N_i} \omega(p + 1)$. Eliminating a set of λ's similar to Λ_5, let's call it Λ_{11}, but for which there exists $i \in \{1, \ldots, k\}$ and a prime factor p of $M_i\lambda + N_i$ with $\omega(p + 1) > 10 \log\log x$, a set whose cardinality is $o(x/M)$ for $x \to \infty$, we get that Z_i' has $O((\log\log x)^2)$ distinct prime factors all of which exceed $y = (\log\log x)^4$, so

$$\frac{\varphi(Z_i')}{Z_i'} = 1 + O\left(\frac{1}{\log\log x}\right). \tag{2.18}$$

Finally, eliminating a subset of λ denoted Λ_{12} similar to Λ_9 and of cardinality $o(x/M)$ as $x \to \infty$, we can assume that $\omega(S_i') < 100 \log\log x$. As in the previous case, this implies that

$$\frac{\varphi(S_i')}{S_i'} = 1 + o(1),$$

as $x \to \infty$. Thus, using (2.18),

$$\frac{(s_\varphi \circ s)(n + i)}{s(n + i)} = 1 - \frac{\varphi(iR_iS_i)}{iR_iS_i} = 1 - \frac{\varphi(i)}{i}\frac{\varphi(R_iZ_i')}{R_iZ_i'}\frac{\varphi(S_i')}{S_i'}$$
$$= (1 + o(1))\left(1 - \frac{\varphi(i)}{i}\frac{\varphi(R_i)}{R_i}\right),$$
$$= (1 + o(1))\left(1 - \frac{\varphi(i)}{i}\frac{R_i}{\sigma(R_i)}\right)$$

as $x \to \infty$, while

$$\frac{s(n+i)}{n+i} = \frac{\sigma(i)}{i}\left(1+\frac{1}{U_i}\right)\frac{\sigma(M_i\lambda+N_i)}{M_i\lambda+N_i}\prod_{q\in\mathcal{Q}_i}\left(1+\frac{1}{q}+\frac{1}{q^2}\right)-1$$
$$= (1+o(1))\left(\frac{\sigma(i)}{i}\frac{\sigma(Q_i)}{Q_i}-1\right).$$

Thus,

$$\frac{(s_\varphi\circ s)(n+i)}{n+i} = (1+o(1))\left(\frac{\sigma(i)}{i}\frac{\sigma(Q_i)}{Q_i}-1\right)\left(1-\frac{\varphi(i)}{i}\frac{R_i}{\sigma(R_i)}\right)$$
$$= (1+o(1))\left(\frac{\sigma(i)}{i}u_i-1\right)\left(1-\frac{\varphi(i)}{iv_i}\right)$$
$$= (1+o(1))\beta_i$$

for $i \in \{1,\dots,k\}$. Since $n+i=(1+o(1))n$, we get that

$$(s\circ s_\varphi)(n+i) = (\alpha_i+o(1))n \quad \text{while} \quad (s_\varphi\circ s)(n) = (\beta_i+o(1))n$$

as $x \to \infty$. This certainly implies that inequalities (1.1) hold for $(f,g) = (s\circ s_\varphi, s_\varphi\circ s)$ and for our large n as $x \to \infty$, which finishes the proof of the theorem.

3 The proof of Theorem 2

For $n > 1$, let $f(n) = s(n)/s_\varphi(n)$. Simple arguments show that for a prime p, we have $f(np) > f(n)$ (one considers the two cases: $p \mid n$, $p \nmid n$). Further $f(p) = 1$. Thus, $f(n) \geq 1$ for all $n > 1$.

Let $\alpha > 1$ be arbitrary. Assume n is squarefree. Then

$$\sigma(n)\varphi(n) = n^2\prod_{p\mid n}\left(1-\frac{1}{p^2}\right),$$

so if n runs over any sequence of squarefree numbers with least prime tending to infinity, we have $\sigma(n)/n \sim n/\varphi(n)$. Since the reciprocal sum of the primes is divergent, it follows that there is a sequence of squarefree integers $n_1 < n_2 < \dots$ such that the least prime factor of n_i tends to infinity as $i \to \infty$, and at the same time, $\sigma(n_i)/n_i \to \alpha$. Then

$$f(n_i) = \frac{\sigma(n_i)/n_i - 1}{1-\varphi(n_i)/n_i} \to \frac{\alpha-1}{1-1/\alpha} = \alpha, \quad \text{as } i \to \infty.$$

This proves the first assertion of the theorem.

We now turn to the second assertion in Theorem 2. If p, q are different primes, note that

$$\frac{(s \circ s_\varphi)(pq)}{(s_\varphi \circ s)(pq)} = \frac{s(p+q-1)}{s_\varphi(p+q+1)}.$$

Thus it suffices to show that the set of limit points of the rationals of the form $s(m-1)/s_\varphi(m+1)$, where m runs over those numbers that are the sum of two distinct primes, is $[0, \infty)$. If we knew the slightly stronger form of Goldbach's conjecture which asserts that all even numbers at least 8 are the sum of two distinct primes, we could assume that m runs over all even numbers at least 8. It turns out this slightly stronger form of Goldbach's conjecture is "almost" true.

Theorem 4 *There is a positive constant c such that if x is sufficiently large, the number of even numbers in $[1, x]$ which are not the sum of two distinct primes is at most x^{1-c}.*

This result is due to Montgomery and Vaughan [9], with later improvements due to Pintz and others, see [10].

Let $\alpha > 0$ be an arbitrary real number. Let x be large, let $m_1 = m_1(x)$ be the product of all of the odd primes to $y := (\log\log x)^{1/2}$, and let $m_2 = m_2(x) < z := e^{e^y}$ be an integer not divisible by any prime $p \le y$ and such that $\sigma(m_2)/m_2 \to \alpha + 1$ as $x \to \infty$. Now let $m \le x$ run over even integers with

$$m \equiv -1 \pmod{m_1}, \quad m \equiv 1 \pmod{m_2}. \tag{3.1}$$

Note that the number of solutions m to x of this system is of magnitude $x/m_1m_2 > x/z^2$, which is huge compared with the exceptional set in Theorem 4. Thus, most of these numbers m are of the form $p+q$ where p, q are distinct primes. We now show that most of these m also satisfy $s(m-1)/s_\varphi(m+1) \to \alpha$ as $x \to \infty$.

It is clear that $\varphi(m+1) = o(m)$ as $x \to \infty$. Write $m - 1 = m_2m_3 = m_2'm_3'$, where m_2' is the largest divisor of $m-1$ supported on the primes dividing m_2. Since the primes in $m-1$ all exceed y, it is clear that

$$\frac{\sigma(m_2')}{m_2'} = (1+o(1))\frac{\sigma(m_2)}{m_2}, \quad \text{as } x \to \infty.$$

Further, $1 \le \sigma(m_3')/m_3' \le \sigma(m_3)/m_3$. For m_1, m_2 fixed, $m_3 \le (x-1)/m_2$ runs through an arithmetic progression with modulus m_1. Let $g(m) = \sum_{p|m} 1/p$. Since no integer $\le x$ is divisible by two primes $p > \sqrt{x}$,

$$\sum_{m_3} g(m_3) \le \sum_{m_3} \frac{1}{\sqrt{x}} + \sum_{y<p\le\sqrt{x}} \frac{1}{p} \sum_{\substack{m_3 \le (x-1)/m_2 \\ p|m_3}} 1.$$

The inner sum is $\ll x/pm_1m_2$. Thus,

$$\sum_{m_3} g(m_3) \ll \frac{x}{m_1m_2y}.$$

We conclude that the number of choices for m_3 with $g(m_3) > 1/y^{1/2}$ is $o(x/m_1m_2)$ as $x \to \infty$. Hence there are $\gg x/m_1m_2$ choices of $m \le x$ where $g(m_3) \le 1/y^{1/2}$. Applying Theorem 4, we may also assume that these numbers m are the sum of two distinct primes. But

$$\frac{\sigma(m_3)}{m_3} \ll e^{g(m_3)},$$

so we may assume that $\sigma(m_3)/m_3 \to 1$ as $x \to \infty$. Putting the above observations together, we have for our numbers m that

$$\frac{s(m-1)}{s_\varphi(m+1)} = \frac{(1+o(1))(\alpha+1)m - m}{m - o(1)m} = (1+o(1))\alpha, \quad \text{as } x \to \infty.$$

This completes the proof of the second assertion in Theorem 2.

For the last assertion of Theorem 2, we again assume n is of the form pq where p, q are distinct primes. Then

$$\frac{(s \circ s)(n)}{(s_\varphi \circ s_\varphi)(n)} = \frac{s(m+1)}{s_\varphi(m-1)},$$

where $m = p + q$. By interchanging "-1" and "1" in the system (3.1), the above argument allows us to complete the proof of the theorem.

4 The proof of Theorem 3

For an integer $n > 20$, let $a(n)$ denote the largest divisor of n supported on the primes to $y(n) := \log\log n/\log\log\log n$. It follows from [7, Lemma 2.1] that on a set of asymptotic density 1 we have $a(n) = a(s(n)) = \gcd(n, s(n))$. Moreover, the same proof shows that on a set of asymptotic density 1, we have $a(n) = a(s_\varphi(n)) = \gcd(n, s_\varphi(n))$. Let

$$h(n) = \sum_{\substack{p \mid ns(n) \\ y(n) < p < (\log n)^2}} \frac{1}{p}.$$

We will show that there is a set $\mathscr{A}$ of asymptotic density 1 such that

$$H_{\mathscr{A}}(x) := \sum_{n \in (x,2x] \cap \mathscr{A}} h(n) = o(x) \quad \text{as } x \to \infty.$$

It will follow that there is a subset $\mathscr{A}'$ of $\mathscr{A}$ of asymptotic density 1 on which $h(n) = o(1)$ as $n \to \infty$.

Let $y = y(x)$, so that

$$H_{\mathscr{A}}(x) \le \sum_{y<p<(\log(2x))^2} \frac{1}{p} \sum_{\substack{n\in(x,2x]\cap\mathscr{A} \\ p|ns(n)}} 1.$$

The contribution to $H_{\mathscr{A}}(x)$ from the case $p \mid n$ is

$$\ll \sum_{y<p<(\log(2x))^2} \frac{x}{p^2} \ll \frac{x}{y} = o(x) \quad \text{as } x \to \infty.$$

Thus, we may concentrate on the case $p \mid s(n)$. Write $n = Pm$, where P is the largest prime factor of n. By a well-known result of de Bruijn, the number of $n \in (x, 2x]$ with $P \le z := x^{1/\log\log x}$ is $\ll x/\log x$, so we may assume that $\mathscr{A}$ captures the condition $P > z$. Fixing a value of $m \le 2x/z$ and a prime $p \in (y, (\log(2x))^2)$, we consider those primes $P \le x/m$ with $p \mid s(Pm)$. Discarding the case where $P^2 \mid n$ as negligible, we have

$$s(Pm) = Ps(m) + \sigma(m).$$

Since $a(n) = a(s(n)) = \gcd(n, s(n))$ may be assumed to hold for members of $\mathscr{A}$, we have $P \nmid \sigma(n)$, so in particular $P \nmid \sigma(m)$. Thus, having $p \mid s(Pm)$ puts P in a residue class mod p. So, ignoring the condition that P is prime, the number of choices for $P \le 2x/m$ is $\ll x/mp$. Hence

$$H_{\mathscr{A}}(x) \ll \sum_{y<p<(\log x)^2} \sum_{m\in(x/P,2x/P]} \frac{x}{mp^2} \ll \sum_{y<p<(\log x)^2} \frac{x}{p^2} \ll \frac{x}{y},$$

which is $o(x)$ as $x \to \infty$.

By an analogous argument, the same holds if we change s to s_φ. Note also that for any $n \in (x, 2x]$, we have

$$\sum_{\substack{p\ge(\log n)^2 \\ p|ns(n)s_\varphi(n)}} \frac{1}{p} \ll \frac{1}{\log x} = o(1) \quad \text{as } x \to \infty.$$

Thus, there is a set $\mathscr{A}$ of asymptotic density 1 such that for $n \in \mathscr{A}$, we have $a(n) = a(s(n)) = a(s_\varphi(n))$ and

$$\sum_{\substack{p>y(n) \\ p|ns(n)s_\varphi(n)}} \frac{1}{p} = o(1) \quad \text{as } n \to \infty.$$

For each fixed $\varepsilon > 0$, let $\mathscr{A}_\varepsilon$ denote the subset of $\mathscr{A}$ consisting of those numbers n where $\epsilon < s(n)/n < 1/\epsilon$. By the continuity of the distribution function for $s(n)/n$, the density of $\mathscr{A} \setminus \mathscr{A}_\epsilon$ tends to 0 as $\epsilon \to 0$. On $\mathscr{A}_\epsilon$ each of $s(n)/n$, $(s \circ s)(n)/s(n)$, and $(s \circ s_\varphi)(n)/s_\varphi(n)$ is asymptotically equal to $s(a(n))/a(n)$. And, each of $s_\varphi(n)/n$, $(s_\varphi \circ s_\varphi)(n)/s_\varphi(n)$, and $(s_\varphi \circ s)(n)/s(n)$ is asymptotically equal to $s_\varphi(a(n))/a(n)$. The various assertions in Theorem 3 now follow.

Acknowledgements This paper was written when F. L. visited Dartmouth College in Summer 2015. He thanks the Mathematics Department there for its hospitality and support. We thank Paul Pollack for suggesting the relevance of [13]. We also thank the referee for a very careful reading.

References

1. A. Balog, C. Pomerance, The distribution of smooth numbers in arithmetic progressions. Proc. Amer. Math. Soc. **115**, 33–43 (1992)
2. N. Doyon, F. Luca, On the local behavior of the Carmichael λ-function. Mich. Math. J. **54**, 283–300 (2006)
3. P. Erdős, On a problem of G. Golomb. J. Austral. Math. Soc. **2**, 1–8 (1961–1962)
4. P. Erdős, A. Granville, C. Pomerance, C. Spiro, On the normal behaviour of the iterates of some arithmetic functions, in *Analytic Number Theory, Proc. Conf. in honor of Paul T. Bateman*, eds. by B. C. Berndt, et al. (Birkhauser, Boston, 1990), pp. 165–204
5. P. Erdős, K. Győry, Z. Papp, On some new properties of functions $\sigma(n)$, $\varphi(n)$, $d(n)$ and $\nu(n)$. Mat. Lapok **28**, 125–131 (1980)
6. F. Luca, C. Pomerance, On some problems of Makowski-Schinzel and Erdős concerning the arithmetical functions φ and σ. Colloq. Math. **92**, 111–130 (2002)
7. F. Luca, C. Pomerance, The range of the sum-of-proper-divisors function. Acta Arith. **168**, 187–199 (2015)
8. M.-O. Hernane, F. Luca, On the independence of $\sigma(\varphi(n))$ and $\varphi(\sigma(n))$. Acta Arith. **138**, 337–346 (2009)
9. H.L. Montgomery, R.C. Vaughan, The exceptional set in Goldbach's problem. Acta Arith. **27**, 353–370 (1975)
10. J. Pintz, Recent results on the Goldbach conjecture, in: *Elementare und analytische Zahlentheorie*, vol. 20, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, (Franz Steiner Verlag, Stuttgar, 2006), pp. 220–254
11. P. Pollack, C. Pomerance, Some problems of Erdős on the sum-of-divisors function. Trans. Am. Math. Soc. Series B **3**, 1–26 (2016). https://doi.org/10.1090/btran10
12. C. Pomerance, On the distribution of amicable numbers. J. Reine Angew. Math. **293/294**, 217–222 (1977)
13. H.N. Shapiro, Addition of functions in probabilistic number theory. Commun. Pure Appl. Math. **26**, 55–84 (1973)
14. T. Xylouris, Über die Nullstellen der Dirichletschen L-Funktionen und die kleinste Primzahl ineiner arithmetischen Progression, Doctoral dissertation, Universiteit Bonn, 2011
15. N. Lebowitz-Lockard, P. Pollack. J. Number. Theor. **180**, 660–672 (2017)

On the Universal Mock Theta Function g_2 and Zwegers' μ-Function

Richard J. McIntosh

Dedicated to Krishnaswami Alladi on the occasion of his 60th birthday

Abstract Kang discovered a formula expressing the universal mock θ-function g_2 in terms of Zwegers' μ-function and a θ-quotient. By modifying the elliptic variables in μ the θ-quotient can be removed from Kang's formula. We also obtain a formula expressing μ in terms of g_2, proving that μ is not more general than g_2, even though it has one more elliptic variable.

Keywords mock theta function · mock modular form · Jacobi form
Appell–Lerch function

2010 Mathematics Subject Classification 11F27 · 11F37 · 11F50

In their survey paper, [4] Gordon and McIntosh observed that the classical mock θ-functions, including those found by Ramanujan, can be expressed in terms of two 'universal' mock θ-functions denoted by g_2 and g_3. In particular, the mock theta functions of odd order can be expressed in terms of g_3 and those of even order in terms of g_2. They also observed that g_3 can be expressed in terms of g_2. For this reason, they referred to g_2 as the universal mock θ-function.

Throughout this paper, the modern notation $a = e^{2\pi i u}$, $b = e^{2\pi i v}$, and $q = e^{2\pi i \tau}$ will be used. The variables u and v are called *elliptic variables* and τ is called the *modular variable*. A Jacobi form is a function of a modular variable and one or more elliptic variables with appropriate transformation properties. A θ-function is a Jacobi form with one elliptic variable.

R. J. McIntosh (✉)
Department of Mathematics and Statistics, University of Regina, Regina,
SK S4S 0A2, Canada
e-mail: mcintosh@math.uregina.ca

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_28

The mock θ-functions g_2 and g_3 can be defined by

$$g_2(u\,;\tau) = g_2(a,q) = \frac{1}{j(q,q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1-aq^n}\,,$$
$$g_3(u\,;\tau) = g_3(a,q) = \frac{1}{j(q,q^3)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-aq^n}\,,$$

where

$$j(v\,;\tau) = j(b,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} b^n = (b\,;q)_\infty (q/b\,;q)_\infty (q\,;q)_\infty\,.$$

These functions are normalized level 2 and level 3 Appell–Lerch functions. The last equality in the definition of j is Jacobi's Triple Product Identity. The arguments in the functions g_2, g_3, j, ϑ, A_k and μ are sometimes given in terms of a, b, and q instead of u, v, and τ. To help avoid confusion we separate the elliptic variables u and v from the modular variable τ by a semicolon. We do not do this when we are using the variables a, b, and q, except in the definition of the q-shifted factorial. Here the q-shifted factorial is defined by

$$(a\,;q^k)_0 = 1\,,$$
$$(a\,;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\cdots(1-aq^{(n-1)k})$$

and

$$(a\,;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk})\,.$$

When $k=1$ we often write $(a)_n$ and $(a)_\infty$ instead of $(a\,;q)_n$ and $(a\,;q)_\infty$, respectively. For non-negative integers n, we have

$$(a\,;q^k)_n = \frac{(a\,;q^k)_\infty}{(aq^{kn};q^k)_\infty}\,,$$

and for other real n we take this as the definition of $(a\,;q^k)_n$. The function g_2 appears in [2, eq. (2.34)] and the function g_3 appears in [5, eq. (2.0) and Theorem 2.0] (also, see [3, Lemma (7.9)]).

The *level k Appell–Lerch function* [1, 8] is defined by

$$A_k(u,v\,;\tau) = A_k(a,b,q) = a^{k/2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{kn(n+1)/2} b^n}{1-aq^n}\,.$$

The sum in this definition is often called a *generalized Lambert series*.

In his proof [10, 11] that a mock θ-function is essentially a mock modular form (the holomorphic part of a real analytic non-holomorphic modular form) Zwegers defined a normalized level 1 Appell–Lerch function by

$$\mu(u, v; \tau) = \mu(a, b, q) = \frac{A_1(a, b, q)}{\vartheta(b, q)},$$

where

$$\vartheta(v; \tau) = \vartheta(b, q) = -ib^{-1/2}q^{1/8}j(b, q).$$

The function μ behaves nearly like a Jacobi form. Its failure to transform exactly like a Jacobi form depends only on the difference $u - v$.

In an important paper S.-Y. Kang [7] obtained the following formulas for g_2 and g_3 in terms of μ:

$$iag_2(a, q) = \frac{\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2u; 2\tau)} + aq^{-1/4}\mu(2u, \tau; 2\tau),$$

$$ia^{3/2}q^{-1/24}g_3(a, q) = \frac{\eta^3(3\tau)}{\eta(\tau)\vartheta(3u; 3\tau)} + aq^{-1/6}\mu(3u, \tau; 3\tau) + a^2q^{-2/3}\mu(3u, 2\tau; 3\tau),$$

where $\eta(\tau) = q^{1/24}(q; q)_\infty$ is the Dedekind η-function. Another formula for g_3 in terms of level 1 Appell–Lerch functions found in [5, Theorem 2.2] (also, see [6, eq. (4.5)]) is

$$j(a^3z, q^3)g_3(a, q) = \frac{(q; q)_\infty^2 j(az, q)^j(z, q^3)}{j(a, q)^j(z, q)} + az^{1/2}q^{-1/2}A_1(zq, a^3z, q^3)$$
$$+ a^{-1}z^{-1/2}q^{1/2}A_1(z^{-1}q, a^{-3}z^{-1}q^3, q^3).$$

Observe that when $z = 1/a$ the θ-quotient vanishes and we obtain

$$j(a^2, q^3)g_3(a, q) = a^{1/2}q^{-1/2}A_1(a^{-1}q, a^2, q^3) + a^{-1/2}q^{1/2}A_1(aq, a^{-2}q^3, q^3). \tag{1}$$

This identity in a different form is given in [6, Proposition 4.2].

After learning about Zwegers work, the author turned his attention toward finding formulas connecting g_2 and μ. Can these functions be expressed in terms of each other? This connection is important because it brings the classical mock θ-functions of Ramanujan, Watson, Gordon, and McIntosh into the family of mock modular forms and Maass wave forms. At first glance, it appears that μ is more general than g_2 because it has two elliptic variables. It is the purpose of this paper to prove that this is not the case.

The transformation formulas for mock θ-functions (mock modular forms) are more complex than those for θ-functions and Jacobi forms. They involve Mordell integrals. (A discussion of the transformation formulas for mock θ-functions is given in [4].) The transformation laws for μ and g_2 can be combined to eliminate the Mordell integrals. It turns out that

$$i\mu(u, v; \tau) + q^{1/8} g_2(a^{1/2}b^{-1/2}q^{1/4}, q^{1/2}) \tag{2}$$

transforms without a Mordell integral. With the aid of computer algebra and numerical analysis, the author discovered that (2) vanishes when $u + v = \tau/2$, leading to the conjecture that

$$ig_2(a, q) = q^{-1/4}\mu(u, \tau - u; 2\tau)\,. \tag{3}$$

Observe that this removes the θ-quotient from Kang's first formula. It also turns out that if we subtract u from the two elliptic arguments of μ in Kang's second formula, then the θ-quotient disappears and we obtain

$$ia^{3/2}q^{-1/24}g_3(a, q) = aq^{-1/6}\mu(2u, \tau - u; 3\tau) + a^2q^{-2/3}\mu(2u, 2\tau - u; 3\tau)\,.$$

This identity is equivalent to (1). A generalization of this property for related level k Appell–Lerch functions is proved in [9].

We now prove (3). Since

$$ig_2(a, q) = \frac{\eta^4(2\tau)}{a\eta^2(\tau)\vartheta(2u; 2\tau)} + q^{-1/4}\mu(2u, \tau; 2\tau)$$

by Kang's first formula, it suffices to show

$$\mu(u, \tau - u; 2\tau) - \mu(2u, \tau; 2\tau) = \frac{q^{1/4}\eta^4(2\tau)}{a\eta^2(\tau)\vartheta(2u; 2\tau)}\,.$$

By Propositions 1.4(7) and 1.3(10) of [11] (also known to Lerch [8]), we have

$$\mu(u+z, v+z) - \mu(u, v) = \frac{\vartheta'(0)\vartheta(u+v+z)\vartheta(z)}{2\pi i\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)} = \frac{i\eta^3(\tau)\vartheta(u+v+z)\vartheta(z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}\,. \tag{4}$$

With $\tau \to 2\tau$, $u \to 2u$, $v \to \tau$, and $z \to -u$, this becomes

$$\begin{aligned}\mu(u, \tau - u; 2\tau) - \mu(2u, \tau; 2\tau) &= \frac{i\eta^3(2\tau)\vartheta(u+\tau; 2\tau)\vartheta(-u; 2\tau)}{\vartheta(2u; 2\tau)\vartheta(\tau; 2\tau)\vartheta(u; 2\tau)\vartheta(\tau - u; 2\tau)}\\ &= \frac{i\eta^3(2\tau)\vartheta(u+\tau; 2\tau)}{\vartheta(2u; 2\tau)\vartheta(\tau; 2\tau)\vartheta(u - \tau; 2\tau)}\,,\end{aligned}$$

since $\vartheta(u)$ is an odd function. By its behavior with respect to the quasi-period [11, Proposition 1.3(2)],

$$\vartheta(u+\tau\,;2\tau) = \vartheta((u-\tau)+2\tau\,;2\tau) = -a^{-1}\vartheta(u-\tau\,;2\tau)\,,$$

and by the identity

$$\vartheta(\tau\,;2\tau) = -iq^{-1/4}j(q,q^2) = -iq^{-1/4}\eta^2(\tau)/\eta(2\tau)\,,$$

we obtain

$$\mu(u,\tau-u\,;2\tau)-\mu(2u,\tau\,;2\tau) = \frac{i\eta^3(2\tau)\vartheta(u+\tau\,;2\tau)}{\vartheta(2u\,;2\tau)\vartheta(\tau\,;2\tau)\vartheta(u-\tau\,;2\tau)} = \frac{q^{1/4}\eta^4(2\tau)}{a\eta^2(\tau)\vartheta(2u\,;2\tau)}\,,$$

which completes the proof of (3).

We can use (3) and (4) to express (2) as a Jacobi form. Observe that

$$\begin{aligned}
&iq^{1/8}g_2(a^{1/2}b^{-1/2}q^{1/4},q^{1/2})-\mu(u,v\,;\tau)\\
&\quad= \mu(\tfrac{u}{2}-\tfrac{v}{2}+\tfrac{\tau}{4}\,,\tfrac{v}{2}-\tfrac{u}{2}+\tfrac{\tau}{4}\,;\tau)-\mu(u,v\,;\tau)\\
&\quad= \mu\big(u-(\tfrac{u}{2}+\tfrac{v}{2}-\tfrac{\tau}{4})\,,v-(\tfrac{u}{2}+\tfrac{v}{2}-\tfrac{\tau}{4})\,;\tau\big)-\mu(u,v\,;\tau)\\
&\quad= \frac{-i\eta^3(\tau)\vartheta(\frac{u}{2}+\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{u}{2}+\frac{v}{2}-\frac{\tau}{4}\,;\tau)}{\vartheta(u\,;\tau)\vartheta(v\,;\tau)\vartheta(\frac{u}{2}-\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{v}{2}-\frac{u}{2}+\frac{\tau}{4}\,;\tau)}\,.
\end{aligned}$$

Hence (2) is equal to the Jacobi form

$$\frac{-\eta^3(\tau)\vartheta(\frac{u}{2}+\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{u}{2}+\frac{v}{2}-\frac{\tau}{4}\,;\tau)}{\vartheta(u\,;\tau)\vartheta(v\,;\tau)\vartheta(\frac{u}{2}-\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{v}{2}-\frac{u}{2}+\frac{\tau}{4}\,;\tau)}\,.$$

It now follows that

$$\begin{aligned}
\mu(u,v\,;\tau) &= iq^{1/8}g_2(a^{1/2}b^{-1/2}q^{1/4},q^{1/2})\\
&\qquad+\frac{i\eta^3(\tau)\vartheta(\frac{u}{2}+\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{u}{2}+\frac{v}{2}-\frac{\tau}{4}\,;\tau)}{\vartheta(u\,;\tau)\vartheta(v\,;\tau)\vartheta(\frac{u}{2}-\frac{v}{2}+\frac{\tau}{4}\,;\tau)\vartheta(\frac{v}{2}-\frac{u}{2}+\frac{\tau}{4}\,;\tau)}\\
&= iq^{1/8}g_2(a^{1/2}b^{-1/2}q^{1/4},q^{1/2})\\
&\qquad-\frac{iq^{1/8}(q;q)_\infty^3\,j(a^{1/2}b^{1/2}q^{1/4},q)\,j(a^{1/2}b^{1/2}q^{-1/4},q)}{j(a,q)\,j(b,q)\,j(a^{1/2}b^{-1/2}q^{1/4},q)\,j(a^{-1/2}b^{1/2}q^{1/4},q)}\,.
\end{aligned}$$

This equation proves that Zwegers' function μ is not more general than g_2. So both μ and g_2 deserve to be called universal mock theta functions. However, with two elliptic variables, the transformation formulas [11, pp. 8–9] for μ are symmetric and more simple than those for g_2.

References

1. P. Appell, Sur les fonctions doublement périodiques de troisième espèce. Annales Scientifiques de l'École Normale Supérieure **1**, 135–164 (1884)
2. Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. Invent. Math. **136**, 497–569 (1999)
3. F.G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11. Trans. AMS **305**, 47–77 (1988)
4. B. Gordon, R.J. McIntosh, A survey of classical mock theta functions, *Partitions, q-Series, and Modular Forms*, vol. 23, Developments in Mathematics (Springer, New York, 2012), pp. 95–144
5. D. Hickerson, A proof of the mock theta conjectures. Invent. Math. **94**, 639–660 (1988)
6. D. Hickerson, E. Mortenson, Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I. Proc. Lond. Math. Soc. **109**, 382–422 (2014)
7. S.-Y. Kang, Mock Jacobi forms in basic hypergeometric series. Compositio Mathematica **145**, 553–565 (2009)
8. M. Lerch, Bemerkungen zur Theorie der elliptischen Funktionen. Jahrbuch über die Fortschritte der Mathematik **24**, 442–445 (1892)
9. R.J. McIntosh, Some identities for Appell–Lerch sums and a universal mock theta function, to appear in Ramanujan J
10. S.P. Zwegers, Mock ϑ-functions and real analytic modular forms, in *q-Series with Applications to Combinatorics, Number Theory, and Physics*, vol. 291, Contemporary Mathematics, ed. by B.C. Berndt, K. Ono (2001), pp. 269–277
11. S.P. Zwegers, Mock Theta Functions, Ph.D. thesis, Universiteit Utrecht, 2002

Mock Theta Function Identities Deriving from Bilateral Basic Hypergeometric Series

James Mc Laughlin

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract The bilateral series corresponding to many of the third-, fifth-, sixth-, and eighth-order mock theta functions may be derived as special cases of ${}_2\psi_2$ series

$$\sum_{n=-\infty}^{\infty} \frac{(a, c; q)_n}{(b, d; q)_n} z^n.$$

Three transformation formulae for this series due to Bailey are used to derive various transformation and summation formulae for both these mock theta functions and the corresponding bilateral series. New and existing summation formulae for these bilateral series are also used to make explicit in a number of cases the fact that for a mock theta function, say $\chi(q)$, and a root of unity in a certain class, say ζ, that there is a theta function $\theta_\chi(q)$ such that

$$\lim_{q \to \zeta} (\chi(q) - \theta_\chi(q))$$

exists, as $q \to \zeta$ from within the unit circle.

This work was partially supported by a grant from the Simons Foundation (#209175 to James Mc Laughlin).

J. Mc Laughlin (✉)
Mathematics Department, West Chester University, 25 University Avenue,
West Chester, PA 19383, USA
e-mail: jmclaughlin2@wcupa.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_29

Keywords Mock theta functions · Basic hypergeometric series · ${}_2\psi_2$ bilateral series · Basic hypergeometric summation formulae · Explicit radial limits for mock theta functions

2010 Mathematics Subject Classification 33D15 · 11F27 · 11B65

1 Introduction

The mock theta functions were introduced to the world by Ramanujan in his last letter to G.H. Hardy ([24, pp. 354–355], [9, pp. 220–223]), in which he also gave examples of mock theta functions of orders three, five, and seven. Ramanujan did not explain precisely what he meant by a mock theta function, and Ramanujan's statements were interpreted by Andrews and Hickerson [5] to mean a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and which satisfies the following two conditions:

(0) For every root of unity ζ, there is a θ-function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(1) There is no single θ-function which works for all ζ; i.e., for every θ-function $\theta(q)$, there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

A similar definition was given by Gordon and McIntosh [15, 17], where they also distinguish between a mock theta function and a "strong" mock theta function. The modern view of mock theta functions is based on the work of Zwegers [29, 30], who showed that the mock theta functions are holomorphic parts of certain harmonic weak Maass forms.

In relation to the results in the present paper, we recall two areas of investigation in the subject of mock theta functions. First, as regards condition (0) above, Folsom, Ono, and Rhoades [13] make this condition explicit for the third-order mock theta function $f(q)$, in that they found a formula for the θ-function $\theta_\zeta(q)$ and an expression for the limit of the difference $f(q) - \theta_\zeta(q)$ as $q \to \zeta$ radially, where ζ is a primitive even-order root of unity (see Theorem 3.4 below). Second, there is the subject of basic hypergeometric transformations of mock theta functions, and summation formula for sums/differences of mock theta functions. Several identities of these types were stated by Ramanujan [24] and were subsequently investigated by Watson [26], and later work was carried out by Andrews [1, 2], and more recently by Gordon and McIntosh [16, 17].

The starting point for the investigation in the present paper is the observation that many of the mock theta functions are special cases of one "side" ($n \geq 0$ or $n < 0$) of certain general bilateral series, bilateral series which in turn derive from the ${}_2\psi_2$ series

$$\sum_{n=-\infty}^{\infty} \frac{(a,c;q)_n}{(b,d;q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a,c;q)_n}{(b,d;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b,q/d;q)_n}{(q/a,q/c;q)_n} \left(\frac{bd}{acz}\right)^n.$$

These include:

Third order—all 9 (Ramanujan, Watson, Gordon, and McIntosh);

Fifth order—8 of 10 (Ramanujan);

Sixth order—8 (Ramanujan);

Eighth order—4 of 8 (Gordon and McIntosh).

A number of transformations and summation formula for the ${}_2\psi_2$ series due to Bailey [6] are combined with the representation of these mock theta functions in terms of the ${}_2\psi_2$ series, together with other existing summation and transformation formulae for q-series, to derive new representations for the mock theta functions, and other q-series identities.

Results in the present paper include:
1) radial limit results for a number of third-, fifth-, sixth-, and eighth-order mock theta functions similar to that of Folsom, Ono, and Rhoades [13] alluded to above,
2) new summation formulae for the bilateral series associated with some of these order mock theta functions,
3) new transformation formulae for some of these mock theta functions deriving from these general bilateral transformations,
4) a number of other summation formulae.

One example of a new summation formula is the following identity for the third-order mock theta function $\phi(q)$:

$$\phi(q) + \sum_{r=1}^{\infty} (-1;q^2)_r q^r = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = \frac{(-q,-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty}.$$

This formula, in turn, implies that if ζ is a primitive even-order $4k$ root of unity, then as q approaches ζ radially within the unit disk,

$$\lim_{q\to\zeta} \left(\phi(q) - \frac{(q^2,-q,-q;q^2)_\infty}{(-q^2,q;q^2)_\infty}\right) = -2\sum_{n=0}^{k-1} (1+\zeta^2)(1+\zeta^4)\ldots(1+\zeta^{2n})\zeta^{n+1}.$$

For the third-order mock theta function $\psi(q)$, there is the transformation

$$\psi(q) = -\sum_{n=0}^{\infty} (q;q^2)_n (-1)^n + \frac{1}{2(q^2;q^2)_\infty^2} \sum_{r=-\infty}^{\infty} q^{2r^2+r}(4r+1)(-1)^r.$$

Note: See the remark at the end of the proof of Theorem 3.2 *about the convergence of the first series on the right.*

As an example of one of the new summation formulae, there is the following:

$$\sum_{r=-\infty}^{\infty}(10r+1)q^{(5r^2+r)/2}=$$
$$\left(\frac{4q(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty}+\frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty}.$$

A number of results of a similar nature may be found throughout the paper.

Remark: The first version of the present paper was written in 2014, and subsequently the author attention was directed (my thanks to the anonymous referee) to a number of recent papers containing similar results, of which the present author was previously unaware (see [7], for example).

In [21], Mortenson derived several identities involving the Appell–Lerch sum

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}\frac{(-1)^r q^{r(r-1)/2}z^r}{1-q^{r-1}xz}, \tag{1.1}$$

(here $j(z;q)=(z,q/z,q;q)_\infty$) and the universal mock theta function $g(x,q)$ (see (3.1)), and some of these were used in [22] to derive explicit radial limits for mock theta functions. As well deriving such radial limits for several particular mock theta functions, in [22] the author also derives a general result for $g(x,q)$, a result which permits an explicit radial limit to be derived for any even-order mock theta function that may be expressed in terms of $g(x,q)$. We will compare results in the present paper with those in [21, 22] in several places throughout the paper.

For example, by applying a formula of Mortenson ([22, Eq. (6.10)]), a different radial limit result is obtained for the eighth-order mock theta function $S_0(q)$ (see (6.18)).

Subsequent to writing the first draft of the present paper, the author was also directed to the recent paper [7], in which the authors also derive explicit radial limits for all of Ramanujan's third- and fifth-order mock theta functions, as well as giving the level and weight information for the theta functions (which are modular forms). The authors in [7] also state, without proof, explicit radial limits for many of the even-order mock theta functions.

In the present paper, we also derive these explicit radial limits using somewhat different methods, but in addition also derive many identities that come from the aforementioned connections with the ${}_2\psi_2$ series.

2 Some required basic hypergeometric formulae

To prove some of the results in the present paper, it is necessary to use a number of transformation and summation formulae for basic hypergeometric series.

$$(a;q)_{-n} := \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{n(n-1)/2}, \tag{2.1}$$

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = (zq, q/z, q^2; q^2)_\infty. \tag{2.2}$$

$$\sum_{n=-\infty}^{\infty} \frac{(a,c;q)_n}{(b,d;q)_n} z^n = \frac{(az, cz, qb/acz, qd/acz; q)_\infty}{(b, d, q/a, q/c; q)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(acz/b, acz/d; q)_n}{(az, cz; q)_n} \left(\frac{bd}{acz}\right)^n. \tag{2.3}$$

$$\sum_{n=-\infty}^{\infty} \frac{(a,c;q)_n}{(b,d;q)_n} z^n = \frac{(b/a, d/c, az, qb/acz; q)_\infty}{(b, q/c, z, bd/caz; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(a, acz/b; q)_n}{(az, d; q)_n} \left(\frac{b}{a}\right)^n. \tag{2.4}$$

$$\sum_{n=-\infty}^{\infty} \frac{(e,f;q)_n}{(aq/c, aq/d;q)_n} \left(\frac{qa}{ef}\right)^n = \frac{(q/c, q/d, aq/e, aq/f; q)_\infty}{(aq, q/a, aq/cd, aq/ef; q)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(c,d,e,f;q)_n}{(1-a)(aq/c, aq/d, aq/e, aq/f;q)_n} \left(\frac{qa^3}{cdef}\right)^n q^{n^2}. \tag{2.5}$$

$$\sum_{n=-\infty}^{\infty} \frac{(q\sqrt{a}, -q\sqrt{a}, b, c, d, e; q)_n}{(\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e; q)_n} \left(\frac{qa^2}{bcde}\right)^n = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}. \tag{2.6}$$

$$\sum_{n=-\infty}^{\infty} \frac{(b,c;q)_n}{(aq/b, aq/c;q)_n} \left(\frac{-qa}{bc}\right)^n = \frac{(aq/bc;q)_\infty (aq^2/b^2, aq^2/c^2, q^2, aq, q/a; q^2)_\infty}{(aq/b, aq/c, q/b, q/c, -qa/bc; q)_\infty}. \tag{2.7}$$

The identity at (2.2) is the famous Jacobi triple product identity. The bilateral transformations at (2.3), (2.4), and (2.5) are all due to Bailey [6]. The identity at (2.6) Bailey's ${}_6\psi_6$ summation formula and (2.7) is a special case of this (see [14, Eq. (II.30), p. 357]).

3 Mock theta functions of the third order

The third-order mock theta functions stated by Ramanujan ([24, pp. 354–355], [9, pp. 220–223]) are the following basic hypergeometric series:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q,-q;q)_n}, \qquad \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n},$$
$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n}, \qquad \psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}.$$

All of the third-order mock theta functions of Ramanujan, as well as those stated later by Watson [26] and Gordon and McIntosh [16], may be expressed in terms of the function $g(x,q)$, where

$$g(x,q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(x,q/x;q)_{n+1}} = x^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x;q)_{n+1}(q/x;q)_n}\right). \tag{3.1}$$

This was shown by Hickerson and Mortenson [18, Eqs. (5.4)–(5.10)] (this function was also defined by Gordon and McIntosh [17], where it was labeled "$g_3(x,q)$"). For completeness, we consider a generalization, namely the series

$$G_3(s,t,q) := 1 + \sum_{n=1}^{\infty} \frac{s^n t^n q^{n^2}}{(sq,tq;q)_n}, \tag{3.2}$$

which was defined in [10, Eq. (7)], and state a number of transformation formulae for this function. Note that the connection with the third-order mock theta functions is that

$$G_3(x,q/x,q) = (1-x)(1-q/x)g(x,q). \tag{3.3}$$

Proposition 3.1. *Let $G_3(s,t,q)$ be as defined at (3.2) above. Then,*

$$G_3(s,t,q) = -\sum_{r=1}^{\infty}(s^{-1},t^{-1};q)_r q^r + \frac{(q/s,q/t;q)_\infty}{(sq,tq;q)_\infty}\sum_{r=-\infty}^{\infty}(s,t;q)_r q^r; \tag{3.4}$$

$$= -\sum_{r=1}^{\infty}(s^{-1}, t^{-1}; q)_r q^r + \frac{(q/t; q)_\infty}{(sq, q; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(t; q)_r (-s)^r q^{r(r+1)/2}}{(tq; q)_r}; \quad (3.5)$$

$$= -\sum_{r=1}^{\infty}(s^{-1}, t^{-1}; q)_r q^r \quad (3.6)$$

$$+ \frac{(q/s, q/t; q)_\infty}{(stq, q/(st), q; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1 - stq^{2r})(s, t; q)_r (st)^{2r} q^{2r^2}}{(1 - st)(sq, tq; q)_r}.$$

Proof. The transformations at (3.4) and (3.5) will follow as special cases of two more general identities. Replace z with zq/ac, let $a, c \to \infty$ and set $b = sq$ and $d = tq$ in, respectively, (2.3) and (2.4), to get that

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(sq, tq; q)_n} = \frac{(sq/z, tq/z; q)_\infty}{(sq, tq; q)_\infty} \sum_{r=-\infty}^{\infty} (z/s, z/t; q)_r \left(\frac{stq}{z}\right)^r, \quad (3.7)$$

$$= \frac{(qs/z; q)_\infty}{(sq, stq/z; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(z/s; q)_r (-s)^r q^{r(r+1)/2}}{(tq; q)_r}. \quad (3.8)$$

Lastly, replace z with st, and use (2.1) on the terms of negative index in the new series on the left sides.

We also prove a generalization of the transformation at (3.6) first, by letting $e, f \to \infty$ in (2.5), and then replacing a with z, c with z/s, and d with z/t, to get

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(sq, tq; q)_n}$$

$$= \frac{(sq/z, tq/z; q)_\infty}{(zq, q/z, stq/z; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1 - zq^{2r})(z/s, z/t; q)_r (zst)^r q^{2r^2}}{(1 - z)(sq, tq; q)_r}. \quad (3.9)$$

The identity at (3.6) follows after replacing z with st. □

The identities (3.4)–(3.6) may be more concisely expressed using the function

$$G_3^*(s, t, q) := \sum_{n=-\infty}^{\infty} \frac{s^n t^n q^{n^2}}{(sq, tq; q)_n} \quad (3.10)$$

as follows

$$G_3^*(s,t,q) = \frac{(q/s,q/t;q)_\infty}{(sq,tq;q)_\infty} G_3^*(s^{-1},t^{-1},q), \tag{3.11}$$

$$= \frac{(q/t;q)_\infty}{(sq,q;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(t;q)_r(-s)^r q^{r(r+1)/2}}{(tq;q)_r}; \tag{3.12}$$

$$= \frac{(q/s,q/t;q)_\infty}{(stq,q/(st),q;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-stq^{2r})(s,t;q)_r(st)^{2r}q^{2r^2}}{(1-st)(sq,tq;q)_r}. \tag{3.13}$$

The identity at (3.4) (or (3.12)) was also proved by Choi [10, Theorem 4], and stated previously by Ramanujan (see [4, Entry 3.4.7]).

We will employ (3.5) to derive some results on explicit radial limits, as mentioned earlier. Before coming to that, we remark that other transformations listed above may be used to derive some new transformations for three of the third order mock theta functions of Ramanujan and one of the third-order mock theta functions of Watson (similar results may be derived for the other third-order mock theta functions of Watson [26] and those of Gordon and McIntosh [16]). Before stating the next theorem, we recall Watson's [26] third-order mock theta function $\nu(q)$, where

$$\nu(q) = \sum_{r=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Theorem 3.2. *If* $|q| < 1$*, then*

$$f(q) = -\sum_{n=1}^{\infty}(-1,-1;q)_n q^n + 4\frac{(-q;q)_\infty^2}{(q;q)_\infty^3} \sum_{r=-\infty}^{\infty} \frac{q^{2r^2+r}(4rq^r+1)}{(1+q^r)^2}. \tag{3.14}$$

$$\phi(q) = -\sum_{n=1}^{\infty}(-1;q^2)_n q^n + 4\frac{(-q^2;q^2)_\infty}{(q;q)_\infty^3} \sum_{r=-\infty}^{\infty} \frac{q^{2r^2+2r}(2rq^{2r}+1)}{(1+q^{2r})^2}. \tag{3.15}$$

$$\nu(q) = -\sum_{n=0}^{\infty}(-q;q^2)_n q^n + 4\frac{(-q;q^2)_\infty}{(q;q)_\infty^3} \sum_{r=-\infty}^{\infty} \frac{q^{2r^2+2r}(r+1)}{(1+q^{2r+1})^2} - 2\frac{(-q;q^2)_\infty}{(q;q)_\infty^3}(-q^4,-q^{12},q^{16};q^{16})_\infty. \tag{3.16}$$

$$\psi(q) = -\sum_{n=0}^{\infty}(q;q^2)_n(-1)^n + \frac{1}{2(q^2;q^2)_\infty^2} \sum_{r=-\infty}^{\infty} q^{2r^2+r}(4r+1)(-1)^r. \tag{3.17}$$

Proof. For (3.14), replace z with z^2, s and t with $-z$ in (3.9), and then let $z \to 1$.

A similar application of (3.9), again with z replaced with z^2, s replaced with iz, and t replaced with $-iz$ and once again letting $z \to 1$ leads to (3.15).

For (3.16), in (3.9) again replace z with z^2, and then replaces s with iz, t with $-iz$, let $z \to \sqrt{q}$ and divide through by $1+q$.

Finally, the transformation at (3.17) follows similarly from (3.9), this time with z replaced with z^2, s replaced with $z/\sqrt{q}$, and t replaced with $-z/\sqrt{q}$ and once again letting $z \to 1$. Note that convergence of the first series on the right of (3.17) is in the Cesàro sense. □

As Watson pointed out in [27, Section 7], certain bilateral series related to fifth-order mock theta functions, which are essentially the sums of pairs of fifth-order mock theta functions, are expressible as theta functions, or combinations of infinite q-products. It seems less well known that the bilateral series associated with two of Ramanujan's third-order mock theta functions are also expressible as infinite products. We also give similar statement for Watson's [26] third-order mock theta function $\nu(q)$.

Theorem 3.3. *If* $|q| < 1$, *then*

$$\phi(q) + \sum_{r=1}^{\infty}(-1;q^2)_r q^r = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = \frac{(-q,-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty}; \tag{3.18}$$

$$\nu(q) + \sum_{r=0}^{\infty}(-q;q^2)_r q^r = \sum_{r=-\infty}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} = 2(-q^2,-q^2;q^2)_\infty (q^4;q^4)_\infty; \tag{3.19}$$

$$\psi(q) + \sum_{r=0}^{\infty}(q;q^2)_r (-1)^r = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \frac{(-q,-q,q^2;q^2)_\infty}{2(q,-q^2;q^2)_\infty}. \tag{3.20}$$

Proof. From (2.7) (replace q with q^2, set $b = -z/t$, $a = -z$, and let $c \to \infty$),

$$\sum_{r=-\infty}^{\infty} \frac{(-z/t;q)_r t^r q^{r(r+1)/2}}{(tq;q)_r} = \frac{(-t^2q^2/z,-zq,-q/z,q^2;q^2)_\infty}{(tq,-tq/z;q)_\infty},$$

and from (3.8) (with $s = -t$),

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(t^2q^2;q^2)_n} = \frac{(-tq/z;q)_\infty}{(-tq,-t^2q/z;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-z/t;q)_r (t)^r q^{r(r+1)/2}}{(tq;q)_r}.$$

Together, these equations imply that

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(t^2q^2;q^2)_n} = \frac{(-zq,-q/z,q^2;q^2)_\infty}{(t^2q^2,-t^2q/z;q^2)_\infty}. \tag{3.21}$$

The identity at (3.18) is now immediate upon setting $z = 1$ and $t^2 = -1$, and that at (3.20) results similarly upon setting $z = 1$ and $t^2 = 1/q$. The identity at (3.19) follows upon setting $z = q, t^2 = -q$, multiplying the resulting product by $1/(1+q)$, and finally performing some elementary q-product manipulations. □

Note that the convergence of the sum added to $\psi(q)$ on the left side of (3.20) is in the Cesàro sense. Note also that comparison of the infinite products on the right sides of (3.18) and (3.20) yields the rather curious identity

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = 2\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{3.22}$$

where, by the previous comment, convergence of the part of the bilateral series on the right consisting of terms of negative index is again in the Cesàro sense.

The summation formulae in the preceding theorem have some interesting implications. First, they allow condition (0) above to be made explicit for some of the third-order mock theta functions. We recall the recent result for $f(q)$ in [13].

Theorem 3.4. *(Folsom, Ono, and Rhoades [13]) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}(f(q) - (-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\ldots(1+\zeta^n)^2\zeta^{n+1}. \tag{3.23}$$

Here,

$$b(q) = \frac{(q;q)_\infty}{(-q;q)_\infty^2}.$$

The infinite product representation of $b(q)$ was not stated in [13] but was stated by Rhoades in [25]. Note that Theorem 3.4 was also proved recently by Zudilin [28].

The following results are immediate upon rearranging the identities in Theorem 3.3, and letting q tend radially to the specified root of unity from within the unit circle, since the other series accompanying each of the mock theta functions in the bilateral sums terminates (the interchange of summation and limit in each of the corresponding series on the right is justified by the absolute convergence of each of these series).

Corollary 3.5. *(i) If ζ is a primitive even-order $4k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(\phi(q) - \frac{(q^2,-q,-q;q^2)_\infty}{(-q^2,q;q^2)_\infty}\right) = -2\sum_{n=0}^{k-1}(1+\zeta^2)(1+\zeta^4)\ldots(1+\zeta^{2n})\zeta^{n+1}. \tag{3.24}$$

(ii) If ζ is a primitive even-order $4k+2$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\nu(q)-2(-q^2;q^2)_\infty^2(q^4;q^4)_\infty\right)=-\sum_{n=0}^{k}(1+\zeta)(1+\zeta^3)\dots(1+\zeta^{2n-1})\zeta^n. \tag{3.25}$$

(iii) If ζ is a primitive odd-order $2k+1$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\psi(q)-\frac{(q^2,-q,-q;q^2)_\infty}{2(-q^2,q;q^2)_\infty}\right)=-\sum_{n=0}^{k}(1-\zeta)(1-\zeta^3)\dots(1-\zeta^{2n-1})(-1)^n. \tag{3.26}$$

Remark: The results in Corollary 3.5 were also proved in [7], using somewhat similar arguments, as were the results in Corollary 4.2 below.

The second implication is that they imply some summation formulae for some of the bilateral series appearing in Theorem 3.2.

Corollary 3.6. *If $|q|<1$, then*

$$\sum_{r=-\infty}^{\infty}\frac{q^{r^2+r}(2rq^r+1)}{(1+q^r)^2}=\frac{(q;q^2)_\infty^4(q;q)_\infty^4}{4} \tag{3.27}$$

$$\sum_{r=-\infty}^{\infty}\frac{q^{2r^2+2r}(r+1)}{(1+q^{2r+1})^2}=\frac{(-q^2;q^2)_\infty^2(q;q)_\infty^3(q^4;q^4)_\infty}{2(-q;q^2)_\infty}+\frac{(-q^4,-q^{12},q^{16};q^{16})_\infty}{2} \tag{3.28}$$

Proof. The first identity (3.27) follows from combining the results at (3.15) and (3.18) and then replacing q^2 with q. The identity at (3.28) follows directly from comparing the identities (3.16) and (3.19). □

4 Mock theta functions of the fifth order

Ramanujan's fifth-order mock theta functions are the following:

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}, \qquad f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q)_n},$$

$$F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n}, \qquad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}},$$

$$\phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n, \qquad \phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}(-q;q^2)_n,$$

$$\psi_0(q)=\sum_{n=0}^{\infty}q^{(n+1)(n+2)/2}(-q;q)_n, \qquad \psi_1(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}(-q;q)_n,$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n (q;q)_n}{(q;q)_{2n}}, \qquad \chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n (q;q)_n}{(q;q)_{2n+1}}.$$

Of interest, here is the fact that that certain combinations of pairs of mock theta functions of order five may be expressed as single bilateral series, and hence in terms of theta products, as was described by Watson in section 7 of [27] (see also the forthcoming book [19], where the proofs of these identities are possibly more transparent than those of Watson [27]). We state these identities directly in terms of q-products, rather than employing the Ramanujan functions $G(q)$ and $H(q)$, as Watson did.

Proposition 4.1. *The following identities hold.*

$$\sum_{n=-\infty}^{\infty} \frac{q^{r^2}}{(-q;q)_r} = f_0(q) + 2\psi_0(q) = 4q\frac{(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty} + \frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}. \tag{4.1}$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{r^2+r}}{(-q;q)_r} = f_1(q) + 2\psi_1(q) = 4\frac{(q^8,q^{12},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty} - \frac{(q,q^4,q^5;q^5)_\infty}{(-q;q)_\infty}. \tag{4.2}$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{4r^2}}{(q^2;q^4)_r} = F_0(q^2) + \phi_0(-q^2) - 1 = q\frac{(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty} + \frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}. \tag{4.3}$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{4r^2+4r}}{(q^2;q^4)_{r+1}} = F_1(q^2) - \frac{\phi_1(-q^2)}{q^2} = \frac{(q^8,q^{12},q^{20};q^{20})_\infty}{q(q^2;q^4)_\infty} - \frac{(q,q^4,q^5;q^5)_\infty}{q(-q;q)_\infty}. \tag{4.4}$$

We note that these summation formulae may be rearranged and used to give explicit radial limits for the difference of certain fifth-order mock theta functions and certain corresponding theta functions, as q tends to certain roots of unity from within the unit circle, in a manner similar to the result of Folsom, Ono, and Rhoades [13] for the third-order mock theta function $f(q)$ stated at (3.23) above, or to the results stated for the third-order mock theta functions $\phi(q)$, $\nu(q)$ and $\psi(q)$ stated in Corollary 3.5. For ease of notation, the statements for $F_0(q)$ and $F_1(q)$ are written in terms of q^2 instead of q.

Corollary 4.2. *(i) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(f_0(q) - \left[4q\frac{(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty} + \frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}\right]\right)$$

$$= -2\sum_{n=0}^{k-1}(1+\zeta)(1+\zeta^2)\dots(1+\zeta^n)\zeta^{(n+1)(n+2)/2}. \qquad (4.5)$$

(ii) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(f_1(q)-\left[4\frac{(q^8,q^{12},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty}-\frac{(q,q^4,q^5;q^5)_\infty}{(-q;q)_\infty}\right]\right)$$
$$= -2\sum_{n=0}^{k-1}(1+\zeta)(1+\zeta^2)\dots(1+\zeta^n)\zeta^{n(n+1)/2}. \qquad (4.6)$$

(iii) If ζ is a primitive even-order $4k+2$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(F_0(q^2)-\left[q\frac{(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty}+\frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}\right]\right)$$
$$= -\sum_{n=1}^{k}(1-\zeta^2)(1-\zeta^6)\dots(1-\zeta^{4n-2})(-1)^n\zeta^{2n^2}. \quad (4.7)$$

(iv) If ζ is a primitive even-order $4k+2$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(F_1(q^2)-\left[\frac{(q^8,q^{12},q^{20};q^{20})_\infty}{q(q^2;q^4)_\infty}-\frac{(q,q^4,q^5;q^5)_\infty}{q(-q;q)_\infty}\right]\right)$$
$$= -\sum_{n=0}^{k}(1-\zeta^2)(1-\zeta^6)\dots(1-\zeta^{4n-2})(-1)^n\zeta^{2n^2+4n}. \qquad (4.8)$$

There are no known unilateral transformations for mock theta functions of the fifth order similar to those that exist for mock theta functions of the third order. However, there are bilateral transformations that may be applied to the bilateral series in Proposition 4.1.

Here, we consider the series

$$G_5^*(w,y,q) = \sum_{n=-\infty}^{\infty}\frac{w^n q^{n^2}}{(y;q)_n}. \qquad (4.9)$$

Identities for this function are not so plentiful as those for $G_3(s,t,q)$ and $G_3^*(s,t,q)$, but two such are given in the next theorem.

Theorem 4.3. *Let $G_5^*(w, y, q)$ be as defined at (4.9). Then,*

$$G_5^*(w, y, q) = \frac{(y/w; q)_\infty}{(y; q)_\infty} \sum_{r=-\infty}^{\infty} (wq/y; q)_r (-y)^r q^{r(r-1)/2}, \tag{4.10}$$

$$= \frac{(y/w; q)_\infty}{(wq, q/w; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1 - wq^{2r})(wq/y; q)_r (-yw^2)^r q^{(5r^2-3r)/2}}{(1 - w)(y; q)_r}. \tag{4.11}$$

Proof. In (2.3) (or (2.4)), replace z with z/ac, then let $a, c \to \infty$ and $d \to 0$. Then replace z with wq and b with y, and (4.10) follows.

For (4.10), let $d, e, f \to \infty$ in (2.5), and then set $a = w$ and $c = wq/y$. □

Remark: For $G_5^*(w, y, q)$ to represent a sum of fifth-order mock theta function, it necessary to have $w = 1$ or $w = q$, and in those cases (4.10) does not provide any nontrivial results (for $w = 1$, the right side is just the series in reverse order).

The identity at (4.11) could be used to derive new expressions for the sums of fifth-order mock theta functions found in Proposition 4.1. However, we instead use it to derive four identities for bilateral series similar to those in Corollary 3.6.

Corollary 4.4. *The following identities hold for $|q| < 1$:*

$$\sum_{r=-\infty}^{\infty} (10r + 1)q^{(5r^2+r)/2} \tag{4.12}$$

$$= \left(\frac{4q(q^4, q^{16}, q^{20}; q^{20})_\infty}{(q^2; q^4)_\infty} + \frac{(q^2, q^3, q^5; q^5)_\infty}{(-q; q)_\infty} \right) \frac{(q; q)_\infty^2}{(-q; q)_\infty},$$

$$\sum_{r=-\infty}^{\infty} (10r + 3)q^{(5r^2+3r)/2} \tag{4.13}$$

$$= \left(\frac{4(q^8, q^{12}, q^{20}; q^{20})_\infty}{(q^2; q^4)_\infty} - \frac{(q, q^4, q^5; q^5)_\infty}{(-q; q)_\infty} \right) \frac{(q; q)_\infty^2}{(-q; q)_\infty},$$

$$\sum_{r=-\infty}^{\infty} (5r + 1)(-1)^r q^{10r^2+4r} \tag{4.14}$$

$$= \left(\frac{q(q^4, q^{16}, q^{20}; q^{20})_\infty}{(q^2; q^4)_\infty} + \frac{(q^2, q^3, q^5; q^5)_\infty}{(-q; q)_\infty} \right) \frac{(q^4; q^4)_\infty^2}{(q^2; q^4)_\infty},$$

$$\sum_{r=-\infty}^{\infty} (5r + 2)(-1)^r q^{10r^2+8r} \tag{4.15}$$

$$= \left(\frac{(q^8, q^{12}, q^{20}; q^{20})_\infty}{(q^2; q^4)_\infty} - \frac{(q, q^4, q^5; q^5)_\infty}{(-q; q)_\infty} \right) \frac{(q^4; q^4)_\infty^2}{(q^2; q^4)_\infty}.$$

Proof. For (4.12), in (4.11) replace w with w^2, set $y = -wq$ and simplify the resulting right side to get

$$\sum_{n=-\infty}^{\infty}\frac{w^{2n}q^{n^2}}{(-wq;q)_n}=\frac{(-q/w;q)_\infty}{(w^2q,q/w^2;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-wq^r)w^{5r}q^{(5r^2-r)/2}}{(1-w)}$$
$$=\frac{(-q/w;q)_\infty}{(w^2q,q^2/w;q)_\infty}$$
$$\times\left(1+\sum_{r=1}^{\infty}\frac{(1-wq^r)w^{5r}q^{(5r^2-r)/2}+(1-wq^{-r})w^{-5r}q^{(5r^2+r)/2}}{1-w}\right)$$
$$=\frac{(-q/w;q)_\infty}{(w^2q,q^2/w;q)_\infty}$$
$$\times\left(1+\sum_{r=1}^{\infty}\frac{w^{-5r}q^{(5r^2-r)/2}}{1-w}((1-wq^r)w^{10r}+(1-wq^{-r})q^r)\right)$$
$$=\frac{(-q/w;q)_\infty}{(w^2q,q^2/w;q)_\infty}$$
$$\times\left(1+\sum_{r=1}^{\infty}w^{-5r}q^{(5r^2-r)/2}\left(-w\frac{1-w^{10r-1}}{1-w}+q^r\frac{1-w^{10r+1}}{1-w}\right)\right).$$

Now let $w\to 1$, noting that the left side above tends to the left side of (4.1), and hence to the right side of (4.1). After using L'Hospital's rule on the terms in the last series on the right side, this series becomes

$$1+\sum_{r=1}^{\infty}q^{(5r^2-r)/2}\left(-(10r-1)+q^r(10r+1)\right)$$
$$=\sum_{r=-\infty}^{\infty}10rq^{(5r^2+r)/2}+\sum_{r=-\infty}^{\infty}q^{(5r^2-r)/2}.$$

The result now follows.

To obtain (4.13), in (4.11) replace w with w^2q, set $y=-wq$ and simplify the resulting right side to get

$$\sum_{n=-\infty}^{\infty}\frac{w^{2n}q^{n^2+n}}{(-wq;q)_n}=\frac{(-1/w;q)_\infty}{(w^2q^2,1/w^2;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-w^2q^{2r+1})w^{5r}q^{(5r^2+3r)/2}}{(1-w^2q)}$$
$$=\frac{(-1/w;q)_\infty(-w^2)}{(w^2q,q/w^2;q)_\infty}\times$$
$$\sum_{r=0}^{\infty}\frac{(1-w^2q^{2r+1})w^{5r}q^{(5r^2+3r)/2}}{(1-w^2)}+\sum_{r=0}^{\infty}\frac{(1-w^2q^{-2r-1})w^{-5r-5}q^{(5r^2+7r+2)/2}}{(1-w^2)}$$
$$=\frac{(-1/w;q)_\infty(-w^2)}{(w^2q,q/w^2;q)_\infty}\times$$

$$\sum_{r=0}^{\infty} w^{5r} q^{(5r^2+3r)/2}\left(\frac{(1-w^{-10r-3})}{(1-w^2)} - w^2 q^{2r+1}\frac{(1-w^{-10r-7})}{(1-w^2)}\right),$$

where the second series in the second right side came from taking the terms of negative index in the series on the first right side, and replacing r with $-r-1$. The identity at (4.13) now follows as previously upon letting $w \to 1$, this time noting that the left side tends to (4.2).

For (4.14) and (4.15), in (4.11) replace (w, y, q) with (w^2, wq^2, q^4) and (w^2q^4, wq^6, q^4), respectively (in the case of (4.15), after making the replacements in (4.11), multiply both sides by $1/(1-wq^2)$). The details are similar to those in the proofs of (4.12) and (4.13), and are omitted. □

5 Mock theta functions of the sixth order

The sixth-order mock theta functions which concern us here are

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \qquad \psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \qquad \sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \qquad \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n},$$

$$\phi_-(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n}, \qquad \psi_-(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n}.$$

The series stated for $\mu(q)$ does not converge, but the sequence of even-indexed partial sums and the sequence of odd-indexed partial sums do converge, and $\mu(q)$ is defined to be the average of these two values.

Andrews and Hickerson [5] proved a number of identities for the sixth-order mock theta functions stated by Ramanujan in the Lost Notebook [23]. Berndt and Chan [8] proved a number of similar identities. The proofs in both of these papers were quite involved, employing both Bailey pairs and the constant term method, and simpler proofs were later given by Lovejoy [20], for four of the identities proved by Andrews and Hickerson [5]. These four identities are listed in the following theorem.

Theorem 5.1. *The following identities hold for* $|q| < 1$.

$$q^{-1}\psi(q^2) + \rho(q) = (-q;q^2)_\infty^2(-q,-q^5,q^6;q^6)_\infty, \tag{5.1}$$

$$\phi(q^2) + 2\sigma(q) = (-q;q^2)_\infty^2(-q^3,-q^3,q^6;q^6)_\infty, \tag{5.2}$$

$$2\phi(q^2) - 2\mu(-q) = (-q;q^2)^2_\infty(-q^3,-q^3,q^6;q^6)_\infty, \tag{5.3}$$

$$2q^{-1}\psi(q^2) + \lambda(-q) = (-q;q^2)^2_\infty(-q,-q^5,q^6;q^6)_\infty, \tag{5.4}$$

To maintain uniformity, we show that, as with the third-order and fifth-order mock theta functions, the bilateral transformations of Bailey at (2.3), (2.4), and (2.5) may be used to express sums of sixth-order mock theta functions as theta functions, and that these identities in turn may likewise be used to examine the limiting behavior of some of these sixth-order mock theta functions as q tends to certain classes of roots of unity from within the unit circle. As above, we begin by stating a number of general bilateral transformations.

Theorem 5.2. *(i) If* $|q|, |bd/azq| < 1$*, then*

$$G_6(a,b,d,z,q) := \sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b,d;q^2)_r} = \frac{(-zq,-qb/az,-qd/az;q^2)_\infty}{(b,d,q^2/a;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(-azq/b,-azq/d;q^2)_r}{(-zq;q^2)_r}\left(\frac{-bd}{azq}\right)^r. \tag{5.5}$$

(ii) If $|q|, |bd/azq|, |b/a| < 1$*, then*

$$\sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b,d;q^2)_r} = \frac{(b/a,-qb/az;q^2)_\infty}{(b,-bd/azq;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(-azq/b,a;q^2)_r}{(d;q^2)_r}\left(\frac{b}{a}\right)^r. \tag{5.6}$$

(iii) If $|q|, |bd/azq| < 1$*, then*

$$\sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b,d;q^2)_r} = \frac{(-bq/az,-dq/az,-qz;q^2)_\infty}{(-bd/aqz,-q^3/az,-aqz;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(1+azq^{4r-1})(a,-azq/b,-azq/d;q^2)_r\,(bdz)^r\,q^{3r^2-4r}}{(1+az/q)(b,d,-zq;q^2)_r}. \tag{5.7}$$

Proof. For (5.5) and (5.6), replace q with q^2, z with $-zq/c$, and then let $c \to \infty$ in (2.3) and (2.4), respectively.

The identity (5.7) is a consequence of replacing q with q^2 in (2.5), and then replacing c with aq^2/b and d with aq^2/d, letting $f \to \infty$, and replacing, in turn, a with $-ze/q$ and finally e with a. □

The sums of various pairs of sixth-order mock theta functions may be expressed in terms of $G_6(a, b, d, z, q)$, and the above theorem may be used to derive some alternative expressions for these sums.

Corollary 5.3. *The following identities hold for* $|q| < 1$.

$$4\sigma(q) + 2\mu(q) = \frac{(q;q^2)_\infty}{(q;q)_\infty^2} \sum_{r=-\infty}^{\infty} (-1)^r (6r+1) q^{r(3r+1)/2}, \tag{5.8}$$

$$\phi(q) + 2\phi_-(q) = \frac{(-q;q)_\infty}{(q^2;q^4)_\infty} \tag{5.9}$$

$$\times \left[2(-q^2;q^4)_\infty^2 (-q^6,-q^6,q^{12};q^{12})_\infty - (q^2;q^4)_\infty^2 (q^6,q^6,q^{12};q^{12})_\infty\right],$$

$$\psi(q) + 2\psi_-(q) = \frac{3q(-q;q)_\infty (q^6,q^6,q^6;q^6)_\infty}{(q^2;q^2)_\infty^2} \tag{5.10}$$

$$2\rho(q) + \lambda(q) = \frac{3(q;q^2)_\infty (q^3,q^3,q^3;q^3)_\infty}{(q;q)_\infty^2} \tag{5.11}$$

Proof. In (5.7), replace z with $-zq^3$ and set $a = zq^2$, $b = zq^3$ and $d = -zq^3$ to get

$$\sum_{r=-\infty}^{\infty} \frac{(zq^2;q^2)_r (-z)^r q^{r^2+3r}}{(zq^3,-zq^3;q^2)_r} = \frac{(1/qz,-1/qz,zq^4;q^2)_\infty}{(-1,1/q^2z^2,q^6z^2;q^2)_\infty}$$
$$\times \sum_{r=-\infty}^{\infty} \frac{q^{3r^2+5r} z^{3r} \left(1-z^2q^{4r+4}\right) \left(q^2z;q^2\right)_r}{\left(1-q^4z^2\right)\left(q^4z;q^2\right)_r},$$
$$= \frac{(q^2/z^2;q^4)_\infty (zq^4;q^2)_\infty z}{2(1-1/z)(-q^2,q^2/z^2,q^6z^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{q^{r(3r+5)} z^{3r} \left(1+zq^{2r+2}\right)}{(1+q^2z)(1+z)}$$

Now multiply both sides by $q^2/(1-z^2q^2)$ and let $z \to -1$, noting that the left side tends to $\sigma(q^2) + \mu(q^2)/2$ using the definitions above and (2.1). On the right side, replace r with $r-1$ and rewrite the resulting series as

$$\sum_{r=-\infty}^{\infty} \frac{(1+zq^{2r})z^{3r}q^{3r^2-r}}{1+z} = \sum_{r=-\infty}^{\infty} \frac{z^{3r}q^{3r^2-r} + z^{3r+1}q^{3r^2+r}}{1+z}$$
$$= \sum_{r=-\infty}^{\infty} \frac{z^{3r}q^{3r^2-r} + z^{-3r+1}q^{3r^2-r}}{1+z} = \sum_{r=-\infty}^{\infty} z^{3r}q^{3r^2-r}\frac{1+z^{-6r+1}}{1+z},$$

where the second equality follows from reversing the order of summation for the second terms in the sum. Now let $z \to -1$ to arrive at

$$\sigma(q^2) + \frac{\mu(q^2)}{2} = \frac{(q^2;q^4)_\infty}{4(q^2;q^2)_\infty^2} \sum_{r=-\infty}^{\infty} (6r+1)(-1)^r q^{3r^2+r}.$$

The identity at (5.8) now follows upon multiplying this last identity by 4 and replacing q with $q^{1/2}$.

Note that the expression for $4\sigma(q) + 2\mu(q)$ deriving from (5.2) and (5.8) together with (5.2) implies that

$$\sum_{r=-\infty}^{\infty} (6r+1)(-1)^r q^{(3r^2+r)/2} = \frac{(q;q)_\infty^2}{(q;q^2)_\infty} \times \left[2(-q;q^2)_\infty^2(-q^3,-q^3,q^6;q^6)_\infty - (q;q^2)_\infty^2(q^3,q^3,q^6;q^6)_\infty\right]. \tag{5.12}$$

Remark: It may be of interest to compare the identity above with that of Fine [12, p. 83]:

$$\sum_{r=-\infty}^{\infty} (6r+1)q^{r(3r+1)/2} = (q;q)_\infty^3(q;q^2)_\infty^2. \tag{5.13}$$

For (5.9), set $a = -zq$, $b = zq$ and $d = zq^2$ in (5.7) to get, after simplifying the right side

$$\sum_{r=-\infty}^{\infty} \frac{(-zq;q^2)_r z^r q^{r^2}}{(zq,zq^2;q^2)_r} = \frac{(q/z,q^2/z,-zq;q^2)_\infty}{(q,q^2/z^2,q^2z^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1+zq^{2r})z^{3r}q^{3r^2-r}}{1+z},$$

Let $z \to -1$ on the left side to get, once again using the definitions above and (2.1),

$$\phi(q) + 2\phi_-(q) = \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2} \sum_{r=-\infty}^{\infty} (6r+1)(-1)^r q^{3r^2+r}.$$

An application of (5.12), with q replaced with q^2, gives the result.

Similarly, for (5.10), replace z with zq^2, a with $-zq$, b with zq^2, and d with zq^3 in (5.7) to get, after once again simplifying the right side, that

$$\sum_{r=-\infty}^{\infty} \frac{(-zq;q^2)_r z^r q^{r^2+2r}}{(zq^2,zq^3;q^2)_r} = \frac{(1/z,q/z,-zq^3;q^2)_\infty}{(q,1/z^2,q^4z^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} z^{3r}q^{3r^2+3r}$$
$$= \frac{(1/z,q/z,-zq^3;q^2)_\infty(-q^6z^3,-1/z^3,q^6;q^6)_\infty}{(q,1/z^2,q^4z^2;q^2)_\infty},$$

where the Jacobi triple product identity (2.2) has been used at the last step. The result now follows after multiplying both sides by $q/(1+q)$ and letting $z \to -1$ as before.

The details of the proof of (5.11) are omitted. Briefly, replace z with zq, a with $-zq^2$, b with zq^3, and d with $-zq^3$ in (5.7), simplify and sum the right side using the Jacobi triple product identity (2.2), let $z \to 1$, multiply both sides by $2/(1-q^2)$, and finally replace q with $q^{1/2}$. □

Remark: Choi [10, p. 370] also gave expressions for each of the sums of sixth-order mock theta functions in Corollary 5.3, but with different combinations of theta functions on the right sides. Yet another version of (5.9) was stated by Ramanujan [23, p. 6 and p. 16] (see also [11, p. 1740]). Different proofs of (5.10) and (5.11) were given by Choi and Kim [11, Theorem 1.4, p. 1742]. The identities in Corollary 5.3 also follow from expressions for the sixth-order mock theta functions in terms of the function $m(x, q, z)$ (see (1.1)) proved by Hickerson and Mortenson in [18], and known results about $m(x, q, z)$.

We note that the identities in Corollary 5.3 may be used to describe the asymptotic behavior of each of the two sixth-order mock theta functions on the left side of each identity, at particular classes of roots of unity. For example, (5.8) may be used in conjunction with (5.12), to make condition (0) in the interpretation by Andrews and Hickerson of a mock theta function explicit for both $\sigma(q)$ at primitive roots of unity of odd-order, and for $\mu(q)$ at primitive roots of unity of even-order. We state the result for just one of each pair of mock theta functions, and leave the result for the other mock theta function of each pair to the reader.

Corollary 5.4. *(i) If ζ is a primitive odd-order $2k+1$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(\sigma(q)-\frac{1}{4}\left[2(-q;q^2)_\infty^2(-q^3,-q^3,q^6;q^6)_\infty-(q;q^2)_\infty^2(q^3,q^3,q^6;q^6)_\infty\right]\right) = -\frac{1}{2}\sum_{n=0}^{k}\frac{(1-\zeta)(1-\zeta^3)\dots(1-\zeta^{2n-1})}{(1+\zeta)(1+\zeta^2)\dots(1+\zeta^n)}(-1)^n. \quad (5.14)$$

(ii) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\phi(q)-\frac{(-q;q)_\infty}{(q^2;q^4)_\infty}\times\left[2(-q^2;q^4)_\infty^2(-q^6,-q^6,q^{12};q^{12})_\infty-(q^2;q^4)_\infty^2(q^6,q^6,q^{12};q^{12})_\infty\right]\right) = -2\sum_{n=1}^{k}\frac{(1+\zeta)(1+\zeta^2)\dots(1+\zeta^{2n-1})}{(1-\zeta)(1-\zeta^3)\dots(1-\zeta^{2n-1})}\zeta^n. \quad (5.15)$$

(iii) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\psi(q)-\frac{3q(-q;q)_\infty(q^6,q^6,q^6;q^6)_\infty}{(q^2;q^2)_\infty^2}\right)$$

$$= -2\sum_{n=1}^{k} \frac{(1+\zeta)(1+\zeta^2)\dots(1+\zeta^{2n-2})}{(1-\zeta)(1-\zeta^3)\dots(1-\zeta^{2n-1})}\zeta^n. \tag{5.16}$$

(iv) If ζ is a primitive odd-order $2k+1$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\rho(q) - \frac{3(q;q^2)_\infty(q^3,q^3,q^3;q^3)_\infty}{2(q;q)_\infty^2}\right) = -\frac{1}{2}\sum_{n=0}^{k} \frac{(1-\zeta)(1-\zeta^3)\dots(1-\zeta^{2n-1})}{(1+\zeta)(1+\zeta^2)\dots(1+\zeta^n)}(-\zeta)^n. \tag{5.17}$$

Before considering eighth-order mock theta functions, we compare the results in the present paper with those implied by an identity of Mortenson ([22, Eq. (6.10)]):

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(x;q)_{n+1}(q/x)_{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2}\frac{q^n(q/x;q)_n(x;q)_n}{(-q;q)_n} = -\frac{j(x;q)}{2J_2}g_3(-x;q)$$
$$+\frac{J_2^3}{J_{1,2}\,j(x^2;q^2)} + \frac{1}{2x}\frac{J_2^{10}j(-x^2;q^2)}{J_1^4J_4^4\,j(x^2;q^2)\,j(-qx^2;q^2)} - \frac{1}{2x}\frac{J_{2,4}^2\,j(x;q)}{j(-x;q)\,j(-qx^2;q^2)}, \tag{5.18}$$

where the first series is $g_2(x,q)$, the universal mock theta function of Gordon and McIntosh [17, Eq. (4.11)], $g_3(x;q)$ is as defined at (3.1), and

$$j(x;q) := (x, q/x, q;q)_\infty, \qquad J_{a,m} := j(q^a;q^m),$$
$$\bar{J}_{a,m} := j(-q^a;q^m), \qquad J_m := J_{m,3m} = (q^m;q^m)_\infty.$$

As Mortenson indicated in [22], if a mock theta function is expressible in terms of $g_2(x,q)$ and combinations of infinite products, then it may be possible to derive a radial limits result for certain classes of roots of unity, and indeed Mortenson derives such results for a second-order mock theta function and one of tenth order, and states that there are many other cases where (5.18) may be applied.

As one way of deriving explicit radial limits, one might hope, after substituting for $g_2(x,q)$ in (5.18), so that this expression now contains a mock theta function, that there is then a class of roots of unity such that as q approaches one of these roots of unity, say ζ, from within the unit circle, the mock theta function becomes unbounded, the term involving $g_3(-x;q)$ vanishes, and the second series on the left terminates. In this case, (5.18) may then be rearranged to give an identity of the form

$$\lim_{q\to\zeta}\ (\text{mock theta function} - \text{theta function}) = \text{finite } q\text{-series in } \zeta,$$

which is the typical form of a radial limits result. For example, if q is replaced with q^6 and x with q^3 in (5.18) and the second identity at [17, Eq. (5.10)], namely

$$\psi(q^4) = \frac{q^3 J_2^2 J_4 J_2 4^2}{J_1 J_3 J_8^2} - q^3 g_2(q^3, q^6) \tag{5.19}$$

is used to substitute for $g_2(q^3, q^6)$, then after some q-product manipulation, we get

$$\psi(q^4) + \frac{q^3 J_{12}^5}{J_6^4} + \frac{J_{12}^{17}}{4 J_6^8 J_{24}^8} - \frac{J_3^4 J_{12}^7}{4 J_6^6 J_{24}^4} - \frac{q^3 J_2^2 J_4 J_{24}^2}{J_1 J_3 J_8^2} - \frac{q^3 J_3^2}{2 J_6 J_{12}} \sum_{n=0}^{\infty} \frac{q^{6n(n+1)}}{(-q^3; q^6)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{6n+3}(q^3, q^3; q^6)_n}{(-q^6; q^6)_n}. \tag{5.20}$$

If $q \to \zeta$, where ζ is a primitive even-order root of unity, then both the series on the right of (5.20) and the last term on the left become unbounded, and there is no radial limit. Unfortunately, for producing explicit radial limits, when $q \to \zeta$, where ζ is a primitive *odd*-order root of unity, while the series on the right of (5.20) terminates, and the last term on the left vanishes, the series for $\psi(q^4)$ also terminates. After eliminating terms that vanish when $q \to \zeta$, where ζ is a primitive $2k + 1$-th root of unity, one gets that

$$\lim_{q \to \zeta} \left(\frac{q^3 J_{12}^5}{J_6^4} + \frac{J_{12}^{17}}{4 J_6^8 J_{24}^8} - \frac{q^3 J_2^2 J_4 J_{24}^2}{J_1 J_3 J_8^2} \right) = \sum_{r=0}^{k} \frac{\zeta^{6r+3}(\zeta^3, \zeta^3; \zeta^6)_r}{2(-\zeta^6; \zeta^6)_r} - \sum_{r=0}^{k} \frac{(-1)^r \zeta^{4(r+1)^2} \left(\zeta^4; \zeta^8\right)_r}{\left(-\zeta^4; \zeta^4\right)_{2r+1}}. \tag{5.21}$$

Curiously, what experiment suggests is that each side is identically zero, and in particular, that if ζ is a primitive $2k + 1$-th root of unity, then

$$\sum_{r=0}^{k} \frac{\zeta^{6r+3}(\zeta^3, \zeta^3; \zeta^6)_r}{2(-\zeta^6; \zeta^6)_r} = \sum_{r=0}^{k} \frac{(-1)^r \zeta^{4(r+1)^2} \left(\zeta^4; \zeta^8\right)_r}{\left(-\zeta^4; \zeta^4\right)_{2r+1}} \quad (= \psi(\zeta^4)) \tag{5.22}$$

holds for all $k \geq 0$. Note that equality in (5.22) does not hold if ζ is replaced with an arbitrary value of q inside the unit circle since the difference is a nonzero function of q:

$$\sum_{r=0}^{k} \frac{q^{6r+3}(q^3, q^3; q^6)_r}{2(-q^6; q^6)_r} - \sum_{r=0}^{k} \frac{(-1)^r q^{4(r+1)^2} \left(q^4; q^8\right)_r}{\left(-q^4; q^4\right)_{2r+1}}$$
$$= \frac{q^3}{2} - q^4 + q^8 + \frac{q^9}{2} - 2q^{12} + \frac{q^{15}}{2} + 2q^{16} - 3q^{20} \ldots.$$

Note also that each of the three individual terms on the left side of (5.21) diverges to ∞ as $q \to \zeta$, even though the combination converges to zero. Of course (5.22) will hold if

$$\frac{q^3 J_{12}^5}{J_6^4} + \frac{J_{12}^{17}}{4 J_6^8 J_{24}^8} - \frac{q^3 J_2^2 J_4 J_{24}^2}{J_1 J_3 J_8^2} = (q; q^2)_\infty \theta(q),$$

where $\theta(q)$ is a function of q that remains bounded as q approaches any primitive odd-order root of unity from within the unit circle. We have not attempted to prove this, nor (5.22).

If Ramanujan's identity (see [17, Eq. (5.8)])

$$2q^{-1}\psi(q^2) + \lambda(-q) = (-q; q^2)_\infty^2 j(-q, q^6)$$

with q replaced with $-q^2$ is used to replace $\psi(q^4)$ in (5.20), and a similar analysis of radial limits is attempted, what experiment also appears to indicate is that

$$\sum_{r=0}^{k} \frac{\zeta^{6r+1}(\zeta^3, \zeta^3; \zeta^6)_r}{(-\zeta^6; \zeta^6)_r} = \sum_{r=0}^{k} \frac{(-1)^r \zeta^{2r} \left(\zeta^2; \zeta^4\right)_r}{\left(-\zeta^2; \zeta^2\right)_r} \quad (= \lambda(\zeta^2)) \tag{5.23}$$

holds, where ζ is a $2k+1$-th primitive root of unity (and again (5.23) does not hold if ζ is replaced with an arbitrary q inside the unit circle). Similar results may be obtained from (5.18) for other sixth-order mock theta functions.

6 Mock theta functions of the eighth order

We consider four of the eight mock theta functions of order eight introduced by Gordon and McIntosh [15]:

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(-q^2; q^2)_n}, \qquad S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q; q^2)_n}{(-q^2; q^2)_n}, \tag{6.1}$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2; q^2)_n}{(-q; q^2)_{n+1}}, \quad T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2; q^2)_n}{(-q; q^2)_{n+1}}. \tag{6.2}$$

As with mock theta functions of other orders, certain sums of eighth-order mock theta functions may be written as single bilateral series, and it is a straightforward consequence of the definitions and (2.1) that

$$S_0(q) + 2T_0(q) = \sum_{r=-\infty}^{\infty} \frac{(-q;q^2)_r q^{r^2}}{(-q^2;q^2)_r}, \tag{6.3}$$

$$S_1(q) + 2T_1(q) = \sum_{r=-\infty}^{\infty} \frac{(-q;q^2)_r q^{r^2+2r}}{(-q^2;q^2)_r}. \tag{6.4}$$

In fact, following the method of the authors in [15], it will be shown that each of these sums has an expression in terms of infinite products. We include the proof here since in [15], the authors omitted the final step of explicitly stating the form of the infinite products (although these expressions were stated by them in [17, Eq. (5.12)], and these expressions with further details of the proof were given by them in [16, Section 4]). As with the identities in Corollary 5.3, the identities in Theorem 6.1 may also be shown to follow from identities proved by Hickerson and Mortenson in [18].

Theorem 6.1. *If* $|q| < 1$, *then*

$$S_0(q^2) + 2T_0(q^2) = \frac{(q^2;q^2)_\infty \left[(q;q^2)_\infty^3 + (-q;q^2)_\infty^3\right]}{2(-q^2;q^2)_\infty}, \tag{6.5}$$

$$S_1(q^2) + 2T_1(q^2) = \frac{(q^2;q^2)_\infty \left[(-q;q^2)_\infty^3 - (q;q^2)_\infty^3\right]}{2q(-q^2;q^2)_\infty}. \tag{6.6}$$

Proof. Let $R_0(q)$ and $R_1(q)$ denote the series on the right side of (6.3) and (6.4), respectively. Next, in (2.6), replace q with q^2, set $a = q$, $b = iq$ and $c = -iq$, and then let $d, e \to \infty$. This leads to

$$\begin{aligned}\frac{R_0(q^2) - qR_1(q^2)}{1-q} &= \sum_{r=-\infty}^{\infty} \frac{(1-q^{4r+1})(-q^2;q^4)_r q^{2r^2}}{(1-q)(-q^4;q^4)_r} \\ &= \frac{(q^3, q, q^2, q; q^2)_\infty}{(iq^2, -iq^2, iq, -iq; q^2)_\infty} = \frac{(q^2;q^2)_\infty (q;q^2)_\infty^3}{(1-q)(-q^2;q^2)_\infty}.\end{aligned}$$

Multiply through by $1-q$ and then replace q with $-q$ to get an expression for $R_0(q^2) + qR_1(q^2)$. The pair of equations may then be solved for, in turn, $R_0(q^2)$ and $R_1(q^2)$, to give the results. □

To make use of the above identities, we consider bilateral sums related to the eighth-order mock theta functions (the eight order equivalent of Theorem 5.2).

Theorem 6.2. *(i) If* $|q| < 1$*, then*

$$G_8(a, b, d, z, q) := \sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b;q^2)_r} = \frac{(-zq, -qb/az;q^2)_\infty}{(b, q^2/a;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(-azq/b;q^2)_r}{(-zq;q^2)_r} (-b)^r q^{r^2-r}. \tag{6.7}$$

(ii) If $|q|, |b/a| < 1$*, then*

$$\sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b;q^2)_r} = \frac{(b/a, -qb/az;q^2)_\infty}{(b;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} (-azq/b, a;q^2)_r \left(\frac{b}{a}\right)^r. \tag{6.8}$$

(iii) If $|q|, |bd/azq| < 1$*, then*

$$\sum_{n=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b;q^2)_r} = \frac{(-bq/az, -qz;q^2)_\infty}{(-q^3/az, -aqz;q^2)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(1+azq^{4r-1})(a, -azq/b;q^2)_r \left(baz^2\right)^r q^{4r^2-4r}}{(1+az/q)(b, -zq;q^2)_r}. \tag{6.9}$$

Proof. Let $d \to 0$ in (5.5), (5.6), and (5.7), respectively. □

The following identities are a consequence of combining the results in Theorems 6.2 and 6.1.

Corollary 6.3. *If* $|q| < 1$*, then*

$$\sum_{r=-\infty}^{\infty} \frac{q^{4r^2}}{(-q^2;q^2)_{2r}} = \frac{(q^2;q^2)_\infty}{2} \left[(q;q^2)_\infty^3 + (-q;q^2)_\infty^3\right], \tag{6.10}$$

$$\sum_{r=-\infty}^{\infty} \frac{q^{4r^2+4r}}{(-q^2;q^2)_{2r+1}} = \frac{(q^2;q^2)_\infty}{2q} \left[(-q;q^2)_\infty^3 - (q;q^2)_\infty^3\right], \tag{6.11}$$

$$\sum_{r=-\infty}^{\infty} (8r+1) q^{8r^2+2r} = \frac{(q^2;q^2)_\infty^3}{2} \left[(q;q^2)_\infty^3 + (-q;q^2)_\infty^3\right], \tag{6.12}$$

$$\sum_{r=-\infty}^{\infty} (8r+3) q^{8r^2+6r} = \frac{(q^2;q^2)_\infty^3}{2q} \left[(-q;q^2)_\infty^3 - (q;q^2)_\infty^3\right], \tag{6.13}$$

Proof. The identity at (6.10) follows upon setting $a = -q$, $b = -q^2$, and $z = 1$ in (6.8), reversing the order of summation in the resulting series on the right, replacing

q with q^2, and using (6.3) in conjunction with (6.5). The identity at (6.11) follows similarly, except that $z = q^2$, and (6.4) is used in conjunction with (6.6).

The identities at (6.12) and (6.13) follow similarly from (6.9). For (6.12), replace a with $-zq$, b with $-zq^2$, and take the limits as $z \to 1$. For (6.13), replace z with zq^2, set $a = -zq$, $b = -zq^2$, and again take the limits as $z \to 1$. The details are omitted. □

The identities in Theorem 6.1 also contain implications for the limiting behavior of each of the four eighth-order mock theta functions that appear in these identities, as q tends to certain classes of roots of unity from within the unit circle. We state these for $S_0(q)$ and $S_1(q)$, as those for $T_0(q)$ and $T_1(q)$ are equally easily derived. To avoid fractional exponents, we state the results for q^2 instead of q.

Corollary 6.4. *(i) If ζ is a primitive even-order $8k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(S_0(q^2) - \frac{(q^2;q^2)_\infty\left[(q;q^2)_\infty^3 + (-q;q^2)_\infty^3\right]}{2(-q^2;q^2)_\infty}\right) = -2\sum_{n=0}^{k-1}\frac{(1+\zeta^4)(1+\zeta^8)\dots(1+\zeta^{4n})}{(1+\zeta^2)(1+\zeta^6)\dots(1+\zeta^{4n+2})}\zeta^{2n^2+6n+4}. \tag{6.14}$$

(ii) If ζ is a primitive even-order $8k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(S_1(q^2) - \frac{(q^2;q^2)_\infty\left[(-q;q^2)_\infty^3 - (q;q^2)_\infty^3\right]}{2q(-q^2;q^2)_\infty}\right) = -2\sum_{n=0}^{k-1}\frac{(1+\zeta^4)(1+\zeta^8)\dots(1+\zeta^{4n})}{(1+\zeta^2)(1+\zeta^6)\dots(1+\zeta^{4n+2})}\zeta^{2n^2+2n}. \tag{6.15}$$

Finally, as was done for sixth-order mock theta functions, we compare the results in the present paper for eighth-order mock theta functions with those implied by Mortenson's identity at (5.18). In that identity, if q is replaced q^8, x is set equal to q and the identity of Gordon and McIntosh [17, p. 125],

$$S_0(-q^2) = \frac{j(-q,q^2)j(q^6,q^{16})}{j(q^2,q^8)} - 2qg_2(q,q^8) \tag{6.16}$$

is used to replace $g_2(q,q^8)$, then

$$S_0(-q^2) - \sum_{n=0}^{\infty} \frac{q^{8n+1}\left(q, q^7; q^8\right)_n}{\left(-q^8; q^8\right)_n} = \frac{qJ_{1,8}}{J_{16}} \sum_{n=0}^{\infty} \frac{q^{8n(n+1)}}{\left(-q, -q^7; q^8\right)_{n+1}} - \frac{2qJ_{16}^3}{J_{8,16}J_{2,16}} + \frac{J_{16,32}^2 J_{1,8}}{\bar{J}_{1,8}\bar{J}_{10,16}} - \frac{J_{16}^{10}\bar{J}_{2,16}}{J_8^4 J_{32}^4 J_{2,16}\bar{J}_{10,16}} + \frac{\bar{J}_{1,2}J_{6,16}}{J_{2,8}}. \tag{6.17}$$

As with sixth-order mock theta functions, when q tends to a primitive root of unity of even-order, the second series on the left side of (6.17) does not terminate, so that the usual kind of explicit radial limit is not obtained. However, when ζ is a primitive root of a certain order, what is obtained is a *convergent* infinite series, thus leading to another type of explicit radial limit. For example, if

$$\zeta_8 = e^{2\pi i/8} = \frac{\sqrt{2} + \sqrt{2}i}{2},$$

a primitive eighth root of unity, then it follows from (6.17) that

$$\lim_{q \to \zeta_8} \left(S_0(-q^2) - \left[-\frac{2qJ_{16}^3}{J_{8,16}J_{2,16}} - \frac{J_{16}^{10}\bar{J}_{2,16}}{J_8^4 J_{32}^4 J_{2,16}\bar{J}_{10,16}} + \frac{\bar{J}_{1,2}J_{6,16}}{J_{2,8}} \right] \right) = \sum_{n=0}^{\infty} \frac{\zeta_8^{8n+1}\left(\zeta_8, \zeta_8^7; q^8\right)_n}{\left(-\zeta_8^8; \zeta_8^8\right)_n} = \zeta_8 \sum_{n=0}^{\infty} \left(\frac{(1-\zeta_8)(1-\bar{\zeta}_8)}{2} \right)^n = \zeta_8 \sum_{n=0}^{\infty} \left(\frac{2-\sqrt{2}}{2} \right)^n = 1 + i. \tag{6.18}$$

Note how this compares with the radial limit given by (6.14):

$$\lim_{q \to \zeta_8} \left(S_0(-q^2) - \frac{\left(\left(-iq; -q^2\right)_\infty^3 + \left(iq; -q^2\right)_\infty^3 \right) \left(-q^2; -q^2\right)_\infty}{2\left(q^2; -q^2\right)_\infty} \right) = 1 + i. \tag{6.19}$$

While the limits are the same, the two theta functions subtracted from $S_0(-q^2)$ are not equal as functions of q.

Here also, as with the sixth-order mock theta functions $\psi(q)$ and $\lambda(q)$, a radial limit is not obtained q tends to a primitive root of unity of odd-order. What is true is that if ζ is a primitive root of unity of order $2k+1$, then

$$\sum_{n=0}^{k}\frac{(-1)^n\zeta^{2n^2}\left(\zeta^2;\zeta^4\right)_n}{\left(-\zeta^4;\zeta^4\right)_n}-\sum_{n=0}^{k}\frac{\zeta^{8n+1}\left(\zeta,\zeta^7;\zeta^8\right)_n}{\left(-\zeta^8;\zeta^8\right)_n}$$
$$=\lim_{q\to\zeta}\left(\frac{\bar{J}_{1,2}J_{6,16}}{J_{2,8}}-\frac{2qJ_{16}^3}{J_{8,16}J_{2,16}}-\frac{J_{16}^{10}\bar{J}_{2,16}}{J_8^4J_{32}^4J_{2,16}\bar{J}_{10,16}}\right), \tag{6.20}$$

and once again, experiment seems to suggest that each side is identically zero when ζ is any primitive root of unity of odd-order. As with the hypotheses suggested by experiment for mock theta functions of sixth order, we have not attempted to prove these assertions.

Results similar to those described above may be derived for other eighth-order mock theta functions.

References

1. G.E. Andrews, On basic hypergeometric series, mock theta functions, and partitions. I. Q. J. Math. Oxford Ser. (2) **17**, 64–80 (1966)
2. G.E. Andrews, On basic hypergeometric series, mock theta functions, and partitions. II. Q. J. Math. Oxford Ser. (2) **17**, 132–143 (1966)
3. G.E. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71 (Cambridge University Press, Cambridge, 1999), pp. xvi+664
4. G.E. Andrews, B.C. Berndt, *Ramanujans Lost Notebook, Part II* (Springer, New York, 2009), pp. xii+418
5. G.E. Andrews, D. Hickerson, Ramanujans, "lost" notebook: the sixth order mock theta functions. Adv. Math. **89**, 60–105 (1991)
6. W.N. Bailey, On the basic bilateral hypergeometric series ${}_2\psi_2$. Q. J. Math. Oxford Ser. (2) **1**, 194–198 (1950)
7. J. Bajpai, S. Kimport, J. Liang, D. Ma, J. Ricci, Bilateral series and Ramanujan's radial limits. Proc. Am. Math. Soc. **143**, 479–492 (2015)
8. B.C. Berndt, S.H. Chan, Sixth order mock theta functions. Adv. Math. **216**(2), 771–786 (2007)
9. B.C. Berndt, R.A. Rankin, *Ramanujan: Letters and Commentary*, History of Mathematics, vol. 9 (American Mathematical Society, Providence, RI, London Mathematical Society, London, 1995), pp. xiv+347
10. Y.-S. Choi, The basic bilateral hypergeometric series and the mock theta functions. Ramanujan J. **24**(3), 345–386 (2011)
11. Y.-S. Choi, B. Kim, Partition identities from third and sixth order mock theta functions. Eur. J. Comb. **33**(8), 1739–1754 (2012)
12. N.J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, 1988), pp. xvi+124
13. A. Folsom, K. Ono, R.C. Rhoades, Mock theta functions and quantum modular forms. Forum Math. Pi **1**(e2), 27 (2013)
14. G. Gasper, M. Rahman, *Basic hypergeometric series*. With a foreword by Richard Askey. 2nd edn. Encyclopedia of mathematics and its applications, vol. 96. (Cambridge University Press, Cambridge, 2004), pp. xxvi+428
15. B. Gordon, R.J. McIntosh, Some eighth order mock theta functions. Lond. Math. Soc. (2) **62**(2), 321–335 (2000)
16. B. Gordon, R.J. McIntosh, Modular transformations of Ramanujan's fifth and seventh order mock theta functions. Ramanujan J. **7**(1–3), 193–222 (2003). Rankin memorial issues

17. B. Gordon, R.J. McIntosh, A survey of classical mock theta functions, *Partitions*, *q-Series, and Modular Forms* (Springer, New York, 2012), pp. 95–144. Dev. Math
18. D.R. Hickerson, E.T. Mortenson, Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I. Proc. Lond. Math. Soc. (3) **109**(2), 382–422 (2014)
19. J. Mc Laughlin, in *Topics and Methods in q-Series*. Monographs in Number Theory, vol. 8 (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017), pp. x+390
20. J. Lovejoy, On identities involving the sixth order mock theta functions. Proc. Amer. Math. Soc. **138**(7), 2547–2552 (2010)
21. E.T. Mortenson, On the dual nature of partial theta functions and Appell–Lerch sums. Adv. Math. **264**, 236–260 (2014)
22. E.T. Mortenson, Ramanujans radial limits and mixed mock modular bilateral q - hypergeometric series. Proc. Edinb. Math. Soc. (2) **59**(3), 787–799 (2016)
23. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*. With an introduction by George E. Andrews (Springer, Berlin; Narosa Publishing House, New Delhi 1988), pp. xxviii+419
24. S. Ramanujan, *Collected papers of Srinivasa Ramanujan*, ed. by G.H. Hardy, P.V. Seshu Aiyar, B.M. Wilson. Third printing of the 1927 original. With a new preface and commentary by Bruce C. Berndt, (AMS Chelsea Publishing, Providence, RI 2000), pp. xxxviii+426
25. R.C. Rhoades, On Ramanujan's definition of mock theta function. Proc. Natl. Acad. Sci. USA **110**(19), 7592–7594 (2013)
26. G.N. Watson, The final problem: an account of the mock theta functions. J. Lond. Math. Soc. **11**, 55–80 (1936)
27. G.N. Watson, The mock theta functions (2). Proc. Lond. Math. Soc. **42**, 274–304 (1937)
28. W. Zudilin, On three theorems of Folsom, Ono and Rhoades. Proc. Am. Math. Soc. **143**(4), 1471–1476 (2015)
29. S. Zwegers, Mock θ-functions and real analytic modular forms, q-series with applications to combinatorics, number theory, and physics. Contemp. Math. **291**, 268–277 (2001)
30. S. Zwegers, *Mock theta functions*, Ph.D. thesis, University of Utrecht, Utrecht, The Netherlands, 2002

Littlewood Polynomials

Hugh L. Montgomery

This paper is dedicated to Krishnaswami Alladi on the occasion of his 60th birthday

Abstract We consider exponential polynomials with restricted coefficients, for example by requiring that they all have absolute value 1, or the even more restricted class considered by Littlewood, namely those whose coefficients are all ± 1. This paper expands on material presented in the Erdős Colloquium delivered on March 18, 2016, as part of the 2016 Gainesville International Number Theory Conference.

Keywords Littlewood polynomials · Shapiro polynomials

2010 Mathematics Subject Classification 42A05

1 Notation and questions

We let

$$\mathcal{F}_N = \Big\{ f(x) = \sum_{n=0}^{N-1} a_n e(nx) \ : \ a_n = \pm 1 \,\forall\, n \Big\}$$

These are what we call *Littlewood polynomials*. Here $e(\theta) = e^{2\pi i \theta}$, as usual. These polynomials are a special subclass of the more general polynomials

Research supported in part by NSF grant DMS-063529.

H. L. Montgomery (✉)
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109–1043, USA
e-mail: hlm@umich.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_30

$$\mathcal{G}_N = \left\{ g(x) = \sum_{n=0}^{N-1} a_n e(nx) \ : \ |a_n| = 1 \,\forall\, n \right\}.$$

This notation is due to Littlewood, but our definition is slightly different, since Littlewood's polynomials have $N + 1$ terms while ours have only N. By Parseval's identity (which is trivial for trigonometric polynomials) it is evident that if $g \in \mathcal{G}_N$, then

$$\int_0^1 |g(x)|^2 \, dx = \sum_{n=0}^{N-1} |a_n|^2 = N,$$

and hence

$$\min_x |g(x)| \le \sqrt{N} \le \max_x |g(x)| \, .$$

We also note that the variance of $|g(x)|^2$ about its mean is

$$\int_0^1 \big(|g(x)|^2 - N\big)^2 \, dx = \int_0^1 |g(x)|^4 \, dx - N^2.$$

Concerning functions of the classes $\mathcal{F}_N$, $\mathcal{G}_N$ we ask questions of the following sort:

Question 1. Does there exist an absolute constant $c > 0$ and a function $g(x)$ such that

$$\min_x |g(x)| \ge c\sqrt{N} \, ?$$

Question 2. Does there exist an absolute constant $C < \infty$ and a function $g(x)$ such that

$$\max_x |g(x)| \le C\sqrt{N} \, ?$$

Question 3. Do there exist absolute constants a, A with $0 < a < A < \infty$ and a function $g(x)$ such that

$$a\sqrt{N} \le |g(x)| \le A\sqrt{N}$$

for all x ?

Question 4. For every $\varepsilon > 0$ does there exist a function $g(x)$ such that

$$\int_0^1 |g(x)|^4 \, dx < (1 + \varepsilon)\sqrt{N} \, ?$$

In each case, we may ask whether the property holds for all N, all sufficiently large N, or for infinitely many N. If $g \in \mathcal{G}_N$, then the frequencies of $e(Kx)g(x)$ lie in the interval $[K, K + N - 1]$, but since $|e(Kx)g(x)| = |g(x)|$ for all x, the situation regarding our questions is unchanged. Thus the length of our polynomial is all that matters.

2 Historical review

In describing what is presently known about our questions, we proceed chronologically. A hundred years ago, Hardy and Littlewood [23] announced the first result in this area, namely that if we take

$$g_N(x) = \sum_{n=1}^{N} e^{in\log n} e(nx),$$

then

$$g_N(x) = O\big(\sqrt{N}\big) \tag{1}$$

uniformly in x. This has applications to Fourier analysis. For example, it is natural to ask how smooth a function should be, in order to assure that its Fourier series is absolutely convergent, i.e., $f \in A(\mathbb{T})$. Bernstein [4] showed that if $f \in \Lambda_\alpha$ for some $\alpha > 1/2$, then $f \in A(\mathbb{T})$. Here Λ_α denotes the Lipschitz class of functions f with period 1 for which there is a constant c such that $|f(x) - f(y)| \le c|x-y|^\alpha$. As to the question of whether the condition $\alpha > 1/2$ might be relaxed, the Hardy–Littlewood example enables us to show that

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{in\log n}}{n} e(nx) \;\in\; \Lambda_{1/2},$$

but $f \notin A(\mathbb{T})$. To see how this is done, put

$$f_N(x) = \sum_{n=1}^{N} \frac{e^{in\log n}}{n} e(nx)\,.$$

By partial summation we see that if $M < N$, then

$$f_N(x) - f_M(x) = -\frac{g_M(x)}{M} + \frac{g_N(x)}{N+1} + \sum_{n=M}^{N} \frac{g_n(x)}{n(n+1)}\,.$$

We use (1) to estimate the three terms on the right-hand side, and let N tend to infinity to see that

$$f(x) = f_M(x) + O\big(M^{-1/2}\big)\,.$$

Now $f_M'(x) = 2\pi i g_M(x) = O\big(M^{1/2}\big)$, so

$$f_M(x+\delta) - f_M(x) = O\big(|\delta| M^{1/2}\big)\,.$$

On taking M comparable to $1/|\delta|$, we deduce that $|f(x+\delta) - f(x)| = O\big(|\delta|^{1/2}\big)$, which is to say that $f \in \Lambda_{1/2}$.

A *Weyl sum* is an exponential sum of the form

$$\sum_{n=1}^{N} e(P(n))$$

where P is a polynomial with real coefficients, say

$$P(x) = c_0 + c_1 x + \cdots + c_d x^d .$$

Hermann Weyl not only established a useful criterion for uniform distribution modulo 1 in terms of exponential sums, he also devised a method for estimating Weyl sums, and in particular showed that if at least one of $c_1, c_2, \ldots, c_d$ is irrational, then the numbers $P(n)$ are uniformly distributed modulo 1. By Weyl's method, we find that if

$$g(x) = \sum_{n=1}^{N} e\big(n^2/N\big)e(nx),$$

then $g(x) = O\big(\sqrt{N}\big)$. This is suggestive of a Gauss sum. If p is an odd prime, then

$$\left|\sum_{n=1}^{p-1}\Big(\frac{n}{p}\Big)e\Big(\frac{an}{p}\Big)\right| = \begin{cases}\sqrt{p} & \text{if } p \nmid a,\\ 0 & \text{if } p|a\,.\end{cases}$$

When we replace a/p by x we obtain a Littlewood polynomial, but it is not uniformly $O\big(\sqrt{p}\big)$. Indeed, Montgomery [34] showed that there is an absolute constant $c > 0$ such that

$$\max_x \left|\sum_{n=1}^{p-1}\Big(\frac{n}{p}\Big)e(nx)\right| > c\sqrt{p}\,\log\log p$$

for all $p > 2$.

H. S. Shapiro [43] defined two families of Littlewood polynomials P_k and Q_k by means of the initial conditions $P_0(x) = Q_0(x) = 1$ and the recurrences

$$P_{k+1}(x) = P_k(x) + e\big(2^k x\big)Q_k(x), \qquad Q_{k+1}(x) = P_k(x) - e\big(2^k x\big)Q_k(x) \tag{2}$$

for $k = 0, 1, 2, \ldots$. Clearly the first 2^k coefficients of $P_{k+1}(x)$ are the coefficients of $P_k(x)$, and they will be the first 2^k coefficients of $P_{k+2}(x)$, $P_{k+3}(x)$, and so on, so the recurrences above define an infinite sequence of numbers $c(n)$ (the *Shapiro coefficients*) with the property that

$$P_k(x) = \sum_{n=0}^{2^k-1} c(n)e(nx), \qquad Q_k(x) = \sum_{n=0}^{2^k-1} c(2^k+n)e(nx)\,.$$

Brillhart and Carlitz [8] showed that if the binary expansion of n is

$$n = \sum_{j\ge 0} b_j 2^j$$

with each $b_j = 0$ or 1, and if

$$a(n) = \sum_{j\ge 0} b_j b_{j+1}$$

is the number of pairs of adjacent 1's in the expansion, then $c(n) = (-1)^{a(n)}$. Thus the $c(n)$ are an example of a 2-automatic sequence, as discussed by Allouche and Shallit [1]. Brillhart [7] observed further that the $c(n)$ are uniquely determined by the properties that

$$c(0) = 1, \qquad c(2n) = c(n), \qquad c(2n+1) = (-1)^n c(n)\,. \tag{3}$$

From the defining recurrence (2) we see that

$$|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 = |P_k(x) + e\big(2^k x\big)Q_k(x)|^2 + |P_k(x) - e\big(2^k x\big)Q_k(x)|^2.$$

The so-called 'law of the parallelogram' asserts that $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$ for arbitrary complex numbers z and w. Thus the above is

$$= 2|P_k(x)|^2 + 2|Q_k(x)|^2.$$

By induction, it follows that

$$|P_k(x)|^2 + |Q_k(x)|^2 = 2^{k+1} \tag{4}$$

for all x and k. Since $|Q_k(x)|^2 \ge 0$, it follows that

$$|P_k(x)| \le 2^{(k+1)/2}. \tag{5}$$

This upper bound is $\sqrt{2}$ times the square root of the length of P_k. From the examples of Hardy–Littlewood and of Weyl, we have seen that our Question 2 has a positive answer with regard to the class $\mathcal{G}_N$. We now see that Question 2 also has a positive answer in the case of the class $\mathcal{F}_N$, at least for N of the form $N = 2^k$. Let

$$T(N;x) = \sum_{n=0}^{N-1} c(n)e(nx)\,. \tag{6}$$

Shapiro [43] used his basic result to show more generally that there is an absolute constant C such that

$$|T(N;x)| \le C\sqrt{N} \tag{7}$$

for all x. Shapiro showed that $C = 2 + 2\sqrt{2} = 4.8284$ is admissible. Later Mendès France and Tenenbaum [33] obtained a better constant, $C = 2 + \sqrt{2} = 3.4142$. For a simple proof of the bound with this value of C, see Exercise 6.5.23 of Montgomery [35]. Saffari [42] has sketched a proof that (7) holds with

$$C = (2 + \sqrt{2})\sqrt{3/5} = 2.6446\,.$$

It has long been conjectured that (7) holds with $C = \sqrt{6} = 2.4495$. This would be best-possible, for it is not hard to show that if we take

$$N = B_r = \frac{8 \cdot 4^r + 1}{3}, \tag{8}$$

then

$$T(B_r;0) = 2^{r+2} - 1 = \sqrt{6B_r} - 1 + O\big(1/\sqrt{B_r}\big)\,.$$

In a paper devoted to unsolved problems, Erdős [16] asked whether there is a constant $c > 1$ such that

$$\max_x |g(x)| > c\sqrt{N}$$

for all $g \in \mathcal{G}_N$. (As we shall see below, this is now known to be false for $\mathcal{G}_N$, but it is thought to be true for $\mathcal{F}_N$.) He also posed our Question 3 for functions of the class $\mathcal{F}_N$, i.e., for Littlewood polynomials.

Rudin [41] rediscovered some of Shapiro's results, and made applications to Fourier analysis.

In 1963, Kahane and Salem (see p. 134 of [27]) discovered that a trigonometric polynomial formed from a short block of Shapiro coefficients is uniformly canceling to the root mean square size. That is, there is an absolute constant C such that

$$\Big|\sum_{n=M}^{M+H-1} e(n)e(nx)\Big| \le C\sqrt{H} \tag{9}$$

for all real x and all integers $M \ge 0$ and $H \ge 1$. This is important, because the classical examples of Hardy–Littlewood and of Weyl do not have this property. Kahane and Salem needed this property for a construction in Fourier analysis. They had $C = 16$, but with more care one can show that the above holds with $C = 2 + 2\sqrt{2}$.

Littlewood [28–30] studied the class $\mathcal{F}_N$ in great detail, and in his book of problems, [31] formulated three conjectures:

Conjecture 1. (JEL) There exist absolute constants a and A with $0 < a < A < \infty$ such that for each N there is an $f \in \mathcal{F}_N$ such that

$$a\sqrt{N} < |f(x)| < A\sqrt{N} \tag{10}$$

for all x.

Conjecture 2. (JEL) For each $\varepsilon > 0$ there is an $N_0(\varepsilon)$ such that if $N > N_0(\varepsilon)$, then there exists an $f \in \mathcal{F}_N$ such that

$$\int_0^1 |f(x)|^4\,dx = (1+\varepsilon)N^2. \tag{11}$$

Conjecture 3. (JEL) The f of Conjecture 1 are exceptional or highly exceptional. Their number is $o\big(2^N\big)$, possibly $O\big(N^3\big)$.

Toward Conjecture 1, Beck [3] showed that there exist trigonometric polynomials f satisfying (10), whose coefficients are roots of unity of order 400. With more work, it should be possible to reduce 400 to a smaller number, possibly even to 3, but his method fails in principle for coefficients ± 1. Odlyzko [38] has recently completed a major numerical study of Littlewood polynomials, which suggests that Conjecture 1 is true, but without identifying a specific sequence of f with the desired properties.

Littlewood [31] showed that the fourth moment of the Shapiro polynomial is

$$\int_0^1 |P_k(x)|^4\,dx \;=\; \frac{4\cdot 2^{2k} - (-1)^k 2^k}{3} \;\sim\; \frac{4}{3}2^{2k}.$$

Thus if the fourth moment is $\geq cN^2$ for all $g \in \mathcal{G}_N$, as Erdős suggested, we must have $c \leq 4/3$.

Kahane [26] proved that there exist $g \in \mathcal{G}_N$ such that

$$\big(1-\varepsilon_N\big)\sqrt{N} \leq |g(x)| \leq \big(1+\varepsilon_N\big)\sqrt{N}$$

with

$$\varepsilon_N \ll \frac{\sqrt{\log N}}{N^{1/17}}.$$

Such trigonometric polynomials are sometimes called 'ultra-flat'. Bombieri and Bourgain [5] have sharpened the above; they showed that one can have

$$\varepsilon_N \ll N^{-1/9+\varepsilon}.$$

With regard to Littlewood's third conjecture, Andrew Odlyzko remarks that while such polynomials are exceptional, and clearly $o\big(2^N\big)$ in number, but if they do exist, then their number cannot be as small as $O\big(N^3\big)$, since one can alter a large number of coefficients without materially affecting the absolute value of the sum.

3 The applied literature

Until comparatively recently, the pure and applied literature developed in complete ignorance of the other. Suppose that $f \in \mathcal{F}_N$, say

$$f(x) = \sum_{n=0}^{N-1} a_n e(nx); \tag{12}$$

thus $a_n = \pm 1$ for all n. Then

$$|f(x)|^2 = N + 2\sum_{k=1}^{N-1} b_k \cos 2\pi kx$$

where

$$b_k = \sum_{n=0}^{N-k-1} a_n a_{n+k}\,.$$

These are the *aperiodic autocorrelation coefficients* of f. Golay [17] defined f_1 and f_2 to form a *complementary pair* if the aperiodic autocorrelation coefficients of f_1 are the negatives of those for f_2. Equivalently,

$$|f_1(x)|^2 + |f_2(x)|^2 = 2N$$

for all x. Golay noted that $f_1(x) = 1 + e(x)$, $f_2(x) = 1 - e(x)$ form a complementary pair. He also used the law of the parallelogram to show that if f_1, f_2 is a complementary pair of length N, then $f_1(x) + e(Nx)f_2(x)$, $f_1(x) - e(Nx)f_2(x)$ is a complementary pair of length $2N$. Thus he had the Shapiro polynomials. He also exhibited a complementary pair of length 10.

By Parseval's identity, we see that

$$\int_0^1 |f(x)|^4\,dx = N^2 + 2\sum_{k=1}^{N-1} b_k^2.$$

Thus the fourth moment is close to N^2 if the b_k are small in mean square. In a series of papers, Golay [17–20, 22] investigated Littlewood polynomials whose aperiodic autocorrelation coefficients are small on average, and in 1972 introduced a precise measure of this, which he called the *merit factor*:

$$F(f) = \frac{N^2}{2\sum_{k=1}^{N-1} b_k^2}.$$

In this language, Littlewood's calculation of 1968 can be interpreted as asserting that $F(P_k) \to 3$ as $k \to \infty$. Littlewood's Conjecture 2 is equivalent to asserting that $F(f)$ can be arbitrarily large. Golay's opinion was different: He thought that merit factors could exceed 12, but not 13. Høholdt and Jensen [24] used the Legendre symbol with a translated argument, $\left(\frac{n+c}{p}\right)$, to show that merit factors approaching 6 can be achieved, and they conjectured that this is best possible. However, Jedwab, Katz, and Schmidt [25] have shown that merit factors can exceed 6.34. Byrnes and Newman [13] showed that if the a_n are independent random variables with $P(a_n = 1) = P(a_n = -1) = 1/2$, then

$$E\left[2\sum_{k=1}^{N-1} b_k^2\right] = N^2 - N,$$

which is to say that

$$E\left[\frac{1}{F}\right] = 1 - \frac{1}{N}\,.$$

For the general Littlewood polynomial as in (12), the aperiodic autocorrelation coefficient b_k is a sum of $N - k$ odd integers. Barker [2] considered polynomials for which these numbers are as small as possible, which is to say that $b_k = 0$ when $N - k$ is even, and $b_k = \pm 1$ when $N - k$ is odd. We note that $-f(x)$ has the same b_k as $f(x)$, and that $f(x + 1/2)$ has aperiodic autocorrelation coefficients $(-1)^k b_k$. Thus by considering $f(x)$, $-f(x)$, $f(x + 1/2)$, $-f(x + 1/2)$ as a group, we may confine our attention to those f for which $a_0 = a_1 = 1$. With this understanding, the coefficients of the known Barker polynomials have signs as follows:

$$\begin{array}{rl}
2 & + + \\
3 & + + - \\
4 & + + + - \\
4 & + + - + \\
5 & + + + - + \\
7 & + + + - - + - \\
11 & + + + - - - + - - + - \\
13 & + + + + + - - + + - + - +
\end{array}$$

Turyn and Storer [44] showed that there are no further Barker codes of odd length. When a Barker code of even length N exists, it follows that there is an $N \times N$ circulant Hadamard matrix. Such matrices are conjectured not to exist for even $N > 4$, so the above list is conjectured to be complete. Borwein and Mossinghoff [6] have shown that there is at most one more Barker code of length $< 4 \cdot 10^{33}$.

4 Shapiro polynomials

From the properties of the Shapiro coefficients $c(n)$ already noted it is easy to show that if k and m are nonnegative and $0 \le n < 2^k$, then

$$c\bigl(2^{k+1}m + n\bigr) = c(m)c(n),$$
$$c\bigl(2^{k+1}m + 2^k + n\bigr) = (-1)^m c(m)c(2^k + n)\,.$$

Thus in an interval $\bigl[m2^k, (m+1)2^k\bigr)$ the coefficients $c(n)$ are the coefficients of $c(m)P_k$ if m is even, and are the coefficients of $c(m)Q_k$ if m is odd.

Much of what we know about the Shapiro polynomials is due to John Brillhart and his collaborators. In particular, Brillhart, Lomont, and Morton [10] noted an interesting relation that arises when the coefficients of P_k are read in reverse order: If $k > 0$, then

$$c\bigl(2^k - 1 - n\bigr) = (-1)^{k+n-1} c(2^k + n)$$

for $0 \le n < 2^k$. Brillhart [7] devised a two-parameter family of identities: For any $j \ge 0$,

$$P_{j+k+1}(x) = P_j(x)P_k(2^{j+1}x) + e(2^j x)Q_j(x)P_k(2^{j+1}x + 1/2)$$

for all $k \ge 0$, and

$$Q_{j+k+1}(x) = P_j(x)Q_k(2^{j+1}x) + e(2^j x)Q_j(x)Q_k(2^{j+1}x + 1/2)$$

for $k \ge 1$. Further interesting results are found in Brillhart, Erdős, and Morton [9], Brillhart, Lomont, and Morton [10], Brillhart and Morton [11], and Brillhart and Morton [12].

For $|z| < 1$ we define the power series

$$P(z) = \sum_{n=0}^{\infty} c(n) z^n\,.$$

Brillhart [7] showed that the unit circle $|z| = 1$ is a natural boundary for the function $P(z)$. From (3) it is clear that

$$P(z) = P(z^2) + zP(-z^2), \qquad P(-z) = P(z^2) - zp(-z^2)\,.$$

Nishioka [36] showed that $P(z)$ and $P(-z)$ are algebraically independent over $\mathbb{C}(z)$, and then applied Mahler's method to prove that if α is algebraic, $0 < |\alpha| < 1$, then $P(\alpha)$ is transcendental; indeed she showed that $P(\alpha)$ and $P(-\alpha)$ are algebraically independent. (See also pp. 155, 158–160 of Nishioka [37].)

In another direction, Mauduit and Rivat [32] have shown that there is a $\theta < 1$ such that

$$\sum_{p \le N} c(p)e(px) \ll N^\theta$$

for $N \ge 2$.

Salem and Zygmund have shown that if the coefficients a_n of a Littlewood polynomial are independent random variables with $P(a_n = 1) = P(a_n = -1) = 1/2$, then

$$\max_x |f(x)| \asymp \sqrt{N \log N}$$

with probability tending to 1 as $N \to \infty$. Since $P_k(x) = O\big(\sqrt{N}\big)$, it is clear that the Shapiro coefficients $c(n)$ do not resemble independent random variables. In particular, from the basic properties already noted it is clear that if $n \equiv 2 \pmod 4$, then $c(n+1) = -c(n)$. Hence there is no n for which $c(n) = c(n+1) = c(n+2) = c(n+3) = c(n+4)$. Let $B(H)$ denote the number of different blocks of length H that occur in the sequence of Shapiro coefficients. We find that $B(H) = 2^H$ for $H = 1, 2, 3, 4$. It is not hard to show that any block $(c(M), c(M+1), \dots, c(M+H-1))$ that occurs will occur with $M \le 26H$. Thus $B(H) \le 26H$ for all H. Blocks can be sorted lexicographically and then counted. A computation of this kind reveals that $B(5) = 24$, $B(6) = 36$, $B(7) = 46$, $B(8) = 56$, and strongly suggests that $B(H) = 8H - 8$ for all $H \ge 8$. This is probably not hard to prove.

Concerning the bound (9) for a sum over a block of H terms of the sequence $c(n)e(nx)$, we consider the question of the best possible constant. Let

$$\mu(H) = \max_{M,x} \frac{\left| \sum_{n=M}^{M+H-1} c(n)e(nx) \right|}{\sqrt{H}}.$$

In Tables 1, 2 we find values of $\mu(H)$. Any block of coefficients of length H will occur with $M \le 26H$, so in the computations we consider M in this range. As to the values of x, we take $x = k/(2K)$ for $0 \le k \le K$ where $K = 100H$.

Among the values in the Tables, we note that $\mu(H)$ is especially large when H is a power of 4:

$$\begin{aligned} \mu(1) &= 1 = 3 - 2 \\ \mu(4) &= 2 = 3 - 1 \\ \mu(16) &= 2.5 = 3 - 0.5 \\ \mu(64) &= 2.75 = 3 - 0.25 \\ \mu(256) &= 2.875 = 3 - 0.125 \end{aligned}$$

Table 1 $\mu(H)$ for $1 \le H \le 128$.

H	$\mu(H)$	M	x	H	$\mu(H)$	M	x	H	$\mu(H)$	M	x	H	$\mu(H)$	M	x
1	1.000	0	0.000	33	2.263	52	0.500	65	2.605	106	0.000	97	2.310	243	0.125
2	1.414	0	0.000	34	2.121	58	0.308	66	2.462	105	0.000	98	2.253	242	0.125
3	1.732	0	0.000	35	2.031	24	0.040	67	2.321	104	0.000	99	2.199	9	0.268
4	2.000	7	0.000	36	2.000	7	0.000	68	2.425	107	0.000	100	2.199	244	0.125
5	1.637	2	0.351	37	1.981	22	0.039	69	2.287	102	0.000	101	2.156	243	0.125
6	1.821	2	0.373	38	1.947	0	0.000	70	2.171	108	0.375	102	2.208	66	0.375
7	1.927	7	0.106	39	2.082	4	0.000	71	2.059	114	0.218	103	2.241	231	0.125
8	2.121	13	0.500	40	2.214	45	0.500	72	2.121	99	0.000	104	2.278	91	0.055
9	1.805	6	0.153	41	2.030	0	0.000	73	1.990	98	0.000	105	2.220	230	0.125
10	1.897	0	0.000	42	2.160	0	0.000	74	1.985	104	0.375	106	2.272	216	0.125
11	2.111	0	0.000	43	2.287	0	0.000	75	2.051	53	0.491	107	2.304	216	0.125
12	2.309	31	0.000	44	2.412	127	0.000	76	2.129	127	0.008	108	2.338	231	0.125
13	1.941	8	0.500	45	2.236	40	0.500	77	2.028	126	0.009	109	2.356	231	0.125
14	2.138	27	0.000	46	2.359	123	0.000	78	2.098	125	0.008	110	2.376	230	0.125
15	2.324	27	0.000	47	2.480	123	0.000	79	2.172	124	0.008	111	2.373	213	0.500
16	2.500	27	0.000	48	2.598	123	0.000	80	2.248	123	0.008	112	2.457	213	0.500
17	2.183	26	0.000	49	2.429	122	0.000	81	2.149	122	0.008	113	2.352	212	0.500
18	1.886	25	0.000	50	2.263	121	0.000	82	2.064	121	0.008	114	2.334	216	0.125
19	1.923	13	0.465	51	2.100	34	0.500	83	2.094	77	0.468	115	2.365	216	0.125
20	2.042	31	0.033	52	2.219	111	0.000	84	2.076	179	0.281	116	2.387	217	0.125
21	1.921	19	0.373	53	2.221	114	0.375	85	2.087	83	0.375	117	2.461	216	0.125
22	2.066	18	0.375	54	2.324	114	0.375	86	2.172	82	0.375	118	2.534	216	0.125
23	2.063	31	0.030	55	2.292	107	0.000	87	2.129	81	0.374	119	2.494	216	0.125
24	2.074	27	0.027	56	2.405	107	0.000	88	2.202	177	0.282	120	2.456	215	0.125
25	2.077	25	0.219	57	2.285	114	0.375	89	2.170	176	0.282	121	2.518	212	0.125
26	2.113	23	0.219	58	2.385	110	0.375	90	2.237	82	0.375	122	2.591	212	0.125
27	2.170	56	0.123	59	2.474	107	0.000	91	2.270	248	0.125	123	2.619	216	0.125
28	2.268	53	0.500	60	2.582	111	0.000	92	2.292	248	0.125	124	2.638	216	0.125
29	2.199	25	0.218	61	2.433	107	0.000	93	2.247	247	0.125	125	2.598	215	0.125
30	2.191	53	0.500	62	2.540	107	0.000	94	2.301	244	0.125	126	2.645	212	0.125
31	2.335	53	0.500	63	2.646	107	0.000	95	2.333	244	0.125	127	2.673	212	0.125
32	2.475	53	0.500	64	2.750	107	0.000	96	2.355	244	0.125	128	2.692	212	0.125

Based on these numerics, we conjecture that

$$\left|\sum_{n=M}^{M+H-1} c(n)e(nx)\right| \le 3\sqrt{H}$$

for all real x, all nonnegative integers M, and all positive integers H.

Table 2 $\mu(H)$ for $129 \leq H \leq 256$.

H	$\mu(H)$	M	x	H	$\mu(H)$	M	x	H	$\mu(H)$	M	x	H	$\mu(H)$	M	x
129	2.652	211	0.125	161	2.286	180	0.500	193	2.663	490	0.000	225	2.501	428	0.375
130	2.604	210	0.125	162	2.200	179	0.500	194	2.585	489	0.000	226	2.467	428	0.375
131	2.538	212	0.125	163	2.166	35	0.195	195	2.506	488	0.000	227	2.417	425	0.375
132	2.557	212	0.125	164	2.186	7	0.000	196	2.571	491	0.000	228	2.435	424	0.375
133	2.518	211	0.125	165	2.102	31	0.195	197	2.494	486	0.000	229	2.407	424	0.375
134	2.472	206	0.125	166	2.173	0	0.000	198	2.416	485	0.000	230	2.445	434	0.375
135	2.384	205	0.125	167	2.244	4	0.000	199	2.339	483	0.000	231	2.468	434	0.375
136	2.312	211	0.125	168	2.315	173	0.500	200	2.404	483	0.000	232	2.506	475	0.055
137	2.325	207	0.125	169	2.231	0	0.000	201	2.328	482	0.000	233	2.457	433	0.375
138	2.344	206	0.125	170	2.301	0	0.000	202	2.252	481	0.000	234	2.495	446	0.375
139	2.258	205	0.125	171	2.371	0	0.000	203	2.192	135	0.125	235	2.544	427	0.000
140	2.174	205	0.125	172	2.440	511	0.000	204	2.240	447	0.000	236	2.604	447	0.000
141	2.139	203	0.125	173	2.357	168	0.500	205	2.236	343	0.188	237	2.556	428	0.375
142	2.091	202	0.125	174	2.426	507	0.000	206	2.251	134	0.125	238	2.593	443	0.000
143	2.091	27	0.000	175	2.495	507	0.000	207	2.294	443	0.000	239	2.652	443	0.000
144	2.167	27	0.000	176	2.563	507	0.000	208	2.357	443	0.000	240	2.711	443	0.000
145	2.077	195	0.125	177	2.480	506	0.000	209	2.283	442	0.000	241	2.641	442	0.000
146	2.062	212	0.125	178	2.399	505	0.000	210	2.232	461	0.469	242	2.571	441	0.000
147	2.076	88	0.010	179	2.317	162	0.500	211	2.286	461	0.469	243	2.574	434	0.375
148	2.045	214	0.268	180	2.385	495	0.000	212	2.280	127	0.125	244	2.591	435	0.375
149	2.085	271	0.177	181	2.304	160	0.500	213	2.330	434	0.375	245	2.645	434	0.375
150	2.036	96	0.229	182	2.372	491	0.000	214	2.387	434	0.375	246	2.698	434	0.375
151	2.063	95	0.229	183	2.439	491	0.000	215	2.359	434	0.375	247	2.670	434	0.375
152	2.109	173	0.500	184	2.507	491	0.000	216	2.332	429	0.375	248	2.667	427	0.000
153	2.063	41	0.195	185	2.426	490	0.000	217	2.382	430	0.375	249	2.692	430	0.375
154	2.095	16	0.000	186	2.493	491	0.000	218	2.439	430	0.375	250	2.745	430	0.375
155	2.169	16	0.000	187	2.559	491	0.000	219	2.461	429	0.375	251	2.765	429	0.375
156	2.242	181	0.500	188	2.626	495	0.000	220	2.479	428	0.375	252	2.782	428	0.375
157	2.155	14	0.000	189	2.546	491	0.000	221	2.451	428	0.375	253	2.754	428	0.375
158	2.228	181	0.500	190	2.612	491	0.000	222	2.489	430	0.375	254	2.791	430	0.375
159	2.300	181	0.500	191	2.677	491	0.000	223	2.511	429	0.375	255	2.818	427	0.000
160	2.372	181	0.500	192	2.742	491	0.000	224	2.539	427	0.000	256	2.875	427	0.000

For nonnegative integers r, let

$$A_r = \frac{5 \cdot 4^r + 1}{3}, \qquad B_r = \frac{8 \cdot 4^r + 1}{3}.$$

This is the same B_r that we defined in (8). It is not hard to show that

$$T(A_r; 0) = 2^r + 1, \qquad T(B_r; 0) = 2^{r+2} - 1.$$

We note that $B_r = A_r + 4^r$. By differencing we deduce that

$$\sum_{n=A_r}^{A_r+4^r-1} c(n) = 3 \cdot 2^r - 2\,.$$

Thus the conjecture is best-possible, if true. The numbers A_r and B_r are famous because $T(H;0)/\sqrt{H}$ is unusually small when $H = A_r$, and unusually large when $H = B_r$. Brillhart and Morton [11] formulate this in precise terms, and show (among other things) that

$$\liminf_{H\to\infty} \frac{T(H;0)}{\sqrt{H}} = \lim_{r\to\infty} \frac{T(A_r;0)}{\sqrt{A_r}} = \sqrt{\frac{3}{5}},$$
$$\limsup_{H\to\infty} \frac{T(H;0)}{\sqrt{H}} = \lim_{r\to\infty} \frac{T(B_r;0)}{\sqrt{B_r}} = \sqrt{6}\,.$$

From Shapiro's bound (5) we know that $0 \le |P_k(x)|^2 \le 2^{k+1}$ for all x. As to the distribution of $|P_k(x)|^2$ within this interval, we have the following conjecture.

Conjecture 4 (*Saffari* ca *1980)*. The distribution of

$$\frac{|P_k(x)|^2}{2^{k+1}}$$

as x runs from 0 to 1 tends toward uniform distribution on [0, 1] as k tends to infinity.

This conjecture is equivalent to the assertion that

$$\int_0^1 |P_k(x)|^{2m}\,dx \sim \frac{2^{m(k+1)}}{m+1}$$

as $k \to \infty$, for each fixed positive integer m. Doche and Habsieger [14] have shown that the above is true for $m = 1, 2, \ldots, 26$.

We now propose a strengthening of Saffari's Conjecture.

Conjecture 5 (*Montgomery* ca 2008). The distribution of

$$\frac{P_k(x)}{2^{(k+1)/2}}$$

as x runs from 0 to 1 tends toward uniform distribution on the unit disc $|z| \le 1$ as k tends to infinity.

For late-breaking information concerning these conjectures, see Ekhad and Zeilberger [15] and Rodgers [40].

Conjecture 5 is equivalent to the assertion that

$$\int_0^1 P_k(x)^m P_k(-x)^n \, dx = \begin{cases} \big(1 + o(1)\big) \dfrac{2^{(k+1)m}}{m+1} & \text{if } m = n, \\ o\big(2^{k(m+n)/2}\big) & \text{if } m \neq n \end{cases}$$

as $k \to \infty$ for each fixed pair m, n of nonnegative integers. It is trivial that

$$\int_0^1 P_k(x)^m \, dx = 1 = o\big(2^{km/2}\big)$$

for each positive integer m. To test the above in a slightly less trivial case, we consider $m = 2, n = 1$. We set

$$A_k = \int_0^1 P_k(x)^2 P_k(-x) \, dx, \qquad G_k = \int_0^1 P_k(x) Q_k(x) Q_k(-x) \, dx,$$
$$B_k = \int_0^1 P_k(x)^2 e(-2^k x) P_k(-x) \, dx, \qquad H_k = \int_0^1 P_k(x) Q_k(x) e(-2^k x) Q_k(-x) \, dx,$$
$$C_k = \int_0^1 P_k(x)^2 Q_k(-x) \, dx, \qquad I_k = \int_0^1 Q_k(x)^2 P_k(-x) \, dx,$$
$$D_k = \int_0^1 P_k(x)^2 e(-2^k x) Q_k(-x) \, dx, \qquad J_k = \int_0^1 Q_k(x)^2 e(-2^k x) P_k(-x) \, dx,$$
$$E_k = \int_0^1 P_k(x) Q_k(x) P_k(-x) \, dx, \qquad K_k = \int_0^1 Q_k(x)^2 Q_k(-x) \, dx,$$
$$F_k = \int_0^1 P_k(x) Q_k(x) e(-2^k x) P_k(-x) \, dx, \qquad L_k = \int_0^1 Q_k(x)^2 e(-2^k x) Q_k(-x) \, dx \, .$$

(Here A_k and B_k have no connection with the A_r and B_r considered earlier in a different context.) From the usual recurrences for the P_k and Q_k we deduce that

$$A_{k+1} = A_k + D_k + 2G_k, \qquad G_{k+1} = A_k - D_k,$$
$$B_{k+1} = 2F_k + I_k + L_k, \qquad H_{k+1} = -I_k + L_k,$$
$$C_{k+1} = A_k - D_k - 2G_k, \qquad I_{k+1} = A_k + D_k - 2G_k,$$
$$D_{k+1} = 2F_k + I_k - L_k, \qquad J_{k+1} = -2F_k + I_k + L_k,$$
$$E_{k+1} = A_k + D_k, \qquad K_{k+1} = A_k - D_k + 2G_k,$$
$$F_{k+1} = -I_k - L_k, \qquad L_{k+1} = -2F_k + I_k - L_k \, .$$

We observe that each of the six integrals A_{k+1}, D_{k+1}, F_{k+1}, G_{k+1}, I_{k+1}, L_{k+1} is a linear combination of the six integrals A_k, D_k, F_k, G_k, I_k, L_k, while the other six integrals are linear combinations of these. Thus

$$\begin{bmatrix} A_{k+1} \\ D_{k+1} \\ F_{k+1} \\ G_{k+1} \\ I_{k+1} \\ L_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} A_k \\ D_k \\ F_k \\ G_k \\ I_k \\ L_k \end{bmatrix}. \tag{13}$$

The characteristic polynomial of this matrix is $p(z) = z^6 - 6z^4 + 24z^2 - 64$. Hence by the Cayley–Hamilton Theorem each of A_k, D_k, F_k, G_k, I_k, and L_k satisfies the linear recurrence

$$u_n = 6u_{n-2} - 24u_{n-4} + 64u_{n-6}\,.$$

The roots of this characteristic polynomial are 2, -2, and $\frac{1}{2}\big(\pm\sqrt{10} \pm i\sqrt{6}\big)$. Since all six roots have modulus 2, we deduce that

$$\int_0^1 P_k(x)^2 P_k(-x)\,dx = O\big(2^k\big) = o\big(2^{3k/2}\big).$$

The curves $P_k(x)$ for $k = 4, \ldots 8$ are plotted in Figures 1, 2 and 3 (a). We note that as k increases, it seems that the part of the plot near the boundary becomes darker than the middle portion. We propose that while there is more ink near the boundary, the quantity $P_k(x)$ does not spend more time near the boundary, but rather that the point $P_k(x)$ tends to move more quickly when it is near the boundary. As a test of this idea, in Figure 3(b) we plot $|P_8|$ against $|P_8'|$. It does indeed seem that $|P_8(x)|$ and $|P_8'(x)|$ are correlated. This is all a little informal. A better visual depiction of the distribution of $P_k(x)$ is provided by a scatter plot, in which we plot $P_k(x)$ at a large number of equally spaced values of x. In Figure 4 we have a plot of P_{10} at 10,000 points. Bernstein's inequality asserts that if T is a trigonometric polynomial with period 1 and degree not exceeding N, then $\|T'\|_\infty \le 2\pi N \|T\|_\infty$. Thus $|P_k(x)| \le \sqrt{8}\pi\, 2^{3k/2}$ for all x. When $k = 10$ we have $N = 1023$, so when $q = 10^4$ the difference between $P_{10}(a/q)$ and $P_{10}((a+1)/q)$ can be of a size comparable to the diameter of the disc in which the values are occurring, and indeed will be of this size for a positive proportion of x. Thus, the numbers $P_{10}(a/q)$ comprise a fairly random sampling of the values of P_{10} when $q = 10^4$. The evaluations of $P_{10}(a/q)$ were not done by summing 1024 terms, but rather by applying the recurrences for P_k and Q_k. We note that

$$\begin{bmatrix} P_k(x) \\ Q_k(x) \end{bmatrix} = \begin{bmatrix} 1 & e\big(2^k x\big) \\ 1 & -e\big(2^k x\big) \end{bmatrix} \begin{bmatrix} 1 & e\big(2^{k-1} x\big) \\ 1 & -e\big(2^{k-1} x\big) \end{bmatrix} \cdots \begin{bmatrix} 1 & e(x) \\ 1 & -e(x) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

When calculated in this way, one evaluation of P_{10} amounts to nine multiplications of 2×2 matrices.

5 Rigorous bounds

In Tables 1, 2 we list the maximum modulus of various trigonometric polynomials. To determine such a maximum modulus we compute the trigonometric polynomial at a discrete set of points. This is fine for purposes of experimentation, but if a rigorous bound is desired, we must consider how much larger the actual maximum is, compared to the computed value. Suppose that T is a real-valued trigonometric polynomial with period 1 and degree not exceeding N. Suppose that $M = \max_x |T(x)|$, and put

$$M(q) = \max_{1 \le a \le q} |T(a/q)|\,.$$

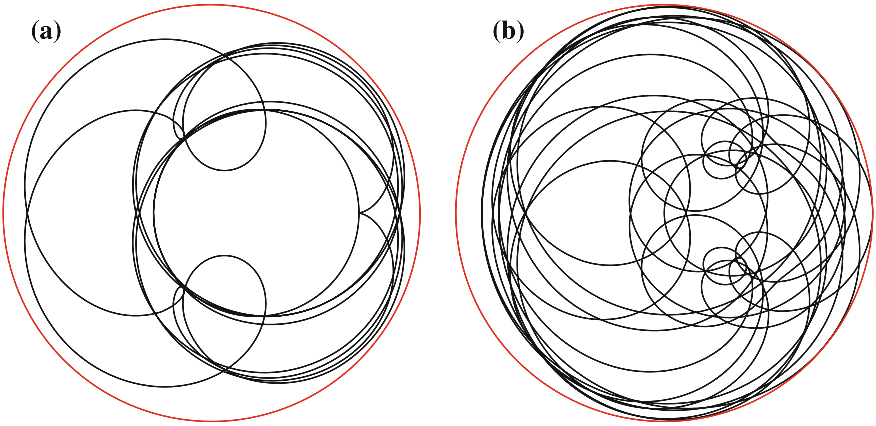


Fig. 1 Plots of (a) $P_4(x)$, (b) $P_5(x)$ for $0 \le x \le 1$.

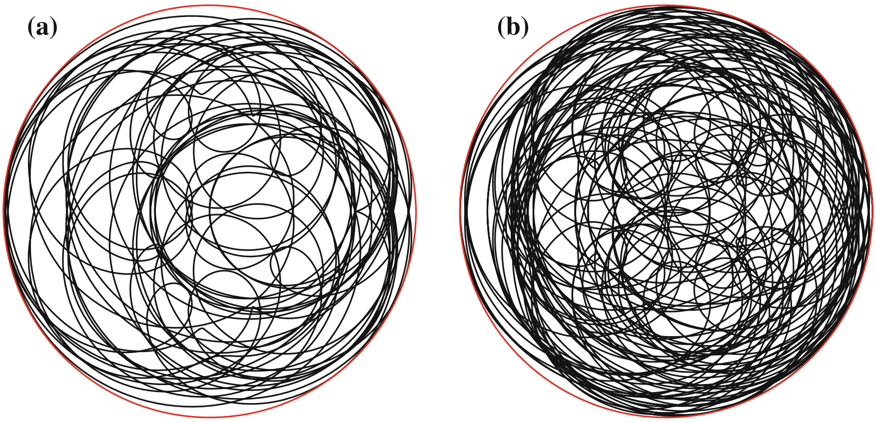


Fig. 2 Plots of (a) $P_6(x)$, (b) $P_7(x)$ for $0 \le x \le 1$.

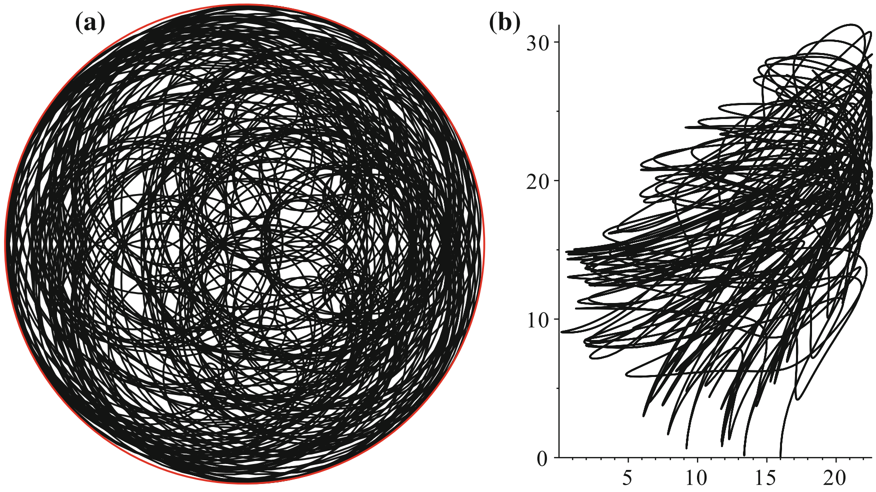


Fig. 3 Plot of (a) $P_8(x)$, (b) $P_8(x)$ vs $P_8'(x)$ for $0 \le x \le 1$.

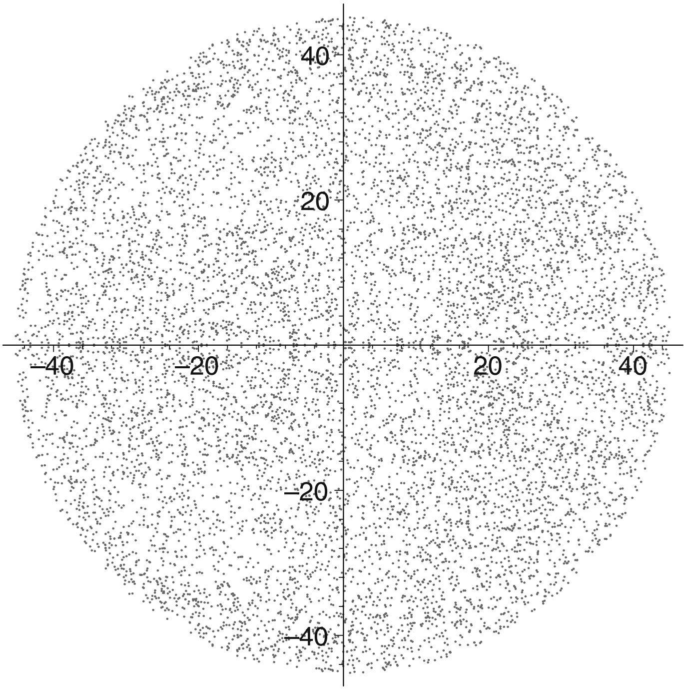


Fig. 4 Plot of $P_{10}(a/q)$ for $a = 1, 2, \ldots, q$, with $q = 10^4$.

Clearly $M(q) \le M$, and we seek an inequality in the reverse direction. For this purpose, a classical result of M. Riesz [39] is the perfect tool. Suppose, as we may, that $T(x_0) = M$. Riesz's result is that

$$T(x) \ge M \cos(2\pi N(x - x_0))$$

for $x_0 - 1/(2N) \le x \le x_0 + 1/(2N)$. That is, the peak that $\cos 2\pi Nx$ has at the origin is as narrow as the peak any trigonometric polynomial of degree at most N can have at its maximum. Choose a so that

$$\left| x_0 - \frac{a}{q} \right| \le \frac{1}{2q} .$$

If $q \ge N$, then $a/q \in [x_0 - 1/(2N), x_0 + 1/(2N)$, and so

$$M \le \frac{M(q)}{\cos \dfrac{\pi N}{q}} .$$

This is best-possible when q is a multiple of N, and is in general quite sharp.

By making a linear change of variable we obtain the following useful variant of the above. If T is a nonnegative trigonometric polynomial with period 1, degree not exceeding N, and $\max_x T(x) = M$, then

$$M \le \frac{M(q)}{\cos^2 \dfrac{\pi N}{2q}} .$$

In connection with Tables 1, 2 we put

$$T(x) = \left| \sum_{n=M}^{M+H-1} c(n) e(nx) \right|^2 .$$

This is a nonnegative trigonometric polynomial of degree H. With $q = 200H$, we see that the actual maxima can be larger than the values stated in the tables by a factor of at most

$$\sec^2 \frac{\pi}{400} = 1.000000015.$$

Acknowledgements The author is also indebted to the referee for valuable suggestions.

References

1. J.-P. Allouche, J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations* (Cambridge University Press, Cambridge, 2003), xvi+571 pp
2. R.H. Barker, Group synchronizing of binary digital systems, in *Communication Theory* (Butterworths Science Publications, London, 1953), pp. 273–287
3. J. Beck, Flat polynomials on the unit circle–note on a problem of Littlewood. Bull. Lond. Math. Soc. **23**, 269–277 (1991)
4. S. Bernstein, Sur la convergence absolue des séries trigonométriques. Commun. Soc. Math. Kharkov **2**(14), 139–144 (1914)
5. E. Bombieri, J. Bourgain, On Kahane's ultraflat polynomials. J. Eur. Math. Soc. **11**, 627–703 (2009)
6. P. Borwein, M. Mossinghoff, Wieferich pairs and Barker sequences. II. LMS J. Comput. Math. **17**, 24–32 (2014)
7. J. Brillhart, On the Rudin–Shapiro polynomials. Duke Math. J. **40**, 335–353 (1973)
8. J. Brillhart, L. Carlitz, Note on the Shapiro polynomials. Proc. Am. Math. Soc. **25**, 114–119 (1970)
9. J. Brillhart, P. Erdős, P. Morton, On sums of Rudin–Shapiro coefficients. II. Pac. J. Math. **107**, 39–69 (1983)
10. J. Brillhart, J.S. Lomont, P. Morton, Cyclotomic properties of the Rudin–Shapiro polynomials. J. Rein. Angew. Math. **288**, 37–65 (1976)
11. J. Brillhart, P. Morton, Über Summen von Rudin–Shapiroschen Koeffizienten. Ill. J. Math. **22**, 126–148 (1978)
12. J. Brillhart, P. Morton, A case study in mathematical research: the Golay–Rudin–Shapiro sequence. Am. Math. Mon. **103**, 854–869 (1996)
13. J.S. Byrnes, D.J. Newman, The L^4 norm of a polynomial with coefficients ± 1. Am. Math. Mon. **97**, 42–45 (1990)
14. C. Doche, L. Habsieger, Moments of the Rudin–Shapiro polynomials. J. Fourier Anal. Appl. **10**, 497–505 (2004)
15. S. B. Ekhad, D. Zeilberger, Integrals involving Rudin–Shapiro polynomials and a sketch of a proof of Saffari's conjecture, preprint, 12 pp (2016)
16. P. Erdős, Some unsolved problems. Mich. Math. J. **4**, 291–300 (1957)
17. M.J.E. Golay, Multislit spectrometry. J. Opt. Soc. Am. **39**, 437–444 (1949)
18. M.J.E. Golay, Static multislit spectrometry and its application to the panoramic display of infraread spectra. J. Opt. Soc. Am. **41**, 468–472 (1951)
19. M.J.E. Golay, A class of finite binary sequences with alternate autocorrelation values equal to zero. IEEE Trans. Inf. Theory **IT-18**, 449–450 (1972)
20. M.J.E. Golay, Sieves for low autocorrelation binary sequences. IEEE Trans. Inf. Theory **IT-23**, 43–51 (1977)
21. M.J.E. Golay, The merit factor of long low autocorrelation binary sequences. IEEE Trans. Inf. Theory **IT-28**(3), 543–549 (1982)
22. M.J.E. Golay, The merit factor of Legendre sequences. IEEE Trans. Inf. Theory **29**, 934–935 (1983)
23. G.H. Hardy, J.E. Littlewood, Some problems of Diophantine approximation: a remarkable trigonometric series. Proc. Natl. Acad. Sci. **2**, 583–586 (1916)
24. T. Høholdt, H.E. Jensen, Determination of the merit factor of Legendre sequences. IEEE Trans. Inf. Theory **34**, 161–164 (1988)
25. J. Jedwab, D.J. Katz, K.-U. Schmidt, Littlewood polynomials with small L^4 norm. Adv. Math. **241**, 127–136 (2013)
26. J.-P. Kahane, Sur les polynômes à coefficients unimodulaires. Bull. Lond. Math. Soc. **12**, 321–342 (1980)
27. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris (1994), 192 pp; Revised Edition, 1994, 243 pp

28. J.E. Littlewood, On the mean values of certain trigonometrical polynomials. J. Lond. Math. Soc. **36**, 307–334 (1961)
29. J.E. Littlewood, On the mean values of certain trigonometrical polynomials II. Ill. J. Math. **6**, 1–39 (1962)
30. J.E. Littlewood, On polynomials $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$. J. Lond. Math. Soc. **41**, 367–376 (1966)
31. J.E. Littlewood, *Some Problems in Real and Complex Analysis* (Health, Lexington, 1968)
32. C. Mauduit, J. Rivat, Prime numbers along Rudin–Shapiro sequences. J. Eur. Math. Soc. **17**, 2595–2642 (2015)
33. M. Mendès France, G. Tenenbaum, Dimension des courbes planes, papiers plies, et suites de Rudin–Shapiro. Bull. Soc. Math. Fr. **109**, 207–215 (1981)
34. H.L. Montgomery, An exponential polynomial formed with the Legendre symbol. Acta Arith. **37**, 375–380 (1980)
35. H.L. Montgomery, *Early Fourier Analysis*. Pure and Applied Undergraduate Texts (American Mathematical Society, Providence, 2015), x+390 pp
36. K. Nishioka, A new approach in Mahler's method. J. Rein. Angew. Math. **407**, 202–219 (1990)
37. K. Nishioka, *Mahler Functions and Transcendence*. LNM (Springer, Berlin, 1996), viii+185 pp
38. A.M. Odlyzko, Search for ultraflat polynomials with plus and minus one coefficients, in *Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham*, ed. by S. Butler, J. Cooper, G. Hurlbert (Cambridge University Press, Cambridge, 2016), 14 pp to appear
39. M. Riesz, Eine trigonometrische Interpolations-formel und einige Ungleichung für Polynome. Jahresbericht des Deutschen Mathematiker-Vereinigung **23**, 354–368 (1914)
40. B. Rodgers, On the distribution of Rudin–Shapiro polynomials, arXiv: 1606.01637 [Math.CA], 12 pp, (2016)
41. W. Rudin, Some theorems on Fourier coefficients. Proc. Am. Math. Soc. **10**, 855–859 (1959)
42. B. Saffari, Une fonction extrémale liée à la suite de Rudin-Shapiro. C. R. Acad. Sci. Paris Sér. I Math **303**, 97–100 (1986)
43. H.S. Shapiro, Extremal problems for polynomials. M.S. thesis, MIT, 1951
44. R. Turyn, J. Storer, On binary sequences. Proc. Am. Math. Soc. **12**, 394–399 (1961)

Trapezoidal Numbers, Divisor Functions, and a Partition Theorem of Sylvester

Melvyn B. Nathanson

To Krishnaswami Alladi on his 60th birthday

Abstract A *partition* of a positive integer n is a representation of n as a sum of a finite number of positive integers (called *parts*). A *trapezoidal number* is a positive integer that has a partition whose parts are a decreasing sequence of consecutive integers, or, more generally, whose parts form a finite arithmetic progression. This paper reviews the relation between trapezoidal numbers, partitions, and the set of divisors of a positive integer. There is also a complete proof of a theorem of Sylvester that produces a stratification of the partitions of an integer into odd parts and partitions into disjoint trapezoids.

Keywords Partitions · Sylvester · Trapezoidal numbers · Divisor functions

2010 Mathematics Subject Classification 05A17 · 11P81 · 11A05 · 11B75

1 Partition theorems of Euler and Sylvester

Let $\mathbf{N}$, $\mathbf{N}_0$, and $\mathbf{Z}$ denote, respectively, the sets of positive integers, nonnegative integers, and integers. A *partition* of a positive integer n is a representation of n as a sum of a finite number of positive integers (called *parts*), written in decreasing order.

Supported in part by a grant from the PSC-CUNY Research Award Program.

M. B. Nathanson (✉)
Department of Mathematics, Lehman College (CUNY),
Bronx, NY 10468, USA
e-mail: melvyn.nathanson@lehman.cuny.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_31

The usual left-justified Ferrers diagram of the partition

$$n = a_1 + a_2 + \cdots + a_k$$

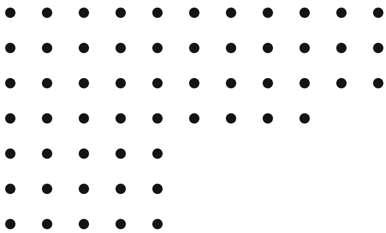
with

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$$

consists of k rows of dots, with a_i dots on row i. For example, the Ferrers diagram of the partition

$$57 = 11 + 11 + 11 + 9 + 5 + 5 + 5$$

is



Perhaps the best-known result about partitions is the following theorem of Euler.

Theorem 1.1 (Euler). *The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Proof Let $p_{odd}(n)$ denote the number of partitions of n into odd parts, and let $p_{dis}(n)$ denote the number of partitions into distinct parts. A deceptively simple proof uses formal power series:

$$\begin{aligned}\sum_{n=0}^{\infty} p_{odd}(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{(1-q^{2n-1})(1-q^{2n})} \\ &= \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} = \prod_{n=1}^{\infty} \frac{(1-q^n)(1+q^n)}{1-q^n} \\ &= \prod_{n=1}^{\infty}(1+q^n) = \sum_{n=0}^{\infty} p_{dis}(n)q^n.\end{aligned}$$

This argument is valid only after one understands infinite products, inversion, and composition of formal power series.

Every positive integer n has a unique g-adic representation in the form $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$, where $\varepsilon_i \in \{0, 1, \ldots, g-1\}$ for $i \in \mathbf{N}_0$ and $\varepsilon_i = 0$ for all sufficiently large i. Glaisher [11] generalized Euler's theorem by using the uniqueness of the g-adic representation. Theorem 1.1 is the special case $g = 2$.

Theorem 1.2 (Glaisher). *Let $g \geq 2$. The number of partitions of n into parts not divisible by g equals the number of partitions of n such that every part occurs less than g times.*

Proof. Every positive integer a can be written uniquely in the form $a = g^{v}s$, where s is not divisible by g. Sylvester calls s the *nucleus* of a. A partition of n in which every part occurs at most $g-1$ times can be written uniquely in the form

$$n = \varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \tag{1}$$

where the parts $a_1, \ldots, a_k$ are pairwise distinct and $\varepsilon_i \in \{1, \ldots, g-1\}$ for $i = 1, \ldots, k$. Let

$$a_i = g^{v_i} s_i = \underbrace{s_i + \cdots + s_i}_{g^{v_i} \text{ summands}}$$

where s_i is the nucleus of a_i. The nuclei $s_1, \ldots, s_k$ are not necessarily distinct. Let $S = \{s_1, \ldots, s_k\}$. For each $s \in S$, let

$$\delta(s) = \sum_{\substack{i \in \{1,\ldots,k\} \\ s_i = s}} \varepsilon_i g^{v_i}.$$

Then

$$\begin{aligned}
n &= \varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \\
&= \varepsilon_1 g^{v_1} s_1 + \cdots + \varepsilon_k g^{v_k} s_k \\
&= \underbrace{s_1 + \cdots + s_1}_{\varepsilon_1 g^{v_1} \text{ summands}} + \cdots + \underbrace{s_k + \cdots + s_k}_{\varepsilon_k g^{v_k} \text{ summands}} \\
&= \sum_{s \in S} \left(\sum_{\substack{i \in \{1,\ldots,k\} \\ s_i = s}} \varepsilon_i g^{v_i} \right) s \\
&= \sum_{s \in S} \delta(s) s.
\end{aligned}$$

Thus, from the partition (1) of n into parts occurring less than g times we have constructed a partition of n as a sum of integers not divisible by g.

Conversely, let $n = \sum_{s \in S} \delta(s)s$ be a partition of n with parts in a set S of integers not divisible by g, and where each $s \in S$ has multiplicity $\delta(s)$. Consider the g-adic representation

$$\delta(s) = \sum_{i \in I_s} \varepsilon_i g^i$$

where $\varepsilon_i \in \{1, \ldots, g-1\}$. If $(i_1, s_1) \neq (i_2, s_2)$, then $g^{i_1}s_1 \neq g^{i_2}s_2$ and so

$$n = \sum_{s \in S} \delta(s)s = \sum_{s \in S} \sum_{i \in I_s} \varepsilon_i g^i s$$

is a partition of n into distinct parts $g^i s$ with multiplicities at most $g-1$. These two partition transformations are inverse maps, and establish a one-to-one correspondence between partitions into parts not divisible by g and parts occurring with multiplicities less than g.

Sylvester [21, Sections 45–46] discovered and proved a different, very beautiful, and insufficiently known generalization of Euler's theorem. We prove this theorem in Section 4.

2 Trapezoidal numbers

For integers $k \in \mathbf{N}$, $t \in \mathbf{N}_0$, and $a \in \mathbf{Z}$, the finite arithmetic progression with length k, difference t, and first term a is the set

$$\{a, a+t, a+2t, \ldots, a+(k-1)t\}. \tag{2}$$

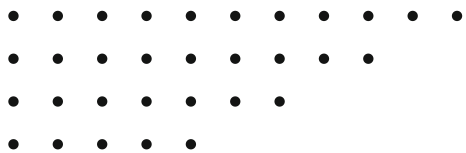
The sum of this arithmetic progression is

$$s_{k,t}(a) = \sum_{i=0}^{k-1}(a+it) = ka + \frac{k(k-1)t}{2}. \tag{3}$$

The integer a is the smallest element of the set (2) because $t \geq 0$.

Let $t \in \mathbf{N}_0$. A positive integer n is a *k-trapezoid with difference t* if it is the sum of a finite arithmetic progression of integers of length k and difference t, that is, if it can be represented in the form (3) for integers $k \in \mathbf{N}$, $t \in \mathbf{N}_0$, and $a \in \mathbf{Z}$. A *trapezoid with difference t* is a k-trapezoid with difference t for some $k \in \mathbf{N}$. A *k-trapezoid* is a k-trapezoid with difference 1. For example, every odd integer is a 2-trapezoid, because $2n-1 = (n-1)+n$. A *trapezoid* is an integer that is a k-trapezoid for some k, that is, an integer that can be represented as the sum of a strictly decreasing sequence of consecutive integers.

A k-trapezoid with difference t is *positive* if $a \geq 1$ and *nonpositive* if $a \leq 0$. If a is positive, then the Ferrers diagram of this partition of n has a trapezoidal shape. For example, $32 = 11 + 9 + 7 + 5$ is a positive 4-trapezoid with difference 2. Its Ferrers diagram is



Every positive integer n has a trivial positive trapezoidal representation with length 1 and difference 1, namely, $n = n$. Sylvester [22] and Mason [16] proved that a positive integer n is a k-trapezoid for some $k \geq 2$ if and only if n is not a power of 2, and that the number of positive trapezoidal representations of n is exactly the number of odd positive divisors of n. Bush [9] extended this result to trapezoidal representations with difference t. We prove their theorems below.

In Section 4 we show how a special case of a partition theorem of Sylvester establishes another bijection between the number of trapezoidal representations of n and the number of positive odd divisors of n.

For every positive integer n, let $\Phi_t(n)$ denote the number of representations of n as a trapezoid with difference t, and let $\Phi_t^+(n)$ denote the number of representations of n as a positive trapezoid with difference t. Thus,

$$\Phi_t(n) = \left|\left\{(k, a) \in \mathbf{N} \times \mathbf{Z} : s_{k,t}(a) = n\right\}\right|$$
$$\Phi_t^+(n) = \left|\left\{(k, a) \in \mathbf{N} \times \mathbf{N} : s_{k,t}(a) = n\right\}\right|$$

For $t = 1$, these functions count partitions into consecutive integers.

Let $d(n)$ denote the number of positive divisors of n, and let $d_1(n)$ denote the number of odd positive divisors of n. Let $d(n, \theta)$ denote the number of positive divisors d of n such that $d < \theta$. If $n/2 < k \leq n$, then

$$d(n, k) = d(n, n) = d(n) - 1.$$

Let $[x]$ denote the integer part of the real number x.

Lemma 2.1. *Let t and n be positive integers. For every positive integer k, there is at most one representation of n as a sum of a k-term arithmetic progression of integers with difference t.*

Proof. This is true because the function $s_{k,t}(a)$ defined by (3) is a strictly increasing function of a.

Theorem 2.1. *Let t be an even positive integer. For every positive integer n,*

$$\Phi_t(n) = d(n) \tag{4}$$

and

$$\Phi_t^+(n) = d(n, \theta) \tag{5}$$

where

$$\theta = \frac{1}{2} + \sqrt{\frac{2n}{t} + \frac{1}{4}}.$$

Proof. For every positive divisor k of n,

$$a_{k,t}(n) = \frac{n}{k} - \frac{(k-1)t}{2}$$

is an integer and

$$s_{k,t}\left(a_{k,t}(n)\right) = \sum_{i=0}^{k-1}\left(\frac{n}{k} - \frac{(k-1)t}{2} + it\right) = n.$$

Moreover, if k and d are distinct positive divisors of n, then $a_{k,t}(n) \neq a_{d,t}(n)$. Thus, $d(n) \leq \Phi_t(n)$.

Conversely, if n is the sum of a k-term arithmetic progression with even difference t and first term a, then

$$n = s_{k,t}(a) = k\left(a + \frac{(k-1)t}{2}\right)$$

and so k is a positive divisor of n and $a = a_{k,t}(n)$. Thus, $\Phi_t(n) \leq d(n)$, and so there is a one-to-one correspondence between the positive divisors of n and representations of n as a sum of a finite arithmetic progression with difference t. This proves (4).

Let $n = \sum_{i=0}^{k-1}(a + it)$. The first term $a = a_{k,t}(n)$ is positive if and only if

$$\frac{n}{k} > \frac{(k-1)t}{2}$$

or, equivalently,

$$k < \frac{1}{2} + \sqrt{\frac{2n}{t} + \frac{1}{4}}.$$

This proves (5).

Lemma 2.2. *Let t be an odd positive integer. Let n be a positive integer, and let $s_{k,t}(a) = n$ for some integer a and some positive integer k. If k is odd, then k is an odd positive divisor of n. If k is even, then $2n/k$ is an odd positive divisor of n.*

Proof. If k is odd, then $(k-1)/2$ is an integer and the identity

$$n = s_{k,t}(a) = ka + \frac{k(k-1)t}{2} = k\left(a + \frac{(k-1)t}{2}\right)$$

implies that k is a positive divisor of n.

If k is even, then $d = 2a + (k-1)t$ is odd and the identity

$$n = \frac{k}{2}(2a + (k-1)t)$$

implies that $2n/k = 2a + (k-1)t$ is an odd positive divisor of n. This completes the proof.

Theorem 2.2. *Let t be an odd positive integer. For every odd positive divisor k of n, there is exactly one representation of n as a sum of a k-term arithmetic progression* *of integers with difference t, and there is exactly one representation of n as a sum of a $(2n/k)$-term arithmetic progression of integers with difference t.*

The number of representations of n as a t-trapezoid is

$$\Phi_t(n) = 2d_1(n).$$

Proof. Let k be an odd positive divisor of n, and let $n = kq$. If $k = 2e + 1$, then

$$n = \sum_{i=-e}^{e} (q + it) \tag{6}$$

is a representation of n as a sum of an arithmetic progression with difference t, length k, and first term

$$a_{k,t}(n) = q - et = \frac{n}{k} - \frac{(k-1)t}{2}. \tag{7}$$

Let

$$b_{k,t}(n) = \frac{n}{2q} - \frac{(2q-1)t}{2} = \frac{k+t}{2} - \frac{nt}{k}. \tag{8}$$

Then $b_{k,t}(n)$ is an integer, and

$$n = \sum_{i=0}^{2q-1} (b_{k,t}(n) + it) \tag{9}$$

is a representation of n as a sum of an arithmetic progression with difference t, length $2q = 2n/k$, and first term $b_{k,t}(n)$. Applying Lemma 2.2, we see that there is a one-to-one correspondence between the odd positive divisors of n and the representations of n as a sum of an arithmetic progression with difference t and odd length, and there is also a one-to-one correspondence between the odd positive divisors of n and the representations of n as a sum of an arithmetic progression with difference t and even length. This completes the proof.

For example, the only odd positive divisor of 1 is 1, and so $\Phi_t(1) = 2d_1(1) = 2$. The two representations of 1 as a sum of a finite arithmetic progression with odd difference t are $1 = 1$ and

$$1 = \left(\frac{1-t}{2}\right) + \left(\frac{1+t}{2}\right).$$

The only odd positive divisor of 2 is 1, and so $\Phi_t(2) = 2d_1(1) = 2$. The two representations of 2 as a sum of a finite arithmetic progression with odd difference t are $1 = 1$ and

$$2 = \left(\frac{1-3t}{2}\right) + \left(\frac{1-t}{2}\right) + \left(\frac{1+t}{2}\right) + \left(\frac{1+3t}{2}\right).$$

The trapezoidal representations with odd difference t of an odd prime p are

$$\begin{aligned} p &= \frac{p-t}{2} + \frac{p+t}{2} \\ &= \sum_{i=0}^{p-1} \left(1 + \frac{(2i-p+1)t}{2} \right) \\ &= \sum_{i=0}^{2p-1} \frac{1 + (2i-2p+1)t}{2}. \end{aligned}$$

Thus, the four trapezoidal representations with difference 3 of the prime 5 are

$$\begin{aligned} 5 &= 1 + 4 \\ &= (-5) + (-2) + 1 + 4 + 7 \\ &= (-13) + (-10) + (-7) + (-4) + (-1) + 2 + 5 + 8 + 11 + 14. \end{aligned}$$

Theorem 2.3. *For every positive integer n,*

$$\Phi_1^+(n) = d_1(n).$$

In particular, $\Phi_1^+(n) = 1$ *if and only if n is a power of 2.*

Equivalently, the positive integer n is a sum of $k \geq 2$ consecutive positive integers if and only if n is not a power of 2.

Proof. Let k be an odd positive divisor of n. The identities

$$a_{k,1}(n) = \frac{n}{k} - \frac{(k-1)}{2} \qquad \text{and} \qquad b_{k,1}(n) = \frac{k+1}{2} - \frac{n}{k}$$

imply that

$$a_{k,1}(n) + b_{k,1}(n) = 1$$

and so exactly one of the integers $a_{k,1}(n)$ and $b_{k,1}(n)$ is positive. Thus, for each odd positive divisor k of n there is exactly one sequence of consecutive positive integers that sums to n. This proves that $\Phi_1^+(n) = d_1(n)$.

Theorem 2.4. *For every odd positive integer t, let*

$$\theta_t(n) = \sqrt{\frac{2n}{t} + \frac{1}{4}} + \frac{1}{2} \qquad \textit{and} \qquad \psi_t(n) = \sqrt{2nt + \left(\frac{t-2}{2}\right)^2} - \left(\frac{t-2}{2}\right).$$

The number of representations of n as a positive trapezoid with difference t is

$$\Phi_t^+(n) = d_1(n) + d_1(n, \theta_t(n)) - d_1(n, \psi_t(n)). \tag{10}$$

Proof. Let k be an odd divisor of n. All of the summands in the length k representation (6) are positive if and only if $a_{k,t}(n) > 0$, or, equivalently, $k < \theta_t(n)$. The number of such divisors is $d_1(n, \theta_t(n))$.

All of the summands in the length $2n/k$ representation (9) are positive if and only if $b_{k,t}(n) > 0$ or, equivalently, $k \geq \psi_t(n)$. The number of such divisors is $d_1(n) - d_1(n, \psi_t(n))$. This completes the proof.

Note that if $t = 1$, then $\theta_1(n) = \psi_1(n)$, and so, for every odd divisor k of n, exactly one of the inequalities $k < \theta_1(n)$ and $k \geq \psi_1(n)$ will hold. This gives another proof that $\Phi_1^+(n) = d_1(n)$.

In a *Comptes Rendus* note in 1883, Sylvester [22] proved that "…le nombre de suites de nombres consécutifs dont la somme est **N** est égal au nombre de diviseurs impairs de **N**." This result (Theorem 2.3) has been rediscovered many times. A special case is in *Number Theory for Beginners* [25] by André Weil: Problem III.4 is to prove that an "integer > 1 which is not a power of 2 can be written as the sum of 2 or more consecutive integers."

MacMahon [15, vol. 2, p. 28] used generating functions to prove Theorem 2.3.

Here is a nice generalization. Let $\Phi_{1,0}^+(n)$ (resp. $\Phi_{1,1}^+(n)$) denote the number of representations of n as the sum of an even (resp. odd) number of consecutive positive integers. Thus,

$$\Phi_1^+(n) = \Phi_{1,0}^+(n) + \Phi_{1,1}^+(n).$$

Andrews, Jiménez-Urroz, and Ono [7] proved analytically that

$$\Phi_{1,0}^+(n) - \Phi_{1,1}^+(n) = d(n, \sqrt{2n}) - d(n, \sqrt{(n/2)}).$$

Chapman [10] gave a combinatorial proof of this result.

3 Hook numbers and the Durfee square

Before describing Sylvester's algorithm, we recall some properties of the Durfee square of a partition of a positive integer n. Let

$$n = r_1 + \cdots + r_k \tag{11}$$

be a partition of n into k positive and decreasing parts. We have $r_1 \geq 1$. Let s be the greatest integer such that $r_s \geq s$. The square array of s^2 dots in the upper left corner of the Ferrers graph is called the *Durfee square* of the partition, and the positive integer s is the *side* of the Durfee square. If $s + 1 \leq i \leq k$, then $r_i \leq r_{s+1} \leq s$ and all of the dots on the ith row of the Ferrers graph lie on the first s columns of the graph. It follows that every dot in the Ferrers graph lies on one of the first s rows or on one of the first s columns of the graph. Therefore, the row numbers $r_1, \ldots, r_s$ and the column numbers $c_1, \ldots, c_s$ determine the partition (11). We extend this observation as follows.

Lemma 3.1. *Let s_1 and s_2 be positive integers, and let $(r_i)_{i=1}^{s_1}$ and $(c_j)_{j=1}^{s_2}$ be sequences of integers such that*

$$r_1 \geq r_2 \geq \cdots \geq r_{s_1} \geq s_2$$

and

$$c_1 \geq c_2 \geq \cdots \geq c_{s_2} \geq s_1.$$

The positive integer

$$n = \sum_{i=1}^{s_1} r_i + \sum_{j=1}^{s_2} c_j - s_1 s_2$$

has a unique partition with parts $r_1, \ldots, r_{s_1}, r_{s_1+1}, \ldots, r_{c_1}$, where, for $i = s_1 + 1, \ldots, c_1$,

$$r_i = \max(j : c_j \geq i).$$

If $s_1 = s_2 = s$, then the Durfee square of this partition has side s, and the row numbers $r_1, \ldots, r_s$ and column numbers $c_1, \ldots, c_s$ determine the partition.

Proof. Note that

$$n = \sum_{i=1}^{s_1} r_i + \sum_{j=1}^{s_2} (c_j - s_1) \geq \sum_{i=1}^{s_1} r_i \geq s_1 s_2.$$

Construct the Ferrers diagram with r_i dots on row i for $i = 1, \ldots, s_1$, and with c_j dots on column j for $j = 1, \ldots, s_2$. The Ferrers diagram has c_1 rows, and so the partition of n has c_1 parts. For $i = s_1 + 1, \ldots, c_1$, there is a dot on the jth column of row i if and only if $j \leq s_2$ and $c_j \geq i$. Therefore, $r_i = \max(j : c_j \geq i) \leq s_2$.

If $s_1 = s_2 = s$, then $r_{s+1} = r_{s_1+1} \leq s_2 \leq r_{s_1} = r_s$, and so this partition has a Durfee square with side s. This completes the proof.

The upper left corner of a Ferrers diagram of a partition contains a unique minimal square array of dots (the Durfee square) whose rows and columns determine the partition. The upper left corner of a Ferrers diagram also contains minimal rectangular arrays of dots whose rows and columns determine the partition. The Ferrers diagram contains a "Durfee rectangle" with sides (s_1, s_2) if

$$r_{s_1+1} \leq s_2 \leq r_{s_1} \quad \text{and} \quad c_{s_2+1} \leq s_1 \leq c_{s_2}.$$

These Durfee rectangles are not unique. For example, the partition

$$23 = 5 + 5 + 4 + 3 + 3 + 2 + 1$$

has Durfee square of side 3, and Durfee rectangles of sides $(s_1, s_2) = (2, 4)$ and $(s_1, s_2) = (5, 2)$.

For $1 \leq i \leq k$ and $1 \leq j \leq r_i$, let $R_{i,j}$ be the set of dots on the ith row that are on and to the right of the jth dot, and let $C_{i,j}$ be the set of dots on the jth column that are on and below the ith dot. The (i, j)*th hook number* is the cardinality of the set $H_{i,j} = R_{i,j} \cup C_{i,j}$. The number of dots on row i is $r_i = |R_{i,1}|$. Denote the number of dots on column j by $c_j = |C_{1,j}|$. We obtain

$$|H_{i,j}| = r_i + c_j - i - j + 1.$$

For $i = 1, \ldots, s$, we define the *diagonal hook number*

$$h_i = |H_{i,i}| = r_i + c_i - 2i + 1.$$

The set of diagonal hooks $\{H_{i,i} : i = 1, \ldots, s\}$ partitions the dots in the Ferrers diagram and produces the *hook partition* of n:

$$n = h_1 + h_2 + \cdots + h_s.$$

Lemma 3.2. *Let $n = r_1 + \cdots + r_k$ be a partition of n, let s be the side of the Durfee square of the Ferrers diagram of this partition, and let $h = h_1 + \cdots + h_s$ be the associated hook partition of n. For $i = 1, \ldots, s - 1$,*

$$h_i - h_{i+1} \geq 2$$

and

$$h_i - h_{i+1} = 2$$

if and only if $r_i = r_{i+1}$ and $c_i = c_{i+1}$.

Proof. For $i = 1, \ldots, s - 1$ we have

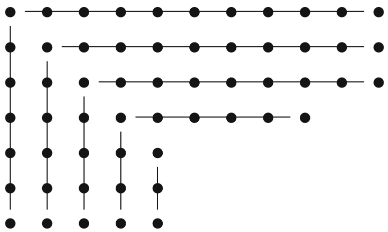
$$\begin{aligned} h_i - h_{i+1} &= (r_i + c_i - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1) \\ &= (r_i - r_{i+1}) + (c_i - c_{i+1}) + 2 \\ &\geq 2. \end{aligned}$$

Moreover, $h_i - h_{i+1} = 2$ if and only if $r_i = r_{i+1}$ and $c_i = c_{i+1}$.

For example, the partition into odd parts

$$57 = 11 + 11 + 11 + 9 + 5 + 5 + 5$$

has the left-justified Ferrers graph



We have $5 = r_5 = r_6 < 6$ and so the Durfee square has side 5 contains $5^2 = 25$ dots. The hook partition is

$$57 = 17 + 15 + 13 + 9 + 3.$$

Note that the hook partition of a partition does not determine the partition. For example, the partitions $5 + 2$ and $4 + 2 + 1$ both have Durfee squares of side 2 and hook partitions $6 + 1$.

Theorem 3.1. *The number of partitions of n into exactly k parts differing by at least 2 is the number of partitions of $n - k^2$ into at most k parts.*

Proof. The first construction converts a partition of $n - k^2$ into at most k parts into a partition of n into exactly k parts differing by at least 2. Let $n > k^2$, and let $n - k^2 = \sum_{i=1}^{k} b_i$ be a partition with $1 \leq r \leq k$ and $b_1 \geq \cdots \geq b_r$. For $r + 1 \leq i \leq k$ we define $b_i = 0$, and for $i = 1, \ldots, k$ we define

$$a_i = b_i + 2(k - i) + 1.$$

It follows that

$$\begin{aligned} a_i - a_{i+1} &= (b_i + 2(k - i) + 1) - (b_{i+1} + 2(k - i - 1) + 1) \\ &= b_i - b_{i+1} + 2 \geq 0 \end{aligned}$$

for $i = 1, \ldots, k - 1$. The identity

$$k^2 = \sum_{i=1}^{k}(2i - 1) = \sum_{i=1}^{k}(2(k - i) + 1)$$

implies that

$$n = \left(n - k^2\right) + k^2 = \sum_{i=1}^{k} (b_i + 2(k - i) + 1) = \sum_{i=1}^{k} a_i.$$

This is a partition of n into exactly k parts differing by at least 2.

The second construction converts a partition of n into exactly k parts differing by at least 2 into a partition of $n - k^2$ into at most k parts. Let $n = \sum_{i=1}^{k} a_i$ be a partition of n into exactly k parts differing by at least 2. We have $a_k \geq 1$. If $1 \leq i \leq k - 1$ and $a_{i+1} \geq 2(k - (i + 1)) + 1$, then

$$a_i \geq a_{i+1} + 2 \geq (2(k - (i + 1)) + 1) + 2 = 2(k - i) + 1.$$

It follows by downward induction that $a_i \geq 2(k - i) + 1$ and so

$$b_i = a_i - (2(k - i) + 1) \geq 0$$

for $i = 1, \ldots, k$. We have

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} (2(k - i) + 1) = n - k^2.$$

This is a partition of $n - k^2$ into at most k parts.

It is straightforward to check that the first and second constructions are inverses of each other. This completes the proof.

Consider a partition of n whose Ferrers diagram has Durfee square of side s. Let $r_1, \ldots, r_s$ be the number of dots on the first s rows of the Ferrers diagram, and let $c_1, \ldots, c_s$ be the number of dots on the first s columns. The *Frobenius symbol* of the partition is the $2 \times s$ matrix

$$\begin{pmatrix} r_1 - 1 & r_2 - 2 & \cdots & r_s - s \\ c_1 - 1 & c_2 - 2 & \cdots & c_s - s \end{pmatrix}.$$

Note that the rows are strictly decreasing sequences of nonnegative integers, and that

$$n = s + \sum_{i=1}^{s} (r_i - 1) + \sum_{i=1}^{s} (c_i - 1).$$

The Frobenius symbol is related to the construction in Lemma 3.1. See Andrews [3, 4].

4 Sylvester's algorithm

Sylvester discovered a graphical algorithm, sometimes called the *fish-hook method*, that transforms a partition of n with odd parts into a partition of n with distinct parts, and showed that this transformation is a bijection between the set of partitions into odd parts and the set of partitions into distinct parts. Moreover, he proved that this transformation has the extraordinary property that if the original partition of n into odd parts contains exactly ℓ different odd integers, then the new partition of n into distinct parts contains exactly ℓ maximal subsequences of consecutive integers.

Here is the algorithm. Let

$$n = a_1 + \cdots + a_k \tag{12}$$

be a partition of n into odd parts, with

$$a_1 \geq \cdots \geq a_k \geq 1 \tag{13}$$

and

$$a_i = 2r_i - 1 \tag{14}$$

for $i = 1, \ldots, k$. Then

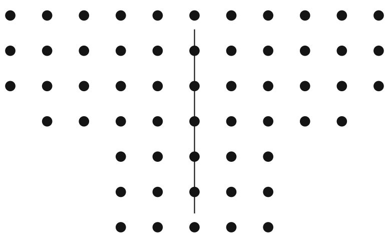
$$r_1 \geq \cdots \geq r_k \geq 1. \tag{15}$$

Because the summands a_i are odd, we can draw a center-justified Ferrers diagram, and divide it into two sub-diagrams. The *major right half* consists of the vertical central line and the dots to its right. The *minor left half* consists of the dots that are strictly to the left of the central line. We compute the hook numbers of the major half, and denote them in decreasing order by $h_1 > h_3 > h_5 > \cdots$. We compute the hook numbers of the minor half, and denote them in decreasing order by $h_2 > h_4 > h_6 > \cdots$. We shall prove that $h_1 > h_2 > h_3 > h_4 > h_5 > h_6 \cdots$, and so the hook numbers create a partition of n into distinct parts.

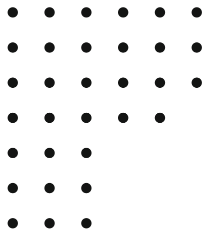
Before proving this statement, we consider an example:

$$57 = 11 + 11 + 11 + 9 + 5 + 5 + 5$$

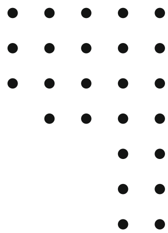
is a partition into odd parts. The center-justified Ferrers diagram is



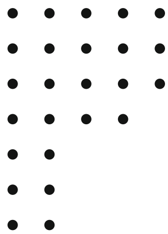
The major half is the Ferrers diagram of the partition $32 = 6 + 6 + 6 + 5 + 3 + 3 + 3$:



The remainder of the original Ferrers diagram is the minor half, associated with the partition $25 = 5 + 5 + 5 + 4 + 2 + 2 + 2$:

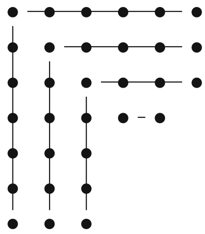


which we rearrange as the Ferrers diagram of the partition $5 + 5 + 5 + 4 + 2 + 2 + 2$:



Note that deleting the first column of the major half produces the minor half.

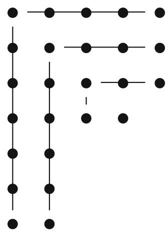
The Durfee square of the major half consists of $4^2 = 16$ vertices. Every dot in this diagram lies on one of the first four rows or on one of the first four columns. We partition the vertices of the major half into the four hooks of the Durfee square



and obtain the hook partition

$$32 = 12 + 10 + 8 + 2.$$

The minor left half is the major half with the left column removed, and the Durfee square of the minor half also consists of 16 vertices. Separating the minor half into hooks, we obtain



with hook partition

$$25 = 11 + 9 + 4 + 1.$$

Notice that not only are the parts in the hook partitions strictly decreasing, but they are also interlaced in magnitude. Their union gives a partition of 57 into distinct parts:

$$57 = 12 + 11 + 10 + 9 + 8 + 4 + 2 + 1.$$

Thus, the original partition with odd parts has been transformed into a partition with distinct parts. We also observe that the original partition of 57 used only the three odd integers 11, 9, and 5, and that the new partition of 57 into distinct parts consists of three maximal decreasing sequences of consecutive integers: (12, 11, 10, 9, 8), (4), and (2, 1).

MacMahon [15, vol. 2, pp. 13–14] contains a description of Sylvester's fish-hook method. Andrews [3, Section 4] uses the Frobenius symbol of a partition to explain the fish-hook method.

5 Sylvester's proof of Euler's theorem

Theorem 5.1. *Let n be a positive integer, let $\mathscr{U}(n)$ be the set of all partitions of n into odd parts, and $\mathscr{V}(n)$ be the set of all partitions of n into distinct parts. The function $f : \mathscr{U}(n) \to \mathscr{V}(n)$ defined by Sylvester's algorithm is a bijection.*

Proof. Consider a partition of n into k odd parts of the form (12)–(15). Let s be the side of the Durfee square of the major half. Every dot in the major half lies on one of the first s rows or on one of the first s columns. For $i = 1, \ldots, s$, the number r_i of dots on the ith row of the major half satisfies

$$r_1 \geq \cdots \geq r_s \geq s \geq r_{s+1}.$$

For $i = 1, \ldots, r_1$, let c_i be the number of dots in the ith column of the major half. Note that $r_{s+1} \leq s$ implies that $c_{s+1} \leq s$, and so

$$k = c_1 \geq \cdots \geq c_s \geq s \geq c_{s+1}.$$

For $i = 1, \ldots, s$, we have the hook numbers

$$h_i = r_i + c_i - 2i + 1. \tag{16}$$

By Lemma 3.2, these numbers satisfy $h_i - h_{i+1} \geq 2$ for $i = 1, \ldots, s - 1$.

The minor half of the original Ferrers diagram is exactly the major half with the first column removed. Therefore, every dot in the minor half lies on one of the first s rows of the minor half or on one of the first $s - 1$ columns of the graph. For $i = 1, \ldots, c_2$, let $r'_i = r_i - 1$ denote the number of dots on the ith row of the minor half. For $i = 1, \ldots, r'_1$, let c'_i denote the number of dots on the ith column of the minor half.

Let s' be the side of the Durfee square of the minor half. Because

$$r'_{s-1} \geq r'_s = r_s - 1 \geq s - 1 \geq r_{s+1} - 1 = r'_{s+1}$$

it follows that $s' = s - 1$ or $s' = s$, Moreover, $s' = s$ if and only if $r_s \geq s + 1$ and $c_{s+1} = s$. Similarly, $s' = s - 1$ if and only if $r_s = s$ and $c_{s+1} = s - 1$.

For $i = 1, \ldots, s'$, there are the hook numbers

$$h'_i = r'_i + c'_i - 2i + 1 = r_i + c_{i+1} - 2i. \tag{17}$$

By Lemma 3.2, we have $h'_i - h'_{i+1} \geq 2$ for $i = 1, \ldots, s' - 1$.

If $s' = s$, then $r_s \geq s + 1$ and $c_{s+1} = s$, and so

$$h'_s = r_s + c_{s+1} - 2s = r_s - s.$$

If $s' = s - 1$, then $r_s = s$. We define $h'_s = 0$, and again have

$$h'_s = r_s - s$$

and

$$\begin{aligned} h'_{s-1} - h'_s = h'_{s-1} &= r_{s-1} + c_s - 2s + 2 \\ &\geq (r_s - s) + (c_s - s) + 2 \\ &\geq 2. \end{aligned}$$

We shall prove that

$$h_1 > h'_1 > h_2 > h'_2 > \cdots > h_s > h'_s \geq 0.$$

For $i = 1, \ldots, s-1$, we have

$$\begin{aligned} h_i - h'_i &= (r_i + c_i - 2i + 1) - (r'_i + c'_i - 2i + 1) \\ &= (r_i - r'_i) + (c_i - c'_i) \\ &= 1 + c_i - c_{i+1} \\ &\geq 1. \end{aligned}$$

Also,

$$\begin{aligned} h_s - h'_s &= (r_s + c_s - 2s + 1) - (r_s - s) \\ &= c_s - s + 1 \geq 1. \end{aligned}$$

For $i = 1, \ldots, s-1$, we have

$$\begin{aligned} h'_i - h_{i+1} &= (r'_i + c'_i - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1) \\ &= (r_i - 1 + c_{i+1} - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1) \\ &= r_i - r_{i+1} + 1 \\ &\geq 1. \end{aligned}$$

Therefore,

$$n = h_1 + h'_1 + h_2 + h'_2 + \cdots + h_{s-1} + h'_{s-1} + h_s + h'_s \tag{18}$$

is a partition into $2s$ or $2s-1$ distinct positive parts, and we have transformed a partition with only odd parts to a partition into distinct parts. We shall prove that this transformation is one-to-one and onto.

Consider a partition of n into $2s$ distinct nonnegative parts:

$$n = h_1 + h'_1 + h_2 + h'_2 + \cdots + h_s + h'_s$$

where

$$h_1 > h'_1 > h_2 > h'_2 > \cdots > h_{s-1} > h'_{s-1} > h_s > h'_s \geq 0.$$

If the number of positive parts is even, then $h'_s \geq 1$. If the number of positive parts is odd, then $h'_s = 0$.

If this partition is constructed by Sylvester's algorithm from a partition of n into odd parts, then there are positive integers $r_1, r_2, \ldots, r_s$ and $c_1, c_2, \ldots, c_s$ such that

$$\begin{aligned}
h_1 &= r_1 + c_1 - 1 \\
h'_1 &= r_1 + c_2 - 2 \\
&\vdots \\
h_i &= r_i + c_i - (2i - 1) \\
h'_i &= r_i + c_{i+1} - 2i \\
&\vdots \\
h_s &= r_s + c_s - (2s - 1) \\
h'_s &= r_s - s.
\end{aligned}$$

Conversely, given the $2s$ parts $h_1, h'_1, \ldots, h'_s$, we can solve these $2s$ equations recursively, and obtain unique integers $r_1, \ldots, r_s, c_1, \ldots, c_s$. For $i = 1, \ldots, s$, the inequality $h_i > h'_i$ implies that

$$r_i + c_i - (2i - 1) > r_i + c_{i+1} - 2i$$

and so

$$c_i \geq c_{i+1}$$

For $i = 1, \ldots, s - 1$, the inequality $h'_i > h_{i+1}$ implies that

$$r_i + c_{i+1} - 2i > r_{i+1} + c_{i+1} - (2i + 1)$$

and so

$$r_i \geq r_{i+1}.$$

Because

$$r_s = h'_s + s \geq s$$

and

$$\begin{aligned} c_s &= h_s - r_s + 2s - 1 \\ &= h_s - (h'_s + s) + 2s - 1 \\ &= h_s - h'_s + s - 1 \\ &\geq s \end{aligned}$$

it follows that $r_1 \geq \cdots \geq r_s \geq s$ and $c_1 \geq \cdots \geq c_s \geq s$ are decreasing sequences of positive integers.

Thus, every partition into odd parts determines a unique partition into distinct parts, and every partition into distinct parts can be obtained uniquely from a partition into odd parts.

For example, consider the partition

$$50 = 22 + 17 + 8 + 3.$$

We have $d = 2$ and

$$\begin{aligned} 22 &= r_1 + c_1 - 1 \\ 17 &= r_1 + c_2 - 2 \\ 8 &= r_2 + c_2 - 3 \\ 3 &= r_2 - 2. \end{aligned}$$

Solving these equations, we obtain

$$\begin{aligned} r_2 &= 5 \\ c_2 &= 6 \\ r_1 &= 13 \\ c_1 &= 10. \end{aligned}$$

Thus, the major half has 10 rows, of lengths

$$\begin{aligned} r_1 &= 13 \\ r_2 &= 5 \\ r_i &= 2 \quad \text{for } i = 3, \ldots, 6 \\ r_i &= 1 \quad \text{for } i = 7, \ldots, 10. \end{aligned}$$

Defining $a_i = 2r_i - 1$ for $i = 1, \ldots, 10$, we obtain the following partition of 50 into odd parts:

$$50 = 25 + 9 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1.$$

Note that the partition into distinct parts consists of four maximal sequences of consecutive integers, and that the corresponding partition into odd parts contains four distinct odd numbers.

Here is another example:

$$31 = 9 + 8 + 7 + 4 + 3.$$

We have $d = 3$ and

$$\begin{aligned} 9 &= r_1 + c_1 - 1 \\ 8 &= r_1 + c_2 - 2 \\ 7 &= r_2 + c_2 - 3 \\ 4 &= r_2 + c_3 - 4 \\ 3 &= r_3 + c_3 - 5 \\ 0 &= r_3 - 3. \end{aligned}$$

Solving these equations, we obtain

$$\begin{aligned} r_3 &= 3 \\ c_3 &= 5 \\ r_2 &= 3 \\ c_2 &= 7 \\ r_1 &= 3 \\ c_1 &= 7. \end{aligned}$$

Thus, the major half has seven rows, of lengths

$$\begin{aligned} r_i &= 3 \quad \text{for } i = 1, 2, 3, 4 \\ r_i &= 2 \quad \text{for } i = 5, 6, 7. \end{aligned}$$

Defining $a_i = 2r_i - 1$ for $i = 1, \ldots, 7$, we obtain the following partition of 31 into odd parts:

$$31 = 5 + 5 + 5 + 5 + 5 + 3 + 3.$$

Note that the partition into distinct parts consists of two maximal sequences of consecutive integers, and that the corresponding partition into odd parts contains two distinct odd numbers.

Another example: The partition into distinct parts

$$30 = 10 + 8 + 7 + 4 + 1$$

is mapped to the following partition into odd parts:

$$30 = 9 + 9 + 5 + 3 + 3 + 1.$$

6 Sylvester's stratification of Euler's theorem

For every positive integer n, let $p_{\text{odd}}(n) = |\mathcal{U}(n)|$, where $\mathcal{U}(n)$ is the set of partitions of n into not necessarily distinct odd parts. Let $p_{\text{dis}}(n) = |\mathcal{V}(n)|$, where $\mathcal{V}(n)$ is the set of partitions of n into distinct parts. Euler proved (Theorem 1.1) that these two sets have the same cardinality, that is, $p_{\text{odd}}(n) = p_{\text{dis}}(n)$. In the proof of Theorem 5.1, we proved that the function $f : \mathcal{U}(n) \to \mathcal{V}(n)$ defined by Sylvester's algorithm is a bijection.

For positive integers n and ℓ, let $\mathcal{U}_\ell(n)$ denote the set of partitions of n into not necessarily distinct odd parts with exactly ℓ distinct odd parts, and let $U_\ell(n) = |\mathcal{U}_\ell(n)|$. We have

$$p_{\text{odd}}(n) = \sum_{\ell=1}^{\infty} U_\ell(n).$$

Similarly, if $\mathcal{V}_\ell(n)$ denotes the set of partitions of n into distinct parts and $V_\ell(n) = |\mathcal{V}_\ell(n)|$, then

$$p_{\text{dis}}(n) = \sum_{\ell=1}^{\infty} V_\ell(n).$$

Sylvester's "stratification" of Euler's theorem is that $U_\ell(n) = V_\ell(n)$ for all positive integers n and ℓ.

For example, the set $\mathcal{U}_3(57)$ contains the partition

$$11 + 11 + 11 + 9 + 5 + 5 + 5$$

which is a partition of 57 into odd parts whose three distinct parts are 11, 9, and 5. Similarly, the set $\mathcal{V}_3(57)$ contains the partition

$$12 + 11 + 10 + 9 + 8 + 4 + 2 + 1$$

which is a partition of 57 with three maximal subsequences of consecutive integers: (12, 11, 10, 9, 8), (4), and (2, 1).

There are three partitions of 5 into odd parts: $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. The partitions with one distinct part are 5 and $1 + 1 + 1 + 1 + 1$, and so $U_1(5) = 2$. The partition with two distinct parts is $3 + 1 + 1$, and so $U_2(5) = 1$.

There are three partitions of 5 into distinct parts: $5 = 4 + 1 = 3 + 2$. The partitions with one maximal subsequence of consecutive integers are 5 and $3 + 2$, and so $V_1(5) = 2$. The partition with two maximal subsequences of consecutive integers is $4 + 1$, and so $V_2(5) = 1$.

The proof of Sylvester's theorem uses the following combinatorial observation.

Lemma 6.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be sets, and let $\{\mathcal{U}_i : i = 1, 2, 3, \ldots\}$ and $\{\mathcal{V}_i : i = 1, 2, 3, \ldots\}$ be partitions of $\mathcal{U}$ and $\mathcal{V}$, respectively. Let $f : \mathcal{U} \to \mathcal{V}$ be a bijection.*

For every positive integer ℓ, let $f_\ell : \mathscr{U}_\ell \to \mathscr{V}$ be the restriction of f to $\mathscr{U}_\ell$. If $f_\ell(\mathscr{U}_\ell) \subseteq \mathscr{V}_\ell$ for all $\ell \in \mathbf{N}$, then $f_\ell : \mathscr{U}_\ell \to \mathscr{V}_\ell$ is a bijection for all $\ell \in \mathbf{N}$.

Proof. Because f is a bijection, it follows that f is one-to-one, and so f_ℓ is one-to-one for all $\ell \in \mathbf{N}$. Let $v \in \mathscr{V}_\ell \subseteq \mathscr{V}$. Because f is onto, there exists $u \in \mathscr{U}$ such that $f(u) = v$. Because $\mathscr{U} = \bigcup_{i=1}^{\infty} \mathscr{U}_i$ is a partition of $\mathscr{U}$, there is a unique integer j such that $u \in \mathscr{U}_j$. Therefore, $v = f(u) = f_j(u) \in \mathscr{V}_j$ and so $v \in \mathscr{V}_\ell \cap \mathscr{V}_j$. Because $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i$ is a partition of $\mathscr{V}$, it follows that $\ell = j$ and $u \in \mathscr{U}_\ell$. Therefore, $f_\ell : \mathscr{U}_\ell \to \mathscr{V}_\ell$ is one-to-one and onto. This completes the proof.

Theorem 6.1. *Let*

$$n = a_1 + \cdots + a_k \tag{19}$$

be a partition of n into k not necessarily distinct odd parts, and let ℓ be the number of distinct odd parts in this partition. The major-minor hook partition consists of exactly ℓ pairwise disjoint maximal sequences of consecutive integers.

Proof. Let

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$$

and, for $i = 1, \ldots, k$, let

$$a_i = 2r_i - 1.$$

We have

$$r_1 \geq r_2 \geq \cdots \geq r_k \geq 1.$$

Let ℓ be the number of distinct odd parts in the partition (19). The proof is by induction on ℓ.

If $\ell = 1$, then $a_i = a_1 = 2r_1 - 1$ for $i = 1, \ldots, k$, and $n = ka_1$. The Ferrers diagram for the partition is a rectangular array consisting of k rows of a_1 dots. The major half of the diagram is a rectangular array consisting of k rows of r_1 dots, and the minor half is a rectangular array consisting of k rows of $r_1 - 1$ dots. The Durfee square of the major half has side $s = \min(k, r_1)$ and the Durfee square of the minor half has side $s' = \min(k, r_1 - 1)$. Let $n = h_1 + h_1' + h_2 + h_2' + \cdots$ be the major-minor hook partition. By Lemma 3.2, we have

$$h_i - h_{i+1} = 2$$

for $i = 1, \ldots, s - 1$, and

$$h_i' - h_{i+1}' = 2$$

for $i = 1, \ldots, s' - 1$. Because

$$h_1 - h_1' = (r_1 + c_1 - 1) - (r_1 + c_1 - 2) = 1$$

it follows that the parts in the major-minor hook partition of n form a strictly decreasing sequence of consecutive integers.

For example, if $n = 21 = 7 + 7 + 7$, then $k = 3$, $r_1 = 4$, and the major-minor hook partition is $21 = 6 + 5 + 4 + 3 + 2 + 1$. If $n = 21 = 3 + 3 + 3 + 3 + 3 + 3 + 3$, then $k = 7$, $r_1 = 2$, and the major-minor hook partition is $21 = 8 + 7 + 6$.

Let $\ell \geq 2$, and assume that the Theorem is true for partitions into at most $\ell - 1$ distinct odd parts. The smallest part in the partition (19) is $a_k = 2r_k - 1$. We also know that $a_k < a_1$ because $\ell \geq 2$. If j is the greatest integer such that $a_k < a_j$, then

$$\begin{aligned} &1 \leq r_k < r_j \\ &a_i = a_k \quad \text{for } i = j+1, \ldots, k \end{aligned}$$

and

$$m = n - (k - j)a_k = a_1 + \cdots + a_j \tag{20}$$

is a partition of m into odd parts with exactly $\ell - 1$ distinct parts. By the induction hypothesis, the Theorem is true for this partition of m.

There are three cases.

Case 1:

$$\boxed{j < r_k < r_j}$$

Because $j < r_j$, both the major and the minor halves of the partition of m have Durfee squares with side j. Let

$$m = g_1 + g'_1 + \cdots + g_j + g'_j \tag{21}$$

be the major-minor hook partition for m, where

$$g_1 > g'_1 > g_2 > \cdots > g_j > g'_j. \tag{22}$$

For $i = 1, \ldots, j$ we have

$$g_i = r_i + j - 2i + 1 \tag{23}$$

$$g'_i = r_i + j - 2i. \tag{24}$$

The partition (20) is a partition of m into odd parts with exactly $\ell - 1$ distinct parts. By the induction hypothesis, the major-minor hook partition (21) consists of exactly $\ell - 1$ pairwise disjoint maximal sequences of consecutive integers.

Because

$$j + 1 \leq r_k = r_{j+1}$$

the Durfee square for the major half of the partition of n has side $s = \min(k, r_k) \geq j + 1$, and the Durfee square for the minor half of the partition of n has side $s' = \min(k, r_k - 1) \geq j$. Let

$$n = h_1 + h'_1 + \cdots + h_j + h'_j + h_{j+1} + \cdots \tag{25}$$

be the major-minor hook partition for n, where

$$h_1 > h'_1 > h_2 > \cdots > h_j > h'_j > h_{j+1} > \cdots .$$

For $i = 1, \ldots, j$ we have

$$h_i = r_i + k - 2i + 1 = g_i + (k - j) \tag{26}$$
$$h'_i = r_i + k - 2i = g'_i + (k - j) \tag{27}$$

It follows that the number of pairwise disjoint maximal sequences of consecutive integers in the sequence $(g_1, \ldots, g'_1, \ldots, g_j, g'_j)$ of parts in the major-minor hook partition for m is equal to the number of pairwise disjoint maximal sequences of consecutive integers in the sequence $(h_1, \ldots, h'_1, \ldots, h_j, h'_j)$. For $i = j + 1, \ldots, s$ we have

$$h_i = r_k + k - 2i + 1$$

and for $i = j + 1, \ldots, s'$ we have

$$h'_i = r_k + k - 2i.$$

We observe that, for $i > j$,

$$h_i - h'_i = h'_i - h_{i+1} = 1$$

and so

$$(h_{j+1}, h'_{j+1}, h_{j+2}, \ldots) \tag{28}$$

is a sequence of consecutive integers. Moreover,

$$h'_j - h_{j+1} = (r_j + k - 2j) - (r_k + k - 2j - 1) = r_j - r_k + 1 \geq 2$$

and so (28) is a maximal sequence of consecutive integers in the major-minor hook partition of n. It follows that the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition of n is exactly one more than the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition of m. By the induction hypothesis, the latter partition consists of $\ell - 1$ maximal disjoint sequences, and so the partition (25) consists of ℓ maximal disjoint sequences.

For example, if

$$n = 49 = 13 + 13 + 9 + 7 + 7$$

then $k = 5$, $\ell = 3$, and

$$j = 3 < r_k = 4 < r_j = 5.$$

We have $k - j = 2$ and

$$m = 35 = 13 + 13 + 9.$$

The major-minor hook partition for m is

$$m = 35 = 9 + 8 + 7 + 6 + 3 + 2$$

and contains two maximal sequences of consecutive integers: (9, 8, 7, 6) and (3, 2). The major-minor hook partition for n is

$$n = 49 = 11 + 10 + 9 + 8 + 5 + 4 + 2$$

and contains three maximal sequences of consecutive integers: (11, 10, 9, 8), (5, 4), and (2). Note that $(11, 10, 9, 8) = (9, 8, 7, 6) + (2, 2, 2, 2)$ and $(5, 4) = (3, 2) + (2, 2)$.

Case 2:

$$\boxed{r_k \leq j < r_j}$$

Because $j < r_j$, the major-minor hook partition of m satisfies the relations (21), (22), (23), and (24).

The major and minor halves of the center-justified Ferrers diagram for the partition (19) of n also have Durfee squares with side j. The associated major-minor hook partition for n, denoted

$$n = h_1 + h'_1 + \cdots + h'_{r_k - 1} + h_{r_k} + h'_{r_k} + h_{r_k + 1} + \cdots + h'_j$$

is a partition into strictly decreasing parts, where

$$h_i = \begin{cases} r_i + k - 2i + 1 & \text{for } i = 1, \ldots, r_k \\ r_i + j - 2i + 1 & \text{for } i = r_k + 1, \ldots, j. \end{cases}$$

$$h'_i = \begin{cases} r_i + k - 2i & \text{for } i = 1, \ldots, r_k - 1 \\ r_i + j - 2i & \text{for } i = r_k, \ldots, j. \end{cases}$$

Applying (23) and (24), we obtain, for $i = 1, \ldots, r_k - 1$,

$$h_i - g_i = h'_i - g'_i = h_{r_k} - g_{r_k} = k - j$$

and, for $i = r_k + 1, \ldots, j$,

$$h_i = g_i = r_i + j - 2I + 1$$

and

$$h'_i = g'_i = r_i + j - 2I.$$

The critical observations are that

$$h'_{r_k} = g'_{r_k} = r_{r_k} + j - 2r_k$$

$$g_{r_k} - g'_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1$$

and

$$\begin{aligned} h_{r_k} - h'_{r_k} &= \left(r_{r_k} + k - 2r_k + 1\right) - \left(r_{r_k} + k - 2r_k\right) \\ &= k - j + 1 \geq 2. \end{aligned}$$

These imply that the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition for n is exactly one more than the number in the major-minor hook partition for m. By the induction hypothesis, the hook partition for m contains exactly $\ell - 1$ such sequences, and so the hook partition for n contains exactly ℓ pairwise disjoint maximal sequences of consecutive integers.

For example, if

$$n = 57 = 11 + 11 + 11 + 9 + 5 + 5 + 5$$

then $k = 7$, $\ell = 3$ and

$$r_7 = 3 < j = 4 < r_j = 5.$$

We have $k - j = 3$ and

$$m = 42 = 11 + 11 + 11 + 9.$$

The major-minor hook partition for m is

$$m = 42 = 9 + 8 + 7 + 6 + 5 + 4 + 2 + 1$$

and contains two maximal sequences of consecutive integers: (9, 8, 7, 6, 5, 4) and (2, 1). The major-minor hook partition for n is

$$n = 57 = 12 + 11 + 10 + 9 + 8 + 4 + 2 + 1$$

and contains three maximal sequences of consecutive integers: (12, 11, 10, 9, 8), (4), and (2, 1). Note that $r_k = r_7 = 3$, $h_3 = 8$, $g_3 = 5$, and $h'_3 = g'_3 = 4$.

Case 3:

$$\boxed{r_k < r_j = j}$$

Because $r_k = r_{j+1} < r_j = j$, it follows that the sides of the Durfee squares of the major halves of the partitions of both m and n are $s = j$. Because $r'_j = r_j - 1 = j - 1$ and $r'_{j-1} = r_{j-1} - 1 \geq r_j - 1 = j - 1$, it follows that the sides of the Durfee squares of the minor halves of the partitions of both m and n are $s' = j - 1$. The hook numbers of the major halves are

$$g_i = r_i + j - 2i + 1 \qquad \text{for } i = 1, \ldots, j$$
$$h_i = \begin{cases} r_i + k - 2i + 1 & \text{if } 1 \leq i \leq r_k \\ r_i + j - 2i + 1 & \text{if } r_k + 1 \leq i \leq j \end{cases}$$

and so

$$h_i - g_i = \begin{cases} k - j & \text{if } 1 \leq i \leq r_k \\ 0 & \text{if } r_k + 1 \leq i \leq j \end{cases}$$

The hook numbers of the minor halves are

$$g'_i = r_i + j - 2i \qquad \text{for } i = 1, \ldots, j - 1$$
$$h'_i = \begin{cases} r_i + k - 2i & \text{if } 1 \leq i \leq r_k - 1 \\ r_i + j - 2i & \text{if } r_k \leq i \leq j \end{cases}$$

and so

$$h'_i - g'_i = \begin{cases} k - j & \text{if } 1 \leq i \leq r_k - 1 \\ 0 & \text{if } r_k \leq i \leq j \end{cases}$$

Because

$$g_{r_k} - g'_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1$$

and

$$h_{r_k} - h'_{r_k} = (r_{r_k} + k - 2r_k + 1) - (r_{r_k} + j - 2r_k) = k - j + 1 \geq 2$$

it follows that the major-minor hook partition for n contains exactly more maximal sequence of consecutive integers than the hook partition for m.

For example, if

$$n = 13 = 9 + 3 + 1$$

then $k = \ell = 3$ and

$$r_3 = 1 < r_j = 2 = j.$$

We have $k - j = 1$ and

$$m = 12 = 9 + 3.$$

The major-minor hook partition for m is

$$m = 12 = 6 + 5 + 1$$

and contains two maximal sequences of consecutive integers: $(6, 5)$ and (1). The major-minor hook partition for n is

$$n = 13 = 7 + 5 + 1$$

and contains three maximal sequences of consecutive integers: (7), (5), and (1).

Case 4:

$$\boxed{r_k < r_j < j}$$

Let s be the side of the Durfee square of the major half of partition of n. The inequality $r_j < j$ implies that $s \leq j - 1$, and so

$$r_k < r_j \leq r_{s+1} \leq s < j.$$

The side of the Durfee square of the major half of the partition of m is also s. Let c_i be the number of dots in the ith column of the Ferrers diagram of the major half of the partition of n. We have

$$g_i = \begin{cases} r_i + j - 2i + 1 & \text{if } 1 \leq i \leq r_j \\ r_i + c_i - 2i + 1 & \text{if } r_j + 1 \leq i \leq s \end{cases}$$

and

$$h_i = \begin{cases} r_i + k - 2i + 1 & \text{if } 1 \leq i \leq r_k \\ r_i + j - 2i + 1 & \text{if } r_k + 1 \leq i \leq r_j \\ r_i + c_i - 2i + 1 & \text{if } r_j + 1 \leq i \leq s. \end{cases}$$

Thus,

$$h_i - g_i = \begin{cases} k - j & \text{if } 1 \leq i \leq r_k \\ 0 & \text{if } r_k + 1 \leq i \leq s. \end{cases}$$

Let $c'_i = c_{i+1}$ be the number of dots in the ith column of the minor half of the partition of n Let s' denote the side of the Durfee square of the minor half of the partition of n. If $r_s \geq s + 1$, then

$$r'_s = r_s - 1 \geq s \geq r_{s+1} > r'_{s+1}$$

and so

$$s' = s.$$

If $r_s = s$, then

$$r'_{s-1} = r_{s-1} - 1 \geq r_s - 1 = s - 1 = r'_s$$

and so

$$s' = s - 1.$$

In both cases we have $r_j - 1 \leq s - 1 \leq s'$ and

$$g'_i = \begin{cases} r_i + j - 2i & \text{if } 1 \leq i \leq r_j - 1 \\ r_i + c'_i - 2i & \text{if } r_j \leq i \leq s' \end{cases}$$

and

$$h'_i = \begin{cases} r_i + k - 2i & \text{if } 1 \leq i \leq r_k - 1 \\ r_i + j - 2i & \text{if } r_k \leq i \leq r_j - 1 \\ r_i + c'_i - 2i & \text{if } r_j + 1 \leq i \leq s'. \end{cases}$$

Thus,

$$h'_i - g'_i = \begin{cases} k - j & \text{if } 1 \leq i \leq r_k - 1 \\ 0 & \text{if } r_k \leq i \leq s'. \end{cases}$$

Because

$$g_{r_k} - g'_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1$$

and

$$h_{r_k} - h'_{r_k} = (r_{r_k} + k - 2r_k + 1) - (r_{r_k} + j - 2r_k) = k - j + 1 \geq 2$$

it follows that the major-minor hook partition for n contains exactly one more sequence of consecutive integers than the hook partition for m.

For example, if

$$n = 50 = 11 + 11 + 9 + 7 + 7 + 5$$

then $k = 6$, $\ell = 4$, and

$$r_6 = 3 < r_5 = 4 < j = 5.$$

We have $k - j = 1$ and

$$m = 45 = 11 + 11 + 9 + 7 + 7.$$

The major-minor hook partition for m is

$$m = 45 = 10 + 9 + 8 + 7 + 5 + 4 + 2$$

and contains three pairwise disjoint maximal sequences of consecutive integers: (10, 9, 8, 7), (5, 4), and (2). The major-minor hook partition for n is

$$n = 50 = 11 + 10 + 9 + 8 + 6 + 4 + 2$$

and contains four disjoint maximal sequences: (11, 10, 9, 8), (6), (4), and (2).

This completes the proof.

Theorem 6.2 (Sylvester). *For all positive integers n and ℓ,*

$$U_\ell(n) = V_\ell(n).$$

Proof By Theorem 6.1, Sylvester's one-to-one and onto function $f : \mathscr{U}(n) \to \mathscr{V}(n)$ maps $\mathscr{U}_\ell(n)$ into $\mathscr{V}_\ell(n)$. We simply apply Lemma 6.1 to complete the proof.

There are several recent proofs of Theorem 6.2, for example, Andrews [1], Andrews and Eriksson [6], and Hirschhorn [13]. V. Ramamani and K. Venkatachaliengar [20] obtained a combinatorial proof. Their method is discussed in Andrews [2, pp. 448–449] and [5, pp. 24–25].

For other recent work on trapezoidal numbers, see Apostol [8], Guy [12], Leveque [14], Moser [17], Pong [18, 19], and Tsai and Zaharescu [23, 24].

7 A Problem

An odd integer is an integer of the form $r + (r - 1)$. Thus, a partition into odd parts is a partition into parts, each of which is a sum of two consecutive integers. A different generalization of Euler's theorem about partitions into odd parts would be a theorem about partitions into parts, each of which is a sum of e consecutive integers, or, equivalently, a sum of e-trapezoids. Thus, we consider positive parts of the form

$$a_i = \sum_{j=0}^{e-1} (r_i - j)$$

with

$$r_i \geq e$$

and partitions of the form

$$n = \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \left(\sum_{j=0}^{e-1} (r_i - j) \right).$$

Interchanging summations, we obtain a partition of n into e parts, each of which inherits a well-defined partition:

$$n = \sum_{j=0}^{e-1} n_j$$

where

$$n_j = \sum_{i=1}^{k} (r_i - j).$$

Partitions into 2-trapezoids (that is, partitions into odd numbers) are equinumerous with partitions into distinct parts. What kind of partition are in one-to-one correspondence with partitions into e-trapezoids for $e \geq 3$?

Acknowledgements I thank the referee for providing many references to the current literature on Sylvester's theorem.

References

1. G.E. Andrews, On generalizations of Euler's partition theorem. Mich. Math. J. **13**, 491–498 (1966)
2. G.E. Andrews, Applications of basic hypergeometric functions. SIAM Rev. **16**, 441–484 (1974)
3. G.E. Andrews, Use and extension of Frobenius' representation of partitions, *Enumeration and Design* (Academic Press, Toronto, 1984), pp. 51–65
4. G.E. Andrews, *Generalized Frobenius Partitions*, vol. 49(301), Memoirs of the American Mathematical Society (American Mathematical Society, London, 1984)
5. G.E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1998). (Reprint of the 1976 original)
6. G.E. Andrews, K. Eriksson, *Integer Partitions* (Cambridge University Press, Cambridge, 2004)
7. G.E. Andrews, J. Jiménez-Urroz, K. Ono, q -series identities and values of certain L-functions. Duke Math. J. **108**, 395–419 (2001)
8. T.M. Apostol, Sums of consecutive integers. Math. Gaz. **87**, 98–101 (2003)
9. L.E. Bush, On the expression of an integer as the sum of an arithmetic series. Am. Math. Mon. **37**, 353–357 (1930)
10. R. Chapman, Combinatorial proofs of q-series identities. J. Combin. Theory Ser. A **99**, 1–16 (2002)
11. J.W.L. Glaisher, A theorem in partitions. Messeng. Math. **12**, 158–170 (1883)
12. R. Guy, Sums of consecutive integers. Fibonacci Quart. **20**(1), 36–38 (1982)
13. M.D. Hirschhorn, Sylvester's partition theorem, and a related result. Mich. Math. J. **21**, 133–136 (1974)
14. W.J. LeVeque, On representations as a sum of consecutive integers. Can. J. Math. **2**, 399–405 (1950)
15. P.A. MacMahon, *Combinatory Analysis* (Cambridge University Press, Cambridge, 1916). (Reprinted by Chelsea Publishing Co., New York, 1960)
16. T.E. Mason, On the representation of an integer as the sum of consecutive integers. Am. Math. Mon. **19**, 46–50 (1912)
17. L. Moser, Notes on number theory. III. On the sum of consecutive primes. Can. Math. Bull. **6**, 159–161 (1963)
18. W.Y. Pong, Sums of consecutive integers. College Math. J. **38**(2), 119–123 (2007)
19. W.Y. Pong, Length spectra of natural numbers. Int. J. Number Theory **5**(6), 1089–1102 (2009)
20. V. Ramamani, K. Venkatachaliengar, On a partition theorem of Sylvester. Mich. Math. J. **19**, 137–140 (1972)
21. J.J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact, and an exodion. Am. J. Math. **5**, 251–330 (1882)
22. J.J. Sylvester, Sur un théorème de partitions. Comptes Rendus Acad. Sci. Paris **96**, 674–675 (1883)
23. M.-T. Tsai, A. Zaharescu, On the sum of consecutive integers in sequences. Int. J. Number Theory **8**(3), 643–652 (2012)
24. M.-T. Tsai, A. Zaharescu, On the sum of consecutive integers in sequences II. Int. J. Number Theory **8**(5), 1281–1299 (2012)
25. A. Weil, *Number Theory for Beginners* (Springer, New York, 1979)

Estimates of $\mathrm{li}(\theta(x)) - \pi(x)$ and the Riemann Hypothesis

Jean-Louis Nicolas

To Krishna Alladi for his sixtieth birthday

Abstract Let us denote by $\pi(x)$ the number of primes $\leqslant x$, by $\mathrm{li}(x)$ the logarithmic integral of x, by $\theta(x) = \sum_{p \leqslant x} \log p$ the Chebyshev function and let us set $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$. Revisiting a result of Ramanujan, we prove that the assertion "$A(x) > 0$ for $x \geqslant 11$" is equivalent to the Riemann Hypothesis.

Keywords Chebyshev function · Riemann Hypothesis · Explicit formula

2010 Mathematics Subject Classification 11N37 · 11M26

1 Introduction

Let us denote by $\pi(x)$ the number of primes $\leqslant x$ and by $\mathrm{li}(x)$ the logarithmic integral of x (see, below, §2.2). It has been observed that, for small x, $\pi(x) < \mathrm{li}(x)$ holds, but Littlewood (cf. [7] or [5, chap. 5]) has proved that, for x tending to infinity, the difference $\pi(x) - \mathrm{li}(x)$ oscillates infinitely many often between positive and negative values.

Research partially supported by CNRS, UMR 5208.

J.-L. Nicolas (✉)
Institut Camille Jordan, Univ. Lyon, Université Claude Bernard Lyon 1,
CNRS UMR 5208, Mathématiques, Bât. Doyen Jean Braconnier,
43 Bd du 11 Novembre 1918, Villeurbanne cedex F-69622, France
e-mail: nicolas@math.univ-lyon1.fr

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_32

Let us set $\theta(x) = \sum_{p \leqslant x} \log p$, the Chebyshev function, and

$$A(x) = \mathrm{li}(\theta(x)) - \pi(x). \tag{1.1}$$

What is the behavior of $A(x)$? In [11, (220), (222), (227), and (228)], under the Riemann Hypothesis (RH), Ramanujan proved that

$$A(x) = \frac{2\sqrt{x} + \sum_\rho x^\rho/\rho^2}{\log^2(x)} + \mathscr{O}\left(\frac{\sqrt{x}}{\log^3(x)}\right) \tag{1.2}$$

where ρ runs over the nontrivial zeros of the Riemann ζ function. Moreover, in [11, (226)], Ramanujan writes under the Riemann Hypothesis

$$\begin{aligned}\left|\sum_\rho \frac{x^\rho}{\rho^2}\right| &\leqslant \sum_\rho \left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\sum_\rho \frac{1}{\rho(1-\rho)} = \sqrt{x}\sum_\rho\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right)\\ &= 2\sqrt{x}\sum_\rho \frac{1}{\rho} = \sqrt{x}(2 + \gamma_0 - \log(4\pi)) = 0.046...\sqrt{x}\end{aligned} \tag{1.3}$$

where γ_0 is the Euler constant and concludes

$$\text{under RH, } \exists\, x_0 \text{ such that, for } x \geqslant x_0,\ \ A(x) \text{ is positive.} \tag{1.4}$$

The aim of this paper is to make these results effective and, in particular, to show that Ramanujan's result (1.4) is true for $x_0 = 11$.

Let us set $\lambda = \sum_\rho \frac{1}{|\rho|^2}$. Under the Riemann Hypothesis, we have (see below (2.26))

$$\lambda = \sum_\rho \frac{1}{|\rho|^2} = \sum_\rho \frac{1}{\rho(1-\rho)} = 0.0461914179322420\ldots \tag{1.5}$$

We shall prove

Theorem 1.1. *Under the Riemann Hypothesis, we have*

$$\limsup_{x\to\infty} \frac{A(x)\log^2(x)}{\sqrt{x}} \leqslant 2 + \lambda = 2.046\ldots, \tag{1.6}$$

$$\liminf_{x\to\infty} \frac{A(x)\log^2(x)}{\sqrt{x}} \geqslant 2 - \lambda = 1.953\ldots, \tag{1.7}$$

$$A(x) \textit{ is positive for } \quad x \geqslant 11, \tag{1.8}$$

$$A(x) \geqslant (2-\lambda)\frac{\sqrt{x}}{\log^2(x)} \quad \textit{for } x \geqslant 37, \tag{1.9}$$

and

$$A(x) \leqslant M \frac{\sqrt{x}}{\log^2(x)} \quad \textit{for } x \geqslant 2, \tag{1.10}$$

where $M = A(3643)(\log^2 3643)/\sqrt{3643} = 5.0643569138\ldots$

Corollary 1.2. *Each of the five assertions* (1.6)–(1.10) *is equivalent to the Riemann Hypothesis.*

Proof. In 1984, Robin (cf. [10, Lemma 2 and (8)] has shown that, if the Riemann Hypothesis does not hold, there exists $b > 1/2$ such that

$$A(x) = \Omega_{\pm}(x^b), \quad \text{i.e. } \limsup_{x\to\infty} \frac{A(x)}{x^b} > 0 \text{ and } \liminf_{x\to\infty} \frac{A(x)}{x^b} < 0$$

and the five assertions of the theorem are no longer satisfied. □

1.1 Notation

$\pi(x) = \sum_{p \leqslant x} 1$ is the prime counting function.

$\Pi(x) = \sum_{p^k \leqslant x} \frac{1}{k} = \sum_{k=1}^{\kappa} \frac{\pi(x^{1/k})}{k}$ with $\kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$.

$\theta(x) = \sum_{p \leqslant x} \log p$ and $\psi(x) = \sum_{p^m \leqslant x} \log p = \sum_{k=1}^{\kappa} \theta(x^{1/k})$ are the Chebyshev functions.

$\Lambda(x) = \begin{cases} \log p & \text{if } x = p^k \\ 0 & \text{if not} \end{cases}$ is the von Mangoldt function.

$\widetilde{\psi}(x) = \psi(x) - \frac{1}{2}\Lambda(x)$ and $\widetilde{\Pi}(x) = \Pi(x) - \frac{\Lambda(x)}{2\log x}$.

$\mathrm{li}(x)$ denotes the logarithmic integral of x (cf. below §2.2).

$L_1(t) = \mathrm{li}(t) - \frac{t}{\log t}$, $L_2(t) = \mathrm{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t}$,

$F_1(t) = \frac{L_1(t)}{t/\log^2 t}$, $F_2(t) = \frac{L_2(t)}{t/\log^3 t}$ $\quad (t > 1)$.

$\widetilde{F}_1(t)$ and $\widetilde{F}_2(t)$ are defined below in (3.16).

$\gamma_0 = 0.57721566\ldots$ is the Euler constant. λ is defined in (1.5), cf. also (2.26).

$\sum_\rho f(\rho) = \lim_{T\to\infty} \sum_{|\Im(\rho)|\leqslant T} f(\rho)$ where $f : \mathbb{C} \to \mathbb{C}$ is a complex function and ρ runs over the nontrivial zeros of the Riemann ζ function.

1.2 *Plan of the article*

In §2, we shall recall some definitions and prove some results that we shall use in the sequel, first, in §2.2, about the logarithmic integral, and, further, in §2.3, about the Riemann ζ function and explicit formulas of the theory of numbers.

In §3, the proof of Theorem 1.1 is given. First, we write $A(x) = A_1(x) + A_2(x)$ with

$$A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x) \quad \text{and} \quad A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x).$$

In §3.1, under the Riemann Hypothesis, an estimate of $A_1(x)$ is given, by applying the explicit formulas. In §3.2, it is shown that $A_2(x)$ depends on the quantity $B(y) = \pi(y) - \theta(y)/\log y$ which is carefully studied.

In §3.3 (resp. §3.4), an effective lower (resp. upper) estimate for $A(x)$ is given when $x \geqslant 10^8$.

In §3.5, for $x < 10^8$, estimates of $A(x)$ are given by numerical computation.

Finally, Theorem 1.1 is proved in two steps, depending on the cases $x \leqslant 10^8$ or $x > 10^8$.

The computations, both algebraic and numerical, have been carried out with Maple. On the website [13], one can find the code and a Maple sheet with the results.

We often implicitly use the following result : for u and v positive, the function

$$t \mapsto \frac{\log^u t}{t^v} \quad \text{is increasing for } 1 \leqslant t \leqslant e^{u/v} \text{and decreasing for } \quad t > e^{u/v}. \tag{1.11}$$

Moreover

$$\max_{t\geqslant 1} \frac{\log^u t}{t^v} = \left(\frac{u}{e\,v}\right)^u. \tag{1.12}$$

2 Preliminary results

2.1 Effective estimates

Without any hypothesis, Platt and Trudgian [9] have shown by computation that

$$\theta(x) < x \text{ for } 0 < x \leq 1.39 \times 10^{17} \tag{2.1}$$

so improving on results of Schoenfeld [12] and Dusart [3]. Under the Riemann Hypothesis, for $x \geqslant 599$, we shall use the upper bounds (cf. [12, (6.3)])

$$|\psi(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \text{and} \quad |\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x. \tag{2.2}$$

2.2 The logarithmic integral

For x real > 1, we define $\mathrm{li}(x)$ as (cf. [1, p. 228])

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0^+}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right) = \int_2^x \frac{dt}{\log t} + \mathrm{li}(2).$$

We have the following values:

x	1	1.45136...	2	3.8464...	8.3	599
$\mathrm{li}(x)$	$-\infty$	0	1.145163...	2.8552...	5.39671...	117.49...

(2.3)

From the definition of $\mathrm{li}(x)$, it follows that

$$\frac{d}{dx}\mathrm{li}(x) = \frac{1}{\log x} \quad \text{and} \quad \frac{d^2}{dx^2}\mathrm{li}(x) = -\frac{1}{x\log^2 x}. \tag{2.4}$$

We also have

$$\mathrm{li}(x) = \gamma_0 + \log(\log(x)) + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n \times n!}$$

(where $\gamma_0 = 0.577\ldots$ is the Euler constant) which implies

$$\mathrm{li}(x) = \log(\log(x)) + \gamma_0 + o(1), \qquad x \to 1^+. \tag{2.5}$$

Let N be a positive integer. For $t > 1$, we have (cf. [13])

$$\int \frac{dt}{\log^N t} = \frac{1}{(N-1)!}\left(\mathrm{li}(t) - \sum_{k=1}^{N-1}(k-1)!\frac{t}{\log^k t}\right) \tag{2.6}$$

and, for $x \to \infty$,

$$\mathrm{li}(x) = \sum_{k=1}^{N} \frac{(k-1)!\, x}{(\log x)^k} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1}}\right). \tag{2.7}$$

Lemma 2.1. *For $t > 1$, we have*

$$L_2(t) = \mathrm{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t} = F_2(t)\left(\frac{t}{\log^3 t}\right) < 4.05\frac{t}{\log^3 t}. \tag{2.8}$$

For $t \geqslant t_0 \geqslant 381$, we have

$$L_2(t) < F_2(t_0)\ \frac{t}{\log^3 t}. \tag{2.9}$$

For $t > 29$, we have

$$L_2(t) > 2\frac{t}{\log^3 t}. \tag{2.10}$$

Proof. let us set (cf. the Maple sheet [13])

$$f_1(t) = (3 - \log t)\ \mathrm{li}(t) + t - \frac{2t}{\log t} - \frac{t}{\log^2 t} = \frac{t^2 F_2'(t)}{\log^2(t)},$$

$$f_2(t) = \frac{t}{\log t} + \frac{t}{\log^2 t} + 2\frac{t}{\log^3 t} - \mathrm{li}(t) = t f_1'(t)$$

and

$$f_3(t) = f_2'(t) = -\frac{6}{\log^4(t)}.$$

Since $f_2'(t) = f_3(t)$ is negative, $f_2(t)$ decreases and vanishes for

$$t_2 = 28.19524\ldots$$

It follows that $f_1'(t) = f_2(t)/t$ is positive for $1 < t < t_2$ and negative for $t > t_2$ so that $f_1(t)$ has a maximum for $t = t_2$,

$$f_1(t_2) = 4.54378\ldots$$

and f_1 vanishes (and so does F_2') in two points

$$t_3 = 3.384879\ldots \qquad t_4 = 380.1544\ldots$$

From (2.5), we get $\lim_{t\to 1^+} F_2(t) = 0$ and the variation of F_2 is given in the following array:

t	1		3.38...		10.39...		380.15...		∞
$F_2(t) = \frac{L_2(t)}{t/\log^3 t}$	0	$\searrow$	$-1.369496\ldots$	$\nearrow$	0	$\nearrow$	$4.040415\ldots$	$\searrow$	2

(2.11)

The proof of (2.8) and (2.9) follows from Array 2.11 and also the proof of (2.10), after deducing from $f_2(t_2) = 0$ that $F_2(t_2) = 2$ holds. □

In the same way, it is possible to study the variation of the function

$$F_1(t) = \frac{L_1(t)}{(t/\log^2 t)} = \frac{\mathrm{li}(t) - \frac{t}{\log t}}{(t/\log^2 t)},$$

The details can be found on [13]. We have

t	1		1.85...		3.8464...		94.6...		∞
$F_1(t) = L_1(t)\left(\frac{\log^2 t}{t}\right)$	0	$\searrow$	$-0.448\ldots$	$\nearrow$	0	$\nearrow$	$1.784\ldots$	$\searrow$	1

(2.12)

Since $L_1(10.3973\ldots) = 1$, Array (2.12) yields

$$t > 10.4 \quad \Longrightarrow \quad L_1(t) = \mathrm{li}(t) - \frac{t}{\log t} > \frac{t}{\log^2 t}. \tag{2.13}$$

The derivative of $\mathrm{li}(t)/t$ is $\frac{t/\log t - \mathrm{li}(t)}{t^2} = -\frac{F_1(t)}{t\log^2 t}$ which, from Array 2.12, is positive for $1 < t < 3.8464$ and negative for $t > 3.8465$. Therefore, we have

$$t > 1 \quad \Longrightarrow \quad \mathrm{li}(t) \leqslant \frac{\mathrm{li}(3.8464\ldots)}{3.8464\ldots}\, t = 0.7423\ldots\, t < \frac{3t}{4}. \tag{2.14}$$

Lemma 2.2. *Let a and x be two real numbers satisfying $\exp(1) \leqslant a < a^3 \leqslant x$. Let κ_1 and κ_2 be two integers such that*

$$2 \leqslant \kappa_1 < \kappa_2 = \left\lfloor \frac{\log x}{\log a} \right\rfloor.$$

Then we have

$$\sum_{k=\kappa_1+1}^{\kappa_2} \frac{1}{k} L_1(x^{1/k}) \leqslant 1.785 \left(4.05 \frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} - L_2(a)\right). \tag{2.15}$$

Proof. Let us set

$$T = \sum_{k=\kappa_1+1}^{\kappa_2} \frac{1}{k} L_1(x^{1/k}).$$

It follows from Array 2.12 that, for $t > 1$, $L_1(t) = tF_1(t)/\log^2 t \leqslant 1.785 \frac{t}{\log^2 t}$ holds and therefore,

$$T \leqslant \frac{1.785}{\log^2 x} \sum_{k=\kappa_1+1}^{\kappa_2} k x^{1/k}.$$

Now, as $x \geqslant \exp(1) > 1$, the function $t \mapsto t x^{1/t}$ is positive and decreasing for $0 < t \leqslant \log x$ so that

$$T \leqslant \frac{1.785}{\log^2 x} \int_{\kappa_1}^{\kappa_2} t x^{1/t} dt \leqslant \frac{1.785}{\log^2 x} \int_{\kappa_1}^{\frac{\log x}{\log a}} t x^{1/t} dt = 1.785 \int_a^{x^{1/\kappa_1}} \frac{du}{\log^3 u}$$

by the change of variable $u = x^{1/t}$. Finally, by (2.6) and (2.8), we get

$$T \leqslant 1.785 \left(L_2(x^{1/\kappa_1}) - L_2(a)\right) \leqslant 1.785 \left(4.05 \frac{x^{1/\kappa_1}}{\log^3(x^{1/\kappa_1})} - L_2(a)\right)$$

which ends the proof of Lemma 2.2. □

Lemma 2.3. *Let $a \geqslant 2.11$ and $x \geqslant a^3$ be real numbers and $\kappa_2 = \left\lfloor \frac{\log x}{\log a} \right\rfloor$. Then we have*

$$\sum_{k=2}^{\kappa_2} \frac{1}{k} x^{1/(2k)} \leqslant \frac{5}{4} x^{1/4}. \tag{2.16}$$

Proof. Let us set

$$T = \sum_{k=2}^{\kappa_2} \frac{1}{k} x^{1/(2k)}.$$

Since $x \geqslant a^3 > 1$, the function $t \mapsto x^{1/(2t)}/t$ is positive and decreasing for $t > 0$ so that

$$T = \frac{1}{2} x^{1/4} + \sum_{k=3}^{\kappa_2} \frac{1}{k} x^{1/(2k)} \leqslant \frac{1}{2} x^{1/4} + \int_2^{\frac{\log x}{\log a}} \frac{x^{1/(2t)}}{t} dt = \frac{1}{2} x^{1/4} + \int_{\sqrt{a}}^{x^{1/4}} \frac{du}{\log u}$$

by the change of variable $u = x^{1/(2t)}$. Finally, by (2.6) and (2.14), we get

$$T \leqslant \frac{1}{2}x^{1/4} + \mathrm{li}(x^{1/4}) - \mathrm{li}(\sqrt{a}) \leqslant \frac{5}{4}x^{1/4} - \mathrm{li}(\sqrt{a})$$

and (2.16) follows since $\sqrt{a} \geqslant \sqrt{2.11} > 1.452$ so that, from Array (2.3), $\mathrm{li}(\sqrt{a}) > 0$ holds. □

Lemma 2.4. *Under the Riemann Hypothesis, for $x \geqslant 599$, one has*

$$\frac{\theta(x) - x}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\theta(x)) - \mathrm{li}(x) \leqslant \frac{\theta(x) - x}{\log x}, \tag{2.17}$$

$$\frac{\psi(x) - x}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\psi(x)) - \mathrm{li}(x) \leqslant \frac{\psi(x) - x}{\log x} \tag{2.18}$$

and

$$\frac{\psi(x) - \theta(x)}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\psi(x)) - \mathrm{li}(\theta(x)) \leqslant \frac{\psi(x) - \theta(x)}{\log x} + \frac{9\log^2 x}{10000}. \tag{2.19}$$

Proof. Let us suppose that $x \geqslant 599$ holds. From (2.2) and (1.11), we get

$$\frac{\psi(x)}{x} \geqslant \frac{\theta(x)}{x} \geqslant \frac{1}{x}\left(x - \frac{\sqrt{x}\log^2 x}{8\pi}\right) = 1 - \frac{\log^2 x}{8\pi\sqrt{x}} \geqslant 1 - \frac{(\log 599)^2}{8\pi\sqrt{599}} > 0.9335. \tag{2.20}$$

Further, for $h > 1 - x$, Taylor's formula and (2.4) yield

$$\mathrm{li}(x + h) = \mathrm{li}(x) + \frac{h}{\log x} - \frac{h^2}{2\,\xi\log^2 \xi}, \tag{2.21}$$

with $\xi \geqslant \min(x, x + h)$. Let us set $h = \theta(x) - x$; we have $h + x = \theta(x) \geqslant \theta(599) > 1$. From (2.20), we get $\xi \geqslant bx$ with $b = 0.9335$ and

$$\begin{aligned}\xi\log^2 \xi &\geqslant bx\log^2(bx) = bx\log^2(x)\left(1 + \frac{\log b}{\log x}\right)^2 \\ &\geqslant bx\log^2(x)\left(1 + \frac{\log b}{\log(599)}\right)^2 \geqslant 0.9135\,x\log^2 x.\end{aligned}$$

From (2.2), it follows that

$$0 \leqslant \frac{h^2}{2\,\xi\log^2 \xi} \leqslant \frac{x\log^4 x}{128\,\pi^2 \xi\log^2 \xi} \leqslant \frac{\log^2 x}{0.9135 \times 128\,\pi^2} < \frac{9\log^2 x}{10000}$$

which, with (2.21), proves (2.17). In the same way, setting $h = \psi(x) - x$ yields (2.18), and (2.19) follows by subtracting (2.17) from (2.18). □

2.3 The Riemann ζ function

We shall use the two explicit formulas valid for $x > 1$

$$\widetilde{\psi}(x) = \psi(x) - \frac{1}{2}\Lambda(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \tag{2.22}$$

and

$$\widetilde{\Pi}(x) = \Pi(x) - \frac{\Lambda(x)}{2\log x} = \operatorname{li}(x) - \sum_{\rho} \operatorname{li}(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t} \tag{2.23}$$

which can be found in many books in analytic number theory, for instance [5, chap. 4]. To Formula (2.23), we prefer the form described in [6, p. 361 and 362, with $R = 0$] :

$$\widetilde{\Pi}(x) = \operatorname{li}(x) - \sum_{\rho} \int_0^\infty \frac{x^{\rho - t}}{\rho - t} dt - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}, \quad x > 1. \tag{2.24}$$

We also have (cf. [4, p. 67] or [2, p. 272])

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2}\log\pi - \log 2 = 0.02309570896612103\ldots \tag{2.25}$$

and (cf. (1.5))

$$\sum_{\rho} \frac{1}{\rho(1-\rho)} = \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) = 2\sum_{\rho} \frac{1}{\rho} = 2 + \gamma_0 - \log(4\pi). \tag{2.26}$$

3 Proof of Theorem 1.1

3.1 Study of $A_1(x) = \operatorname{li}(\psi(x)) - \Pi(x)$

Under the Riemann Hypothesis, we write

$$\gamma = \Im\rho \quad \text{i.e.} \quad \rho = \frac{1}{2} + i\gamma.$$

Lemma 3.1. *Under the Riemann Hypothesis, we have*

$$\sum_\rho \frac{1}{|\gamma|^3} \leqslant \frac{1}{300}.$$

Proof. It is possible to get better estimates for the sum $\sum_\rho \frac{1}{|\gamma|^3}$, but, for our purpose, the above upper bound will be enough. By observing that

$$|\rho|^2 = \rho(1-\rho) = \frac{1}{4} + \gamma^2$$

and that the first zero of $\zeta(s)$ is $1/2 + 14.134725\ldots i$ (cf. [4, p. 96] or the extended tables of [8]), we get

$$\sum_\rho \frac{1}{\gamma^2} = \sum_\rho \frac{1 + 1/(4\gamma^2)}{1/4 + \gamma^2} \leqslant \sum_\rho \frac{1 + 1/(4 \times 14.134^2)}{1/4 + \gamma^2} \leqslant \frac{800}{799} \sum_\rho \frac{1}{\rho(1-\rho)}.$$

Further, from (2.26), we get

$$\sum_\rho \frac{1}{|\gamma|^3} \leqslant \frac{1}{14.134} \sum_\rho \frac{1}{\gamma^2} \leqslant \frac{800}{799 \times 14.134} \sum_\rho \frac{1}{\rho(1-\rho)} = 0.00327\ldots$$

which completes the proof of Lemma 3.1. □

Lemma 3.2. *For* $x > 1$*, under the Riemann Hypothesis, we have*

$$\sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho - t} dt = \sum_\rho \frac{x^\rho}{\rho \log x} + \sum_\rho \frac{x^\rho}{\rho^2 \log^2 x} + K(x)$$

with

$$|K(x)| \leqslant \frac{2}{300} \frac{\sqrt{x}}{\log^3 x}. \tag{3.1}$$

Proof. By partial integration, one has

$$\int_0^\infty \frac{x^{\rho-t}}{\rho - t} dt = \frac{x^\rho}{\rho \log x} + \frac{x^\rho}{\rho^2 \log^2 x} + \frac{2}{\log^2 x} \int_0^\infty \frac{x^{\rho-t}}{(\rho - t)^3} dt$$

and

$$\left| \int_0^\infty \frac{x^{\rho-t}}{(\rho - t)^3} dt \right| \leqslant \frac{1}{|\Im\rho|^3} \int_0^\infty x^{1/2-t} dt = \frac{1}{|\Im\rho|^3} \frac{\sqrt{x}}{\log x}$$

so that we get

$$|K(x)| = \left|\sum_\rho \frac{2}{\log^2 x}\int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3}dt\right| \leqslant \frac{2\sqrt{x}}{\log^3 x}\sum_\rho \frac{1}{|\Im\rho|^3}$$

and (3.1) follows from Lemma 3.1. □

Proposition 3.3. *Under the Riemann Hypothesis, for $x \geqslant 599$, we have*

$$A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x) = \sum_\rho \frac{x^\rho}{\rho^2\log^2 x} + J(x)$$

with

$$-0.0009\log^2 x - \frac{2}{300}\frac{\sqrt{x}}{\log^3 x} \leqslant J(x) \leqslant \frac{2}{300}\frac{\sqrt{x}}{\log^3 x} + \log 2. \tag{3.2}$$

Proof. Let us write

$$\mathrm{li}(\psi(x)) = \mathrm{li}(x) + \frac{\psi(x)-x}{\log x} + J_1(x) = \mathrm{li}(x) + \frac{\widetilde{\psi}(x) - x + \Lambda(x)/2}{\log x} + J_1(x)$$

with, from (2.18), for $x \geqslant 599$,

$$-0.0009\log^2 x \leqslant J_1(x) \leqslant 0. \tag{3.3}$$

Therefore, from (2.22) and (2.24), we have

$$\begin{aligned} A_1(x) &= \mathrm{li}(x) + \frac{1}{\log x}\left(-\sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) + \frac{1}{2}\Lambda(x)\right) \\ &+ J_1(x) - \left(\mathrm{li}(x) - \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}dt + \int_x^\infty \frac{dt}{t(t^2-1)\log t} - \log 2 + \frac{\Lambda(x)}{2\log x}\right) \\ &= \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}dt - \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + J_1(x) + J_2(x) + J_3(x) \end{aligned}$$

with

$$J_2(x) = \log 2 - \frac{\log(2\pi)}{\log x} \quad \text{and} \quad J_3(x) = -\frac{\log(1-1/x^2)}{2\log x} - \int_x^\infty \frac{dt}{t(t^2-1)\log t}.$$

Further, from Lemma 3.2, one gets

$$A_1(x) = \sum_\rho \frac{x^\rho}{\rho^2\log^2 x} + J(x) \tag{3.4}$$

with

$$J(x) = K(x) + J_1(x) + J_2(x) + J_3(x) \tag{3.5}$$

and $K(x)$ is as in Lemma 3.2.

It remains to bound $J_2(x) + J_3(x)$. We have

$$J_3(x) = \int_x^\infty \frac{1}{t(t^2-1)} \left(\frac{1}{\log x} - \frac{1}{\log t} \right) dt$$

which, for $x \geqslant 599$, implies

$$\begin{aligned} 0 \leqslant J_3(x) &\leqslant \frac{1}{\log x} \int_x^\infty \frac{dt}{t(t^2-1)} = \frac{\log(1+1/(x^2-1))}{2\log x} \\ &\leqslant \frac{1}{2(x^2-1)\log x} < \frac{\log(2\pi)}{\log x} \end{aligned}$$

and $0 < J_2(x) + J_3(x) < \log 2$. Therefore, (3.2) results from (3.1), (3.3), (3.4), and (3.5). □

3.2 *Study of* $A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x)$

For $y \geqslant 2$, let us set

$$B(y) = \pi(y) - \frac{\theta(y)}{\log y} = \sum_{p \leqslant y} \left(1 - \frac{\log p}{\log y} \right).$$

Note that $B(y)$ is nonnegative. If $q < q'$ are two consecutive primes, $B(y)$ is increasing and continuous on $[q, q')$ and

$$\lim_{y \to q',\ y<q'} B(y) = \pi(q) - \frac{\theta(q)}{\log q'} = \pi(q') - 1 - \frac{\theta(q') - \log q'}{\log q'} = B(q')$$

so that $B(y)$ is continuous and increasing for $y \geqslant 2$. In the two following lemmas, we give estimates of $B(y)$.

Lemma 3.4. *Let y be a real number satisfying $y_0 = 8.3 \leqslant y \leqslant 1.39 \times 10^{17}$. We have*

$$B(y) \leqslant L_1(y) = \mathrm{li}(y) - \frac{y}{\log y} \tag{3.6}$$

while, if $y \geqslant y_1 = 599$, under the Riemann Hypothesis, we have

$$B(y) \leqslant L_1(y) + \frac{\sqrt{y}}{4\pi}. \tag{3.7}$$

Under the Riemann Hypothesis, for $y \geqslant y_2 = 2903$, we have

$$B(y) \geqslant L_1(y) - \frac{\sqrt{y}}{4\pi}. \tag{3.8}$$

Proof. By Stieljes's integral, one has

$$\pi(y) = \int_{2^-}^{y} \frac{d[\theta(t)]}{\log t} = \frac{\theta(y)}{\log y} + \int_2^y \frac{\theta(t)}{t\log^2 t}dt. \tag{3.9}$$

Further, we have

$$B(y) = \int_2^y \frac{\theta(t)}{t\log^2 t}dt = \int_2^{y_0} + \int_{y_0}^{y} \frac{\theta(t)}{t\log^2 t}dt = B(y_0) + \int_{y_0}^{y} \frac{\theta(t)}{t\log^2 t}dt. \tag{3.10}$$

By (2.1) and (2.6), for $y \leqslant 1.39 \times 10^{17}$, we get

$$\int_{y_0}^{y} \frac{\theta(t)}{t\log^2 t}dt \leqslant \int_{y_0}^{y} \frac{1}{\log^2 t}dt = \mathrm{li}(y) - \frac{y}{\log y} - \mathrm{li}(y_0) + \frac{y_0}{\log y_0} = L_1(y) - L_1(y_0),$$

so that (3.10) yields $B(y) \leqslant L_1(y) + B(y_0) - L_1(y_0)$, which proves (3.6), since $B(y_0) - L_1(y_0) = -0.001379\ldots < 0$ (cf. [13].

Replacing y_0 by y_1 in (3.10) yields

$$B(y) = B(y_1) + \int_{y_1}^{y} \frac{\theta(t)dt}{t\log^2 t} = B(y_1) - L_1(y_1) + L_1(y) + T(y, y_1) \tag{3.11}$$

with $T(y, y_1) = \int_{y_1}^{y} \frac{\theta(t)-t}{t\log^2 t}dt$ and, from (2.2) ,

$$|T(y, y_1)| \leqslant \int_{y_1}^{y} \frac{\sqrt{t}\log^2 t}{8\pi t\log^2 t}dt = \frac{\sqrt{y} - \sqrt{y_1}}{4\pi}. \tag{3.12}$$

From (3.11) and (3.12), it follows that

$$B(y) \leqslant L_1(y) + \frac{\sqrt{y}}{4\pi} + B(y_1) - L_1(y_1) - \frac{\sqrt{y}_1}{4\pi}$$

which proves (3.7), since $B(y_1) - L_1(y_1) - \frac{\sqrt{y}_1}{4\pi} = -4.80566\ldots < 0$.

In the same way than the one used to get (3.11), for $y \geqslant y_2$, we obtain

$$B(y) = B(y_2) - L_1(y_2) + L_1(y) + T(y, y_2) \geqslant L_1(y) - \frac{\sqrt{y}}{4\pi} + B(y_2) - L_1(y_2) + \frac{\sqrt{y}_2}{4\pi}$$

and as $B(y_2) - L_1(y_2) + \frac{\sqrt{y_2}}{4\pi} = 0.00671\ldots > 0$, this completes the proof of Lemma 3.4. □

Let us set

$$\varepsilon(y) = \begin{cases} 0 & \text{if } y \leqslant 1.39 \times 10^{17} \\ 1 & \text{if } y > 1.39 \times 10^{17}. \end{cases}$$

It follows from (3.6) and (3.7) that, under the Riemann Hypothesis, one has

$$B(y) \leqslant L_1(y) + \varepsilon(y)\frac{\sqrt{y}}{4\pi} \quad \text{for} \quad y \geqslant 8.3. \tag{3.13}$$

Proposition 3.5. *Under the Riemann Hypothesis, for $x \geqslant 599$, we have*

$$A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x) = \sum_{k=2}^{\kappa} \frac{1}{k} B(x^{1/k}) + U(x) \tag{3.14}$$

with

$$\kappa := \left\lfloor \frac{\log x}{\log 2} \right\rfloor \quad \textit{and} \quad |U(x)| \leqslant \frac{9 \log^2 x}{10000}. \tag{3.15}$$

Proof. From (2.19), for $x \geqslant 599$, we get

$$\mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) = \frac{\theta(x) - \psi(x)}{\log x} + U(x) \text{ with } |U(x)| \leqslant \frac{9 \log^2 x}{10000}.$$

From the definition of $\psi(x)$ and $\Pi(x)$, this implies

$$A_2(x) = \sum_{k=2}^{\kappa} \left(\frac{\pi(x^{1/k})}{k} - \frac{\theta(x^{1/k})}{\log x} \right) + U(x)$$

which, via the definition of B, proves (3.14). □

It is convenient to introduce the notation

$$\widetilde{F}_2(t) = \begin{cases} 4.05 & \text{if } t \leqslant 381 \\ F_2(t) & \text{if } t > 381 \end{cases} \quad \text{and} \quad \widetilde{F}_1(t) = \begin{cases} 1.785 & \text{if } t \leqslant 95 \\ F_1(t) & \text{if } t > 95 \end{cases} \tag{3.16}$$

so that, from Arrays (2.11) and (2.12), for $t > 1$, $\widetilde{F}_2(t)$ and $\widetilde{F}_1(t)$ are nonincreasing and we have

$$L_2(t) = \frac{t F_2(t)}{\log^3 t} \leqslant \frac{t \widetilde{F}_2(t)}{\log^3 t} \quad \text{and} \quad L_1(t) = \frac{t F_1(t)}{\log^2 t} \leqslant \frac{t \widetilde{F}_1(t)}{\log^2 t}. \tag{3.17}$$

Lemma 3.6. *Let us set $a = 10.4$. For $x > 10^8$, we set $\kappa = \lfloor \frac{\log x}{\log 2} \rfloor$, $\kappa_2 = \lfloor \frac{\log x}{\log a} \rfloor$ and let κ_1 be an integer satisfying $3 \leqslant \kappa_1 < \kappa_2$. Then, under the Riemann Hypothesis, we have*

$$\sum_{k=2}^{\kappa} \frac{B(x^{1/k})}{k} \leqslant \frac{2\sqrt{x}}{\log^2 x} + \frac{4\sqrt{x}}{\log^3 x}\widetilde{F}_2(\sqrt{x}) + \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x}\widetilde{F}_1(x^{1/k}) + \frac{7.23\,\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} + 2.35 + 0.94\frac{\sqrt{x}}{\log^5 x}.$$

Proof. For $2 \leqslant k \leqslant \kappa_2$ we have $x^{1/k} \geqslant x^{1/\kappa_2} \geqslant x^{(\log a)/\log x} = a$, and, under the Riemann Hypothesis, it follows from (3.13) that

$$B(x^{1/k}) \leqslant L_1(x^{1/k}) + \varepsilon(x^{1/k})\frac{x^{1/(2k)}}{4\pi}$$

which implies that

$$\sum_{k=2}^{\kappa} \frac{B(x^{1/k})}{k} \leqslant T_1 + T_2 + T_3 + T_4 + T_5$$

with

$$T_1 = \frac{1}{2}L_1(\sqrt{x}), \quad T_2 = \sum_{k=3}^{\kappa_1} \frac{L_1(x^{1/k})}{k}, \quad T_3 = \sum_{k=\kappa_1+1}^{\kappa_2} \frac{L_1(x^{1/k})}{k},$$

$$T_4 = \sum_{k=\kappa_2+1}^{\kappa} \frac{B(x^{1/k})}{k}, \quad T_5 = \sum_{k=2}^{\kappa_2} \varepsilon(x^{1/k})\frac{x^{1/(2k)}}{4k\pi}.$$

From the definition of L_1, L_2, F_1, F_2 and from (3.17), one has

$$T_1 = \frac{L_2(\sqrt{x})}{2} + \frac{\sqrt{x}}{2\log^2\sqrt{x}} = \frac{2\sqrt{x}}{\log^2 x} + 4\frac{\sqrt{x}F_2(\sqrt{x})}{\log^3 x} \leqslant \frac{2\sqrt{x}}{\log^2 x} + 4\frac{\sqrt{x}\widetilde{F}_2(\sqrt{x})}{\log^3 x}$$

and

$$T_2 = \sum_{k=3}^{\kappa_1} \frac{L_1(x^{1/k})}{k} = \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x}F_1(x^{1/k}) \leqslant \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x}\widetilde{F}_1(x^{1/k}).$$

From Array (2.11), $L_2(10.4)$ is positive, so that, from Lemma 2.2 with $a = 10.4$, we have

$$T_3 \leqslant 1.785 \left(4.05 \frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} - L_2(10.4)\right)$$
$$\leqslant 1.785 \times 4.05 \frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} \leqslant \frac{7.23\ \kappa_1^3 x^{1/\kappa_1}}{\log^3 x}.$$

For $k \geqslant \kappa_2 + 1 > (\log x)/\log a$, we have $x^{1/k} < a$; since $y \mapsto B(y)$ is nondecreasing, we have $B(x^{1/k}) \leqslant B(a) = B(10.4) = 1.7166\ldots < 1.72$ and

$$T_4 \leqslant 1.72 \sum_{k=\kappa_2+1}^{\kappa} \frac{1}{k} \leqslant 1.72 \int_{\kappa_2}^{\kappa} \frac{dt}{t} \leqslant 1.72 \int_{\frac{\log x}{\log a}-1}^{\frac{\log x}{\log 2}} \frac{dt}{t}$$
$$= 1.72 \left(\log\left(\frac{\log x}{\log 2}\right) - \log\left(\frac{\log(x/a)}{\log a}\right)\right)$$
$$= 1.72 \left(\log\left(\frac{\log a}{\log 2}\right) + \log\left(\frac{\log x}{\log(x/a)}\right)\right)$$
$$\leqslant 1.72 \left(\log\left(\frac{\log a}{\log 2}\right) + \left(\frac{\log x}{\log(x/a)} - 1\right)\right)$$
$$= 1.72 \left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(x/a)}\right)$$
$$\leqslant 1.72 \left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(10^8/a)}\right) = 2.34449\ldots$$

Since $\varepsilon(t)$ is nondecreasing and vanishes for $x \leqslant 10^{17}$, from Lemma 2.3, one gets

$$T_5 = \sum_{k=2}^{\kappa_2} \varepsilon(x^{1/k}) \frac{x^{1/(2k)}}{4k\pi} \leqslant \varepsilon(\sqrt{x}) \sum_{k=2}^{\kappa_2} \frac{x^{1/(2k)}}{4k\pi} \leqslant \frac{5}{16\pi} \varepsilon(\sqrt{x}) x^{1/4}$$
$$= \frac{5}{16\pi} \varepsilon(\sqrt{x}) \frac{\sqrt{x}}{\log^5 x} \frac{\log^5 x}{x^{1/4}} < \frac{5}{16\pi} \frac{\sqrt{x}}{\log^5 x} \frac{\log^5 10^{34}}{10^{34/4}} = 0.93\ldots \frac{\sqrt{x}}{\log^5 x},$$

which completes the proof of Lemma 3.6. □

3.3 A lower bound for $A(x)$

Proposition 3.7. *Under the Riemann Hypothesis, for $x \geqslant 9 \times 10^6$, we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x} \left(2 - \lambda + \frac{1}{\log x} \left(7.993 - \frac{\log^3 x}{8\pi x^{1/4}} - \frac{18}{10000} \frac{\log^5 x}{\sqrt{x}}\right)\right). \quad (3.18)$$

Proof. Since $B(y)$ is nonnegative, from (3.14) and (3.15), we get, for $x \geqslant 599$

$$A_2(x) \geqslant \frac{1}{2} B(\sqrt{x}) - \frac{9 \log^2 x}{10000}.$$

As $x \geqslant 2903^2$, we may apply (3.8) which yields

$$\begin{aligned} A_2(x) &\geqslant \frac{1}{2}\left(L_1(\sqrt{x}) - \frac{x^{1/4}}{4\pi}\right) - \frac{9\log^2 x}{10000} \\ &= \frac{1}{2}\left(\frac{\sqrt{x}}{\log^2\sqrt{x}} + L_2(\sqrt{x}) - \frac{x^{1/4}}{4\pi}\right) - \frac{9\log^2 x}{10000}. \end{aligned}$$

Now, as $x > 29^2$, by (2.10), it follows

$$\begin{aligned} A_2(x) &\geqslant \frac{1}{2}\left(\frac{\sqrt{x}}{\log^2\sqrt{x}} + \frac{2\sqrt{x}}{\log^3\sqrt{x}} - \frac{x^{1/4}}{4\pi}\right) - \frac{9\log^2 x}{10000} \\ &= \frac{\sqrt{x}}{\log^2 x}\left(2 + \frac{8}{\log x} - \frac{\log^2 x}{8\pi x^{1/4}} - \frac{9\log^4 x}{10000\sqrt{x}}\right). \end{aligned}$$

From Proposition 3.3, one has:

$$A_1(x) \geqslant -\left|\sum_\rho \frac{x^\rho}{\rho^2 \log^2 x}\right| - 0.0009\log^2 x - \frac{2}{300}\frac{\sqrt{x}}{\log^3 x}$$

so that $A(x) = A_1(x) + A_2(x)$ satisfies

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \sum_\rho \frac{1}{|\rho^2|} + \frac{8 - 2/300}{\log x} - \frac{\log^2 x}{8\pi x^{1/4}} - \frac{18\log^4 x}{10000\sqrt{x}}\right)$$

which, via 1.5, implies (3.18). □

Corollary 3.8. *Under the Riemann Hypothesis, for* $x \geqslant 10^8$, *we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right). \tag{3.19}$$

Proof. From (1.11), the functions $x \mapsto \frac{\log^3 x}{x^{1/4}}$ and $x \mapsto \frac{\log^5 x}{\sqrt{x}}$ are decreasing for $x \geqslant 10^8$ and therefore, we have

$$7.993 - \frac{\log^3 x}{8\pi x^{1/4}} - \frac{18}{10000}\frac{\log^5 x}{\sqrt{x}} \geqslant 7.993 - \frac{\log^3 10^8}{8\sqrt[4]{10^8}\,\pi} - \frac{18}{10000}\frac{\log^5 10^8}{\sqrt{10^8}} = 5.124\ldots$$

(cf. [13]). □

3.4 An upper bound for $A(x)$

Proposition 3.9. *Under the Riemann Hypothesis, for $x \geqslant 10^8$, we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{Q(\kappa_1, x)}{\log x}\right) \tag{3.20}$$

where κ_1 is an integer satisfying $3 \leqslant \kappa_1 < \lfloor \frac{\log x}{\log 10.4} \rfloor$ and

$$\begin{aligned} Q(\kappa_1, x) = 4\widetilde{F}_2(\sqrt{x}) + \frac{2}{300} + \frac{3.05\ \log^3 x}{\sqrt{x}} + \sum_{k=3}^{\kappa_1} \frac{k\widetilde{F}_1(x^{1/k}) \log x}{x^{1/2-1/k}} \\ + \frac{7.23\ \kappa_1^3}{x^{1/2-1/\kappa_1}} + \frac{0.94}{\log^2 x} + \frac{9 \log^5 x}{10000\sqrt{x}} \end{aligned} \tag{3.21}$$

with $\widetilde{F}_2$ and $\widetilde{F}_1$ defined in (3.16).

Proof. From Proposition 3.3 and (1.5), for $x \geqslant 599$, we have

$$A_1(x) \leqslant \lambda \frac{\sqrt{x}}{\log^2 x} + \frac{2}{300} \frac{\sqrt{x}}{\log^3 x} + 0.7$$

while, from Proposition 3.5, we have

$$A_2(x) \leqslant \sum_{k=2}^{\kappa} \frac{1}{k} B(x^{1/k}) + \frac{9\ \log^2 x}{10000}.$$

Therefore, from Lemma 3.6, we get the upper bound (3.20) for $A(x) = A_1(x) + A_2(x)$. □

Corollary 3.10. *Under the Riemann Hypothesis, for $x \geqslant 10^8$, we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{25.22}{\log x}\right). \tag{3.22}$$

Proof. We choose $\kappa_1 = 5$ and observe that, from (3.16) and (1.11), all the terms of the right-hand side of (3.21) are positive and nonincreasing for $x \geqslant 10^8$ so that $Q(5, x) \leqslant Q(5, 10^8) = 25.2119\ldots$ (cf. [13]). □

Corollary 3.11. *Under the Riemann Hypothesis, for x tending to infinity, we have*

$$\frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{7.993 + o(1)}{\log x}\right) \leqslant A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{8.007 + o(1)}{\log x}\right). \tag{3.23}$$

Proof. The lower bound of (3.23) follows from Proposition 3.7. From Array (2.11), from (2.13) and from (3.16), one sees in (3.21) that $\lim_{x\to\infty} \widetilde{F}_2(\sqrt{x}) = 2$ and $\lim_{x\to\infty} \widetilde{F}_1(x^{1/3}) = 1$ so that (3.21) yields $\lim_{x\to\infty} Q(3, x) = 8 + 2/300$ and the upper bound of (3.23) follows from Proposition 3.9 with $\kappa_1 = 3$. □

3.5 *Numerical computation*

Let us denote by p^- and p^+ the primes surrounding the prime p.

Proposition 3.12. *For $x < 1.39 \times 10^{17}$, $A(x)$ is nondecreasing. There exists infinitely many primes p for which $A(p) < A(p^-)$ holds.*

Proof. Let us consider a prime p satisfying $3 \leqslant p < 1.39 \times 10^{17}$. From (2.1), one has

$$A(p) - A(p^-) = \mathrm{li}(\theta(p)) - \mathrm{li}(\theta(p^-)) - 1 = -1 + \int_{\theta(p^-)}^{\theta(p)} \frac{dt}{\log t}$$
$$> -1 + \frac{\theta(p) - \theta(p^-)}{\log \theta(p)} = \frac{\log p}{\log \theta(p)} - 1 > 0.$$

From Littlewood (cf. [7] or [5, chap. 5]), we know that there exists $C > 0$ and a sequence of values of x going to infinity such that

$$\theta(x) \geqslant x + C\sqrt{x} \log\log\log x.$$

Let p be the largest prime $\leqslant x$. For x and p large enough, one has

$$\theta(p) = \theta(x) \geqslant x + C\sqrt{x} \log\log\log x > p + \log p$$

and

$$A(p) - A(p^-) < \frac{\log p}{\log \theta(p^-)} - 1 = \frac{\log p}{\log(\theta(p) - \log p)} - 1 < 0$$

which completes the proof of Proposition 3.12. □

Remark. In [9, p. 8], Platt and Trudgian have proved the existence of u satisfying $727 < u < 728$ and $\theta(e^u) - e^u > 10^{152}$. If P is the largest prime $\leqslant e^u$, this implies

$$\theta(P) = \theta(e^u) > e^u + 10^{152} > P + u \geqslant P + \log P$$

and $A(P) < A(P^-) + \dfrac{\log P}{\log(\theta(P) - \log P)} - 1 < A(P^-)$.

Proposition 3.13. *(i) For* $11 \leqslant x \leqslant 1.39 \times 10^{17}$ *we have*

$$A(x) > 0. \tag{3.24}$$

(ii) Under the Riemann Hypothesis, for $x \geqslant 2$ *we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log x}\left(2 + \lambda + \frac{27.7269\ldots}{\log x}\right) \tag{3.25}$$

with equality for $x = 33647$.
(iii) Under the Riemann Hypothesis, for $x \geqslant 520\,878$ *we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{25.22}{\log x}\right). \tag{3.26}$$

(iv) For $2 \leqslant x \leqslant 10000$ *we have*

$$A(x) \leqslant 5.0643\ldots \frac{\sqrt{x}}{\log^2 x}. \tag{3.27}$$

with equality for $x = 3643$.
(v) Under the Riemann Hypothesis, for $x \geqslant 84.11$ *we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right). \tag{3.28}$$

(vi) For $37 \leqslant x < 89$ *we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}(2 - \lambda). \tag{3.29}$$

Proof. First, for $x \geqslant 2$, we define $C(x)$ and $c(x)$ by

$$A(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{C(x)}{\log x}\right) \quad \text{and} \quad A(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{c(x)}{\log x}\right)$$

so that

$$C(x) = (\log x)\left(\frac{A(x)\log^2 x}{\sqrt{x}} - 2 - \lambda\right)$$

and

$$c(x) = (\log x)\left(\frac{A(x)\log^2 x}{\sqrt{x}} - 2 + \lambda\right).$$

(i) (3.24) follows from Proposition 3.12 and $A(11) = 0.1301\ldots$ Note that $A(7) = -0.1541 < 0$ (cf. [13]).

(ii) If $x \geqslant 10^8$, (3.25) follows from Corollary 3.10.

If $2 \leqslant x < 409$, from (1.12), one has $(\log^2 x)/\sqrt{x} \leqslant 16/e^2$ and, from Proposition 3.12, $A(x) \leqslant A(401) \leqslant 2.52$ so that

$$C(x) = (\log x)\left(A(x)\frac{\log^2 x}{\sqrt{x}} - 2 - \lambda\right) \leqslant (\log 409)\left(2.52\,\frac{16}{e^2} - 2 - \lambda\right) < 20.51$$

which proves (3.25).

If $409 \leqslant x < 10^8$, let p be the largest prime $\leqslant x$. As $409 > e^6$ holds, from (1.11), for $x \in [p, p^+)$, the function $x \mapsto (\log x)\left(A(p)\frac{\log^2 x}{\sqrt{x}} - 2 - \lambda\right)$ is decreasing, which implies

$$C(x) \leqslant C(p) \tag{3.30}$$

and, by computation,

$$\max_{409 \leqslant x \leqslant 10^8} C(x) = \max_{409 \leqslant p < 10^8} C(p) = C(33647) = 27.7269\ldots$$

which completes the proof of (3.25).

(iii) For $x \geqslant 10^8$, (3.26) follows from Corollary 3.10.

We compute $p_0 = 520\,867$ the largest prime $< 10^8$ such that $C(p_0) \geqslant 25.22$. For $p_0^+ = 520\,889 \leqslant x < 10^8$, we denote by p the largest prime $\leqslant x$ and, from (3.30), one has $C(x) \leqslant C(p) < 25.22$, which implies (3.26). Then, one calculates

$$\lim_{x \to p_0^+,\, x < p_0^+} C(x) = (\log p_0^+)\left(A(p_0)\frac{\log^2 p_0^+}{\sqrt{p_0^+}} - 2 - \lambda\right) = 25.21964\ldots$$

As the above value is < 25.22, we have to solve the equation $C(t) = 25.22$ for $p_0 \leqslant t < p_0^+$ and find $t = 520\,877.54\ldots$

(iv) For $t \geqslant 1$ the function $t \mapsto (\log^2 t)/\sqrt{t}$ is maximal for $t = e^4 = 54.59\ldots$ where its value is $16/e^2 = 2.16\ldots$ (cf. (1.11) and (1.12)). As $A(x)$ is nondecreasing, for $x < 59$, we have

$$A(x)\frac{\log^2 x}{\sqrt{x}} \leqslant \frac{16}{e^2}A(53) = \frac{16}{e^2}1.155\ldots = 2.501\ldots$$

For $p \geqslant 59$ and $p \leqslant x < p^+$, one has

$$A(x)\frac{\log^2 x}{\sqrt{x}} = A(p)\frac{\log^2 x}{\sqrt{x}} \leqslant A(p)\frac{\log^2 p}{\sqrt{p}}$$

and we compute the maximum of $A(p)\frac{\log^2 p}{\sqrt{p}}$ for $59 \leqslant p < 10000$ which is equal to $5.064\ldots$ for $p = 3643$.

(v) Let us set

$$f(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right).$$

For $x \geqslant 10^8$, $A(x) > f(x)$ follows from Corollary 3.8.

Let p be a prime satisfying $e^6 < 409 \leqslant p < 10^8$. For $p \leqslant x < p^+$, one has $A(x) = A(p)$,

$$c(x) = (\log x)\left(A(p)\frac{\log^2 x}{\sqrt{x}} - 2 + \lambda\right),$$

$$c'(x) = \frac{A(p)(\log^2 x)(6 - \log x) - 2(2-\lambda)\sqrt{x}}{2x^{3/2}} < 0$$

so that $c(x)$ is decreasing and

$$c(x) \geqslant \widetilde{c}(p) \overset{def}{=\!=} \lim_{x \to p^+,\, x < p^+} c(x) = (\log p^+)\left(A(p)\frac{\log^2 p^+}{\sqrt{p^+}} - 2 + \lambda\right).$$

Therefore, for $409 \leqslant x < 10^8$ one has $c(x) \geqslant \min_{409 \leqslant p < 10^8} \widetilde{c}(p)$ and, by computation, one gets

$$\min_{409 \leqslant p < 10^8} \widetilde{c}(p) = \widetilde{c}(409) = 15.3735\ldots$$

which implies $A(x) > f(x)$.

The function f is decreasing on $(1, x_1 = 111.55\ldots]$ and increasing for $x \geqslant x_1$ (cf. [13]). Therefore, for $1 < a < b$, the upper bound of f on the interval $[a, b)$ is $\max(f(a), f(b))$. We have $A(84.1) = A(83) < f(84.1)$ while, for $84.11 \leqslant x < 89$, $A(x) = A(83) > \max(f(84.11), f(89)) \geqslant f(x)$ holds.

For $89 \leqslant p \leqslant 401 = 409^-$, one checks that $A(p) > \max(f(p), f(p^+))$ holds which shows that $A(x) > f(x)$ for $89 \leqslant x < 409$ and completes the proof of (3.28).

(vi) From (1.11), the function $\varphi(t) = (\log^2 t)/\sqrt{t}$ is increasing for $1 \leqslant t \leqslant e^4 = 54.598\ldots$ and decreasing for $t \geqslant e^4$ so that, for $1 < a < b$, the lower bound of φ on the interval $[a, b)$ is $\min(\varphi(a), \varphi(b))$.

Let p be a prime satisfying $11 \leqslant p \leqslant 83$. From (i), one has $A(p) > 0$ and, for $x \in [p, p^+)$,

$$A(x)\frac{\log^2 x}{\sqrt{x}} = A(p)\frac{\log^2 x}{\sqrt{x}} \geqslant A(p)\min(\varphi(p), \varphi(p^+)).$$

To prove (3.29), it remains to check that $A(p)\min(\varphi(p), \varphi(p^+)) > 2 - \lambda$ holds for $37 \leqslant p \leqslant 83$. □

3.6 Proof of Theorem 1.1

Proof. The proof of (1.6) follows from Corollary 3.10 while Corollary 3.8 yields (1.7).

The proof of (1.8) results of Proposition 3.13, (i) and (v).

Inequality (1.9) results of Proposition 3.13, (v) and (vi).

If $x \leqslant 10000$, Inequality (1.10) follows from Proposition 3.13, (iv), while for $x > 10000$, Proposition 3.13, (ii), implies

$$\begin{aligned} A(x) &\leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{27.7269\ldots}{\log x}\right) \\ &\leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{27.7269\ldots}{\log 10000}\right) = 5.0566\ldots \frac{\sqrt{x}}{\log^2 x} \end{aligned}$$

which ends the proof of Theorem 1.1. □

Acknowledgements I am pleased to thank Marc Deléglise for his computations and for several discussions about this paper.

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications Inc., New-York, 1964)
2. H. Cohen, *Number Theory*, Analytic and Modern Tools (Springer, Berlin, 2007)
3. P. Dusart, Explicit estimates of some functions over primes. Ramanujan J. (to appear)
4. H.M. Edwards, *Riemann's Zeta Function* (Academic Press, 1974)
5. A.E. Ingham, *The distribution of prime numbers. Cambridge Mathematical Library* (Cambridge University Press, Cambridge, 1990), Reprint of the 1932 original. With a foreword by Vaughan, R.C
6. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen, I*, 2nd edn. (Chelsea, New York, 1953)
7. J.E. Littlewood, Sur la distribution des nombres premiers. C. R. Acad. Sci. Paris Sér. I Math. **158**, 1869–1872 (1914)
8. A. Odlyzko, Table of zeros of the Riemann zeta function, http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html. Accessed 14 Nov 2017
9. D.J. Platt, T. Trudgian, On the first sign change of $\theta(x) - (x)$. Math. Comput. **85**(299), 1539–1547 (2016)
10. G. Robin, Sur la différence $\mathrm{Li}(\theta(x)) - \pi(x)$. Annales Faculté des Sciences Toulouse **6**, 257–268 (1984)
11. S. Ramanujan, Highly composite numbers. Proc. Lond. Math. Soc. **14**, 347–409 (1915). Collected papers, Cambridge University Press, 78–128 (1927)
12. L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II. Math. Comput. **30**, 337–360 (1976)
13. http://math.univ-lyon1.fr/homes-www/~nicolas/lithetax.html (2016). Accessed 14 Nov 2017

Partition-Theoretic Formulas for Arithmetic Densities

Ken Ono, Robert Schneider and Ian Wagner

In celebration of Krishnaswami Alladi's 60th birthday

Abstract If $\gcd(r, t) = 1$, then Alladi proved the Möbius sum identity

$$-\sum_{\substack{n \geq 2 \\ p_{\min}(n) \equiv r \pmod{t}}} \mu(n) n^{-1} = \frac{1}{\varphi(t)}.$$

Here $p_{\min}(n)$ is the smallest prime divisor of n. The right-hand side represents the proportion of primes in a fixed arithmetic progression modulo t. Locus generalized this to Chebotarev densities for Galois extensions. Answering a question of Alladi, we obtain analogs of these results to arithmetic densities of subsets of positive integers using q-series and integer partitions. For suitable subsets $\mathcal{S}$ of the positive integers with density $d_{\mathcal{S}}$, we prove that

$$-\lim_{q \to 1} \sum_{\substack{\lambda \in \mathscr{P} \\ \mathrm{sm}(\lambda) \in \mathcal{S}}} \mu_{\mathscr{P}}(\lambda) q^{|\lambda|} = d_{\mathcal{S}},$$

This research was partially supported by NSF grant DMS 1601306. The first author thanks the support of the Asa Griggs Candler Fund, and the second author thanks the support of an Emory University Woodruff Fellowship.

K. Ono · R. Schneider · I. Wagner (✉)
Dept. Mathematics and Computer Science, Emory University,
30322 Atlanta, Georgia
e-mail: iwagner@emory.edu

K. Ono
e-mail: ono@mathcs.emory.edu

R. Schneider
e-mail: robert.schneider@emory.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_33

where the sum is taken over integer partitions λ, $\mu_{\mathscr{P}}(\lambda)$ is a partition-theoretic Möbius function, $|\lambda|$ is the size of partition λ, and $\mathrm{sm}(\lambda)$ is the smallest part of λ. In particular, we obtain partition-theoretic formulas for even powers of π when considering power-free integers.

Keywords Arithmetic densities · Partition-theoretic Möbius function

2010 Mathematics Subject Classification 05A17 · 11P82

1 Introduction and Statement of Results

The fact $\sum_{n=1}^{\infty} \mu(n)/n = 0$, where $\mu(n)$ is the Möbius function, can be reformulated as

$$-\sum_{n=2}^{\infty} \frac{\mu(n)}{n} = 1.$$

For notational convenience define $\mu^*(n) := -\mu(n)$. This equation above can be interpreted as the statement that 100% of integers $n \geq 2$ are divisible by at least one prime. This idea was used by Alladi [1] to prove that if $\gcd(r, t) = 1$, then

$$\sum_{\substack{n \geq 2 \\ p_{\min}(n) \equiv r \pmod{t}}} \frac{\mu^*(n)}{n} = \frac{1}{\varphi(t)}. \tag{1.1}$$

Here $\varphi(t)$ is Euler's phi function, and $p_{\min}(n)$ is the smallest prime factor of n.

Alladi has asked [2] for a partition-theoretic generalization of this result. We answer his question by obtaining an analog of a generalization that was recently obtained by Locus [7]. Locus began by interpreting Alladi's theorem as a device for computing densities of primes in arithmetic progressions. She generalized this idea, and proved analogous formulas for the Chebotarev densities of Frobenius elements in unions of conjugacy classes of Galois extensions.

We recall Locus's result. Let S be a subset of primes with Dirichlet density, and define

$$\mathfrak{F}_S(s) := \sum_{\substack{n \geq 2 \\ p_{\min}(n) \in S}} \frac{\mu^*(n)}{n^s}. \tag{1.2}$$

Suppose K is a finite Galois extension of $\mathbb{Q}$ and p is an unramified prime in K. Define

$$\left[\frac{K/\mathbb{Q}}{p}\right] := \left\{\left[\frac{K/\mathbb{Q}}{\mathfrak{p}}\right] : \mathfrak{p} \subseteq \mathscr{O}_K \text{ is a prime ideal above } p\right\},$$

where $\left[\frac{K/\mathbb{Q}}{\mathfrak{p}}\right]$ is the Artin symbol (for example, see Chapter 8 of [8]), and $\mathscr{O}_K$ is the ring of integers of K. It is well known that $\left[\frac{K/\mathbb{Q}}{p}\right]$ is a conjugacy class C in $G = \mathrm{Gal}(K/\mathbb{Q})$. If we let

$$S_C := \left\{ p \text{ prime} : \left[\frac{K/\mathbb{Q}}{p}\right] = C \right\}, \tag{1.3}$$

then Locus proved (see Theorem 1 of [7]) that

$$\mathfrak{F}_{S_C}(1) = \frac{\#C}{\#G}.$$

Remark 1.1. Alladi's formula (1.1) is the cyclotomic case of Locus' Theorem.

We now turn to Alladi's question concerning a partition-theoretic analog. A *partition* is a finite non-increasing sequence of positive integers, say

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}),$$

where $\ell(\lambda)$ denotes the length of λ, i.e. the number of parts. The size of λ is $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}$, i.e., the number being partitioned. Furthermore, we let $\mathrm{sm}(\lambda) := \lambda_{\ell(\lambda)}$ denote the smallest part of λ (resp. $\mathrm{lg}(\lambda) := \lambda_1$ the largest part of λ). We require the partition-theoretic Möbius function

$$\mu_{\mathscr{P}}(\lambda) := \begin{cases} 0 & \text{if } \lambda \text{ has repeated parts} \\ (-1)^{\ell(\lambda)} & \text{otherwise} \end{cases} \tag{1.4}$$

examined in [11], where $\mathscr{P}$ denotes the set of integer partitions. Notice that $\mu_{\mathscr{P}}(\lambda) = 0$ if λ has any repeated parts, which is analogous to the vanishing of $\mu(n)$ for integers n which are not square-free. In particular, the parts in partition λ play the role of prime divisors of n in this analogy. We define $\mu^*_{\mathscr{P}}(\lambda) := -\mu_{\mathscr{P}}(\lambda)$ as in Locus' theorem, for aesthetic reasons.

The table below offers a complete description of the objects which are related with respect to this analogy. However, it is worthwhile to first compare the generating functions for $\mu(n)$ and $\mu_{\mathscr{P}}(\lambda)$. Using the Euler product for the Riemann zeta function, it is well known that the Dirichlet generating function for $\mu(n)$ is

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \sum_{m=1}^{\infty} \mu(m) m^{-s}. \tag{1.5}$$

To obtain the generating function for $\mu_{\mathscr{P}}(\lambda)$, we recall the q-Pochhammer symbol

$$(a; q)_n := \prod_{m=0}^{n-1} (1 - aq^m). \tag{1.6}$$

For $|q| < 1$, the q-Pochhammer symbol with $n = \infty$ is naturally defined by

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n. \tag{1.7}$$

Using this symbol, it is clear that the generating function for $\mu_{\mathscr{P}}(\lambda)$ is

$$(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{\lambda} \mu_{\mathscr{P}}(\lambda) q^{|\lambda|}.$$

By comparing the generating functions for $\mu(n)$ and $\mu_{\mathscr{P}}(\lambda)$, we see that prime factors and integer parts of partitions are natural analogs of each other. The following table offers the identifications that make up this analogy.

Natural number m	Partition λ
Prime factors of m	Parts of λ
Square-free integers	Partitions into distinct parts
$\mu(m)$	$\mu_{\mathscr{P}}(\lambda)$
$p_{\min}(m)$	$\mathrm{sm}(\lambda)$
$p_{\max}(m)$	$\lg(\lambda)$
m^{-s}	$q^{\lvert\lambda\rvert}$
$\zeta(s)^{-1}$	$(q;q)_\infty$
$s = 1$	$q \to 1$

Remark 1.2. There are further analogies between multiplicative number theory and the theory of integer partitions. For instance, in [3] Alladi and Erdős exploited a bijection between prime factorizations of integers and partitions into prime parts, to study an interesting arithmetic function; recently, the first two authors have shown that many theorems in multiplicative number theory are special cases of much more general partition-theoretic phenomena [10, 11].

Suppose that $\mathcal{S}$ is a subset of the positive integers with arithmetic density

$$\lim_{X\to\infty} \frac{\#\{n \in \mathcal{S} : n \le X\}}{X} = d_{\mathcal{S}}.$$

The partition-theoretic counterpart to (1.2) is

$$F_{\mathcal{S}}(q) := \sum_{\substack{\lambda\in\mathscr{P}\\ \mathrm{sm}(\lambda)\in\mathcal{S}}} \mu^*_{\mathscr{P}}(\lambda) q^{|\lambda|}. \tag{1.8}$$

To state our results, we define

$$\mathcal{S}_{r,t} := \{n \in \mathbb{Z}^+ : n \equiv r \pmod{t}\}. \tag{1.9}$$

These sets are simply the positive integers in an arithmetic progression r modulo t.

Our first result concerns the case where $t = 2$. Obviously, the arithmetic densities of $\mathcal{S}_{1,2}$ and $\mathcal{S}_{2,2}$ are both $\frac{1}{2}$. The theorem below offers a formula illustrating these densities and also offers curious lacunary q-series identities expressed in terms of theta series.

Theorem 1.1. *Assume the notation above.*

1. *The following q-series identities are true:*

$$F_{\mathcal{S}_{1,2}}(q) = \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2},$$

$$F_{\mathcal{S}_{2,2}}(q) = 1 + \sum_{n=1}^{\infty}(-1)^{n}q^{n^2} - \sum_{m=-\infty}^{\infty}(-1)^{m}q^{\frac{m(3m-1)}{2}}.$$

2. *We have that*

$$\lim_{q\to 1} F_{\mathcal{S}_{1,2}}(q) = \lim_{q\to 1} F_{\mathcal{S}_{2,2}}(q) = \frac{1}{2}.$$

Remark 1.3. An identity which implies the first part of Theorem 1.1 was obtained earlier by Andrews and Stenger [5]. This identity and a generalization is also discussed in Examples 2 and 3 on p. 156-157 of [4].

Remark 1.4. The limits in Theorem 1.1 are understood as q tends to 1 from within the unit disk.

Example 1. *For complex z in the upper-half of the complex plane, let $q(z) := \exp\left(-\frac{2\pi i}{z}\right)$. Therefore, if $z \to 1$ in the upper-half plane, then $q(z) \to 1$ in the unit disk. The table below displays a set of such z beginning to approach 1 and the corresponding values of $F_{\mathcal{S}_{1,2}}(q(z))$.*

z	$F_{\mathcal{S}_{1,2}}(q(z))$
$1 + .10i$	0.458233...
$1 + .09i$	0.471737...
$1 + .08i$	0.482784...
$1 + .07i$	0.491003...
$1 + .06i$	0.496296...
$1 + .05i$	0.498998...
$1 + .04i$	0.499919...
$1 + .03i$	0.500048...
$1 + .02i$	0.500024...
$1 + .01i$	0.500006...

The first claim in Theorem 1.1 offers an immediate combinatorial interpretation. Let $D^{+}_{\text{even}}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part even, and let $D^{+}_{\text{odd}}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part odd. Similarly, let $D^{-}_{\text{even}}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part even, and let $D^{-}_{\text{odd}}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part odd. To make this precise, for integers k let $\omega(k) := \frac{k(3k-1)}{2}$ be the index k *pentagonal number*.

Corollary 1.2. *Assume the notation above.*

1. *For partitions into distinct parts whose smallest part is odd, we have*

$$D^{+}_{\text{odd}}(n) - D^{-}_{\text{odd}}(n) = \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n \text{ is an even square} \\ -1 & \text{if } n \text{ is an odd square.} \end{cases}$$

2. *For partitions into distinct parts whose smallest part is even, we have*

$$D^{+}_{\text{even}}(n) - D^{-}_{\text{even}}(n)$$
$$= \begin{cases} -1 & \text{if } n \text{ is an even square and not a pentagonal number} \\ 1 & \text{if } n \text{ is an odd square and not a pentagonal number} \\ 1 & \text{if } n \text{ is an even index pentagonal number and not a square} \\ -1 & \text{if } n \text{ is an odd index pentagonal number and not a square} \\ 0 & \text{otherwise.} \end{cases}$$

Question 1. *It would be interesting to obtain a combinatorial proof of Corollary 1.2 which is in the spirit of Franklin's proof of Euler's Pentagonal Number Theorem (see pages 10-11 of [4]).*

Our proof of Theorem 1.1 makes use of the q-Binomial Theorem and some well-known q-series identities. It is natural to ask whether such a relation holds for general sets $\mathcal{S}_{r,t}$. The following theorem shows that Theorem 1.1 is indeed a special case of a more general phenomenon.

Theorem 1.3. *If* $0 \leq r < t$ *are integers and* $\gcd(m,t) = 1$*, then we have that*

$$\lim_{q \to \zeta} F_{\mathcal{S}_{r,t}}(q) = \frac{1}{t},$$

where ζ *is a primitive* m*th root of unity.*

Remark 1.5. The limits in Theorem 1.3 are understood as q tends to ζ from within the unit disk.

Obviously, these results hold for any set $\mathcal{S}$ of positive integers that is a finite union of arithmetic progressions. It turns out that this theorem can also be used to compute arithmetic densities of more complicated sets arising systematically from a careful study of arithmetic progressions. We focus on the sets of positive integers $\mathcal{S}_{\mathrm{fr}}^{(k)}$ which are kth power-free. In particular, we have that

$$\mathcal{S}_{\mathrm{fr}}^{(2)} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, \ldots\}.$$

It is well known that the arithmetic densities of these sets are given by

$$\lim_{X\to+\infty} \frac{\#\left\{1 \le n \le X \ : \ n \in \mathcal{S}_{\mathrm{fr}}^{(k)}\right\}}{X} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)}.$$

To obtain partition-theoretic formulas for these densities, we first compute a partition-theoretic formula for the density of

$$\mathcal{S}_{\mathrm{fr}}^{(k)}(N) := \{n \ge 1 \ : \ p^k \nmid n \text{ for every } p \le N\}. \tag{1.10}$$

Theorem 1.4. *If $k, N \ge 2$ are integers, then we have that*

$$\lim_{q\to 1} F_{\mathcal{S}_{\mathrm{fr}}^{(k)}(N)}(q) = \prod_{p\le N \text{ prime}} \left(1 - \frac{1}{p^k}\right).$$

The constants in Theorem 1.4 are the arithmetic densities of positive integers that are not divisible by the kth power of any prime $p \le N$, namely $\mathcal{S}_{\mathrm{fr}}^{(k)}(N)$. Theorem 1.4 can be used to calculate the arithmetic density of $\mathcal{S}_{\mathrm{fr}}^{(k)}$ by letting $N \to +\infty$.

Corollary 1.5. *If $k \ge 2$, then*

$$\lim_{q\to 1} F_{\mathcal{S}_{\mathrm{fr}}^{(k)}}(q) = \frac{1}{\zeta(k)}.$$

Furthermore, if $k \ge 2$ is even, then

$$\lim_{q\to 1} F_{\mathcal{S}_{\mathrm{fr}}^{(k)}}(q) = (-1)^{\frac{k}{2}+1} \frac{k!}{B_k \cdot 2^{k-1}} \cdot \frac{1}{\pi^k},$$

where B_k is the kth Bernoulli number.

This paper is organized as follows. In Section 2.1 we discuss the q-Binomial Theorem, which will be an essential tool for our proofs, as well as a duality principle for partitions related to ideas of Alladi. In Section 2.2 we will use the q-Binomial Theorem to prove results related to Theorem 1.3. Section 3 will contain the proofs of all of the theorems, and Section 4 will contain some nice examples.

2 The q-Binomial Theorem and its consequences

In this section we recall elementary q-series identities, and we offer convenient reformulations for the functions $F_{\mathcal{S}}(q)$.

2.1 Easy nuts and bolts

Let us recall the classical q-Binomial Theorem (see [4] for proof).

Lemma 2.1. *For $a, z \in \mathbb{C}$, $|q| < 1$ we have the identity*

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \cdot z^n.$$

We recall the following well-known q-product identity (for proof, see page 6 of [6]).

Lemma 2.2. *Using the above notations, we have that*

$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}.$$

The following elementary lemma plays a crucial role in this paper.

Lemma 2.3. *If $\mathcal{S}$ is a subset of the positive integers, then the following are true:*

$$F_{\mathcal{S}}(q) = \sum_{n\in\mathcal{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}) = (q;q)_\infty \cdot \sum_{\substack{\lambda\in\mathscr{P} \\ \mathrm{lg}(\lambda)\in\mathcal{S}}} q^{|\lambda|}.$$

Remark 2.1. Lemma 2.3 may be viewed as a partition-theoretic case of Alladi's duality principle, originally stated in [1] as a relation between functions on smallest versus largest prime divisors of integers, which he gave in full generality in a lecture [2].

Proof. By inspection, we see that

$$F_{\mathcal{S}}(q) = \sum_{\substack{\lambda\in\mathscr{P} \\ \mathrm{sm}(\lambda)\in\mathcal{S}}} \mu^*_{\mathscr{P}}(\lambda) q^{|\lambda|} = \sum_{n\in\mathcal{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}).$$

By factoring out $(q;q)_\infty$ from each summand, we find that

$$F_{\mathcal{S}}(q) = \sum_{n\in\mathcal{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}) = (q;q)_\infty \cdot \sum_{n\in\mathcal{S}} \frac{q^n}{(q;q)_n}$$
$$= (q;q)_\infty \cdot \sum_{\substack{\lambda\in\mathscr{P} \\ \mathrm{lg}(\lambda)\in\mathcal{S}}} q^{|\lambda|}.$$

2.2 Case of $F_{\mathcal{S}_{r,t}}(q)$

Here we specialize Lemma 2.3 to the sets $\mathcal{S}_{r,t}$. The next lemma describes the q-series $F_{\mathcal{S}_{r,t}}(q)$ in terms of a finite sum of quotients of infinite products. To prove this lemma we make use of the q-Binomial Theorem.

Lemma 2.4. *If t is a positive integer and $\zeta_t := e^{2\pi i/t}$, then*

$$F_{\mathcal{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t}\left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q;q)_\infty}\right].$$

Proof. From Lemma 2.3, we have that

$$F_{\mathcal{S}_{r,t}}(q) = (q;q)_\infty \cdot \sum_{n=0}^{\infty} \frac{q^{tn+r}}{(q;q)_{tn+r}}.$$

By applying the q-Binomial Theorem (see Lemma 2.1) with $a=0$ and $z=\zeta_t^m q$, we find that

$$\frac{1}{t}\left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q;q)_\infty}\right] = \frac{1}{t}\left[\sum_{m=1}^{t}\sum_{n=0}^{\infty} \frac{\zeta_t^{m(n-r)} q^n}{(q;q)_n}\right].$$

Due to the orthogonality of roots of unity we have

$$\sum_{m=1}^{t} \zeta_t^{m(n-r)} = \begin{cases} t & \text{if } n \equiv r \pmod{t} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, this sum allows us to sieve on the sum in n leaving only those summands with $n \equiv r \pmod{t}$, namely the series

$$\sum_{n=0}^{\infty} \frac{q^{tn+r}}{(q;q)_{tn+r}}.$$

Therefore, it follows that

$$F_{\mathcal{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t}\left[\sum_{m=1}^{t}\sum_{n=0}^{\infty}\frac{\zeta_t^{m(n-r)}q^n}{(q;q)_n}\right].$$

Lemma 2.5. *If a and m are positive integers and ζ is a primitive mth root of unity, then*

$$\lim_{q\to 1}\frac{(q;q)_\infty}{(\zeta^a q;q)_\infty} = \begin{cases}1 & \text{if } m \mid a\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since $(aq;q)_\infty^{\pm 1}$ is an analytic function of q inside the unit disk (i.e., of $q := e^{2\pi i z}$ with z in the upper half-plane) when $|a| \le 1$, the quotient on the left-hand side of Lemma 2.5 is well-defined as a function of q (resp. of z), and we can take limits from inside the unit disk. When $m \mid a$, the q-Pochhammer symbols cancel and the quotient is identically 1. When $m \nmid a$, then $(q;q)_\infty$ clearly vanishes as $q \to 1$ while $(\zeta^a q;q)_\infty$ is non-zero; thus the quotient is zero.

3 Proofs of our results

3.1 *Proof of Theorem 1.1*

Here we prove the first part of Theorem 1.1; we defer the proof of the second part until the next section because it is a special case of Theorem 1.3.

Proof. Here we prove the first part of Theorem 1.1. By Lemma 2.4 we have

$$\begin{aligned}F_{\mathcal{S}_{1,2}}(q) &= (q;q)_\infty \cdot \frac{1}{2}\left[\frac{1}{(q;q)_\infty} - \frac{1}{(-q;q)_\infty}\right]\\ &= \frac{1}{2}\left[1 - \frac{(q;q)_\infty}{(-q;q)_\infty}\right]\\ &= \frac{1}{2}\left[1 - \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\right].\end{aligned}$$

Lemma 2.2 now implies that

$$F_{\mathcal{S}_{1,2}}(q) = \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}.$$

To prove the $F_{\mathcal{S}_{2,2}}(q)$ identity, first recall that $\sum_{\lambda\in\mathscr{P}}\mu^*_{\mathscr{P}}(\lambda)q^{|\lambda|} = -(q;q)_\infty$. From this we know $F_{\mathcal{S}_{1,2}}(q) + F_{\mathcal{S}_{2,2}}(q) = 1 - (q;q)_\infty$. Using the identity for $F_{\mathcal{S}_{1,2}}(q)$ and Euler's Pentagonal Number Theorem completes the proof.

Proof. Now we prove Corollary 1.2.

Part 1. This claim follows easily from the first part of Theorem 1.1. The reader should recall that $F_{\mathcal{S}_{1,2}}(q)$ is the generating function for $\mu^*_{\mathscr{P}}(\lambda) = -\mu_{\mathscr{P}}(\lambda)$.

Part 2. This corollary is not as immediate as the first part. Of course, we must classify the integer pairs m and n for which $n^2 = m(3m-1)/2$. After simple manipulation, we find that this holds if and only if

$$(6m-1)^2 - 6(2n)^2 = 1.$$

In other words, we require that $(x, y) = (6m-1, 2n)$ be a solution to Pell's equation

$$x^2 - 6y^2 = 1.$$

It is well known that all of the positive solutions to Pell's equation are of the form (x_k, y_k), where

$$x_k + \sqrt{6} \cdot y_k = (5 + 2\sqrt{6})^k.$$

Using this description and the elementary congruence properties of (x_k, y_k), one easily obtains the second part of Corollary 1.2.

3.2 *Proof of Theorem 1.3*

Here we prove the general limit formulas for the arithmetic densities of $\mathcal{S}_{r,t}$.

Proof. Now we prove Theorem 1.3. From Lemma 2.4 we have

$$F_{\mathcal{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t}\left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q; q)_\infty}\right].$$

We stress that we can take a limit here because we have a finite sum of functions which are analytic inside the unit disk. Using Lemma 2.5 we see that

$$\lim_{q\to 1} \frac{(q;q)_\infty}{(\zeta_t^m q; q)_\infty} = \begin{cases} 1 & \text{if } m = t \\ 0 & \text{otherwise.} \end{cases}$$

From this we have

$$\lim_{q\to 1} F_{\mathcal{S}_{r,t}}(q) = \frac{1}{t}.$$

The proof for $q \to \zeta$ where ζ is a primitive mth root of unity with $\gcd(m,t) = 1$ follows the exact same steps.

3.3 Proofs of Theorem 1.4 and Corollary 1.5

Here we will prove Theorem 1.4 and Corollary 1.5 by building up kth power-free sets using arithmetic progressions. We prove Theorem 1.4 first.

Proof. Here we prove Theorem 1.4. The set of numbers not divisible by p^k for any prime $p \leq N$ can be built as a union of sets of arithmetic progressions. Therefore, for a given fixed N we only need to understand divisibility by p^k for all primes $p \leq N$. Because the divisibility condition for each prime is independent from the other primes, we have

$$F_{\mathcal{S}_{\mathrm{fr}}^{(k)}(N)}(q) = \sum_{\substack{0 \leq r < M \\ p^k \nmid r}} F_{\mathcal{S}_{r,M}}(q),$$

where $M := \prod_{\substack{p \leq N \\ \text{prime}}} p^k$. We have a finite number of summands, and the result now follows immediately from Theorem 1.3.

Next, we will prove Corollary 1.5.

Proof. Here we prove Corollary 1.5. For fixed N define

$$\zeta_N(k) := \prod_{p \leq N \text{prime}} \left(\frac{1}{1 - p^k} \right),$$

so $\lim_{q \to 1} F_{\mathcal{S}_{\mathrm{fr}}^{(k)}(N)}(q) = \frac{1}{\zeta_N(k)}$. It is clear $\lim_{N \to \infty} \zeta_N(k) = \zeta(k)$. It is in this sense that we say $\lim_{q \to 1} F_{\mathcal{S}_{\mathrm{fr}}^{(k)}}(q) = \frac{1}{\zeta(k)}$.

4 Examples

Example 2. *In the case of $\mathcal{S}_{1,3}$, which has arithmetic density 1/3, Theorem 1.3 holds for any mth root of unity where $3 \nmid m$. The two tables below illustrate this as q approaches $\zeta_1 = 1$ and $\zeta_4 = i$, respectively.*

q	$F_{\mathcal{S}_{1,3}}(q)$
0.70	0.340411885...
0.75	0.335336994...
0.80	0.333552814...
0.85	0.333331545...
0.90	0.333333329...
0.95	0.333333333...

q	$F_{\mathcal{S}_{1,3}}(q)$
$0.70i$	$\approx 0.034621 + 0.793781i$
$0.75i$	$\approx 0.057890 + 0.802405i$
$0.80i$	$\approx 0.097030 + 0.771774i$
$0.85i$	$\approx 0.167321 + 0.674712i$
$0.90i$	$\approx 0.294214 + 0.454400i$
$0.95i$	$\approx 0.424978 + 0.067775i$
$0.97i$	$\approx 0.376778 - 0.016187i$
$0.98i$	$\approx 0.340170 + 0.005772i$
$0.99i$	$\approx 0.332849 + 0.000477i$

Example 3. *The table below illustrates Theorem 1.4 for the set* $\mathcal{S}_{\text{fr}}^{(2)}(5)$, *which has arithmetic density* $16/25 = 0.64$. *These are the positive integers which are not divisible by 4, 9 and 25. Here we give numerics for the case of* $F_{\mathcal{S}_{\text{fr}}^{(2)}(5)}(q)$ *as* $q \to 1$ *along the real axis.*

q	$F_{\mathcal{S}_{\text{fr}}^{(2)}(5)}(q)$
0.90	0.615367...
0.91	0.619346...
0.92	0.625991...
0.93	0.631607...
0.94	0.631748...
0.95	0.631029...
0.96	0.638291...
0.97	0.639893...

Example 4. *Here we approximate the density of* $\mathcal{S}_{\text{fr}}^{(4)}$, *the fourth power-free positive integers. Since* $\zeta(4) = \pi^4/90$, *it follows that the arithmetic density of* $\mathcal{S}_{\text{fr}}^{(4)}$ *is* $90/\pi^4 \approx 0.923938...$. *Here we choose* $N = 5$ *and compute the arithmetic density of* $\mathcal{S}_{\text{fr}}^{(4)}(5)$, *the positive integers which are not divisible by* $2^4, 3^4$, *and* 5^4. *The density of this set is* $208/225 \approx 0.924444...$. *This density is fairly close to the density of fourth power-free integers because the convergence in the* N *aspect is significantly faster for fourth power-free integers than for square-free integers, as discussed above.*

q	$F_{\mathcal{S}_{\text{fr}}^{(4)}(5)}(q)$
0.90	0.934926...
0.91	0.936419...
0.92	0.936718...
0.93	0.935027...
0.94	0.931517...
0.95	0.925619...
0.96	0.921062...
0.97	0.925998...
0.98	0.924967...

References

1. K. Alladi, Duality between prime factors and an application to the prime number theorem for arithmetic progressions. J. Number Theory **9**, 436–451 (1977)
2. K. Alladi, A duality between the largest and smallest prime factors via the Moebius function and arithmetical consequences. Emory Univ. Number Theory Seminar (2017)
3. K. Alladi, P. Erdös, On an additive arithmetic function. Pac. J. Math. **71.2**, 275–294 (1977)
4. G.E. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, 1998)
5. G.E. Andrews, A. Stenger, Solutions to advanced problems # 5865. Am. Math. Monthly **80**, 1147–1148 (1973)
6. N.J. Fine, Basic hypergeometric series and applications. Math. Surv. Monogr. (1988)
7. M. Locus, A new formula for Chebotarev densities (2017). arXiv:1703.08194
8. D.A. Marcus, *Number Fields*, vol. 1995 (Springer, New York, 1977)
9. K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, vol. 102, CBMS Regional Conference Series in Mathematics (American Mathematical Society, Providence, 2004). Published for the Conference Board of the Mathematical Sciences, Washington, DC
10. K. Ono, L. Rolen, R. Schneider, Explorations in the theory of partition zeta functions, in *Exploring the Riemann Zeta Function*, 190 years from Riemann's Birth, ed. by H. Montgomery, A. Nikeghbali, M. Rassias (Springer, Berlin). (To appear)
11. R. Schneider, Arithmetic of partitions and the q-bracket operator. Proc. Am. Math. Soc. **145**(5), 1953–1968 (2017)

A New Witness Identity for $11 \mid p(11n + 6)$

Peter Paule and Cristian-Silviu Radu

Dedicated to our friend Krishna Alladi on his 60th birthday

Abstract Let $p(n)$ be the number of partitions of the positive integer n. A new q-series identity is presented which witnesses Ramanujan's observation that $11|p(11n+6)$ for all $n \geq 0$ at one glance. This identity can be derived in a natural way by applying an algorithm to present subalgebras of the polynomial ring $\mathbb{K}[z]$ as $\mathbb{K}[z]$-modules.

Keywords Ramanujan partition congruences · Subalgebra presentations of modular function spaces · Computer algebra algorithms for modular functions

2010 Mathematics Subject Classification 11P83 · 11F03 · 13P10 · 68W30

1 Introduction

Recall the celebrated partition congruences observed by Ramanujan [11]:

The research of Radu was supported by the strategic program "Innovatives OÖ 2010 plus" by the Upper Austrian Government in the frame of project W1214-N15-DK6 of the Austrian Science Fund (FWF). Both authors were supported by grant SFB F50-06 of the Austrian Science Fund (FWF)

P. Paule · C.-S. Radu (✉)
Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria
e-mail: sradu@risc.uni-linz.ac.at
P. Paule
e-mail: ppaule@risc.uni-linz.ac.at

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_34

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \tag{1.1}$$

and

$$p(11n+6) \equiv 0 \pmod{11}, \; n \geq 0, \tag{1.2}$$

where $p(n)$ counts the number of partitions of the positive integer n.

For the congruences in (1.1) Ramanujan [11] provided clever elementary proofs based on Euler's pentagonal number theorem and on Jacobi's identity for the third power of Dedekind's eta-function. Also in [11], Ramanujan presented two q-series identities which in explicit fashion witness the stated divisibilities by 5 and 7:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7 \prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q \prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{1.4}$$

Bruce Berndt's commentary in [5, pp. 372–375] on Ramanujan's paper gives the history of proofs of Ramanujan's congruences. In particular, Berndt points to the fact that (1.4) in Ramanujan's original paper is stated without any proof and also that Ramanujan only briefly sketches a proof of (1.3)—an identity greatly admired by Hardy; see Hardy's remark in [5, p. xxxv].

Concerning Ramanujan's congruence (1.2) a witness identity like (1.3) or (1.4), i.e., presenting $\sum_{n\geq 0} p(11n+6)q^n$ as a linear combination of eta-quotients, has been found only recently by Radu [10] with the help of his Ramanujan-Kolberg Algorithm. As the main theorem of this note we present a new and simpler witness identity but built in different fashion:

Theorem 1.1. *Let*

$$t := q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12}, \tag{1.5}$$

and

$$f := q\, t \prod_{k=1}^{\infty} (1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6)q^n. \tag{1.6}$$

Then

$$\begin{aligned} f^5 &= 5 \cdot 11^4 f^4 + 11^4(-2 \cdot 5 \cdot 11^4 + 251\, t) f^3 \\ &\quad + 11^3(2 \cdot 5 \cdot 11^9 + 2 \cdot 3 \cdot 5 \cdot 11^5 \cdot 31\, t + 4093\, t^2) f^2 \\ &\quad + 11^4(-5 \cdot 11^{12} + 2 \cdot 5 \cdot 11^8 \cdot 17\, t - 2^2 \cdot 3 \cdot 11^3 \cdot 1289\, t^2 + 3 \cdot 41\, t^3) f \\ &\quad + 11^5(11^4 + t)(11^{11} - 3 \cdot 7 \cdot 11^7\, t + 11^2 \cdot 1321\, t^2 + t^3). \end{aligned} \tag{1.7}$$

The divisibility $11|p(11n + 6)$ follows immediately from the fact that all coefficients of powers of q on the right hand side are integers containing 11 as a factor. This property clearly carries over to f since f^5 is an element of an integral domain—regardless whether the q-series/products involved are considered as formal Laurent series or as analytic functions.

The rest of this article is structured as follows. In Section 2 we put the witness identity (1.7) into the algebraic framework of presenting subalgebras of the polynomial ring $\mathbb{K}[z]$ as finitely generated $\mathbb{K}[z]$-modules. To apply the machinery for deriving the witness identity (1.7) in algorithmic fashion, basic facts from modular functions are needed; these are provided in Section 3. One of the consequences of this setting is a computational verification of (1.7). Section 4 describes an algorithm to compute the desired module presentation for the case of polynomials. The analogous algorithm for modular functions is discussed in Section 5, including the algorithmic derivation of the witness identity (1.7).

2 Presenting Subalgebras of the polynomial ring $\mathbb{K}[z]$

By $\mathbb{K}[z]$ we denote the ring of univariate polynomials in z with coefficients from a field $\mathbb{K}$. In our context it is useful to keep in mind that $\mathbb{K}[z]$ is also a vector space over $\mathbb{K}$; sometimes we emphasize this fact by saying that $\mathbb{K}[z]$ is a $\mathbb{K}$-algebra[1] For example, we can consider the $\mathbb{K}$-algebra generated by given polynomials $f_0, \dots, f_n \in \mathbb{K}[z]$,

$$\mathbb{K}[f_0, f_1, \dots, f_n] := \Big\{ \sum_{j_0, j_1, \dots, j_n \geq 0} c_{j_0, j_1, \dots, j_n} f_0^{j_0} f_1^{j_1} \dots f_n^{j_n} \Big\};$$

i.e., the elements are polynomials in the f_j with coefficients $c_{j_0, j_1, \dots, j_n} \in \mathbb{K}$.

We will consider also slight variations of this setting. For example, the right hand side of (1.7) can be considered as an element of the $\mathbb{Q}$-algebra $\mathbb{Q}[t, f]$. Here t and f are not polynomials but Laurent series in q. More precisely, the right hand side of (1.7) has a particular structure; namely, it is an element of

$$\mathbb{Q}[t] + \mathbb{Q}[t]f + \cdots + \mathbb{Q}[t]f^4 := \{p_0(t) + p_1(t)f + \cdots + p_4(t)f^4 : p_j(t) \in \mathbb{Q}[t]\}.$$

Obviously this is a subalgebra of $\mathbb{Q}[t, f]$. Conversely, the relation (1.7) guarantees that all the elements of $\mathbb{Q}[t, f]$ are contained in this subalgebra. As a consequence, the witness identity (1.7) has also an *algebraic* meaning; namely,

$$\mathbb{Q}[t, f] = \mathbb{Q}[t] + \mathbb{Q}[t]f + \cdots + \mathbb{Q}[t]f^4. \tag{2.1}$$

In other words, (2.1) tells us that the algebra $\mathbb{Q}[t, f]$ can be presented as a module over the polynomial ring $\mathbb{Q}[t]$ with module generators $1, f, \dots, f^4$.

---

[1] For us a $\mathbb{K}$-algebra R is a commutative ring R with 1 which is also a vector space over $\mathbb{K}$.

We note that this module is freely generated[2] owing to fact that the orders $\mathrm{ord}(f^j)$ are pairwise different. As usual, for a formal Laurent series, resp. meromorphic function, $F(q) = \sum_{n=\ell}^{\infty} F_n q^n$ with $F_\ell \neq 0$, its order is defined as $\mathrm{ord}(F(q)) := \ell$.

It is well-known that subalgebras of $\mathbb{K}[z]$ are finitely generated:

Theorem 2.1 ([4]). *Let $A \neq \mathbb{K}$ be a subalgebra of $\mathbb{K}[z]$ and let n be the degree of the polynomial of smallest positive degree in A. Then A can generated by a set of not more than $n + 1$ elements.*

Gale's proof is elegant and short, but non-constructive. Nevertheless, its underlying idea is based upon presenting a subalgebra as a finitely generated $\mathbb{K}[t]$-module as in (2.1). Almost 60 years after Gale's paper, Radu[3] in [10] introduced a constructive version of this approach.

Remark 2.1. Presentations like (2.1) also solve the problem to decide subalgebra membership. In computer algebra usually such problems are solved by constructing a convenient basis. To this end, for *multivariate* polynomial rings, SAGBI ("Subalgebra Analogs to Gröbner Bases for Ideals") bases are considered. This concept was introduced by Kapur and Madlener [7] and independently by Robbiano and Sweedler [12]. They also present a method for computing such bases given a set of generators for a subalgebra of a multivariate polynomial ring. In general this method is not algorithmic, but in the *univariate* case $\mathbb{K}[z]$ it can be shown to terminate after a finite number of steps. In this context, Anna Torstensson [13] gave a careful algorithm analysis for the case when the subalgebra is generated by two polynomials. Radu's algorithm works completely different to the SAGBI mechanism. The algorithm computes a Noether normalization (e.g., [3, Theorem 30]) of a finitely generated $\mathbb{K}$-subalgebra of $\mathbb{K}[z]$ to solve the subalgebra membership problem in the case of *univariate* polynomial rings.

In Section 4 we explain the main algorithmic ideas used to establish (1.7), resp. (2.1). Before doing so, we set up the required algebraic/analytic frame for these equalities.

3 Modular Functions Background

Modular functions provide a convenient mathematical environment for the objects t and f in Theorem 1.1. In this context we view $q = q(\tau)$ as a function on the upper half complex plane $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ defined by $q(\tau) := \exp(2\pi i \tau)$. The congruence subgroup $\Gamma_0(N)$ of the modular group is defined as

[2] I.e., every element $p_0(t) + p_1(t)f + \cdots + p_4(t)f^4$ in the module has uniquely determined coefficients $p_j(t) \in \mathbb{Q}[t]$.

[3] Not knowing about Gale's paper at that time.

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

A holomorphic function g defined on $\mathbb{H}$ is called a modular function for $\Gamma_0(N)$ if (i) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = g(\tau),\ \tau \in \mathbb{H}, \tag{3.1}$$

and (ii) if for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a Laurent series expansion with *finite principal part* such that for all $\tau \in \mathbb{H}$ sufficiently close to $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$:

$$g(\tau) = \sum_{n=-\infty}^{\infty} c_n(\gamma) e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma} \tag{3.2}$$

where

$$w_\gamma := \min\left\{ h \in \mathbb{N} \setminus \{0\} : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_0(N)\gamma \right\}.$$

Note. $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(N)$ act on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau := \frac{a\tau+b}{c\tau+d}$, and also on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by defining $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\infty := \frac{a}{c}$ and $\frac{a}{0} := \infty$.

If $m = m(\gamma)$ is the smallest index such that $c_m(\gamma) \neq 0$ in (3.2), then we call m the γ-order of g at $\tau = \frac{a}{c}$; notation: $m = \mathrm{ord}^{\gamma}_{\mathrm{a/c}}(\mathrm{g})$. If $\frac{a}{c} = \gamma_1\infty = \gamma_2\infty$ for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, then $\mathrm{ord}^{\gamma_1}_{\mathrm{a/c}}(\mathrm{g}) = \mathrm{ord}^{\gamma_2}_{\mathrm{a/c}}(\mathrm{g})$. This leads to define the order of a modular function g at a point $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$, called a "cusp of g", by

$$\mathrm{ord}_{\mathrm{a/c}}(\mathrm{g}) := \mathrm{ord}^{\gamma}_{\mathrm{a/c}}(\mathrm{g})$$

for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = \frac{a}{c}$.

The action of $\Gamma_0(N)$ maps cusps onto cusps, and it turns out that for each $N \in \mathbb{N} \setminus \{0\}$ the set of cusps $\mathbb{Q} \cup \{\infty\}$ splits only into finitely many orbits. A cusp $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ is called a critical point of a modular function g if $\mathrm{ord}_{\mathrm{a/c}}(\mathrm{g}) < 0$. This property is invariant under the action of $\Gamma_0(N)$ owing to the fact that for two cusps $\frac{a}{c}$ and $\frac{a'}{c'}$ from the same orbit, i.e., $\frac{a'}{c'} = \gamma\frac{a}{c}$ for some $\gamma \in \Gamma_0(N)$, one has

$$\mathrm{ord}_{\mathrm{a'/c'}}(\mathrm{g}) = \mathrm{ord}_{\mathrm{a/c}}(\mathrm{g}).$$

The $\Gamma_0(N)$-orbit of the cusp $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ is denoted by $[\frac{a}{c}]$. An orbit $[\frac{a}{c}]$ where $\frac{a}{c}$ is a critical point of g is called critical orbit of g.

The set of modular functions for $\Gamma_0(N)$ forms a $\mathbb{C}$-algebra denoted by $M(N)$. An important subalgebra is

$$M^{\infty}(N) := \{g \in M(N) : g \text{ is constant, or } [\infty] \text{ is the only critical orbit of } g\}.$$

This, in view of (3.2), gives a normal form presentation for any modular function $g \in M^{\infty}(N)$. Namely, we take the Laurent series expansion at the cusp ∞ using $\gamma := \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) = \mathrm{Id}$:

$$g(\tau) = \sum_{n=\mathrm{ord}_{\infty}(g)}^{\infty} c_n q^n. \tag{3.3}$$

Note that $q = q(\tau) = \exp(2\pi i \tau)$, $c_n := c_n(\mathrm{Id})$, and $w_{\gamma} = w_{\mathrm{Id}} = 1$; if g is a constant then $\mathrm{ord}_{\infty}(g) = 0$ and $c_n = 0$ for all $n \geq 1$. We call this unique Laurent series the *q-series presentation* of g.

This setting provides a convenient mathematical environment for the objects t and f in Theorem 1.1.

Lemma 3.1. *Let t and f be defined as in* (1.5) *and* (1.6) *of Theorem 1.1 where $q = q(\tau) := \exp(2\pi i \tau)$, $\tau \in \mathbb{H}$. Then*

$$t, f \in M^{\infty}(11),$$

and the q-series/products in (1.5) *and* (1.6) *correspond to the Laurent series expansions of t and f at the cusp ∞ as in* (3.3)*. Consequently,*

$$\mathrm{ord}_{\infty}(t) = -5 \textit{ and } \mathrm{ord}_{\infty}(f) = -4.$$

Proof. The statement follows immediately from

$$q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M^{\infty}(11) \text{ and } q \prod_{k=1}^{\infty} (1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n \in M^{\infty}(11),$$

which, for instance, is proven in [10].

Despite the analytic setting, to decide equality of two functions in $M^{\infty}(N)$ can be done in purely algebraic and finitary fashion.

Lemma 3.2. *Let g and h be in $M^{\infty}(N)$ with q-series presentations*

$$g(\tau) = \sum_{n=\mathrm{ord}_{\infty}(g)}^{\infty} a_n q^n \textit{ and } h(\tau) = \sum_{n=\mathrm{ord}_{\infty}(h)}^{\infty} b_n q^n.$$

Then $g = h$ if and only if $\mathrm{ord}_{\infty}(g) = \mathrm{ord}_{\infty}(h) =: \ell$ *and*

$$(a_{\ell}, \ldots, a_{-1}, a_0) = (b_{\ell}, \ldots, b_{-1}, b_0).$$

Proof. The lemma is just a special case of Liouville's theorem for any compact Riemann surface: a holomorphic function on a compact Riemann surface must be constant. Also see [9, Sect. 6], or any introductory text on modular functions.

In other words, if $g(\tau) = \sum_{n=\mathrm{ord}_\infty(g)}^{\infty} a_n q^n \in M^\infty(N)$, the coefficients a_n, $n \geq 1$, are uniquely determined by those of the principal part and a_0. Algebraically this corresponds to an isomorphic embedding of $\mathbb{C}$-vector spaces:

$$\varphi : M^\infty(N) \to \mathbb{C}[z],$$
$$\sum_{n=\mathrm{ord}_\infty(g)}^{\infty} a_n q^n \mapsto a_{\mathrm{ord}_\infty(g)} z^{-\mathrm{ord}_\infty(g)} + \cdots + a_{-1} z + a_0. \tag{3.4}$$

In computationally feasible cases the zero test for $g - h \stackrel{?}{=} 0$ of Lemma 3.2 trivializes the task of proving identities between modular functions.

Example 3.1. Knowing from Lemma 3.1 that t and f are in $M^\infty(11)$ the proof of the witness identity (1.7) can be left to a computer algebra package: First one specifies the input for t and f as, for instance, in In[16] and In[17] of Section 5. Then applying a simplification command as, for instance, `Simplify` in Mathematica reduces the difference of the expressions on the left and on the right hand side of (1.7) to `O[q]^2`; this means, to 0 in view of Lemma 3.2.

4 Algorithm "MODULE GENERATORS" for Polynomials

In view of the isomorphic vector space embedding φ defined in (3.4), we present an algorithm to compute a suitable module presentation as in (2.1).

ALGORITHM 4.1 ("MODULE GENERATORS: polynomial case").
Given non-constant polynomials $t := f_0, f_1, \ldots, f_n \in \mathbb{K}[z]$, *the algorithm computes* $g_1, \ldots, g_k \in \mathbb{K}[z]$ *such that*

$$\mathbb{K}[t, f_1, \ldots, f_n] = \mathbb{K}[t] + \mathbb{K}[t]\, g_1 + \cdots + \mathbb{K}[t]\, g_k. \tag{4.1}$$

To explain a basic ingredient of the algorithm, we first consider the problem of finding convenient presentations of finitely generated additive submonoids of $\mathbb{N} = \{0, 1, \ldots\}$.

Example 4.1. Consider the submonoid M generated by 6, 9, and 20; i.e.,

$$M = \{6a + 9b + 20c : a, b, c \in \mathbb{N}\}.$$

The presentation of M which is most relevant for our purpose is a set partition of M into residue classes modulo t:

$$[M]_i := \{x \in M : x \equiv i \pmod{t}\}.$$

For example, choosing $t := 6$ we have that M is the disjoint union of $[M]_0, \dots, [M]_5$ where $[M]_0 = 0 + 6\mathbb{N} = \{0 + 6m : m \in \mathbb{N}\}$, and

$$[M]_1 = 49 + 6\mathbb{N}, [M]_2 = 20 + 6\mathbb{N}, [M]_3 = 9 + 6\mathbb{N}, [M]_4 = 40 + 6\mathbb{N}, [M]_5 = 29 + 6\mathbb{N}.$$

Example 4.2. How does one compute the elements 49, 20, etc.? For example, 49 is the smallest element of the form $1 + 6a$ that can be represented in the form $9b + 20c$, $a, b, c \in \mathbb{N}$. There are various tools to solve such linear Diophantine problems; e.g., the Omega package [1, 2] written in Mathematica:

```
In[1]:= << RISC`Omega.m`
```

Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © RISC-JKU

```
In[2]:= OEqR[OEqSum[x^a y^b z^c, -6a + 9b + 20c == 1, λ]]
```

Out[2]= $\frac{x^8 y z^2}{(1 - x^3 y^2)(1 - x^{10} z^3)}$

Out[2] gives the rational form of the generating function $\sum x^a y^b z^c$ of all non-negative integer triples (a, b, c) satisfying $-6a + 9b + 20c = 1$; $(8, 1, 2)$ corresponds to $9 \cdot 1 + 20 \cdot 2 = 1 + 6 \cdot 8 = 49$.

Next we consider what happens when to a given submonoid of $\mathbb{N}$ another generator is added[4].

Example 4.3. Consider the submonoid M^+ generated by 4, 6, 9, and 20; i.e.,

$$M = \{4a + 6b + 9c + 20d : a, b, c, d \in \mathbb{N}\} := \langle 4, 6, 9, 20 \rangle. \tag{4.2}$$

Note. Subsequently it will be convenient to use a short hand notation for the monoid which lists its generators as on the right side of (4.2).

Keeping the choice $t := 6$, we need to update the residue classes. Doing so, we obtain $M^+ = [M^+]_0 \cup \dots \cup [M^+]_5$ where $[M^+]_0 = 0 + 6\mathbb{N}$, and

$$[M^+]_1 = 13 + 6\mathbb{N}, [M^+]_2 = 8 + 6\mathbb{N}, [M^+]_3 = 9 + 6\mathbb{N}, [M^+]_4 = 4 + 6\mathbb{N}, [M^+]_5 = 17 + 6\mathbb{N}.$$

A simple but relevant observation is that for each j the smallest element in $[M^+]_j$ is less or equal to smallest element in $[M]_j$.

These elementary facts about monoids are used to compute the desired module presentations.

Example 4.4. As specified in the input/output description of Algorithm 4.1 we compute a module presentation of the subalgebra $\mathbb{Q}[t, f_1, f_2]$ of $\mathbb{Q}[z]$ where

```
In[3]:= t = z^6 - 1; f1 = z^9 + 2; f2 = z^20 + 1;
```

[4]An entertaining application of this situation is shown in the Numberphile video "How to order 43 Chicken McNuggets": www.youtube.com/watch?v=vNTSugyS038

Obviously the monoid $\langle 6, 9, 20\rangle$ generated by the degrees of t, f_1, and f_2 is a subset of the set of all possible degrees arising from the polynomials in $\mathbb{Q}[t, f_1, f_2]$. Consequently, in view of Example 4.1, it is a natural idea to choose as monoid generators $g_j \in \mathbb{Q}[t, f_1, f_2]$ such that

$$(\deg(g_1), \deg(g_2), \deg(g_3), \deg(g_4), \deg(g_5)) \equiv (1, 2, 3, 4, 5) \pmod 6. \quad (T1)$$

Note that here, as in Example 4.1, we decided to go modulo 6 which is the smallest degree of the given polynomials t, f_1, and f_2. For setting up presentations of monoids in general, this choice is free and can be adapted to the context.

To establish

$$\mathbb{Q}[t, f_1, f_2] = \mathbb{Q}[t] + \mathbb{Q}[t]\, g_1 + \cdots + \mathbb{Q}[t]\, g_5 \quad (4.3)$$

we need to show:

$$\mathbb{Q}[t] + \mathbb{Q}[t]\, g_1 + \cdots + \mathbb{Q}[t]\, g_5 \text{ is a subalgebra of } \mathbb{Q}[t, f_1, f_2]; \text{ and} \quad (T2a)$$

$$f_1, f_2 \in \mathbb{Q}[t] + \mathbb{Q}[t]\, g_1 + \cdots + \mathbb{Q}[t]\, g_5. \quad (T2b)$$

Task (T1). By Examples 4.1 and 4.2 we know that $49 = 1 \cdot 9 + 2 \cdot 20$ is the smallest element in the monoid $\langle 6, 9, 20\rangle$ which is congruent to 1 modulo 6. This suggests to take

In[4]:= $\mathbf{g_1 = f_1 f_2^2}$.

With the same reasoning we choose also the remaining elements:

In[5]:= $\mathbf{g_2 = f_2;\ g_3 = f_1;\ g_4 = f_2^2;\ g_5 = f_1 f_2;}$

This choice satisfies *(T1)*:

$$\begin{aligned}(\deg(g_1), \deg(g_2), \deg(g_3), \deg(g_4), \deg(g_5)) &= (49, 20, 9, 40, 29)\\ &\equiv (1, 2, 3, 4, 5) \pmod 6,\end{aligned}$$

and settles also *Task (T2b)*.

In view of In[4] and In[5] *Task (T2a)* amounts to show

$$g_i g_j \in \mathbb{Q}[t] + \mathbb{Q}[t]\, g_1 + \cdots + \mathbb{Q}[t]\, g_5 \text{ for all } i, j \in \{1, \ldots, 5\}.$$

This is checked computationally. For example,

In[6]:= $\mathbf{\{g_3^2, g_3^2 - t^3\}//Expand}$

Out[6]= $\{4 + 4z^9 + z^{18}, 5 - 3z^6 + 4z^9 + 3z^{12}\}$

In[7]:= $\mathbf{\{g_3^2 - t^3, g_3^2 - t^3 - 3t^2\}//Expand}$

```
Out[7]= {5 − 3z^6 + 4z^9 + 3z^12, 2 + 3z^6 + 4z^9}
In[8]:= {g3^2 − t^3 − 3t^2, g3^2 − t^3 − 3t^2 − 4g3}//Expand
Out[8]= {2 + 3z^6 + 4z^9, −6 + 3z^6}
In[9]:= {g3^2 − t^3 − 3t^2 − 4g3, g3^2 − t^3 − 3t^2 − 4g3 − 3(t − 1)}//Expand
Out[9]= {−6 + 3z^6, 0}
```

In other words, we obtained

$$g_3^2 = t^3 + 3t^2 + 3(t-1) + 4g_3 \in \mathbb{Q}[t] + \mathbb{Q}[t]\, g_1 + \cdots + \mathbb{Q}[t]\, g_5.$$

Such reductions work for all $g_i g_j$. To give another example,

$$g_2 g_4 = p - 3g_2 + 3g_4$$

for

$$p = t^{10} + 10t^9 + 45t^8 + 120t^7 + 210t^6 + 252t^5 + 210t^4 + 120t^3 + 45t^2 + 10t + 2.$$

In contrast to Example 4.4, in general it is not true that

$$\{\deg g(z) : g(z) \in \mathbb{K}[t, f_1, \ldots, f_n]\} = \langle \deg t(z), \deg f_1(z), \ldots, \deg f_n(z)\rangle;$$

see the next example.

Example 4.5. Consider the subalgebra $\mathbb{Q}[t, f_1, f_2, f_3]$ of $\mathbb{Q}[z]$ where

```
In[10]:= t = z^6 − 1; f1 = z^9 + 2; f2 = z^20 + 1; f3 = z^18 + z^4;
```

Observe that

$$\langle \deg t(z), \deg f_1(z), \deg f_2(z), \deg f_3(z)\rangle = \langle 6, 9, 20, 18\rangle = \langle 6, 9, 20\rangle$$
$$\neq \{\deg h(z) : g(z) \in \mathbb{Q}[t, f_1, f_2, f_3]\};$$

for instance,

$$h(z) := f_3 - (t^3 + 3t^2 + 3t) = z^4 + 1 \in \mathbb{Q}[t, f_1, f_2, f_3] \tag{4.4}$$

and therefore, $\deg(h) = \deg(z^4 + 1) = 4 \neq \langle 6, 9, 20\rangle$. Consequently, to obtain a general algorithm as specified in Algorithm 4.1, we need to modify the procedure from Example 4.4 as follows: we update the given data by considering $\mathbb{Q}[t, f_1, f_2, f_3, f_4]$ instead of $\mathbb{Q}[t, f_1, f_2, f_3]$ by adding explicitly the "new" element from (4.4):

```
In[11]:= f4 = z^4 + 1;
```

Recall Example 4.3 where the element 4 was added to the monoid $\langle 6, 9, 20\rangle$; this had the effect that for the resulting monoid $\langle 4, 6, 9, 20\rangle$ the smallest elements in the representing residue classes modulo 6 changed from (49, 20, 9, 40, 29) to (13, 8, 9, 4, 17). For termination reasons of the algorithm it is important to note that by adding a new element to the monoid the new smallest elements are less or equal than their predecessors in the respective residue classes.

To obtain a module representation for $\mathbb{Q}[t, f_1, f_2, f_3, f_4]$ we utilize this observation when updating the generators g_j accordingly. This means, we now choose $G_j \in \mathbb{Q}[t, f_1, f_2, f_3, f_4]$ such that

```
In[12]:=  G1 = f1f4; G2 = f4^2; G3 = f1; G4 := f4; G5 = f1f4^2;
```

This choice satisfies condition *(T1)* from above:

$$(\deg(G_1), \deg(G_2), \deg(G_3), \deg(G_4), \deg(G_5)) = (13, 8, 9, 4, 17)$$
$$\equiv (1, 2, 3, 4, 5) \pmod 6.$$

Because of

```
In[13]:=  f2 − G4^5 + 5G2^2 − 10G4^3 + 10G2 − 5G4//Expand

Out[13]=  0
```

and (4.4) it also satisfies condition *(T2b)*. Again a computational check verifies condition *(T2a)*; for example:

```
In[14]:=  G1^2 − (t^3 + 3t^2 + 3t − 3)G2 − 4G5//Expand

Out[14]=  0
```

From Example 4.5 we can obtain a complete picture of the Algorithm 4.1, namely: Out of the subalgebra generators $f_0, \dots, f_n \in \mathbb{K}[z]$ we choose a non-constant element $t := f_0$ which fixes the modulus $\deg(t)$ for all the steps of the algorithm. It also determines the module structure

$$\mathbb{K}[t] + \mathbb{K}[t]\, g_1 + \cdots + \mathbb{K}[t]\, g_k$$

for the first step of the algorithm, where $k = \deg(t) - 1$ is the number of non-constant module generators $g_1, \dots, g_k \in \mathbb{K}[z]$. Whenever it happens, as in Example 4.5, that during the module-reduction with respect to $g_1, \dots, g_k$ multiplied by powers of t an element $h \in \mathbb{K}[t, f_1, \dots, f_n]$ arises with

$$\deg(h) \neq \langle \deg(t), \deg(f_1), \dots, \deg(f_n)\rangle,$$

then we update to new generators $G_1, \dots, G_k$ as in Example 4.5. Since in each such update-step the degrees of the G_j are less or equal to those of the corresponding g_j (at least one degree has to be smaller in case of an update!), the algorithm terminates after a finite number of steps.

5 Algorithm "MODULE GENERATORS" for Modular Functions

Algorithm 4.1 carries over from polynomials to modular functions by the linear embedding φ defined in (3.4). We present the version given in [9].

ALGORITHM 5.1 (*"***MODULE GENERATORS: modular function case***"*)**.** *Given non-constant modular functions* $t := f_0, f_1, \dots, f_n \in M^{\infty}(N)$ *with* $m := -\mathrm{ord}_{\infty}(\mathrm{t})$ *and*

$$\gcd\left(\mathrm{ord}_{\infty}(\mathrm{t}), \mathrm{ord}_{\infty}(\mathrm{f_2}), \dots, \mathrm{ord}_{\infty}(\mathrm{f_n})\right) = 1,$$

the algorithm computes $g_1, \dots, g_{m-1} \in M^{\infty}(N)$ *such that*

$$\mathbb{C}[t, f_1, \dots, f_n] = \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \dots + \mathbb{C}[t]\, g_{m-1}. \tag{5.1}$$

The gcd-condition and also the steps of the algorithm are explained in detail in [9]. In fact, exchanging the polynomial degrees with negative orders, the algorithm works completely analogous to the case of polynomials.

As an illustration we sketch the derivation of our 11-witness identity (1.7).

Example 5.1. Consider $\mathbb{C}[t, f]$ with $t, f \in M^{\infty}(N)$ as in (1.5) and (1.6) of our main Theorem (1.1). We have $m = -\mathrm{ord}_{\infty}(t) = 5$, hence we expect $m - 1 = 4$ module generators in addition to the constant function 1. Observing that

$$\begin{aligned}(-\deg(f), -\deg(f^2), -\deg(f^3), -\deg(f^4)) &= (4, 8, 12, 16)\\ &\equiv (4, 3, 2, 1) \pmod 5,\end{aligned}$$

we choose

$$(g_1, g_2, g_3, g_4) := (f, f^2, f^3, f^4).$$

This matches condition *(T1)* above; condition *(T2b)* is trivially satisfied. Owing to the fact that $g_i g_j = f^{i+j}$, to verify condition *(T2a)* reduces to showing that

$$f^5 \in \mathbb{C}[t] + \mathbb{C}[t]\, f + \mathbb{C}[t]\, f^2 + \mathbb{C}[t]\, f^3 + \mathbb{C}[t]\, f^4.$$

But this is a *straightforward* computational exercise. In fact, anyone being familiar with Example 4.4 can easily accomplish this task; in other words, can easily "discover" herself/himself the witness identity (1.7) just by applying the reduction process to f^5 with respect to the given t and f.

We restrict to present only the first steps of this computational "discovery" of (1.7).

In[15]:= $\mathbf{Tquot[k\_] := \dfrac{1 - q^k}{1 - q^{11k}};}$

In[16]:= $\mathbf{t = \dfrac{1}{q^5}Product[Series[Tquot[k]^{12}, q, 0, 26], k, 1, 26]}$

Out[16]= $\frac{1}{q^5} - \frac{12}{q^4} + \frac{54}{q^3} - \frac{88}{q^2} - \frac{99}{q} + 540 - 418q - 648q^2 + \cdots - 22176q^{20} + 61656q^{21} + O[q]^{22}$

In[17]:= **f = qtSeries[(1 − q^11)(1 − q^22), q, 0, 21] ∗ Sum[PartitionsP[11n + 6]q^n, n, 0, 21]**

Out[17]= $\frac{11}{q^4} + \frac{165}{q^3} + \frac{748}{q^2} + \frac{1639}{q} + 3553 + 4136q + 6347q^2 + \cdots + 12738q^{16} - 51216q^{17} + O[q]^{18}$

In[18]:= $\mathbf{F = \frac{f}{11};}$

In[19]:= $\mathbf{F^5}$

Out[19]= $\frac{1}{q^{20}} + \frac{75}{q^{19}} + \frac{2590}{q^{18}} + \ldots + \frac{298958660282220}{q} + 530018316923711 + 877706745683995q + O[q]^2$

In[20]:= $\mathbf{F^5 - t^4}$

Out[20]= $\frac{123}{q^{19}} + \frac{1510}{q^{18}} + \frac{69935}{q^{17}} + \cdots + 530018316923711 + 877706745683995q + O[q]^2$

In[21]:= $\mathbf{F^5 - t^4 - 3*41*t^3F}$

Out[21]= $\frac{4093}{q^{18}} + \frac{54929}{q^{17}} + \frac{570947}{q^{16}} + \cdots + 530565611750339 + 877363195058527q + O[q]^2$

In[22]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2}$

Out[22]= $\frac{30371}{q^{17}} + \frac{1008898}{q^{16}} + \frac{12509585}{q^{15}} + \cdots + 536556550241327 + 873666097417069q + O[q]^2$

In[23]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3}$

Out[23]= $\frac{6655}{q^{16}} + \frac{573782}{q^{15}} - \frac{16074850}{q^{14}} + \cdots + 552225581222579 + 867953372178310q + O[q]^2$

In[24]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3 - 11^3*5*F^4}$

Out[24]= $\frac{174482}{q^{15}} - \frac{26869260}{q^{14}} + \frac{438710910}{q^{13}} + \cdots - 2294272165605596 - 3831090632203670q + O[q]^2$

In[25]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3 - 11^3*5*F^4 -}$
$\mathbf{11^2*2*7*103*t^3}$

Out[25]= $\frac{20587908}{q^{14}} + \frac{335068602}{q^{13}} + \frac{3501794450}{q^{12}} - \cdots - 3781753922516174q + O[q]^2$

In[26]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3 - 11^3*5*F^4 -}$
$\mathbf{11^2*2*7*103*t^3 + 11^3*2^2*3*1289*t^2F}$

Out[26]= $\frac{149777430}{q^{13}} + \frac{2678278130}{q^{12}} + \frac{7440556200}{q^{11}} - \cdots - 3884781928756850q + O[q]^2$

In[27]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3 - 11^3*5*F^4 -}$
$\mathbf{11^2*2*7*103*t^3 + 11^3*2^2*3*1289*t^2F - 11^5*2*3*5*31*tF^2}$

Out[27]= $-\frac{17715610}{q^{12}} - \frac{797202450}{q^{11}} - \frac{13641019700}{q^{10}} - \cdots - 3863023611723320q + O[q]^2$

In[28]:= $\mathbf{F^5 - t^4 - 3*41t^3F - 4093t^2F^2 - 11^2*251*tF^3 - 11^3*5*F^4 -}$
$\mathbf{11^2*2*7*103*t^3 + 11^3*2^2*3*1289t^2F - 11^5*2*3*5*31*tF^2 +}$
$\mathbf{11^6*2*5*F^3}$

Out[28]= $\frac{1931001490}{q^{10}} - \frac{9903025990}{q^{9}} + \frac{619514881700}{q^{8}} - \cdots + 6359340881620540q + O[q]^2$

Some remarks are in place. As input for t and f we take their truncated q-series expansions with one more term than needed to decide equality; i.e., up to $O(q^2)$. In order to keep coefficient growth within bounds we work with $F := f/11$ instead of f. Starting with the reduction of F^5 each reduction step works with respect to a uniquely determined power product $t^a F^b$. The last reduction displayed shows a jump from order -12 to order -10. By looking at the monoid $\langle 5, 4, 8, 12, 16\rangle$ this can be explained by the fact that 11 is the largest integer not contained in this monoid[5]. The reduction process stops when the witness identity (1.7) is fully revealed.

6 Conclusion

The algorithm we applied to derive the witness identity (1.7) is a powerful tool also in much more general situations when dealing with q-series identities in the context of modular functions. For example, it plays an essential role in Radu's algorithmic approach to Ramanujan-Kolberg type identities [10]. There, in order to produce generators of the multiplicative monoid of eta quotients in $M^{\infty}(N)$, a Diophantine system of linear equations, resp. inequalities, has to be solved over the integers, resp. non-negative integers, depending on the setting. In the non-negative integer setting this translates to the problem of determining (interior) lattice points in certain cones. This aspect plays a fundamental role in the extensive monograph by Köhler [8]; namely, to determine whether eta quotients, being modular forms of weight k, are holomorphic at their cusps. Apart from the fact that $k = 0$ in our context, Radu's machinery solves the linear Diophantine problem as a preprocessing step to subalgebra reduction; i.e., to finding module presentations.

In [10], among other things, the Ramanujan-Kolberg algorithm was used to derive a witness identity for $11|p(11n+6)$ of completely different character than (1.7); namely, where $\sum_{n\geq 0} p(11n+6)q^n$ is expressed as a $\mathbb{Q}$-linear combination of eta-quotients. In order to conclude $11|p(11n+6)$ from this presentation, additional "massage" like "freshman's dream relations" is needed. However, Ralf Hemmecke [6] obtained an identity which presents $\sum_{n\geq 0} p(11n+6)q^n$ as a essentially *integer*-linear combination which in direct fashion shows the divisibility by 11. This was done by generalizing the algorithm for deciding membership in a $\mathbb{Q}$-subalgebra of $\mathbb{Q}[z]$ to an algorithm that decides membership in a $\mathbb{Z}$-subalgebra of $\mathbb{Z}[z]$. Such kind of results indicate additional potential for using variants of this algorithm to obtain suitable identities which witness divisibility.

Acknowledgements The authors thank the anonymous referee for carefully reading the manuscript and for the suggestions to improve it. Special thanks go to Frank Garvan for very careful editing.

---

[5]Such a number is called Frobenius number. Note that 1, 2, 3, 6, and 7 are the other integers not contained in the monoid.

References

1. G.E. Andrews, P. Paule, A. Riese, MacMahon's partition analysis III: the omega package. Euro. J. Comb. **22**, 887–904 (2001)
2. G.E. Andrews, P. Paule, A. Riese, MacMahon's partition analysis VI: a new reduction algorithm. Ann. Comb. **5**, 251–270 (2001)
3. D.S. Dummit, R.M. Foote, *Abstract Algebra*, 3rd edn. (Wiley, New York, 2005)
4. D. Gale, Subalgebras of an algebra with a single generator are finitely generated. Proc. Am. Math. Soc. **8**, 929–930 (1957)
5. G. Hardy, P. Aiyar, B. Wilson (eds.), *Collected Papers of Srinivasa Ramanujan* (Cambridge University Press, Cambridge, 1927). Reprint: AMS Chelsea Publishing (2000)
6. R. Hemmecke Dancing Samba with Ramanujan Partition Congruences. preprint, 14 pages (2016). (To appear in: J. Symb. Comput.)
7. D. Kapur, K. Madlener, *A Completion Procedure for Computing a Canonical Basis for a k-Subalgebra*, Computers and Mathematics (Springer, Cambridge, MA, 1989), pp. 1–11
8. G. Köhler, *Eta Products and Theta Series Identities* (Springer, Berlin, 2011)
9. P. Paule, C.S. Radu, Partition analysis, modular functions, and computer algebra, *Recent Trends in Combinatorics*, The IMA Volumes in Mathematics (Springer, Berlin, 2016), pp. 511–544
10. C.S. Radu, An algorithmic approach to Ramanujan-Kolberg identities. J. Symb. Comput. **68**, 1–33 (2014)
11. S. Ramanujan, Some properties of $p(n)$, the number of partitions of n. Proc. Camb. Philos. Soc. **19**, 207–210 (1919)
12. L. Robbiano, M. Sweedler Subalgebra bases, in *Commutative Algebra (Salvador 1988)*. Lecture Notes in Mathematics, vol. 1430. (Springer, Berlin, 1990), pp. 61–87
13. A. Torstensson, Canonical bases for subalgebras on two generators in the univariate polynomial ring. Beiträge Algebra Geom. **43**(2), 565–577 (2002)

On the Representations of a Positive Integer by Certain Classes of Quadratic Forms in Eight Variables

B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh

Dedicated to Professor Krishnaswami Alladi on the occasion of his 60th birthday

Abstract In this paper we use the theory of modular forms to find formulas for the number of representations of a positive integer by a certain class of quadratic forms in eight variables, viz., forms of the form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + b_1(x_5^2 + x_5x_6 + x_6^2) + b_2(x_7^2 + x_7x_8 + x_8^2)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, $b_1 \leq b_2$ and a_i's $\in \{1, 2, 3\}$, b_i's $\in \{1, 2, 4\}$. We also determine formulas for the number of representations of a positive integer by the quadratic forms $(x_1^2 + x_1x_2 + x_2^2) + c_1(x_3^2 + x_3x_4 + x_4^2) + c_2(x_5^2 + x_5x_6 + x_6^2) + c_3(x_7^2 + x_7x_8 + x_8^2)$, where $c_1, c_2, c_3 \in \{1, 2, 4, 8\}$, $c_1 \leq c_2 \leq c_3$.

Keywords quadratic forms in eight variables · representation numbers of quadratic forms · modular forms of one variable

2010 Mathematics Subject Classification Primary 11F25 · Secondary 11A25 11F1

Brundaban Sahu is partially funded by SERB grant SR/FTP/MS-053/2012.

B. Ramakrishnan (✉) · A. K. Singh
Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi 211 019, Allahabad, India
e-mail: ramki@hri.res.in

A. K. Singh
e-mail: anupsingh@hri.res.in

B. Sahu
School of Mathematical Sciences, National Institute of Science Education and Research, HBNI, Via-Jatni, Bhubaneswar, Khurda 752 050, Odisha, India
e-mail: brundaban.sahu@niser.ac.in

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_35

1 Introduction

In this paper, we consider the problem of finding the number of representations of the following quadratic forms in eight variables given by

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + b_1(x_5^2 + x_5x_6 + x_6^2) + b_2(x_7^2 + x_7x_8 + x_8^2), \quad (1)$$

where the coefficients $a_i \in \{1, 2, 3\}, 1 \leq i \leq 4$ and $b_1, b_2 \in \{1, 2, 4\}$. Without loss of generality, we can assume that $a_1 \leq a_2 \leq a_3 \leq a_4$ and $b_1 \leq b_2$. In [3], A. Alaca et al. considered similar types of quadratic forms in four variables, which are either sums of four squares with coefficients 1, 2, 3, 4, or 6 (7 such forms) or direct sum of the sums of two squares with coefficients 1 or 3 and the quadratic form $x^2 + xy + y^2$ with coefficients 1, 2, or 4 (6 such forms). They used theta function identities to determine the representation formulas for these 13 quadratic forms. In our recent work [17], we constructed bases for the space of modular forms of weight 4 for the group $\Gamma_0(48)$ with character, and used modular forms techniques to determine the number of representations of a natural number n by certain octonary quadratic forms with coefficients 1, 2, 3, 4, 6. Finding formulas for the number of representations for octonary quadratic forms with coefficients 1, 2, 3, or 6 were considered by various authors using several methods (see, for example [1, 2, 4–7]). In the present work, we adopt similar (modular forms) techniques to obtain the representation formulas. We show directly that the theta series corresponding to each of the quadratic form considered belongs to the space of modular forms of weight 4 on $\Gamma_0(24)$ with some character (depending on the coefficients). Now, by constructing a basis for the space of modular forms $M_4(\Gamma_0(24), \chi)$ we find the required formulas. Here χ is either the trivial Dirichlet character modulo 24 or one of the primitive Dirichlet characters (modulo m) $\chi_m = \left(\frac{m}{\cdot}\right), m = 8, 12, 24$. Since $M_4(\Gamma_0(24), \chi) \subseteq M_4(\Gamma_0(48), \chi)$, where χ is a Dirichlet character modulo 24, we get the required explicit bases from the basis of modular forms $M_4(\Gamma_0(48), \psi)$, where ψ is a Dirichlet character modulo 48, which was constructed in [17].

In the second part of the paper, we consider the quadratic forms of eight variables given by:

$$(x_1^2 + x_1x_2 + x_2^2) + c_1(x_3^2 + x_3x_4 + x_4^2) + c_2(x_5^2 + x_5x_6 + x_6^2) + c_3(x_7^2 + x_7x_8 + x_8^2), \quad (2)$$

where $c_1 \leq c_2 \leq c_3$ and $c_1, c_2, c_3 \in \{1, 2, 4, 8\}$. We note that for the c_i's in the list, each of the quadratic form represents a theta series which belong to the space $M_4(\Gamma_0(24))$. Therefore, using our methods adopted for the earlier case, we also determine explicit formulas for the number of representations of a natural number by these class of quadratic forms.

The total number of such quadratic forms given by (1) with coefficients $a_i \in \{1, 2, 3\}$ and $b_i \in \{1, 2, 4\}$ is 90. Each quadratic form in this list is denoted as a sextuple $(a_1, a_2, a_3, a_4, b_1, b_2)$ and we list them in Table 1. We also put them in four classes corresponding to each of the modular forms space $M_4(\Gamma_0(24), \chi)$. Similarly, we list the quadratic forms (total 19) given by (2) in Table 2. In this case, all the corresponding

Table 1 List of quadratic forms in 8 variables given in (1)

$(a_1, a_2, a_3, a_4, b_1, b_2)$	space
(1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 2), (1, 1, 1, 1, 1, 4), (1, 1, 1, 1, 2, 2), (1, 1, 1, 1, 2, 4), (1, 1, 1, 1, 4, 4), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 1, 2), (1, 1, 2, 2, 1, 4), (1, 1, 2, 2, 2, 2), (1, 1, 2, 2, 2, 4), (1, 1, 2, 2, 4, 4), (1, 1, 3, 3, 1, 1), (1, 1, 3, 3, 1, 2), (1, 1, 3, 3, 1, 4), (1, 1, 3, 3, 2, 2), (1, 1, 3, 3, 2, 4), (1, 1, 3, 3, 4, 4), (2, 2, 2, 2, 1, 1), (2, 2, 2, 2, 1, 2), (2, 2, 2, 2, 1, 4), (2, 2, 2, 2, 2, 2), (2, 2, 2, 2, 2, 4), (2, 2, 2, 2, 4, 4), (2, 2, 3, 3, 1, 1), (2, 2, 3, 3, 1, 2), (2, 2, 3, 3, 1, 4), (2, 2, 3, 3, 2, 2), (2, 2, 3, 3, 2, 4), (2, 2, 3, 3, 4, 4), (3, 3, 3, 3, 1, 1), (3, 3, 3, 3, 1, 2), (3, 3, 3, 3, 1, 4), (3, 3, 3, 3, 2, 2), (3, 3, 3, 3, 2, 4), (3, 3, 3, 3, 4, 4)	$M_4(\Gamma_0(24))$
(1, 1, 1, 2, 1, 1), (1, 1, 1, 2, 1, 2), (1, 1, 1, 2, 1, 4), (1, 1, 1, 2, 2, 2), (1, 1, 1, 2, 2, 4), (1, 1, 1, 2, 4, 4), (1, 2, 2, 2, 1, 1), (1, 2, 2, 2, 1, 2), (1, 2, 2, 2, 1, 4), (1, 2, 2, 2, 2, 2), (1, 2, 2, 2, 2, 4), (1, 2, 2, 2, 4, 4), (1, 2, 3, 3, 1, 1), (1, 2, 3, 3, 1, 2), (1, 2, 3, 3, 1, 4), (1, 2, 3, 3, 2, 2), (1, 2, 3, 3, 2, 4), (1, 2, 3, 3, 4, 4)	$M_4(\Gamma_0(24), \chi_8)$
(1, 1, 1, 3, 1, 1), (1, 1, 1, 3, 1, 2), (1, 1, 1, 3, 1, 4), (1, 1, 1, 3, 2, 2), (1, 1, 1, 3, 2, 4), (1, 1, 1, 3, 4, 4), (1, 2, 2, 2, 1, 1), (1, 2, 2, 3, 1, 2), (1, 2, 2, 3, 1, 4), (1, 2, 2, 3, 2, 2), (1, 2, 2, 3, 2, 4), (1, 2, 2, 3, 4, 4), (1, 3, 3, 3, 1, 1), (1, 3, 3, 3, 1, 2), (1, 3, 3, 3, 1, 4), (1, 3, 3, 3, 2, 2), (1, 3, 3, 3, 2, 4), (1, 3, 3, 3, 4, 4)	$M_4(\Gamma_0(24), \chi_{12})$
(1, 1, 2, 3, 1, 1), (1, 1, 2, 3, 1, 2), (1, 1, 2, 3, 1, 4), (1, 1, 2, 3, 2, 2), (1, 1, 2, 3, 2, 4), (1, 1, 2, 3, 4, 4), (2, 2, 2, 3, 1, 1), (2, 2, 2, 3, 1, 2), (2, 2, 2, 3, 1, 4), (2, 2, 2, 3, 2, 2), (2, 2, 2, 3, 2, 4), (2, 2, 2, 3, 4, 4), (2, 3, 3, 3, 1, 1), (2, 3, 3, 3, 1, 2), (2, 3, 3, 3, 1, 4), (2, 3, 3, 3, 2, 2), (2, 3, 3, 3, 2, 4), (2, 3, 3, 3, 4, 4)	$M_4(\Gamma_0(24), \chi_{24})$

theta series belong to $M_4(\Gamma_0(24))$. Among the cases in Table 1, the following 18 cases (for $i, j \in \{1, 2, 4\}$, the cases (t, t, t, t, i, j), $t = 1, 3$ and $(1, 1, 3, 3, i, j)$) were considered in [11]. The cases (1, 1, 1, 1, 1, 1) and (1, 1, 1, 1, 1, 2) were also considered in [22]. The methods used in their works is different from our method and the corresponding formulas are also different from ours (they differ in the cusp form parts). In Table 2, we do not consider the case (1, 1, 1, 1) as the formula is already known (see [21, Theorem 17.4]). It was shown that $s_8(n) = 24\sigma_3(n) + 216\sigma_3(n/3)$. In our notation (see §3) $s_8(n) = M(1, 1, 1, 1; n)$. Also, the cases (1, 2, 2, 4) and (1, 2, 4, 8) have been proved in [10] by using convolution sums method. The same method was used in [11] to get the cases (1, 1, 1, 2), (1, 1, 1, 4), (1, 1, 4, 8), (1, 2, 2, 2), (1, 2, 4, 4), and (1, 2, 8, 8). The case (1, 1, 1, 2) was also considered in [22].

The paper is organized as follows. In §2, we present the theorems proved in this article and in §3 we give some preliminary results which are needed in proving the theorems. In §4, we give a proof of our theorems using the theory of modular forms.

Table 2 List of quadratic forms in (2) indicated by $(1, c_1, c_2, c_3)$.

$(1, c_1, c_2, c_3)$	space
(1, 1, 1, 2), (1, 1, 1, 4), (1, 1, 1, 8), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 2, 8), (1, 1, 4, 4), (1, 1, 4, 8), (1, 1, 8, 8), (1, 2, 2, 2), (1, 2, 2, 4), (1, 2, 2, 8), (1, 2, 4, 4), (1, 2, 4, 8), (1, 2, 8, 8), (1, 4, 4, 4), (1, 4, 4, 8), (1, 4, 8, 8), (1, 8, 8, 8)	$M_4(\Gamma_0(24))$

2 Statement of results

Let $\mathbf{N}$, $\mathbf{N}_0$ and $\mathbf{Z}$ denote the set of positive integers, nonnegative integers and integers, respectively. For $(a_1, a_2, a_3, a_4, b_1, b_2)$ as in Table 1, we define

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) := \\ \# \left\{ (x_1, \ldots, x_8) \in \mathbf{Z}^8 \middle| \, n = \sum_{i=1}^{4} a_i x_i^2 + b_1(x_5^2 + x_5x_6 + x_6^2) + b_2(x_7^2 + x_7x_8 + x_8^2) \right\}.$$

to be the number of representations of n by the quadratic form (1). Note that $N(a_1, a_2, a_3, a_4, b_1, b_2; 0) = 1$. The formulas corresponding to Table 1 are stated in the following theorem. Formulas are divided into four parts each corresponding to one of the four spaces of modular forms $M_4(\Gamma_0(24), \chi)$.

Theorem 2.1 *Let $n \in \mathbf{N}$.*
(i) *For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1* corresponding to the space $M_4(\Gamma_0(24))$, we have

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \alpha_i A_i(n), \tag{3}$$

where $A_i(n)$ are the Fourier coefficients of the basis elements f_i defined in §4.1 and the values of the constants α_is are given in Table 3.
(ii) For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_8)$, we have

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \beta_i B_i(n), \tag{4}$$

where $B_i(n)$ are the Fourier coefficients of the basis elements g_i defined in §4.2 and the values of the constants β_i's are given in Table 4.
(iii) For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_{12})$, we have

Table 3 (Theorem 2.1 (i))

$(a_1a_2a_3a_4, b_1b_2)$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
(1111, 11)	$\frac{7}{75}$	$\frac{-7}{100}$	$\frac{-9}{25}$	$\frac{-28}{75}$	$\frac{27}{100}$	0	$\frac{36}{25}$	0	$\frac{-72}{5}$	$\frac{-288}{5}$	0	0	0	12	0	0
(1111, 12)	$\frac{13}{300}$	$\frac{-13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$\frac{-27}{200}$	0	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0	0
(1111, 14)	$\frac{7}{300}$	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	0	$\frac{36}{25}$	0	$\frac{-18}{5}$	$\frac{72}{5}$	0	0	0	12	0	0
(1111, 22)	$\frac{7}{300}$	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$\frac{-48}{5}$	0	0	0	0	0	0
(1111, 24)	$\frac{13}{1200}$	$\frac{-13}{400}$	$\frac{9}{400}$	$\frac{26}{75}$	$\frac{-27}{400}$	0	$\frac{18}{25}$	0	$\frac{12}{5}$	$\frac{96}{5}$	0	0	0	3	0	0
(1111, 44)	$\frac{7}{1200}$	$\frac{7}{400}$	$\frac{-9}{400}$	$\frac{-28}{75}$	$\frac{-27}{400}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	3	0	0
(1122, 11)	$\frac{7}{150}$	$\frac{-7}{150}$	$\frac{-9}{50}$	$\frac{7}{300}$	$\frac{9}{50}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{-36}{5}$	-48	$\frac{-768}{5}$	-3	-81	6	36	9
(1122, 12)	$\frac{13}{600}$	$\frac{-13}{600}$	$\frac{9}{200}$	$\frac{-13}{600}$	$\frac{-9}{200}$	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$	$\frac{24}{5}$	12	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	-3	-6	$\frac{-9}{2}$
(1122, 14)	$\frac{7}{600}$	$\frac{-7}{600}$	$\frac{-9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{-9}{5}$	6	$\frac{-48}{5}$	$\frac{-3}{2}$	$\frac{-81}{2}$	6	0	$\frac{9}{2}$
(1122, 22)	$\frac{7}{600}$	$\frac{-7}{600}$	$\frac{-9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{6}{5}$	-12	$\frac{-288}{5}$	0	0	0	12	0
(1122, 24)	$\frac{13}{2400}$	$\frac{-13}{2400}$	$\frac{9}{800}$	$\frac{-13}{600}$	$\frac{-9}{800}$	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$	$\frac{6}{5}$	12	$\frac{96}{5}$	0	0	$\frac{3}{2}$	-6	0
(1122, 44)	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	$\frac{7}{300}$	$\frac{9}{800}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	6	$\frac{-48}{5}$	0	0	$\frac{3}{2}$	0	0
(1133, 11)	$\frac{2}{75}$	$\frac{-1}{30}$	$\frac{6}{25}$	$\frac{8}{75}$	$\frac{-3}{10}$	0	$\frac{24}{25}$	0	$\frac{48}{5}$	$\frac{288}{5}$	0	0	0	0	0	0
(1133, 12)	$\frac{1}{60}$	$\frac{-1}{120}$	$\frac{-3}{20}$	$\frac{-2}{15}$	$\frac{3}{40}$	0	$\frac{6}{5}$	0	0	0	0	0	0	6	0	0
(1133, 14)	$\frac{1}{150}$	$\frac{-1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$\frac{-3}{25}$	0	$\frac{24}{25}$	0	$\frac{42}{5}$	$\frac{168}{5}$	0	0	0	0	0	0
(1133, 22)	$\frac{1}{150}$	$\frac{-1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$\frac{-3}{25}$	0	$\frac{24}{25}$	0	$\frac{12}{5}$	$\frac{48}{5}$	0	0	0	0	0	0
(1133, 24)	$\frac{1}{240}$	$\frac{1}{240}$	$\frac{-3}{80}$	$\frac{-2}{15}$	$\frac{-3}{80}$	0	$\frac{6}{5}$	0	0	0	0	0	0	3	0	0
(1133, 44)	$\frac{1}{600}$	$\frac{-1}{120}$	$\frac{3}{200}$	$\frac{8}{75}$	$\frac{-3}{40}$	0	$\frac{24}{25}$	0	$\frac{18}{5}$	$\frac{48}{5}$	0	0	0	0	0	0

(continued)

Table 3 (continued)

$(a_1a_2a_3a_4, b_1b_2)$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
(2222, 11)	$\frac{7}{400}$	$\frac{91}{1200}$	$\frac{-27}{400}$	$\frac{-7}{100}$	$\frac{-117}{400}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	$\frac{-36}{5}$	$\frac{-312}{5}$	$\frac{-768}{5}$	-3	-81	9	36	9
(2222, 12)	$\frac{13}{800}$	$\frac{-247}{2400}$	$\frac{27}{800}$	$\frac{13}{200}$	$\frac{-171}{800}$	$\frac{26}{75}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{84}{5}$	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	$\frac{-9}{2}$	-6	$\frac{-9}{2}$
(2222, 14)	$\frac{7}{800}$	$\frac{-133}{2400}$	$\frac{-27}{800}$	$\frac{7}{100}$	$\frac{171}{800}$	$\frac{-28}{75}$	$\frac{-27}{100}$	$\frac{36}{25}$	$\frac{-18}{5}$	$\frac{-24}{5}$	$\frac{-48}{5}$	$\frac{-3}{2}$	$\frac{-81}{2}$	$\frac{9}{2}$	0	$\frac{9}{2}$
(2222, 22)	0	$\frac{7}{75}$	0	$\frac{-7}{100}$	$\frac{-9}{25}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	0	$\frac{-72}{5}$	$\frac{-288}{5}$	0	0	0	12	0
(2222, 24)	0	$\frac{13}{300}$	0	$\frac{-13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$\frac{-27}{200}$	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0
(2222, 44)	0	$\frac{7}{300}$	0	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$\frac{-48}{5}$	0	0	0	0	0
(2233, 11)	$\frac{1}{75}$	$\frac{-1}{75}$	$\frac{3}{25}$	$\frac{-1}{150}$	$\frac{-3}{25}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{24}{5}$	48	$\frac{768}{5}$	10	18	0	-24	-6
(2233, 12)	$\frac{1}{120}$	$\frac{-1}{120}$	$\frac{-3}{40}$	$\frac{1}{120}$	$\frac{3}{40}$	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$	0	-12	-96	$\frac{-7}{2}$	$\frac{-9}{2}$	3	6	$\frac{9}{2}$
(2233, 14)	$\frac{1}{300}$	$\frac{-1}{300}$	$\frac{3}{100}$	$\frac{-1}{150}$	$\frac{-3}{100}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{21}{5}$	18	$\frac{48}{5}$	$\frac{5}{2}$	$\frac{63}{2}$	0	-12	$\frac{-3}{2}$
(2233, 22)	$\frac{1}{300}$	$\frac{-1}{300}$	$\frac{3}{100}$	$\frac{-1}{150}$	$\frac{-3}{100}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{6}{5}$	12	$\frac{288}{5}$	1	-9	0	0	-3
(2233, 24)	$\frac{1}{480}$	$\frac{-1}{480}$	$\frac{-3}{160}$	$\frac{1}{120}$	$\frac{3}{160}$	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$	0	0	0	-2	-18	$\frac{3}{2}$	6	0
(2233, 44)	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	$\frac{-1}{150}$	$\frac{-3}{400}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{9}{5}$	6	$\frac{48}{5}$	$\frac{-1}{2}$	$\frac{9}{2}$	0	0	$\frac{-3}{2}$
(3333, 11)	$\frac{1}{75}$	$\frac{-1}{100}$	$\frac{-7}{25}$	$\frac{-4}{75}$	$\frac{21}{100}$	0	$\frac{28}{25}$	0	$\frac{24}{5}$	$\frac{96}{5}$	0	0	0	4	0	0
(3333, 12)	$\frac{1}{300}$	$\frac{-1}{200}$	$\frac{13}{100}$	$\frac{2}{75}$	$\frac{-39}{200}$	0	$\frac{26}{25}$	0	$\frac{16}{5}$	$\frac{32}{5}$	0	0	0	2	0	0
(3333, 14)	$\frac{1}{300}$	0	$\frac{-7}{100}$	$\frac{-4}{75}$	0	0	$\frac{28}{25}$	0	$\frac{6}{5}$	$\frac{-24}{5}$	0	0	0	4	0	0
(3333, 22)	$\frac{1}{300}$	0	$\frac{-7}{100}$	$\frac{-4}{75}$	0	0	$\frac{28}{25}$	0	$\frac{-4}{5}$	$\frac{16}{5}$	0	0	0	0	0	0
(3333, 24)	$\frac{1}{1200}$	$\frac{-1}{400}$	$\frac{13}{400}$	$\frac{2}{75}$	$\frac{-39}{400}$	0	$\frac{26}{25}$	0	$\frac{4}{5}$	$\frac{32}{5}$	0	0	0	-1	0	0
(3333, 44)	$\frac{1}{1200}$	$\frac{1}{400}$	$\frac{-7}{400}$	$\frac{-4}{75}$	$\frac{-21}{400}$	0	$\frac{28}{25}$	0	$\frac{-6}{5}$	$\frac{-24}{5}$	0	0	0	1	0	0

Table 4 (2.1 (ii))

$(a_1a_2a_3a_4, b_1b_2)$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}
(1112, 11)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{6656}{451}$	$\frac{27648}{451}$	$\frac{168}{451}$	$\frac{11448}{451}$	$\frac{-2496}{451}$	$\frac{-17280}{451}$	$\frac{24}{41}$	$\frac{936}{41}$	$\frac{144}{41}$	$\frac{-384}{41}$	$\frac{4032}{41}$	$\frac{-48}{41}$
(1112, 12)	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{3584}{451}$	$\frac{-6912}{451}$	$\frac{480}{451}$	$\frac{-2052}{451}$	$\frac{-2688}{451}$	$\frac{1728}{451}$	$\frac{-60}{41}$	$\frac{216}{41}$	$\frac{-108}{41}$	$\frac{-2112}{41}$	$\frac{-1440}{41}$	$\frac{288}{41}$
(1112, 14)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{-54}{41}$	$\frac{702}{41}$	$\frac{1632}{41}$	$\frac{576}{41}$	$\frac{-228}{41}$
(1112, 22)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{684}{41}$	$\frac{-36}{41}$	$\frac{-2304}{41}$	$\frac{576}{41}$	$\frac{264}{41}$
(1112, 24)	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{896}{451}$	$\frac{-1728}{451}$	$\frac{-66}{41}$	$\frac{-108}{451}$	$\frac{-1344}{451}$	$\frac{-864}{451}$	$\frac{-42}{41}$	$\frac{-90}{41}$	$\frac{306}{41}$	$\frac{336}{41}$	$\frac{-576}{41}$	$\frac{-36}{41}$
(1112, 44)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{416}{451}$	$\frac{1728}{451}$	$\frac{-1182}{451}$	$\frac{1728}{451}$	$\frac{624}{451}$	$\frac{-4320}{451}$	$\frac{-66}{41}$	$\frac{252}{41}$	$\frac{288}{41}$	$\frac{-816}{41}$	$\frac{-288}{41}$	$\frac{96}{41}$
(1222, 11)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{3328}{451}$	$\frac{13824}{451}$	$\frac{8468}{451}$	$\frac{6264}{451}$	$\frac{-15264}{451}$	$\frac{-10368}{451}$	$\frac{-352}{41}$	$\frac{-708}{41}$	$\frac{516}{41}$	$\frac{3584}{41}$	$\frac{13536}{41}$	$\frac{-660}{41}$
(1222, 12)	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{1792}{451}$	$\frac{-3456}{451}$	$\frac{9598}{451}$	$\frac{-756}{451}$	$\frac{-16224}{451}$	0	$\frac{-458}{41}$	$\frac{-1956}{41}$	$\frac{168}{41}$	$\frac{2800}{41}$	$\frac{5040}{41}$	$\frac{-420}{41}$
(1222, 14)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{-6955}{451}$	$\frac{216}{41}$	$\frac{7632}{451}$	$\frac{-5184}{451}$	$\frac{473}{41}$	$\frac{1749}{41}$	$\frac{57}{41}$	-56	-144	$\frac{357}{41}$
(1222, 22)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{10634}{451}$	$\frac{216}{41}$	$\frac{-14016}{451}$	$\frac{-5184}{451}$	$\frac{-634}{41}$	$\frac{-2310}{41}$	$\frac{426}{41}$	112	288	$\frac{-750}{41}$
(1222, 24)	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{448}{451}$	$\frac{-864}{451}$	$\frac{-3182}{451}$	$\frac{216}{451}$	$\frac{6096}{451}$	$\frac{-1296}{451}$	$\frac{166}{41}$	$\frac{474}{41}$	$\frac{6}{41}$	$\frac{-896}{41}$	$\frac{-3384}{41}$	$\frac{156}{41}$
(1222, 44)	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{208}{451}$	$\frac{864}{451}$	$\frac{2381}{451}$	$\frac{1404}{451}$	$\frac{-2880}{451}$	$\frac{-3888}{451}$	$\frac{-151}{41}$	$\frac{-681}{41}$	$\frac{219}{41}$	$\frac{1400}{41}$	$\frac{2520}{41}$	$\frac{-219}{41}$
(1233, 11)	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{2560}{451}$	$\frac{-18432}{451}$	$\frac{488}{451}$	$\frac{-6192}{451}$	$\frac{-1600}{451}$	$\frac{6912}{451}$	$\frac{-112}{41}$	$\frac{-480}{41}$	$\frac{432}{41}$	$\frac{704}{41}$	$\frac{-3456}{41}$	$\frac{-24}{41}$
(1233, 12)	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{1024}{451}$	$\frac{11520}{451}$	$\frac{-1280}{451}$	$\frac{5220}{451}$	$\frac{-256}{451}$	$\frac{-8640}{451}$	$\frac{-20}{41}$	$\frac{312}{41}$	$\frac{348}{41}$	$\frac{-512}{41}$	$\frac{1440}{41}$	$\frac{24}{41}$
(1233, 14)	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{-66}{41}$	$\frac{306}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
(1233, 22)	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{180}{41}$	$\frac{60}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
(1233, 24)	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{256}{451}$	$\frac{2880}{451}$	$\frac{-1238}{451}$	$\frac{180}{41}$	$\frac{128}{451}$	$\frac{-4320}{451}$	$\frac{-50}{41}$	$\frac{330}{41}$	$\frac{150}{41}$	-16	0	$\frac{72}{41}$
(1233, 44)	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{160}{451}$	$\frac{-1152}{451}$	$\frac{-1072}{451}$	$\frac{288}{451}$	$\frac{-400}{451}$	$\frac{-1728}{451}$	$\frac{-52}{41}$	$\frac{222}{41}$	$\frac{90}{41}$	$\frac{-976}{41}$	$\frac{-576}{41}$	$\frac{126}{41}$

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \gamma_i C_i(n), \tag{5}$$

where $C_i(n)$ are the Fourier coefficients of the basis elements h_i defined in §4.3 and the values of the constants γ_i's are given in Table 5.
(iv) For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_{24})$, we have

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \delta_i D_i(n), \tag{6}$$

where $D_i(n)$ are the Fourier coefficients of the basis elements F_i defined in §4.4 and the values of the constants δ_i's are given in Table 6.

Now we consider, the class of quadratic forms given by (2). For $(1, c_1, c_2, c_3)$ as in Table 2, we define

$$M(1, c_1, c_2, c_3; n) := \\ \# \left\{ (x_1, \ldots, x_8) \in \mathbf{Z}^8 \middle| \; n = (x_1^2 + x_1x_2 + x_2^2) + \sum_{t=1}^{3} c_t(x_{2t+1}^2 + x_{2t+1}x_{2t+2} + x_{2t+2}^2) \right\}$$

to be the number of representations of n by the quadratic form (2). Note that $M(1, c_1, c_2, c_3; 0) = 1$. The formulas corresponding to Table 2 are stated in the following theorem.

Theorem 2.2 *Let* $n \in \mathbf{N}$.
For each entry $(1, c_1, c_2, c_3; n)$ *in Table* 2, we have

$$M(1, c_1, c_2, c_3; n) = \sum_{i=1}^{16} \nu_i A_i(n), \tag{7}$$

where $A_i(n)$ are the Fourier coefficients of the basis elements f_i defined in §4.1 and the values of the constants ν_i's are given in Table 7.

Remark 2.1. Since one can write down the exact formulas using the explicit Fourier coefficients of the basis elements and using the coefficients tables given in each of the cases, we have not stated explicit formulas in the theorems (due to large number of such formulas). However, in §5 (at the end of the Tables), we give some sample formulas corresponding to each case.

Table 5 (2.1 (iii))

$(a_1a_2a_3a_4, b_1b_2)$	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{15}	γ_{16}
(1113, 11)	$\frac{1}{23}$	$\frac{288}{23}$	$\frac{32}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{84}{23}$	$\frac{720}{23}$	$\frac{336}{23}$	$\frac{864}{23}$	0	0	0	0
(1113, 12)	$\frac{1}{23}$	$\frac{144}{23}$	$\frac{-16}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{156}{23}$	$\frac{-48}{23}$	$\frac{-168}{23}$	$\frac{-456}{23}$	0	0	0	0
(1113, 14)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{186}{23}$	$\frac{600}{23}$	$\frac{-228}{23}$	$\frac{372}{23}$	0	0	0	0
(1113, 22)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{96}{23}$	0	0	0	0
(1113, 24)	$\frac{1}{23}$	$\frac{36}{23}$	$\frac{-4}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{114}{23}$	$\frac{84}{23}$	$\frac{-120}{23}$	$\frac{-156}{23}$	0	0	0	0
(1113, 44)	$\frac{1}{23}$	$\frac{18}{23}$	$\frac{2}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{108}{23}$	$\frac{156}{23}$	$\frac{-162}{23}$	$\frac{42}{23}$	0	0	0	0
(1223, 11)	0	$\frac{144}{23}$	$\frac{16}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{162}{23}$	$\frac{264}{23}$	$\frac{-1188}{23}$	$\frac{420}{23}$	$\frac{192}{23}$	$\frac{864}{23}$	$\frac{-4704}{23}$	$\frac{-5760}{23}$
(1223, 12)	0	$\frac{72}{23}$	$\frac{-8}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{120}{23}$	$\frac{96}{23}$	$\frac{336}{23}$	$\frac{-384}{23}$	$\frac{-240}{23}$	$\frac{-912}{23}$	$\frac{4368}{23}$	$\frac{2784}{23}$
(1223, 14)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{144}{23}$	$\frac{480}{23}$	$\frac{-504}{23}$	$\frac{312}{23}$	$\frac{-360}{23}$	$\frac{1416}{23}$	$\frac{-1944}{23}$	$\frac{-1344}{23}$
(1223, 22)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{6}{23}$	$\frac{-72}{23}$	$\frac{-228}{23}$	$\frac{36}{23}$	$\frac{192}{23}$	$\frac{-240}{23}$	$\frac{-1392}{23}$	$\frac{-1344}{23}$
(1223, 24)	0	$\frac{18}{23}$	$\frac{-2}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{30}{23}$	$\frac{24}{23}$	$\frac{84}{23}$	$\frac{-96}{23}$	$\frac{36}{23}$	$\frac{-360}{23}$	$\frac{504}{23}$	$\frac{576}{23}$
(1223, 44)	0	$\frac{9}{23}$	$\frac{1}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{36}{23}$	$\frac{120}{23}$	$\frac{-126}{23}$	$\frac{78}{23}$	$\frac{-84}{23}$	$\frac{312}{23}$	$\frac{-840}{23}$	$\frac{-240}{23}$
(1333, 11)	$\frac{1}{23}$	$\frac{96}{23}$	$\frac{-32}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{260}{23}$	$\frac{352}{23}$	$\frac{-544}{23}$	$\frac{-352}{23}$	0	0	0	0
(1333, 12)	$\frac{1}{23}$	$\frac{48}{23}$	$\frac{16}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{116}{23}$	$\frac{160}{23}$	$\frac{-40}{23}$	$\frac{104}{23}$	0	0	0	0
(1333, 14)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{170}{23}$	$\frac{16}{23}$	$\frac{-292}{23}$	$\frac{-340}{23}$	0	0	0	0
(1333, 22)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{32}{23}$	$\frac{16}{23}$	$\frac{-16}{23}$	$\frac{-64}{23}$	0	0	0	0
(1333, 24)	$\frac{1}{23}$	$\frac{12}{23}$	$\frac{4}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{26}{23}$	$\frac{76}{23}$	$\frac{32}{23}$	$\frac{116}{23}$	0	0	0	0
(1333, 44)	$\frac{1}{23}$	$\frac{6}{23}$	$\frac{-2}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{44}{23}$	$\frac{-68}{23}$	$\frac{-22}{23}$	$\frac{-130}{23}$	0	0	0	0

Table 6 (2.1 (iv))

$(a_1a_2a_3a_4, b_1b_2)$	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}
(1123, 11)	$\frac{1}{261}$	$\frac{256}{29}$	$\frac{-256}{261}$	$\frac{-1}{29}$	$\frac{1808}{87}$	$\frac{656}{29}$	$\frac{-2056}{87}$	$\frac{-3808}{29}$	$\frac{-4144}{29}$	$\frac{736}{3}$	$\frac{472}{3}$	$\frac{-41984}{87}$	$\frac{-1096}{87}$	$\frac{-968}{87}$
(1123, 12)	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{128}{261}$	$\frac{1}{29}$	$\frac{208}{87}$	$\frac{-32}{29}$	$\frac{-284}{87}$	$\frac{-368}{29}$	$\frac{1048}{29}$	$\frac{-6224}{87}$	$\frac{-7100}{87}$	$\frac{21248}{87}$	$\frac{8}{3}$	$\frac{500}{87}$
(1123, 14)	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{-84}{29}$	$\frac{-114}{29}$	$\frac{450}{29}$	$\frac{1800}{29}$	$\frac{-324}{29}$	$\frac{-4200}{29}$	$\frac{-2814}{29}$	$\frac{-3072}{29}$	$\frac{318}{29}$	$\frac{414}{29}$
(1123, 22)	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{264}{29}$	$\frac{60}{29}$	$\frac{-420}{29}$	$\frac{-1680}{29}$	$\frac{-1368}{29}$	$\frac{2064}{29}$	$\frac{1884}{29}$	$\frac{-3072}{29}$	$\frac{-204}{29}$	$\frac{-108}{29}$
(1123, 24)	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{32}{261}$	$\frac{1}{29}$	$\frac{860}{87}$	$\frac{218}{29}$	$\frac{-970}{87}$	$\frac{-2296}{29}$	$\frac{-628}{29}$	$\frac{7976}{87}$	$\frac{-778}{87}$	$\frac{4288}{87}$	$\frac{-622}{87}$	$\frac{-10}{3}$
(1123, 44)	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{-16}{261}$	$\frac{-1}{29}$	$\frac{16}{87}$	$\frac{-176}{29}$	$\frac{244}{87}$	$\frac{592}{29}$	$\frac{-152}{29}$	$\frac{-6992}{87}$	$\frac{-3404}{87}$	$\frac{-1024}{87}$	$\frac{292}{87}$	$\frac{620}{87}$
(2223, 11)	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{-128}{261}$	$\frac{-1}{29}$	$\frac{5480}{261}$	$\frac{1472}{87}$	$\frac{-15016}{261}$	$\frac{-2992}{87}$	$\frac{-23584}{87}$	$\frac{79664}{261}$	$\frac{102248}{261}$	$\frac{-194048}{261}$	$\frac{-116}{9}$	$\frac{-3704}{261}$
(2223, 12)	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{64}{261}$	$\frac{1}{29}$	$\frac{-160}{261}$	$\frac{-640}{87}$	$\frac{7172}{261}$	$\frac{-5656}{87}$	$\frac{12320}{87}$	$\frac{-50824}{261}$	$\frac{-47284}{261}$	$\frac{96640}{261}$	$\frac{1076}{261}$	$\frac{964}{261}$
(2223, 14)	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{-466}{87}$	$\frac{4970}{261}$	$\frac{-1888}{87}$	$\frac{-628}{87}$	$\frac{-24496}{261}$	$\frac{-10030}{261}$	$\frac{-44672}{261}$	$\frac{494}{261}$	$\frac{394}{261}$
(2223, 22)	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{1100}{87}$	$\frac{-9124}{261}$	$\frac{2288}{87}$	$\frac{-10024}{87}$	$\frac{50672}{261}$	$\frac{32252}{261}$	$\frac{-44672}{261}$	$\frac{-1072}{261}$	$\frac{-1172}{261}$
(2223, 24)	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{16}{261}$	$\frac{1}{29}$	$\frac{-748}{261}$	$\frac{518}{87}$	$\frac{-2470}{261}$	$\frac{2936}{87}$	$\frac{-1156}{87}$	$\frac{11192}{261}$	$\frac{-8830}{261}$	$\frac{21088}{261}$	$\frac{578}{261}$	$\frac{562}{261}$
(2223, 44)	$\frac{1}{261}$	$\frac{8}{29}$	$\frac{-8}{261}$	$\frac{-1}{29}$	$\frac{-340}{261}$	$\frac{224}{87}$	$\frac{-604}{261}$	$\frac{1520}{87}$	$\frac{-1936}{87}$	$\frac{5840}{261}$	$\frac{-6388}{261}$	$\frac{-7328}{261}$	$\frac{284}{261}$	$\frac{244}{261}$
(2333, 11)	$\frac{1}{261}$	$\frac{256}{87}$	$\frac{256}{261}$	$\frac{1}{87}$	$\frac{-13360}{261}$	$\frac{-3520}{87}$	$\frac{20816}{261}$	$\frac{12608}{29}$	$\frac{20960}{87}$	$\frac{-219712}{261}$	$\frac{-113968}{261}$	$\frac{133120}{261}$	$\frac{15464}{261}$	$\frac{16384}{261}$
(2333, 12)	$\frac{1}{261}$	$\frac{128}{87}$	$\frac{-128}{261}$	$\frac{-1}{87}$	$\frac{11168}{261}$	$\frac{2312}{87}$	$\frac{-14212}{261}$	$\frac{-27920}{87}$	$\frac{-14968}{87}$	$\frac{129296}{261}$	$\frac{45212}{261}$	$\frac{-64256}{261}$	$\frac{-9856}{261}$	$\frac{-8948}{261}$
(2333, 14)	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{3836}{261}$	$\frac{338}{87}$	$\frac{-4126}{261}$	$\frac{-2968}{29}$	$\frac{-2092}{87}$	$\frac{21368}{261}$	$\frac{-10174}{261}$	$\frac{32512}{261}$	$\frac{-2530}{261}$	$\frac{-1514}{261}$
(2333, 22)	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{-5560}{261}$	$\frac{-1228}{87}$	$\frac{6836}{261}$	$\frac{4688}{29}$	$\frac{7304}{87}$	$\frac{-72592}{261}$	$\frac{-24268}{261}$	$\frac{32512}{261}$	$\frac{5300}{261}$	$\frac{6316}{261}$
(2333, 24)	$\frac{1}{261}$	$\frac{32}{87}$	$\frac{-32}{261}$	$\frac{-1}{87}$	$\frac{-940}{261}$	$\frac{-274}{87}$	$\frac{842}{261}$	$\frac{2584}{87}$	$\frac{-76}{87}$	$\frac{-12808}{261}$	$\frac{2666}{261}$	$\frac{-14528}{261}$	$\frac{878}{261}$	$\frac{1882}{261}$
(2333, 44)	$\frac{1}{261}$	$\frac{16}{87}$	$\frac{16}{261}$	$\frac{1}{87}$	$\frac{1088}{261}$	$\frac{128}{87}$	$\frac{-2140}{261}$	$\frac{-1120}{29}$	$\frac{-808}{87}$	$\frac{11168}{261}$	$\frac{5204}{261}$	$\frac{7360}{261}$	$\frac{-1156}{261}$	$\frac{-4}{9}$

Table 7 (2.2)

$(1, c_1, c_2, c_3)$	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	ν_9	ν_{10}	ν_{11}	ν_{12}	ν_{13}	ν_{14}	ν_{15}	ν_{16}
(1, 1, 1, 2)	$\frac{3}{40}$	$\frac{-1}{5}$	$\frac{-27}{40}$	0	$\frac{9}{5}$	0	0	0	0	0	0	0	0	0	0	0
(1, 1, 1, 4)	$\frac{3}{100}$	$\frac{-9}{100}$	$\frac{27}{100}$	$\frac{4}{25}$	$\frac{-81}{100}$	0	$\frac{36}{25}$	0	$\frac{54}{5}$	$\frac{432}{5}$	0	0	0	0	0	0
(1, 1, 1, 8)	$\frac{3}{160}$	$\frac{-9}{160}$	$\frac{-27}{160}$	$\frac{9}{80}$	$\frac{81}{160}$	$\frac{-1}{5}$	$\frac{-81}{80}$	$\frac{9}{5}$	0	0	0	$\frac{-27}{4}$	$\frac{243}{4}$	0	81	$\frac{81}{4}$
(1, 1, 2, 2)	$\frac{1}{50}$	$\frac{2}{25}$	$\frac{9}{50}$	0	$\frac{18}{25}$	0	0	0	$\frac{36}{5}$	0	0	0	0	0	0	0
(1, 1, 2, 4)	$\frac{1}{80}$	$\frac{1}{16}$	$\frac{-9}{80}$	$\frac{-1}{5}$	$\frac{-9}{16}$	0	$\frac{9}{5}$	0	0	0	0	0	0	9	0	0
(1, 1, 2, 8)	$\frac{1}{200}$	$\frac{1}{40}$	$\frac{9}{200}$	$\frac{-9}{100}$	$\frac{9}{40}$	$\frac{4}{25}$	$\frac{-81}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	$\frac{144}{5}$	$\frac{1152}{5}$	9	81	0	0	0
(1, 1, 4, 4)	$\frac{1}{200}$	$\frac{3}{200}$	$\frac{9}{200}$	$\frac{2}{25}$	$\frac{27}{200}$	0	$\frac{18}{25}$	0	$\frac{54}{5}$	$\frac{216}{5}$	0	0	0	0	0	0
(1, 1, 4, 8)	$\frac{1}{320}$	$\frac{3}{320}$	$\frac{-9}{320}$	$\frac{1}{16}$	$\frac{-27}{320}$	$\frac{-1}{5}$	$\frac{-9}{16}$	$\frac{9}{5}$	0	0	0	$\frac{9}{4}$	$\frac{-81}{4}$	$\frac{9}{4}$	27	$\frac{27}{4}$
(1, 1, 8, 8)	$\frac{1}{800}$	$\frac{3}{800}$	$\frac{9}{800}$	$\frac{3}{200}$	$\frac{27}{800}$	$\frac{2}{25}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{36}{5}$	$\frac{234}{5}$	$\frac{576}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
(1, 2, 2, 2)	$\frac{1}{40}$	$\frac{-3}{20}$	$\frac{-9}{40}$	0	$\frac{27}{20}$	0	0	0	0	0	0	0	0	0	0	0
(1, 2, 2, 4)	$\frac{1}{100}$	$\frac{-7}{100}$	$\frac{9}{100}$	$\frac{4}{25}$	$\frac{-63}{100}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	0	0	0
(1, 2, 2, 8)	$\frac{1}{160}$	$\frac{-7}{160}$	$\frac{-9}{160}$	$\frac{9}{80}$	$\frac{63}{160}$	$\frac{-1}{5}$	$\frac{-81}{80}$	$\frac{9}{5}$	0	0	0	$\frac{-9}{4}$	$\frac{81}{4}$	0	9	$\frac{27}{4}$
(1, 2, 4, 4)	$\frac{1}{160}$	$\frac{-1}{32}$	$\frac{-9}{160}$	$\frac{-1}{10}$	$\frac{9}{32}$	0	$\frac{9}{10}$	0	0	0	0	0	0	$\frac{9}{2}$	0	0
(1, 2, 4, 8)	$\frac{1}{400}$	$\frac{-1}{80}$	$\frac{9}{400}$	$\frac{-1}{20}$	$\frac{-9}{80}$	$\frac{4}{25}$	$\frac{-9}{20}$	$\frac{36}{25}$	$\frac{9}{10}$	$\frac{27}{5}$	$\frac{72}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
(1, 2, 8, 8)	$\frac{1}{640}$	$\frac{-1}{128}$	$\frac{-9}{640}$	$\frac{-3}{160}$	$\frac{9}{128}$	$\frac{-1}{10}$	$\frac{27}{160}$	$\frac{9}{10}$	0	0	0	$\frac{-9}{8}$	$\frac{81}{8}$	$\frac{27}{8}$	$\frac{9}{2}$	$\frac{27}{8}$
(1, 4, 4, 4)	$\frac{1}{400}$	$\frac{-9}{400}$	$\frac{9}{400}$	$\frac{3}{25}$	$\frac{-81}{400}$	0	$\frac{27}{25}$	0	$\frac{27}{5}$	$\frac{54}{5}$	0	0	0	0	0	0
(1, 4, 4, 8)	$\frac{1}{640}$	$\frac{-9}{640}$	$\frac{-9}{640}$	$\frac{7}{80}$	$\frac{81}{640}$	$\frac{-1}{5}$	$\frac{-63}{80}$	$\frac{9}{5}$	0	0	0	$\frac{9}{8}$	$\frac{-81}{8}$	$\frac{9}{8}$	0	$\frac{27}{8}$
(1, 4, 8, 8)	$\frac{1}{1600}$	$\frac{-9}{1600}$	$\frac{9}{1600}$	$\frac{1}{40}$	$\frac{-81}{1600}$	$\frac{2}{25}$	$\frac{9}{40}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{36}{5}$	$\frac{36}{5}$	$\frac{9}{4}$	$\frac{81}{4}$	0	0	0
(1, 8, 8, 8)	$\frac{1}{2560}$	$\frac{-9}{2560}$	$\frac{-9}{2560}$	$\frac{9}{320}$	$\frac{81}{2560}$	$\frac{-3}{20}$	$\frac{-81}{320}$	$\frac{27}{20}$	0	0	0	$\frac{27}{32}$	$\frac{-243}{32}$	$\frac{81}{32}$	0	$\frac{81}{32}$

3 Preliminaries

In this section, we present some preliminary facts on modular forms. For $k \in \frac{1}{2}\mathbf{Z}$, let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k for the congruence subgroup $\Gamma_0(N)$ with character χ and $S_k(\Gamma_0(N), \chi)$ be the subspace of cusp forms of weight k for $\Gamma_0(N)$ with character χ. We assume $4|N$ when k is not an integer and in that case, the character χ (which is a Dirichlet character modulo N) is an even character. When χ is the trivial (principal) character modulo N, we shall denote the spaces by $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$, respectively. Further, when $k \geq 4$ is an integer and $N = 1$, we shall denote these vector spaces by M_k and S_k, respectively.

For an integer $k \geq 4$, let E_k denote the normalized Eisenstein series of weight k in M_k given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n,$$

where $q = e^{2i\pi z}$, $\sigma_r(n)$ is the sum of the rth powers of the positive divisors of n, and B_k is the kth Bernoulli number defined by $\frac{x}{e^x - 1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m$.

The classical theta function which is fundamental to the theory of modular forms of half-integral weight is defined by

$$\Theta(z) = \sum_{n\in\mathbf{Z}} q^{n^2}, \tag{8}$$

and is a modular form in the space $M_{1/2}(\Gamma_0(4))$. Another function which is mainly used in our work is the Dedekind eta function $\eta(z)$ and it is given by

$$\eta(z) = q^{1/24}\prod_{n\geq 1}(1 - q^n). \tag{9}$$

An eta-quotient is a finite product of integer powers of $\eta(z)$ and we denote it as follows:

$$\prod_{i=1}^{s}\eta^{r_i}(d_i z) := d_1^{r_1}d_2^{r_2}\cdots d_s^{r_s}, \tag{10}$$

where d_i's are positive integers and r_i's are nonzero integers.

We denote the theta series associated to the quadratic form $x^2 + xy + y^2$ by

$$\mathscr{F}(z) = \sum_{x,y\in\mathbf{Z}} q^{x^2+xy+y^2}. \tag{11}$$

This function is referred to as the Borweins' two-dimensional theta function in the literature. By [18, Theorem 4], it follows that $\mathscr{F}(z)$ is a modular form in

$M_1(\Gamma_0(3), \chi_{-3})$. Here and in the sequel, for $m < 0$, the character χ_m is the odd Dirichlet character modulo $|m|$ given by $\left(\frac{-m}{\cdot}\right)$.

In the following, we shall present some facts about modular forms of integral and half-integral weights, which we shall be using in our proof. We state them as lemmas, whose proofs follow from elementary theory of modular forms (of integral and half-integral weights).

Lemma 1. *(Duplication of modular forms)*
If f is a modular form in $M_k(\Gamma_0(N), \chi)$, then for a positive integer d, the function $f(dz)$ is a modular form in $M_k(\Gamma_0(dN), \chi)$, if k is an integer and it belongs to the space $M_k(\Gamma_0(dN), \chi\chi_d)$, if k is a half-integer.

Lemma 2. *For positive integers r, r_1, r_2, d_1, d_2, we have*

$$\Theta^r(d_1 z) \in \begin{cases} M_{r/2}(\Gamma_0(4d_1), \chi_{d_1}) & \text{if } r \text{ is odd,} \\ M_{r/2}(\Gamma_0(4d_1), \chi_{-4}) & \text{if } r \equiv 2 \pmod 4, \\ M_{r/2}(\Gamma_0(4d_1)) & \text{if } r \equiv 0 \pmod 4. \end{cases} \tag{12}$$

For odd positive integers r_1, r_2, we have

$$\Theta^{r_1}(d_1 z) \cdot \Theta^{r_2}(d_2 z) \in \begin{cases} M_{\frac{r_1+r_2}{2}}(\Gamma_0(4[d_1, d_2]), \chi_{(-d_1 d_2)}) & \text{if } r_1 + r_2 \equiv 2 \pmod 4, \\ M_{\frac{r_1+r_2}{2}}(\Gamma_0(4[d_1, d_2]), \chi_{(d_1 d_2)}) & \text{if } r_1 + r_2 \equiv 0 \pmod 4. \end{cases} \tag{13}$$

Lemma 3. *If $f_i \in M_{k_i}(\Gamma_0(M_i), \psi_i)$, $i = 1, 2$, then the product $f_1 \cdot f_2$ is a modular form in $M_{k_1+k_2}(\Gamma_0(M), \psi_1\psi_2)$, where $M = \mathrm{lcm}(M_1, M_2)$.*

Lemma 4. *The vector space $M_k(\Gamma_1(N))$ is decomposed as a direct sum:*

$$M_k(\Gamma_1(N)) = \oplus_\chi M_k(\Gamma_0(N), \chi), \tag{14}$$

where the direct sum varies over all Dirichlet characters modulo N if the weight k is a positive integer and varies over all even Dirichlet characters modulo N, $4|N$, if the weight k is half-integer. Further, if k is an integer, one has $M_k(\Gamma_0(N), \chi) = \{0\}$, if $\chi(-1) \neq (-1)^k$. We also have the following decomposition of the space into subspaces of Eisenstein series and cusp forms:

$$M_k(\Gamma_0(N), \chi) = \mathscr{E}_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi), \tag{15}$$

where $\mathscr{E}_k(\Gamma_0(N), \chi)$ is the space generated by the Eisenstein series of weight k on $\Gamma_0(N)$ with character χ.

Lemma 5. *By the Atkin–Lehner theory of newforms, the space $S_k(\Gamma_0(N), \chi)$ can be decomposed into the space of newforms and oldforms:*

$$S_k(\Gamma_0(N), \chi) = S_k^{new}(\Gamma_0(N), \chi) \oplus S_k^{old}(\Gamma_0(N), \chi), \tag{16}$$

where the above is an orthogonal direct sum (with respect to the Petersson scalar product) and

$$S_k^{old}(\Gamma_0(N), \chi) = \bigoplus_{\substack{r|N, r<N \\ rd|N}} S_k^{new}(\Gamma_0(r), \chi)|B(d). \tag{17}$$

In the above, $S_k^{new}(\Gamma_0(N), \chi)$ is the space of newforms and $S_k^{old}(\Gamma_0(N), \chi)$ is the space of oldforms and the operator $B(d)$ is given by $f(z) \mapsto f(dz)$.

Lemma 6. *Suppose that χ and ψ are primitive Dirichlet characters with conductors M and N, respectively. For a positive integer k, let*

$$E_{k,\chi,\psi}(z) := c_0 + \sum_{n\geq 1}\left(\sum_{d|n} \psi(d)\cdot \chi(n/d) d^{k-1}\right) q^n, \tag{18}$$

where

$$c_0 = \begin{cases} 0 & \text{if } M > 1, \\ -\frac{B_{k,\psi}}{2k} & \text{if } M = 1, \end{cases}$$

and $B_{k,\psi}$ denotes generalized Bernoulli number with respect to the character ψ. Then, the Eisenstein series $E_{k,\chi,\psi}(z)$ belongs to the space $M_k(\Gamma_0(MN), \chi/\psi)$, provided $\chi(-1)\psi(-1) = (-1)^k$ and $MN \neq 1$. When $\chi = \psi = 1$ (i.e., when $M = N = 1$) and $k \geq 4$, we have $E_{k,\chi,\psi}(z) = -\frac{B_k}{2k}E_k(z)$, where E_k is the normalized Eisenstein series of integer weight k as defined before. We refer to [16, 20] for details.

We give a notation to the inner sum in (18):

$$\sigma_{k-1;\chi,\psi}(n) := \sum_{d|n} \psi(d)\cdot \chi(n/d) d^{k-1}. \tag{19}$$

For more details on the theory of modular forms of integral and half-integral weights, we refer to [8, 9, 12, 16, 18, 19].

4 Proofs of Theorems

In this section, we shall give a proof of our results. As mentioned in the introduction, we shall be using the theory of modular forms.

The basic functions for the two types of quadratic forms considered in this paper are $\Theta(z)$ and $\mathscr{F}(z)$. To each quadratic form in (1) with coefficients $(a_1, a_2, a_3, a_4, b_1, b_2)$ as in Table 1, the associated theta series is given by

$$\Theta(a_1 z)\Theta(a_2 z)\Theta(a_3 z)\Theta(a_4 z)\mathscr{F}(b_1 z)\mathscr{F}(b_2 z).$$

Using Lemmas 1 and 2 along with the fact that $\mathscr{F}(z) \in M_1(\Gamma_0(3), \chi_{-3})$, it follows that the above product is a modular form in $M_4(\Gamma_0(24), \chi)$, where the character χ is one of the four characters that appear in Table 1 and it is determined by the coefficients a_1, a_2, a_3, a_4. As remarked earlier, the theta series corresponding to the form $x^2 + xy + y^2$ is given by (11) and it belongs to the space $M_1(\Gamma_0(3), \chi_{-3})$. Therefore, the associated modular form corresponding to the quadratic forms defined by (2) is given explicitly by

$$\mathscr{F}(z)\mathscr{F}(c_1 z)\mathscr{F}(c_2 z)\mathscr{F}(c_3 z).$$

Again by using Lemmas 1, 2, and 3, it follows that the above product is a modular form in $M_4(\Gamma_0(24))$. Therefore, in order to get the required formulae for $N(a_1, a_2, a_3, a_4, b_1, b_2; n)$ and $M(1, c_1, c_2, c_3; n)$ we need a basis for the above spaces of modular forms of level 24. (We have used the L-functions and modular forms database [13] and [15] to get some of the cusp forms of weight 4.)

4.1 A basis for $M_4(\Gamma_0(24))$ and proof of Theorem 2.1(i).

The vector space $M_4(\Gamma_0(24))$ has dimension 16 and we have $\dim_{\mathbb{C}} \mathscr{E}_4(\Gamma_0(24)) = 8$ and $\dim_{\mathbb{C}} S_4(\Gamma_0(24)) = 8$. For $d = 6, 8, 12$ and 24, $S_4^{new}(\Gamma_0(d))$ is one-dimensional. Let us define some eta-quotients and use them to give an explicit basis for $S_4(\Gamma_0(24))$. Let

$$f_{4,6}(z) = 1^2 2^2 3^2 6^2 := \sum_{n\geq 1} a_{4,6}(n) q^n, \quad f_{4,8}(z) = 2^4 4^4 := \sum_{n\geq 1} a_{4,8}(n) q^n, \tag{20}$$

$$f_{4,12}(z) = 1^{-1} 2^2 3^3 4^3 6^2 12^{-1} - 1^3 2^2 3^{-1} 4^{-1} 6^2 12^3 := \sum_{n\geq 1} a_{4,12}(n) q^n, \tag{21}$$

$$f_{4,24}(z) = 1^{-4} 2^{11} 3^{-4} 4^{-3} 6^{11} 12^{-3} := \sum_{n\geq 1} a_{4,24}(n) q^n. \tag{22}$$

We use the following notation in the sequel. For a Dirichlet character χ and a function f with Fourier expansion $f(z) = \sum_{n\geq 1} a(n) q^n$, we define the twisted function $f \otimes \chi(z)$ as follows.

$$f \otimes \chi(z) = \sum_{n\geq 1} \chi(n) a(n) q^n. \tag{23}$$

A basis for the space $M_4(\Gamma_0(24))$ is given in the following proposition.

Proposition 4.1 *A basis for the Eisenstein series space $\mathscr{E}_4(\Gamma_0(24))$ is given by*

$$\{E_4(tz), t|24\} \tag{24}$$

and a basis for the space of cusp forms $S_4(\Gamma_0(24))$ *is given by*

$$\left\{f_{4,6}(t_1z), t_1|4;\ f_{4,8}(t_2z), t_2|3;\ f_{4,12}(t_3z), t_3|2;\ f_{4,24}\otimes\chi_4(z)\right\}. \tag{25}$$

Together they form a basis for $M_4(\Gamma_0(24))$.

For the sake of simplicity in the formulae, we list these basis elements as $\{f_i(z)|1\le i\le 16\}$, where $f_1(z)=E_4(z)$, $f_2(z)=E_4(2z)$, $f_3(z)=E_4(3z)$, $f_4(z)=E_4(4z)$, $f_5(z)=E_4(6z)$, $f_6(z)=E_4(8z)$, $f_7(z)=E_4(12z)$, $f_8(z)=E_4(24z)$, $f_9(z)=f_{4,6}(z)$, $f_{10}(z)=f_{4,6}(2z)$, $f_{11}(z)=f_{4,6}(4z)$, $f_{12}(z)=f_{4,8}(z)$, $f_{13}(z)=f_{4,8}(3z)$, $f_{14}(z)=f_{4,12}(z)$, $f_{15}(z)=f_{4,12}(2z)$, $f_{16}(z)=f_{4,24}\otimes\chi_4(z)$.
For $1\le i\le 16$, we denote the Fourier coefficients of the basis functions $f_i(z)$ as

$$f_i(z)=\sum_{n\ge1}A_i(n)q^n.$$

We are now ready to prove the theorem. Noting that all the 36 cases corresponding to the trivial character in Table 1, the resulting functions belong to the space of modular forms of weight 4 on $\Gamma_0(24)$ with trivial character (using Lemmas 1 to 3). So, we can express these theta functions as a linear combination of the basis given in Proposition 4.1 as follows.

$$\Theta(a_1z)\Theta(a_2z)\Theta(a_3z)\Theta(a_4z)\mathscr{F}(b_1z)\mathscr{F}(b_2z)=\sum_{i=1}^{16}\alpha_i f_i(z), \tag{26}$$

where α_i's are some explicit constants. Comparing the n-th Fourier coefficients on both the sides, we get

$$N(a_1,a_2,a_3,a_4,b_1,b_2;n)=\sum_{i=1}^{16}\alpha_i A_i(n).$$

Explicit values for the constants α_i, $1\le i\le 16$ corresponding to these 36 cases are given in Table 3.

4.2 A basis for $M_4(\Gamma_0(24),\chi_8)$ and proof of Theorem 2.1(ii).

The vector space $M_4(\Gamma_0(24),\chi_8)$ has dimension 14 and we have $\dim_{\mathbf{C}}\mathscr{E}_4(\Gamma_0(24),\chi_8))=4$ and $\dim_{\mathbf{C}}S_4(\Gamma_0(24),\chi_8))=10$. For $d=6$ and 12, $S_4^{new}(\Gamma_0(d),\chi_8)=\{0\}$. Also $S_4^{new}(\Gamma_0(8),\chi_8)$ is two-dimensional and $S_4^{new}(\Gamma_0(24),\chi_8)$ is six-dimensional. In order to give explicit basis for this space, we define the following

$$E_{4,\mathbf{1},\chi_8}(z) \ = \ \frac{11}{2} + \sum_{n\geq 1} \sigma_{3;\mathbf{1},\chi_8}(n)q^n, \quad E_{4,\chi_8,\mathbf{1}}(z) \ = \ \sum_{n\geq 1} \sigma_{3;\chi_8,\mathbf{1}}(n)q^n. \tag{27}$$

$$f_{4,8,\chi_8;1}(z) = 1^{-2}2^{11}4^{-3}8^2 := \textstyle\sum_{n\geq 1} a_{4,8,\chi_8;1}(n)q^n, \tag{28}$$

$$f_{4,8,\chi_8;2}(z) \ = 1^2 2^{-3}4^{11}8^{-2} := \textstyle\sum_{n\geq 1} a_{4,8,\chi_8;2}(n)q^n. \tag{29}$$

For the space of Eisenstein series, we use the basis elements of $\mathscr{E}_4(\Gamma_0(8),\,\chi_8)$ given in (27). A basis for $S_4^{new}(\Gamma_0(8),\,\chi_8)$ is given in (28) and (29). The following six eta-quotients span the space $S_4^{new}(\Gamma_0(24),\,\chi_8)$.

$$f_{4,24,\chi_8;1}(z) = 1^2 2^1 3^{-4} 4^1 6^{10} 8^2 12^{-4} := \sum_{n\geq 1} a_{4,24,\chi_8;1}(n)q^n, \tag{30}$$

$$f_{4,24,\chi_8;2}(z) = 1^1 2^3 3^{-1} 4^1 6^4 8^{-1} 24^1 := \sum_{n\geq 1} a_{4,24,\chi_8;2}(n)q^n, \tag{31}$$

$$f_{4,24,\chi_8;3}(z) = 1^{-1} 2^4 3^1 6^3 8^1 12^1 24^{-1} := \sum_{n\geq 1} a_{4,24,\chi_8;3}(n)q^n, \tag{32}$$

$$f_{4,24,\chi_8;4}(z) = 1^{-2} 2^4 4^2 6^1 8^2 12^1 := \sum_{n\geq 1} a_{4,24,\chi_8;4}(n)q^n, \tag{33}$$

$$f_{4,24,\chi_8;5}(z) = 2^1 3^{-2} 4^1 6^4 12^2 24^2 := \sum_{n\geq 1} a_{4,24,\chi_8;5}(n)q^n, \tag{34}$$

$$f_{4,24,\chi_8;6}(z) = 1^{-6} 2^{14} 6^1 8^{-2} 12^1 := \sum_{n\geq 1} a_{4,24,\chi_8;6}(n)q^n. \tag{35}$$

A basis for the space $M_4(\Gamma_0(24),\,\chi_8)$ is given in the following proposition.

Proposition 4.2 *A basis for the space $M_4(\Gamma_0(24),\,\chi_8)$ is given by*

$$\big\{E_{4,\mathbf{1},\chi_8}(tz),\ E_{4,\chi_8,\mathbf{1}}(tz), t|3;\ f_{4,8,\chi_8;1}(t_1 z),\ f_{4,8,\chi_8;2}(t_1 z),\ t_1|3;\ f_{4,24,\chi_8;1}(z),\ f_{4,24,\chi_8;2}(z),\ f_{4,24,\chi_8;3}(z),\ f_{4,24,\chi_8;4}(z),\ f_{4,24,\chi_8;5}(z),\ f_{4,24,\chi_8;6}(z)\big\},$$

where $E_{4,\mathbf{1},\chi_8}(z)$, and $E_{4,\chi_8,\mathbf{1}}(z)$ are defined in (27), $f_{4,8,\chi_8;i}(z)$, $i=1,2$ are defined in (28), (29) and $f_{4,24,\chi_8;j}(z)$, $1\leq j\leq 6$ are defined by (30) – (35).

For the sake of simplifying the notation, we shall list the basis in Proposition 4.2 as

$$g_i(z) = \sum_{n\geq 1} B_i(n)q^n,\ \ 1\leq i\leq 14,$$

where $g_1(z) = E_{4,\mathbf{1},\chi_8}(z)$, $g_2(z) = E_{4,\mathbf{1},\chi_8}(3z)$, $g_3(z) = E_{4,\chi_8,\mathbf{1}}(z)$, $g_4(z) = E_{4,\chi_8,\mathbf{1}}(3z)$, $g_5(z) = f_{4,8,\chi_8;1}(z)$, $g_6(z) = f_{4,8,\chi_8;1}(3z)$, $g_7(z) = f_{4,8,\chi_8;2}(z)$, $g_8(z) = f_{4,8,\chi_8;2}(3z)$, $g_9(z) = f_{4,24,\chi_8;1}(z)$, $g_{10}(z) = f_{4,24,\chi_8;2}(z)$, $g_{11}(z) = f_{4,24,\chi_8;3}(z)$, $g_{12}(z) = f_{4,24,\chi_8;4}(z)$, $g_{13}(z) = f_{4,24,\chi_8;5}(z)$, $g_{14}(z) = f_{4,24,\chi_8;6}(z)$, We now prove Theorem 2.1(ii). In this case, for all the 18 sextuples corresponding to the χ_8 character space (in Table 1), the resulting products of theta functions are modular forms of weight 4 on $\Gamma_0(24)$ with character χ_8 (By Lemmas 1 to 3). So, we can express these products of theta functions as a linear combination of the basis given in Proposition 4.2:

$$\Theta(a_1z)\Theta(a_2z)\Theta(a_3z)\Theta(a_4z)\mathscr{F}(b_1z)\mathscr{F}(b_2z) = \sum_{i=1}^{14} \beta_i g_i(z). \tag{36}$$

Comparing the nth Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \beta_i B_i(n).$$

Explicit values for the constants β_i, $1 \le i \le 14$ corresponding to these 18 cases are given in Table 4.

4.3 A basis for $M_4(\Gamma_0(24), \chi_{12})$, and proof of Theorem 2.1(iii).

The dimension of the space, in this case, is 16, with $\dim_{\mathbf{C}} \mathscr{E}_4(\Gamma_0(24), \chi_{12}) = 8$ and $\dim_{\mathbf{C}} S_4(\Gamma_0(24), \chi_{12}) = 8$. The old class is spanned by the space $S_4^{new}(\Gamma_0(12), \chi_{12})$, which is 4 dimensional with spanning functions given by the following four eta-quotients:

$$f_{4,12,\chi_{12};1}(z) = 2^{-1}3^4 4^2 6^5 12^{-2}, \quad f_{4,12,\chi_{12};2}(z) = 3^4 4^3 6^{-2} 12^3, \tag{37}$$
$$f_{4,12,\chi_{12};3}(z) = 2^2 3^4 4^{-1} 6^{-4} 12^7, \quad f_{4,12,\chi_{12};4}(z) = 1^4 4^{-1} 6^{-2} 12^7. \tag{38}$$

We write the Fourier expansions of these forms as $f_{4,12,\chi_{12};j}(z) = \sum_{n\ge1} a_{4,12,\chi_{12};j}(n) q^n$, $1 \le j \le 4$. In the following proposition, we give a basis for the space $M_4(\Gamma_0(24), \chi_{12})$.

Proposition 4.3 *A basis for the space $M_4(\Gamma_0(24), \chi_{12})$ is given by*

$$\left\{E_{4,\mathbf{1},\chi_{12}}(tz), E_{4,\chi_{12},\mathbf{1}}(tz), E_{4,\chi_{-4},\chi_{-3}}(tz), E_{4,\chi_{-3},\chi_{-4}}(tz), t|2; \; f_{4,12,\chi_{12};j}(t_1z), t_1|2, 1 \le j \le 4\right\},$$

where the Eisenstein series in the basis are defined by (18).

Let us denote, the 16 basis elements in the above proposition as follows. $\{h_i(z) | 1 \le i \le 16\}$, where $h_1(z) = E_{4,\mathbf{1},\chi_{12}}(z)$, $h_2(z) = E_{4,\chi_{12},\mathbf{1}}(z)$, $h_3(z) =$

$E_{4,\chi_{-4},\chi_{-3}}(z)$, $h_4(z) = E_{4,\chi_{-3},\chi_{-4}}(z)$, $h_5(z) = E_{4,\mathbf{1},\chi_{12}}(2z)$, $h_6(z) = E_{4,\chi_{12},\mathbf{1}}(2z)$, $h_7(z) = E_{4,\chi_{-4},\chi_{-3}}(2z)$, $h_8(z) = E_{4,\chi_{-3},\chi_{-4}}(2z)$,$h_{8+j}(z) = f_{4,12,\chi_{12};j}(z)$, $1 \le j \le 4$, $h_{12+j}(z) = f_{4,12,\chi_{12};j}(2z)$, $1 \le j \le 4$.

To prove Theorem 2.1(iii), we consider the case of 18 sextuples corresponding to the character χ_{12} in Table 1. The resulting products of theta functions are modular forms of weight 4 on $\Gamma_0(24)$ with character χ_{12} (once again we use Lemmas 1 to 3 to get this). So, we can express each of these products of theta functions as a linear combination of the basis given in Proposition 4.3 as follows.

$$\Theta(a_1z)\Theta(a_2z)\Theta(a_3z)\Theta(a_4z)\mathscr{F}(b_1z)\mathscr{F}(b_2z) = \sum_{i=1}^{16} \gamma_i g_i(z). \qquad (39)$$

Comparing the nth Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \gamma_i C_i(n).$$

Explicit values of the constants γ_i, $1 \le i \le 16$ corresponding to these 18 cases are given in Table 5.

4.4 *A basis for $M_4(\Gamma_0(24), \chi_{24})$ and proof of Theorem 2.1(iv).*

We have $\dim_{\mathbf{C}} M_4(\Gamma_0(24), \chi_{24}) = 14$ and $\dim_{\mathbf{C}} \mathscr{E}_4(\Gamma_0(24), \chi_{24}) = 4$. To get the span of the Eisenstein series space $\mathscr{E}_4(\Gamma_0(24), \chi_{24})$, we use the Eisenstein series $E_{4,\chi,\psi}(z)$ defined in (18), where $\chi, \psi \in \{\mathbf{1}, \chi_{-8}, \chi_{-12}, \chi_{24}\}$. Note that for d = 6, 8 and 12, $S_4^{new}(\Gamma_0(d), \chi_{24}) = \{0\}$ and the space $S_4^{new}(\Gamma_0(24), \chi_{24})$ is spanned by the following ten eta-quotients (notation as in (10)):

$$f_{4,24,\chi_{24};1}(z) = 3^{-2}6^7 8^3 12^3 24^{-3}, \qquad f_{4,24,\chi_{24};2}(z) = 3^2 4^7 6^{-3} 8^{-2} 12^4,$$
$$f_{4,24,\chi_{24};3}(z) = 3^2 4^{-3} 6^1 8^6 12^2, \qquad f_{4,24,\chi_{24};4}(z) = 3^2 6^{-3} 8^3 12^5 24^1,$$
$$f_{4,24,\chi_{24};5}(z) = 3^2 4^{-1} 6 8^2 24^4, \qquad f_{4,24,\chi_{24};6}(z) = 3^2 4^2 6^{-3} 8^{-1} 12^3 24^5,$$
$$f_{4,24,\chi_{24};7}(z) = 3^2 4^1 6^1 8^{-2} 12^{-2} 24^8, \qquad f_{4,24,\chi_{24};8}(z) = 1^1 3^{-1} 6^1 8^{-2} 12^1 24^8,$$
$$f_{4,24,\chi_{24};9}(z) = 2^2 3^6 4^1 6^{-3} 8^2, \qquad f_{4,24,\chi_{24};10}(z) = 3^2 4^3 6^5 12^{-4} 24^2.$$

We write the Fourier expansions as $f_{4,24,\chi_{24};j}(z) = \sum_{n\ge 1} a_{4,24,\chi_{24};j}(n)q^n$. We now give a basis for the space $M_4(\Gamma_0(24), \chi_{24})$ in the following proposition.

Proposition 4.4 *The following functions span the space* $M_4(\Gamma_0(24), \chi_{24})$.

$$\left\{E_{4,\mathbf{1},\chi_{24}}(z),\, E_{4,\chi_{24},\mathbf{1}}(z),\, E_{4,\chi_{-8},\chi_{-3}}(z),\, E_{4,\chi_{-3},\chi_{-8},}(z),\, f_{4,24,\chi_{24};j}(z),\, 1 \le j \le 10\right\}. \tag{40}$$

We list these basis elements as $\{F_i(z) | 1 \le i \le 14\}$, where $F_1(z) = E_{4,\mathbf{1},\chi_{24}}(z)$, $F_2(z) = E_{4,\chi_{24},\mathbf{1}}(z)$, $F_3(z) = E_{4,\chi_{-8},\chi_{-3}}(z)$, $F_4(z) = E_{4,\chi_{-3},\chi_{-8},}(z)$, $F_5(z) = f_{4,24,\chi_{24};1}(z)$, $F_6(z) = f_{4,24,\chi_{24};2}(z)$, $F_7(z) = f_{4,24,\chi_{24};3}(z)$, $F_8(z) = f_{4,24,\chi_{24};4}(z)$, $F_9(z) = f_{4,24,\chi_{24};5}$

(z), $F_{10}(z) = f_{4,24,\chi_{24};6}(z)$, $F_{11}(z) = f_{4,24,\chi_{24};7}(z)$, $F_{12}(z) = f_{4,24,\chi_{24};8}(z)$, $F_{13}(z) = f_{4,24,\chi_{24};9}(z)$, $F_{14}(z) = f_{4,24,\chi_{24};10}(z)$.
As in the previous cases, we denote the Fourier coefficients of these basis functions by

$$F_i(z) = \sum_{n \ge 1} D_i(n) q^n, \;\; 1 \le i \le 14.$$

To get the formula in Theorem 2.1(iv), we note that for all the 18 sextuples corresponding to the character χ_{24} in Table 1, the resulting functions belong to the space $M_4(\Gamma_0(24), \chi_{24})$, by using Lemmas 1 to 3. So, as before, we express these theta functions as linear combinations of the basis elements:

$$\Theta(a_1z)\Theta(a_2z)\Theta(a_3z)\Theta(a_4z)\mathscr{F}(b_1z)\mathscr{F}(b_2z) = \sum_{i=1}^{14} \delta_i g_i(z). \tag{41}$$

Comparing the n-th Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \delta_i D_i(n).$$

Explicit values of the constants δ_i, $1 \le i \le 14$ corresponding to these 18 cases corresponding to character χ_{24} are given in Table 6.

4.5 *Proof of Theorem 2.2*

This theorem is corresponding to Table 2 and in this case all the product functions

$$\mathscr{F}(z)\mathscr{F}(c_1z)\mathscr{F}(c_2z)\mathscr{F}(c_3z)$$

belong to the space $M_4(\Gamma_0(24))$. Therefore, proceeding as in the proof of Theorem 2.1(i), we express these theta functions as linear combinations of the basis elements:

$$\mathscr{F}(z)\mathscr{F}(c_1z)\mathscr{F}(c_2z)\mathscr{F}(c_3z) = \sum_{i=1}^{16} \nu_i f_i(z). \tag{42}$$

Comparing the nth Fourier coefficients on both the sides, we get

$$M(1, c_1, c_2, c_3; n) = \sum_{i=1}^{16} \nu_i A_i(n).$$

The constants ν_i, $1 \le i \le 16$ corresponding to the 19 cases of table 2 are given in Table 7.

5 Sample formulas

In this section, we shall give explicit formulas for a few cases from Tables 1 and 2.

First two formulas of Theorem 2.1(i):

$$\begin{aligned} N(1,1,1,1,1,1;n) &= \frac{112}{5}\sigma_3(n) - \frac{84}{5}\sigma_3(n/2) - \frac{432}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) \\ &+ \frac{324}{5}\sigma_3(n/6) + \frac{1728}{5}\sigma_3(n/12) - \frac{72}{5}a_{4,6}(n) - \frac{288}{5}a_{4,6}(n/2) + 12a_{4,12}(n), \end{aligned}$$

$$\begin{aligned} N(1,1,1,1,1,2;n) &= \frac{52}{5}\sigma_3(n) - \frac{78}{5}\sigma_3(n/2) + \frac{108}{5}\sigma_3(n/3) + \frac{416}{5}\sigma_3(n/4) \\ &- \frac{162}{5}\sigma_3(n/6) + \frac{864}{5}\sigma_3(n/12) + \frac{48}{5}a_{4,6}(n) + \frac{96}{5}a_{4,6}(n/2) - 6a_{4,12}(n). \end{aligned}$$

First two formulas of Theorem 2.1(ii):

$$\begin{aligned} N(1,1,1,2,1,1;n) &= -\frac{26}{451}\sigma_{3;\mathbf{1},\chi_8}(n) + \frac{108}{451}\sigma_{3;\mathbf{1},\chi_8}(n/3) + \frac{6656}{451}\sigma_{3;\chi_8,\mathbf{1}}(n) \\ &+ \frac{27648}{451}\sigma_{3;\chi_8,\mathbf{1}}(n/3) + \frac{168}{451}a_{4,8,\chi_8;1}(n) + \frac{11448}{451}a_{4,8,\chi_8;1}(n/3) - \frac{2496}{451}a_{4,8,\chi_8;2}(n) \\ &- \frac{17280}{451}a_{4,8,\chi_8;2}(n/3) + \frac{24}{41}a_{4,24,\chi_8;1}(n) + \frac{936}{41}a_{4,24,\chi_8;2}(n) + \frac{144}{41}a_{4,24,\chi_8;3}(n) \\ &- \frac{384}{41}a_{4,24,\chi_8;4}(n) + \frac{4032}{41}a_{4,24,\chi_8;5}(n) - \frac{48}{41}a_{4,24,\chi_8;6}(n), \end{aligned}$$

$$N(1,1,1,2,1,2;n) = \frac{28}{451}\sigma_{3;\mathbf{1},\chi_8}(n) + \frac{54}{451}\sigma_{3;\mathbf{1},\chi_8}(n/3) + \frac{3584}{451}\sigma_{3;\chi_8,\mathbf{1}}(n) - \frac{6912}{451}\sigma_{3;\chi_8,\mathbf{1}}(n/3) + \frac{480}{451}a_{4,8,\chi_8;1}(n) - \frac{2052}{451}a_{4,8,\chi_8;1}(n/3) - \frac{2688}{451}a_{4,8,\chi_8;2}(n) + \frac{1728}{451}a_{4,8,\chi_8;2}(n/3) - \frac{60}{41}a_{4,24,\chi_8;1}(n) + \frac{216}{41}a_{4,24,\chi_8;2}(n) - \frac{108}{41}a_{4,24,\chi_8;3}(n) - \frac{2112}{41}a_{4,24,\chi_8;4}(n) - \frac{1440}{41}a_{4,24,\chi_8;5}(n) + \frac{288}{41}a_{4,24,\chi_8;6}(n).$$

First two formulas of Theorem 2.1(iii):

$$N(1,1,1,3,1,1;n) = \frac{1}{23}\sigma_{3;\mathbf{1},\chi_{12},}(n) + \frac{288}{23}\sigma_{3;\chi_{12},\mathbf{1},}(n) + \frac{32}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n) + \frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n) + \frac{84}{23}a_{4,12,\chi_{12};1}(n) + \frac{720}{23}a_{4,12,\chi_{12};2}(n) + \frac{336}{23}a_{4,12,\chi_{12};3}(n) + \frac{864}{23}a_{4,12,\chi_{12};4}(n),$$

$$N(1,1,1,3,1,2;n) = \frac{1}{23}\sigma_{3;\mathbf{1},\chi_{12},}(n) + \frac{144}{23}\sigma_{3;\chi_{12},\mathbf{1}}(n) - \frac{16}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n) - \frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n) + \frac{156}{23}a_{4,12,\chi_{12};1}(n) - \frac{48}{23}a_{4,12,\chi_{12};2}(n) - \frac{168}{23}a_{4,12,\chi_{12};3}(n) - \frac{456}{23}a_{4,12,\chi_{12};4}(n).$$

First two formulas of Theorem 2.1(iv):

$$N(1,1,2,3,1,1;n) = \frac{1}{261}\sigma_{3;\mathbf{1},\chi_{24}}(n) + \frac{256}{29}\sigma_{3;\chi_{24},\mathbf{1}}(n) - \frac{256}{261}\sigma_{3;\chi_{-8},\chi_{-3}}(n) - \frac{1}{29}\sigma_{3;\chi_{-3},\chi_{-8}}(n) + \frac{1808}{87}a_{4,24,\chi_{24};1}(n) + \frac{656}{29}a_{4,24,\chi_{24};2}(n) - \frac{2056}{87}a_{4,24,\chi_{24};3}(n) - \frac{3808}{29}a_{4,24,\chi_{24};4}(n) - \frac{4144}{29}a_{4,24,\chi_{24};5}(n) + \frac{736}{3}a_{4,24,\chi_{24};6}(n) + \frac{472}{3}a_{4,24,\chi_{24};7}(n) - \frac{41984}{87}a_{4,24,\chi_{24};8}(n) - \frac{1096}{87}a_{4,24,\chi_{24};9}(n) - \frac{968}{87}a_{4,24,\chi_{24};10}(n),$$

$$N(1,1,2,3,1,2;n) = \frac{1}{261}\sigma_{3;\mathbf{1},\chi_{24}}(n) + \frac{128}{29}\sigma_{3;\chi_{24},\mathbf{1}}(n) + \frac{128}{261}\sigma_{3;\chi_{-8},\chi_{-3}}(n) + \frac{1}{29}\sigma_{3;\chi_{-3},\chi_{-8}}(n) + \frac{208}{87}a_{4,24,\chi_{24};1}(n) - \frac{32}{29}a_{4,24,\chi_{24};2}(n) - \frac{284}{87}a_{4,24,\chi_{24};3}(n) - \frac{368}{29}a_{4,24,\chi_{24};4}(n) + \frac{1048}{29}a_{4,24,\chi_{24};5}(n) - \frac{6224}{87}a_{4,24,\chi_{24};6}(n) - \frac{7100}{87}a_{4,24,\chi_{24};7}(n) + \frac{21248}{87}a_{4,24,\chi_{24};8}(n) + \frac{8}{3}a_{4,24,\chi_{24};9}(n) + \frac{500}{87}a_{4,24,\chi_{24};10}(n).$$

First two formulas of Theorem 2.2:

$$M(1,1,1,2;n) = 18\sigma_3(n) - 48\sigma_3(n/2) - 162\sigma_3(n/3) + 432\sigma_3(n/6),$$

$$M(1,1,1,4;n) = \frac{36}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/2) + \frac{324}{5}\sigma_3(n/3) + \frac{192}{5}\sigma_3(n/4) - \frac{972}{5}\sigma_3(n/6) + \frac{1728}{5}\sigma_3(n/12) + \frac{54}{5}a_{4,6}(n) + \frac{432}{5}a_{4,6}(n/2).$$

Acknowledgements We have used the open-source mathematics software SAGE (www.sagemath.org) to perform our calculations. Anup Kumar Singh thanks the Department of Mathematics, NISER, Bhubaneswar (where part of this work was done) for their warm hospitality and support. Finally, we thank the referee for correcting the typos in two sample formulas and for pointing out the works [11, 22], where similar works have been carried out.

References

1. S. Alaca, Y. Kesicioğlu, Representations by certain octonary quadratic forms with coefficients are 1, 2, 3 and 6. Integers **15** (2015)
2. A. Alaca, M.N. Kesicioğlu, Representations by octonary quadratic forms with coefficients 1, 2, 3 or 6. Int. J. Number Theory (2016). https://doi.org/10.1142/S1793042117500385, arXiv:1603.07780 [math.NT]
3. S. Alaca, Y. Kesicioğlu, Representations by certain octonary quadratic forms whose coefficients are 1, 2, 3 and 6. Int. J. Number Theory **10**, 133–150 (2014)
4. S. Alaca, K.S. Williams, The number of representations of a positive integer by certain octonary quadratic forms. Funct. Approx. **43**, 45–54 (2010)
5. A. Alaca, S. Alaca, K.S. Williams, Seven octonary quadratic forms. Acta Arith. **135**, 339–350 (2008)
6. A. Alaca, S. Alaca, M. Lemire, K.S. Williams, Theta function identities and representations by certain quaternary quadratic forms. Int. J. Number Theory **4**, 219–239 (2008)
7. A. Alaca, S. Alaca, K.S. Williams, Fourteen octonary quadratic forms. Int. J. Number Theory **6**, 37–50 (2010)
8. A.O.L. Atkin, J. Lehner, Hecke operators on $\Gamma_0(m)$. Math. Ann. **185**, 134–160 (1970)
9. N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, vol. 97, 2nd edn., Graduate Texts in Mathematics (Springer, Berlin, 1993)
10. B. Köklüce, The representation numbers of certain octonary quadratic forms. Int. J. Number Theory **9**(5), 1125–1139 (2013)
11. B. Köklüce, Representation numbers of two octonary quadratic forms. Int. J. Number Theory **9**(7), 1641–1648 (2013)
12. W.-W. Li, Newforms and functional equations. Math. Ann. **212**, 285–315 (1975)
13. LMFDB, The database of L-functions, modular forms and related objects, http://www.lfmdb.org/
14. G.A. Lomadze, Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$. Acta Arith. **54**, 9–36 (1989). (in Russian)
15. Y. Martin, Multiplicative η-quotients. Trans. Am. Math. Soc. **348**, 4825–4856 (1996)
16. T. Miyake, *Modular forms* (Springer, Berlin, 1989)
17. B. Ramakrishnan, B. Sahu, A.K. Singh, On the number of representations of certain octonary quadratic forms with coefficients 1, 2, 3, 4 and 6. Preprint, June 2016

18. B. Schoeneberg, *Elliptic Modular, Functions: An Introduction, Die Grundlehren der Mathematischen Wissenschaften*, vol. 203 (Springer, New York, 1974). Translated from the German by J.R. Smart and E.A. Schwandt
19. G. Shimura, On modular forms of half-integral weight. Ann. Math. **97**, 440–481 (1973)
20. W. Stein, *Modular Forms, a Computational Approach*, vol. 79, Graduate Texts in Mathematics (American Mathematical Society, Providence, 2007)
21. K.S. Williams, *Number Theory in the Spirit of Liouville*, vol. 76, London Mathematical Student Texts (Cambridge University Press, Cambridge, 2011)
22. E.X.W. Xia, O.X.M. Yao, On the representations of integers by certain quadratic forms. Int. J. Number Theory **9**(1), 189–204 (2013)

A Note on Thue Inequalities with Few Coefficients

N. Saradha and Divyum Sharma

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Let $F(X, Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \in \mathbb{Z}[X, Y]$ be a form of degree $r \geq 3$, irreducible over $\mathbb{Q}$, and having at most $s + 1$ nonzero coefficients. Mueller and Schmidt showed that the number of solutions of the Thue inequality

$$|F(X, Y)| \leq h$$

is $\ll s^2 h^{2/r}(1 + \log h^{1/r})$. They conjectured that s^2 may be replaced by s. In this note we show some instances when s^2 may be improved.

Keywords Thue equations · Thue inequalities
Large, medium and small solutions

2010 Mathematics Subject Classification Primary 11J25 · 11D59 · Secondary 11D45

N. Saradha (✉) · D. Sharma
School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Mumbai 400 005, India
e-mail: saradha@math.tifr.res.in

D. Sharma
e-mail: divyum@math.tifr.res.in

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_36

1 Introduction

Let $F(X, Y)$ be a form of degree $r \geq 3$ with integer coefficients, irreducible over $\mathbb{Q}$, and having at most $s + 1$ nonzero coefficients. Write

$$F(X, Y) = \sum_{i=0}^{s} a_i X^{r_i} Y^{r-r_i} \tag{1.1}$$

with $0 = r_0 < r_1 < \ldots < r_s = r$. Let D, H, and M denote the discriminant, naive height, and Mahler height of $F(X, 1)$, respectively. For $h \geq 1$, consider the Thue inequality

$$|F(X, Y)| \leq h. \tag{1.2}$$

Let $N_F(h)$ denote the number of integer solutions (x, y) of (1.2). Bombieri modified a conjecture of Siegel and asked if $N_F(h)$ could be bounded by a function depending only on s and h. (See Mueller and Schmidt [8, p. 208]). Toward this, Schmidt [10] proved that

$$N_F(h) \ll \sqrt{rs}\, h^{2/r}(1 + \log h^{1/r}). \tag{1.3}$$

Throughout this note, the constants implied by $\ll$ are absolute. The modified Siegel's conjecture was shown to be true in the case $s = 1$ by Hyyrö [3], Evertse [1], and Mueller [6]. The case $s \geq 2$ was considered by Mueller and Schmidt in [7] and [8]. They proved that

$$N_F(h) \ll s^2 C(r, h) \tag{1.4}$$

where $C(r, h) = h^{2/r}(1 + \log h^{1/r})$. From a result of Mahler [5], it is known that the factor $h^{2/r}$ in $C(r, h)$ is unavoidable while the logarithmic factor was improved by Thunder when h is large, see [11] and [12].

When s is as large as r, (1.4) is weaker than (1.3). It was conjectured in [8] that it may possible to replace the factor s^2 above by s. In [9], some results were given where the factor s^2 was improved. For instance, the following results were proven.

(i) We always have $|r_i - r_w| \geq |i - w| \geq 1$ for $i \neq w$. Suppose $|r_i - r_w| \geq c_1|i - w|$ with $c_1 \geq 1$, an absolute constant. Then,

$$N_F(h) \ll s^{1+\frac{1}{c_1}}\, C(r, h).$$

Thus, the exponent of s is < 2 whenever $c_1 > 1$.

(ii) Suppose $|r_i - r_w| \geq \frac{1}{3}|i - w| \log |i - w|$. Then,

$$N_F(h) \ll s \log^3 s\, C(r, h).$$

In another direction, it was shown that if the coefficients of $F(X,Y)$ satisfy

$$\left|\frac{a_i}{a_0}\right|^{1/r_i} \leq \left|\frac{a_s}{a_0}\right|^{1/r_s} \quad \text{for } i = 1, \ldots, s-1 \tag{1.5}$$

and $r \geq \max(4s, s\log^3 s)$, then

$$N_F(h) \ll s(\log s)\, C(r,h). \tag{1.6}$$

If $r < \max(4s, s\log^3 s)$, then by (1.3), we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r,h).$$

Thus, under the condition (1.5), we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r,h).$$

In particular, the above estimate holds whenever $|a_0| = |a_s| = H$ where H is the maximum of the absolute values of the coefficients of F. Hence, the estimate holds for forms F with coefficients ± 1.

Let q_1 be the smallest integer, $0 \leq q_1 \leq s$, with $|a_{q_1}| = H$ and let q_2 be the largest integer, $0 \leq q_2 \leq s$, with $|a_{q_2}| = H$. The condition (1.5) implies that

$$(q_1, q_2) \in \{(0,0), (0,s), (s,s)\}.$$

In this note, we shall consider a few more cases of (q_1, q_2). Throughout, we use the following assumptions A.

A: (a) $r \geq \max(4s, s\log^3 s)$.
(b) $|a_i|^{r_{q_1}} \leq |a_{q_1}|^{r_i} |a_0|^{r_{q_1} - r_i}$ for $0 \leq i \leq q_1$.
(c) $|a_i|^{r - r_{q_2}} \leq |a_{q_2}|^{r - r_i} |a_s|^{r_i - r_{q_2}}$ for $q_2 \leq i \leq s$.

Note that A(b) holds trivially if $q_1 = 0$ and A(c) holds trivially if $q_2 = s$. We prove the following result.

Theorem 1.1. *Suppose that the assumption A holds. Then,* (1.6) *is valid in the following three cases:*

(i) $q_1 = 0$, $0 < q_2 < s$ *and* $H \leq |a_s|^{\frac{r}{\max(s, r_{q_2})}}$;
(ii) $q_2 = s$, $0 < q_1 < q_2$ *and* $H \leq |a_0|^{\frac{r}{\max(s, r - r_{q_1})}}$;
(iii) $q_1 \neq 0$, $q_2 \neq s$ *and* $H \leq \min\left(|a_0|^{\frac{r-s}{r}} |a_s|^{\frac{r_{q_1}}{r}}, |a_0|^{\frac{r - r_{q_2}}{r}} |a_s|^{\frac{r-s}{r}}\right)$.

Remark.

(a) When $r < \max(4s, s\log^3 s)$, we use (1.3) to obtain

$$N_F(h) \ll s(\log s)^{3/2}\, C(r, h).$$

Therefore, under the conditions of the theorem, we have

$$N_F(h) \ll s(\log s)^{3/2}\, C(r, h).$$

(b) We may assume that s is large, as otherwise inequality (1.4) is sufficient.

2 Preliminaries

Let $F(X, Y)$ be given by (1.1). Let $X_1, X_2 > 0$. Divide the solutions (x, y) of (1.2) into three sets as

$$\max(|x|, |y|) > X_1; \max(|x|, |y|) \le X_1 \text{ and } \min(|x|, |y|) \ge X_2;$$
$$\min(|x|, |y|) < X_2.$$

Denote the number of primitive solutions in these sets by $P_{lar}(X_1)$, $P_{med}(X_1, X_2)$, and $P_{sma}(X_2)$, respectively. If $X_2 > X_1$, put $P_{med}(X_1, X_2) = 0$. Let $P(h)$ be the number of primitive solutions of (1.2). Thus,

$$P(h) = P_{lar}(X_1) + P_{med}(X_1, X_2) + P_{sma}(X_2).$$

We can bound $N_F(h)$ by finding an upper estimate for $P(h)$. Choose numbers a, b with $0 < a < b < 1$. Define

$$t = \sqrt{2/(r + a^2)}, \lambda = 2/((1 - b)t),$$

$$A = \frac{1}{a^2}\left(\log M + \frac{r}{2}\right).$$

Further, we put

$$B = \frac{2^r r^{r/2} M^r h}{\sqrt{|D|}}, R = e^{800\log^3 r},$$

$$Y_E = (2B\sqrt{|D|})^{1/(r-\lambda)}(4e^A)^{\lambda/(r-\lambda)}, Y_W = R^{1/(r-\lambda)} Y_E$$

$$Y_S = (4^r(rs)^{2s} R^s h)^{\frac{1}{r-2s}}.$$

We take

$$X_1 = Y_W \text{ and } X_2 = Y_S.$$

Mueller and Schmidt [8] had shown that

$$P_{lar}(Y_W) \ll s. \tag{2.1}$$

To estimate the number of small solutions, we use the following lemma from [8].

Lemma 2.1. *[8, Lemma 18]*
Let $F(X, Y)$ be given by (1.1) and let $r \geq 4s$. Then for any $Y \geq 1$, we have

$$P_{sma}(Y) \ll (rs^2)^{2s/r} h^{2/r} + s\ Y.$$

The next lemma is a consequence of the above lemma.

Lemma 2.2. *Let $F(X, Y)$ be given by (1.1). Then,*

$$P_{sma}(Y_S) \ll s\ h^{2/r} \textit{whenever } r \geq s \log^3 s.$$

For dealing with the medium solutions, we use the Archimedean Newton polygon of the polynomial $F(X, 1)$. This is the lower boundary of the convex hull of the points $P_i = (r_i, -\log|a_i|), 0 \leq i \leq s$. Let $L_{i,j}$ denote the line joining P_i to P_j with slope $\sigma(i, j)$, say. Further, let

$$\hat{F}(X, 1) = a_s + a_{s-1} X^{r_s - r_{s-1}} + \cdots + a_1 X^{r_s - r_1} + a_0 X^{r_s}$$

be the reciprocal polynomial of $F(X, 1)$. Let $Q_i = (r_s - r_i, -\log|a_i|)$ and let $L'_{i,j}$ denote the line joining Q_i to Q_j with slope $\sigma'(i, j)$.

Lemma 2.3. *Suppose that the coefficients of $F(X, Y)$ satisfy the assumptions A(b) and A(c). Then, the edges of the Archimedean Newton polygon of $F(X, 1)$ are L_{0,q_1}, L_{q_1,q_2}, and $L_{q_2,s}$. Further, $\sigma(q_1, q_2) = 0$ and every root α of $F(x, 1)$ satisfies*

$$\frac{1}{2} e^{\sigma(0,q_1)} < |\alpha| < 2e^{\sigma(q_2,s)}. \tag{2.2}$$

Every root β of $\hat{F}(X, 1)$ satisfies

$$\frac{1}{2} e^{-\sigma(q_2,s)} < |\beta| < 2e^{-\sigma(0,q_1)}.$$

Proof. Put

$$\sigma_1 = \sigma(0, q_1) \text{ and } \sigma_2 = \sigma(q_2, s).$$

By A(b), we have

$$\sigma(0, i) \geq \sigma_1 \text{ for } 0 \leq i \leq q_1.$$

By A(c), we have

$$\sigma(q_2, i) \geq \sigma_2 \text{ for } q_2 \leq i \leq s.$$

Hence, the Archimedean Newton polygon consists of L_{0,q_1} and $L_{q_2,s}$ as the left most edge and right most edge, respectively. Since the height of the polynomial is attained at a_{q_1} and at a_{q_2}, we see that $\sigma(q_1, q_2) = 0$ and $\sigma(q_1, i) \geq 0$ for $q_1 \leq i \leq q_2$. Thus, L_{q_1,q_2} is the third edge. Now we prove (2.2). By the convexity of the Newton polygon, we have

$$\sigma_2 \geq \sigma(i, s) \text{ for } 0 \leq i \leq s. \tag{2.3}$$

Let $z = e^{\sigma_2 w}$ with $|w| \geq 2$. Then,

$$|F(z, 1)| \geq |a_s| e^{\sigma_2 r_s} \left(|w|^{r_s} - \frac{|a_{s-1}|}{|a_s|} e^{\sigma_2 (r_{s-1} - r_s)} |w|^{r_{s-1}} - \cdots - \frac{|a_0|}{|a_s|} e^{-\sigma_2 r_s} \right).$$

By (2.3), we have

$$\frac{|a_i|}{|a_s|} e^{\sigma_2 (r_i - r_s)} \leq 1.$$

Hence,

$$|F(z, 1)| \geq |w|^{r_s} - |w|^{r_{s-1}} - \cdots - 1 > 0$$

since $|w| \geq 2$. Thus, every root α of $F(X, 1)$ has $|\alpha| < 2e^{\sigma_2}$.

To get the lower bound, we use the reciprocal polynomial $\hat{F}(X, 1)$. The Archimedean Newton polygon of this polynomial has edges L'_{s,q_2}, L'_{q_2,q_1}, and $L'_{q_1,0}$. Arguing as above, we find that every root β which is the inverse of some root α of $F(X, 1)$ satisfies $|\beta| \leq 2e^{\sigma'(q_1,0)}$, where $\sigma'(q_1, 0)$ is the slope of $L'_{q_1,0}$. Hence,

$$|\alpha| \geq \frac{1}{2} e^{-\sigma'(q_1,0)}.$$

Now the result follows on noticing that $\sigma'(q_1, 0) = -\sigma_1$. □

The Archimedean Newton polygons when conditions (i), (ii), or (iii) of Theorem 1.1 hold, are shown in Figure 1.

Another tool needed to estimate the medium solutions is the Diophantine approximation property. Let S be the set of roots $\alpha_1, \cdots, \alpha_r$ of $f(z) = F(z, 1)$ and S^* the set of roots of $F(1, z)$. Then, $S^* = \{\alpha_1^{-1}, \cdots, \alpha_r^{-1}\}$. Let (x, y) be a solution of (1.2) with $y \neq 0$. Define

$$d\left(S, \frac{x}{y}\right) = \min_{1 \leq i \leq r} \left| \alpha_i - \frac{x}{y} \right|.$$

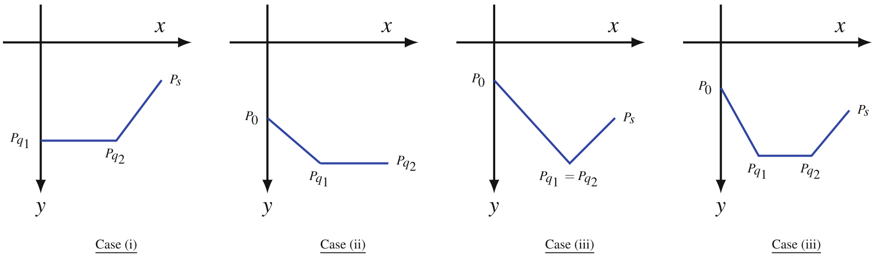


Fig. 1 Archimedean Newton polygon.

It was shown in [8, Lemma 7], that there exists $S_1 \subseteq S$ with $|S_1| \ll s$ such that

$$d\left(S_1, \frac{x}{y}\right) \leq R\, d\left(S, \frac{x}{y}\right). \tag{2.4}$$

Suppose that $d(S, x/y) = |\alpha - x/y|$ for some $\alpha \in S$. If $f^{(u)}(\alpha) \neq 0$ for some u with $1 \leq u \leq r$, then by [8, Lemma 10], we have

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{r}{2}\left(\frac{2^r h}{|f^{(u)}(\alpha)||y|^r}\right)^{1/u}. \tag{2.5}$$

Let e, h be two nonnegative integers. Let $(e)_h$ be the Pochhammer symbol defined as

$$(e)_h = \begin{cases} 0 & \text{if } e = 0 \\ 1 & \text{if } h = 0 \\ e(e-1)\cdots(e-h+1) & \text{otherwise.} \end{cases}$$

Using the explanations given in [8, p. 223–231], we can obtain

$$\sum_{u=1}^{s} E_u^{(s)} \alpha^u f^{(u)}(\alpha) = a_s \alpha^{r_s} \prod_{0 \leq i < j \leq s} (r_i - r_j) \tag{2.6}$$

where

$$E_u^{(s)} = (-1)^{s+u} \det \begin{pmatrix} 1 & \cdots & 1 \\ (r_0)_1 & \cdots & (r_{s-1})_1 \\ \vdots & \vdots & \vdots \\ (r_0)_{u-1} & \cdots & (r_{s-1})_{u-1} \\ (r_0)_{u+1} & \cdots & (r_{s-1})_{u+1} \\ \vdots & \vdots & \vdots \\ (r_0)_s & \cdots & (r_{s-1})_s \end{pmatrix}.$$

From [8, Eqns (6.12) & (6.13)], we get that

$$|E_u^{(s)}| \le 2^s (s^2 r)^{s-1} \prod_{0 \le i < j \le s} (r_j - r_i).$$

We also refer to [9] for more details. Using the above estimate for $|E_u^{(s)}|$ in (2.6), we find that there exists u with $1 \le u \le s$ such that

$$|f^{(u)}(\alpha)| \ge |a_s||\alpha|^{r-u} 2^{-s} (s^2 r)^{-(s-1)} s^{-1}.$$

The following lemma is now immediate from (2.4) and (2.5).

Lemma 2.4. *There exists a set $S_1 \subseteq S$ with $|S_1| \ll s$, such that for some $\alpha \in S_1$, we have*

$$\left|\alpha - \frac{x}{y}\right| \le \frac{rR}{2} \left(\frac{s(rs^2)^{s-1} 2^{r+s} h}{|y|^r |a_s||\alpha|^{r-u}} \right)^{1/u}.$$

A similar inequality holds with (x, y) replaced by (y, x) for some set $S_2 \subseteq S^*$ of roots with $|S_2| \le s$.

The following is a lemma on counting the number of elements in a set satisfying some gap conditions. (See [9, Lemma 2.1(i)]).

Lemma 2.5. *Let $n \ge 2$ and let $U = \{u_1, \cdots, u_n\}$ be a set together with a map $T : U \to \mathbb{R}^*$ such that*

$$A_1 \le T(u_1) \le T(u_2) \le \cdots \le T(u_n)$$

and

$$T(u_i) \ge \beta T(u_{i-1})^\gamma \ for\ 2 \le i \le n\ with\ \beta > 0, \gamma \ge 2.$$

Let

$$\kappa = \begin{cases} 2\ if\ \beta > 1 \\ 1\ if\ \beta \le 1. \end{cases}$$

Suppose that $T(u_n) \le B_1$ and $A_1 \beta^{1/(\kappa(\gamma-1))} > 1$. Then,

$$n \le 1 + \frac{1}{\log \gamma} \log \left(\frac{\log B_1}{\log A_1 + (\log \beta)/(\kappa(\gamma - 1))} \right).$$

Proof. By induction, we get

$$\begin{aligned} T(u_n) &\ge \beta^{1+\gamma+\cdots+\gamma^{n-2}} T(u_1)^{\gamma^{n-1}} \\ &\ge (\beta^{1/(\kappa(\gamma-1))} T(u_1))^{\gamma^{n-1}}. \end{aligned} \tag{2.7}$$

Since $T(u_n) \leq B_1$, from (2.7), we get

$$(\beta^{1/(\kappa(\gamma-1))}T(u_1))^{\gamma^{n-1}} \leq B_1.$$

Taking logarithms twice, we get the assertion of the lemma. □

3 Proof of Theorem 1.1

By (2.1) and Lemma 2.1, it is enough to estimate $P_{med}(Y_W, Y_S)$. We shall consider S_1 from Lemma 2.4. The argument for S_2 is similar. We claim that

$$|a_s\alpha^{r-u}| \geq H^{u/r}2^{-(r-u)}.$$

We prove this when condition (ii) of the theorem holds. The other cases are similar. By Lemma 2.3,

$$\begin{aligned} |a_s\alpha^{r-u}| &> |a_s|e^{(r-u)\sigma_1}2^{-(r-u)} \\ &= |a_s|e^{(r-u)\frac{(-\log H+\log|a_0|)}{rq_1}}2^{-(r-u)}. \end{aligned}$$

Thus, our claim is true if

$$|a_s||a_0|^{\frac{r-u}{rq_1}} \geq H^{\frac{r-u}{rq_1}+\frac{u}{r}}.$$

Since $q_2 = s$, we have $|a_s| = H$. Therefore, the above inequality holds if

$$|a_0| \geq H^{\frac{r-rq_1}{r}},$$

which is true by our assumption. Hence, the claim follows. Thus, by Lemma 2.4, for $y \geq Y_S$,

$$\left|\alpha - \frac{x}{y}\right| < \frac{rR}{2H^{1/r}}\left(\frac{s2^{2r}(rs^2)^{s-1}h}{y^r}\right)^{1/s}. \tag{3.1}$$

Let $U = \{(x_1, y_1), \cdots, (x_\nu, y_\nu)\}$ be the set of all solutions of (3.1) with $\gcd(x_i, y_i) = 1$ and

$$Y_S \leq y_1 \leq \cdots \leq y_\nu \leq Y_W.$$

Suppose $\nu \geq 2$. Then,

$$\frac{1}{y_i y_{i+1}} \leq \left|\frac{x_i}{y_i} - \frac{x_{i+1}}{y_{i+1}}\right| \leq \left|\alpha - \frac{x_i}{y_i}\right| + \left|\alpha - \frac{x_{i+1}}{y_{i+1}}\right|$$

$$\leq \frac{K}{2y_i^{r/s}} + \frac{K}{2y_{i+1}^{r/s}} \leq \frac{K}{y_i^{r/s}},$$

where

$$K = R(rs)^2 4^{r/s} h^{1/s} H^{-1/r}.$$

Thus, we have

$$y_{i+1} \geq K^{-1} y_i^{r/s-1}.$$

We apply Lemma 2.5 with $T((x_i, y_i)) = y_i$, $\beta = 1/K$, $\gamma = \frac{r-s}{s}$, $A_1 = Y_S$ and $B_1 = Y_W$. Note that $\gamma = r/s - 1 \geq \max(3, \log^3 s - 1)$. Also, $R \geq 4(rs)^4$. Further, $\log Y_W \ll \sqrt{r} + \log H + \log h^{1/r}$. Hence, by Lemma 2.5, we get

$$\begin{aligned}
\nu &\ll 1 + \frac{1}{\log \gamma} \log \left(\frac{\log Y_W}{\log Y_S + \frac{\log \beta}{\kappa(\gamma - 1)}} \right) \\
&\ll \frac{1}{\log \gamma} \log \left(\frac{2(r-2s)(\sqrt{r} + \log H + \log h^{1/r})}{\log H} \right) \\
&\ll \frac{\log r + \log(1 + \log h^{1/r})}{\log \gamma}.
\end{aligned}$$

Suppose $r \ll s^3$. Then,

$$\nu \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s} \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s}.$$

If $r \gg s^3$, then

$$\nu \ll 1 + \frac{\log(1 + \log h^{1/r})}{\log r} \ll \frac{\log s + \log(1 + \log h^{1/r})}{\log s}.$$

Thus,

$$P_{med}(Y_W, Y_S) \ll s \left(\frac{\log s + \log(1 + \log h^{1/r})}{\log\log\ s} \right).$$

We combine the above inequality with (2.1) and Lemma 2.2 to get

$$P(h) \ll s(\log s) h^{2/r}.$$

Using a partial summation argument, it was shown in [8, p. 212] that

$$N_F(h) \ll P(h) + h^{1/r} r^{-1} \sum_{n=1}^{h-1} P(n) n^{-1-(1/r)}.$$

Substituting our estimate for $P(h)$, we obtain the result of the theorem.

References

1. J.-H. Evertse, On the equation $ax^n - by^n = c$. Compositio Math. **47**, 288–315 (1982)
2. J.-H. Evertse, K. Győry, *Thue Inequalities with a Small Number of Solutions*. The Mathematical Heritage of C.F. Gauss (World Scientific Publishing Company, Singapore, 1991), pp. 204–224
3. S. Hyyrö, Über die Gleichung $ax^n - by^n = c$ und das Catalansche problem. Ann. Acad. Sci. Fenn. Ser. AI **355**, 1–50 (1964)
4. K. Mahler, An application of Jensen's formula to polynomials. Mathematika **7**, 98–100 (1960)
5. K. Mahler, Zur approximation algebraischer Zahlen III. Acta Math. **62**, 91–166 (1934)
6. J. Mueller, Counting solutions of $|ax^r - by^r| \leq h$. Q. J. Math. Oxf. **38**(2), 503–513 (1987)
7. J. Mueller, W.M. Schmidt, Trinomial Thue equations and inequalities. J. Rein. Angew. Math. **379**, 76–99 (1987)
8. J. Mueller, W.M. Schmidt, Thue's equation and a conjecture of Siegel. Acta Math. **160**, 207–247 (1988)
9. N. Saradha, D. Sharma, Contributions to a conjecture of Mueller and Schmidt on Thue inequalities, to appear in proceedings–Mathematical Sciences, Indian Acad. Sci. **127**(4), 565–584
10. W.M. Schmidt, Thue equations with few coefficients. Trans. Am. Math. Soc. **303**, 241–255 (1987)
11. J.L. Thunder, The number of solutions to cubic Thue inequalities. Acta Arith. **66**(3), 237–243 (1994)
12. J.L. Thunder, On Thue inequalities and a conjecture of Schmidt. J. Number Theory **52**, 319–328 (1995)

Basic Hypergeometric Summations from Rook Theory

Michael J. Schlosser and Meesue Yoo

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract We employ a one-variable extension of q-rook theory to give combinatorial proofs of some basic hypergeometric summations, including the q-Pfaff–Saalschütz summation and a ${}_4\phi_3$ summation by Jain.

Keywords Basic hypergeometric series · q-Pfaff–Saalschütz summation · Jain summation · Rook numbers · q-analogues · Alpha-parameter model · Matchings

2010 Mathematics Subject Classification Primary 05A19 · Secondary 05A15 05A30 · 11B65 · 11B73 · 11B83

Partially supported by FWF Austrian Science Fund grant F50-08, and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (SRC-AORC) (No. 2016R1A5A1008055).

M. J. Schlosser (✉) · M. Yoo
Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1,
A-1090 Vienna, Austria
e-mail: michael.schlosser@univie.ac.at

M. Yoo
AORC center, Sungkyunkwan University, Seoul 16419, Republic of Korea
e-mail: meesue.yoo@skku.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_37

1 Introduction

The theory of q-series has a prominent history. It made its first appearance in a combinatorial study by Euler on partitions of numbers. Among the first q-series identities were explicit summations for two different q-analogues of the exponential series. These identities were later unified by Gauß, Heine and Cauchy who, all three independent from each other, discovered and proved the nonterminating q-binomial theorem. This initiated the systematic study of q-hypergeometric series, or synonymously, basic hypergeometric series, "basic" referring to the *base* q, as objects of their own interest, separate from combinatorics. While in the early days only a small number of mathematicians studied the combinatorics of q-series (most notably, J. J. Sylvester in the nineteenth century, and P. A. MacMahon and I. Schur in the early twentieth century, to name just a few figures whose research had big impact), the situation rapidly changed in the 1960s when B. Gordon found extensions of the Rogers–Ramanujan identities with combinatorial interpretations, after which many more people entered the scene. See Andrews' book chapter [1] for an account of the history of q-series and partitions. Further, the preface of Gasper and Rahman's textbook [5] provides a brief history of basic hypergeometric series, and the book itself contains further background on the subject.

Basic hypergeometric series appear from time to time in combinatorial studies. It is particularly instructive to see combinatorial proofs of q-series identities. Having a combinatorial interpretation of an identity at hand leads to a better understanding, since one gets a feeling why the identity is true. Now, focusing on combinatorial interpretations and given a reasonably simple identity, it is by all means legitimate to ask the question: is there a combinatorial proof for it? For instance, for the well-known q-Pfaff–Saalschütz summation,

$$\sum_{k=0}^{n} \frac{(a, b, q^{-n}; q)_k}{(q, c, abq^{1-n}/c; q)_k} q^k = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}, \tag{1.1}$$

(see Section 2 for the notation) several combinatorial proofs are known [2, 8, 9, 16, 21, 22]. But what about a combinatorial proof of the following summation by Jain [12] (which is a q-analogue of a ${}_3F_2$ summation by Bailey [3] and can be written as a summation for a specific ${}_4\phi_3$ series)?

$$\sum_{k=0}^{n} \frac{(a, b; q)_k (q^{-2n}; q^2)_k}{(q, q^{-2n}; q)_k (abq; q^2)_k} q^k = \frac{(aq, bq; q^2)_n}{(q, abq; q^2)_n}. \tag{1.2}$$

In this paper, we present a new combinatorial proof of (1.1) and also, to the best of our knowledge, a first combinatorial proof of (1.2), in addition to a few similar results. We achieve this by employing a specific one-variable extension of q-rook theory with an extra variable a, which we shall refer to as $(a; q)$-rook theory for short. (The $(a; q)$-rook numbers that we are dealing with in this paper are actually special cases of the elliptic rook numbers which we recently have considered in [18–20]. While the elliptic rook numbers satisfy nice identities, they don't factorize in general. However, for particular boards, the $(a; q)$-rook numbers do factorize into closed forms. This is the reason why we focus on the $(a; q)$-case here which is still more general than the q-case.)

Already earlier, Haglund [10] has made out intimate connections between rook theory and (basic) hypergeometric series. In particular, he showed that a big class of (q-)rook numbers generally admit a representation in terms of (basic) hypergeometric series of Karlsson–Minton type. In our case, we are on the one hand (for three different rook models) working with $(a; q)$-rook numbers, i.e. we add an extra parameter to the q-rook numbers. On the other hand, we are looking at very special situations, obtained by restricting to special boards, where the $(a; q)$-rook numbers nicely factorize. Since the $(a; q)$-rook numbers satisfy certain product formulas, we are thus able to obtain explicit summations, by substituting the factorized forms in the product formulas.

In Section 2, we recall standard q-series notation and introduce the special $(a; q)$-weights that we use. Section 3 is devoted to $(a; q)$-rook theory. We explain all the ingredients we need for the $(a; q)$-extensions of the different rook models that we work with, namely, the standard model, the (more general) alpha-parameter model and the matching model. From these, we deduce basic hypergeometric summations as applications.

2 Standard q-notation and $(a; q)$-weights

For a parameter q, called the *base*, and variable u, the q-shifted factorial is defined by

$$(u; q)_0 = 1, \qquad \text{and} \qquad (u; q)_n = (1 - u)(1 - uq) \dots (1 - uq^{n-1}).$$

(The index n can also be ∞, then the product is an infinite product in which case one requires $|q| < 1$, for convergence.) For brevity, we frequently use the notation

$$(a_1, \dots, a_m; q)_n = (a_1; q)_n \dots (a_m; q)_n.$$

The q-number of z is defined as

$$[z]_q = \frac{1 - q^z}{1 - q}.$$

We now introduce *(a;q)-weights* which include an additional variable a. Define

$$w_{a;q}(k) = \frac{(1 - aq^{2k+1})}{(1 - aq^{2k-1})} q^{-1},$$

$$W_{a;q}(k) = \frac{(1-aq^{2k+1})}{(1-aq)} q^{-k},$$

$$[z]_{a;q} = \frac{(1-q^z)(1-aq^z)}{(1-q)(1-aq)} q^{1-z},$$

for any value k, which we call the *small* weights, *big* weights and the $(a;q)$-*number* of z, respectively. Note that in the limit case $a \to \infty$, we recover the q-weights

$$\lim_{a\to\infty} w_{a;q}(k) = q, \quad \lim_{a\to\infty} W_{a;q}(k) = q^k, \quad \lim_{a\to\infty} [z]_{a;q} = \frac{1-q^z}{1-q} = [z]_q.$$

For a positive integer k, we have

$$W_{a;q}(k) = \prod_{i=1}^{k} w_{a;q}(i).$$

Other useful properties are

$$[y+z]_{a;q} = [y]_{a;q} + W_{a;q}(y)[z]_{aq^{2y};q},$$

and

$$W_{a;q}(k+n) = W_{a;q}(k) W_{aq^{2k};q}(n).$$

Remark 2.1. This $(a;q)$-weight was first defined in [17] to generalize the binomial theorem for noncommuting variables. That is, in the unital algebra $\mathbb{C}_{a;q}[x,y]$ over $\mathbb{C}$ defined by the following commutation relations:

$$\begin{aligned} yx &= \frac{(1-aq^3)}{(1-aq)} q^{-1} xy, \\ xa &= qax, \\ ya &= q^2 ay, \end{aligned}$$

the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a;q} x^k y^{n-k}$$

holds, where the $(a;q)$-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a;q} := \frac{(q^{1+k}, aq^{1+k}; q)_{n-k}}{(q, aq; q)_{n-k}} q^{k(k-n)} = \frac{[n]_{a;q}!}{[k]_{a;q}![n-k]_{a;q}!}, \tag{2.1}$$

with the $(a;q)$-factorials being defined by

$$[0]_{a;q}! = 1, \qquad [n]_{a,q}! = [n]_{a,q}[n-1]_{a,q}!.$$

The $(a;q)$-binomial coefficients are symmetric in $(k, n-k)$ (whereas the more general elliptic extension of (2.1), considered in [16, 17], is not). They satisfy the two recursions

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a;q} = \begin{bmatrix} n \\ k \end{bmatrix}_{a;q} + \frac{(1-aq^{2n+2-k})}{(1-aq^k)} q^{k-n-1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a;q},$$
$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a;q} = \frac{(1-aq^{n+1+k})}{(1-aq^{n+1-k})} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_{a;q} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a;q},$$

which, together with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{a;q} = 1, \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{a;q} = 0, \qquad \text{for } k > n \text{ or } k < 0,$$

determine them uniquely. For $a \to \infty$ the $(a;q)$-binomial coefficients reduce to the usual q-binomial coefficients.

3 $(a;q)$-Rook theory

For an introduction to classical rook theory, see [4]. A lot of material on generalized rook theory which we survey is borrowed from our papers [18–20] on elliptic rook theory. As mentioned in the introduction, the $(a;q)$-case is just a special case of the elliptic case which admits particularly attractive closed formulas. We utilize the closed formulas from the $(a;q)$-extensions of different rook models to derive some concrete basic hypergeometric summations.

3.1 $(a;q)$-Extension of the standard model

Let $\mathbb{N}$ denote the set of positive integers. We consider a finite subset of the $\mathbb{N} \times \mathbb{N}$ grid which we refer to as a *board*, and label the columns and rows by $1, 2, \ldots,$ from the left and from the bottom, respectively. We use (i, j) to denote the cell in the intersection of the column i and the row j.

Let $B(b_1, b_2, \ldots, b_n)$ denote the set of cells

$$B = B(b_1, \ldots, b_n) = \{(i, j) \mid 1 \le i \le n,\ 1 \le j \le b_i\},$$

for nonnegative integer b_i's, for all i. If a board B can be represented by the set $B(b_1, \dots, b_n)$ with non-decreasing integer sequence $0 \le b_1 \le \cdots \le b_n$, then the board $B = B(b_1, \dots, b_n)$ is called a *Ferrers board*. Given a Ferrers board, we say that we *place k nonattacking rooks* in B by choosing a k-subset of B such that no two elements have a common coordinate. Let $\mathcal{N}_k(B)$ denote the set of all nonattacking placements of k rooks in B. Note that $|\mathcal{N}_k(B)|$ is the original k-th rook number defined in [13].

Given a rook placement $P \in \mathcal{N}_k(B)$, a rook in P is said to *cancel* all the cells to the right in the same row and all the cells below it in the same column. Let $U_B(P)$ denote the set of cells in $B - P$ which are not cancelled by any rook in P. We define the $(a; q)$-analogue of the k-th rook number by assigning the small weights $w_{a;q}(j)$ to the respective cells in $U_B(P)$, depending on their position and the configuration of rooks.

Definition 3.1. Given a Ferrers board $B = B(b_1, \dots, b_n)$, let the k-th $(a; q)$-rook number be

$$r_k(a, q; B) = \sum_{P \in \mathcal{N}_k(B)} wt(P), \tag{3.1}$$

where

$$wt(P) = \prod_{(i,j) \in U_B(P)} w_{a;q}(i - j - r_{(i,j)}(P)),$$

and $r_{(i,j)}(P)$ counts the number of rooks in P positioned in the north-west region of (i, j).

This $(a; q)$-analogue of the rook numbers satisfy the following product formula which was proved with original rook numbers, i.e. in the $a \to \infty$, $q \to 1$ case, by Goldman, Joichi, and White [7].

Theorem 3.1 ([18]). For any Ferrers board $B = B(b_1, \dots, b_n)$, we have

$$\prod_{i=1}^{n} [z + b_i - i + 1]_{aq^{2(i-1-b_i)};q} = \sum_{k=0}^{n} r_{n-k}(a, q; B) \prod_{j=1}^{k} [z - j + 1]_{aq^{2(j-1)};q}.$$

Remark 3.1. In [18], we prove Theorem 3.1 with more general rook numbers. That is, the rook numbers $r_k(a, b; q, p; B)$ and the weights used to define r_k in (3.1) are elliptic (i.e. meromorphic and doubly periodic) and include two more parameters b and p. The $(a; q)$-rook numbers can be obtained from the elliptic ones by letting $p \to 0$ and $b \to 0$.

By distinguishing the cases when there is a rook or not in the last column, we obtain a recursion for the $(a; q)$-rook numbers.

Proposition 3.1. *Let B be a Ferrers board with l columns of height at most m, and $B \cup m$ denote the board obtained by adding the $(l + 1)$-st column of height m to the right of B. Then, for integer k with $1 \le k \le l + 1$, we have*

$$r_k(a, q; B \cup m) = \\ W_{aq^{2(l-m)};q}(m-k) r_k(a, q; B) + [m-k+1]_{aq^{2(l-m)};q} r_{k-1}(a, q; B),$$

assuming the conditions

$$\begin{aligned} r_k(a, b; q, p; B) &= 0 \quad \text{for} \quad k < 0 \quad \text{or} \quad k > l, \text{and} \\ r_0(a, b; q, p; B) &= 1 \quad \text{for} \quad l = 0, \text{i.e. for } B \text{ being the empty board.} \end{aligned}$$

In the case of a rectangular shape board $B = [l] \times [m]$, where $[n] := \{1, 2, \ldots, n\}$, the $(a; q)$-rook number has a closed form expression which can be proved by the recursion in Proposition 3.1:

$$r_k(a, q; [l] \times [m]) = \\ q^{\binom{k+1}{2}-lm} \begin{bmatrix} l \\ k \end{bmatrix}_q \frac{[m]_q!}{[m-k]_q!} \frac{(aq^{l-m-k}; q)_k (aq^{1+2l-2m}; q^2)_{m-k}}{(aq^{1-2m}; q^2)_m}. \tag{3.2}$$

For more details, including the omitted proofs, see [18].

3.1.1 *r*-Restricted Lah numbers

The r-restricted Lah numbers count the number of placements of the elements $1, 2, \ldots, n$ into k nonempty tubes of linearly ordered elements such that $1, 2, \ldots, r$ are in distinct tubes (cf. [15] or [14]). These numbers admit a rook theoretic interpretation when B is the board $\mathsf{L}_n^{(r)} = [n+r-1] \times [n-r]$. In [18, Subsection 3.4], we have established a correspondence between the rook configurations P of $n-k$ nonattacking rooks on $\mathsf{L}_n^{(r)}$ and the set of placements T of the elements $1, 2, \ldots, n$ into k nonempty tubes of linearly ordered elements such that the first r numbers $1, 2, \ldots, r$ are in distinct tubes. For the full description of the correspondence, refer to [18].

For the Ferrers board $B = \mathsf{L}_n^{(r)}$, the product formula in Theorem 3.1 becomes

$$\begin{aligned} &\prod_{i=1}^{n-r} [z+n-i]_{aq^{2(i-n)};q} \prod_{i=1}^{r} [z-i+1]_{aq^{2(i-1)};q} \\ &= \sum_{k=r}^{n} r_{n-k}(aq^{2(1-r)}, q; \mathsf{L}_n^{(r)}) \prod_{j=1}^{k} [z-j+1]_{aq^{2(j-1)};q}, \end{aligned} \tag{3.3}$$

after doing certain shifts of variables and cancellation of factors. We define an $(a; q)$-analogue of the r-restricted Lah numbers by

$$\mathscr{L}_{n,k}^{(r)}(a, q) := r_{n-k}(aq^{2(1-r)}, q; \mathsf{L}_n^{(r)}).$$

It can be shown that $\mathscr{L}_{n,k}^{(r)}(a,q)$ satisfy the following recursion:

$$\mathscr{L}_{n+1,k}^{(r)}(a,q) = W_{aq^{-2n};q}(n+k-1)\,\mathscr{L}_{n,k-1}^{(r)}(a,q) + [n+k]_{aq^{-2n};q}\,\mathscr{L}_{n,k}^{(r)}(a,q),$$

assuming the initial conditions

$$\begin{aligned} \mathscr{L}_{n,k}^{(r)}(a,q) &= 0 \qquad \text{for } k < r-1 \text{ or } k > n, \\ \mathscr{L}_{r-1,r-1}^{(r)}(a,q) &= 1 \qquad \text{(an artificial but felicious initial condition).} \end{aligned}$$

Since the board $\mathsf{L}_n^{(r)}$ is of rectangular shape, (3.2) gives the closed form formula for $\mathscr{L}_{n,k}^{(r)}(a,q)$, namely,

$$\mathscr{L}_{n,k}^{(r)}(a,q) = q^{\binom{k}{2}-\binom{n}{2}-n(k-1)+2\binom{r}{2}} \begin{bmatrix} n+r-1 \\ k+r-1 \end{bmatrix}_q \frac{[n-r]_q!}{[k-r]_q!} \times \frac{(aq^{1-n+k};q)_{n-k}(aq^{1+2r};q^2)_{k-r}}{(aq^{3-2n};q^2)_{n-r}}. \tag{3.4}$$

Combining (3.4) with the product formula (3.3) gives a combinatorial proof of the q-Pfaff-Saalschütz sum, in the following form:

Proposition 3.2.

$$\frac{(q^{z+r};q)_n(a^{-1}q^{r-z};q)_n}{(a^{-1};q)_n(q^{2r};q)_n} = \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{r-z};q)_k(aq^{z+r};q)_k}{(q;q)_k(q^{2r};q)_k(aq^{1-n};q)_k} q^k. \tag{3.5}$$

Proof. If we replace $r_{n-k}(aq^{2(1-r)},q;\mathsf{L}_n^{(r)})$ by the closed form given in (3.4), we obtain

$$\begin{aligned} &\frac{(q^{z+r};q)_{n-r}(aq^{z-n+1};q)_{n-r}}{(1-q)^{n-r}(aq^{3-2n};q^2)_{n-r}} \frac{(q^{z-r+1};q)_r(aq^z;q)_r}{(1-q)^r(aq;q^2)_r} q^{-nz+\frac{1}{2}n(3-n)+r^2-r} \\ &= \sum_{k=r}^{n} q^{\binom{k}{2}-\binom{n}{2}-n(k-1)+2\binom{r}{2}-kz+\binom{k+1}{2}} \\ &\qquad \times \frac{(q;q)_{n+r-1}}{(q;q)_{n-k}(q;q)_{k+r-1}} \frac{(q;q)_{n-r}}{(q;q)_{k-r}(1-q)^{n-k}} \\ &\qquad \times \frac{(q^{z-k+1};q)_k(aq^z;q)_k}{(1-q)^k(aq;q^2)_k} \frac{(aq^{1-n+k};q)_{n-k}(aq^{1+2r};q^2)_{k-r}}{(aq^{3-2n};q^2)_{n-r}}. \end{aligned} \tag{3.6}$$

Then (3.5) is the result of simplifying (3.6) with appropriate shifts of n and k. □

Remark 3.2. If we perform the substitution $A = aq^{z+r}$, $B = q^{r-z}$ and $C = q^{2r}$ in (3.5), we get

$$\frac{(C/A, C/B;q)_n}{(C, C/AB;q)_n} = {}_3\phi_2\left[\begin{matrix} A, B, q^{-n} \\ C, ABC^{-1}q^{1-n} \end{matrix}; q, q\right] \tag{3.7}$$

which is the q-Pfaff-Saalschütz summation [5, (II.12)], written in standard basic hypergeometric form (cf. [5]). The problem with this substitution is that whereas a and z are general parameters, r is not. To show that (3.5), where r is a nonnegative integer, is actually equivalent to the general case where r is any complex number, works by a standard polynomial argument. If we multiply both sides of (3.5) by $(q^{2r}; q)_n$ and formally replace q^r by x, we obtain a polynomial equation in x of degree $2n$ which is valid for $x = q^r$, for $r = 0, 1, 2 \ldots$ (i.e. for more than $2n$ values), thus must be true for all complex x.

As mentioned in the introduction, there exist also other combinatorial proofs of the ${}_3\phi_2$ summation. Among the references we have listed, Yee's paper [21] is remarkable as the proof there establishes the full q-Pfaff-Saalschütz summation at once, and no appeal to a polynomial argument is needed.

3.2 $(a; q)$-Extension of the alpha-parameter model

In [6], Goldman and Haglund introduced generalized rook models, called *i-creation model* and *alpha-parameter model*, which we briefly introduce first.

Given a board B, a *file placement* of k rooks is a k-subset of B such that no two cells lie in the same column, that is, there can be two or more rooks in the same row, but each column contains at most one rook. Let $\mathscr{F}_k(B)$ denote the set of all k-file placements. Given a Ferrers board B and a file placement $P \in \mathscr{F}_k(B)$, we assign weights to the rows containing rooks as follows. If there are u rooks in a given row, then the weight of this row is

$$\begin{cases} 1 & \text{if } 0 \leq u \leq 1, \\ \alpha(2\alpha - 1)(3\alpha - 2)\cdots((u-1)\alpha - (u-2)), & \text{if } u \geq 2. \end{cases}$$

The weight of a placement P, $wt(P)$, is the product of the weights of all the rows. Then for a Ferrers board B, set

$$r_k^{(\alpha)}(B) = \sum_{P \in \mathscr{F}_k(B)} wt(P).$$

Note that for $\alpha = 0$, $r_k^{(0)}(B)$ reduces to the original rook number. If α is a positive integer i, $r_k^{(i)}(B)$ is the *i-creation* rook number which counts the number of i-creation rook placements of k rooks on B. The i-creation rook placement is defined as follows: we first choose the columns to place the rooks. Then as we place rooks from left to right, each time a rook is placed, i new rows are created drawn to the right end and immediately above where the rook is placed.

In this setting, Goldman and Haglund [6] proved the *α-factorization theorem*: given a Ferrers board $B = B(b_1, \ldots, b_n)$,

$$\prod_{j=1}^{n}(z+b_j+(j-1)(\alpha-1))$$
$$=\sum_{k=0}^{n} r_k^{(\alpha)}(B)z(z+\alpha-1)\cdots(z+(n-k-1)(\alpha-1)).$$

Furthermore, Goldman and Haglund defined a q-analogue of $r_k^{(\alpha)}(B)$ by assigning q-weights to the cells in B. Here, we describe the $(a;q)$-extension of their result which involves the use of the extra variable a in the weights of the cells.

Given a Ferrers board B and a rook placement $P \in \mathscr{N}_k(B)$, for each cell $c \in B$, let $v(c)$ be the number of rooks strictly to the left of, and in the same row as c and $r_c(P)$ be the number of rooks in the north-west region of c. Then define the weight of c to be

$$wt_\alpha(c) = \begin{cases} 1, & \text{if there is a rook above and in the same column as } c, \\ [(\alpha-1)v(c)+1]_{aq^{2(-j+(\alpha-1)(1-i+r_c(P)))};q}, & \text{if } c \text{ contains a rook,} \\ W_{aq^{2(-j+(\alpha-1)(1-i+r_c(P)))};q}((\alpha-1)v(c)+1), & \text{otherwise.} \end{cases}$$

The weight of the rook placement P is defined to be the product of the weights of all cells:

$$wt_\alpha(P) = \prod_{c\in B} wt_\alpha(c).$$

We define an $(a;q)$-analogue of $r_k^{(\alpha)}(B)$ by setting

$$r_k^{(\alpha)}(a,q;B) = \sum_{P\in\mathscr{F}_k(B)} wt_\alpha(P).$$

With this $r_k^{(\alpha)}(a,q;B)$, we can also prove an $(a;q)$-analogue of the α-factorization theorem.

Theorem 3.2. *For any Ferrers board $B = B(b_1, b_2, \ldots, b_n)$, we have*

$$\prod_{j=1}^{n}[z+b_j+(j-1)(\alpha-1)]_{aq^{-2(b_j+(j-1)(\alpha-1))};q}$$
$$=\sum_{k=0}^{n} r_{n-k}^{(\alpha)}(a,q;B)\prod_{i=1}^{k}[z+(i-1)(\alpha-1)]_{aq^{-2(i-1)(\alpha-1)};q}. \tag{3.8}$$

Proof. Let us extend the board by attaching z rows of width n below the board B, denoted by B_z, and compute

$$\sum_{P\in\mathscr{F}_n(B_z)} wt_\alpha(P)$$

in two different ways. The left-hand side of (3.8) is the result of computing the above weight sum columnwise, and the right-hand side can be obtained by computing the weight of the cells in B and the cells in the extended part separately. For the details, see [20]. □

Remark 3.3. In [19], the authors have constructed a general rook theory model utilizing an augmented rook board which can be specialized to all the known rook theory models. The product formula in Theorem 3.2 was also obtained in [19, (4.17)] but by using a different approach.

In the case $\alpha = 2$ and the board is of the staircase shape $St_n = B(0, 1, 2, \ldots, n-1)$, $r_k^{(2)}(a, q; St_n)$ has a closed form expression (see [18]).

$$r_k^{(2)}(a, q; St_n) = q^{-\binom{n+k}{2}+k(k+2)} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \times \prod_{j=1}^{k} [2j-1]_q \frac{(aq; q^{-2})_{n-k}(aq^{1-2n}; q^2)_k}{(aq; q^{-4})_n}. \tag{3.9}$$

We now give combinatorial proofs of two special ${}_4\phi_3$ summations.

Proposition 3.3.

$$\frac{(q^{z+2}, a^{-1}q^{2-z}; q^2)_n}{(q, a^{-1}q^3; q^2)_n} = \sum_{k=0}^{n} \frac{(q^{-n}, -q^{-n}, q^{z+1}, a^{-1}q^{1-z}; q)_k}{(q, q^{-2n}, a^{-1/2}q^{3/2}, -a^{-1/2}q^{3/2}; q)_k} q^k, \tag{3.10}$$

and

$$\frac{(q^{z+2}, aq^{z-2n}; q^2)_n}{(q^{z+1}, aq^{z-n}; q)_n} = \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+1}, a^{1/2}q^{-n-1/2}, -a^{1/2}q^{-n-1/2}; q)_k}{(q, -q, q^{-z-n}, aq^{z-n}; q)_k} q^k. \tag{3.11}$$

Proof. If we use the closed form expression for $r_{n-k}^{(2)}(a, q; St_n)$ of (3.9) in (3.8) for $b_j = j - 1$, we get

$$\begin{aligned} &\frac{(q^z; q^2)_n}{(1-q)^n} \frac{(aq^z; q^{-2})_n}{(aq; q^{-4})_n} q^{-n(z+1)} \\ &= \sum_{k=0}^{n} q^{-k(z+1)} \frac{(q; q)_{2n-k-1}}{(q; q)_{2n-2k}(q; q)_{k-1}} \frac{(q; q^2)_{n-k}}{(1-q)^{n-k}} \\ &\quad \times \frac{(aq; q^{-2})_k (aq^{1-2n}; q^2)_{n-k}}{(aq; q^{-4})_n} \frac{(q^z; q)_k}{(1-q)^k} \frac{(aq^z; q^{-1})_k}{(aq; q^{-2})_k} \end{aligned}$$

which after some elementary manipulations simplifies to (3.10).

Similarly, (3.11) is the result of simplifying

$$\prod_{j=1}^{n}[z+2(j-1)]_{aq^{-4(j-1)};q}=\sum_{k=0}^{n}r_k^{(2)}(a,q;St_n)\prod_{i=1}^{n-k}[z+i-1]_{aq^{-2(i-1)};q},$$

after replacing $r_k^{(2)}(a,q;St_n)$ by the closed form expression in (3.9). □

The summation in (3.10) is equivalent (up to an obvious substitution of variables) to Jain's ${}_4\phi_3$ summation (1.2) mentioned in the introduction.

The summation in (3.11) can be also verified by the following terminating q-analogue of Whipple's ${}_3F_2$ sum [5, (II.19)]:

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{-n},q^{n+1},C,-C\\E,C^2q/E,-q\end{matrix};q,q\right]\\&=\frac{(Eq^{-n},Eq^{n+1},C^2q^{1-n}/E,C^2q^{n+2}/E;q^2)_\infty}{(E,C^2q/E;q)_\infty}q^{\binom{n+1}{2}},\end{aligned}$$

where we take $C=a^{1/2}q^{-n-1/2}$ and $E=q^{-z-n}$, and apply

$$\begin{aligned}&\frac{(q^{-z-2n},q^{1-z},aq^{z-2n},aq^{z+1};q^2)_\infty}{(q^{-z-n},aq^{z-n};q)_\infty}q^{\binom{n+1}{2}}\\&=\frac{(q^{-z-2n};q^2)_n}{(q^{-z-n};q)_n}\frac{(q^{-z},q^{1-z};q^2)_\infty}{(q^{-z};q)_\infty}\frac{(aq^{z-2n};q^2)_n}{(aq^{z-n};q)_n}\frac{(aq^z,aq^{z+1};q^2)_\infty}{(aq^z;q)_\infty}q^{\binom{n+1}{2}}\\&=\frac{(q^{z+2};q^2)_n}{(q^{z+1};q)_n}\frac{(aq^{z-2n};q^2)_n}{(aq^{z-n};q)_n}.\end{aligned}$$

The two summations in (3.10) and (3.11) are actually equivalent to each other; one follows from the other by reversing the sum (i.e. substituting the summation index $k\mapsto n-k$).

3.3 $(a;q)$-Rook theory for matchings

Haglund and Remmel [11] extended the rook theory by considering partial matchings as opposed to considering partial permutations in the original rook theory, and for which they consider the shifted board B_{2n} pictured in Figure 1.

For each perfect matching M of K_{2n} consisting of n pairwise vertex disjoint edges in K_{2n}, where K_{2n} is the complete graph on the set of vertices $\{1,2,\ldots,2n\}$, let

$$P_M=\{(i,j)\mid i<j\text{ and }\{i,j\}\in M\},$$

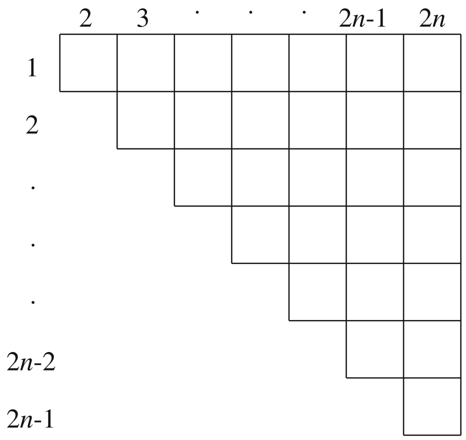


Fig. 1 B_{2n}.

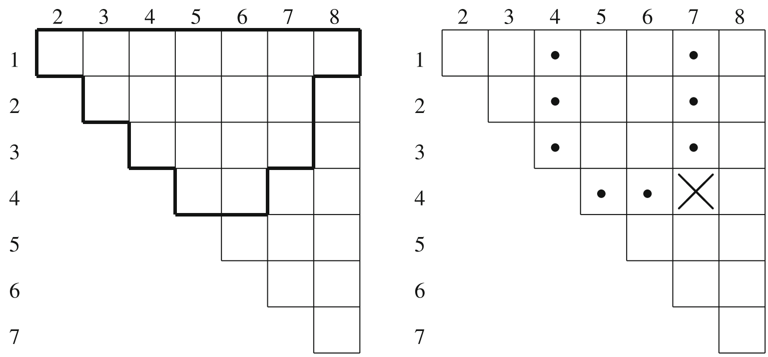


Fig. 2 The shifted Ferrers board $B = (7, 5, 4, 2, 0, 0, 0) \subseteq B_8$, and the cells cancelled by a rook in (4, 7) on B_8.

where (i, j) denotes the square in row i and column j of B_{2n} according to the labelling of rows and columns pictured in Figure 1. A rook placement in B_{2n} is defined to be a subset of some P_M for a perfect matching M of K_{2n}.

Given a board $B \subseteq B_{2n}$, we let $\mathscr{M}_k(B)$ denote the set of k element rook placements in B. In this setting, we let $B(a_1, a_2, \dots, a_{2n-1})$ denote the following set of cells in B_{2n}:

$$B(a_1, a_2, \dots, a_{2n-1}) = \{(i, i + j) \mid 1 \le i \le 2n - 1, 1 \le j \le a_i\}.$$

It is called a *shifted Ferrers board* if $2n - 1 \ge a_1 \ge a_2 \ge \cdots \ge a_{2n-1} \ge 0$, and the non-zero entries of a_i's are strictly decreasing. A rook in (i, j) with $i < j$ in a rook placement *cancels* all cells (i, s) in B_{2n} with $i < s < j$ and all cells (t, j) and (t, i) with $t < i$. See Figure 2 for a specific example of a shifted Ferrers board and the cells being cancelled by a rook on the shifted board B_8.

Definition 3.2. Given a shifted Ferrers board $B = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$ and a rook placement $P \in \mathscr{M}_k(B)$, define

$$m_k(a, q; B) = \sum_{P \in \mathscr{M}_k(P)} wt_m(P),$$

where

$$wt_m(P) = \prod_{(i,j) \in U_B(P)} w_{a;q}(\hat{i} + \hat{j} - 1 - 2r_{(i,j)}(P) - s_{(i,j)}(P)),$$

$U_B(P)$ denotes the set of cells in B which are neither cancelled by rooks nor contain any rooks in P, $r_{(i,j)}(P)$ is the number of rooks in P positioned south-east of (i, j) such that the two columns cancelled by those rooks are to the right of the column j, $s_{(i,j)}(P)$ is the number of rooks in P which are in the south-east region of (i, j) such that only one cancelled column is to the right of column j, and $\hat{i} := 2n - i$.

We also have a product formula involving $m_k(a, q; B)$, which is an $(a; q)$-analogue of the product formula proved by Haglund and Remmel [11].

Theorem 3.3. *Given a shifted Ferrers board $B = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$, we have*

$$\prod_{i=1}^{2n-1} [z + a_{2n-i} - 2i + 2]_{aq^{2(2i-2-a_{2n-i})};q}$$
$$= \sum_{k=0}^{n} m_k(a, q; B) \prod_{j=1}^{2n-1-k} [z - 2j + 2]_{aq^{4j-4};q}. \tag{3.12}$$

In the case of the full board $B_{2n} = B(2n - 1, 2n - 2, \ldots, 2, 1)$, $m_k(a, q; B_{2n})$ has a closed form

$$m_k(a, q; B_{2n}) = q^{k^2 - \binom{2n}{2}} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \prod_{j=1}^{k} [2j - 1]_q \frac{(aq^{4n-2k-3}; q^2)_{2n-k-1}}{(aq^{-1}; q^4)_{2n-k-1}}, \tag{3.13}$$

which can be verified by the following recursion:

$$m_k(a, q; B_N) = [N - 2k + 1]_{aq^{2(N-3)};q}\, m_{k-1}(a, q; B_{N-1})$$
$$+ W_{aq^{2(N-3)};q}(N - 2k - 1)\, m_k(a, q; B_{N-1}).$$

Replacing $m_k(a, q; B)$ in (3.12) for $a_i = 2n - i$ by (3.13) gives a special case of the q-Pfaff-Saalschütz sum.

Proposition 3.4. *We have*

$$\frac{(q^{z-n+\frac{3}{2}}, aq^{z-\frac{1}{2}}; q)_n}{(q^{z-2n+2}, aq^{z+n-\frac{1}{2}}; q)_n} = \sum_{k=0}^{n} \frac{(q^{-n}, q^{-n+\frac{1}{2}}, q^{\frac{5}{2}-2n}/a; q)_k}{(q, q^{z-2n+2}, q^{2-2n-z}/a; q)_k} q^k. \tag{3.14}$$

Proof. Putting the closed form expression for $m_k(a,q;\,B_{2n})$ in (3.12) for $a_i = 2n - i$ gives

$$\frac{(q^{z+1};q^{-1})_{2n-1}(aq^{z-1};q)_{2n-1}}{(1-q)^{2n-1}(aq^{-1};q^2)_{2n-1}}$$
$$= \sum_{k=0}^{n} q^{2k(k-2n+1)+kz}\frac{(q;q)_{2n}}{(q;q)_{2k}(q;q)_{2n-2k}}\frac{(q;q^2)_k}{(1-q)^k}$$
$$\times\frac{(aq^z, aq^{4n-2k-3};q^2)_{2n-k-1}}{(aq, aq^{-1};q^4)_{2n-k-1}}\frac{(q^z;q^{-2})_{2n-k-1}}{(1-q)^{2n-k-1}}$$

which simplifies to

$$\frac{(q^{-z-2}, aq^{z-2};q)_{2n}}{(q^{-z-2}, aq^{z-2};q^2)_{2n}}q^{n(2n-1)} = \sum_{k=0}^{n}\frac{(q^{-2n}, q^{1-2n}, q^{5-4n}/a;q^2)_k}{(q^2, q^{z-4n+4}, q^{4-4n-z}/a;q^2)_k}q^{2k}.$$

The identity can now be obtained by replacing $q^2 \to q$ and $z/2 \to z$. □

Remark 3.4. The identity (3.14) (proved combinatorially) is actually the

$$A = q^{-n+\frac{1}{2}},\quad B = q^{\frac{5}{2}-2n}/a,\quad \text{and}\ \ C = q^{z-2n+2}$$

special case of the q-Pfaff-Saalschütz sum (3.7).

References

1. G.E. Andrews, Partitions, in *Combinatorics: Ancient and Modern*, ed. by R. Wilson, et al. (Oxford University Press, Oxford, 2013), pp. 205–229
2. G.E. Andrews, D.M. Bressoud, Identities in combinatorics III: further aspects of ordered set sorting. Discret. Math. **49**, 223–236 (1984)
3. W.N. Bailey, On the sum of a terminating ${}_3F_2(1)$. Q. J. Math. Oxf. Ser. **4**(2), 237–240 (1953)
4. F. Butler, M. Can, J. Haglund, J.B. Remmel, Rook theory notes, book project, http://www.math.ucsd.edu/~remmel/files/Book.pdf
5. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edn., Encyclopedia of Mathematics and Its Applications (Cambridge University Press, Cambridge, 2004)
6. J. Goldman, J. Haglund, Generalized rook polynomials. J. Comb. Theory Ser. A **91**, 509–530 (2000)
7. J.R. Goldman, J.T. Joichi, D.E. White, Rook theory. IV. Orthogonal sequences of rook polynomials. Stud. Appl. Math. **56**, 267–272 (1976/1977)
8. I.P. Goulden, A bijective proof of the q-Saalschütz theorem. Discret. Math. **57**, 39–44 (1985)
9. V.W. Guo, J. Zeng, A combinatorial proof of a symmetric q-Pfaff–Saalschütz identity. Electron. J. Comb. **12**, #N2 (2005)
10. J. Haglund, Rook theory and hypergeometric series. Adv. Appl. Math. **17**, 408–459 (1996)
11. J. Haglund, J.B. Remmel, Rook theory for perfect matchings. Adv. Appl. Math. **27**, 438–481 (2001)

12. V.K. Jain, Some transformations of basic hypergeometric functions, II. SIAM J. Math. Anal. **12**, 957–961 (1981)
13. I. Kaplansky, J. Riordan, The problem of the rooks and its applications. Duke Math. J. **13**, 259–268 (1946)
14. T. Mansour, M. Schork, *Commutation Relations, Normal Ordering, and Stirling Numbers* (Chapman and Hall/CRC Press, Boca Raton, 2015)
15. G. Nyul, G. Rácz, The r-Lah numbers. Discret. M. **338**, 1660–1666 (2015)
16. M.J. Schlosser, Elliptic enumeration of nonintersecting lattice paths. J. Comb. Theory Ser. A **114**, 505–521 (2007)
17. M.J. Schlosser, A noncommutative weight-dependent generalization of the binomial theorem, arxiv.org/abs/1106.2112
18. M.J. Schlosser, M. Yoo, Elliptic rook and file numbers. Electron. J. Comb. **24**(1), P1.31 (2017)
19. M.J. Schlosser, M. Yoo, An elliptic extension of the general product formula for augmented rook boards. Eur. J. Comb. **58**, 247–266 (2016)
20. M.J. Schlosser, M. Yoo, Elliptic extensions of the alpha-parameter model and the rook model for matchings. Adv. Appl. Math. **84**, 8–33 (2017)
21. A.-J. Yee, Combinatorial proofs of identities in basic hypergeometric series. Eur. J. Comb. **29**, 1365–1375 (2008)
22. D. Zeilberger, A q-Foata proof of the q-Saalschütz identity. Euro. J. Comb. **8**, 461–463 (1987)

Overpartitions and Singular Overpartitions

Seunghyun Seo and Ae Ja Yee

This paper is dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract Singular overpartitions, which were defined by George Andrews, are overpartitions whose Frobenius symbols have at most one overlined entry in each row. In his paper, Andrews obtained interesting combinatorial results on singular overpartitions, one of which relates a certain type of singular overpartition with a subclass of overpartitions. In this paper, we provide a combinatorial proof of Andrews's result, which answers one of his open questions.

Keywords Partitions · Overpartitions · Frobenius symbols · Singular overpartitions · Dyson's map · Wright's map

2010 Mathematics Subject Classification 05A17 · 11P81

S. Seo—The first author was partially supported by a research grant of Kangwon National University in 2015.
A.J. Yee—The second author was partially supported by a grant (#280903) from the Simons Foundation.

S. Seo
Department of Mathematics Education, Kangwon National University,
Chuncheon, Kangwon-do 24341, Republic of Korea
e-mail: shyunseo@kangwon.ac.kr

A. J. Yee (✉)
Department of Mathematics, The Pennsylvania State University, University Park,
PA 16802, USA
e-mail: auy2@psu.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_38

1 Introduction

A Frobenius symbol for n is a two-rowed array [2, 10]:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix}$$

where $\sum_{t=1}^{\delta}(a_t + b_t + 1) = n$, $a_1 > a_2 > \cdots > a_\delta \geq 0$, and $b_1 > b_2 > \cdots > b_\delta \geq 0$. There is a natural one-to-one correspondence between the Frobenius symbols for n and the ordinary partitions of n (see Section 2.2). Thus a Frobenius symbol for n is another representation of an ordinary partition of n.

An overpartition of n is a partition in which the first occurrence of a part may be overlined [5]. For an overparition, one can define the corresponding Frobenius symbol by allowing overlined entries in a similar way. It should be noted that the Frobenius symbol of an overpartition is defined in a different way in [5, 7].

Recently, George Andrews introduced a certain subclass of overpartitions, namely singular overpartitions which are Frobenius symbols with at most one overlined entry in each row [3]. For integers k, i with $k \geq 3$ and $1 \leq i < k$, Andrews found interesting combinatorial and arithmetic properties of (k, i)-singular overpartitions, which are singular overpartitions with some restrictions subject to k and i. Because of the complexity of the restrictions, we defer the exact definition to Section 2.4. One of the main results of Andrews in [3] is the following.

Theorem 1.1 (Andrews, [3]). *The number of (k, i)-singular overpartitions of n equals the number of overpartitions of n in which no part is divisible by k and only parts congruent to $\pm i$ mod k may be overlined.*

Equivalently,

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n = \prod_{n=0}^{\infty} \frac{(1 + q^{nk+i})(1 + q^{(n+1)k-i})}{(1 - q^{nk+1})(1 - q^{nk+2}) \cdots (1 - q^{nk+k-1})},$$

where $\overline{Q}_{k,i}(n)$ is the number of (k, i)-singular overpartitions of n.

Andrews concluded his paper with four open questions. The first question is to prove Theorem 1.1 bijectively. The primary purpose of this paper is to provide an answer to the first question. His second question is indeed a special case of the first one. Consequently, the second question will be settled as well. In addition, we obtain a refined version of Theorem 1.1, namely Theorem 4.1 in Section 4.

This paper is organized as follows. In Section 2, necessary definitions and maps are reviewed. In Section 3, an enumeration formula for subclasses of (k, i)-singular overpartitions is given (see Theorem 3.1), and finally, a combinatorial proof of Theorem 1.1 will be presented in Section 4.

2 Preliminaries

In this section, we provide some definitions and bijections that are needed in later sections.

2.1 Definitions

For a partition or overpartition λ, we write it as $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$. We denote by $|\lambda|$ the sum of parts, and by $\ell(\lambda)$ the number of parts.

The conjugate λ' of a partition λ is the partition resulting from the reflection of the Ferrers graph of λ about the main diagonal.

For a positive integer k, we also define $k\lambda$ as the partition whose parts are k times each part of λ. For instance, let $\lambda = (3, 3, 2, 1)$. Then $5\lambda = (15, 15, 10, 5)$.

Let λ and μ be two partitions. Then we define the union $\lambda \cup \mu$ as the partition consisting of all the parts of λ and μ.

We denote the number of partitions of n by $p(n)$. We also denote the partition with no parts by $\emptyset$. For further standard definitions, see [1].

2.2 Frobenius symbol

Recall the definition of a Frobenius symbol for n in Introduction. For a partition λ of n, let δ be the largest integer such that $\lambda_\delta - \delta \geq 0$, i.e., δ is the side of the Durfee square of λ. We now consider the following two-rowed array:

$$\begin{pmatrix} \lambda_1 - 1 & \lambda_2 - 2 & \cdots & \lambda_\delta - \delta \\ \lambda'_1 - 1 & \lambda'_2 - 2 & \cdots & \lambda'_\delta - \delta \end{pmatrix}.$$

Clearly, this satisfies the conditions for Frobenius symbols for n, and this is reversible. Thus there is a unique Frobenius symbol associated with λ. For instance, the associated Frobenius symbol of the partition (7, 5, 5, 3, 2, 2, 1) is

$$\begin{pmatrix} 6 & 3 & 2 \\ 6 & 4 & 1 \end{pmatrix}.$$

2.3 (k, i)-parity blocks and anchors

Throughout this paper, we assume that k and i are integers such that $k \geq 3$ and $1 \leq i \leq k - 1$.

For a partition λ, by abuse of the notation, we will denote its Frobenius symbol by λ. A column $\begin{matrix} a_t \\ b_t \end{matrix}$ of λ is called (k, i)-positive if $a_t - b_t \geq k - i - 1$ and called (k, i)-negative if $a_t - b_t \leq -i + 1$. If $-i + 2 \leq a_t - b_t \leq k - i - 2$, we call the column (k, i)-neutral.

If two columns are both (k, i)-positive or both (k, i)-negative, we shall say that they have the same parity.

We now divide λ into (k, i)-parity blocks. These are sets of contiguous columns maximally extended to the right, where all the entries have either the same parity or neutral.

We shall say that a parity block is neutral if all columns are neutral. Owing to the maximality condition this can only occur if all the columns of λ are neutral. In all other cases, we shall say that a block is positive (or negative) if it contains no negative (or positive, resp.) columns. For instance, consider the following Frobenius symbol:

$$\begin{pmatrix} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{pmatrix}.$$

The (3, 1)-parity blocks are

$$\left(\begin{array}{c|c|ccc|ccc|c|ccc|cc|cc} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{array}\right),$$

and the (5, 2)-parity blocks are

$$\left(\begin{array}{ccccc|ccccccc|cc|cc} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{array}\right). \tag{2.1}$$

For a non-neutral block, we now define its anchor as the first non-neutral column.

2.4 (k, i)-singular overpartitions

We are ready to define (k, i)-singular overpartitons. A Frobenius symbol is (k, i)-singular if it satisfies one of the following conditions:

- there are no overlined entries;
- if there is one overlined entry on the top row, then it occurs in the anchor of a positive block;
- if there is one overlined entry on the bottom row, then it occurs in the anchor of a negative block;
- if there are two overlined entries, then they occur in adjacent anchors with one on the top row of the positive block and the other on the bottom row of the negative block.

For the Frobenius symbol in (2.1), the following are all the (5, 2)-singular with exactly two overlined entries:

$$\begin{pmatrix} 31 & 28 & \overline{27} & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & \overline{19} & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & \overline{6} & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & \overline{19} & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & \overline{6} & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & \overline{2} & 0 \end{pmatrix}.$$

2.5 Dyson map

For a partition λ, the rank of λ is the largest part minus the number of parts, i.e.,

$$rank(\lambda) := \lambda_1 - \ell(\lambda).$$

We remark that the rank of λ is equal to $a_1 - b_1$ if $\begin{matrix} a_1 \\ b_1 \end{matrix}$ is the first column of the Frobenius symbol of λ. For convenience, we define $rank(\emptyset) = 0$.

Freeman Dyson defined a map to prove a symmetry in partitions [6]. Here, we use the description of the Dyson map in [4, 8]. For a partition λ of n with $rank(\lambda) \le r$, we subtract 1 from each part of λ and then add a part of size $r - 1 + \ell(\lambda)$. We call this the Dyson map and denote it by d_r.

Remark 2.1. It can be easily checked that $d_r(\lambda)$ is a partition of $n + r - 1$ with $rank(d_r(\lambda)) \ge r - 2$.

We now describe the Dyson map with Frobenius symbol for later use. Consider

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix},$$

with $rank(\lambda) \le r$. When $\delta \ge 2$, $d_r(\lambda)$ is given by

- $\begin{pmatrix} b_1 + r - 1 & a_1 - 2 & \cdots & a_{\delta-2} - 2 & a_{\delta-1} - 2 & a_\delta - 2 \\ b_2 + 2 & b_3 + 2 & \cdots & b_\delta + 2 & 1 & 0 \end{pmatrix}$ if $a_\delta \ge 2$,
- $\begin{pmatrix} b_1 + r - 1 & a_1 - 2 & \cdots & a_{\delta-2} - 2 & a_{\delta-1} - 2 \\ b_2 + 2 & b_3 + 2 & \cdots & b_\delta + 2 & 1 \end{pmatrix}$ if $a_\delta = 1$,
- $\begin{pmatrix} b_1 + r - 1 & a_1 - 2 & \cdots & a_{\delta-2} - 2 \\ b_2 + 2 & b_3 + 2 & \cdots & b_\delta + 2 \end{pmatrix}$ if $a_\delta = 0, a_{\delta-1} = 1$,
- $\begin{pmatrix} b_1 + r - 1 & a_1 - 2 & \cdots & a_{\delta-2} - 2 & a_{\delta-1} - 2 \\ b_2 + 2 & b_3 + 2 & \cdots & b_\delta + 2 & 0 \end{pmatrix}$ if $a_\delta = 0, a_{\delta-1} \ge 2$.

When $\delta = 1$, $d_r(\lambda)$ is given by

- $\begin{pmatrix} b_1 + r - 1 & a_1 - 2 \\ 1 & 0 \end{pmatrix}$ if $a_1 \geq 2$,
- $\begin{pmatrix} b_1 + r - 1 \\ 1 \end{pmatrix}$ if $a_1 = 1$,
- $\begin{pmatrix} b_1 + r - 1 \\ 0 \end{pmatrix}$ if $a_1 = 0$ and $b_1 \geq 1 - r$,
- $\emptyset$ if $a_1 = 0$ and $b_1 = -r$.

Note that the upper left entry of $d_r(\lambda)$ is $b_1 + r - 1$ if $d_r(\lambda)$ is nonempty.

2.6 Shift and Shifted Conjugate

Given an integer u, a shift map s_u is defined as follows:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix} \xrightarrow{s_u} \begin{pmatrix} a_1 - u & a_2 - u & \cdots & a_\delta - u \\ b_1 + u & b_2 + u & \cdots & b_\delta + u \end{pmatrix}.$$

Similarly, a shifted conjugate map c_u is defined as follows:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix} \xrightarrow{c_u} \begin{pmatrix} b_1 - u & b_2 - u & \cdots & b_\delta - u \\ a_1 + u & a_2 + u & \cdots & a_\delta + u \end{pmatrix}.$$

Remark 2.2. It follows from the definitions that $s_u^{-1} = s_{-u}$ and $c_u^{-1} = c_u$. Also, for the Frobenius symbol of the empty partition $\emptyset$, we define $s_u(\emptyset) = c_u(\emptyset) = \emptyset$.

2.7 Wright map

The Wright map is a bijection between pairs of partitions into distinct parts and pairs of an ordinary partition and a triangular number (see [8, 9, 11]). However, in this paper, we give a modified version of the map using Frobenius symbol for our purpose. We denote the map by φ.

Let μ^1 be a partition into distinct parts congruent to i mod k and μ^2 be a partition into distinct parts congruent to $(k - i)$ mod k, namely

$$\mu^1 = (ka_1 + i, ka_2 + i, \ldots, ka_{s+m} + i),$$
$$\mu^2 = (kb_1 + (k - i), kb_2 + (k - i), \ldots, kb_s + (k - i)),$$

where $a_1 > a_2 > \cdots > a_{s+m} \geq 0$ and $b_1 > b_2 > \cdots > b_m \geq 0$.

Suppose that $m \geq 0$. We consider the following Frobenius symbol

$$\mu = \begin{pmatrix} a_{1+m} & a_{2+m} & \dots & a_{s+m} \\ b_1 & b_2 & \dots & b_s \end{pmatrix}.$$

We now take $\nu = (a_1 - m + 1, a_2 - m + 2, \dots, a_m)$. Then since $a_1 > a_2 > \cdots$, it is clear that ν is a partition. We define $\varphi(\mu^1, \mu^2) = (k(\nu \cup \mu), m)$.

For example, let $k = 5$ and $i = 2$. If $\mu^1 = (37, 27, 22, 7)$ and $\mu^2 = (18, 13)$, then $m = 2$, and we obtain

$$\mu = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = (5, 3, 2, 2), \quad \nu = (6, 5).$$

Thus

$$\varphi(\mu^1, \mu^2) = (5(6, 5, 5, 3, 2, 2), 2) = ((30, 25, 25, 15, 10, 10), 2).$$

Similarly, if $m < 0$, we consider the following Frobenius symbol

$$\mu = \begin{pmatrix} b_{1-m} & b_{2-m} & \dots & b_s \\ a_1 & a_2 & \dots & a_{s+m} \end{pmatrix}.$$

We now take $\nu = (b_1 + m + 1, b_2 + m + 2, \dots, b_{-m})$. Then since $b_1 > b_2 > \cdots$, it is clear that ν is a partition. We define $\varphi(\mu^1, \mu^2) = (k(\nu \cup \mu)', m)$.

We can easily check that $|\mu^1| + |\mu^2| = k(|\nu| + |\mu|) + k\binom{m}{2} + im$, and we omit the details.

Remark 2.3. We note that the Wright map proves that the number of such pairs of partitions μ^1, μ^2 with $|\mu^1| + |\mu^2| = n$ and $\ell(\mu^1) - \ell(\mu^2) = m$ is

$$p\left(\frac{n - k\binom{m}{2} - im}{k}\right).$$

3 Singular overpartitions and dotted blocks

3.1 Dotted parity blocks

We now introduce another representation of a (k, i)-singular overpartition. We will use this representation throughout this paper.

Let λ be a (k, i)-singular overpartition. We first separate all the columns before the first anchor to form a block. By the definition of parity blocks, we see that these columns must be all neutral if exist. We denote this block by E. For the blocks of the remaining columns, we denote each of them by P and N if its anchor is positive and negative, respectively.

If there is exactly one overlined entry in λ, we put a dot on the top of each of the blocks between the first non-neutral block and the block of the overlined entry. If there are two overlined entries in λ, then we put a dot on the top of each block between the second non-neutral block and the block of the last overlined entry.

It is clear that a Frobenius symbol λ is (k, i)-singular if

S1. there are no dotted blocks, or
S2. there are consecutive dotted blocks starting from the first non-neutral block, or
S3. there are consecutive dotted blocks starting from the second non-neutral block.

For instance, if a sequence of parity blocks is $EPNPN$, then the following are all singular:

$$EPNPN,$$
$$E\dot{P}NPN,\ E\dot{P}\dot{N}PN,\ E\dot{P}\dot{N}\dot{P}N,\ E\dot{P}\dot{N}\dot{P}\dot{N},$$
$$EP\dot{N}PN,\ EP\dot{N}\dot{P}N,\ EP\dot{N}\dot{P}\dot{N}.$$

Since there is a one-to-one correspondence between (k, i)-singular overpartitions and Frobenius symbols with a sequence of parity blocks satisfying S1, S2, or S3, we will use the latter form from now on.

3.2 (k, i)-singular overpartitions with m dotted blocks

The following theorem is one of our main results.

Theorem 3.1. *Let m be a positive integer.*

1. *The number of (k, i)-singular overpartitions of n with exactly m dotted blocks and the last dotted block negative is*

$$p\left(n - k\binom{m}{2} - im\right).$$

2. *The number of (k, i)-singular overpartitions of n with exactly m dotted blocks and the last dotted block positive is*

$$p\left(n - k\binom{m+1}{2} + im\right).$$

Note that singular overpartitions with no dotted blocks are just ordinary partitions, which with Theorem 3.1 yields the following corollary.

Corollary 3.1. *The number of (k, i)-singular overpartitions of n is*

$$\sum_{m=-\infty}^{\infty} p\left(n - k\binom{m}{2} - im\right).$$

Proof. For $m < 0$, the number of (k, i)-singular overpartitions of n with exactly $|m|$ dotted blocks and the last block positive is

$$p\left(n - k\binom{m}{2} - im\right),$$

which completes the proof.

To prove Theorem 3.1, we will construct a bijection in Section 3.3. However, we first need the following two lemmas.

Lemma 3.1. *Given integers f, g, h with $g \geq 1$, $2g \geq f + 1$, $h \geq f$, consider two Frobenius symbols*

$$L = \begin{pmatrix} a_1 & \cdots & a_{t-1} \\ b_1 & \cdots & b_{t-1} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \alpha_1 & \cdots & \alpha_\tau \\ \beta_1 & \cdots & \beta_\tau \end{pmatrix} \neq \emptyset,$$

such that

i) $a_y - b_y \geq f$ *for all* $1 \leq y \leq t - 1$,
ii) $\alpha_1 - \beta_1 \leq f - 2g + 1$,
iii) $a_{t-1} > \alpha_1 + g - 1$,
iv) $b_{t-1} > \beta_1 - g + 1 \geq 0$.

Let

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1\delta'} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2\delta'} \end{pmatrix} := c_{g-f+1}(L)\, d_{f-2g+1}(R),$$

where the first $t - 1$ columns of μ are from $c_{g-f+1}(L)$ and the rest are from $d_{f-2g+1}(R)$. Then the following are true.

(1) μ *is a Frobenius symbol.*
(2) $\mu_{1y} - \mu_{2y} \leq f - 2g - 2$ *for all* $1 \leq y \leq t - 1$ *and* $\mu_{1t} - \mu_{2t} \geq f - 2g - 1$ *if* μ_{1t} *and* μ_{2t} *exist.*
(3) $rank(\mu) \leq -h + 2f - 2g - 2$ *if* $L \neq \emptyset$ *and* $rank(L) \geq h$.
(4) *The correspondence from* (L, R) *to* μ *is reversible.*
(5) $|L| + |R| - |\mu| = 2g - f$.

Proof. First note that since $rank(R) = \alpha_1 - \beta_1 \leq f - 2g + 1$, $d_{f-2g+1}(R)$ is well defined. We now prove each of the five statements.

(1) If $L = \emptyset$, then $\mu = d_{f-2g+1}(R)$ is obviously a Frobenius symbol because the Dyson map is defined on partitions. Now assume $L \neq \emptyset$. Then the last column of $c_{g-f+1}(L)$ is $\begin{matrix} b_{t-1} - g + f - 1 \\ a_{t-1} + g - f + 1 \end{matrix}$.

- If $d_{f-2g+1}(R) \neq \emptyset$, then its first column is $\begin{matrix} \beta_1 + f - 2g \\ \gamma \end{matrix}$, where γ is 0, 1, or $\beta_2 + 2$ (see Section 2.5). Since $b_{t-1} > \beta_1 - g + 1$, we have

$$b_{t-1} - g + f - 1 > \beta_1 + f - 2g.$$

Also, since $a_{t-1} - b_{t-1} \geq f$ and $b_{t-1} > \beta_1 - g + 1$, it follows that

$$a_{t-1} + g - f + 1 \geq b_{t-1} + g + 1 > \beta_1 + 2 > \beta_2 + 2 \geq \gamma.$$

Thus $\mu = c_{g-f+1}(L)\, d_{f-2g+1}(R)$ is a Frobenius symbol.

- If $d_{f-2g+1}(R) = \emptyset$, then $R = \begin{pmatrix} 0 \\ 2g - f - 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$. Since $b_{t-1} > \beta_1 - g + 1$,

$$b_{t-1} - g + f - 1 \geq \beta_1 - 2g + f + 1 = 0.$$

Also, since $a_{t-1} > \alpha_1 + g - 1$ and $2g \geq f + 1$,

$$a_{t-1} + g - f + 1 \geq \alpha_1 + 2g - f + 1 \geq 2.$$

Thus $\mu = c_{g-f+1}(L)$ is a Frobenius symbol.

(2) For $1 \leq y \leq t - 1$, since $a_y - b_y \geq f$,

$$\mu_{1y} - \mu_{2y} = (b_y - a_y) - 2g + 2f - 2 \leq -f - 2g + 2f - 2.$$

Also, by Remark 2.1,

$$\mu_{1t} - \mu_{2t} = rank(d_{f-2g+1}(R)) \geq (f - 2g + 1) - 2.$$

(3) If $rank(L) \geq h$ then $a_1 - b_1 \geq h$. Thus

$$rank(\mu) = \mu_{11} - \mu_{21} = (b_1 - a_1) - 2g + 2f - 2 \leq -h - 2g + 2f - 2.$$

(4) Consider a Frobenius symbol $\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1\delta'} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2\delta'} \end{pmatrix}$ satisfying (2). Set

$$L' = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1(t-1)} \\ \mu_{21} & \cdots & \mu_{2(t-1)} \end{pmatrix} \quad \text{and} \quad R' = \begin{pmatrix} \mu_{1t} & \cdots & \mu_{1\delta'} \\ \mu_{2t} & \cdots & \mu_{2\delta'} \end{pmatrix}.$$

Then $L = c_{g-f+1}(L')$ and $R = d^{-1}_{f-2g+1}(R')$ are desired Frobenius symbols, so (4) holds.

(5) Finally, since $|c_{g-f+1}(L)| = |L|$ and $|d_{f-2g+1}(R)| = |R| - (2g - f)$, (5) holds true.

Lemma 3.2. *Given integers f, g, h with $g \geq 1$, $2g \geq f+1$, $h \leq f$, consider two Frobenius symbols*

$$L = \begin{pmatrix} a_1 & \cdots & a_{t-1} \\ b_1 & \cdots & b_{t-1} \end{pmatrix} \quad \textit{and} \quad R = \begin{pmatrix} \alpha_1 & \cdots & \alpha_\tau \\ \beta_1 & \cdots & \beta_\tau \end{pmatrix} \neq \emptyset,$$

such that

i) $a_y - b_y \leq f$ *for all* $1 \leq y \leq t-1$,
ii) $\alpha_1 - \beta_1 \leq f - 2g + 1$,
iii) $a_{t-1} > \beta_1 - g + f + 1$,
iv) $b_{t-1} > \alpha_1 + g - f - 1 \geq 0$.

Let

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1\delta'} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2\delta'} \end{pmatrix} := s_{g+1}(L)\, d_{f-2g+1}(R),$$

where the first $t-1$ columns of μ are from $s_{g+1}(L)$ and the rest are from $d_{f-2g+1}(R)$. Then the following are true.

(1) μ *is a Frobenius symbol.*
(2) $\mu_{1y} - \mu_{2y} \leq f - 2g - 2$ *for all* $1 \leq y \leq t-1$ *and* $\mu_{1t} - \mu_{2t} \geq f - 2g - 1$ *if* μ_{1t} *and* μ_{2t} *exist.*
(3) $rank(\mu) \leq h - 2g - 2$ *if* $L \neq \emptyset$ *and* $rank(L) \leq h$.
(4) The correspondence from (L, R) *to* μ *is reversible.*
(5) $|L| + |R| - |\mu| = 2g - f$.

Proof. First note that since $rank(R) = \alpha_1 - \beta_1 \leq f - 2g + 1$, $d_{f-2g+1}(R)$ is well defined. We now prove each of the five statements.

(1) If $L = \emptyset$, then $\mu = d_{f-2g+1}(R)$ is obviously a Frobenius symbol since the Dyson map is defined on partitions. Now assume $L \neq \emptyset$. Then the last column of $s_{g+1}(L)$ is $\begin{matrix} a_{t-1} - g - 1 \\ b_{t-1} + g + 1 \end{matrix}$.

- If $d_{f-2g+1}(R) \neq \emptyset$, then its first column is $\begin{matrix} \beta_1 + f - 2g \\ \gamma \end{matrix}$, where γ is 0, 1, or $\beta_2 + 2$ (see Section 2.5). Since $a_{t-1} > \beta_1 - g + f + 1$, we have

$$a_{t-1} - g - 1 > \beta_1 + f - 2g.$$

Also, since $a_{t-1} - b_{t-1} \leq f$ and $a_{t-1} > \beta_1 - g + f + 1$, it follows that

$$b_{t-1} + g + 1 \geq a_{t-1} - f + g + 1 > \beta_1 + 2 > \beta_2 + 2 \geq \gamma.$$

Thus $\mu = s_{g+1}(L)\, d_{f-2g+1}(R)$ is a Frobenius symbol.
- If $d_{f-2g+1}(R) = \emptyset$, then $R = \begin{pmatrix} 0 \\ 2g - f - 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$. Since $a_{t-1} > \beta_1 - g + f + 1$,

$$a_{t-1} - g - 1 \geq \beta_1 - 2g + f + 1 = 0.$$

Also, since $b_{t-1} > \alpha_1 + g - f - 1$ and $2g \geq f + 1$,

$$b_{t-1} + g + 1 \geq \alpha_1 + 2g - f + 1 \geq 2.$$

Thus $\mu = s_{g+1}(L)$ is a Frobenius symbol.

(2) For $1 \leq y \leq t-1$, since $a_y - b_y \leq f$,

$$\mu_{1y} - \mu_{2y} = (a_y - b_y) - 2g - 2 \leq f - 2g - 2.$$

Also, by Remark 2.1,

$$\mu_{1t} - \mu_{2t} = rank(d_{f-2g+1}(R)) \geq (f - 2g + 1) - 2.$$

(3) If $rank(L) \leq h$, then $a_1 - b_1 \leq h$. Thus

$$rank(\mu) = \mu_{11} - \mu_{21} = (a_1 - b_1) - 2g - 2 \leq h - 2g - 2.$$

(4) Consider a Frobenius symbol $\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1\delta'} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2\delta'} \end{pmatrix}$ satisfying (2). Set

$$L' = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1(t-1)} \\ \mu_{21} & \cdots & \mu_{2(t-1)} \end{pmatrix} \quad \text{and} \quad R' = \begin{pmatrix} \mu_{1t} & \cdots & \mu_{1\delta'} \\ \mu_{2t} & \cdots & \mu_{2\delta'} \end{pmatrix}.$$

Then $s_{-g-1}(L')$ and $d^{-1}_{f-2g+1}(R')$ are the desired Frobenius symbols. So (4) holds.

(5) Finally, since $|s_{g+1}(L)| = |L|$ and $|d_{f-2g+1}(R)| = |R| - (2g - f)$, (5) holds true.

3.3 Bijection ψ_m

Assume that m is a positive integer. We now construct a bijection ψ_m between the (k, i)-singular overpartitions of n with m dotted blocks and the partitions of n', where $n' = n - k\binom{m}{2} - im$ if the last dotted block is negative, and $n' = n - k\binom{m+1}{2} + im$ if the last dotted block is positive. This proves Theorem 3.1. By symmetry, it is sufficient to show the case that the last dotted block is negative.

Let λ be a (k, i)-singular overpartition in which there are exactly m dotted blocks and the last dotted block is negative.

First let D_1 be the union of the last dotted block and the blocks on the right of the last dotted block if any. From the right to left, denote each of the unchosen dotted blocks by D_v for $1 < v \leq m$. Let D_{m+1} be the union of the blocks on the left of D_m if any. For example, consider a (5, 2)-singular overpartition

$$\lambda = \left(\begin{array}{cc|cc|ccc|c|cc} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 6 & 1 & 0 \\ 30 & 28 & 25 & 20 & 19 & 11 & 8 & 4 & 2 & 0 \end{array} \right),$$

with its sequence of dotted blocks $E\dot{P}\dot{N}PN$. Then we have

$$D_3 = \begin{pmatrix} 31 & 28 \\ 30 & 28 \end{pmatrix}, \ D_2 = \begin{pmatrix} 27 & 22 \\ 25 & 20 \end{pmatrix}, \ D_1 = \begin{pmatrix} 18 & 9 & 8 & 6 & 1 & 0 \\ 19 & 11 & 8 & 4 & 2 & 0 \end{pmatrix}.$$

We then define $\Gamma_1, \ldots, \Gamma_{m+1}$ and $\psi_m(\lambda)$ as follows.

- Set $\Gamma_1 = D_1$.
- For $1 \le v \le m$, set

$$\Gamma_{v+1} = \begin{cases} c_{i+wk}(D_{v+1})\, d_{1-i-(v-1)k}(\Gamma_v) & \text{if } v = 2w+1 \text{ for some } w \ge 0, \\ s_{wk}(D_{v+1})\, d_{1-i-(v-1)k}(\Gamma_v) & \text{if } v = 2w \text{ for some } w > 0. \end{cases}$$

- Define $\psi_m(\lambda) = \Gamma_{m+1}$.

Now we will inductively show that, for each $1 \le v \le m$, Γ_v is a partition satisfying

$$rank(\Gamma_v) \le 1 - i - (v-1)k. \tag{3.1}$$

First, since $\Gamma_1 = D_1$ and the first column of D_1 is (k, i)-negative, Γ_1 is a partition satisfying (3.1). Assume that for $1 \le v < m$, $\Gamma_1, \Gamma_2, \ldots, \Gamma_v$ are well defined and satisfy (3.1). Consider Γ_{v+1}.

Case 1: Suppose $v = 2w + 1$ for some $w \ge 0$. Then we can write $\Gamma_{v+1} = \Gamma_{2w+2}$ as

$$\Gamma_{2w+2} = c_{i+wk}(D_{2w+2})\, d_{1-i-2wk}(\Gamma_{2w+1}).$$

In Lemma 3.1, set $f = 2 - i$, $g = wk + 1$, $h = k - i - 1$, and $L = D_{2w+2}$, $R = \Gamma_{2w+1}$. Clearly f, g, h satisfy $g \ge 1$, $2g \ge f + 1$, $h \ge f$.
Let us check the four conditions of the lemma. First, note that D_{2w+2} is not (k, i)-negative because the last dotted block is negative and the parity is alternating, so Condition i) holds true. From the assumption (3.1),

$$rank(\Gamma_{2w+1}) \le 1 - i - 2wk,$$

so Condition ii) holds true. Lastly, let $\begin{matrix} x_1 \\ x_2 \end{matrix}$ and $\begin{matrix} z_1 \\ z_2 \end{matrix}$ be the last column of D_{2w+2} and the first column of D_{2w+1}, respectively. Since $D_{2w+2}D_{2w+1}$ forms a Frobenius symbol, we know

$$x_1 > z_1, \quad x_2 > z_2.$$

Also, since $\Gamma_{2w+1} = s_{wk}(D_{2w+1})d_{1-i-(2w-1)k}(\Gamma_{2w})$, the first column of Γ_{2w+1} is $\begin{matrix} z_1 - wk \\ z_2 + wk \end{matrix}$. Thus, we have

$$x_1 > (z_1 - wk) + (wk + 1) - 1,$$
$$x_2 > (z_2 + wk) - (wk + 1) + 1 \geq 0,$$

which verify Conditions iii) and iv). Since all the four conditions in Lemma 3.1 are satisfied, by Statement (3) of Lemma 3.1, Γ_{2w+2} is a Frobenius symbol satisfying (3.1).

Case 2: Suppose $v = 2w$ for some $w \geq 1$. Then we can write $\Gamma_{v+1} = \Gamma_{2w+1}$ as

$$\Gamma_{2w+1} = s_{wk}(D_{2w+1})\, d_{1-i-(2w-1)k}(\Gamma_{2w}).$$

In Lemma 3.2, set $f = k - i - 2$, $g = wk - 1$, $h = 1 - i$, and $L = D_{2w+1}$, $R = \Gamma_{2w}$. Clearly, f, g and h satisfy $g \geq 1$, $2g \geq f + 1$, $h \leq f$. Note that D_{2w+1} is not (k, i)-positive,

$$rank(\Gamma_{2w}) \leq 1 - i - (2w - 1)k,$$

and $D_{2w+1}D_{2w}$ forms a Frobenius symbol. Thus, in the same way as Case 1, we can see all the four conditions in Lemma 3.2 are satisfied. Therefore, by Statement (3) of Lemma 3.2, Γ_{2w+1} is a Frobenius symbol satisfying (3.1).

We now have that Γ_m is a Frobenius symbol satisfying (3.1) from the induction. Also, D_{m+1} is a Frobenius symbol. We can easily check that D_{m+1} and Γ_m satisfy the conditions for Lemmas 3.1 or 3.2. Thus, Γ_{m+1} is a Frobenius symbol by Statement (1) of each lemma.

Let us then check the weight difference. By Statement (5) of each of Lemmas 3.1 and 3.2, we have

$$|D_{v+1}\Gamma_v| - |\Gamma_{v+1}| = i + (v - 1)k \tag{3.2}$$

for $v = 1, \ldots, m$. By (3.2), we have

$$\begin{aligned}
|\lambda| &= |D_{m+1}D_m \cdots D_5D_4D_3D_2D_1| \\
&= |D_{m+1}D_m \cdots D_5D_4D_3D_2\Gamma_1| \\
&= |D_{m+1}D_m \cdots D_5D_4D_3\Gamma_2| + i \\
&= |D_{m+1}D_m \cdots D_5D_4\Gamma_3| + (i + k) + i \\
&= |D_{m+1}D_m \cdots D_5\Gamma_4| + (i + 2k) + (i + k) + i \\
&\vdots \\
&= |\Gamma_{m+1}| + \sum_{v=1}^{m} \big(i + k(v - 1)\big).
\end{aligned}$$

Thus

$$|\Gamma_{m+1}| = n - k\binom{m}{2} - im.$$

By Statement (4) of Lemma 3.1 and Lemma 3.2, each process of producing Γ_{v+1} is reversible. Therefore, ψ_m is indeed a bijection. The inverse map will be given after the following example.

Example 3.1. Consider a (5, 2)-singular overpartition

$$\lambda = \left(\begin{array}{cc|ccc|ccccccc|cc|cc} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{array}\right),$$

with its sequence of dotted blocks $EP\dot{N}\dot{P}\dot{N}$. Note that $k = 5$, $i = 2$, and $m = 3$. We have the following Γ_v for $v = 1, 2, 3, 4$:

- $\Gamma_1 = D_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$,
- $\Gamma_2 = c_2(D_2)\, d_{-1}(\Gamma_1) = \left(\begin{array}{cc:c} 2 & 1 & 0 \\ 8 & 6 & 2 \end{array}\right)$,
- $\Gamma_3 = s_5(D_3)\, d_{-6}(\Gamma_2) = \left(\begin{array}{ccccccc:cc} 13 & 11 & 9 & 8 & 4 & 3 & 2 & 1 & 0 \\ 24 & 21 & 20 & 17 & 15 & 13 & 12 & 8 & 4 \end{array}\right)$,
- $\Gamma_4 = c_7(D_4)\, d_{-11}(\Gamma_3) = \left(\begin{array}{ccccc:cccccccc} 23 & 21 & 18 & 17 & 13 & 12 & 11 & 9 & 7 & 6 & 2 & 1 & 0 \\ 38 & 35 & 34 & 32 & 30 & 23 & 22 & 19 & 17 & 15 & 14 & 10 & 6 \end{array}\right)$,

where the dotted lines are put to separate the concatenated two arrays in each Γ_v. Here Γ_4 is the ordinary partition corresponding to the (5, 2)-singular overpartition λ.

We now briefly describe the inverse of ψ_m. Let μ be an ordinary partition.

- Set $\Gamma_{m+1} = \mu$.
- For $v = m, \ldots, 1$, let $\begin{matrix} a_t \\ b_t \end{matrix}$ be the first column of Γ_{v+1} such that $a_t - b_t \geq -(v-1)k - i - 1$. If there exists no such t, then we define t to be $1 + \ell(\Gamma_{v+1})$, where $\ell(\Gamma_{v+1})$ denotes the number of columns of Γ_{v+1}. Split Γ_{v+1} into two arrays L_v and R_v by choosing the first $t - 1$ columns for L_v and the rest for R_v. Set

$$D_{v+1} = \begin{cases} c_{i+wk}(L_v) & \text{if } v = 2w + 1 \text{ for some } w \geq 0, \\ s_{-wk}(L_v) & \text{if } v = 2w \text{ for some } w > 0, \end{cases}$$

  and $\Gamma_v = d^{-1}_{1-i-(v-1)k}(R_v)$.
- Define $\psi_m^{-1}(\mu) = D_{m+1} \cdots D_2 D_1$.

In the following example, we present how ψ_m^{-1} works with the partition obtained in Example 3.1.

Example 3.2. Let

$$\mu = \begin{pmatrix} 23 & 21 & 18 & 17 & 13 & 12 & 11 & 9 & 7 & 6 & 2 & 1 & 0 \\ 38 & 35 & 34 & 32 & 30 & 23 & 22 & 19 & 17 & 15 & 14 & 10 & 6 \end{pmatrix}.$$

Note that we know that $k = 5$, $i = 2$, and $m = 3$. Below a dotted line is used to separate the two arrays in each step. Also, to the right of Γ_v, $a_t - b_t$ is given to indicate t, and to the right of $D_{v+1}\Gamma_v$, two maps that are applied to get D_{v+1} and Γ_v are given.

- $\Gamma_4 = L_4 R_4 = \left(\begin{array}{ccccc:cccccccc} 23 & 21 & 18 & 17 & 13 & 12 & 11 & 9 & 7 & 6 & 2 & 1 & 0 \\ 38 & 35 & 34 & 32 & 30 & 23 & 22 & 19 & 17 & 15 & 14 & 10 & 6 \end{array}\right)$, $a_6 - b_6 \geq -13$,
- $D_4 \Gamma_3 = \left(\begin{array}{ccccc:cccccccc} 31 & 28 & 27 & 25 & 23 & 13 & 11 & 9 & 8 & 4 & 3 & 2 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 24 & 21 & 20 & 17 & 15 & 13 & 12 & 8 & 4 \end{array}\right)$, c_7, d_{-11}^{-1}
- $\Gamma_3 = L_3 R_3 = \left(\begin{array}{ccccccc:cc} 13 & 11 & 9 & 8 & 4 & 3 & 2 & 1 & 0 \\ 24 & 21 & 20 & 17 & 15 & 13 & 12 & 8 & 4 \end{array}\right)$, $a_8 - b_8 \geq -8$,
- $D_3 \Gamma_2 = \left(\begin{array}{ccccccc:ccc} 18 & 16 & 14 & 13 & 9 & 8 & 7 & 2 & 1 & 0 \\ 19 & 16 & 15 & 12 & 10 & 8 & 7 & 8 & 6 & 2 \end{array}\right)$, s_{-5}, d_{-7}^{-1},
- $\Gamma_2 = L_2 R_2 = \left(\begin{array}{cc:c} 2 & 1 & 0 \\ 8 & 6 & 2 \end{array}\right)$, $a_3 - b_3 \geq -3$,
- $D_2 \Gamma_1 = \left(\begin{array}{cc:cc} 6 & 4 & 1 & 0 \\ 4 & 3 & 2 & 0 \end{array}\right)$, c_2, d_{-1}^{-1}
- $D_1 = \Gamma_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$.

Thus we recover the singular overpartition λ in Example 3.1:

$$\lambda = \begin{pmatrix} 31 & 28 & 27 & 25 & 23 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 10 & 8 & 7 & 4 & 3 & 2 & 0 \end{pmatrix},$$

where

$$D_4 = \begin{pmatrix} 31 & 28 & 27 & 25 & 23 \\ 30 & 28 & 25 & 24 & 20 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 18 & 16 & 14 & 13 & 9 & 8 & 7 \\ 19 & 16 & 15 & 12 & 10 & 8 & 7 \end{pmatrix}, D_2 = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Let us give another (simple but nontrivial) example.

Example 3.3. Consider a partition

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that we have information $m = 2$, but arbitrary k and i. Then,

- $\Gamma_3 = L_3 R_3 = \left(\begin{array}{:c} 0 \\ 0 \end{array}\right)$, $a_1 - b_1 = 0 \geq -i - k - 1$, so $t = 1$,

- $D_3\Gamma_2 = \begin{pmatrix} \vdots & 0 \\ \vdots & i+k \end{pmatrix}$, s_{-k}, d^{-1}_{1-i-k},
- $\Gamma_2 = L_2R_2 = \begin{pmatrix} 0 & \vdots \\ i+k & \vdots \end{pmatrix}$, $a_1 - b_1 = -i-k < -i-1$, so $t = 2$,
- $D_2\Gamma_1 = \begin{pmatrix} k & \vdots & 0 \\ i & \vdots & i-1 \end{pmatrix}$, c_i, d^{-1}_{1-i},
- $D_1 = \Gamma_1 = \begin{pmatrix} 0 \\ i-1 \end{pmatrix}$.

Thus we can restore

$$\lambda = \begin{pmatrix} k & 0 \\ i & i-1 \end{pmatrix},$$

where

$$D_3 = \emptyset,\ \ D_2 = \begin{pmatrix} k \\ i \end{pmatrix},\ D_1 = \begin{pmatrix} 0 \\ i-1 \end{pmatrix}.$$

4 The question of Andrews

Let $\overline{C}_{k,i}(n)$ be the number of overpartitions of n in which no parts are multiples of k and only parts congruent to $\pm i$ mod k can be overlined. Theorem 1.1 says

$$\overline{Q}_{k,i}(n) = \overline{C}_{k,i}(n).$$

For any integer m, we let $\overline{C}_{k,i}(n,m)$ be the number of overpartitions counted by $\overline{C}_{k,i}(n)$ such that the number of overlined parts congruent to i mod k minus the number of overlined parts congruent to $-i$ mod k equals m.

Also, for $m \geq 0$, let $\overline{Q}_{k,i}(n,m)$ be the number of (k,i)-singular overpartitions of n with exactly m dotted blocks and the last dotted block negative. For $m < 0$, let $\overline{Q}_{k,i}(n,m)$ be the number of (k,i)-singular overpartitions of n with exactly $|m|$ dotted blocks and the last dotted block positive.

We will prove the following refined version of Theorem 1.1:

Theorem 4.1. *For any integer m,*

$$\overline{Q}_{k,i}(n,m) = \overline{C}_{k,i}(n,m).$$

Proof. Let π be an overpartition counted by $\overline{C}_{k,i}(n)$. We first divide the parts of π into three partitions μ^1, μ^2, γ as follows: μ^1 is the partition consisting of the overlined parts of π that are congruent to i mod k, μ^2 is the partition consisting of the overlined parts of π that are congruent to $-i$ mod k, and γ is the partition consisting of the nonoverlined parts of π. Clearly, $\pi = \mu^1 \cup \mu^2 \cup \gamma$ and $\ell(\mu^1) - \ell(\mu^2) = m$.

Recall the Wright map φ from Section 2.7. Let $(\kappa, m) = \varphi(\mu^1, \mu^2)$. Then κ is a partition into multiples of k. Clearly, $\kappa \cup \gamma$ is a partition of $n - k\binom{m}{2} - im$. Thus we have

$$\overline{C}_{k,i}(n, m) = p\left(n - k\binom{m}{2} - im\right),$$

which with Theorem 3.1 completes the proof.

Remark 4.1. Since only φ and ψ_m are used, the proof above is bijective, which answers to the question of Andrews.

Lastly, we illustrate how to combine φ and ψ_m to relate a (k, i)-singular overpartition of n to an overpartition counted by $\overline{C}_{k,i}(n)$ in an example.

Example 4.1. Let λ be the $(5, 2)$-singular overpartition of 469 given in Example 3.1, i.e.,

$$\lambda = \left(\begin{array}{cc|ccc|ccccccc|cc|cc} 31 & 28 & 27 & 25 & 22 & 18 & 16 & 14 & 13 & 9 & 8 & 7 & 6 & 4 & 1 & 0 \\ 30 & 28 & 25 & 24 & 20 & 19 & 16 & 15 & 12 & 11 & 8 & 7 & 4 & 3 & 2 & 0 \end{array}\right),$$

with its sequence of dotted blocks is $EP\dot{N}\dot{P}\dot{N}$. Then we know from Example 3.1 that $m = 3$ and

$$\begin{aligned}\psi_3(\lambda) = \mu &= \begin{pmatrix} 23 & 21 & 18 & 17 & 13 & 12 & 11 & 9 & 7 & 6 & 2 & 1 & 0 \\ 38 & 35 & 34 & 32 & 30 & 23 & 22 & 19 & 17 & 15 & 14 & 10 & 6 \end{pmatrix} \\ &= (24, 23, 21^2, 18^3, 17, 16^2, 13^9, 12^3, 11^3, 9, 8, 7^2, 5^6, 4, 3, 1^2),\end{aligned}$$

where the power of a number indicates the number of occurrences of the number as a part. We now take the multiples of 5 in μ to form the partition $(5, 5, 5, 5, 5, 5)$. Then

$$\varphi^{-1}((5, 5, 5, 5, 5, 5), 3) = ((17, 12, 7, 2), (13)).$$

Thus, we obtain the following overpartition π counted by $\overline{C}_{5,2}(469)$:

$$\pi = (24, 23, 21^2, 18^3, \overline{17}, 17, 16^2, 13^9, \overline{13}, \overline{12}, 12^3, 11^3, 9, 8, \overline{7}, 7^2, 4, 3, \overline{2}, 1^2).$$

Acknowledgements The authors would like to thank the anonymous referee for valuable comments and suggestions.

References

1. G.E. Andrews, *The Theory of Partitions* (Addison–Wesley, Reading, 1976). Reissued: Cambridge University Press, Cambridge, 1998
2. G.E. Andrews, Generalized Frobenius partitions. Mem. AMS **49** (1984)
3. G.E. Andrews, Singular overpartitions. Int. J. Number Theory **11**, 1523–1533 (2015)
4. C. Boulet, Dyson's new symmetry and generalized Rogers–Ramanujan identities. Ann. Comb. **14**, 159–191 (2010)

5. S. Corteel, J. Lovejoy, Overpartitions. Trans. Am. Math. Soc. **356**, 1623–1635 (2004)
6. F.J. Dyson, A new symmetry of partitions. J. Combin. Theory **7**, 56–61 (1969)
7. J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition. Ann. Comb. **9**, 321–334 (2005)
8. I. Pak, Partition bijections, a survey. Ramanujan J. **12**, 5–75 (2006)
9. E.M. Wright, An enumerative proof of an identity of Jacobi. J. Lond. Math. Soc. **40**, 55–57 (1965)
10. A.J. Yee, Combinatorial proofs of generating function identities for F-partitions. J. Combin. Theory A **102**, 217–228 (2003)
11. A.J. Yee, Truncated Jacobi triple product theorem. J. Combin. Theory A **130**, 1–14 (2015)

A Classical q-Hypergeometric Approach to the $A_2^{(2)}$ Standard Modules

Andrew V. Sills

Dedicated to Krishna Alladi on the occasion of his 60th birthday

Abstract This is a written expansion of the talk delivered by the author at the International Conference on Number Theory in Honor of Krishna Alladi for his 60th Birthday, held at the University of Florida, March 17–21, 2016. Here, we derive Bailey pairs that give rise to Rogers–Ramanujan type identities, the product sides of which are known to be the principally specialized characters of the $A_2^{(2)}$ standard modules $(\ell - 2i + 2)\Lambda_0 + (i-1)\Lambda_1$ for any level ℓ, and $i = 1, 2$.

Keywords Basic hypergeometric series · Rogers–Ramanujan identities
Capparelli identities · Affine Lie algebras · Bailey pairs

2010 Mathematics Subject Classification 33D15 · 17B67 · 11P84

1 Notation and Motivation

1.1 q-series notation and classical results

Let q denote a formal variable. The standard notation for the infinite rising q-factorial is

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j).$$

A. V. Sills (✉)
Georgia Southern University, Statesboro, Georgia
e-mail: ASills@georgiasouthern.edu

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_39

In order to allow for positive and negative values of n, we define the finite rising q-factorial as

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

We will also use the abbreviations $(q)_n$ and $(q)_\infty$ for $(q;q)_n$ and $(q;q)_\infty$, respectively. Additionally,

$$(a_1, a_2, \ldots, a_r; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n$$

and

$$(a_1, a_2, \ldots, a_r; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty.$$

The bilateral basic hypergeometric series is given by

$${}_r\psi_r \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z \right] := \sum_{n \in \mathbf{Z}} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots b_r; q)_n} z^n.$$

For $m, n \in \mathbb{Z}$, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

We will require the following classical results: Jacobi's triple product identity [17, p. 15, Eq. (1.6.1)].

$$\sum_{n \in \mathbf{Z}} (-1)^n z^n q^{n^2} = (q/z, zq, q^2; q^2)_\infty, \tag{1.1}$$

and a result that follows from (1.1) and the quintuple product identity ([16], cf. [17, p. 147, ex. 5.6])

$$(-qz^3, -q^2z^{-3}, q^3; q^3)_\infty - z(-qx^{-3}, -q^2z^3, q^3; q^3)_\infty = (q/z, z, q; q)_\infty (q/z^2, qz^2; q^2)_\infty. \tag{1.2}$$

In fact, Eq. (1.2) is called "The quintuple product identity; second version" in [10, p. 19, Theorem 1.3.18].

Herein, we will only need to consider z of the form $z = \pm q^r$ for some $r \in \frac{1}{2}\mathbf{Z}$.

1.2 Certain affine Lie algebras and their connection to q-series

Let $\mathfrak{g}$ denote the affine Lie algebra $A_1^{(1)}$ or $A_2^{(2)}$ and let h_0, h_1 denote the usual basis of a maximal toral subalgebra T of $\mathfrak{g}$. Let d denote the degree derivation of $\mathfrak{g}$ and let $\tilde{T} := T \oplus \mathbf{C}d$. For all dominant integral $\lambda \in \tilde{T}^*$, there is a unique irreducible, integrable, highest weight module $L(\lambda)$, assuming (without loss of generality) that $\lambda(d) = 0$. Also, $\lambda = s_0 \Lambda_0 + s_1 \Lambda_1$ where Λ_0 and Λ_1 are the fundamental weights, given by $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$; s_0 and s_1 are nonnegative integers. For $A_1^{(1)}$, the canonical central element is $c = h_0 + h_1$, and for $A_2^{(2)}$, the canonical central element is $c = h_0 + 2h_1$. The *level* $\lambda(c) = \lambda_{\mathfrak{g}}(c)$ of $L(\lambda)$ is

$$\lambda(c) = \begin{cases} s_0 + s_1 & \text{if } \mathfrak{g} = A_1^{(1)}, \\ s_0 + 2s_1 & \text{if } \mathfrak{g} = A_2^{(2)}, \end{cases}$$

(cf. [20], [23]). For brevity, it is common to refer to $L(\lambda) = L(s_0\Lambda_0 + s_1\Lambda_1)$ as the "(s_0, s_1)-module".

Additionally ([23]), there is an infinite product $F_{\mathfrak{g}}$ associated with $\mathfrak{g}$, sometimes called the "fudge factor", which needs to be divided out of the the principally specialized character $\chi(L(\lambda)) = \chi(s_0\Lambda_0 + s_1\Lambda_1)$, in order to obtain the quantities of interest here. The fudge factor $F_{\mathfrak{g}}$ is given by

$$F_{\mathfrak{g}} = \begin{cases} (q;q^2)_\infty^{-1} & \text{if } \mathfrak{g} = A_1^{(1)}, \\ \Big((q;q^6)_\infty (q^5;q^6)_\infty\Big)^{-1} & \text{if } \mathfrak{g} = A_2^{(2)}. \end{cases}$$

Also, $\mathfrak{g}$ has a certain infinite-dimensional Heisenberg subalgebra known as the principal Heisenberg vacuum subalgebra $\mathfrak{s}$ (consult [24] for the construction of $A_1^{(1)}$ and [21] for that of $A_2^{(2)}$). As demonstrated in [25], the principal character $\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1))$, where $\Omega(\lambda)$ is the vacuum space for $\mathfrak{s}$ in $L(\lambda)$, is

$$\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1)) = \frac{\chi(L(s_0\Lambda_0 + s_1\Lambda_1))}{F_{\mathfrak{g}}}, \tag{1.3}$$

where $\chi(L(\lambda))$ is the principally specialized character of $L(\lambda)$.

By [23] applied to (1.3) in the case of $A_1^{(1)}$, the standard modules of odd level correspond to Andrews' analytic generalization of the Rogers–Ramanujan identities [3], known as the "Andrews–Gordon identity", and the partition theoretic generalization of the Rogers–Ramanujan identities due to B. Gordon [18]. Bressoud's even modulus counterpart to the Andrews–Gordon identity [12, p. 15, Eq. (3.4)] and its partition theoretic counterpart [11, p. 64, Theorem, $j = 0$ case] was explained vertex operator theoretically in [26] and [27] to correspond to the standard modules of even level in $A_1^{(1)}$.

The combined Andrews–Gordon–Bressoud identity (for both even and odd moduli) and its correspondence to the level ℓ standard modules of $A_1^{(1)}$ can be stated compactly as

$$\chi(\Omega((\ell+1-i)\Lambda_0+(i-1)\Lambda_1)) = \sum_{n_1\geqq n_2\geqq\cdots\geqq n_{k-1}\geqq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+n_{i+1}+\cdots+n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}(-q)_{[2|\ell]n_{k-1}}} = \frac{(q^i,q^{\ell+2-i},q^{\ell+2};q^{\ell+2})_\infty}{(q)_\infty}, \qquad (1.4)$$

where $k := k(\ell) = 1 + \lfloor \ell/2 \rfloor$, $1 \le i \le k$, and

$$[P] := \begin{cases} 1 \text{ if } P \text{ is true,} \\ 0 \text{ if } P \text{ is false} \end{cases},$$

is the "Iverson bracket" notation introduced by K.E. Iverson in [19] and advocated for by D.E. Knuth in [22].

A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ form a *Bailey pair* with respect to a if

$$\beta_n(a,q) = \sum_{s=0}^{n} \frac{\alpha_n(a,q)}{(q)_{n-s}(aq;q)_{n+s}}.$$

[5, p. 25–26]; cf. [9, pp. 2, 5].

It is well known that identities of Rogers–Ramanujan type may be derived by the insertion of Bailey pairs into limiting cases of Bailey's lemma [5, p. 25, Thm. 3.3; p. 27, Eq. (3.33)] such as

$$\sum_{n=0}^{\infty} a^n q^{n^2}\beta_n(a,q) = \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty} a^n q^{n^2}\alpha_n(a,q), \qquad (1.5)$$

and setting a equal to a power of q.

An efficient method for deriving (1.4) for odd ℓ is via the Bailey lattice [2], which is an extension of the Bailey chain ([4]; cf. [5, §3.5, pp. 27ff]) built upon the "unit Bailey pair"

$$\beta_n(1,q) = \delta_{n0},$$

where δ_{ij} is the Kronecker δ-function,

$$\alpha_n(1,q) = \begin{cases} 1 & \text{if } n = 0 \\ (-1)^n q^{n(n-1)/2}(1+q^n) & \text{if } n > 0. \end{cases}$$

Similarly, for even ℓ, (1.4) follows from a Bailey lattice built upon the Bailey pair

$$\beta_n(1,q) = \frac{1}{(q^2;q^2)_n},$$

$$\alpha_n(1,q) = \begin{cases} 1 & \text{if } n = 0 \\ (-1)^n 2q^{n^2} & \text{if } n > 0. \end{cases}$$

Thus, the standard modules of $A_1^{(1)}$ correspond to two interlaced instances of the Bailey lattice.

In contrast, the standard modules of $A_2^{(2)}$ are not as well understood, and a uniform q-series and partition correspondence analogous to what is known for $A_1^{(1)}$ has to date remained beyond our reach.

As with $A_1^{(1)}$, there are $1 + \lfloor \frac{\ell}{2} \rfloor$ inequivalent level ℓ standard modules associated with the Lie algebra $A_2^{(2)}$, but the principal characters for the level ℓ standard modules are given by instances of the quintuple product identity (1.2) (rather than the triple product identity) divided by $(q)_\infty$:

$$\chi(\Omega((\ell-2i+2)\Lambda_0 + (i-1)\Lambda_1)) \\ = \frac{(q^i, q^{\ell+3-i}, q^{\ell+3}; q^{\ell+3})_\infty (q^{\ell+3-2i}, q^{\ell+2i+3}; q^{2\ell+6})_\infty}{(q)_\infty}, \tag{1.6}$$

where $1 \le i \le 1 + \lfloor \frac{\ell}{2} \rfloor$; see [23].

2 Bailey pairs for $A_2^{(2)}$

Let $(\alpha_n^{(\ell,i)}, \beta_n^{(\ell,i)})$ denote the Bailey pair which, upon insertion into (1.5) with $a = 1$, gives the principally specialized character of the $A_2^{(2)}$ standard module corresponding to the highest weight $(\ell - 2i + 2)\Lambda_0 + (i-1)\Lambda_1$.

The strategy is to choose $\alpha_n^{(\ell,i)}$ that gives rise to the desired product side, and use a standard transformation to find the corresponding $\beta_n^{(\ell,i)}$.

2.1 Bailey pairs for $\chi(\Omega(\ell\Lambda_0))$

$$\alpha_n = \alpha_n^{(\ell,1)}(1,q) = \begin{cases} 1 & \text{if } n=0 \\ q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r} + q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r} & \text{if } n=3r>0 \\ -q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r} & \text{if } n=3r+1 \\ -q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r} & \text{if } n=3r-1 \end{cases}$$

$$\begin{aligned}
\beta_n^{(\ell,1)}(1,q) &= \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,1)}(1,q)}{(q)_{n-s}(q)_{n+s}} \\
&= \frac{\alpha_0}{(q)_n^2} + \sum_{r\geq 1} \frac{\alpha_{3r}}{(q)_{n-3r}(q)_{n+3r}} + \sum_{r\geq 0} \frac{\alpha_{3r+1}}{(q)_{n-3r-1}(q)_{n+3r+1}} \\
&\quad + \sum_{r\geq 1} \frac{\alpha_{3r-1}}{(q)_{n-3r+1}(q)_{n+3r-1}} \\
&= \frac{1}{(q)_n^2} + \sum_{r\geq 1} \frac{q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r} + q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r}(q)_{n+3r}} \\
&\quad + \sum_{r\geq 0} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r-1}(q)_{n+3r+1}} + \sum_{r\geq 1} \frac{q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r}}{(q)_{n-3r+1}(q)_{n+3r-1}} \\
&= \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r}(q)_{n+3r}} - \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r-1}(q)_{n+3r+1}} \\
&= \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r}(q)_{n+3r+1}} \Big((1-q^{n+3r+1}) - (1-q^{n-3r}) \Big) \\
&= \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r}}{(q)_{n-3r}(q)_{n+3r+1}} (q^{n-3r} - q^{n+3r+1}) \\
&= \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r+n-3r}}{(q)_{n-3r}(q)_{n+3r+1}} (1-q^{6r+1}) \\
&= \frac{q^n}{(q)_n(q)_{n+1}} \sum_{r\in\mathbf{Z}} \frac{q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-9)r}(1-q^{6r+1})(q^{n-3r+1};q)_{3r}}{(q^{n+2};q)_{3r}} \\
&= \frac{q^n}{(q)_n(q^2;q)_n} \sum_{r\in\mathbf{Z}} \frac{(1-q^{6r+1})(q^{-n};q)_{3r}}{(1-q)(q^{n+2};q)_{3r}} (-1)^r q^{\frac{3}{2}(\ell-6)r^2+\frac{1}{2}(\ell-6)r}.
\end{aligned} \tag{2.1}$$

For each $\ell = 1, 2, 3, \ldots$, the series expression in (2.1) is a limiting case of a very well-poised bilateral basic hypergeometric series.

For example, we have

$$\frac{q^{-n}(q)_n(q)_{n+1}}{1-q}\beta^{(\ell,1)}(1,q)$$
$$= \begin{cases} \lim\limits_{e\to 0} {}_8\psi_8\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e, e, e \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e}, \frac{q^4}{e}, \frac{q^4}{e} \end{matrix}; q^3, \frac{q^{3n+6}}{e^3}\right] & \text{if } \ell = 3 \\ \lim\limits_{e\to 0} {}_8\psi_8\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e, e, -q^2 \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e}, \frac{q^4}{e}, -q^2 \end{matrix}; q^3, \frac{-q^{3n+4}}{e^2}\right] & \text{if } \ell = 4 \\ \lim\limits_{e\to 0} {}_6\psi_6\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e} \end{matrix}; q^3, \frac{q^{3n+2}}{e}\right] & \text{if } \ell = 5 \\ {}_6\psi_6\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, -q^2 \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, -q^2 \end{matrix}; q^3, -q^{3n}\right] & \text{if } \ell = 6. \\ \lim\limits_{e\to \infty} {}_6\psi_6\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e} \end{matrix}; q^3, \frac{q^{3n+2}}{e}\right] & \text{if } \ell = 7 \\ \lim\limits_{e\to \infty} {}_8\psi_8\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e, e, -q^2 \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e}, \frac{q^4}{e}, -q^2 \end{matrix}; q^3, \frac{-q^{3n+4}}{e^2}\right] & \text{if } \ell = 8 \\ \lim\limits_{e\to \infty} {}_8\psi_8\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, e, e, e \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \frac{q^4}{e}, \frac{q^4}{e}, \frac{q^4}{e} \end{matrix}; q^3, \frac{q^{3n+6}}{e^3}\right] & \text{if } \ell = 9 \end{cases}$$

Observe that the easiest cases are $\ell = 5, 6, 7$, as these are instances of Bailey's summable bilateral very well-poised ${}_6\psi_6$ [8, Eq. (4.7)]; cf. [17, p. 357, Eq. (II.33)]. Indeed, Slater evaluated the cases $\ell = 5$ and 7 [33, p. 464, Eqs. (3.4) and (3.3) resp.], while McLaughlin and Sills evaluated the case $\ell = 6$ [28, p. 772, Table 3.1, line (P2)].

We have

$$\beta_n^{(5,1)}(1,q) = \frac{q^{n^2}}{(q)_{2n}}, \tag{2.2}$$

$$\beta_n^{(6,1)}(1,q) = \frac{q^n(-1;q^3)_n}{(-1;q)_n(q)_{2n}}, \tag{2.3}$$

$$\beta_n^{(7,1)}(1,q) = \frac{q^n}{(q)_{2n}}. \tag{2.4}$$

To evaluate $\beta_n^{(\ell,1)}(1,q)$ for levels $\ell = 3, 4, 8, 9$, we can use the following identity [17, p. 147, exercise 5.11], analogous to Bailey's ${}_6\psi_6$ sum: for nonnegative integer n,

$$
{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, c, d, e, f, aq^{-n}, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, q^{n+1}, aq^{n+1} \end{matrix}; q, \frac{a^2q^{2n+2}}{cdef}\right]
$$

$$
= \frac{(aq, \frac{q}{a}, \frac{aq}{cd}, \frac{aq}{ef}; q)_n}{(\frac{q}{c}, \frac{q}{d}, \frac{aq}{e}, \frac{aq}{f}; q)_n} {}_4\psi_4\left[\begin{matrix} e, f, \frac{aq^{n+1}}{cd}, q^{-n} \\ \frac{aq}{c}, \frac{aq}{d}, q^{n+1}, \frac{ef}{aq^n} \end{matrix}; q, q\right]. \tag{2.5}
$$

Notice that for level $\ell = 1$ and 2, $q^{-n}(q)_n(q^2;q)_n\beta_n^{(\ell,1)}$ is a limiting case of a ${}_{10}\psi_{10}$, while for $\ell > 6$, it is a limiting case of a ${}_r\psi_r$ with $r = \ell - 1 + (1 + (-1)^\ell)/2$. More precisely, if ℓ is even and $\ell \geqq 6$,

$$
q^{-n}(q)_n(q^2;q)_{n+1}\beta_n^{(\ell,1)}(1,q)
$$

$$
= \lim_{e\to\infty} {}_\ell\psi_\ell\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, \overbrace{e, e, \ldots, e}^{\ell-6}, -q^2 \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \underbrace{\frac{q^4}{e}, \frac{q^4}{e}, \ldots, \frac{q^4}{e}}_{\ell-6}, -q^2 \end{matrix}; q^3, \frac{-q^{3n+2\ell-12}}{e^{\ell-6}}\right],
$$

while if ℓ is odd and $\ell > 6$,

$$
q^{-n}(q)_n(q^2;q)_{n+1}\beta_n^{(\ell,1)}(1,q)
$$

$$
= \lim_{e\to\infty} {}_{\ell-1}\psi_{\ell-1}\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, \overbrace{e, e, \ldots, e}^{\ell-6} \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \underbrace{\frac{q^4}{e}, \frac{q^4}{e}, \ldots, \frac{q^4}{e}}_{\ell-6} \end{matrix}; q^3, \frac{q^{3n+2\ell-12}}{e^{\ell-6}}\right].
$$

Then, to obtain the series and product expressions for $\chi(\Omega(\ell\Lambda_0))$, one inserts the Bailey pair $\left(\alpha_n^{(\ell,1)}(1,q), \beta_n^{(\ell,1)}(1,q)\right)$ into (1.5) with $a = 1$, and upon applying (1.1) and (1.2), we find that

$$
\sum_{m=0}^{\infty} q^{9m^2}\left(q^{-6m+1}\beta_{3m-1}^{(\ell,1)}(1,q) + \beta_{3m}^{(\ell,1)}(1,q) + q^{6m+1}\beta_{3m+1}^{(\ell,1)}(1,q)\right)
$$

$$
= \frac{(q, q^{\ell+2}, q^{\ell+3}; q^{\ell+3})_\infty (q^{\ell+1}, q^{\ell+5}; q^{2\ell+6})}{(q)_\infty}. \tag{2.6}
$$

And thus in (2.6), we have a uniform series-product identity for the principally specialized character of the $(\ell, 0)$ standard module of $A_2^{(2)}$ for any ℓ.

To express the $\beta_n^{(\ell,1)}$ as a multisum for arbitrary ℓ, one may employ the Andrews–Baxter–Forrester bilateral very well-poised q-hypergeometric summation formula [5, p. 83, Eq. (8.56)]; cf. [7, Appendix B, pp. 261–265].

The level 3 case will be considered in detail in the next section.

2.2 Bailey pairs for $\chi(\Omega((\ell-2)\Lambda_0+\Lambda_1))$

$$\alpha_n = \alpha_n^{(\ell,2)}(1,q) = \begin{cases} 1 & \text{if } n=0 \\ q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(9-\ell)r} + q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(9-\ell)r} & \text{if } n=3r>0 \\ -q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell+3)r+1} & \text{if } n=3r+1 \\ -q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell+3)r+1} & \text{if } n=3r-1 \end{cases}.$$

The calculation of $\beta_n^{(\ell,2)}$ parallels that of $\beta_n^{(\ell,1)}$. The details of the $\ell=7$ case are given by Slater [33, p. 464].

$$\beta_n^{(\ell,2)}(1,q) = \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,2)}(1,q)}{(q)_{n-s}(q)_{n+s}} = \frac{1}{(q)_n(q^2;q)_n} \sum_{r\in\mathbf{Z}} \frac{(1-q^{6r+1})(q^{-n};q)_{3r}}{(1-q)(q^{n+2};q)_{3r}} (-1)^r q^{\frac{3}{2}(\ell-6)r^2+\frac{1}{2}(\ell-6)r}. \tag{2.7}$$

Notice that

$$q^n \beta_n^{(\ell,2)}(1,q) = \beta_n^{(\ell,1)}(1,q). \tag{2.8}$$

And so it follows that the series and product expressions for $\chi(\Omega((\ell-2)\Lambda_0+\Lambda_1))$ are

$$\sum_{m=0}^{\infty} q^{9m^2}\left(q^{-6m+1}\beta_{3m-1}^{(\ell,2)}(1,q) + \beta_{3m}^{(\ell,2)}(1,q) + q^{6m+1}\beta_{3m+1}^{(\ell,2)}(1,q)\right) = \frac{(q^2,q^{\ell+1},q^{\ell+3};q^{\ell+3})_\infty (q^{\ell-1},q^{\ell+7};q^{2\ell+6})}{(q)_\infty}, \tag{2.9}$$

a general identity corresponding to the $(\ell-2,1)$ module.

3 Level 3

Let us consider the case $\ell=3$ in detail. This level is of particular interest as it was the study of the the level 3 standard modules of $A_2^{(2)}$ that led S. Capparelli to discover two new Rogers–Ramanujan type partition identities [1, 6, 14, 15]. See [32, §3] for some historical notes.

From the $(3, 0)$-module, Capparelli conjectured (and later proved [15], although the first proof was due to Andrews [6]) the following partition identity. A *partition* λ of an integer n is a finite weakly decreasing sequence $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers that sum to n; each λ_i is called a *part* of the partition λ.

Theorem 3.1 (Capparelli's first partition identity).
Let $c_1(n)$ denote the number of partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ of n wherein

- $\lambda_i \neq 1$ *for* $i = 1, 2, \dots, l$,
- $\lambda_i - \lambda_{i+1} \geq 2$, *for* $i = 1, 2, \dots, l-1$,
- $\lambda_i - \lambda_{i+1} = 2$ *only if* $\lambda_i \equiv 1 \pmod 3$,
- $\lambda_i - \lambda_{i+1} = 3$ *only if* $\lambda_i \equiv 0 \pmod 3$.

Let $c_2(n)$ denote the number of partitions of n into distinct parts $\not\equiv \pm 1 \pmod 6$. Let $c_3(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 3 \pmod{12}$. Then, $c_1(n) = c_2(n) = c_3(n)$ for all n.

From the $(1, 1)$-module, Capparelli obtained the companion identity:

Theorem 3.2 (Capparelli's second partition identity).
Let $d_1(n)$ denote the number of partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ of n wherein

- $\lambda_i \neq 2$ *for* $i = 1, 2, \dots, l$,
- $\lambda_i - \lambda_{i+1} \geq 2$, *for* $i = 1, 2, \dots, l-1$,
- $\lambda_i - \lambda_{i+1} = 2$ *only if* $\lambda_i \equiv 1 \pmod 3$,
- $\lambda_i - \lambda_{i+1} = 3$ *only if* $\lambda_i \equiv 0 \pmod 3$.

Let $d_2(n)$ denote the number of partitions of n into distinct parts $\not\equiv \pm 2 \pmod 6$.
Then, $d_1(n) = d_2(n)$ for all n.

3.1 $\chi(\Omega(3\Lambda_0))$

In order to use (2.5), we need to consider three cases, $n = 3m, 3m + 1$, and $3m - 1$.

In (2.5), replace q by q^3; then set $a = q$, $n = m$, $f = q^{2-3m}$, and $c = d = e$, to obtain

$$\begin{aligned}
\beta_{3m}^{(3,1)}(1,q) &= \frac{q^n(1-q)}{(q)_n(q)_{n+1}} \lim_{e\to 0} {}_8\psi_8 \left[\begin{matrix} q^{-7/2}, -q^{7/2}, e, e, e, q^{2-3m}, q^{1-3m}, q^{-3m} \\ q^{1/2}, -q^{1/2}, \frac{q^4}{e}, \frac{q^4}{e}, \frac{q^4}{e}, q^{3m+2}, q^{3m+3}, q^{3m+4} \end{matrix} ; q^3, q^3 \right] \\
&= \frac{q^{3m}(1-q)}{(q)_{3m}(q)_{3m+1}} \lim_{e\to 0} \frac{(q^4, q^2, \frac{q^4}{e^2}, \frac{q^{3m+2}}{e}; q^3)_m}{(\frac{q^3}{e}, \frac{q^3}{e}, \frac{q^4}{e}, q^{3m+2}; q^3)_m} \\
&\quad \times {}_4\psi_4 \left[\begin{matrix} e, q^{2-3m}, \frac{q^{3m+4}}{e^2}, q^{-3m} \\ \frac{q^4}{e}, \frac{q^4}{e}, q^{3m+3}, eq^{1-6m} \end{matrix} ; q^3, q^3 \right] \\
&= \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{\frac{3}{2}m^2 - \frac{1}{2}m - \frac{1}{2}r - 3mr + \frac{3}{2}r^2}}{(q^2;q^3)_{2m}(q^3;q^3)_{m+r}(q;q^3)_{m-r}(q^3;q^3)_{m-r}} \\
&= \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2 + \frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}}. \qquad (3.1)
\end{aligned}$$

In (2.5), replace q by q^3; then set $a = q$, $n = m$, $f = q^{-1-3m}$, and $c = d = e$, to obtain

$$\begin{aligned}
\beta_{3m+1}^{(3,1)}(1,q) &= \frac{q^n(1-q)}{(q)_n(q)_{n+1}} \lim_{e\to 0} {}_8\psi_8 \left[\begin{matrix} q^{-7/2}, -q^{7/2}, e, e, e, q^{-1-3m}, q^{1-3m}, q^{-3m} \\ q^{1/2}, -q^{1/2}, \frac{q^4}{e}, \frac{q^4}{e}, \frac{q^4}{e}, q^{3m+2}, q^{3m+5}, q^{3m+4} \end{matrix} ; q^3, q^3 \right] \\
&= \frac{q^{3m+1}(1-q)}{(q)_{3m+1}(q)_{3m+2}} \lim_{e\to 0} \frac{(q^4, q^2, \frac{q^4}{e^2}, \frac{q^{3m+5}}{e}; q^3)_m}{(\frac{q^3}{e}, \frac{q^3}{e}, \frac{q^4}{e}, q^{3m+5}; q^3)_m} \\
&\quad \times {}_4\psi_4 \left[\begin{matrix} e, q^{-1-3m}, \frac{q^{3m+4}}{e^2}, q^{-3m} \\ \frac{q^4}{e}, \frac{q^4}{e}, q^{3m+3}, eq^{-2-6m} \end{matrix} ; q^3, q^3 \right] \\
&= \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{\frac{3}{2}m^2 + \frac{7}{2}m - \frac{7}{2}r - 3mr + \frac{3}{2}r^2 + 1}}{(q^2;q^3)_{2m+1}(q^3;q^3)_{m+r}(q;q^3)_{m-r+1}(q^3;q^3)_{m-r}} \\
&= \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2 + \frac{7}{2}r + 1}(q^2;q^3)_r}{(q^2;q^3)_{2m+1}(q^3;q^3)_{2m-r}(q)_{3r+1}}. \qquad (3.2)
\end{aligned}$$

For convenience, let us define the abbreviation

$$\sigma(m,r) := \frac{(-1)^r q^{\frac{3}{2}r^2 + \frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}}, \qquad (3.3)$$

so that we have immediately

$$\beta_{3m}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \sigma(m,r),$$

and with a bit of elementary algebra,

$$\beta_{3m+1}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \frac{\sigma(m,r)}{1-q^{6m+2}} \left(\frac{1}{1-q^{3r+1}} - 1 \right),$$

for $m \geqq 0$.

The author could not find a direct substitution into (2.5), analogous to the $n = 3m$ and $n = 3m+1$ cases, which yields the $n = 3m-1$ case. So we resort to an alternate method to obtain the $n = 3m-1$ case.

From the Paule–Riese `qZeil.m` *Mathematica* package available for download at
`http://www.risc.jku.at/research/combinat/software/qZeil/index.php`
and documented in [30], one can find that $\beta_n^{(3,1)}(1,q)$ satisfies the recurrence

$$\beta_n = \frac{-q^2+q^{2n}+q^{2n+1}}{q^2(1-q^{2n})(1-q^{2n-1})}\beta_{n-1} - \frac{1}{(1-q^{2n})(1-q^{2n-1})}\beta_{n-2} \tag{3.4}$$

as certified by the rational function

$$\frac{q^{-n-6r-2}\left(q^{3r}-q^n\right)\left(q^{3r+1}-q^n\right)\left(q^{3r+2}-q^n\right)}{\left(q^n-1\right)\left(q^n+1\right)\left(q^{2n}-q\right)\left(q^{6r+1}-1\right)}.$$

Setting $n = 3m+1$ in (3.4) and rearranging, we see how to express β_{3m-1} in terms of the two known expressions β_{3m} and β_{3m+1}:

$$\begin{aligned}
\beta_{3m-1} &= -(1-q^{6m+2})(1-q^{6m+1})\beta_{3m+1} - (1-q^{6m}-q^{6m+1})\beta_{3m} \\
&= -(1-q^{6m+2})(1-q^{6m+1}) \sum_{r=0}^{2m} \frac{\sigma(m,r)}{1-q^{6m+2}} \left(\frac{1}{1-q^{3r+1}} - 1 \right) \\
&\qquad + (1-q^{6m+1})\beta_{3m} - (1-q^{6m}-q^{6m+1})\beta_{3m} \\
&= -(1-q^{6m+1}) \sum_{r=0}^{2m} \frac{\sigma(m,r)}{1-q^{3r+1}} + q^{6m}\beta_{3m} \\
&= \sum_{r=0}^{2m} \sigma(m,r) \left(q^{6m} - \frac{1-q^{6m+1}}{1-q^{3r+1}} \right),
\end{aligned} \tag{3.5}$$

for $m \geq 1$.

Inserting $\left(\alpha_n^{(3,1)}(1,q), \beta_n^{(3,1)}(1,q)\right)$ into (1.5) with $a = 1$, and applying (1.1) and (1.2), we find that

$$\sum_{m=0}^{\infty} q^{9m^2}\left(q^{-6m+1}\beta_{3m-1}^{(3,1)}(1,q) + \beta_{3m}^{(3,1)}(1,q) + q^{6m+1}\beta_{3m+1}^{(3,1)}(1,q)\right)$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{r=0}^{2m} q^{9m^2}\sigma(m,r)\left(q^{1-6m}\left(q^{6m}-\frac{1-q^{6m+1}}{1-q^{3r+1}}\right)+1\right.\\
&\qquad\left.+\frac{q^{6m+1}}{1-q^{6m+2}}\left(1-\frac{1}{1-q^{3r+1}}\right)\right)\\
&= \frac{(q,q^5,q^6;q^6)_\infty(q^4,q^8;q^{12})_\infty}{(q)_\infty} = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}\\
&= (-q^2;q^2)_\infty(-q^3;q^6)_\infty. \qquad (3.6)
\end{aligned}
$$

The series expansion of $(q^2, q^3, q^9, q^{10}; q^{12})_\infty^{-1}$ in (3.6) is quite different than others that have appeared in the literature, due to Alladi, Andrews, and Gordon [1, pp. 648–649, Lemma 2(b)] (cf. [31, p. 399, Eq. (1.3)]), the author [31, p. 399, Eq. (1.4) and Eq. (1.5)], and Bringmann and Mahlburg [13].

3.2 $\chi(\Omega(\Lambda_0 + \Lambda_1))$

In light of (2.8), it is trivial to obtain $\beta_n^{(3,2)}(1,q)$ from $\beta_n^{(3,1)}(1,q)$, and upon inserting $\left(\alpha_n^{(3,2)}(1,q), \beta_n^{(3,2)}(1,q)\right)$ into (1.5) with $a = 1$, and applying (1.1) and (1.2), we find that

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{r=0}^{2m} q^{9m^2-3m}\sigma(m,r)\left(q^{2-6m}\left(q^{6m}-\frac{1-q^{6m+1}}{1-q^{3r+1}}\right)+1\right.\\
&\qquad\left.+\frac{q^{6m}}{1-q^{6m+2}}\left(1-\frac{1}{1-q^{3r+1}}\right)\right)\\
&\qquad= \frac{(q^2,q^4,q^6;q^6)_\infty(q^2,q^{10};q^{12})_\infty}{(q)_\infty} = (-q;q^2)_\infty(-q^6;q^6)_\infty. \qquad (3.7)
\end{aligned}
$$

3.3 *Nandi's recent work on level* 4

It should be noted that recently D. Nandi, in his Ph.D. thesis [29] conjectured the partition identities corresponding to the three inequivalent level 4 standard modules (4, 0), (2, 1), and (0, 2). These identities, while still in the spirit of the Rogers–Ramanujan and Capparelli identities, involve difference conditions that are *much* more complicated than anything that has been considered previously in the theory of partitions. It is no wonder that after Capparelli's discoveries for level 3, it took a quarter century to successfully perform the analogous feat for level 4.

4 Bailey pairs for levels 3 through 9 summarized

4.1 Level 3

$$\beta_{3m}^{(3,1)}(1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{\frac{3}{2}m^2-\frac{1}{2}m+3mr+\frac{3}{2}r^2-\frac{1}{2}r}(q^2;q^3)_{2m}}{(q)_{6m}(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2-\frac{1}{2}r}}{(q;q^3)_{2m}(q^2;q^3)_r(q^3;q^3)_{2m-r}(q^3;q^3)_r}$$

$$\beta_{3m+1}^{(3,1)}(1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{\frac{3}{2}m^2+\frac{5}{2}m+3mr+\frac{3}{2}r^2+\frac{5}{2}r+1}(q^2;q^3)_{2m}}{(q)_{6m+1}(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{5}{2}r}}{(q;q^3)_{2m+1}(q^2;q^3)_{r+1}(q^3;q^3)_{2m-r}(q^3;q^3)_r}$$

$$\beta_{3m-1} = -(1-q^{6m+1})(1-q^{6m+2})\beta_{3m+1} - (1-q^{6m}-q^{6m+1})\beta_{3m}$$

4.2 Level 4

$$\beta_{3m}^{(4,1)}(1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2+3mr+\frac{3}{2}r^2-\frac{1}{2}r}(-q^2;q^3)_r(q^2;q^3)_{2m}}{(q)_{6m}(-q^2;q^3)_m(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2-3mr+3r^2}}{(-q;q^3)_{m-r}(q;q^3)_{2m}(-q^2;q^3)_m(q^2;q^3)_r(q^3;q^3)_{2m-r}(q^3;q^3)_r}$$

$$\beta_{3m+1}^{(4,1)}(1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2+3m+3mr+\frac{3}{2}r^2+\frac{5}{2}r+1}(-q^2;q^3)_r(q^2;q^3)_{2m}}{(q)_{6m+1}(-q^2;q^3)_m(q^2;q^3)_{m+r+1}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2-3mr+3r^2+3r+1}}{(-q;q^3)_{m-r}(q;q^3)_{2m+1}(-q^2;q^3)_m(q^2;q^3)_{r+1}(q^3;q^3)_{2m-r}(q^3;q^3)_r}$$

$$q^{3m}\beta_{3m-1} = -(1-q^{6m+1})(1-q^{6m+2})(1+q^{3m})\beta_{3m+1} \\ - q^{3m}(1+q^{3m})(1-q^{3m}-q^{3m+1})\beta_{3m}$$

4.3 Level 5

$$\beta_{3m}^{(5,1)}(1,q) = \frac{q^{9m^2}}{(q)_{6m}},$$

$$\beta_{3m+1}^{(5,1)}(1,q) = \frac{q^{9m^2+6m+1}}{(q)_{6m+2}},$$

and

$$q^{6m-1}\beta_{3m-1} = (1-q^{6m})(1-q^{6m-1})\beta_{3m},$$

which, together, simplifies to Eq. (2.2).

4.4 Level 6

$$\beta_{3m}^{(6,1)}(1,q) = \frac{q^{3m}(-1;q)_{3m}}{(-1;q^3)_{3m}(q)_{6m}},$$

$$\beta_{3m+1}^{(6,1)}(1,q) = \frac{q^{3m+1}(-1;q^3)_{3m+1}}{(q)_{6m+2}(-1;q)_{3m+1}},$$

and

$$(q+q^{6m-1}-q^{3m})\beta_{3m-1} = (1-q^{6m})(1-q^{6m-1})\beta_{3m},$$

and thus (2.3) holds.

4.5 Level 7

$$\beta_{3m}^{(7,1)}(1,q) = \frac{q^{3m}}{(q)_{6m}},$$

$$\beta_{3m+1}^{(7,1)}(1,q) = \frac{q^{3m+1}}{(q)_{6m+2}},$$

and

$$q\beta_{3m-1} = (1-q^{6m})(1-q^{6m-1})\beta_{3m},$$

and thus (2.4) holds.

4.6 Level 8

$$\beta_{3m}^{(8,1)}(1,q) = \sum_{r=-m}^{m} \frac{q^{\frac{3}{2}r^2+\frac{1}{2}r+3m}(-q^2;q^3)_r(q^2;q^3)_{2m}}{(q)_{6m}(-q^2;q^3)_m(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{q^{3m^2+2m+3r^2+r-6mr}}{(-q;q^3)_{m-r}(q;q^3)_{2m}(-q^2;q^3)_m(q^2;q^3)_r(q^3;q^3)_{2m-r}(q^3;q^3)_r},$$

$$\beta_{3m+1}^{(8,1)}(1,q) = \sum_{r=-m}^{m} \frac{q^{\frac{3}{2}r^2+\frac{1}{2}r+3m+1}(-q^2;q^3)_r(q^2;q^3)_{2m}}{(q)_{6m+1}(-q^2;q^3)_m(q^2;q^3)_{m+r+1}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{q^{3m^2+2m+3r^2+r-6mr+1}}{(-q;q^3)_{m-r}(q;q^3)_{2m+1}(-q^2;q^3)_m(q^2;q^3)_{r+1}(q^3;q^3)_{2m-r}(q^3;q^3)_r},$$
$$q^2\beta_{3m-1} = -(1-q^{6m+1})(1-q^{6m+2})(1+q^{3m})\beta_{3m+1} + q(1+q+q^{3m}-q^{6m+1})\beta_{3m}.$$

4.7 Level 9

$$\beta_{3m}^{(9,1)}(1,q) = \sum_{r=-m}^{m} \frac{q^{3r^2+r+3m}(q^2;q^3)_{2m}}{(q)_{6m}(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{q^{3m^2+2m+3r^2+r-6mr}}{(q;q^3)_{2m}(q^2;q^3)_r(q^3;q^3)_{2m-r}(q^3;q^3)_r},$$

$$\beta^{(9,1)}_{3m+1}(1,q) = \sum_{r=-m}^{m} \frac{q^{3r^2+r+3m+1}(q^2;q^3)_{2m}}{(1-q^{3m+1})(1-q^{3m+2})(q)_{6m}(q^2;q^3)_{m+r}} \begin{bmatrix} 2m \\ m+r \end{bmatrix}_{q^3}$$
$$= \sum_{r=0}^{2m} \frac{q^{3m^2+2m+3r^2+r-6mr+1}}{(1-q^{3m+1})(1-q^{3m+2})(q;q^3)_{2m}(q^2;q^3)_r(q^3;q^3)_{2m-r}(q^3;q^3)_r},$$

$$q^3\beta_{3m-1} = -(1-q^{6m+1})(1-q^{6m+2})\beta_{3m+1} + q(1+q-q^{6m+1})\beta_{3m}.$$

5 Conclusion and Open Questions

It is the hope of the author that the results presented here will help to provide some insight into the structure of $A_2^{(2)}$ that can be exploited by vertex operator algebraists. Questions of course remain. For instance, as pointed out by Ole Warnaar during the question-and-answer period following my talk at the Alladi 60 conference, it is not at all clear how the series expressions in (3.6) and (3.7) enumerate the partition functions $c_1(n)$ and $d_1(n)$, respectively. It would be very nice indeed if this connection could be established. The referee echoed this same concern, pointing out that the double sums in (3.6) and (3.7) involve complicated, highly non-hypergeometric and non-positive terms, and this is only at level 3. Christian Krattenthaler pointed out that it was conceivable that there are *other* families of Bailey pairs that could give rise to identities with the same product sides.

Indeed, the approach employed here has limitations, and it may be that this is not the best way to approach this topic. The referee suggested that in the increasingly unwieldy multisum expressions, using only q-notation masks what is really going on, and that it is conceivable that the sum sides could be more meaningfully expressed in terms of some specialized orthogonal polynomials. While the above is admittedly true, the choice of the α_n employed here is motivated by classical work; in particular, the level 5 and 7 identities and the corresponding Bailey pairs coincide with the work of Slater [33, 34]. Further the availability of the Andrews–Baxter–Forrester transformation to express the β_n as a multisum for *any* level ℓ is an encouraging sign that this may point to a fruitful direction in the effort to better understand $A_2^{(2)}$ as a whole.

Acknowledgements The author thanks Jim Lepowsky for assistance with the exposition and the referee for carefully reading the manuscript and providing a number of useful suggestions for improvement.

References

1. K. Alladi, G.E. Andrews, B. Gordon, Refinements and generalizations of Capparelli's conjecture on partitions. J. Algebra **174**, 636–658 (1995)
2. A.K. Agarwal, G.E. Andrews, D.M. Bressoud, The Bailey lattice. J. Indian Math. Soc. (N.S.) **51**, 57–73 (1987)
3. G.E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli. Proc. Nat. Acad. Sci. USA **71**, 4082–4085 (1974)
4. G.E. Andrews, Multiple series Rogers–Ramanujan type identities. Pac. J. Math. **114**, 267–283 (1984)
5. G.E. Andrews, *Q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, vol. 66, CBMS Regional Conferences Series in Mathematics (American Mathematical Society, Providence, 1986)
6. G.E. Andrews, Schur's theorem, Capparelli's conjecture and q-trinomial coefficients, *The Rademacher legacy to mathematics (University Park, PA, 1992)*, vol. 166, Contemporary Mathematics (American Mathematical Society, Providence, 1994), pp. 141–154
7. G.E. Andrews, R.J. Baxter, P.J. Forrester, Eight-vertex SOS model and generalized Rogers–Ramanujan type identities. J. Stat. Phys. **35**, 193–266 (1984)
8. W.N. Bailey, Series of hypergeometric type which are infinite in both directions. Quart. J. Math. (Oxford) **7**, 105–115 (1936)
9. W.N. Bailey, Identities of the Rogers–Ramanujan type. Proc. Lond. Math. Soc. **2**(50), 1–10 (1949)
10. B.C. Berndt, *Number Theory in the Spirit of Ramanujan* (American Mathematical Society, Providence, 2006)
11. D.M. Bressoud, A generalization of the Rogers–Ramanujan identities for all moduli. J. Combin. Theory Ser. A **27**, 64–68 (1979)
12. D.M. Bressoud, Analytic and combinatorial generalizations of the Rogers–Ramanujan identities. Mem. Am. Math. Soc. **24**(227), 1–54 (1980)
13. K. Bringmann, K. Mahlburg, False theta functions and companions to Capparelli's identities. Adv. Math. **278**, 121–136 (2015)
14. S. Capparelli, Vertex operator relations for affine algebras and combinatorial identities. Ph.D. thesis, Rutgers University, 1988
15. S. Capparelli, A construction of the level 3 modules for the affine Lie algebra $A_2^{(2)}$ and a new combinatorial identity of the Rogers–Ramanujan type. Trans. Am. Math. Soc. **348**, 481–501 (1996)
16. R. Fricke, *Die Elliptischen Funktionen und ihre Anwendungen. Ers Teil* (Teubner, Leipzig, 1916)
17. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edn. (Cambridge University Press, Cambridge, 2004)
18. B. Gordon, A combinatorial generalization of the Rogers–Ramanujan identities. Am. J. Math. **83**, 393–399 (1961)
19. K.E. Iverson, *A Programming Language* (Wiley, New York, 1962)
20. V.G. Kac, *Infinite Dimensional Lie Algebras*, 3rd edn. (Cambridge University Press, Cambridge, 1990)
21. V.G. Kac, D.A. Kazhdan, J. Lepowsky, R.L. Wilson, Realization of the basic representations of the Euclidean Lie algebras. Adv. Math. **42**, 83–112 (1981)
22. D.E. Knuth, Two notes on notation. Am. Math. Monthly **99**, 403–422 (1992)
23. J. Lepowsky, S. Milne, Lie algebraic approaches to classical partition identities. Adv. Math. **29**, 15–59 (1978)
24. J. Lepowsky, R.L. Wilson, Construction of the affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. **62**, 43–53 (1978)
25. J. Lepowsky, R.L. Wilson, A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities. Adv. Math. **45**, 21–72 (1982)

26. J. Lepowsky, R.L. Wilson, The structure of standard modules I: universal algebras and the Rogers–Ramanujan identities. Invent. Math. **77**, 199–290 (1984)
27. J. Lepowsky, R.L. Wilson, The structure of standard modules II: the case $A_1^{(1)}$, principal gradation. Invent. Math. **79**, 417–442 (1985)
28. J. McLaughlin, A.V. Sills, Ramanujan–Slater type identities related to the moduli 18 and 24. J. Math. Anal. Appl. **344**, 765–777 (2008)
29. D. Nandi, Partition identities arising from the standard $A_2^{(2)}$ modules of level 4. Ph.D. thesis, Rutgers University, 2014
30. P. Paule, A. Riese, A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping, in Special Functions, q-Series and Related Topics. Fields Inst. Commun. **14**, 179–210 (1997)
31. A.V. Sills, On series expansions of Capparelli's infinite product. Adv. Appl. Math. **33**, 397–408 (2004)
32. A.V. Sills, Rademacher-type formulas for restricted partition and overpartition functions. Ramanujan J. **23**, 253–264 (2010)
33. L.J. Slater, A new proof of Rogers transformation of infinite series. Proc. Lond. Math. Soc. **2**(53), 460–475 (1951)
34. L.J. Slater, Further identities of the Rogers–Ramanujan type. Proc. Lond. Math. Soc. **2**(54), 147–167 (1952)

Generalized Mertens Sums

Gérald Tenenbaum

To Krishna Alladi, half-way,
as a token of a life-long friendship

Abstract Extending a classical estimate of Mertens for the sum of the reciprocals of the first primes, we provide an explicit remainder formula for products of an arbitrary, but fixed, number of primes.

Keywords Mertens' constant · Meissel–Mertens constant · Kronecker constant Hadamard-La Vallée-Poussin constant · Sums of reciprocals of primes

2010 Mathematics Subject Classification Primary 11N25 · 11N37 · Secondary 11N05

Set

$$S_k(x) := \sum_{p_1 \cdots p_k \leqslant x} \frac{1}{p_1 \cdots p_k} \qquad (x \geqslant 2),$$

where p_j denotes a prime number. It is a well-known result of Mertens that

$$S_1(x) = \log_2 x + c_1 + O\Big(\frac{1}{\log x}\Big) \qquad (x \geqslant 3),$$

G. Tenenbaum (✉)
Institut Élie Cartan, Université de Lorraine, BP 70239,
54506 Vandœuvre-lès-Nancy Cedex, France
e-mail: gerald.tenenbaum@univ-lorraine.fr

G. E. Andrews and F. Garvan (eds.), *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proceedings in Mathematics & Statistics 221, https://doi.org/10.1007/978-3-319-68376-8_40

with (see, e.g., [3], p. 18)

$$c_1 := \gamma - \sum_p \Big\{ \log\Big(\frac{1}{1-1/p}\Big) - \frac{1}{p}\Big\} \approx 0.261497. \qquad (1)$$

Here, and in the sequel, γ is Euler's constant, p stands for a prime number, and $\log_2$ denotes the twofold iterated logarithm. The number c_1 is called Mertens' constant, also known as the Meissel–Mertens, or the Kronecker, or the Hadamard-La Vallée-Poussin constant.

In [1], [2], Popa used elementary techniques to derive similar asymptotic formulae in the cases $k = 2$ and 3, with a main term equal to a polynomial of degree k in $\log_2 x$ and a remainder term $\ll (\log_2 x)^k / \log x$. In this note, we investigate the general case. We define classically $\varGamma$ as the Euler gamma function.

Theorem 1. *Let $k \geqslant 1$. We have*

$$S_k(x) = P_k\big(\log_2 x\big) + O\bigg(\frac{(\log_2 x)^k}{\log x}\bigg) \qquad (x \geqslant 3),$$

where $P_k(X) := \sum_{0\leqslant j\leqslant k} \lambda_{j,k} X^j$, and

$$\lambda_{j,k} := \sum_{0\leqslant m\leqslant k-j} \binom{k}{m,\, j,\, k-m-j} (c_1-\gamma)^{k-m-j} \Big(\frac{1}{\varGamma}\Big)^{(m)}(1) \qquad (0 \leqslant j \leqslant k).$$

Proof. Write $P(s) := \sum_p 1/p^s$, so that we have

$$P(s) = \log\zeta(s) - g(s), \qquad g(s) := \sum_{m\geqslant 2} \frac{1}{m} \sum_p \frac{1}{p^{ms}}$$

in any simply connected zero-free region of the zeta function where the series $g(s)$ converges. (Here, $\log\zeta(s)$ is the branch that is real for real $s > 1$.) Moreover, for $s+1$ in the same region, we have

$$P(s+1) = \log(1/s) + h(s),$$

with $h(s) = \log\{s\zeta(s+1)\} - g(s+1)$ and where $\log(1/s)$ is understood as the principal branch. The function $h(s)$ is clearly holomorphic in a disk around $s = 0$.

Now, for any $c > 0$, we have

$$S_k(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} P(s+1)^k x^s \frac{\mathrm{d}s}{s} \qquad \big(x \in \mathbb{R}^+ \smallsetminus \mathbb{N}\big). \qquad (2)$$

By following, *mutatis mutandis*, the argument of the Selberg–Delange method (see [3], chaps. II.5 & II.6), we readily obtain

$$S_k(x) = \frac{1}{2\pi i}\int_{\mathcal{H}} \Big\{\log\Big(\frac{1}{s}\Big) + h(0)\Big\}^k x^s \frac{\mathrm{d}s}{s} + O\Big(\frac{(\log_2 x)^k}{\log x}\Big) \qquad (x \geqslant 2),$$

where $\mathcal{H}$ is a Hankel contour around $\mathbb{R}^-$, positively oriented.

We also observe that, by (1), we have

$$h(0) = -\sum_p \Big\{\log\Big(\frac{1}{1-1/p}\Big) - \frac{1}{p}\Big\} = c_1 - \gamma.$$

It remains to compute

$$I_m(x) := \frac{1}{2\pi i}\int_{\mathcal{H}} \Big\{\log\frac{1}{s}\Big\}^m x^s \frac{\mathrm{d}s}{s} \qquad (m \geqslant 0).$$

To this end, we consider Hankel's formula (see, e.g., [3], th. II.0.17)

$$\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{x^s}{s^{1+z}}\mathrm{d}s = \frac{(\log x)^z}{\Gamma(z+1)} \qquad (z \in \mathbb{C})$$

and derive

$$I_m(x) = \sum_{0\leqslant j\leqslant m}\binom{m}{j}(\log_2 x)^j\Big(\frac{1}{\Gamma}\Big)^{(m-j)}(1).$$

Rearranging the terms, we arrive at the announced formula for $P_k(X)$. □

Specialization. Noting that

$$\begin{aligned}
&(1/\Gamma)'(1) = \gamma, \quad (1/\Gamma)''(1) = \gamma^2 - \tfrac{1}{6}\pi^2,\\
&(1/\Gamma)'''(1) = 2\zeta(3) - \tfrac{1}{2}\pi^2\gamma + \gamma^3,\\
&(1/\Gamma)^{(4)}(1) = \tfrac{1}{60}\pi^4 + 8\gamma\zeta(3) + \pi^2\gamma^2 + \gamma^4,
\end{aligned}$$

as may be deduced from classical formulae for the logarithmic derivative of the Euler function (see, e.g., [3], chap. II.0), we find

$$\begin{aligned}
&P_1(X) = X + c_1, \qquad P_2(X) = (X+c_1)^2 - \tfrac{1}{6}\pi^2,\\
&P_3(X) = (X+c_1)^3 - \tfrac{1}{2}\pi^2(X+c_1) + 2\zeta(3),\\
&P_4(X) = (X+c_1)^4 - \pi^2(X+c_1)^2 + 8\zeta(3)(X+c_1) + \tfrac{1}{60}\pi^4.
\end{aligned}$$

Remark. By retaining, in the integrand of (2), the first $N+1$ terms of the Taylor expansion of $h(s)$ at the origin, the above method readily yields, for arbitrary integer

$N \geqslant 0$, an asymptotic formula of the type

$$S_k(x) = \sum_{0 \leqslant j \leqslant N} \frac{P_{j,k}(\log_2 x)}{(\log x)^j} + O\left(\frac{(\log_2 x)^k}{(\log x)^{N+1}}\right)$$

where $P_{j,k}$ is an explicit polynomial of degree k.

Acknowledgements The author wishes to express warm thanks to Dumitru Popa for providing his works on the subject and for subsequent interesting conversations on the problem.

References

1. D. Popa, A double Mertens type evaluation. J. Math. Anal. Appl. **409**(2), 1159–1163 (2014)
2. D. Popa, A triple Mertens evaluation. J. Math. Anal. Appl. **444**(1), 464–474 (2016)
3. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, vol. 163, Graduate Studies in Mathematics (American Mathematical Society, Providence, 2015)