Chapter 12
Making Implicit Algebraic Thinking Explicit: Exploiting National Characteristics of German Approaches

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Abstract German mathematics teaching-units in primary school lack explicit algebra learning environments. Then again, many national characteristics of teachers' attitudes and beliefs, everyday school life in mathematics classes, and deep-seated approaches that expect children to communicate and argue about mathematical findings, provide favorable prerequisites for algebra. Moreover, the contents taught have the potential to address algebraic thinking if approached from a new perspective. Yet, teachers and children are mostly unaware of the algebraic potential of certain tasks. This chapter includes three studies with a special explicit focus on possible key ideas, children's abilities, and challenges offered by tasks. These evaluated ideas illustrate in interweaving perspectives feasible approaches that enable teachers to integrate algebraic thinking into their classroom culture. Moreover, the implicitly given opportunities revealed by the special focus of each study are hoped to lead to a sensible acceptance of algebraic thinking in primary math classes and its curriculum.

Keywords Awareness · Generalization · Properties · Relational thinking
Key ideas · Patterns · Structures

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12.1 Introduction

Algebraic thinking is an important branch of mathematics from the start (Brownell et al. 2014). Working on numbers and operations is twofold (Müller and Wittmann 1984). On the one hand, numbers are regarded as digits or strings in a place value system. This allows carrying out algorithms and calculating discrete solutions. On the other hand, numbers and operations form an algebraic structure with special properties. This allows thinking about patterns, terms, and equations as special objects (e.g., Kieran 1981; Sfard 1991; Tall et al. 2001). Both of these perspectives on numbers and operations are crucial for a substantial mathematical education from the very start. Yet, the latter is almost neglected in daily school life in Germany. The authors of this chapter wonder why the various implicit possibilities for implementing and supporting algebra in primary mathematics are largely unknown to teachers and not taken up in textbooks and syllabi.

In the next section three different aspects are identified and described which may pave the way for a sensible and deliberate teaching and learning of algebra in German primary schools. This analysis of the special situation in Germany therefore frames the offered approaches. In particular, the issues and opportunities raised by the national characteristics are focused on. Afterwards in Sects. 12.3–12.5, the interwoven perspectives underpinning three research studies, and their respective theoretical frames, methods, and results, are outlined in detail. The topics of these studies are at the core of recent research that Kieran et al. (2016) identify as “a focus on mathematical relations, patterns, and arithmetical structures” (p. 10). The perspectives evince optional ways of implementing algebra and algebraic thinking in daily school life.

12.2 Issues and Opportunities for Supporting Early Algebra in German Primary Mathematics

The situation of German primary mathematics concerning early algebra is ambivalent. On the one hand, algebraic topics have no tradition and still no explicit place in German primary school curricula and teaching-units, unlike in some other countries (e.g., NCTM 2000). German primary curricula and standards mention algebra in a very limited way, if at all. Also there are only a few national research studies on algebraic thinking in the lower grades (e.g., Akinwunmi 2012; Gerhard 2013; Lenz 2016; Nührenbörger and Schwarzkopf 2016) and the ‘Early Algebra’ movement has not (yet) spread in Germany. At the same time, we also face the same didactical problems concerning this topic that are mentioned internationally (Malle 1993), for example, weak conceptions of variables (Franke and Wynands 1991; Specht 2009). On the other hand, when we take a closer look at the German primary mathematics classroom, we can identify many characteristics that offer opportunities to support algebraic thinking.

Astonishingly, primary classroom interaction and teachers' attitudes towards teaching and learning mathematics are very coherent throughout the country in spite of the fact that there are different curricula in the federal states, but all based on a common national standard (KMK 2004). Primary maths in German classrooms is very up-to-date concerning many fruitful teaching and learning principles (Krauthausen and Scherer 2007; Radatz et al. 1996; Schütte 2008; Steinweg 2014a). At least regarding the following three themes, a common ground can be identified.

- *Teacher attitudes and practices*: Teaching is no longer understood as passing on knowledge but as supporting individual and constructive life-long-processes. For instance, it is common to support children's individual solving strategies in terms of framing learning as discovery (Winter 1991). This goes hand-in-hand with seeing each child as an individual being with individual needs, abilities, and experiences (Bauersfeld 1983). With this in mind teachers also take into account prior knowledge, and misconceptions or mistakes are treated in classroom interaction as learning opportunities. Individuals' reactions and solutions to tasks are valued and integrated into the classroom discussion and interaction (e.g., Gallin 2012; Kühnel 1916/1966; Selter 1998).
- *Tasks and problem posing*: The teaching units offered are mainly substantial learning environments (Wittmann 1998), closely linked to mathematics as the science of patterns. Tasks are embedded in learning environments and are therefore related to each other. So-called operative variations (Wittmann 1985) of arithmetical tasks build up patterns and offer opportunities to discover mathematical relations among them. Most German textbooks and worksheets offer tasks with mathematically sound patterns to be spotted and to be described.
- *Expectations on learners*: The expected reactions in classroom interaction differ from just giving numerical answers. The German national standards (KMK 2004) expect children of all ages to communicate and to argue mathematically. This includes commenting on solutions, describing one's own solving processes, detecting patterns, defending different approaches, explaining certain patterns, and so forth. Primary teachers are very much aware of the importance of these so-called process competencies and try to support them in the classroom (Walther et al. 2008).

In summary, German norms around daily classroom interaction and common beliefs about teaching and learning reveal bright opportunities for early algebra. Yet, the algebraic potential of patterns and structures is fairly unknown to both teachers and children. Algebraic thinking is mostly understood as a content of secondary school, being very abstract and possessing no links to primary maths. This might originate from a lack of knowledge about the nature of algebraic thinking as well as a traditional view of algebra. Kaput et al. (2008b, p. xviii) call this belief the "algebra-as-we-were-taught-it, [which] follows arithmetic-as-we-were-taught-it." In an already overfilled curriculum it is understandable that primary school teachers might have some reservations about this alleged new additional content. As a consequence, teachers do not promote algebraic thinking explicitly in primary maths class.

Because of this situation, we believe that it is of high importance to promote researchers' and teachers' awareness concerning algebraic thinking and to make the implicit algebraic thinking that is already present in German classrooms explicit. The objective is to encourage teachers to integrate algebraic thinking into their classroom. Moreover, the aim is to enable teachers to become aware of their already addressing algebraic thinking in their maths class. This might finally lead to an acceptance of algebraic thinking in the primary math class and its curriculum.

In this chapter we present three different research studies, which build on three perspectives regarding the integration of algebraic thinking in primary maths classrooms. Each study, which is grounded in the national characteristics of German primary mathematics teaching, has a separate focus:

1. A focus on topics and contents of primary school mathematics that contain opportunities for promoting algebraic thinking. Section 12.3 clarifies the nature of algebraic thinking by describing four essential key ideas.
2. A focus on the algebraic competencies of primary school children. Section 12.4 provides insight into the potential of children's generalization processes with respect to the development of algebraic thinking.
3. A focus on the design of tasks and problem posing. Section 12.5 describes the challenges of task design and how tasks might promote algebraic thinking even in very young children.

12.3 Study I: All Eyes on Key Ideas

One perspective concerning the integration of algebraic thinking in the primary maths classroom takes as its starting point the mathematical topics and contents taught in primary school. In spite of the fact that algebra is not mentioned explicitly in curricula and syllabi, many topics are related to algebraic ideas and content fields. From this perspective, algebra is not a new content to add but a content field to be identified within already taught topics.

12.3.1 Theoretical Framework

The theoretical framework of this study includes two different aspects: an analytical and a constructive part. The analysis given in the first subsection tries to differentiate between the terms *pattern* and *structure* from a mathematical point of view. The constructive part, which is presented in the second subsection, suggests an option for categorizing topics into key ideas.

12.3.1.1 Patterns and Structures

Often mathematics itself is described as the science of patterns (Devlin 1997). In this view, all mathematical theories arise from patterns spotted. Even axioms characterize patterns to build on. Not surprisingly, teaching and learning about patterns and structures is no special topic but fundamental for all mathematics lessons. “Mathematics ‘makes sense’ because its patterns allow us to generalize our understanding from one situation to another” (Brownell et al. 2014, p. 84).

Becoming aware of patterns allows us to see sense in mathematics and to appreciate its beauty. This awareness is at least twofold. On the one hand, seeking patterns can be classified as meta-cognitive; on the other hand, there is a cognitive component of awareness that is characterized by “knowledge of structure” (Mulligan and Mitchelmore 2009, p. 38).

Patterns can be described as “any predictable regularity, usually involving numerical, spatial or logical relationships” (Mulligan and Mitchelmore 2009, p. 34). Constructing a pattern of numbers or shapes by making up a rule or a certain operative variation (Wittmann 1985) of a given number or task is a very creative process. If, for instance, the pattern of a number sequence is creatively made up, the regularity then is fixed and can be used, continued, and described (Steinweg 2001).

In this research study *structure* is understood as mathematical structure and not as a category system to describe the individual pattern awareness of children (on different uses of the term structure c.f. Rivera 2013; also Kieran in this volume). Mason et al. (2009) recommend “to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced” (p. 10). They point out the difference between the spotting of (singular) relations and the use of the given example as such for the general structure with certain properties:

> Recognising a relationship amongst two or more objects is not in itself structural or relational thinking, which, for us, involves making use of relationships as instantiations of properties. Awareness of the use of properties lies at the core of structural thinking. We define structural thinking as a disposition to use, explicate and connect these properties in one’s mathematical thinking. (Mason et al. 2009, pp. 10–11)

Hence, detecting structures, in contrast to patterns, requires mathematical knowledge about objects and operations. The relation between mathematical objects is essentially determined by mathematical structures (Wittmann and Müller 2007). Awareness of structures often suffers from the fact that structures are mentioned only briefly and only formulated in ‘rules’ in mathematics lessons. Unfortunately, these condensed statements are not an appropriate tool to become aware of the logical structures and properties of mathematical objects and relations, which are fundamental for mathematics.

Sufficient knowledge of mathematical structures is crucial for both teachers and children. Only well trained teachers are able to understand the mathematical structures and to make them accessible for children (Chick and Harris 2007; Devlin

1997). One approach for obtaining access to mathematical structures lies in explicit learning environments that enable children to explore, use, describe, and even prove patterns originating from underlying structures (Steinweg 2014b).

12.3.1.2 Algebraic Key Ideas

The main issue, worked on in this study, is to become aware of and to appreciate algebraic topics in primary class interaction. Hence, the most important question is which mathematical ideas are key, when it comes to algebraic thinking. The international research discourses provides several possibilities concerning the framing of algebra in primary school mathematics, for example, NCTM (2000). Besides standards and curricula various research projects outline different approaches or major ideas of algebraic thinking. Kaput (2008, p. 11), for example, identifies three strands of algebra, which are generalized arithmetic, functional thinking, and the application of modeling languages.

In the German context outlined above further detailing of algebraic content has not yet occurred. The initial step has to build on the existing terms used in syllabi and standards in order to receive broad acceptance and to make an impact on daily school lessons and mathematics textbooks. This possible link is the content area ‘patterns and structures’, which is given in the national standards (KMK 2004). Mathematics in primary school offers various opportunities to become aware of algebra as a mathematical background, that is, mathematical structures. In order to encourage sensitivity to important learning opportunities, the common topics are re-structured according to key ideas of algebraic thinking (Steinweg 2017).

(1) Patterns (& Structures)
(2) Property Structures
(3) Equivalence Structures
(4) Functional Structures

The first idea is briefly described above (also see Sect. 12.4). The second lies in the properties of numbers (e.g., parity, divisibility) and operations (e.g., commutativity, associativity, and distributivity). Examples of this key idea are presented below. The third key idea holds learning opportunities in evaluating, preserving, or construing equivalence in given correct or incorrect equations by assessing terms, and so on. The main issue here is to overcome the urge to solve equations but to focus on the relation of given numbers, sums, differences, products, or quotients (Steinweg 2006).

> Inviting children to find ‘quick ways’ to do arithmetic calculations such as adding the same to both numbers to reach an easier calculation (47–38 = 49–40) and the many variants, can be an entry into appreciating structure. (Mason et al. 2009, p. 14)

The last key idea involves learning environments on functional structures, (i.e., mainly proportional) relations, and co-variation aspects, for example, ‘number and partner number’ (Steinweg 2003)—also used in the study described in Sect. 12.4.

12.3.2 Methodology

The research design follows a constructive approach against the background of mathematics education as a design science (Wittmann 1995). In the research project (Steinweg 2013) learning environments that are suitable for the key ideas outlined above are designed and evaluated. Learning environments provide—besides some implementation ideas—in particular, tangible examples of common tasks in order to uncover their algebraic potential. Each learning environment includes various tasks in a booklet to be handed out to the children along with further mathematical background and educational information for teachers in a teacher's guide. The teachers participated in an introductory meeting in which the tasks and possible teaching arrangements—given in the guidelines—were discussed. They committed themselves to implement all of the 'extra' tasks in the booklet among the usual textbook tasks in daily classroom work over a period of 10 months. The frequency, intensity, and depth of the use of the learning environments were to be decided freely by the teachers. There was no specific focus on the child-teacher-interaction while working on the tasks—with the exception of some mathematics lessons randomly visited by the researcher. The research therefore focused on the question:

> *Does the implementation of 'new' tasks structured by key ideas* via *learning environments show any effects on children's algebraic competencies?*

Six German primary school classes with 144 children from 2nd to 4th grade (on average 7- to 9-year-olds) participated in the project. Additionally, two children per class took part in video-recorded interviews throughout the project period.

In the results presented here, we focus on distributivity as one element of the key idea 'property structures'. The main challenge is to see equations and expressions in a meta-perspective way. For instance, in the expression $2 \times 8 + 5 \times 8$ children have to spot the specific 'internal semantic' (Kieran 2006, p. 32). Only if the common factor is identified as an important component in the products can the 'variable' factors be summed up. For a start the two products have to be regarded as objects in a sum and then the two different factors can be added to create a new product (7×8), which is equal to the sum of two products. Of course, it is always possible to take a procedural perspective and to calculate expressions to determine the specific result (product, sum, etc.). This arithmetical perspective is very much supported in primary mathematics. The change in perception of expressions and equations is therefore crucial and challenging.

12.3.3 Research Results on the Example of Distributivity

Tasks can be designed in such a way as to take advantage of the natural urge to calculate (Fig. 12.1); for example, summing up multiplication table results yields a new sequence that can be identified in the 3rd line of the table as consisting of the

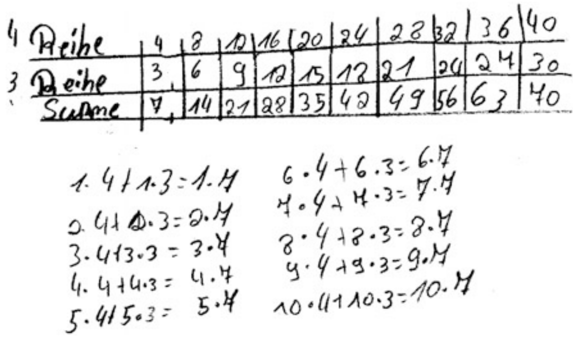


Fig. 12.1 Exploring distributive structures in symbolic representations

sum of the addends (cf. table with addends and sums in Fig. 12.1). This may, at first, be a surprising result for the children. If other examples are tested and in a next step the addends are rediscovered as products, as given in the lower part of Fig. 12.1, the underlying general idea can become more and more clear.

Besides tasks in symbolic representations, rectangle areas as a representation of multiplication (length by width) are used as well in the tasks of the given booklet. Such rectangles are provided by the teachers as representations on worksheets or 'actively' made up by the children by cutting out sections of grid paper. If rectangles are accepted as multiplication representations, manipulating these rectangles by cutting and re-interpreting the two part-rectangles as multiplications can be the next step to explore and understand distributivity (Fig. 12.2).

As the main research question aims to evaluate the effects of the implementation of the learning environments, results of a pre- and post-test are of interest. The results of the test item $10 \times 5 - 4 \times 5 = \_\_\_ \times \_\_\_$ (corresponding to distributivity) are herein documented by way of example (see Table 12.1).

Most likely, the children participating in the project had already experienced derive-and-combine-strategies for solving multiplication tasks in class. This approach to the multiplication tables, which is used in German mathematics in primary school, is somewhat peculiar. There is no longer 'doing tables,' but working on core tasks (e.g., doubles, times 5, times 10) and the use of derive-and-combine-strategies to solve other multiplications. Only core tasks

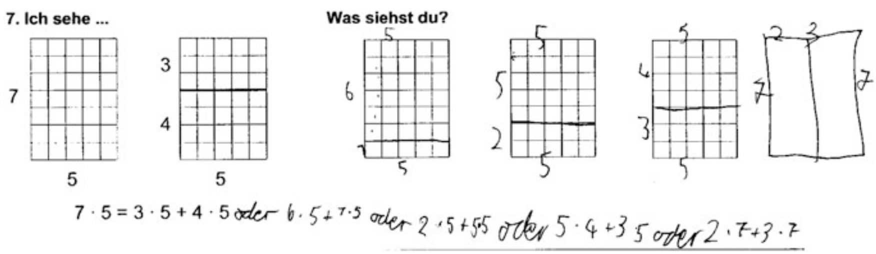


Fig. 12.2 Exploring distributive structures by interpreting rectangles as multiplications

Table 12.1 Results for solving $10 \times 5 - 4 \times 5 = \_\_\_ \times \_\_\_$

Category	Pre-test (n = 135) (%)	Post-test (n = 133) (%)
'Proceptual'	1.5	32
Procedural	32.5	35
No answer given	66	33

should be known by heart as facts (sometimes known as 'helping facts' in the Anglo-Saxon literature). For example, in order to solve 7×8 the children are encouraged to combine the known facts 2×8 and 5×8. This combination is possible because of distributivity. Even so, the task item was found to be quite hard to handle for the participating children in the pre-test (Table 12.1).

Prior to the project two-thirds of the children had no idea what to fill in the blanks. Only in very few cases were children able to combine the two given multiplications referred to in Table 12.1 into 6×5 and thereby make use of the structure (what we are naming the 'proceptual' or algebraic perspective). After participating in the project, one-third of the children were able to give this answer. Another third of them responded with a result such as 3×10, which is fitting because of the equivalent result 30 (the "procedural" or arithmetical perspective). Despite the fact that these results are still far from being satisfactory, the increase in the numbers of children using an algebraic perspective is considerable.

12.3.4 Discussion

The project gives an initial indication that it is possible to foster algebraic thinking by providing sound learning environments. The challenges offered to the children support effects on understanding and increased performance on algebraic tasks. Yet, the impact of learning environments alone is not enough to support all children. Teachers' instructions and interaction in classroom discussions as well as the specific role of representations have to be focused on in further studies.

As mentioned above, the project provided no binding specifications to teachers regarding how to focus on distributivity, but offered different opportunities to explore this mathematical structure via the 'new' tasks in learning environments. As a "good balance between skill and insight, between acting and thinking, is […] crucial" (Drijvers et al. 2011, p. 22), further effort should focus on exploring the differences between procedural and structural/conceptual work on tasks.

The developed *key ideas* may function as *bridges and guiding principles* between arithmetical and algebraic topics. If common arithmetical strategies—like derive-and-combine—are seen from a different angle, they actually are algebraic. This has to be made more explicit to both teachers and children. From a meta-perspective view the procedures performed are determined by mathematical structure and the properties of operations, that is, by algebra. Last but not least, this 'new' perspective and awareness implies "better understanding of rules and procedures" (Banerjee and Subramaniam 2012, p. 364).

12.4 Study II: All Eyes on Children's Algebraic Thinking

In this section, we explore how to exploit the potential of German primary mathematics classroom culture by making explicit the algebraic character of children's daily mathematical communication and reasoning. We believe that algebraic thinking is already taking place in the present maths lessons implicitly due to the national characteristics described earlier. It is then necessary to clarify the nature of algebraic thinking and to support its recognition in students' actions and communications. We illustrate this by focusing on the *generalization of patterns*—one of the most important parts of algebraic thinking (Kieran et al. 2016).

12.4.1 Theoretical Framework

"Patterns and Structures" are fundamental content in German maths classes—starting from the primary school level or even earlier. To discover, to describe, and to reason about patterns are essential activities according to the national primary mathematics standards (KMK 2004). Working on patterns and structures holds great opportunities for algebraic thinking as it can evoke children's generalization processes, which are considered essential for algebraic thinking (Kaput 2008; Mason et al. 2005). Unfortunately, teachers are mostly unaware that such opportunities exist.

Generalizing mathematical patterns is one of the main approaches to algebra and also to the introduction of variables. Mason and Pimm (1984) describe *generalizing* as "seeing the general in the particular." The concepts of variables as *general numbers* (or indeterminates, see Freudenthal 1973) and as *varying numbers* (variables in functional relations, see Freudenthal 1983) are powerful tools for generalization. Thus, the use of variables enables students to communicate, to reason, to explore, and to solve problems on a general level (Malle 1993). Variables can therefore be introduced as meaningful and necessary signs in the context of generalization.

Generalizing is an important part of any mathematics classroom in which the focus is laid on patterns and structures—thus also in German primary maths class. Patterns and structures have to be constructed actively by the learners by interpreting the given mathematical signs (Steinbring 2005). In order to achieve this, the students' challenge is to see something general in the particular (Steinbring 2005). Whenever learners communicate about mathematics, including when they talk about regularities, structures, and relations, they find inevitable the need to generalize. But before they are introduced to algebraic symbols and conventional signs for their generalizations, they face the problem of trying to say something general without having the necessary tools, such as variables. They are thus compelled to find their own fitting signs that can represent their explored mathematical patterns and structures. In the last decade, research has focused on the competencies of young children in the field of the emergence of algebraic thinking. Studies have revealed promising findings. Young learners are able to generalize and reason about patterns, number relations, and arithmetic laws (e.g., Bastable and Schifter 2008; Cooper and Warren 2011; Schliemann et al. 2007). Radford (2003) claims the importance of natural language as well as non-symbolic forms of generalization (e.g., by means of gestures).

In order to exploit the algebraic character of primary school students' communication of patterns and structures, a study was conducted to explore their individual generalization processes. The study focused on the following research questions:

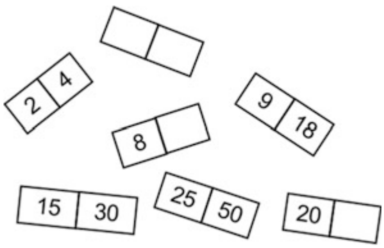
> *How and with which linguistic resources do primary school students generalize mathematical patterns? How do students develop variable concepts by generalizing mathematical patterns?*

12.4.2 Methodology

The presented interview study (Akinwunmi 2012) investigated the generalizing processes of primary school children. Thirty participating fourth graders (approximately 9–10 years) were engaged in three different task formats that included tasks that are known to focus on the exploration of patterns:

(1) "Think of a number" (e.g., Mason et al. 1985; Sawyer 1964): Children explore and explain the structure that lies in the following task. "Think of a number. Add 4. Add 8. Subtract the number you thought of. Subtract 2. The result is 10."
(2) "partner numbers" (Steinweg 2003): Children explore and describe the relationship of pairs of numbers (Fig. 12.3) and fill in some missing values.
(3) "growing patterns" (e.g., Orton 1999, see example on the next page).

Fig. 12.3 Example for "partner numbers"



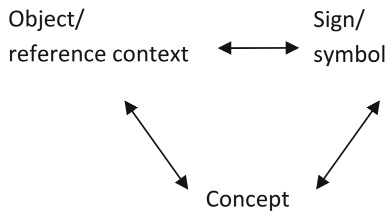


Fig. 12.4 The epistemological triangle (Steinbring 2005, p. 22)

The interviews focused on individuals' oral and written descriptions and explanations of patterns. The students were chosen from three different schools and included a heterogeneous range of achievements in mathematics according to their teachers. All interviews, each of an approximate duration of 45 min, were conducted by one of the authors; they were videotaped and transcribed.

The data were analyzed by a group of researchers by means of the epistemological triangle (Fig. 12.4) based on Steinbring's (2005) theory of the construction of new mathematical knowledge in classroom interaction.

The epistemological triangle can be used to reconstruct the referential mediation between mathematical signs/symbols and the reference contexts that serve for the interpretation of the signs. Steinbring (2005) describes the interdependence among the three entities by explaining that, "the referential mediation is steered by conceptual mathematical knowledge and at the same time, conceptual mathematical knowledge emerges in the referential mediation" (p. 179).

The study presented here reconstructs the development of variable concepts by observing the construction of knowledge by means of new referential mediations between mathematical signs and reference contexts.

12.4.3 Research Results

The presentation of the results is divided into two subsections. The first subsection (12.4.3.1) gives an insight into the analysis of an exemplary generalization process from an epistemological oriented perspective by revealing students' development of the variable concept. The second subsection (12.4.3.2) gives an overview of the children's forms of generalization and presents their linguistic tools from a semiotic perspective.

12.4.3.1 Epistemologically-Oriented Analysis of the Generalization Process

The epistemologically-oriented analysis of children's individual generalization processes is illustrated below with reference to an interview sequence with Lars, a student who worked on the growing pattern of "The L-Numbers" (Fig. 12.5).

First, Lars was asked to continue the pattern and to calculate the required number of squares for L1 to L10, L20, L100, and for other figures. His calculations showed that he split the figures into two sections: the vertical squares (including the linking square), which total one more than the term number of the sequence, and the horizontal squares which equal the term number. Using variables, his strategy could be described by the explicit formula $(n + 1) + n$. When requested to explain his strategy for calculating the required number of squares for any term of the sequence, his first statement involved citing an example: "I calculate, for example, 5 + 4. That's how I get the result." When asked to write down a description of his strategy, he drew a figure (Fig. 12.6) and explained it as illustrated below.

Lars' description of 'The L-Numbers':

Lars:	So this is five *(points to the five vertical squares)* and this is five (*points to the five horizontal squares including the linking square in the left corner*). So and ehm (*adds the two arrows and the plus sign to his figure*).
Interviewer:	Great. Now can you explain to me in detail, what exactly you mean (*points to the arrows*)?
Lars:	This downwards (*moves his pen alongside the vertical squares*) plus this (*moves his pen alongside the horizontal squares*), I calculate.
Interviewer:	Ah, ok. Good.

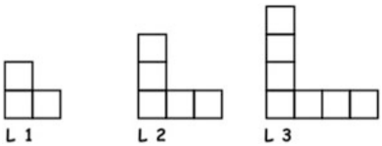


Fig. 12.5 Growing pattern of "The L-numbers"

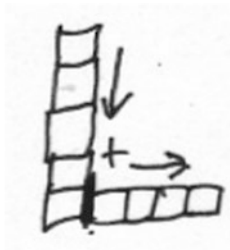


Fig. 12.6 Lars' description of the pattern

Lars: And ehm, this here (*points to the linking square in the corner of the 'L'*) belongs to the downwards. That's why I put a thicker line there (*retraces the line between the vertical and the horizontal squares*).

In this interview scene Lars used a drawing of the fourth figure of the sequence to describe the structure that he saw in the growing pattern. To this concrete figure he added a vertical and a horizontal arrow and a plus sign. With his explanation of the arrows "this downwards plus this, I calculate," he pointed out that these expressions represent the summands of the addition of the two parts of the 'L'-figure. The words "downwards" and "this" can therefore be construed as word variables as they referred to the varying number of squares in the two parts of Lars pattern. They enable Lars to describe the general structure of the 'L-numbers' beyond his first example, 5 + 4. He took these signs from a geometrical context as they initially indicated a direction within the given figure. In this new context they now referred to the varying amount of needed squares for one part of the figure. Thus Lars constructed a new referential mediation between the used words and arrows and the general structure of "The L-Numbers" (Fig. 12.7).

It is this mediation that is characteristic of the concept of variable as a general or varying number in the way that it includes the semiotic nature of the relation between one unifying symbolic object and its referring to multiple instances. Creating that symbolic object lies at the very heart of generalization (Kaput et al. 2008a, p. 20). It is important to note that this object is not necessarily a symbolic variable in the form of a letter. A 'variable' here appears in the form of words, signs, or symbols. We agree with Radford (2011, p. 311) that algebra can be considered as a "particular way of thinking that, instead of being characterized by alphanumeric signs, is rather characterized by the specific manner in which it attends to the objects of discourse." Therefore, we can say that the process of generalizing mathematical patterns is fostering students' concept of variable by naturally establishing this kind of mediation.

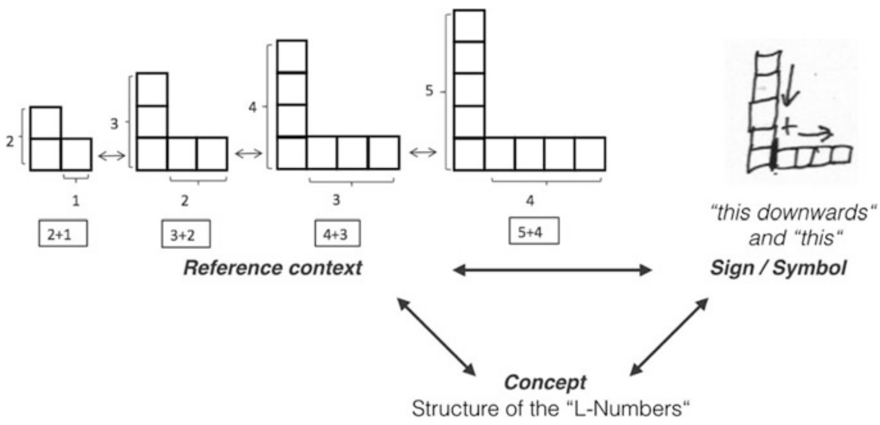


Fig. 12.7 Interpreting Lars' drawing and his explanation with the epistemological triangle

12.4.3.2 Children's Linguistic Forms of Generalization

Across the study's different task formats and the various individual reactions, different linguistic forms of generalization could be identified that served the students as tools for generalization (Table 12.2).

We found that children mixed and combined the above forms even within one description. Although the first four forms of generalizing are limited in terms of creating generally valid statements because the description does not apply to all objects in the pattern, they nevertheless illustrate the general character of the pattern. As such they also present possibilities for learners to express statements that can be understood as "general" in the classroom discussion.

12.4.4 Discussion

The analysis of the interviews of which we could just present a brief insight above shows that when asked to explore and especially requested to describe mathematical patterns and structures, children felt the necessity to generalize in order to be able to communicate about their discoveries in the way that Lars did. The individual use of signs or symbols with variable character originated from the motivation to refer to a mathematical structure in general and to describe it beyond the visible objects of the pattern. Learners spontaneously used signs or symbols drawn from other contexts. In the new context of generalization, they served as variables (as Lars' adverb of direction "downwards", the deictic expression "this," as well as the arrows in the described interview scene). In the individual process of generalizing, children constructed new referential mediations between the used signs and symbols and the general structure. Thus, this mediation shaped the concept of variable. The linguistic forms of generalization occurred in the

Table 12.2 Children's linguistic forms of generalization

Forms of generalization	Description of the category	Illustration by means of the term x^2
Stating one example	Students use one example and explicitly indicate this as an example	"For example it's three times three."
Listing several examples	Students list several examples and sometimes refer to a continuation	"It's one times one, two times two, three times three and so on."
Quasi-variables	Students use concrete numbers combined with a generalizing expression	"I always calculate three times three."
Conditional sentences	Students phrase conditional sentences	"If it is three, then I calculate three times three."
Variables	Students use words or signs with variable character	"You have to calculate the number times the same number." "?·?"

interaction while working on mathematical patterns and structures and took on the role of variables in the context of generalizations. They enabled the learners to describe mathematical patterns and structures in general and therefore served for the propaedeutic development of variables as general or as varying numbers.

We believe that it is important for teachers to be able to identify and support students' attempts to generalize as they themselves are a key to children's algebraic thinking and to the development of the variable concept. Teachers have to understand that these generalization processes take place in primary math class when students communicate about mathematical patterns and structures. The linguistic differentiation among forms of generalization presented above can help to open teachers' eyes and ears to generalization processes that occur in classroom communication and therefore aid in nurturing the awareness of algebraic thinking.

12.5 Study III: All Eyes on Tasks

This section proposes to expand the scope of common tasks used in maths lessons. Dealing with variables and establishing relationships as important aspects of algebraic thinking were addressed by a task design appropriate for children from 5 years on. In addition to addressing different aspects of variables, the tasks also promoted relational thinking. An interview study tried to find out which relations children describe between known and unknown quantities, represented as marbles and boxes.

12.5.1 Theoretical Framework

Algebra focuses not only on procedures, which are directly operable, but also and very importantly on the concepts that are represented in equations as relations between numbers, objects, or variables (e.g., Steinweg 2013). Relational thinking especially describes this way of thinking and therefore is an important part of algebraic thinking. Relational thinking refers to the recognition and use of relationships among numbers, sets, and relations. It enriches the learning of arithmetic and can be a foundation for smoothing the transition to algebra (e.g., Carpenter et al. 2005).

Another important part of algebra and the emergence of algebraic thinking is that of variables. While "variables" can be hard to define, several authors mention different aspects of variables with the aim of clarifying the field. At least, three different kinds of variables can be defined: *Unknowns* describe a specific, but undetermined number, whose value can be evaluated. For instance, in the equation $25 + x = 30$, x can be determined (e.g., Freudenthal 1973; Usiskin 1988). *Variables* describe a range of unspecified values and a relationship between two sets of values (Küchemann 1981). In this way, variables appear in statements about functional relationships. *General numbers* describe indeterminate numbers which appear in generalizations, such as descriptions of properties of a set as in $a + b = b + a$ (e.g., Freudenthal 1973). Some research on children's understanding of variables also emphasizes *quasi-variables*, which by the use of examples expresses a basic structure in general terms (e.g., Fujii and Stephens 2001). Related to this work on variables is the substantial empirical research on relational thinking about numbers and operations in symbolic equations (e.g., Carpenter et al. 2005; Steinweg 2013; Stephens and Wang 2008). We note however that, in some of the tasks used in these latter studies, namely 28 + 32 = 27 + __, students still have the opportunity to calculate and, thus, there may be no need to use relational thinking.

In contrast, Stephens and Wang (2008) used the following task to investigate 6th and 7th-graders' relational thinking:

$$18 + \underset{\text{Box A}}{\square} = 20 + \underset{\text{Box B}}{\square}$$

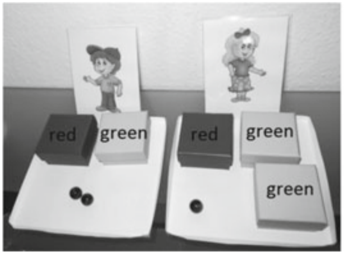
Students had to put numbers in the boxes named A and B to make the sentence correct. Tasks with more than one unknown quantity seemed to have the potential to push students to use and show relational thinking, instead of using computational methods to find a solution. Examples of tasks with more than one unknown and which are represented with concrete objects can also be found in Affolter et al. (2003) and Schliemann et al. (2007). In these studies, boxes containing rods and marbles were used in order to provide access to variables in equation situations. To encourage student's relational thinking, concrete materials would seem to have an advantage, especially for younger students, even as young as kindergarteners. However, studies on relational thinking with concrete material and unknown quantities are few. Therefore the following study addressed the question:

How do young children describe relations between known and unknown quantities that are represented with concrete materials?

12.5.2 Methodology

To find out what competences children already have for dealing with unknown quantities, clinical interviews (e.g., Selter and Spiegel 1997) with 82 children aged 5–10 years (kindergarten and elementary school) were conducted and videotaped. The underlying concept was to "translate" different kinds of equations with variables into a representation that young children could handle. Therefore known and unknown quantities were represented with concrete materials in the form of marbles and boxes. The boxes represented unknown quantities because their content was unknown. To make the task accessible, the following story was told: "Here you see two children. They are playing with marbles. Some marbles are packed up in different colored boxes and some marbles are separate. Boxes with the same color always contain the same amount of marbles."

Task of type B



How many marbles have to be in the green box, so that both children have the same amount of marbles?

Following task of type C

How many marbles have to be in the lime-green box, so that both children have the same amount of marbles?

Fig. 12.8 Tasks and interview questions (Lenz 2016, p. 176) (The labels designating the colors were subsequently added to accommodate black-and-white publishing constraints.)

The main study included 12 tasks on four levels of difficulty (Lenz 2016). In this section of the chapter two tasks are chosen as examples because the transition between tasks of type B and C (shown in Fig. 12.8) was found to be especially interesting. In tasks of type B children can answer with a concrete number. For instance, there is one marble in every green box (see Fig. 12.8). The green box represents an aspect of variables that can be identified as an unknown. The content of the red boxes is unknown and their determination is not necessary for the solution of the task. In contrast, the boxes in task C (see Fig. 12.8) take on another role. They can be seen as variables that represent a functional dependency. Since the amount of marbles in both boxes is unknown, no concrete number can be given. It can only be said that the lime-green box contains one marble more than the other box. This describes the relationship between the two unknown quantities.

12.5.3 Research Results

To gain insight into students' work, the following transcript shows how the 4th-grader Rick (11 years old) dealt with the consecutive tasks of type B and C.

Interviewer: How many marbles have to be in the green box, so that both children have the same amount of marbles? (task B, Fig. 12.8)
Rick: One.
Interviewer: And how did you get that?
Rick: Because…one plus one (*points one after another to the girl's green boxes*) plus this one marble (*points to the girl's marble*) are three. And here (*points to the boy's green box*) is also one marble in, plus

	the two loose marbles. And that's then the same (*points one after another to the girl's red box and the boy's red box*).

Rick gave the correct answer immediately and justified the number of marbles in the green boxes. He also pointed to the two red boxes and named them as "the same" without having to know the number of marbles contained.

Interviewer:	How many marbles have to be in the lime-green box, so that both children have the same amount of marbles? (task C, Fig. 12.8)
Rick:	Two marbles.
Interviewer:	And how did you get that?
Rick:	Because I think there is one marble in (*points to the girl's box*), plus the loose marble are two marbles. Then there (*points to the boy's box)* just can be two marbles, because you have to get the same result.
…	[The interviewer gives different examples of amounts of marbles for the different boxes. Rick gives the corresponding number of marbles of the other box.]
Interviewer:	Can you say in general, how to indicate the number of marbles in the boy's box?
Rick:	You have to, uh, here is any number of marbles inside (*points to the girl's box*) plus the one marble (*points to the girl's single marble*), then there must not be as many as in this box (*points to the boy's box*), but one more in there.

Rick was confronted with a task in which both unknowns depended on each other. In order to give an answer to the interviewer, he mentioned discrete values for both boxes. In response to the interviewer's further questions, he was able to state a general relationship: he described the amount of marbles in the girl's box as "any number," which can be interpreted as a general number.

The responses of the other children covered a broad spectrum. We evaluated their various responses in two ways—according to the nature of the relationship that they expressed between the two quantities and according to the way in which they were handling the unknowns. The categories that were used in the evaluations were partly based on the distinctions described in the theoretical framework above and partly on other distinctions that emerged from the children's responses.

12.5.3.1 A First Evaluation: Relationship Between the Quantities

Regarding the answers to the task of type C, some children directly described a relationship between the two quantities in the boxes, as was the case with the 4th-grader Luca: "In the green box is always one marble more than in the orange box." Other children referred to the dependency between the amounts of marbles in the boxes, as did the 4th-grader Kathy: "It depends on how many marbles are in the orange box." Here, Kathy did not specify the relationship between the amounts of marbles in the boxes, but did have a sense of the dependency. Other children neither described a relation nor referred to the dependency between the amounts of marbles in the boxes. The 2nd-grader Lena (8 years old) mentioned specific numbers for the amounts of marbles in both boxes: “In the green box are three marbles and in the orange box are two marbles.” Other children wanted to shake the boxes to hear how many marbles were inside.

12.5.3.2 A Second Evaluation: Handling the Unknowns

The children’s answers were also classified according to how they treated the unknowns. In some cases, the amounts of marbles in the boxes were seen as *general numbers*, that is, the amount of marbles in one box was considered a generalized indeterminate number in relation to the amount of marbles in the other box. As noted above, Luca said: “In the green box is always one marble more, than in the orange box.” Here the amount of marbles is undetermined; it is always one marble more—no matter how many are actually in it.

In other cases, the amounts of marbles in the boxes were seen as *quasi-variables*: the children recognized the relationship between the amounts of marbles in the two boxes, but rather than stating a general description they mentioned specific numbers. The six-year old kindergartener Adam said: “…if there are eight or nine marbles in the orange box, then I take one marble more, that’s nine or ten marbles for the green box.”

For others, the amounts of marbles in the boxes were seen as *variables* where the amounts of marbles in the boxes depended on each other. As mentioned above, the 4th-grader Kathy explained: “It depends on how many marbles are in the orange box.” Further requests showed that she could handle the variation of numbers as a functional relationship, even if she did not specify it in terms of a static relationship.

The amount of marbles in the boxes was seen by others as an *absolute number* in that they referred to a specific number of marbles in the box, partially without taking the two related boxes into consideration. Clara from kindergarten (6 years old) answered: “Four…because the box is so small, there just fit four marbles in.”

Lastly, the amount of marbles in the boxes was seen as an *undeterminable*: Children said that the amount of marbles could not be defined. Axel (a 2nd grader) said: “I’m not a clairvoyant”; Rob (another 2nd grader) said: “I have to open the box.”

12.5.4 Discussion

The task design shows how algebraic thinking can be built on a concrete level. The boxes as representations for unknowns offer a possibility to get in touch with variables at an early stage. Relational thinking can be stimulated at this early stage by leaving the numerical values ambiguous. The tasks look simple at first glance and are visually very similar. However, they allow the construction from simple to mathematically complex contexts. They are therefore suitable for working from kindergarten to the secondary level and for addressing different aspects of variables while promoting relational thinking at the same time. In particular, the difference between the task types B and C marks a special breaking-point in the use of variables. Their roles change from an unknown that can be determined to a variable whose value cannot be known but can be described as a relation. Hence, tasks of type C strengthen the use of relational thinking since relationships between the sets have to be established. Different approaches to the solution of the tasks can also be made clear by operating on the tangible material (boxes and marbles). For example, in task B, both red boxes can be removed in order to clarify their irrelevance for the solution of the task. In later grades, it is possible to transfer the underlying structures to the formal level. Placeholders, symbols, or letters can replace the real boxes. Thus, with regard to the variables as well as with regard to the establishment of relationships, different changes in the levels of representation can take place.

12.6 Conclusions

This chapter aimed to explore how existing characteristics of German mathematics teaching could serve as opportunities to promote early algebraic thinking. Though a national perspective, it may serve as a framework for many other countries facing comparable issues and obstacles on the way to supporting algebraic thinking. The common aim of our research community is to provide fruitful learning environments and therefore learning opportunities for children regarding algebraic themes.

The above outlined ideas aim to overcome apparent stumbling blocks that cannot be attributed to children but to the given framing of mathematics lessons. Children are very capable of generating sound and viable reactions to algebraic challenges. Hence, we tried to emphasize three evaluated and promising approaches for supporting children's algebraic competencies. The common denominator of the three viewpoints that were presented lies in the existing implicit opportunities that have to be made explicit. This includes creating sensitivity to the algebraic potential of the mathematical content already taught, encountering children's abilities, and paying attention to the nature of the challenges created when designing tasks. If teachers, researchers, and curricula developers are aware of the potential of already daily used tasks (Sect. 12.3), the rich scope of children's abilities (Sect. 12.4), and the great effect of minor changes in problem posing (Sect. 12.5), then children will benefit sufficiently.

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