Chapter 4
Seeking, Using, and Expressing Structure in Numbers and Numerical Operations: A Fundamental Path to Developing Early Algebraic Thinking

Carolyn Kieran

Abstract The dominant focus on generalizing in the development of algebraic thinking has to a large extent obscured the process of seeing structure. While generalization-oriented activity remains highly important in algebra and early algebra, and in fact includes a structural component, equal attention needs to be paid to the complementary process of looking through mathematical objects and to decomposing and recomposing them in various structural ways. With the aim of instigating greater attention to structure and elaborating more widely on its meaning with respect to developing early algebraic thinking, this chapter explores the notion of structure and structural activity from various perspectives, and then presents a research-based example of 12-year-olds seeking structure within an activity involving factors, multiples, and divisors.

Keywords Structure · Early algebraic thinking · Properties · Structural equivalence · Number and numerical operations · Multiplication and division

4.1 Introduction

High school algebra involves working with generalized forms. The ability to see structure in these forms is crucial to being successful in algebraic transformational activity and to making sense of those transformations. While generalization has long been considered the heart of school algebra (e.g., Kaput 2008; Mason 1996), this focus on the process of generalizing has to a large extent obscured the process of seeing structure, even if generalizing does includes a structural component. While imbuing algebraic and early algebraic activity with generalization-oriented

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tasks remains a highly important aspect in developing students' algebraic thinking, equal attention needs to be paid to the complementary process of *looking through mathematical objects*, such as the expression $x^6 - 1$ or the number 989, and to decomposing and recomposing them in various structural ways (e.g., seeing that $x^6 - 1$ can be decomposed into $(x^3)^2 - 1$ or into $(x^2)^3 - 1$ and factored accordingly, or seeing that 989 can be decomposed into, for example, the structural expressions $9 \times 109 + 8$ or $9 \times 110 - 1$, or even $9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$). As the latter example suggests with its decomposition of 989 in terms of the *division algorithm* theorem, or according to place-value, as well as any number of other structural decompositions, attention to looking through mathematical objects at the primary and lower middle school levels means developing awareness of the possible and various ways of structuring number and the numerical operations of arithmetic. However, "students' experiences in learning arithmetic only rarely foster an appreciation of structure" (Arcavi et al. 2017, p. 53). Similarly, Mason (2016) has argued that looking at something structurally is an often-overlooked aspect of algebraic thinking. This chapter explores the notion of structure and structural activity from various mathematical, theoretical, and empirical perspectives, and then presents a research-based example of 12-year-old students seeking, using, and expressing structure within a selected domain of arithmetic activity, namely that involving multiplication, division, factors, multiples, and divisors.

4.2 Viewing Structure from Various Perspectives

Structure is without doubt one of the big ideas of mathematics (e.g., Kuntze et al. 2011; Mason et al. 2009) and is to be found everywhere in mathematics. A relevant example is drawn from Blanton and Kaput's (2004, p. 142) definition of algebraic thinking where they emphasize the foundational notion of structure: "[algebraic thinking is] a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationships." However, *structure* is often treated within the mathematics education community as if it were tantamount to an undefined term; it is further assumed that there is universal agreement on its meaning (Mason et al. 2010). That this may be problematic, in particular for mathematical teaching practice and research in early algebra with 5- to 12-year-olds, became obvious at the Early Algebra Topic Study Group at ICME-13 in Hamburg in July 2016 when one of the participants asked the others what they meant when they used the term *structure*. As participants attempted to express the notion of structure relative to the various content areas of early algebra, their responses suggested some uncertainty and a tendency to focus rather narrowly on the basic properties of arithmetic. With the aim of instigating greater attention to structure and elaborating more widely on its meaning with respect to the development of early algebraic thinking, this first section of this chapter examines and pulls together various perspectives on structure. It addresses structure and generalization, structure in numbers and numerical operations, and structure in figural patterns and functions.

4.2.1 Structure and Generalization

For Blanton et al. (2011), and in line with Kaput (2008), the essence of early algebra lies in generalizing mathematical ideas, representing and justifying generalizations in multiple ways, and reasoning with generalizations. They define generalizing as follows:

> Generalizing is the process by which we identify structure and relationships in mathematical situations. … It can refer to identifying relationships between quantities that vary in relation to each other. It can also mean lifting out and expressing arithmetic structure in operations on the basis of repeated, regular observations of how these operations behave. (p. 9)

This characterization of generalizing links it closely with the processes of "identifying, lifting out, and expressing arithmetic structure." In other words, generalizing in arithmetic involves identifying the structural.

And, conversely, the structural involves identifying the general, according to Mason et al. (2009):

> We take *mathematical structure* to mean the identification of general properties which are instantiated in particular situations as relationships between elements; these elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. Usually it is helpful to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced. … When a relationship is seen as instantiation of a property, the relation becomes (part of) a structure. (p. 10)

For Mason et al. (2009), attending to properties lies at the core of structural thinking, the latter of which they define as a disposition to use, explicate, and connect these properties in one's mathematical thinking. If a relationship between two or more objects is not seen as exemplifying some general property, then that relationship is not in itself related to structural thinking. Further, they assert that:

> Structural appreciation lies in the sense of generality, which in turn is based on basic properties of arithmetic such as commutativity, associativity, distributivity and the properties of the additive and multiplicative identities 0 and 1, together with the understanding that addition and subtraction are inverses of each other, as are multiplication and division. (p. 15)

In this intertwining of the structural and the general, one is led to ask whether discussion of the structural aspects of the various activities that are engaged in within early algebra and which aim at developing algebraic thinking in 5- to 12-year-olds could benefit from being expanded beyond the basic properties of arithmetic. What are some of the other structural properties that could be said to be included in activity involving numbers and numerical operations? Does the discussion need to be broadened even more when referring to structuring activity involving figural patterns—one of the most widespread approaches to developing algebraic thinking in early algebra? Is the notion of *structure* the same across the various content domains of early algebra (for these content domains, see Kieran et al. 2016, p. 10)? With these questions in mind, we move next to exploring the notion of structure in numbers and numerical operations, and follow this with exploring the notion of structure in figural-patterning and function-oriented activity.

4.2.2 Structure in Numbers and Numerical Operations

For basic notions on structure in mathematics and the activity of structuring in arithmetic, we turn first to Freudenthal (1991). He points out that the system of whole numbers constitutes an *order structure* where addition can be derived from the order in the structure, such that for each pair of numbers a third, its sum, can be assigned. The relations of this system are of the form $a + b = c$, which one calls an *addition structure*. In his book, *Didactical Phenomenology of Mathematical Structures,* Freudenthal (1983, pp. 112–113) describes the *multiplicative structure* of the natural numbers in terms that comprise more than the act of multiplying. It is the whole complex of relations $a \times b = c$, possibly also expressed as $c/b = a$, and complemented by $a \times b \times c = d$, $a \times b = d/c$, and all other relations one would like to consider in this context. It encompasses such properties as commutativity, associativity, distributivity, equivalence of $a \times b = c$ and $c/b = a$, and many more properties of this kind. But, according to Freudenthal, the structure of the natural numbers also allows for prescribing c in the relation $a \times b = c$ and asking for its splittings into two factors. Freudenthal asserts further that c can be split into its prime factors, with divisors and multiples being other means of structuring. As well, tying the order structure to the multiplicative structure yields the property that, given the product, increasing one factor means decreasing the other.

It is of interest that in Freudenthal's discussion of structure there is not just one all-encompassing *structure*. He refers, for example, to order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples, and so on. And these different but related structures have properties—in fact, *many* properties based on these structures, not simply the basic properties of arithmetic that are often referred to as the field properties. We notice too that Freudenthal also uses the phrasing, *means of structuring*, which puts forward the notion of alternative structurings that can be deduced from the basic structures. Freudenthal's perspective serves to broaden considerably the dimensions of any discussion related to characterizing structures and structuring activity within the mathematics of arithmetic and where the development of algebraic thinking is a goal.

Structures and structuring activity have naturally enough been a preoccupation of past research in school algebra and algebra learning (e.g., Kieran 1989, 2006a;

Warren et al. 2016). Certain aspects of this research are applicable to the present aim of enlarging the discussion of structures and structural activity in number and the numerical operations of arithmetic. In their research on structure, Hoch and Dreyfus (2004) define *algebraic structure* as follows:

> Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure. (p. 50)

One of the examples they provide is the expression $30x^2 - 28x + 6$ that students come to see as having a quadratic structure, which in turn allows it to be transformed into an equivalent factorized expression involving two linear terms. Their definition alerts us to the aspect of internal order, as well as to its possible structural decompositions. Warren (2003), in a paper on the role of arithmetic structure in the transition from arithmetic to algebra, and in line with earlier research of Morris (1999), similarly contends that knowledge of mathematical structure is knowledge about mathematical objects and the relationship between the objects and the properties of those objects. She states that:

> Mathematical structure is concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality). (Warren 2003, p. 123)

In addition to the field properties, we note that Warren includes mention of equivalence and equality properties as well as order properties.

From the research that has documented difficulties experienced by beginning algebra students with recognizing structure in algebraic expressions and equations, we obtain further insights for an enlarged perspective on structure (for overviews of this research, see Kieran 1992, 2007; for an alternative point of view on structure, see Kirshner 2001). Linchevski and Livneh (1999), who coined the phrase "structure sense," maintain that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. These researchers thereupon insist that instruction be designed to foster the development of structure sense by providing experience with equivalent structures of expressions ("equivalent structures of expressions" being sometimes referred to—in, e.g., Mason et al. 1985—as "equivalent expressions" or equivalent "forms") and with their decomposition and recomposition. Hoch and Dreyfus (2005, 2006) have also reported that very few of the secondary-level students they observed had a sense of algebraic structure, that is, very few could: "(i) recognize a familiar structure in its simplest form, (ii) deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form, and (iii) choose appropriate manipulations to make best use of structure" (2006, p. 306). Demby (1997) too found that algebra students were poor at identifying structure, in particular, the properties they use when they transform algebraic expressions—

despite having been taught how to use these properties. Thus, these algebra researchers suggest that improving attention to structure with younger students needs to go beyond focusing on the basic properties and should include experience with equivalence of compound and simple forms, that is, with equivalence expressed through decomposition, recomposition, and substitution, as well as with recognizing equivalence to familiar structures.

Some of these recommendations have been further unpacked in various proposals related to the development of algebraic thinking within arithmetic. For example, Ellemor-Collins and Wright (2009, p. 53) claim that *structuring numbers* means "organising numbers more formally: establishing regularities in numbers, relating numbers to other numbers, and constructing symmetries and patterns in numbers." For Subramaniam and Banerjee (2011, p. 91), "numerical expressions must be viewed not merely as encoding instructions to carry out a sequence of binary operations, but as revealing a particular operational composition of a number … how quantities or numbers combine." Slavit (1999) emphasizes the importance of being able to break an operation into its base components, of knowledge of operation facts, and of understanding the relationships between the operations. Asghari and Khosroshahi (2016, p. 1) argue that "mathematical thinking involving equality among young learners can comprise both an operational and a structural conception and that the operational conception has a side that is productively linked to the structural conception." Schwarzkopf (2015, p. 14, citing Winter 1982) advances the notion that "understanding an equality between two mathematical terms means understanding that the terms are different representations of the same mathematical object" (e.g., 5 + 4 = 2 + 7)—a perspective on structure that is similar to the relational thinking approach to equalities promoted by Carpenter et al. (2003) and that includes attention to the role played by substitution in conceptions of equality (Jones et al. 2012).

In their research with elementary school children, Malara and Navarra (2016) point to the importance of expressing structural aspects of number in transparent, non-canonical ways, as illustrated by their example of 10-year-old students representing the sum of 5 and its successor: One student offered the expression "5 + 6", but a classmate argued that her own representation of "5 + 5 + 1" was clearer and more transparent because it expressed the functional relationship between a number and its successor. Similar transparency is stressed by Carraher et al. (2006), who have used the *N*-number line representation to help students focus on the structure of numbers and the relation between a number and its numerical neighbours.

The kind of "structural transparency" advocated by Malara and Navarra and by Carraher et al. is also emphasized by Arcavi et al. (2017), who argue that students' compulsion to calculate numerical answers can make it difficult for them to see patterns and mathematical structure. They describe an activity based on the well-known arithmetical sequence of triangular numbers arranged in dot formation. Arcavi et al. state that it is much easier to see the structure of the numerical sequence, and to generalize it, by observing the pattern in the uncalculated expressions 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, …, than by looking only at the total numbers of dots 1, 3, 6, 10, … for each member of the sequence. Expressing the total numbers of dots by means of their “unclosed” sums allows for seeing that the sequence is the sum of the natural numbers. Collis (1975), many years ago, brought attention to the importance of “acceptance of lack of closure” in the development of algebraic thinking (i.e., being able to accept, say, 8 + 4 as a number with the same legitimacy as its calculated value of 12). As pointed out by Mason et al. (2009), “by working on tasks which focus on the nature of the relation rather than on calculation, students’ attention is drawn to structural aspects as properties which apply in many instances” (p. 15).

Other research-based suggestions related to developing elementary school students’ structuring experiences with numbers and operations in arithmetic have included, to name just a few: activity with quasi-variables that brings out the additive inverse and additive identity properties (Fujii and Stephens 2001); the role of students’ drawings to illustrate, for example, the doubling-halving property of multiplicative factors (Russell et al. 2011); multiplication-table tasks where students are encouraged to seek reasons why certain cells are the same (Neagoy 2015) and to articulate the structures underpinning the tables (Hewitt 1998; Mason et al. 2005); numerical tasks involving equivalence and compensation within addition and subtraction (as well as within multiplication and division), such as for example transforming 298 + 57 to 300 + 55 so as to make calculations easier (Baek 2008; Blanton et al. 2011; Britt and Irwin 2011); and the “three dice guessing” activity where students decompose by partition all the combinatorial possibilities from 3 to 18 (Wittmann 2016).

In the spirit of Freudenthal (1983, 1991), and as reflected in the research literature exemplified above, *structure* as it pertains to number and numerical operations at the elementary and early middle school levels encompasses many means of structuring—structuring according to factors, multiples, powers of 10, evens and odds, sums of 10, prime decomposition, and many more—such structurings often expressed in decomposed, uncalculated form. These structurings have properties, such as the basic properties of arithmetic, but also a multitude of other properties such as the successor property, the sum of consecutive odd numbers property, the sum of even and odd numbers property, equivalence and equality properties, and so on. To conclude this section, we would argue that the inclusion of such additional means of structuring and their properties within our discussions of structure related to number and numerical operations allows for a broader conceptualization of a fundamental aspect of early algebraic thinking and its development.

4.2.3 Structure in Figural Patterns and Functions

Number and numerical operations are not the only content included in early algebra research and teaching practice. Patterning and functions are also integral to this area of study. So what do we mean by *structure* in figural patterns and functions? As will be argued later in this section, the structure of number and numerical operations remains a central component even within these additional focus areas. One point that needs, however, to be brought out beforehand is that the term *structure*, in general, as it relates to figural patterns needs to be distinguished from the term *structure*, in general, as it relates to number and numerical operations. In activity that involves seeking structures in numbers and numerical operations, the structures are inherent to the numbers and numerical operations—a consequence of the axioms. Such is not the case with patterns. As pointed out by Carraher et al. (2008), pattern is not an acknowledged, much less well-defined, concept in mathematics. Patterns can be extended mathematically in any way that one wishes. There is no inherent structure to be uncovered and then generalized (Mason et al. 2010). Patterning involves the search for some regularity and the *imposing* of a certain structure. This imposing of structure affords some predictability to the pattern and so allows for generalizing beyond the provided set of examples of the pattern. Mason et al. (2009) remind us that many "mathematical-looking" tasks involve asking students to extend "patterns" and to predict the *n*th term; they emphasize that there must be some prior agreement or articulation of the actual underlying structure that generates the given sequence in order for a pattern to be considered a mathematical task—an articulation that researchers of figural patterning activity are generally careful to provide by means of the story context that accompanies the pattern (see, e.g., Moss et al. 2008, p. 157).

Once the actual framing of the underlying structure has been set out, the process of generalizing within patterning is considered to involve searching for some invariant property in a set of objects. In fact, distinguishing between what is invariant and what it is that is varying constitutes a crucial first stage in the activity of patterning (Kieran 2006b; Mason et al. 2005). According to Rivera (2013), and in line with Mason et al. (2009), the development of structural thinking within patterning activity involves the recognition of relationships of similarity and difference within a structure, followed by the perceiving of properties that characterize the objects being analyzed, and then by reasoning on the basis of the identified properties.

While many different types of patterns have been used in early algebraic activity, one of the most widely used types is that of the growing figural, or geometric, pattern. Based upon his extensive research on the development of algebraic thinking with 7- to 9-year-olds, Radford (2011) has argued that:

> Generally speaking, to extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one *spatial* and the other *numerical*; from the spatial structure emerges a sense of the figures' *spatial position*, whereas their numerosity emerges from a numerical structure. (p. 19)

In one of his studies, 7- and 8-year-olds were presented with the pattern shown in Fig. 4.1. Radford describes how one of the children, Carlos, when asked to draw the 5th term, very carefully produced the drawing shown in Fig. 4.2—one that did not conform to the two-row configuration of the given pattern. Carlos's geometric shape for Term 5 did not help him figure out its numerosity. On the other hand, other children who did attend to shape were still not attuned to numerosity, and vice versa.

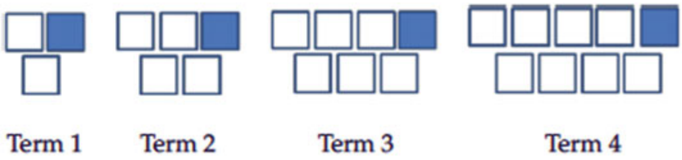

Fig. 4.1 The first four terms of a sequence given to the students in a Grade 2 class (Radford 2011)

Fig. 4.2 Carlos's drawing of the 5th term (Radford 2011)

Radford has emphasized that the linkage of spatial and numerical structures in figural patterns constitutes an important aspect of the development of algebraic thinking.

The day after the observations involving Carlos, the teacher discussed the row-wise geometric structure of the pattern with the children and thereby helped them to make links between the numerical and geometric structures. When she then asked the children about the number of squares in Term 25, one child, Mary, volunteered: "25 on the bottom, and 25 on the top plus 1." Radford (2012) explains the children's progress as follows:

> They became aware of the fact that the counting process can be based on a *relational idea*: to link the number of the figure to relevant parts of it (e.g., the squares on the bottom row). … The terms appear now not as a mere bunch of ordered rectangles but as something susceptible to being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. … This cultural transformation of the eye is not specific to Grade 2 students. It reappears in other parts of the students' developmental trajectory. It reappears, later on, when students deal with factorization, where discerning structural *syntactic forms* becomes a pivotal element in recognizing common factors or prototypical expressions. (pp. 216–217)

The point here is that the eye must learn to look for structural features in a variety of mathematical objects, structural features that involve decomposing and recomposing (see also Radford 2010). Malara and Navarra (2016) hinted at the structural transformation of the eye within purely numerical activity in their description of the young student who came to recognize that the number 6 could be decomposed into 5 + 1, thereby allowing one to see and to express the successor property of number. We shall return to this notion of the "structural eye" later on, near the end of the chapter.

Patterning is widely used in early algebra research studies that explore how children in the elementary grades come to think about and represent functional relationships, in particular, functions with a linear structure, but also exponential and quadratic structures (e.g., Cooper and Warren 2011; Rivera and Becker 2011). Rivera (2013), for example, describes in detail the structuring processes engaged in by various aged children across a variety of functional patterning tasks. There exists as well a substantial amount of research related to children's development of algebraic thinking in the context of functions that does not involve patterning tasks, but rather functional problem situations (e.g., Blanton et al. 2015; Carraher et al. 2006). Within this extensive body of research literature devoted to the theme of algebraic reasoning within patterning and functional activity is an understated aspect that pertains directly to our preceding discussion of structure in number and numerical operations. It concerns the explicit expression of sequences of operations.

An example of this aspect is drawn from the research of Moss and London McNab (2011), who aimed at developing 7- and 8-year-old students' awareness of linear functional relationships by means of both numeric and geometric (figural) patterns. The research method they employed involved first using geometric patterns (tile arrays) and then numeric (function machine) patterns and then moving back and forth between the two. They found that a bridging occurred between the two types of patterns that was enabled by the idea of a function rule: "It was the specific movement back and forth between the two representations, geometric and numeric, that ultimately supported the students to gain not only flexibility with, but also a structural sense of, two-part linear functions [i.e., $y = mx + b$]" (p. 296). In particular, they claim that it was the explicit expressing of the sequences of operations that corresponded to the functional structure of the numeric patterns that eventually came to be seen as a common thread in both the geometric and numeric patterns. As well, students became aware that expressions such as $15 \times 2 + 3$ were equivalent to $15 + 15 + 3$ by means of the parallel geometric and numeric structures and without necessarily calculating the totals for each. The emphasis on the role played by the explicit expressing of the sequences of operations in the Moss and London McNab study reminds us of the point made earlier by Arcavi et al. (2017) regarding the structure-developing role that can be played by observing the uncalculated expressions of the successive terms of a pattern sequence. Other types of activity involving explicit sequences of uncalculated expressions playing a similar role have been noted in research on "think of a number games" (e.g., Cedillo and Kieran 2003) and "tracking arithmetic" tasks (Mason 2017; Mason et al. 2005)—where students are encouraged to represent explicitly and in uncomputed form the operations that are applied to the thought-of numbers (sometimes represented as clouds) so as to more easily detect the properties being applied throughout and thereby explain the final results.

To recapitulate, the structures involved in figural patterns clearly include a numerical component (in addition to a spatial component) and, vice versa, the numerical aspects of the patterns are structural in nature. Recall Mary's response to the question posed to her in Radford's (2011) research regarding the number of squares in the 25th term of the pattern: "25 plus 25 plus 1"—a structural response that was expressed by a numerical decomposition, one that corresponded to her spatial decomposition of the figure. And decomposed numerical expressions constituted students' functional activity in the Moss and London McNab (2011) study. The point being made here is that structuring experiences involving decomposed numerical expressions are central not only to the content area of number and numerical operations, but also to the content areas of patterning and functions. We view such structuring experiences as fundamental to the development of early algebraic thinking.

4.3 Seeking, Using, and Expressing Structure in Numbers and Numerical Operations by 12-Year-Olds

This next section of the chapter presents a researched example of 12-year-old students engaged in structuring activity with numbers and numerical operations. The study (Kieran and Guzmán 2005) involved three classes of students from a Mexican private school in a task-based calculator environment and focused on the ways in which they sought, used, and expressed structures related to multiplication, division, factors, multiples, and divisors.

4.3.1 The "Five Steps to Zero" Problem: The Tasks and Learning Environment

Seeking and using structure involving multiplication and division has received less attention in the research literature than has addition and subtraction. Even less appears on the ways in which computing tools might be harnessed in the development of structural thinking in this area. In our study, we integrated a combined task-technique-theory perspective that was based on the so-called instrumental approach to tool use (Artigue 2002). Within this approach, mathematical concepts are considered to co-develop while the learner is perfecting his/her techniques with the tool. According to Lagrange (2000, p. 17, our translation and our emphasis): "The new instruments of mathematical work are of interest … because they permit students to *develop new techniques that constitute a bridge between tasks and theories.*" If techniques can constitute a bridge between tasks and the emergence of theoretical knowledge, then it is by looking at the techniques that students develop with the aid of their technological instruments, in response to certain tasks, that we obtain a window into the evolution of their structural awareness.

The tasks that we developed were based on the "Five Steps to Zero" problem (Williams and Stephens 1992; see Fig. 4.3—note that all whole numbers from 1 to 1000, with the exception of 851 and 853, which require six steps, can be brought down to zero in five or fewer steps). Successfully tackling this task situation, with the constraint of using only the whole numbers from 1 to 9 and only one operation per line, involves developing techniques for decomposing numbers (prime or

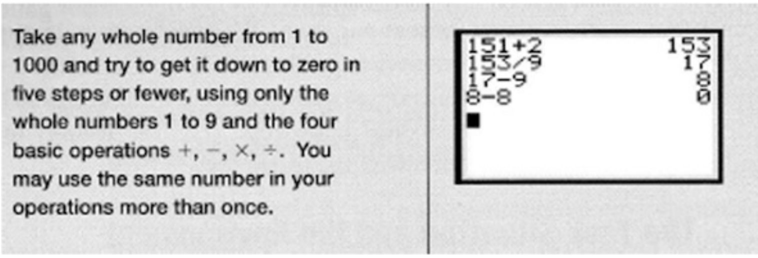

Fig. 4.3 The basic problem, "Five Steps to Zero" (adapted from Williams and Stephens 1992), accompanied by an example (151), displayed on the multiline screen of the calculator

composite) into other numbers in the same neighbourhood (not more than 9 away from the given number) that have divisors not larger than 9 so as to reach zero in five or fewer steps.

In the example illustrated in Fig. 4.3, the given number 151—a prime—was first converted into a non-prime number, followed by a test to see if the result was divisible by 9 (*divisible* being understood to mean *divisible without remainder*). As an alternative to the approach displayed in Fig. 4.3, a student might first subtract 1 from 151, and then divide 150 by divisors such as 2, 3, and 5, eventually arriving at 5, which can then be brought to zero by subtracting 5. Since students were encouraged to use as few steps as possible, this task provided fertile ground for learning, for example, the structural property that if a number has both a and b as divisors, then it is also divisible by $a \times b$, as long as $a \times b$ does not exceed 9.

Theoretical awareness of this property has been reported to be difficult for many students to develop. Past research has shown that, for example, just because students were able to find that $a \times b = ab$, they did not then state that ab is a multiple of b before first dividing ab by b (Vergnaud 1988). In another study involving older students, Zazkis and Campbell (1996, p. 542) asked: "Consider the number $M = 3^3 \times 5^2 \times 7$. Is M divisible by 7? Explain. Is M divisible by 5, 2, 9, 63, 11, 15? Explain." Students' understanding of divisibility and prime decomposition was found to be so poor that the researchers argued that developing a conceptual understanding of divisibility and factorization, which is essential in the development of conceptual understanding of the multiplicative structure of numbers, should be happening in the middle grades. We were thus interested in investigating how the tasks and tools that we designed to be used with the middle-grade students of our study would increase their structural awareness of factors, multiples, and divisibility.

We developed a set of ten activity sheets (see Fig. 4.4) involving tasks based on the "Five Steps to Zero" problem on which students worked over a period of one week (five classes of fifty minutes each). When students finished one activity sheet and handed it in, they were given the next one.

Several considerations were included in designing the tasks for the 10 activity sheets. Task #1 involved an even number (144) with several divisors below

1. Take the number 144. Write as many ways as you can for bringing 144 to zero, using as few steps as possible.
2. Take the number 151. Write as many ways as you can for bringing 151 to zero, using as few steps as possible.
3. Take the number 732. Write as many ways as you can for bringing 732 to zero, using as few steps as possible.
4. Describe your strategies for minimizing the number of steps.
5. Here is a solution proposed by a pupil for bringing 432 to zero: 432/2 = 216; 216/2 = 108; 108/2 = 54; 54/3 = 18; 18/3 = 6; 6 – 6 = 0. Show a way of bringing 432 to zero in fewer steps. Explain your strategy. Do you think it will always work? Why?
6. Here is a strategy proposed by a pupil for bringing 731 to zero: 731 + 7 = 738; 738/9 = 82; 82 – 1 = 81; 81/9 = 9; 9 – 9 = 0. Show a way of bringing 731 to zero with fewer steps. Explain your strategy.
7. The number 266 has as its divisors 2, 7, and 19. In other words, 266 = 2 × 7 × 19. What is the best strategy for bringing 266 to zero? Why? Explain why your strategy is the best.
8. Here is the strategy proposed by a pupil for bringing 499 to zero: 499 + 1 = 500; 500/5 = 100; 100/5 = 20; 20/5 = 4; 4 – 4 = 0. Show a way of bringing 499 to zero in fewer steps. Explain your strategy.
9. What do you consider to be the best strategies for bringing numbers down to zero?
10. Think of a number that your classmates would find difficult to bring to zero in five or fewer steps. Write down why you think it would be a hard number. Show the solution you found for your hard number.

Fig. 4.4 Tasks of the 10 activity sheets prepared to accompany the “Five Steps to Zero” problem

10—thus a fairly accessible number to start with. Note also that, by asking students to record on paper the solutions they were trying with their calculator, we would be able to trace how their strategies were evolving. Task #2 involved a prime number, which could be handled by means of the addition or subtraction of some number in order to have a composite that could be divided by numbers less than 10. Would students aim for an even number, a number ending in 5 or 0, or something else? Task #3 began with 732, which could be brought to zero in five steps if one began with a division by 3, 4, or 6; but a four-step solution required adjusting the given number so as to have a multiple of 9. After the first three open-ended tasks, the very important Task #4 asked students to describe in writing their techniques for minimizing the number of steps to be taken to reach zero. Task #5 presented a six-step approach for bringing 432 to zero. All of the divisors used in the given task were 2s and 3s. Would the students spontaneously think of combining some of the given divisors—for example, the first three 2s to yield a divisor of 8, and the last two 3s for a divisor of 9—so as to reduce the number of steps from six to three? Task #6, which was designed to provide some experience with multiples of 9, illustrated a five-step method for bringing 731 to zero that led off with the conversion of 731 to 738. Would students come to see that if 738 is divisible by 9, so too is 738 − 9, and that this would save a step because the resulting quotient is immediately divisible by 9? Task #10, the final task, formed the basis for a follow-up competition in class.

The class was to be divided into three groups; each group settling on one of the "hard" numbers proposed and defended by its group members. Three rounds of competition involving two teams trying to solve the "hard" number of the third team were to take place. This competition and the individual justifications written down by each student as to their "hard" number were to provide further evidence of the evolution in their structuring activity during the week.

Each student was equipped with a graphing calculator. While the graphing capability of the calculator was not used, the larger screen of this kind of calculator made it possible for the students to record and observe at a glance all their steps toward zero—as opposed to the small one-line screen of a simple four-operation calculator. The calculator permitted students to carry out each of the basic operations in one step. Without having to keep track of all the intermediate moves that would normally capture their attention in a paper-and-pencil environment, they were free to focus on structural aspects. It is noted that the calculators that were used in the study did not provide the complete factorization of a number; thus, it was not possible to know instantly whether a given whole number was a multiple of some other number, or even whether or not it was prime—a constraint that was capitalized on for our study.

The teachers introduced the main task situation as follows. They began with the example of 360 and illustrated with the classroom view-screen (a room-size projection of the screen of the calculator that was hooked up to the view-screen device) that they could get down to zero in the following way: 360/2, 180/2, 90/3, 30/6, $5 - 5$. The teachers then requested volunteers to come forward to show how they might get to zero in fewer than five steps. After that, students were asked to suggest their own starting numbers, say, larger than 200, which other students came forward to solve. The students then began to work on the tasks of the activity sheets, either individually or in pairs, but each student filled in his/her own activity sheets.

Regularly during the week, individual students were invited to come forward to the classroom view-screen and to work out a task using that device. This allowed both the researchers and the classroom teacher to observe directly the nature of the approaches that the students were trying out in response to the problem tasks. It is important to note that the students whose work was being recorded on the view-screen had not already arrived at a solution beforehand, but were in fact allowing us to witness all the false starts, dead ends, and various structural relations they were trying to find. During the week that followed the classroom part of the study, four students representing a range of mathematical ability (as rated by the classroom teacher) from each of the participating classes were individually interviewed. A pre-test had also been given to the students prior to the study to inquire into their knowledge of divisors, multiples, and primes. The interviews, in conjunction with the pre-test results, gave the researchers the opportunity to explore at closer range the nature of the structural awarenesses that students had developed over the course of the previous week. In the sections that follow, we present and discuss samples of students' work that are representative of the ways in which the techniques of the three classes of students evolved.

4.3.2 The Emergence of Structurally-Oriented Techniques

The techniques that students used at the beginning of the week's activities tended to be based on simple criteria for divisibility, such as dividing by 5 if the number ended in 0 or 5, or dividing by 2 if the number was even. This is illustrated by the work of Marianne with the given number 151 from the second activity sheet (see Fig. 4.5). Her two recorded attempts suggested two different decompositions of 151: $5 \times 30 + 1$ and $5 \times 31 - 4$, both handled by means of inverse operations. She was clearly aiming at converting the given number 151 into a multiple of 5. We wonder whether she noticed the structural property that when two adjacent multiples of 5 (i.e., 150 and 155) are divided by 5, the two quotients that are obtained (i.e., 30 and 31) are consecutive numbers.

The initial techniques of Marianne evolved, just as they did for her classmates. On the third activity sheet with 732 (see Fig. 4.6), she showed a shift toward trying to find larger divisors.

On the fourth activity sheet, in describing the techniques that had emerged thus far for her, Marianne wrote (translated from Spanish):

> Divide by the largest divisor possible from 1 to 9; if there are no divisors, then add or subtract to obtain another number where the division is possible. After dividing, look at the result and test whether division is again possible. If not, repeat the previous procedure until arriving at a number less than 9 and finish the procedure with a subtraction.

Nicolas offers us another example of how pupils in this study were evolving from more basic techniques to that of trying to find the largest divisor possible. Having unsuccessfully tested whether 9 or 8 was a divisor of 930 (see lines 2 and 3 of Fig. 4.7), Nicolas's next efforts centered on finding another number in the neighbourhood of the given number 931 for which he could use large divisors

L1: 151 – 1	150
L2: 150/5	30
L3: 30/5	6
L4: 6 – 6	0

L1: 151 + 4	155
L2: 155/5	31
L3: 31 – 1	30
L4: 30/5	6
L5: 6 – 6	0

Fig. 4.5 Two consecutive attempts by Marianne to bring 151 to zero (Note that line numbers have been added to make it easier to refer to specific lines of the screen display)

L1: 732/6	122
L2: 122/2	61
L3: 61 + 3	64
L4: 64/8	8
L5: 8 – 8	0

L1: 732/4	183
L2: 183 – 3	180
L3: 180/9	20
L4: 20/5	4
L5: 4 – 4	0

Fig. 4.6 Two attempts by Marianne with the given number 732

L1: 931 – 1	930	L26: 931 + 5	936
L2: 930/9	103.33	L27: 936/9	104
L3: 930/8	116.25	L28: 104/8	13
L4: 930/5	186	L29: 13 – 9	4
L5: 186/9	20.66	L30: 4 – 4	0
L6: 186/8	23.25		

Fig. 4.7 Various attempts by Nicolas to find suitable decompositions of 931

L1: 9 × 86	774	L10: 9 × 106	954
L4: 9 × 97	873	L11: 9 × 108	972
L6: 9 × 99	891	L12: 971 + 1	972
L7: 9 × 105	945	L13: 972/9	108
L8: 9 × 110	990	L14: 108/6	18
L9: 9 × 107	963	L15: 18/9	2
		L16: 2 – 2	0

Fig. 4.8 Marianne's shift to multiplication in her search for an appropriate structuring of 971

throughout. In attempting to find numbers in the vicinity of 931 that were divisible by 9, had Nicolas noticed the structural property that within every interval of 9 numbers there is exactly one number that is divisible by 9? His later work was to confirm that he had indeed discovered this property.

Toward the end of the week's activities, several students began to make structural breakthroughs. Their focus became more controlled in that instead of using successive trial and error with the divisors 9, 8, and 7, they started to search for techniques oriented around the use of the factor 9. The challenge of the activity had become that of finding a structural way to convert the given number into a multiple of 9 so as to arrive at zero in the fewest number of steps possible. Marianne, for example, wrote on her sixth activity sheet that she wanted "to subtract or add in order to arrive at a number divisible by 9; if you divide by the largest number, even if you do a subtraction or yet an addition, you will reach zero more rapidly."

With Marianne, it was not until the last day of the week, when she was using the classroom view-screen and was given the number 971 to bring down to zero, that we witnessed the structural technique that she had developed (see Fig. 4.8). Marianne began at once to search for a number in the vicinity of 971 by using the product of two factors, one of which was 9. She was in fact working in reverse, using multiplication rather than division, so as to try and arrive at a multiple of 9 in the neighbourhood of her starting number. Once she had found two multiples that were on either side of the starting number (see Lines 8 and 9 of Fig. 4.8), she successively refined her search until she reached a multiple of 9 that was within 9 units of 971 (see Line 11). The structure of multiplication with subtraction ($9 \times 108 - 1$) was then converted to addition with division (Lines 12 and 13).

A related technique that involved the structural interplay between dividing by 9 and using multiples of 9 emerged for another student, Mara, near the end of the week. While she was at the front of the class using the view-screen, a classmate suggested she try 731 as her initial number. After a few unsuccessful tries involving the search for neighbouring numbers that could be divided by 9, she seemed suddenly to notice that she could take the whole-number part of the quotient, which she rounded up to 82, and did a reverse multiplication (see Lines 16 and 17 of Fig. 4.9). The product told her immediately how much of a structural adjustment needed to be made to the initial number. We note that, had she truncated rather than rounding up the quotient, she would have saved a couple of steps in that 81 would have allowed an immediate subsequent division by 9. Another student, Pablo, had developed a similar technique (see Fig. 4.10). By multiplying 9 with 103 (Line 2), Pablo then inferred that the remainder on trial dividing 931 by 9 was 4, thereby leading to a structural decomposition of 931 as $9 \times 103 + 4$.

During the week following the classroom study, when individual interviews were held with some of the students, a revealing conversation took place with Nicolas. When asked what he would do if a given initial number was not divisible on the first step by a number between 2 and 9, he answered that he would add or subtract. So we continued by asking him how he figured out the amount that he needed to add or subtract, to which he responded that he had a certain “technique” (see Fig. 4.11 for the transcript of the relevant segment of the interview; I is the Interviewer and N is Nicolas).

Notice (Episode 36 of Fig. 4.11) that Nicolas is trying to control three factors at a time, all of them in the range of 2 to 9, in his search for a product in the neighbourhood of 431. On screen line L4, he enters the following multiple of 9: $9 \times 9 \times 5$, and sees that it yields 405. So, he then decides to adjust the second and third factors simultaneously. He decreases the 9 to 8 and increases the 5 to 6, entering $9 \times 8 \times 6$ into the calculator. This numerical expression produces the result 432, just 1 more than the given number (see screen line L5). He clearly realizes that $9 \times 8 \times 6 - 1 = 431$. This decomposition will allow him to bring 431

L1: 731 + 1	732	L16: 731/9	81.22
L2: 732/9	81.33	L17: 9 × 82	738
. . .		L18: 731 + 7	738
L10: 731 – 8	723	L19: 738/9	82
L11: 723/9	80.33	L20: 82/2	41

Fig. 4.9 Mara’s shift from dividing to the reverse operation of multiplying the divisor by the rounded-up quotient

L1: 931/9	103.44	L3: 931 – 4	927
L2: 9 × 103	927	L4: 927/9	103

Fig. 4.10 Pablo’s similar shift from dividing to multiplying in decomposing 931

32.	N:	Because (pause) well, I also have a "technique" that I use. First I do a multiplication, say, $9 \times 9 \times 3$ or something like that to arrive at another number, and I look at that number.
33.	I:	Let's see, repeat that for me one more time.
34.	N:	For example, if I have the number 571 and I multiply 9×9, it gives 81.
35.	I:	Let us say that I give you the number 431.
36.	N:	OK, so I go (and he picks up the calculator): L1: 9×9 81 L2: 81×3 243 L3: $9 \times 9 \times 4$ 324 L4: $9 \times 9 \times 5$ 405 So, like that, I arrive more quickly.
37.	I:	But I said 431. With this strategy that you have just described, how do you begin?
38.	N:	First, $9 \times 9 \times$ something, no? Until arriving close to the number. For example (he again picks up the calculator): L5: $9 \times 8 \times 6$ 432
39.	I:	Yes, I told you 431.
40.	N:	So, 431 plus 1, divided by 6, divided by 8, and so on.
41.	I:	Let's see.
42.	N:	(Nicolas picks up the calculator): L6: 431 + 1 432 L7: 432/6 72 L8: 72/8 9 L9: 9 – 9 0 And there it is!

Fig. 4.11 Segment of transcript from the interview with Nicolas where he describes his "technique" and illustrates it with the given number 431

to zero in four steps by means of inverse operations where each of the factors will be treated as divisors, except for the last one, which will be subtracted so as to arrive at zero. The complex of structural relations between multiplication and division and between addition and subtraction, as well as a structuring according to divisors and multiples, have all been expressed in Nicolas's mathematical work.

4.3.3 Analysis of the Evolution of Students' Structuring Activity

Some of the most powerful structural explorations that occurred during the week of activity on the "Five Steps to Zero" tasks involved the search for multiples of 9. Since students wanted to arrive at zero in the fewest number of steps possible, their initial techniques soon evolved into attempts to discover whether the given number was divisible by 9, or whether any numbers in the close vicinity (i.e., within 9 on either side of the given number) were. But how to find the right numbers in the close vicinity was the question. Furthermore, the students seemed unaware of the criterion for divisibility by 9 (i.e., sum the digits to see if the total is a multiple of 9) or how this test might be used to locate a multiple of 9 in the neighbourhood of the given number.

4.3.3.1 Variants of the *Division Algorithm*

The structural awareness that emerged for many students involved variants of the *division algorithm.* According to this theorem, any whole number can be decomposed and expressed as the product of two whole numbers plus remainder, that is, "For any $b > 0$ and a, there exist unique integers c and d with $0 \leq d < b$ such that $a = b \times c + d$" (e.g., $989 = 9 \times 109 + 8$). Even though students were not taught this theorem, their work showed the different structural means by which they tried to obtain the value of c and thereby illustrated the ways in which they were beginning to think structurally about division with remainder—even if not always articulated explicitly. One variant of the division algorithm reflected in their work was the following: "For any $b > 0$ and a, there exist unique integers c and d with $0 \leq d < b$ such that $a = b \times c - \text{d}$" when the decomposition of the initial number a led to using the multiple of b on the higher side of a rather than on the lower side (e.g., $989 = 9 \times 110 - 1$).

But we also witnessed other structural "variants" of the division algorithm. For example, the techniques of Mara and Pablo evolved to take the form of carrying out a trial division by 9, followed by the multiplication of the truncated or rounded-up quotient with 9 in order to see how far the product was from the initial number—a structural approach that we named the "division algorithm invoking trial division" (e.g., $989/9 = 109.8888889$, followed by $9 \times 109 = 981$). Since 989 can thereby be decomposed into $9 \times 109 + 8$, their approach for bringing 989 to zero involved using the inverses of addition and multiplication as in $989 - 8 = 981$, $981/9 = 109$, and so on. An implicit variant of this technique involved *looking at the size of the decimal portion of the quotient, without actually carrying out the related multiplication of 9 with the truncated or rounded-up quotient,* to provide a structural clue as to how close the given number was to a multiple of 9.

Another variant of the division algorithm was based on what we named the "division algorithm invoking trial multiplication." This approach, observed with Marianne, involved carrying out trial multiplications in order to find an appropriate value of c, as in, for example, the structural relation, $989 = 9 \times c + d$ (e.g., $9 \times 106 = 954$, $9 \times 108 = 972$, $9 \times 109 = 981$)—the latter multiplication clearly bringing the solver into the interval that is within 9 of the given number 989, thereby allowing for decomposing 989 as $9 \times 109 + 8$.

While the searches by Marianne for multiples of 9 always involved two factors, the technique that Nicolas came to develop involved a complete decomposition of the given initial number into three factors or more, accompanied where necessary by an addend-adjustment. His technique is one that we named "trial multiplication involving more than two factors" and synthesized as $a = b \times c \times d \times e \pm k$, where b, c, d, and e are whole number factors between 2 and 9, and k is a whole number addend or subtrahend such that $0 \leq k \leq 9$. In Nicolas's above work with the given number 431, he was able to analyze his trial sequence of operations, $9 \times 9 \times 5$, which had yielded 405, and adjust the expression in such a way that, by decreasing the second 9 to 8 and increasing the 5 to 6, the result ($9 \times 8 \times 6$) would be slightly larger than 405. The degree of control he showed not only in generating the multiple factors approach but also in changing the specific decomposition from $9 \times 9 \times 5$ to $9 \times 8 \times 6$ hinted at the structural eye that he was beginning to develop for number and numerical operations.

4.3.3.2 Developing a Structural Eye for Decomposing Number

Developing an eye for structure is surely a long process that needs to reinvent itself with every new type of mathematical object that is encountered. While the "Five Steps to Zero" activity lasted only a week, and there was no follow-up opportunity to see whether the ways in which students sought structure within that activity would carry through to their everyday mathematical work, the shifts in structural techniques that we observed suggest that the students had indeed begun to develop a structural eye for the multiplicative decomposition of number. This was highlighted in, for example, Mara's noticing that she could replace several trial divisions by just one, followed by multiplication of the rounded-up quotient by the divisor. From our observations, we conjecture that the shifts that emerged were partially motivated by students' lack of satisfaction with the initial trial-and-error methods they were using—dissatisfaction that pushed them to "develop a technique," to use Nicolas's words.

As was seen in the students' early trial divisions with the large divisors 8 and 9, the quotients usually contained decimals, which "got in the way" of reaching zero quickly. In the search for more effective means of tackling the tasks, Mara, as noted above, came to realize that she could "clean up" the decimal quotient and multiply it by the divisor to arrive immediately at a product that was in the required range of the given number. Marianne became aware that she could supplant her "messy" trial divisions with more focused multiplications involving two factors, one of which was 9. Nicolas came to develop a method that allowed him to arrive at a decomposition of the given number consisting of a complete set of factors plus any required additive adjustment. According to Subramaniam and Banerjee (2011), a refined, structural understanding of operational composition includes accurate judgments about relational and transformational aspects, such as judging how the contribution of one part of the expression will change if some of the numbers involved in the expression change. Such judgments were reflected in the compensations made by Nicolas in arriving at his complete decomposition of 431 (in Episodes 36–38 of Fig. 4.11) during the post-study interview.

One last point needs to be made with respect to students' beginning to develop a structural eye: It concerns the role of the tasks and the calculator. In Radford's

(2012) study with younger children, the teacher played a key role in helping them come to see the spatial structure of the pattern and to coordinate this with the numerical structure. In our study, the teachers played much more of an observational role. On the other hand, the inherent challenge of the “Five Steps to Zero” problem, as well as the wording of the task questions and the actual numbers used (Fig. 4.4), is likely to have contributed to encouraging the students to think more deeply about the structures that combine multiplication and division. Furthermore, had it not been for the presence of the calculator, the tasks that led to the development of students’ structurally-based techniques, although doable, would surely have been less feasible. The calculators with their multiline screens permitted students to analyze successive results for possible indications of numerical structure. The classroom view-screen also enabled the sharing of newly discovered techniques. In sum, the nature of the tasks, the technological tool, students’ collaborative work in pairs and in teams, as well as their own personal determination to find satisfying techniques to meet the challenges of the “Five Steps to Zero” problem, are considered to have all contributed to constituting an emergent, culturally-shared activity that underpinned the evolution of their “structural eye” and shaped the movement of their structural growth. But for such movement to further develop into what could be referred to as persistent structure-oriented practice in mathematical activity would require, in our opinion, “the mathematical work of the teacher in pressing students, provoking, supporting, pointing, and attending with care” (Bass and Ball 2003, p. vii).

4.4 Concluding Remarks

The aim of this chapter has been to instigate greater attention to *structure* and to elaborate more broadly on its meaning with respect to number and numerical operations in the development of early algebraic thinking. As characterized by Freudenthal (1983, 1991), *structure* encompasses the whole web of relations associated with the order, addition, and multiplication structures. These basic structures provide the foundation for multiple additional means of structuring. Furthermore, countless properties, in addition to the oft-cited basic properties of arithmetic, are generated by these structures. But, early algebra involves more than number and numerical operations; it also involves patterning activity and the development of functional thinking. Thus, other structures enter into play such as the spatial structures of figural patterns and various functional structures, namely linear, quadratic, and exponential. Nevertheless, a common aspect of all of these various approaches to early algebraic activity is the multitude of properties and means of structuration that are related to number and numerical operations.

To contribute to illustrating some of the ways in which we might elaborate more broadly on the meaning of structure within number and numerical operations and extend the discussion of the structural properties associated with arithmetical activity, a study (Kieran and Guzmán 2005) on the “Five Steps to Zero” problem was presented. It involved three classes of 12-year-olds who were observed as they generated multiple structural decompositions of the numbers they were given in their problem-solving activity. While students' structural decompositions of the given numbers were not unique, they all displayed an order structure, additive structure, multiplicative structure, and a structure combining the inverse relations between multiplication and division within the *division algorithm*; but their technical approaches also evolved to express structurings according to the related elements of factors, divisors, multiples, and remainder on dividing.

Structural properties that were explicitly indicated in the students' work included the following:

- If a number has both a and b as divisors, then it is also divisible by $a \times b$.
- When two adjacent multiples of a number n are divided by n, then the two quotients that are obtained are consecutive (e.g., "738 and 729 are two adjacent multiples of 9; when they are both divided by 9, the quotients are the consecutive numbers 82 and 81").
- Within every interval of n numbers, there is exactly one number divisible by n (e.g., "In the 9-number interval from 735 to 743 inclusive, there is exactly one number divisible by 9").
- If adding n to a number x yields a multiple of m, then so too will subtracting $m - n$ (e.g., "If adding 1 to 989 yields a multiple of 9, then so too will subtracting 8, due to the resulting difference of 9 between the two adjusted numbers 990 and 981").

Students' early solving strategies evolved in ways that entailed a genesis from the use of trial and error to more deliberate structuring according to the network of relations between multiplication and division. That the evolution occurred speaks powerfully for the initial use of trial-and-error methods that sparked the rise of stronger and more controlled techniques. The details that were provided of the ways in which students attempted to seek, use, and express alternative structures within the "Five Steps to Zero" problem contribute to better understanding how students of this age can come to extract certain multiplication/division structures within numerical activity.

The problem situation itself was pivotal to students' structural growth. But the "Five Steps to Zero" problem should not be considered only in the form in which it was used within this research study. This problem with the challenge of arriving at zero in a restricted number of steps has a generic quality to it. It could easily be adapted for younger students by changing the range of numbers, the range of operations, the number of steps, and even the permissible numbers to be used within the operations, such as even or odd numbers only, or just multiples of, say, 3, and so on. Researchers and practitioners could create variants of this problem, perhaps even for use without a calculator, as a means of developing students' awareness of structure for number and numerical operations at various grade levels in elementary school.

We consider the structuring activity described in this chapter to be a fundamental path to developing students' early algebraic thinking. As mentioned in the introductory remarks, high school algebra requires the ability to see structure within generalized forms. Algebra researchers have argued that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. They have thus offered several recommendations as to how instruction in arithmetic might be designed to foster the development of structure sense. A central suggestion has been that of providing experience with decomposition and recomposition of numerical expressions and with their structural equivalence. The activity involving the "Five Steps to Zero" problem has included exactly this type of experience—experience in ways of thinking that will be of value for the later structuring demands to be made in secondary school algebra (e.g., see Guzmán et al. 2010, for algebra students' work involving the structural relation that links factorizability, polynomial division with/without remainder, and cancelling terms in the simplification of expressions such as $(4x + 4y)/(x + y)$ and $(3x + 4y)/(x + y)$). In sum, we concur with Subramaniam and Banerjee (2011, p. 101), when they state that: "Numerical expressions emerge as a domain for reasoning and for developing an understanding of the structure of symbolic representation." More specifically, we contend that developing an understanding of the structure of number and numerical operations by means of various property-based, structural decompositions is vital to the emergence of early algebraic thinking. Indeed, to conclude we would argue that there is a dual face to activity that promotes early algebraic thinking: one face looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards "seeing through mathematical objects" and drawing out relevant structural decompositions.

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