Chapter 6
Characteristics of Korean Students' Early Algebraic Thinking: A Generalized Arithmetic Perspective

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Abstract This chapter reports two studies that examined the early algebraic thinking of Korean students. Firstly, it deals with students' understanding of the equal sign, expressions, and equations as they progress through elementary school. Secondly, it investigates how third graders respond to diverse assessment items related to early algebraic thinking. The overall results show high percentages of correct answers. Whereas a majority of students showed a tendency to use computation, a detailed analysis of strategies used by students indicated some were capable of employing a structural approach. This chapter closes with discussions of the development of early algebraic thinking through the mathematics curriculum and the relationship between computational proficiency and algebraic thinking.

Keywords Early algebraic thinking · Equal sign · Expression · Equation
Variable

6.1 Introduction

Various studies in early algebra have been conducted on the nature, process, learning, and teaching of algebraic thinking (Kieran et al. 2016). Such studies demonstrate young students' algebraic thinking with the support of well-designed intervention programs promoting early algebraic thinking. This chapter reports two studies that examined the early algebraic thinking of Korean students. As early

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algebraic thinking has not yet been explicitly mentioned in the national mathematics curriculum of Korea, the results of the studies would be expected to reveal both the successes and the difficulties of algebraic thinking development under the current elementary mathematics curriculum. As such, this chapter is expected to contribute in two ways to the monograph: (a) As little has been known in international contexts of the algebraic thinking of Korean students, this chapter adds new and informative data to the field; and (b) by interpreting students' performance in relation to the current mathematics curriculum, this chapter urges intentional interest in improving the current mathematics curriculum to foster early algebraic thinking.

6.2 Background to the Study

6.2.1 A Generalized Arithmetic Perspective on Algebraic Thinking

Building on the identification of three strands of algebra by Kaput (2008), most of the early algebra research adopts a generalized arithmetic perspective with an emphasis on structures and relations arising in arithmetic, and a functional perspective with an emphasis on functions, co-variations, and changes. Note that the term *generalized arithmetic* has been used within the context of early algebra in a broad sense to include the properties and relations arising in arithmetical operations, without necessarily using letter-symbolic notations. As such, a generalized arithmetic perspective on content "not only includes number/quantity, operations, properties, equality, and related representations and diagrams, but also can include variables, expressions, and equations" (Kieran et al. 2016, p. 12).

As arithmetic has been regarded as the main context for early algebraic thinking (Carpenter et al. 2003; Kieran 2014), many studies have been conducted to probe children's understanding of the equal sign, expressions, and equations (Molina and Ambrose 2008; Stephens et al. 2013). It has been well documented that many students regard the equal sign as an operator to perform a calculation or as a signal to write down the answer that comes next (Kieran 1981).

The development of a *relational* understanding of the equal sign, which interprets the equal sign as a symbol to represent an equivalence relation between two expressions rather than as an operator, has been emphasized as fundamental to early algebraic thinking (Blanton et al. 2011; Knuth et al. 2006). Specifically, Matthews et al. (2012) developed a construct map for students' various conceptions of the equal sign in terms of four levels: (a) students at the *rigid operational level* are successful with typical equations having operations on the left side of the equal sign; (b) students at the *flexible operational level* are successful with atypical equations having operations on the right side of the equal sign or no operations; (c) students at the *basic relational level* are successful with equations having operations on both sides and accept a relational definition of the equal sign; and (d) students at the *comparative relational level* compare the expressions on the both sides of the equal sign and consistently generate a relational interpretation of the equal sign. The researchers designed a comprehensive set of tasks to assess students' understanding of the equal sign and ultimately of mathematical equality. The tasks were given to 224 students in Grades 2–6. Results showed that students had some difficulty when all operations were on the right side of the equal sign and experienced greater difficulty when operations were on both sides of the equal sign. Even students who successfully solved the items requiring a relational understanding of the equal sign tended to fail to generate a relational definition of the equal sign in words. An important finding of this study is that the children with an advanced understanding of the equal sign tended to solve difficult equations, which suggests a link between knowledge of the equal sign and algebraic thinking.

Byrd et al. (2015) focused on how a specific misconception of the equal sign may hinder students' learning of early algebra. The researchers differentiated the interpretations of the equal sign in three ways: (a) arithmetic-specific (e.g., "it means when you add something, you get the total"); (b) non-relational (e.g., "end of question", "a symbol to let you know the answer is next"); and (c) relational (e.g., "something is equivalent to something else"). Children who interpreted the equal sign in arithmetic-specific terms showed lower performance in solving early algebra items than those who defined the equal sign in a non-relational way but without using arithmetic-specific words. The negative effects of an arithmetic-specific view of the equal sign on early algebra learning occurred more for the fifth graders than for the third graders. This implies that an arithmetic-specific interpretation needs to be replaced by a relational or at least another non-relational view before students learn mathematical equivalence and its concomitant concepts in upper elementary grades.

An understanding of the different meanings of variable, coupled with a relational understanding of the equal sign, is fundamental in early algebra (Blanton et al. 2011; Usiskin 1988). A meaning of variable that is frequently used for lower graders at the elementary school level is that of a fixed but unknown number. However, it is not always easy for students to understand this prevalent meaning of variable, and seems to depend on the forms and structures in which it is used. For instance, according to Matthews et al. (2012), the items with letters as variables (e.g., $13 = n + 5$) proved more difficult than those with a similar format but without a letter variable (e.g., $8 = 6 + \square$). Note that students were able to easily solve equations with operations on the left side of the equal sign, but the use of variables rendered a dramatic increase in difficulty. In particular, equations with multiple instances of the unknowns on both sides of the equal sign such as $m + m + m = m + 12$ proved more difficult than the item asking for a relational definition of the equal sign. Students are expected to interpret algebraically the equations in which variables appear. Regarding the equation above, students need to realize that 'm' may be subtracted from each side, and that the simplified equation $m + m = 12$ or $2 \times m = 12$ may be divided by 2 on each side.

A variable can be used to express generalizations beyond specific numerical instances at the elementary school level (Blanton et al. 2011). For instance, while working with basic addition facts, young students can conjecture the commutative property of addition beyond particular number sentences. Young students are able to attend to general aspects by treating the specific numbers as *quasi-variables* (Fujii and Stephens 2008). Furthermore, such a property can be expressed through words (e.g., the sum of two numbers is the same regardless of the order of the numbers) or symbols (e.g., $\square + \triangle = \triangle + \square$ or $a + b = b + a$). The ability to use variables to represent a number in a generalized pattern is powerful for students to communicate their mathematical ideas succinctly (Brizuela et al. 2015).

A variable can also be employed to represent the relationship between two co-varying quantities (Blanton et al. 2011). However, children have difficulties in understanding or representing unknown quantities and tend to assign specific numerical values to solve a problem with unknown quantities (Carraher and Schliemann 2007). In conclusion, an understanding of the multiple meanings of variable and the ability to employ variables to express mathematical relationships or situations are significant in fostering students' algebraic thinking.

6.2.2 Development of Early Algebraic Thinking Through Instruction

Recent studies demonstrate that children are able to engage successfully with diverse aspects of essential algebraic ideas, and their ability can be enhanced through appropriate instruction with a well-developed curriculum. Recently some researchers have begun to compare students in an intervention program promoting early algebraic thinking with their counterparts being instructed with a typical arithmetic-based curriculum.

For instance, Britt and Irwin (2011) endeavored to promote students' algebraic thinking in arithmetic in their New Zealand Numeracy Project. Students with a new curriculum developed by the project were more successful than their counterparts with a conventional curriculum in solving all test items. These included not only simple compensation in addition but also complex equivalence with fractional values. The researchers emphasized that the newly developed curriculum led students to understand the underlying algebraic structure of arithmetic. By conducting a longitudinal study including students aged 12–14 the researchers demonstrated that sustained exposure to algebraic thinking in elementary school leads to more sophisticated generalization with the special symbols of algebra in intermediate school.

More recently, Blanton et al. (2015) demonstrated that, as early as grade 3, students are capable of developing algebraic thinking skills, when they are supported by appropriate tasks and teacher intervention that foster such thinking for a substantial period of time. The participants were third graders, 39 students from intervention groups and 67 counterparts from non-intervention groups. Whereas the former received specifically designed 19 one-hour early algebra lessons throughout the school year, the latter were taught by typical instruction. The study found that students in the intervention group statistically outperformed the non-intervention group in the post-test. Students in the intervention group were better in overcoming their misconceptions about the equal sign and noticing the underlying structure in equations, which helped them determine if the two sides of the equation had the same value without computation. More importantly, only students in the intervention group began to use an *unwind strategy* connected to inverse operations (e.g., to find the value of n in $3 \times n + 2 = 8$, students subtracted 2 from 8, and then divided 6 by 3 to yield 2), though they had not been formally taught this strategy.

Another noteworthy aspect of the Blanton et al. (2015) results was that as many as 74% of the students in the intervention group were able to model the problems that involved unknown quantities with variable notation, even though these students had assigned a specific numerical value to the unknown before participating in the program. The students were able to connect the variable notation across a series of problem situations and used it more frequently than words to represent the relationship between unknown quantities. This study showed that early and sustained exposure to algebraic thinking has a positive impact on students' use of variables.

6.3 Study 1: Students' Understanding of the Equal Sign, Expressions, and Equations

6.3.1 Overview

Given the importance of students' understanding of the equal sign as a basis for developing algebraic ideas, this section reports a study that examined such understanding (Kim et al. 2016). Assessment items from Matthews et al. (2012) were used. Because the items were developed on the basis of prior studies, this allowed for increased validity and reliability in examining students' comprehensive understanding of the equal sign, expressions, and equations. Students aged 7–12 years (i.e., from Grade 2 to Grade 6) were included to investigate whether their understanding of the equal sign, expressions, and equations improves as their grade levels increase following exposure to the current elementary mathematics curriculum.

6.3.2 Method

6.3.2.1 Participants

The participants for this study were from three elementary schools in two provinces. Overall academic abilities and socio-economic levels of students in the selected schools were considered as average in Korea. As this study investigated how students at different grades understand equivalence, we included students in Grades 2–6. Two classrooms for each grade in each of the selected schools were randomly chosen. A total of 695 students were included for the study: 135 second graders, 140 third graders, 140 fourth graders, 144 fifth graders, and 136 sixth graders.

6.3.2.2 Assessment Items

As already mentioned, assessment items from Matthews et al. (2012, pp. 345–347) were used. This instrument includes 27 items of three types: (a) *equation structure items,* such as deciding whether a given number sentence is true or false (e.g., 31 + 16 = 16 + 31 True/False/Don't Know), (b) *equal sign items,* such as asking students to write the meaning of the equal sign, and (c) *equation solving items,* such as finding the unknown number in a given equation (e.g., □ + 2 = 6 + 4). Among equation structure items, specifically *advanced relational reasoning items* are included, such as asking students to solve a given problem without direct computation (e.g., 17 + 12 = 29 is true. Is 17 + 12 + 8 = 29 + 8 true or false? How do you know?). There are also nine items that ask students to describe their answer or explain their solution process (e.g., What does the equal sign (=) mean? Can it mean anything else?). Some minor revisions of the original items were necessary. Specifically, for second graders, numbers less than 30 were used and letter variables were replaced by non-letter variables (e.g., "10 = △ + 6" in place of "10 = z + 6").

6.3.2.3 Data Collection and Analysis

The students in this study solved the assessment items in 40 min and all students' written responses were analyzed. Each item was scored either 0 for incorrect answer or 1 for correct answer. For the nine explanation items, three sub-categories were further used: (a) relational thinking, (b) computation, and (c) incomplete or incorrect explanation. "Relational thinking" here indicates that students explained their solution method by using the structure of the given equation or expression. To emphasize, we employed these sub-categories even for incorrect answers, because our purpose was to investigate the nature of students' understanding. Examples of student responses are included below in the results section.

After responses were coded according to correctness for all items and strategy use for the explanation items, they were analyzed quantitatively using the SPSS 12.0 program. Specifically, ANOVA and post hoc tests[1] were conducted to examine any significant differences for grades.

[1]An ANOVA test tells you whether you have an overall difference between your groups, but it does not tell you which specific groups differed—post hoc tests do.

6.3.3 Main Results

6.3.3.1 Students' Overall Performance

Figure 6.1 shows the results of students' performance for all items. The horizontal axis refers to items and the vertical axis refers to the percentages of correct answers from all grades. Note that Fig. 6.1 displays the percentages only for correctness, regardless of the strategies that students used. The results show that students were quite successful in almost all items (see the following sections for a detailed analysis of selected items). Using ANOVA, a significant difference for grade was found, $F(4, 688) = 125.838$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades. This implies that students' understanding of the equal sign, expressions, and equations improves as their grade levels increase until the fifth grade.

6.3.3.2 Students' Understanding of Equation Structure

Items from 1a to 2b ask students to decide if the given equation is true, whereas Items 3–8 ask for relational thinking. The percentages of correct answers for the latter were lower than those for the former. Generally speaking, the percentages of correct answers for equation structure items increased according to grade levels. Using ANOVA, a significant difference for grade was found, $F(4, 689) = 125.688$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades.

An analysis of the explanation that students wrote for Items 3–8 indicates that less than 35% of the students got the correct answer based on relational thinking. For instance, regarding Item 3 in Table 6.1, only 33.1% of the students justified the correct answer by relational thinking. For instance, some students wrote: "68 is larger than 67 by 1 and 85 is smaller than 86 by 1. So 67 + 86 is the same as 68 + 85." Others justified as follows: "67 + 86 is the same as 68 + 85, because 67 + 1 + 86 − 1 = 68 + 85. Here adding 1 and subtracting 1 makes the answer the

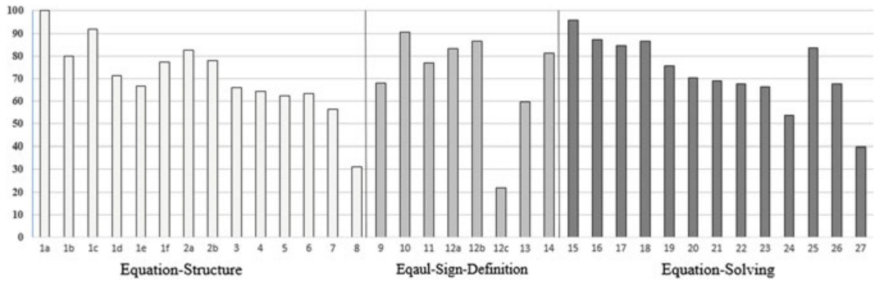

Fig. 6.1 Students' overall performance with respect to correctness

Table 6.1 Item 3 and students' responses

Item 3 *Without adding* 67 + 86, can you tell if the number sentence below is true or false? 67 + 86 = 68 + 85. How do you know? (Note: 7 + 4 = 8 + 3 for Grade 2)							
Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	88 (67.2)	58 (41.4)	34 (24.5)	21 (14.6)	25 (18.4)	226 (32.8)
	Relational	0 (0)	2 (1.4)	2 (1.4)	0 (0)	0 (0)	4 (0.6)
	Computational	1 (0.8)	0 (0)	3 (2.2)	0 (0)	0 (0)	4 (0.6)
Correct	Incomplete	13 (9.9)	19 (13.6)	14 (10.1)	24 (17.4)	24 (17.6)	94 (13.6)
	Relational	1 (0.8)	44 (31.4)	55 (39.6)	65 (45.1)	64 (47.1)	228 (33.1)
	Computational	28 (21.4)	17 (12.1)	31 (22.3)	34 (23.6)	23 (16.9)	133 (19.3)

same." Still others wrote: "67 + 86 = 68 − 1 + 85 + 1" and drew circles over '−1' and '+1.' The thinking of these students can be described, more formally, as using the associativity and commutativity properties of addition, $68 + 85 = (67 + 1) + (86 - 1) = 67 + 86 + (1 - 1) = 67 + 86$. However, even for upper graders, the percentage of those using relational thinking based on the algebraic structure of arithmetic was less than 50%. About 20% of the students in almost all grades used a computational strategy, even though the item explicitly states: "without adding 67 + 86." Some students got the correct answer, but gave incomplete or incorrect explanations, including "67 + 86 = 68 + 85 is true because it is the same" or "if you add each number, then $6 + 7 + 8 + 6 = 27$ and $6 + 8 + 8 + 5 = 27$, the number is the same." Regarding the incorrect answers, the most common response type was that of incomplete or incorrect explanations, such as "67 + 86 = 68 + 85 is false because the addends are different respectively." A rare example of an incorrect answer using relational thinking included, "67 < 68 and 86 > 85, so 67 + 86 = 68 + 85 is false."

Item 8 in Table 6.2 was the most difficult item among the equation structure items because it includes multiplication and the unknown number □. Note also that the new multiplier 8 is multiplied from the left in both sides of the first equation. Whereas the majority of students employed a computational strategy or gave an incomplete explanation of the strategy used to get the answer (e.g., "You know it if you just see it"), only 4.2% of the students were able to use relational thinking in their solution process (e.g., "There is the same 8 on both sides of $8 \times 2 \times \square = 8 \times 58$, so you simply divide by 8 only to get $2 \times \square = 58$").

6.3.3.3 Students' Understanding of the Equal Sign

Items 9 through 14 deal with the meaning of the equal sign. The percentages of correct answers to these items were high except for Item 12c. The percentages of correct answers for the equal sign items highly correlated with grade level and, in fact, a significant difference for grade was found using ANOVA, $F(4, 690) = 42.013$,

Table 6.2 Item 8 and students' responses

Item 8
Is the number that goes in the box the same number in the following two number sentences?
$2 \times \square = 58$, $8 \times 2 \times \square = 8 \times 58$ (Note: $2 + \square = 10$, $5 + 2 + \square = 5 + 10$ for Grade 2)
How do you know?

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	98 (74.8)	129 (92.1)	98 (70.5)	80 (55.6)	58 (42.6)	463 (67.2)
	Relational	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
	Computational	0 (0)	0 (0)	10 (7.2)	0 (0)	0 (0)	10 (1.5)
Correct	Incomplete	19 (14.5)	10 (7.1)	7 (5.0)	33 (22.9)	15 (11.0)	84 (12.2)
	Relational	3 (2.3)	0 (0)	1 (0.7)	4 (2.8)	21 (15.4)	29 (4.2)
	Computational	11 (8.4)	1 (0.7)	23 (16.5)	27 (18.8)	42 (30.9)	103 (14.9)

Table 6.3 Item 12 and students' responses

Item 12
Is this a good definition of the equal sign? Circle *True* or *False*
a. The equal sign means the same as. True False
b. The equal sign means add. True False
c. The equal sign means the answer to the problem. True False

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Correct	Item 12a	70 (53.4)	113 (80.7)	121 (87.1)	139 (96.5)	132 (97.1)	574 (83.3)
	Item 12b	97 (74.0)	128 (91.4)	124 (89.2)	114 (79.2)	133 (97.8)	595 (86.4)
	Item 12c	30 (22.9)	31 (22.1)	30 (21.6)	39 (27.1)	21 (15.4)	151 (21.9)

$p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades.

The most challenging Item 12 (see Table 6.3) asks students to determine whether the given definition of the equal sign is *true* or *false*. The percentages of correct answers for Items 12a and 12b were quite high from Grade 3 onward. However, only 21.9% of the students answered correctly for Item 12c. In other words, the majority of students have an understanding that the equal sign means "the same as", and that the equal sign does not mean "add". However, at the same time, many students thought that the equal sign means "the answer to the problem". Maybe students thought that it means to 'do' instead of to 'add,' as this might also mean subtract, multiply, or divide, but they seemed to agree with part c. More importantly, this non-relational thinking regarding the equal sign was persistent across all grade levels.

6.3.3.4 Students' Equation-Solving

Items 15–27 ask students to find the unknown number in the given equation. The percentages of correct answers for equation solving items were high except for Items 24 and 27. Again, the percentages of correct answers increased as grade levels rose. Once again, a significant difference for grade was found using ANOVA, $F(4, 689) = 95.288$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades. This may be related to the elementary mathematics curriculum by which whole number operations are dealt with in multiple contexts up to the fourth grade but in the fifth and sixth grades mainly fraction and decimal operations are dealt with.

Whereas items 15–24 use the symbol variable □, the last three items use letter variables. Simply using a letter variable did not increase item difficulty, as shown in the results for Items 17 and 25 (see Table 6.4). However, multiple instances of a letter as a variable made it difficult for students, specifically second and third graders, to solve the given problem, as shown in the results for Item 26. In fact, the Korean elementary mathematics textbooks provide little opportunity for students to experience multiple occurrences of the variable.

Item 24 in Table 6.5 examines whether students use their understanding of the equal sign and equation structure to solve a given equation with relatively large numbers. Note that we changed the original equation, 43 + □ = 48 + 76, in Matthews et al. (2012) into 47 + □ = 48 + 76. This was done in order to see whether students were able to employ relational thinking for the close numbers 47 and 48, and to interpret the result for Item 24 in relation to the result for Item 3 (67 + 86 = 68 + 85) among the equation structure items. The results for Item 24 show that only 35.6% of the students solved the equation by relational thinking. Some students explicitly wrote their reasoning process (e.g., "The equal sign (=) here means the same as, therefore, the expressions are to be the same. Because 47 is less than 48 by 1, □ should be larger than 76 by 1, so the answer is 77."). Others used a computational strategy with an incorrect use of the equal sign (e.g., "47 + □ = 124 − 47 = 77, 48 + 76 = 124") or wrote an incomplete or incorrect

Table 6.4 Selected equation-solving items and students' responses

Item 17. Find the number that goes in the box. 8 = 6 + □
Item 25. Find the value of z. In other words, what value of z will make the following number sentence true? Circle your answer.
$10 = z + 6$ (Note: 10 = △ + 6 for Grade 2)
Item 26. Find the value of n. $n + n + n + 2 = 17$
(Note: ☆ + ☆ + ☆ + 2 = 17 for Grade 2)

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Correct	Item 17	79 (60.3)	116 (82.9)	118 (84.9)	139 (96.5)	130 (95.6)	582 (84.5)
	Item 25	80 (61.1)	111 (79.3)	120 (86.3)	137 (95.1)	128 (94.1)	576 (83.6)
	Item 26	40 (30.5)	79 (56.4)	111 (79.9)	119 (82.6)	118 (86.8)	467 (67.8)

Table 6.5 Item 24 and students' responses

Directions: *Find the number that goes in each box. You can try to find a shortcut so you don't have to do all the adding. Show your work and write your answer in the box.*
Item 24. 47 + ☐ = 48 + 76. How do you know? (Note: 7 + ☐ = 8 + 6 for Grade 2)

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	110 (84.0)	69 (49.3)	56 (40.3)	41 (28.5)	29 (21.3)	305 (44.3)
	Relational	0 (0)	1 (0.7)	4 (2.9)	1 (0.7)	2 (1.5)	8 (1.2)
	Computational	0 (0)	1 (0.7)	5 (3.6)	1 (0.7)	0 (0)	7 (1.0)
Correct	Incomplete	6 (4.6)	20 (14.3)	8 (5.8)	23 (16.0)	17 (12.5)	73 (10.6)
	Relational	5 (3.8)	42 (30.0)	56 (40.3)	68 (47.2)	74 (54.4)	245 (35.6)
	Computational	10 (7.6)	7 (5.0)	10 (7.2)	10 (6.9)	14 (10.3)	51 (7.4)

explanation (e.g., "It seems that I can't explain it without adding the numbers."). We also found that the results for Item 24 were quite similar to the results for Item 3 (see Table 6.1), implying that students' understanding of the equation structure seems to influence their equation-solving abilities.

6.4 Study 2: Diverse Aspects of Early Algebraic Thinking in Third Graders

6.4.1 Overview

A multitude of studies have documented that elementary students can successfully develop essential algebraic ideas. This section reviews a study that examined third graders' early algebraic thinking (Pang and Choi 2016). Early algebra has not been explicitly addressed in the national elementary mathematics curriculum in Korea. We wondered how students not exposed to a specific intervention program or curriculum fostering such thinking processes would respond to the diverse assessment items related to early algebraic thinking.

In order to better understand our students' performance in the international context, we adapted Blanton et al. (2015)'s study for at least three reasons. Firstly, because Blanton et al. (2015) document the data from both the nonintervention group and the early algebra intervention group, we can locate our students' performance against both groups. Secondly, third graders participated in the study of Blanton et al. (2015). It is reasonable to examine the algebraic thinking of third-graders in our study, considering that it would be useful to examine the capability of these lower grade students with respect to engaging in early algebraic ideas. At the same time, these students have been sufficiently exposed to the elementary mathematics curriculum so as to enable the researchers to interpret their performance in relation to the curricular experience. Thirdly, the test items in Blanton et al. (2015) are sufficiently comprehensive in that they include *big ideas* in early algebra such as equivalence and equations, generalized arithmetic, functional thinking, and variables. The use of such items was expected to reveal multiple aspects of students' algebraic thinking that were developed while using the regular mathematics curriculum.

6.4.2 Methods

6.4.2.1 Participants

The third-grade students in this study were from seven elementary schools in four provinces. Overall academic abilities and socio-economic levels of students in the selected schools were considered as average in Korea. A written assessment was distributed to a total of 220 students. Unfortunately, 23 students did not answer the items asking for their explanation or justification. They were mostly in the one classroom in which the teacher did not emphasize the need to do so. After excluding the data from these 23 students, the data from the remaining 197 students were analyzed.

6.4.2.2 Assessment Items

The assessment items were from Blanton et al. (2015, pp. 83–86). One item among the original 11 items, dealing with proportional reasoning, was not included because it is not appropriate for Korean third grade students. Careful translation of the 10 items was conducted and a pilot test was administered in one third-grade classroom. A few revisions were necessary. Item 4, written in sentences, was changed into the form of a dialogue so as to make it more understandable, while keeping the meaning of the original item (see Sect. 6.4.3.3 for the detailed revision). A critical issue involving variable notation was raised. In Blanton et al. (2015), concepts associated with variables were integrated into the instruction and students were expected to be able to use letter variables to represent an unknown quantity in different problem contexts. However, in Korea, variable notation without letter symbols is used in the textbooks or workbooks for lower graders: For instance, (a) a variable as a fixed unknown number: as in $5 + \square = 7$ or $9 - \square = 5$ and (b) a variable as a tool for generalization: as in $\heartsuit + 0 = \heartsuit$, $0 + \heartsuit = \heartsuit$. Given this, in keeping with the original assessment items for comparison purposes, we developed supplementary items with the use of non-letter variable notation (see Sect. 6.4.3.5 for an example).

6.4.2.3 Data Collection and Analysis

The students in this study solved the assessment items in 40 min. A total of 197 students' written responses were analyzed for correctness for the items that have only one correct answer. The strategies employed by students were initially analyzed according to the coding scheme in Blanton et al. (2015). Whenever different strategies or responses emerged, new codes were assigned such as "correct answer with incomplete explanation" and "incorrect answer with an error to be noted". Criteria for determining correctness or strategy use are mentioned with the examples in the following results section.

In addition, unstructured interviews with nine students were conducted to investigate their reasoning processes in detail. For instance, the interviewees included students who answered correctly without explanation, students who used different strategies for similar assessment items, or students who used an apparently new strategy. The interviews were audiotaped and transcribed, which served to identify diverse aspects of those students' algebraic thinking.

6.4.3 Main Results

6.4.3.1 Comparison of Students' Overall Performance

Table 6.6 shows students' overall performance with the items rated in terms of percentage correct. Notice that four items (i.e., Items 3a, 3b, 4a, and 8b) were not included here because they were analyzed only for strategy use as in Blanton et al. (2015, p. 87). A cautionary note in reading Table 6.6 is that our main purpose was to better understand our students' overall performance in international contexts. Due to limited space, the results for some items are included in subsequent sections.

For most items, the Korean students performed as well as, or only slightly worse than, students in the Blanton et al. intervention group, and much higher than students in their non-intervention group. These items included figuring out a missing value in the equation (e.g., $7 + 3 = \square + 4$), evaluating an equivalence relationship (e.g., $12 + 3 = 15 + 4$ True/False), generalizing the commutative property of addition, and selecting a generalized algebraic expression on the basis of particular examples (e.g., $a - a = 0$ from $8 - 8 = 0$ to $12 - 12 = 0$). Regarding Items 5 and 9, the Korean students performed far better than students in the Blanton et al. intervention group, although they experienced substantial difficulties in Items 7 and 10. What follows is a detailed analysis of the strategies students employed on selected items.

Table 6.6 Comparison of students' overall performance by percentage of correct response

Item		Korean (N = 197)	Blanton et al. (2015)'s posttest correct	
			Non-intervention (N = 67)	Intervention (N = 39)
1	a	69.0	3.2	84.2
	b	70.0	1.6	84.2
2	a	74.1	31.7	86.8
	b	73.6	9.5	84.2
	c	75.1	14.3	89.5
4b		66.3	34.9	73.7
5		73.6	4.8	36.8
6		84.2	57.1	89.5
7	a	16.2	12.7	73.7
	b	15.2	7.9	63.2
	c	4.5	3.2	39.5
8a		84.7	49.2	89.5
9		85.7	28.6	52.6
10	a	76.1	52.4	86.8
	b	29.9	41.3	78.9
	c	47.7	7.9	31.6
	d	4.5	0.0	15.8
	e	47.2	41.3	55.3

6.4.3.2 Students' Understanding of Equivalence

The percentages of correct responses for the items addressing students' understanding of the equal sign, expressions, and equations were high. But when we analyzed their solution strategies, we found that the students solved the items by computation (coded as *computational* strategy) more often than noticing the underlying structure in the equation without computing (coded as *structural* strategy). For instance, whereas a majority of students responded correctly to Item 2b, 64.9% of them used a computational strategy and only 4.5% of the students used a structural strategy (see Table 6.7). The tendency of employing a computational strategy was lower for Item 2c, but still served as a main foundation for determining if the two sides of the equation had the same value. About 20% of the students consistently added the numbers to the left of the equal sign to get the solution (coded as *operational* strategy).

We wondered whether students were unable to use any structural strategy, even when they had a relational understanding of the equal sign. We interviewed some students who answered for Item 3 that the meaning of the symbol "=" in the number sentence 3 + 4 = 7 is "same as" or "equal," but who used only a computational strategy to find a missing value or to determine equivalence. As reflected in Episode 1, the student initially approached the item computationally, but was able to use a structural strategy when asked to solve it without computation.

Table 6.7 Item 2 and students' strategy use

Item 2 Circle True or False and explain your choice. b. 57 + 22 = 58 + 21. True False How do you know? c. 39 + 121 = 121 + 39. True False How do you know?		
Strategy	Example or explanation	Frequency (%)
Structural	Item 2b: True, because 58 is one more than 57 and 21 is one less than 22.	9 (4.5*)
	Item 2c: True, because 121 + 39 is 39 + 121 in reverse.	66 (33.5)
Computational	Item 2b: True, because 57 + 22 = 79 and 58 + 21 = 79.	128 (64.9)
	Item 2c: True, because 39 + 121 = 160 and 121 + 39 = 160.	71 (36.0)
Operational	Item 2b: False, because 57 + 22 = 79, not 58.	45 (22.8)
	Item 2c: False, because 39 + 121 is not 121.	41 (20.8)

*The sum of percentages does not reach 100 because the table includes main strategies

Episode 1 Emergent use of structural strategy versus tendency to compute.

Interviewer (I): (points to the student's written response *12 + 3 = 15, 15 + 4 = 19* for Item 2a: "12 + 3 = 15 + 4 True False How do you know?") *Do you necessarily need to compute 15 and 19 to solve this item?*

Student (S): *Probably.*

I: *Can it be done without computation?*

S: *I can solve it by comparison.*

I: *Okay. Why don't you compare?*

S: *If you compare 3 and 4, 3 is less. If you compare 12 and 15, 12 is less. So this part* (pointing to 12 + 3) is less.

I: *Okay. How about this* (pointing to Item 2b)?

S: *57 plus 22 is 79 and 58 plus 21 is also 79.*

I: *Without computation? How would you explain by comparison as you just did?*

S: *It becomes the same if I give 1 from 58, so I know it without calculation.*

I: *Right, very good! How about this* (pointing to Item 2c)? *Do you have to add these two numbers* (pointing to 121 and 39) *to find out 160?*

S: *I know right away because it just switches the positions of the numbers.*

In the episode above, the student went back to use a computational strategy for Item 2b, even after she had just solved Item 2a without computation. When asked to explain by comparison, however, the student was able to use a structural strategy (i.e., "give 1 from 58"). She then continued to use a structural strategy for Item 2c by justifying with a fundamental property of number and operations.

6.4.3.3 Students' Understanding of the Commutative Property of Addition

Item 2c in Table 6.7 examines students' understanding of equivalence, but it also reflects the commutative property of addition. As described, students' tendency of using a computational strategy for Item 2b was decreased for Item 2c. More significantly, only 3% of the students continued to use a computational strategy for Item 4a, whereas almost half of them used a structural strategy (see Table 6.8). This may be related to the slight but important difference between Item 2c and Item 4. Note that both items involve the commutative property of addition. However, Item 2c asks students to evaluate whether the given equation is true or false, and to justify their reasoning. In contrast, Item 4 does not include the equal sign and, more importantly, encourages students to reason without computation.

For Item 4b, 66.3% of the students answered that Yuna's thinking will work for all numbers. In fact, the majority of the students justified it by describing the commutative property of addition in words. It is not surprising because they had already learned it through their mathematics textbook. What is interesting here is that about 10% of the students justified their answer by writing another example (e.g., 1 + 2 = 2 + 1). We wondered whether the students were capable of thinking beyond particular instances to generalize the fundamental property. Episode 2 is an interview with a student who wrote a single instance for Item 4b.

Table 6.8 Item 4 and students' strategy use

<table>
<tr><td>Item 4 (original)
Marcy's teacher asks her to figure out "23 + 15." She adds the two numbers and gets 38. The teacher then asks her to figure out "15 + 23." Marcy already knows the answer.</td><td colspan="2">Item 4 (revised)
The following is the dialogue between Yuna and her teacher.
Teacher: Yuna, what is 23 + 15?
Yuna: If I add 23 and 15, I get 38.
Teacher: Then, what is 15 + 23?
Yuna: I already know it without computation!</td></tr>
<tr><td colspan="3">a. How does Yuna know?
b. Do you think this will work for all numbers? If so, how do you know?</td></tr>
<tr><td>Strategy</td><td>Example</td><td>Frequency (%)</td></tr>
<tr><td>Structural</td><td>Item 4a: It is the same as 23 + 15, because only the numbers are switched.</td><td>96 (48.7)</td></tr>
<tr><td>Computational</td><td>Item 4a: 23 + 15 = 38 and 15 + 23 = 38.</td><td>6 (3.0)</td></tr>
</table>

Episode 2 Generalization beyond particular instances regarding the commutative property of addition.

I: (reads Item 4a) *How did Yuna know?*
S: *Because only the positions of the numbers were switched.*
I: *Okay, you wrote 21 + 22. What does it mean?*
S: *21 + 22 is the same as 22 + 21.*
I: *Then, is it okay to use 3 and 4 instead of 21 and 22?*
S: *Yes!*
I: *Why?* (After no response from the student, interviewer continued) *Then, let's say that the first number is* $\square$ *and the second number is* $\triangle$. *Can we say* $\square + \triangle = \triangle + \square$*?*
S: *Yes.*
I: *Why do you think so?*
S: *Because it just switches the position of the figures.*

As reflected in Episode 2, the student answered with the single instance of 21 + 22. But he knew that the numbers could be changed to other numbers, in fact, any numbers. In other words, he expressed the generalization in terms of specific numbers. Given the generalized representation of the symbols $\square$ and $\triangle$, the student was able to justify his thinking in words.

6.4.3.4 Students' Understanding of Equations

Item 9 examines how students solve a simple linear equation and justify their answer (see Table 6.9). Note that we changed the original equation $3 \times n + 2 = 8$ to $3 \times \square + 2 = 8$, because letter variables are not taught in Korea until Grade 6.

A noticeable result was that the percentage of correct answers for Korean students (i.e., 85.7%) was the highest for Item 9, which was much higher than for that of the Blanton et al. intervention group (i.e., 52.6%), as was seen in Table 6.6. To emphasize, mathematics textbooks in Korea do not deal with equations with two operations until Grade 3. More interestingly, most students used a different strategy (coded as *intuitive use of number facts*) than either the "Guess and Test" or "Unwind" strategies. According to Blanton et al. (2015, p. 57), the use of the Guess and Test strategy means that the student works through the equation in a forward manner, substitutes value(s) in for the variable and computes to see if the value is correct, and the Unwind strategy refers to the student working backward through constraints in the equation, inverting operations. Our students worked through the equation in a forward manner and seemed to notice the underlying structure of the given equation as a whole. Instead of substituting values for the variable (e.g., 3 and then 2, or arbitrarily initially choosing 2), our students started with the fact that (a certain number) +2 is 8, so the number must be 6. Then the question becomes easier because the original item turns into $3 \times \square = 6$. A noteworthy aspect is that students seem to be capable of seeing $3 \times \square$ as an object, which makes it easier for them to notice the structure of the equation. In this process, students could have

Table 6.9 Item 9 and students' strategy use

Item 9 Find the value of □ in the following equation. How did you get your answer? $3 \times \square + 2 = 8$		
Strategy	Example	Frequency (%)
Guess and test	$3 \times 3 + 2 = 11$, $3 \times 2 + 2 = 8$	5 (2.5)
Unwind	$8 - 2 = 6$, $6 \div 3 = 2$	2 (1.0)
Intuitive use of number facts	(A certain number) + 2 is 8, so the number must be 6. $3 \times \square = 6$, $3 \times 2 = 6$. $3 \times 2 = 6$, $6 + 2 = 8$	150 (76.1)

subtracted 2 from 8 to get 6 and then divided the 6 by 3 to get 2 (i.e., the unwind strategy). However, our students did not invert the operations, but employed familiar number facts in an intuitive manner (i.e., $3 \times 2 = 6$, $6 + 2 = 8$). In order to understand better students' thinking processes, we interviewed some of them who simply wrote "$3 \times 2 = 6$, $6 + 2 = 8$" for Item 9. Note that we gave them extra simple equations on the spot to trace their thinking.

Episode 3.1 Using 'equation sense' based on number facts.

I: *Could you explain how you solved this* (pointing to Item 9)?
S1: *3 times 2 is 6 and 6 plus 2 is 8.*
I: *Aha, how about this problem* (writing $7 \times \square + 3 = 24$)?
S1: *Some number less than 24, the product of 7 and a certain number should be less than 24 but at the same time close enough to 24. 7, 3, 21 and plus 3 is 24.*
I: *Okay, I will give you another problem with larger numbers.* (Writes $12 \times \square + 1 = 49$.)
S1: *12 times 4 is 48, here* (pointing to $12 \times \square$) *is 48 and plus 1 is 49.*
I: *Why did you think this* (pointing to $12 \times \square$) *is 48?*
S1: *Because 49 minus 1 is 48, then I thought 12 times what number makes 48.*

Episode 3.2 Noticing the structure in a linear equation

I: *How did you find 2 for Item 9* (pointing to $3 \times \underline{2} + 2 = 8$)?
S2: *I left 'plus 2' alone. 3 times a certain number, that product, that number plus 2 is to be 8. As 3 is there, I thought of the '3 times table'. 3, 2, 6 and 6 plus 2 is 8.*
I: *Okay, why don't you solve this problem* (writing $7 \times \square + 3 = 24$)?
S2: *3.*
I: *How did you know so quickly?*
S2: *This is the same. You just need to see a certain number plus 3. 7, 3, 21 and then plus 3 is 24.*
I: *Now, I will give you larger numbers.* (Writes $13 \times \square + 5 = 31$)
S2: *2!*
I: *Wow! How did you know the answer so quickly?*

S2: *It is the same, too. It says a certain number plus 5 is 31. 31 minus 5 is 26. So 13 times 2 is 26.*

As reflected in the episodes above, it was clear that the students were capable of noticing the structure of a given equation as a whole. On the basis of the understanding of the equal sign and expressions, S1 knew that the left side of the equation (i.e., $7 \times \square + 3$) should be 24 and, as there is '+3', "the product of 7 and a certain number should be less than 24 but at the same time close enough to 24." For another equation with larger numbers, S1 partially used an *unwind* strategy by inverting '+1' as '−1.' However, she still solved the equation in a forward manner; not by $48 \div 12 = 4$ but by $12 \times 4 = 48$, even though the number 12 is not in the common times table (the 2–9 times table). It was also obvious that S2 in Episode 3.2 was able to see $3 \times \square$ as an object, when he called it "that product, that number." His interpretation of numbers and expressions as objects was consistent. He explained both $7 \times \square$ in the equation $7 \times \square + 3 = 24$ and $13 \times \square$ in $13 \times \square + 5 = 31$ as 'a certain number.' S2 also partially used an *unwind* strategy when converting the addition of 5 to the subtraction of 5 (from 31) in $13 \times \square + 5 = 31$.

Given that both S1 and S2 partially used an *unwind* strategy, we might have coded their responses as the "Unwind" strategy. However, we chose not to do so in order to emphasize the noticing of the equation structure as a whole, rather than focusing on employing inverse operations step by step. Also worth noting is that students solved the equations very quickly in a forward manner through familiar number facts or "equation sense." In this respect, students might not use an inverse operation when they initially solve the equation, but merely mention it later to justify their answer.

6.4.3.5 Students' Understanding of Algebraic Expressions with Variable Notation

Item 7 examines students' understanding of algebraic expressions and, in particular, how they represent unknown quantities with variables (see Table 6.10). Note the original item was slightly altered because pennies are not used in daily life in Korea. 'Coins' were addressed instead of the specific coin penny, but kept the critical aspect of the item, that is to say, an indeterminate amount of coins.

Item 7 was the most challenging problem for our students. The percentage of correct answer was the lowest among the 10 assessment items (see Table 6.6) and, in fact, the percentage of "no response" answers was about 38%. The most frequent strategy students used was to assign a specific numerical value to the unknown quantity, although the item specifically says that the quantity is not known. Slightly less than 30% of the students assigned arbitrary numerical values to the unknowns of Items 7a, 7b, and 7c. In contrast, about 20% of the students assigned specific numerical values that were related to one another (see the strategy *value-related* in Table 6.10). Students used this strategy in a consistent way for Item 7a, 7b, and 7c.

Table 6.10 Item 7 and students' strategy use

Item 7 Hajun and Yejun have coins in their piggy banks, and the kinds of coins are the same. They know that their piggy banks each contain the same number of coins, but they don't know how many. Yejun also has 8 coins in his hand. a. How would you represent the number of coins Hajun has? b. How would you represent the number of coins Yejun has? c. How would you represent the total number of coins they have?		
Strategy	Example	Frequency (%)
Variable	Item 7a: □	25 (12.6)
Variable-related	Item 7b: □ + 8	23 (11.6)
	Item 7c: □ + □ + 8	9 (4.5)
Value-related*	Item 7a: 8	34 (17.2)
	Item 7b: 16	38 (19.2)
	Item 7c: 24	41 (20.8)

*This strategy code was given only to the responses in which a specific numerical value was assigned and interrelated with the unknowns of Items 7a, 7b, and 7c, such as in the provided example

Another strategy students used was to assign a non-letter variable (i.e., □) to the unknown quantity (coded as the *variable* strategy). What is important here is whether students were able to connect their representations in Items 7b and 7c to their representation of Item 7a (coded as the *variable-related* strategy). The majority of students related their representation in Item 7b to that in Item 7a, but did not relate their representation in 7c to those in Items 7a and 7b. In other words, the students who represented the number of coins Hajun has by □ (Item 7a) tended to keep their non-letter variable to represent the number of coins Yejun has by □ + 8 (Item 7b). However, they had difficulties in representing the combined number of coins Hajun and Yejun have as □ + □ + 8. Students instead assigned either a numerical value or □ which is assumed to simply represent that the combined number of coins is unknown.

In Korea, students are taught to represent the unknown number mostly by □ from the first grade. We wondered how students' responses would change if we provided them with variables. Against students' difficulties with Item 7c, we provided students with supplementary items in which both the number of coins Hajun has, and the number of coins Yejun has, are represented in the form of variables (see Table 6.11). We focused on those strategies in which students used a variable to represent the combined number of coins (i.e., the sum of responses related to 7a and 7b in the original items) and to flexibly operate with expressions involving such variable notation.

As shown in Table 6.11, when the specific variables were provided in the items, the percentage of correct answers increased in comparison to 4.5% for Item 7c, as was seen in Table 6.10. What is even more noticeable here is that it was easier for students to represent the unknown quantity as □ + △ than as □ + □ + 8. As the representation □ + □ + 8 includes two additions with the same symbol,

Table 6.11 Supplementary item 7 and students' strategy use

Supplementary Item 7 (S7). Hajun and Yejun have coins in their piggy banks, and the kinds of coins are the same. They know that their piggy banks each contain the same number of coins, but they don't know how many. Yejun also has 8 coins in his hand.		
a. The number of coins Hajun has is □ and the number of coins Yejun has is □ + 8. How would you describe the total number of coins Hajun and Yejun have?		
b. The number of coins Hajun has is □ and the number of coins Yejun has is △. How would you describe the total number of coins Hajun and Yejun have?		
Strategy	Example	Frequency (%)
Variable-related	Item S7a: □ + □ + 8	34 (17.2)
	Item S7b: □ + △	49 (24.8)

students seemed to attempt to compute further. Some students wrote it without the plus sign (i.e., □□8) or put a rectangle to show the result of '□ + □' (i.e., $\boxed{\quad\quad}$ 8). Others wrote '□ + □ + 8 = ?,' implying that it is not an object per se but something to be calculated.

6.5 Discussion and Implications

6.5.1 Development of Early Algebraic Thinking Through a Curriculum

Given the importance of early algebraic thinking, specific content domains aiming at fostering such thinking skills have emerged in various curricula, such as "patterning and algebra" (Ontario Ministry of Education 2005), "operations and algebraic thinking" (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010), and "number and algebra" (New Zealand Ministry of Education 2009). As aforementioned, the Korean national elementary mathematics curriculum does not include early algebra or algebraic thinking as a specific content domain. However, the two studies reported in this chapter indicate that students are capable of developing essential algebraic ideas from a generalized arithmetic perspective through the current curriculum. Specifically, a promising result was that except for a few items, our students' overall performance was similar to that of students in the intervention group in the Blanton et al. (2015) study. This means that new content areas are not necessarily needed in the current curriculum to induce early algebraic thinking and to make it accessible to students (McNeil et al. 2015). Early algebraic thinking can instead be fostered as a specific form of thinking while students learn typical content areas.

Another important result, as shown in Study 1, is that students' overall understanding of the equal sign, expressions, and equations evolves as their grade levels go up until the fifth grade. This tendency was consistent across different types of assessment items. Given the difficulties that lower graders such as Grades 2 and 3 experienced, however, specific pedagogical attention is needed. For instance, the equal sign is addressed in the first grade in Korea but only about half of the second graders in Study 1 understood that the equal sign means "the same as." More importantly, about 80% of the second graders had the misconception that the equal sign means "the answer to the problem." Note that this misconception persists even in upper grades. Such misconception must be related to curricular materials and instruction in which students see and use the equal sign (McNeil et al. 2015). According to Ki and Cheong (2008), our textbooks use equations mostly in a standard format (i.e., all operations are on the left side of the equation and the answer comes after the equal sign). Considering the importance of relational understanding of the equal sign as an essential idea for algebraic thinking (Blanton et al. 2011), diverse types of equations need to be utilized in curricular materials and instruction from the earliest grades.

The results of Study 2 also indicate our students' weaknesses in understanding algebraic expressions and representing the unknown quantities with variables. The students tended to assign a specific numerical value to the unknown quantity. Even the students who were able to use a non-letter variable had difficulty in connecting such a representation to other related contexts in a consistent way. Although variables have multiple meanings, they are frequently addressed in the current Korean curricular materials beginning in the first grade mainly as a fixed unknown quantity associated with missing-value problems (Pang et al. 2017). Variables to represent the relationships between varying quantities are addressed only from the fourth grade. Radford (2014, p. 260) postulates a framework for characterizing algebraic thinking in terms of three key notions: (a) *indeterminacy:* not-known numbers are involved in the given problem, (b) *denotation:* the indeterminate numbers are named or symbolized in various ways such as with gestures, words, or alphanumeric signs, and (c) *analyticity*: the indeterminate quantities are operated with as if they were known numbers. In order to increase our students' exposure to these key notions, improvement is needed in those parts of the current mathematics curriculum dealing with numbers and operations, in developing a relational understanding of equality, and in writing expressions or equations with variables to represent diverse problem contexts.

6.5.2 Computational Proficiency and Algebraic Thinking

Special attention in this chapter was given to a generalized arithmetic perspective in a broad sense so as to include equivalence, expressions, equations, and inequalities. A common and significant result of Studies 1 and 2 was the finding that our students tend to use a computational strategy in examining an equation structure or in finding an unknown number in an equation, even when the assessment items explicitly ask them not to use direct computation. Korean students are confident in computation; so it may be easy for them to calculate in solving a given problem or to use such computational ability when asked to justify their answers. On one hand, relational thinking or a structural approach over computation is desirable in dealing with mathematical equivalence. Our students need to be further instructed to notice the underlying structure of expressions or equations before jumping into calculation to get the correct answer. On the other hand, computational proficiency does not need to be discouraged in favor of early algebraic thinking. As shown in Episode 1, students with relational understanding of the equal sign are capable of using a structural strategy despite their tendency to compute. More interestingly, our students' computational proficiency seems to help them find the missing value in simple linear equations such as $3 \times \square + 2 = 8$, due to the way they think about such equations. On the basis of familiar number facts our students immediately noticed the structure of the given equation with two operations by regarding $3 \times \square$ as an object. As reflected in Episodes 3.1 and 3.2, the students were able to apply their algebraic reasoning about relationships to solve other equations with larger numbers.

To emphasize, arithmetic is a main context for early algebraic thinking. This study shows that students can be exposed to algebraic ideas as they develop the computational proficiency emphasized in arithmetic. The issue is then for teachers to elicit and foster students' early algebraic thinking through questioning with an emphasis on mathematical structure and relationships while they learn typical mathematical topics (e.g., Can you decide if the given equation is true or false without computation? Can you find the missing value in the equation without computation? What are the unknown numbers or quantities in the context and how can you represent them? Do you think this particular property of number and operations will work for all numbers?).

To conclude, this chapter is expected to provide information on Korean students' early algebraic thinking that develops by means of the current elementary mathematics curriculum. This chapter also shows that specific algebraic ideas need to be intentionally fostered in the curriculum from the earlier grades, because these ideas are not naturally developed in students as they progress through elementary school.

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