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Carolyn Kieran *Editor*

Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds

The Global Evolution of an Emerging Field of Research and Practice

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Introduction

Carolyn Kieran

This volume results from the activity of Topic Study Group 10 (TSG 10) on the Teaching and Learning of Early Algebra, which unfolded at the ICME-13 conference held in Hamburg, Germany, in July 2016. In preparation for that activity, a pre-conference monograph presenting a topical survey of Early Algebra research (Kieran et al. 2016) was published by Springer. As described in that monograph,

> The core of recent research in early algebra has been a focus on mathematical relations, patterns, and arithmetical structures, with detailed attention to the reasoning processes used by young students, aged from about 6 to 12 years, as they come to construct these relations, patterns, and structures—processes such as noticing, conjecturing, generalizing, representing, and justifying. Intertwined with the study of the ways in which these processes are engaged in are the two main mathematical content areas of generalized arithmetic (i.e., number/quantity, operations, properties) and functions. (p. 10)

The monograph highlighted how the field of early algebra has gradually come to be more clearly delineated since the early 2000s, bringing with it more comprehensive views and theoretical framings of algebraic thinking. Thus, the contents of the monograph, which aimed to provide an accurate reflection of the evolution of the field, set the stage for TSG 10 contributors to link their newest work with the foundational and more recent advances in this area, as well as to signal its further evolution.

The Hamburg conference was the first of the quadrennial ICME conferences to include within its roster of topic study groups one that was dedicated to the theme of early algebra. The impressive attendance at TSG 10, in addition to the international spread of the authors of papers and posters that were presented and discussed, attested to the worldwide emergence of early algebra as a significant field of research and practice. The diverse range of topics that were covered—several of them little explored, even unexplored, up to now—indicated the ways in which early algebra is being conceptualized in various parts of the world and, very importantly, signaled the demand for this volume.

The authors of promising papers presented at the TSG 10 sessions were invited to extend their papers for inclusion in this volume. Other highly respected scholars in the field of early algebra were also encouraged to write chapters. The result is a set of 17 chapters that represents the work of both experienced and younger researchers (including a chapter with a teacher as coauthor) from 13 countries: Australia, Canada, Germany, Ireland, Italy, Korea, New Zealand, Singapore, South Africa, Spain, Turkey, UK, and USA—this internationalism of the field giving rise to the subtitle of the volume, *The global evolution of an emerging field of research and practice.*

The scope of the volume encompasses multiple aspects related to the development of algebraic thinking at the primary and lower middle grades, stemming from both longitudinal programs of research as well as from single shorter-term studies. Some of the aspects that are highlighted include the following: (i) theoretical perspectives such as the structural, the linguistic, the analytic, and the expression of generality; (ii) the emergence of symbolic algebraic thinking; (iii) children's algebraic thinking within current curricular content and the potential of those curricula to address algebraic thinking; (iv) functional approaches focusing on the use of story problems, patterning, and function-machine tasks; (v) generalized arithmetic approaches involving work with fractions, operations as objects, and equality concepts; and (vi) the development of practicing and preservice teachers' actions to promote algebraic thinking. The reader will perhaps notice special attention to the aspect of *structure*, an aspect that the TSG 10 participants argued could benefit from more emphasis on research and practice related to early algebraic thinking.

The volume is organized into three parts: the first on theoretical perspectives, the second on learning, and the third on teaching. While most of the chapters included herein encompass all three of these themes, the one that was foregrounded determined the part of the volume into which the chapter was included.

The first part (Chaps. 1–5), which is titled *Theoretical perspectives for developing early algebraic thinking,* focuses primarily on the theoretical frameworks that have been developed to underpin several existing longitudinal programs of research on the teaching and learning of early algebraic thinking, but includes also examples of the empirical studies conducted within these same programs. The contributors to the five chapters found in Part I comprise the following:

- Luis Radford, whose longitudinal investigations rest on a characterization of algebraic thinking based on its *analytical* nature and attention to the *semiotic systems* through which students express the mathematical variables involved. He presents the results of his program of research on the emergence of symbolic algebraic thinking in young Canadian students in the context of pattern generalization.
- Maria Blanton, and her team of eight coauthor/coresearchers, Bárbara Brizuela, Ana Stephens, Eric Knuth, Isil Isler, Angela Murphy Gardiner, Rena Stroud, Nicole Fonger, and Despina Stylianou, who describe the framework that guides their work—one encompassing the core aspects of making and expressing generalizations in increasingly formal and conventional symbol systems. They describe elementary-school-aged USA students' algebraic thinking within the

content strands of generalized arithmetic, functional relations, and equivalence and equality.

- Nicolina Malara and Giancarlo Navarra, whose ArAl research program involving practicing teachers in Italy is based on the view that early algebraic thinking involves the progressive construction of the algebraic language and consequently building in students an attitude of looking for regularities, relationships, and properties, and expressing them first in natural, and then in algebraic, language.
- Carolyn Kieran, who argues that the dominant focus on generalizing in the development of algebraic thinking has to a large extent obscured the process of seeing structure. She explores the notion of structure and structural activity from various perspectives, and then presents a research-based example of Mexican students' seeking structure within an activity involving factors, multiples, and divisors.
- David Carraher and Analúcia Schliemann, who identify algebraic thinking with the formulation of and the operation upon relations, particularly functional relations. They illustrate by means of examples drawn from their several research studies conducted in the USA how functions offer opportunities for bringing out the algebraic nature of arithmetic and for supporting the introduction of variables.

The second part of the volume (Chaps. 6–11), which is titled *Learning to think algebraically in primary and lower middle school,* explores the development of algebraic thinking among 5- to 12-year-olds in various countries around the world, with some studies including recourse to the literal symbols of algebra and others not so. The latter studies focus on conceptualizing equality, generalizing fractional operations, and shape patterning; the former on functional story situations for young learners, function-machine tasks, and assessing the algebraic thinking of students taught within the framework of an existing national curriculum. It is noted that the chapters that are found in Parts I and III of this volume also deal with learning to think algebraically and serve to broaden the range of examples that are presented in this second part. The contributors to the six chapters found in Part II comprise the following:

- JeongSuk Pang and JeongWon Kim, who present two studies that examined the algebraic thinking of young Korean students (Grades 2–6) taught from the current mathematics curriculum that includes some content related to early algebra. The Korean students' overall performance was found to compare well with that of students taught within specialized intervention programs of research in other countries.
- Swee Fong Ng, who introduced function-machine tasks to young Singaporean students from each of the primary Grades 1–6. She found that those students who could best make sense of the covariation between the inputs and outputs, and state the relationship between them in the form of a rule, were those who also had a sound knowledge of their number facts.

- Ralph Schwarzkopf, Marcus Nührenbörger, and Carolin Mayer, who offer a perspective on algebraic reasoning and the understanding of arithmetic equalities, without the latter needing to be formally established in the form of equations. Pairs of 10-year-olds (in Germany), who were confronted with tasks that involved computing chains with boxes and arrows, came to reflect on the given structures with arithmetical operations becoming the central algebraic objects in their discussions of the equalities.
- Aisling Twohill, who investigated the strategies that 9- to 10-year-old students attending Irish schools adopted when asked to construct general terms for shape patterns. Catalysts for assisting the students in coming to view the pattern relationships *explicitly* rather than *recursively* included teacher prompts, student interactions, and concrete materials.
- Catherine Pearn and Max Stephens, who focus on how students find an unknown whole, when given a known fractional part of the whole and its equivalent quantity. They show how 11- and 12-year-old Australian students, who have yet to meet formal algebraic notation, create algebraic meaning and syntax through their solutions of these fraction problems.
- Marta Molina, Rebecca Ambrose, and Aurora del Rio, who present findings from a teaching experiment on the initial understandings that primary Spanish students (6- and 8-year-olds) demonstrated when first introduced to the use of letters to stand for indeterminate varying quantities in functional story situations. Based on their results, they recommend the introduction of alphanumeric symbols from the first grade.

The third part of the volume (Chaps. 12–17), which is titled *Teaching for the development of early algebraic thinking,* focuses on the critical role of teachers and the actions they carry out to spur the growth of their students' early algebraic thinking, from prompting structuring and generalizing activity to developing sensitivity to the algebraic potential of the regular mathematical content. Professional development and preservice education with the aim of cultivating teaching practice in early algebra are also featured. The contributors to the six chapters found in Part III comprise the following:

- Anna Steinweg, Kathrin Akinwunmi, and Denise Lenz, who argue that German teachers' cultural–instructional characteristics put them in good stead to address early algebraic thinking if the current curriculum were to be approached from a new perspective—a perspective that focuses on key ideas related to patterns and structures, on children's existing abilities, and on the potential of specific tasks.
- Deborah Schifter, who presents USA classroom episodes that illustrate teachers and students working together on tasks involving generalizations in the contexts of arithmetic and functions. She points to the teachers' actions that draw students' attention to operations as distinct objects and to the structures associated with these operations.

- John Mason, who argues that the critical feature for promoting algebraic thinking is the opportunities noticed by teachers for calling upon learners' powers to express and manipulate generalities. He outlines specific pedagogic actions that focus on the expression of generality as the core of algebraic thinking, including examples of task contexts that invoke reasoning both with and without using numbers.
- Susanne Strachota, Eric Knuth, and Maria Blanton, who analyze the nature of the "algebrafied" instruction in a USA classroom that promotes the generalizing of structural relationships. They found that the manner in which the teacher responded to a student's generalization by asking for clarification and by prompting the other students to build upon the previously stated generalization initiated a cycle of generalizing actions that was extremely productive.
- Jodie Hunter, Glenda Anthony, and David Burghes, who draw on the findings from a classroom-based case study in New Zealand to show how professional development can lead to shifts in teacher practice. Careful task design and enactment, teacher questioning, and noticing and responding to student reasoning were all central to facilitating students to make conjectures, justify, and generalize.
- Sharon Mc Auliffe and Cornelis Vermeulen, who focus on preservice teachers' (PSTs) learning to teach functional thinking within early algebra in South Africa. They point to the challenges the PSTs faced during their practicum in bringing students to generalize relationships, especially generalizing on the basis of argumentation.

The *Concluding* section of the volume highlights cross-cutting issues and compelling ideas presented in the various chapters, as well as offering a few remarks with a view to looking ahead.

Acknowledgements

I wish to express my appreciation to the other members of the TSG 10 organizing team: JeongSuk Pang (cochair), Swee Fong Ng, Deborah Schifter, and Anna Susanne Steinweg, for their unflagging collaboration leading up to the unfolding of the TSG 10 activities at the ICME-13 conference, as well as for their contributions to our post-conference discussions that served to shape the present volume. I also express my gratitude to the authors of this volume for their cooperation in adhering to a rather strict timeline for its preparation.

Reference

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Part I
Theoretical Perspectives for Developing Early Algebraic Thinking

Chapter 1
The Emergence of Symbolic Algebraic Thinking in Primary School

Luis Radford

Abstract This chapter presents the results of a longitudinal investigation on the emergence of symbolic algebraic thinking in young students in the context of sequence generalization. The investigation rests on a characterization of algebraic thinking based on its *analytic* nature and a careful attention to the *semiotic systems* through which students express the mathematical variables involved. Attention to the semiotic systems and their interplay led us to identify non-symbolic and symbolic (alphanumeric) early algebraic generalizations and the students' evolving intelligibility of the variables and their relationships, and mathematical sequence structure. The results shed some light on the transition from non-symbolic to symbolic algebraic thinking in primary school.

Keywords Early algebra · Semiotic systems · Pattern generalization · Algebraic generalizations

1.1 Introduction

Over the years, the teaching and learning of algebra has consistently figured as one of the prominent research areas in mathematics education. Recently, research on early algebra has gained an increasing interest (see, e.g., Ainley 1999; Cai and Knuth 2011; Kaput 1998; Kaput et al. 2008b; Rivera 2010; Vergel 2015). Some of the main initial ideas behind the early algebra movement are to determine: (a) whether or not young students can really start learning algebra (Carraher and Schliemann 2007) and (b) if an early exposure to elementary algebraic concepts can alleviate the very well-known difficulties that adolescents encounter in algebra in secondary education (Blanton et al. 2017). Such ideas run against the traditional curricular conception that algebra can only be learned after the students have a

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sufficient knowledge of arithmetic, which excluded, until recently, algebra from primary school in many curricula around the world.

To merely envision the idea of exposing young students 5–12 years old to algebra instruction requires us, however, to revisit from new theoretical perspectives several key issues frequently discussed in the 1980s, such as the nature of algebraic thinking (Bednarz et al. 1996; Filloy and Rojano 1989; Kieran 1989a; Wagner and Kieran 1989). The efforts that have been made to come to terms with these and other concomitant issues have often led researchers to a sense of awareness that there are still many important things to investigate and learn. For instance, Carraher and Schliemann (2007, p. 676) remark that "the analysis of algebraic thinking is still in its infancy." And so is the analysis of the genetic relationship between algebra and arithmetic, and the role of signs in arithmetic and algebraic thinking.

Let us pause a moment and consider the role of signs. In his historical investigations, Damerow (1996) notes that ancient Egyptian and Babylonian arithmetic thinking arose as a result of *operating with signs* in order to systematically solve elementary problems involving counting and measuring. The earliest simplest numeric configurations were created by *iterating* a sign for the unit. These "numerals" had the purpose of facilitating the systematic calculation of additions and subtractions. As a result, from the outset, the concepts of addition and subtraction were consubstantial with the representations of the involved numbers. Embedded in practical activities oriented to the solution of administrative and other societal problems, the constructive-additive representation of numbers went hand in hand with the emergence of an elementary cognitive arithmetic additive structure. Later on we find the introduction of new signs to replace strings of sign-unit iterations and the ensuing rules of symbol-substitution, leading eventually to a positional numerical system. Within the possibilities of the historically developed cognitive additive-symbolic structures, new signs (e.g., signs for fractions) and operations (e.g., duplicating unit fractions, as in Egypt) became available. Contemporary elementary (school) arithmetic thinking depends no less on mathematical signs than the cultures of the past: it rests on a symbolic positional numerical system and sign-based algorithmic procedures for the basic arithmetic operations. A developmental approach to school arithmetic thinking would be impossible without attending to the role of signs. And so is the case of algebraic thinking in general, and early algebraic thinking in particular.

However, the exact role of signs in algebraic thinking remains a matter of contention among mathematics educators. In early algebra research it is not unusual (even if only implicitly) to see the use of alphanumeric symbolism as the trademark of algebraic thinking. Such a theoretical position is nonetheless untenable from a cultural-historical developmental viewpoint. The invention of alphanumeric symbolism is, indeed, a relatively historical recent event. It goes back to the work of 16th and 17th century mathematicians such as Rafael Bombelli, René Descartes, and François Viète. Equating the use of alphanumeric symbolism with algebraic thinking would amount to maintaining that algebra did not exist before the Western early modern period. Yet, 9th century Arabian mathematicians (like Al-Khwarizmi) and hundreds of Renaissance masters of abacus recognized and referred to their work as *algebraic*. So is the case of the 1544 "Libro e trattato della praticha d'alcibra" [Book and treatise of the practice of algebra] of the Sienese mathematician Gori (1984). You can go through the book page after page, line after line, word after word, and you will see no alphanumeric formulas or equations. You will see algebraic problem solving procedures expressed in words.

To better understand what can be termed as "algebraic" a more nuanced position is hence required. Mason et al. (1985), on the one hand, and Kaput et al. (2008a) on the other, offer a conception of algebra that is linked to the idea of generalization. For Mason et al.:

> Generality is the lifeblood of mathematics and algebra is the language in which generality is expressed. In order to learn the language of algebra, it is necessary to have something you want to say. You must perceive some pattern or regularity, and then try to express it succinctly so that you can communicate your perception to someone else, and use it to answer specific questions. (1985, p. 8)

Here, perceptual activity acquires a primordial role. They say: "Seeing, saying and recording form an important sequence in all maths lessons which applies particularly to all of the Roots of Algebra" (1985, p. 28). In this view, full symbolization—i.e., symbolization based on alphanumeric signs—is not required to start thinking algebraically: "Full symbolization should only come much later" (1985, p. 24).

Kaput et al. also link algebra to the expression of generalization: "We regard a symbolization activity as algebraic if it involves symbolization in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbol systems" (2008a, p. 49).

Although both perspectives on algebra revolve around the idea of generalization, they do not ascribe the same role to signs. While for Mason and collaborators the alphanumeric symbolism is not a condition for thinking algebraically, for Kaput and collaborators, in order for a symbolic activity to be called algebraic, full (i.e., alphanumeric) symbolization is required. Those activities in which generalization is expressed through other symbol systems are not considered genuinely algebraic: they are termed "quasi-algebraic" (Kaput et al. 2008a, p. 49). Along this line of thought, Blanton et al. argue that "algebraic reasoning ultimately involves reasoning with perhaps the most ubiquitous cultural artifact of algebra—the conventional symbol system based on variable notation" (2017, p. 182), which provides the rationale to attend to alphanumeric symbolism as early as Grade 1.

Perhaps we can better appreciate the differences between the aforementioned perspectives on algebra if we see them in terms of their conception about the role that signs play in cognition. Mason and collaborators' perspective draws on an empiricist philosophy of language and symbols, one proponent of which was the 17th century British philosopher John Locke. For him, the relationship between cognition and signs is based on an epistemological schema that can be represented as follows:

Sensation → Ideas → Words

Within this schema, for Locke the purpose of language is to communicate ideas between individuals: "communication … is the chief end of language" (Locke 1825, p. 315). Within this context, "Words do not play a significant role in generating concepts since language enters the process post facto, after our ideas have been formed. Ideas come first: words follow" (Hardcastle 2009, p. 186). Or as Mason et al. say, "You must perceive some pattern or regularity, and then try to express it succinctly so that you can communicate your perception to someone else" (Mason et al. 1985, p. 8).

Kaput and collaborators also draw on an empiricist philosophy of language, but of a different kind—one that goes back to the 18th century French Enlightened tradition that had Étienne Bonnot de Condillac as one of its proponents. In Condillac's account signs are more than tools of communication: language and signs acquired a cognitive role in mastering human psychological functions. Referring to memory and imagination, Condillac argued that

> by the assistance of signs he [the individual] can recall at will, he revives, or at least is often able to revive, the ideas that are attached to them. In due course he will gain greater command of his imagination as he invents more signs, because he will increase the means of exercising it. (Condillac 2001, p. 40)

We see that Condillac appears as a precursor of Vygotsky's concept of signs as mediators of psychological functions. It is precisely this concept of mediation that allows Kaput and collaborators to see a continuous connection between sign and ideas:

> Ideas, especially generalizations, grow out of our attempts to express them to ourselves and others, and our attempts to express them give rise to symbolizations that in turn help build and fill out the ideas, folding back into those ideas so that conceptualization and symbolization become inseparable. (Kaput et al. 2008a, p. 21)

One of the difficulties with the second perspective on algebra discussed above is the restrictive view that emanates from its requirement that thinking be expressed through the alphanumeric symbolism (or notations). I already mentioned that, from a cultural-historical developmental viewpoint, such a requirement may prove to be very limiting, in particular to approaches to early algebra. Such a requirement may lead to the failure to recognize non-symbolic forms of thinking as genuinely algebraic. Such a requirement may also lead to the attribution of an algebraic nature to forms of thinking that are in fact arithmetic. What is often overlooked is the fact that contemporary school arithmetic thinking resorts to alphanumeric symbolism too. The generalization $a + b = b + a$, which results from noticing that, for example, $2 + 3 = 3 + 2$, $1 + 6 = 6 + 1$, etc., may be considered as a genuine *arithmetic* generalization.

One of the difficulties with the first perspective on algebra was identified by Kaput et al.: "People have sometimes criticized inclusive views of algebraic reasoning on the grounds that it becomes difficult to distinguish thinking algebraically from thinking mathematically or (just plain) thinking" (2008b, p. xxi). Indeed, some researchers in the 1980s, like Kieran, expressed concerns about the difficulties of such an "inclusive" perspective: "For some authors (e.g., Open University 1985), the main idea of algebra is that it is a means of representing and manipulating generality and, thus, they see algebraic thinking everywhere—even in the recording of geometric transformations" (Kieran 1989a, p. 170). Certainly, by equating generalizing and algebraic thinking, it becomes difficult, if not impossible, to distinguish between an algebraic form of generalizations and other forms of mathematical generalizations (in particular arithmetic generalizations). As Kieran noted, "Generalization is neither equivalent to algebraic thinking, nor does it even require algebra" (1989a, p. 165). From research on animal cognition we know that chimpanzees, as well as birds, can start distinguishing between "edible" and "inedible" concrete items. They generalize their concrete experience and come to form what we humans would term the concept of "edible" (for details, see Radford 2011). Yet, we could hardly say that the chimps' generalization is an algebraic one.

To sum up, I have pointed out one difficulty arising from each one of the two perspectives on algebraic thinking that I have been discussing. An additional common difficulty is the fact that they reduce arithmetic thinking to mere computation. In other words, arithmetic thinking turns out to be reduced to procedural and mechanic calculation. I want to argue that this is a too restrictive view on arithmetic thinking. There are generalizations in arithmetic too. There may be very sophisticated arithmetic generalizations in the early grades that we are not even aware of, given the limiting view of arithmetic thinking that has been often adopted in early algebra research.

To move forward, we need to overcome the enduring conflation of algebraic thinking and notation use on the one hand, and the conflation of algebraic thinking and generalization on the other hand. Two points may be convenient to consider in this endeavor. First, notations are neither a necessary nor a sufficient condition for algebraic thinking (Radford 2014). Second, generalization is a common attribute of human thinking and cannot consequently capture the specificity of algebraic thinking. Our question is: What is it then that characterizes algebraic thinking?

The suggestion that I want to make draws from Kieran's (1989a) work on the one hand, and the work of Bednarz and Janvier (1996) and Filloy et al. (2007) on the other. I start from Kieran's 1989a paper and the idea that "For algebraic thinking to be different from generalization, I propose that a necessary component in the use of algebraic symbolism is to reason about and to express that generalization" (Kieran 1989a, p. 165). I want to make two points.

The first point: I want to take a very broad view on what counts as algebraic symbolism. In this view, I suggest that genuine algebraic symbolism includes the alphanumeric symbolism but also non-conventional semiotic systems—like natural language, which is mentioned in Kieran's paper, as well as gestures, rhythm, and other semiotic resources through which, as recent research shows, students signify generality (Radford et al. 2017).

The second point: There is something that remains unspecified in Kieran's proposal, namely what is meant by "to reason about and to express that generalization" (Kieran 1989a, p. 165). The reasoning that underpins the students' algebraic activity has to be specified. It cannot be *any* form of reasoning. It has to be *algebraic*. But what is *it*? It is at this point that I bring in the work of Bednarz and Janvier (1996) and Filloy et al. (2007). The Montreal team and the Mexican team have shown that one of the characteristics of algebraic thinking is its *analytic* nature (see, e.g., Bednarz et al. 1992; Filloy and Rojano 1989).

My suggestion is that algebraic thinking

- resorts to:
  - (a) indeterminate quantities and
  - (b) idiosyncratic or specific culturally and historically evolved modes of representing/symbolizing these indeterminate quantities and their operations,
- and deals with:
  - (c) indeterminate quantities in an *analytical* manner.

(a) Indeterminate quantities refer to the fact that the situation the students tackle in an algebraic manner involves more than given numbers or other mathematical entities. Indeterminate quantities can be unknowns, variables, parameters, generalized numbers, etc.

(b) As mentioned previously, although indeterminate quantities can be expressed through alphanumeric symbolism, they can also be expressed through other semiotic systems, without detriment to the algebraic nature of thinking. Naturally, alphanumeric symbolism constitutes a powerful semiotic system. With a very precise syntax and an extremely condensed system of meanings, alphanumeric symbolism offers a tremendous array of possibilities to carry out calculations in an efficient manner—calculations that may be difficult, if not impossible, to carry out with other semiotic systems (gestures, for instance, or even natural language). Yet, from an early algebra perspective, in the students' first contact with the historically evolved form of algebraic thinking conveyed in contemporary curricula, alphanumeric symbolism may not be required. The students can also resort to idiosyncratic or non-traditional modes of representing/symbolizing the indeterminate quantities and their operations.

(c) The indeterminate quantities and their operations are handled in an *analytic* manner. That is to say, although these quantities are not known, they are added, subtracted, multiplied, divided, etc. as if they were known—as Descartes says "without making a distinction between known and unknown [numbers]" (Descartes 1954, p. 8).

The adjective *analytic* comes from the noun analysis, which the ancient mathematician Pappus explained as the movement from what is given to what is sought (Rideout 2008). It is in this sense that algebra is considered by Viète as an analytic art where you make deductions; that is, you work from what is admitted "through the consequences [of that assumption]" (Viète 1983, p. 11). It is true that Viète introduced letters in a systematic way to solve problems algebraically. Certainly, he was aware of what he was accomplishing. Yet, he did not call his work "algebra with letters." What was distinctively algebraic for him was something else: the analytic manner in which we think when we think algebraically. Hence, the title of his work is *The Analytic Art* (Viète 1983).

Let me consider the equation $2x + 2 = 10 + x$. In the perspective on algebraic thinking that I am outlining here, a solution by trial and error would not be considered as algebraic, even if the task includes indeterminate numbers and the students are working with notations. In a solution based on trial and error, the students are resorting to arithmetic concepts only. By contrast, if the students deduce from $2x + 2 = 10 + x$ that $2x = 8 + x$ (by subtracting 2 from both sides of the equation), etc., we can say that the students are thinking algebraically. They are working through the consequences of assuming that $2x + 2$ is equal to $10 + x$. Likewise, in pattern generalization, an algebraic generalization entails *deducing* a formula from some terms of a given sequence. That the formula be expressed or not in alphanumeric symbolism is irrelevant. Notice that the fact that the general term of the sequence be expressed in alphanumeric symbolism does not imply at all that the generalization is the result of thinking algebraically about the sequence. In Radford (2006), I discuss the way in which some groups of students tackle the generalization of a figural sequence made up of two rows (see Fig. 1.1).

The students resorted to a trial and error method: "times 2 plus 1", "times 2 plus 2" or "times 2 plus 3" and checked their validity on a few cases. This form of thinking does not qualify as algebraic. Another group of students suggested: "$n \times 2$ (+3)". When I asked how they arrived at their formula, their answer was: "We found it by accident." Although the students' way of thinking about the sequence involves indeterminate quantities and alphanumeric symbolism, the formula was not deduced, but guessed. This is an example of arithmetic generalization—a simple one. It is not an example of algebraic generalization.

The theoretical perspective on algebraic thinking that I present here might be of particular interest to early algebra research. Indeed, the criterion about analyticity—i.e., the specific *analytic* calculation with/on unknown quantities—offers an operational principle to distinguish arithmetic and algebraic thinking. The theoretical perspective recognizes the importance of the alphanumeric semiotic system, but does not confine algebraic thinking to it. It opens the door to the investigation of non-symbolic (i.e., non-alphanumeric) forms of early algebraic thinking. And it allows us to envision, under a new light, the educational problem of the transition from a non-symbolic form of algebraic thinking to a symbolic one. Some of my

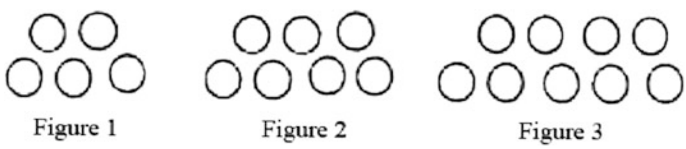

Fig. 1.1 The sequence of figures given to the students in a Grade 8 class (13–14 years old)

previous research has been devoted to the investigation of the emergence of early forms of non-symbolic algebraic thinking (Radford 2011, 2012). Focusing on pattern generalization, in this chapter I deal with the problem of the transition from non-symbolic to symbolic forms of algebraic thinking.

1.2 A Longitudinal Investigation of Early Algebraic Thinking

1.2.1 Research Methodology

The investigation that I report here was part of a six-year longitudinal research program in which Grade 2 students were followed as they moved from Grade 2 (7- to 8-year-old students) to Grade 6 (11- to 12-year-old students). In our research the primary interest is in understanding the development of students' algebraic thinking in situ. This starting premise is congruent with the fundamental principle of sociocultural research that stresses the link between cognition and context (Cole 1996). Drawing on the dialectic materialist theory of objectification (Radford 2008a), cognition can only be studied in *movement*; that is, through the activity in which it unfolds. In our case it is *classroom activity* (Radford 2015). As a result, our focus is the mathematics lessons.

We designed a flexible teaching-researching agenda committed to meeting two main goals. First, we sought to create the conditions that would allow the students to encounter the algebraic concepts stipulated by the curriculum. This was a practical concern framed by the political educational context of Ontario (Radford 2010a). Second, we wanted to deepen our understanding of the emergence and development of students' algebraic thinking, the difficulties that the students encounter as they engage in the practice of algebra, and the possible ways to overcome them. The longitudinal research was characterized by a continuous loop, which is represented in Fig. 1.2.

The arrows in Fig. 1.2 (and the whole Fig. 1.2) should not be understood in the empiricist sense of a clear-cut set of steps that assume that educational phenomena

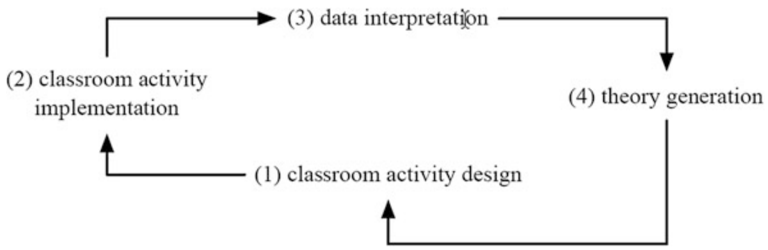

Fig. 1.2 Methodology of the longitudinal research (from Radford 2010a, p. 38)

obeys specific laws that are describable and whose variables can be controlled. In opposition to the Galilean paradigm of "teaching experiments," we consider methodology as an inquisitional and reflective practice, a philosophical practice in fact, and adopt a social science paradigm that

> conceives of the educational phenomena as messy and context sensitive. [It is a paradigm whose] claims are not backed up by some immutable laws whose existence is asserted by a confrontation of the laws and empirical facts. Rather, general assertions are sustained by actual references that may guide further action through a reflective stance. (Radford and Sabena 2015, p. 158)

1.2.2 Data Collection and Participants

Our participants were 21 7- to 8-year-old students of a Grade 2 class in a public school in Sudbury, Ontario. In Grade 5 the class had 29 students and 31 in Grade 6. Data were collected through two one-week videotaped sessions per year, although we kept contact with the teacher during the year in order to exchange ideas and discuss the teacher's and the students' achievements and challenges about the teaching and learning of algebra. Each year, each one of the 10 videotaped lessons lasted 100 min. We had four cameras in the classroom to videotape a small group of students with each. In addition to the videotapes, we kept a copy of activity sheets, homework and individual written assessments (see below) of the videotaped groups as well as of the remaining groups of the class in order to broaden, complement, and enrich our videotaped data.

1.2.3 Task Design

Before each one-week videotaped session, the teacher and the research team (the author of this chapter and graduate and undergraduate students) participated in joint task design research meetings. The joint task design included a careful conception and production of

(a) pattern generalization problems for the students to solve in class,
(b) homework sheets, and
(c) individual written assessments.

During the joint task design sessions, videotaped classroom activities, transcripts, and copies of students' sheets were discussed with the teacher (who changed from year to year) to highlight previous years' students' accomplishments and challenges.

1.2.4 Data Analysis

Our data analysis revolved around a multimodal approach that included fine-grained video-analysis (often short episodes subjected to frame to frame scrutiny) with special attention to gesture, language, perception, and symbol-use to account for non-conventional forms of signifying mathematical generality.

Problems of increased difficulty appeared as the students moved from grade to grade (for example, generalizations of figural sequences showing non-consecutive terms (e.g., Terms 1, 3 and 5); generalization of figural sequences where variables are organized in tables, and numeric (i.e., non-figural) sequences without geometric-spatial clues). Using the modern algebraic symbolism, almost all sequences corresponded to the formula $y = ax + b$ (with $a \in Z$, and $b \in N$). From Grade 3 on, a "core problem" remained invariable each year to better appreciate the students' progress. Because of space limitations and the fact that activities surrounding alphanumeric algebra appeared in Grade 4 for the first time, this chapter revolves mainly around this core problem and what happened in Grades 4, 5, and 6.

1.3 The Core Problem: "The Tireless Ant"

The core problem featured an ant that found a container with one crumb in it. The ant collected two crumbs each day, so that at the end of Day 1 the ant had 3 crumbs in the container; at the end of Day 2, it had 5 crumbs; at the end of Day 3, it had 7 crumbs, etc. A drawing (see Fig. 1.3a) was included in the activity sheet. Working in small groups of three or four, the students were invited to draw the container for Days 4 and 5, and then to find out the number of crumbs on Day 33. Then there was a question dealing with the writing of a message for another student. I shall return to the message question below.

The question of drawing the container for Days 4 and 5 was intended to investigate the students' evolving awareness of the mathematical structure of the sequence, and the semiotic means to which they resort to make the structure

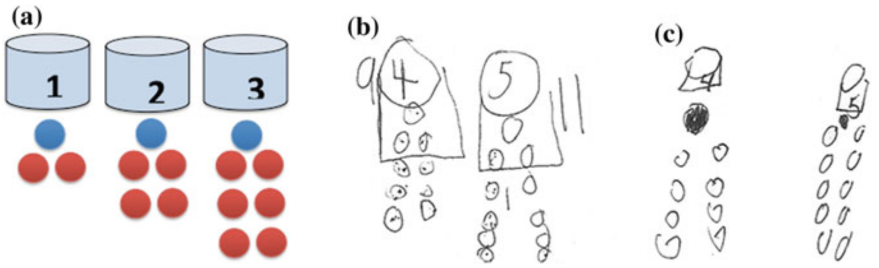

Fig. 1.3 The first terms of the sequence and examples from students' extension of the sequence

apparent. The question about Day 33 should reveal the students' actual generalizing process. The manner in which the students draw and talk about the content of the container on Days 4 and 5 provides us, indeed, with clues about their developing awareness of sequence structure.

1.4 Grade Four

Figure 1.3b, c show two typical answers of Grade 4 students. They come from one of the four small groups we videotaped. By looking at the drawings alone, there seems to be no difference. However, the video analysis shows that the underlying generalizing logic is not the same. Jay's drawing (Fig. 1.3b) is based on the recurrent relation mentioned in the statement of the problem: two crumbs are added each day. Jay says: "For Day 4 we draw the number again [i.e., Day 3]. After that we will add … [two crumbs]." He draws the crumbs by rows. Alex, by contrast, perceives the term globally. Visually, he recognizes figural *parts* of the term as *key parts* to make the drawings. Thus, after hearing Jay's utterance, he moves close to Jay's sheet and, pointing at the left column of Term 3, says: "There, there is 3 there at Day 3 (at the same time he counts successively the circles), plus (pointing at the initial crumb) the one on the top. So, we must always draw the number of days like this (pointing at the left column) plus one on top."

The "recurrent" and the "global" approaches (illustrated by Jay and Alex, respectively) are predominant in Grade 4. The first one is based on the recurrent relation between consecutive terms. The second approach goes beyond what is explicitly stated in the problem. It deals with the expression of a mathematical relationship between two variables: the number of the day and *visual key parts* of the term (the number of crumbs on the columns of the term). This approach requires a specific perceptual activity and a finer interpretation. Yet, we see Alex's difficulties to express verbally the key parts. They are referred to through pointing gestures. The awareness of the term structure seems to remain to a large extent visual: the perceived thing seems to remain inexpressible in the realm of language; it is hence expressed otherwise—by resorting to another semiotic system: the dynamic and frugal semiotic system of gestures. All in all, the grasping of the structure unfolds in a process of perceptual semiosis through language, gestures, the pictorial sign of terms, and visual activity.

But there is an additional point that needs to be discussed: the role of the temporal adverb "always" in the second part of Alex's utterance. "So, we must always draw the number of days like this (pointing at the left column) plus one on top." The temporal adverb "always" is what bestows the phenomenon under discussion with its full generality. What Alex has just perceived does not apply to Day 3 only. This is corroborated by the absence of specific numbers in the second part of Alex's utterance. Alex is not talking about Term 3 only. He is talking about *all* terms of the sequence. This is why, when the group moves to the question of drawing the container for Day 5, the question was quickly answered. Catherine said: "There are 5 on the side." For the first time, the visual key part of the term is explicitly named. It is named the "side." While naming the visual key part of the term, Catherine makes two straight sliding gestures, meaning the two columns of Term 5. It is as if the name alone is not enough to convey unambiguously its reference. Catherine resorts, hence, to gestures to complement the emerging meaning. At this point the teacher comes by to check on the students' work. Jay has just finished drawing Day 5, still row by row. Talking to the teacher, Catherine addresses the question of Day 33, and quickly says: "So you put 33 and 33" while again making two sliding gestures. Taking into account the first crumb in the calculations, Catherine and Alex say "67." Jay says "yes," and switching to the global perception of the terms, adds: "It is the same number of all things." Alex replies: "underneath each side there is the number of things, so 33 … plus 33 plus the one on top."

1.5 Factual Generalizations

In the previous section we see the students noticing a structure in the first given terms (Days 1, 2, and 3), and generalizing it to all terms of the sequence. More precisely, the students started by grasping a *commonality* noticed on the first three given terms (Days 1, 2, and 3), which have been perceived as having "sides." Then, the students generalized this commonality to all subsequent terms and were able to use the commonality to provide a direct expression of any term of the sequence. The generalized commonality is what Peirce (1958, 2.270) called an *abduction*—i.e., something only plausible. In the last part of the generalization process this abduction became the warrant to deduce expressions of remote elements of the sequence. Direct expression of the terms of the sequence requires the elaboration of a formula (that is, a *rule* or *method*) based on the variables involved. The analytic trait that, as I suggested above, is required for the generalization to be algebraic is to be found in the passage where Alex contends that "we must *always* draw the number of days like this (pointing at the left column) plus one on top." The analytic trait is manifested in the *deduction* that Alex expresses in his utterance (as opposed to an induction). All things kept the same (i.e., the tireless ant always adding two crumbs each day), Alex can deduce that "underneath each side there is the number of things, so 33 … plus 33 plus the one on top." Although the students have not used alphanumeric symbolism, the students' generalization is genuinely algebraic in nature.

I have taken some time to analyze the students' generalization, as it shows an example of algebraic generalization that is not based on the alphanumeric symbolism. In previous work I have called this type of generalization *factual generalization* (Radford 2011). The adjective *factual* means that the variables of the formula appear in a *tacit* form. The formula is expressed through particular instances of the variable (the variable is instantiated in specific numbers or "facts") in the form of a *concrete rule* ("33 plus 33, plus the one on top"). This concrete rule empowers the students to deal with any specific term of the sequence (e.g., Terms 100, 500). To make sure, I came to see the group and asked about Day 60. Catherine answered: "we would do 60 on one side, 60 on the other …" Alex interrupted and added: "and the one on top."

1.6 Writing a Message: Contextual Generalizations

Our path towards symbolism was based on the following question: The students were asked to write a message for another student to tell her how to quickly calculate the number of crumbs in the container for a certain day. The number of the day was drawn from a box containing cards, each one with a big number on it. First, the teacher drew a card and showed it to the students. The card had the number 100 on it. Here is an excerpt of the discussion in Alex's group.

1. Alex: We put the number on both sides … and, and one on top and add all that (he writes)
2. Teacher: I am going to read (reading) "We put the number on both sides." Which numbers?
3. Alex: The number of the day … (talking to his group-mates) we write the number on both sides of the day, write the number of the day.
4. Jay: And you have to add both days.
5. Catherine: (Interrupting) Together.
6. Jay: Ah yeah! OK it's like … one must add … the two days together and add another … another day!
7. Alex: I don't get it. I do not understand what you are saying … no offence man, but …

In Turn 2, the teacher asks the students to specify which numbers they are talking about. In Turn 6, Jay mixes the number of crumbs and the number of days.

This dialogue highlights some of the difficulties that the students found in articulating in a clear manner the variables and their relationship at the level of language. These difficulties appeared also in the students' written messages. The messages were essentially of the same form: a drawing with some calculations and a short text. Figure 1.4a, b show a paradigmatic drawing and a text. In the drawing, the student identifies the container as "Jour 100" (Day 100). He explains that the black circle is the initial crumb ("miette"). Towards the right of Fig. 1.4a, he writes: "One adds 100 on each side." The text brings forward the spatial context in an explicit manner (see Fig. 1.4b). It reads: "One remarks 100 on each side. One adds it to arrive to the answer and one adds the crumb that he found. At the end one makes a calculation. Here is how to solve this problem."

We see that in both the student's written text and the oral discussion (see previous excerpt), the relationship between the variables remains unclear. It is as if the formula has not yet completely entered the realm of verbal thinking.

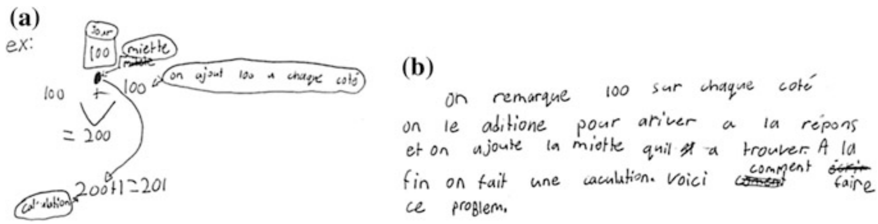

Fig. 1.4 A paradigmatic example of the student's written text

Yet, despite the challenge of putting the formula in words, the dialogue above shows that the students have moved to a new layer of generality. Although the object of discourse in the previous dialogue is Term 100 of the sequence, we see the students engaged in a discussion where numbers start receding to the background (see the students' dialogue above). The students' attention moves to the variables and their relationship, which, bit by bit, become the central object of discourse.

1.7 The Emergence of Symbolic Algebraic Thinking in Grade Four

After a general discussion, the teacher again drew a card from the box but hid the number on it from the students. She said: "I draw a card, I do not tell you the number. I put the card in an envelope and I will send it to a student. What do you write in the message to this student now?" Marika replied: "As Catherine and Alex said, it is twice in the line [i.e., the "side"], it is the same thing day and crumbs, and you have to put twice, and then you have to add the crumb on top that it [the ant] has already found."

Marika's utterance offers an example of a *contextual generalization*. That is, a generalization whose formula is based on spatial and other deictic terms (here, "sides" and "top"). The deictics endow the variables with a meaning deeply related to the spatial or other contextual clues of the terms of the sequence (Radford 2011).

After a general discussion about Marika's formula, the teacher moved towards the didactic agenda: the search for a symbolic formula. The teacher said: "Now I do not want you to put a phrase. I want you to write down a calculation." Dylaina suggested to use a letter, but formulates the message as if the number was known: "You put r for the number of days and you put on each side and it is equal to 200, then you add 1 and it is equal to 201." The teacher reminded the students that the number in the envelope is not known. A student went to the blackboard and suggested to use the sign "#" for the number of the day; other students suggested the signs "?" and "__." The students' formula on the blackboard was: $2 \times \_\_ = \_\_ + 1 = \_\_$ (see Fig. 1.5a). The teacher asked if they could use letters instead. The students suggested *a*, *b*, and *c*, so the formula was transformed into

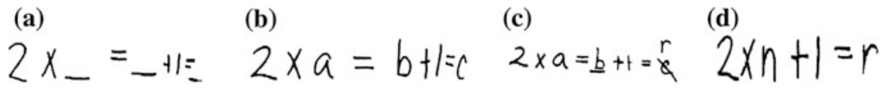

Fig. 1.5 The first symbolic expressions in Grade 4

2xa = b + 1 = c (see Fig. 1.5b). The 100-minute math lesson ended up with the teacher asking the students to reflect on the meaning of each letter.

The next day the class came back to the formula $2 \times a = b + 1 = c$. Since the last number is the answer ("réponse" in French), the students suggested replacing "*c*" with "*r*" (see Fig. 1.5c). The teacher started a new thread in the conversation.

1. Teacher: I will write something on the blackboard and I want you to tell me if I can do this (she writes on the blackboard; see Fig. 1.5d).
2. Students: Yes!
3. Teacher: I need someone to explain … Lola, would you like to explain?
4. Lola: Because 2 times the number plus 1 equals the answer.
5. Teacher: Ok. And *n*, what does it represent?
6. Lola: It represents 100, 101, etc.
7. Teacher: Ok. And plus 1, what does it represent?
8. Lola: It represents the first crumb.

The teacher then asked if other formulas were possible. Alex suggested: "*n* plus *n* equals plus 1 equals *r*."

Generally speaking, the class made substantial progress towards the production of alphanumeric formulas. However, although the produced formulas start moving away from the recourse to the spatial deictics that are the hallmark of contextual generalizations, the formulas exhibit something that manifests itself as one of the greatest obstacles in becoming fluent with the alphanumeric symbolism and the meaning of the symbolic formula, namely the strong tendency to *calculate sub-totals*. The alphanumeric formula expresses the algebraic calculations in a *global* manner. It focuses on the structure. The students' tendency to calculate sub-totals reminds us of Davis's (1975) "process-product dilemma" (Kieran 1989b, p. 41). The "process-product" dilemma refers to the difficulty in considering an expression such as "$x + 3$" as an answer. In our interpretation, what this dilemma means is that the emphasis in the alphanumeric formula is not on the numbers themselves but *on the operations*. We move here to an altogether new realm of generality—*symbolic generality*. In this level of generality, the novelty is not only the introduction of alphanumeric symbolism, but a whole reconceptualization of numerical operations.

1.8 Grade Five

In Grade 5 the students again tackled the Ant Problem. This time the mathematical structure was easily perceived:

1. Catherine: So, we can do the first crumb first … the crumb he found first …
2. Alex: (Interrupting) And then there are 4, 4 on each side (he makes two gestures in the air meaning the two sides), [and] 1.
3. Catherine: Ah yeah, because the number of the … day equals the ones on the two sides … So 4 crumbs on each side …
4. Alex: And then, for 5, it's 5 on each side.

Alex mentions right away the spatial deictic "side," which is also the reference of his gestures. Although not yet perfect, the linguistic relationship between the variables is much better articulated than in Grade 4 and the mathematical structure of the terms is much better ascertained. And as the variables' relationship enters the realm of language, the gestural activity recedes into the background. This is why, when the students tackled in Grade 5 the question of Day 33, gestures were no longer required. The answer came without difficulty. Alex said: "So 33 plus 33 equal 66, plus 1."

This passage provides us with a neat example of *semiotic contraction*; that is, the mechanism that consists of making a choice between what counts as relevant and irrelevant. In semiotic contraction there is a reorganization of the semiotic recourses that help the students to direct their attention to those aspects that appear to be most significant. In general, semiotic contraction is an indicator of a deeper level of consciousness and intelligibility (Radford 2008b).

The students' deeper level of consciousness and intelligibility of the mathematical structure of the sequence was also manifested in the flexibility and creativity that the students showed in dealing with the ant context. Alex challenged his teammates Catherine and Andrew (who joined Alex and Catherine's group in Grade 5, while Jay went to work with another group), with the question of finding the crumbs in day 103: "OK. And if it is the 103rd day, how many pieces [crumbs] is he going to have?" Andrew replied immediately: "207." Andrew went even further and said: "Um, anything ["n'importe quoi"] plus anything equals um" He explains: "I do it the other way: I give you the answer, but I do not give you the numbers."

During a general discussion, the teacher visually and discursively emphasized the relationship between variable "Day" and "number of crumbs" in the container. The teacher said: "If I want to draw Day 6, I have to put one crumb (she draws 1 crumb) and how many circles should I put here on my left column? (She makes a vertical sliding gesture where the crumbs/circles will be drawn).

After that, the group came back to Andrew's challenge.

1. Catherine: OK, so, it is day 201, no … there are 201 crumbs. Which day is it?"
2. Alex: 100! Woo!
3. Catherine: Now, challenge me!

During a series of consecutive challenges that the students enjoyed very much, they came to realize that the challenging number had to be an odd number. Also important from a developmental viewpoint is the fact that in the course of these challenges, for the first time, the students linguistically identified the variables in an explicit and proper manner. The consequence was that the contextual generalizations that they were producing were much more refined than those produced in Grade 4.

When it came to write the message as a sequence of operations, the students resorted to similar symbolic formulas as those they proposed in Grade 4; that is, formulas that include sub-totals. During the general discussion, another student, Janelle, wrote on the blackboard the equation that the class came up with last year: ___ × 2 = ___ + 1 = ____. Figure 1.6a, b show two more examples.

These figures suggest that students have become more and more conscious that different signs are required to represent different numbers. Thus, in Fig. 1.6a, Gavin explains: "We did a multiplication. We did the mysterious number times 2, equals a mysterious number, and there (pointing at the "?" sign on the second row; see Fig. 1.6a) plus 1 equals a mysterious number. Now, this (encircling the second and the third "?" signs) are the same thing, the same number." Figure 1.6b is even more explicit about the fact that each sign stands for a different number. In Fig. 1.6c, the teacher invites the class to use letters. Alex suggests using "a," "l," and "e" (the first three letters of his name), while Christiane suggests "n" for number, "r" for "réponse" (i.e., answer), and "vr" for "**v**raie **r**éponse" (i.e., the real answer). To close the 100-minute lesson, the teacher asked the students if they had learned something new. Théo answered: "We can put any letter as long as the numbers are different … 'Cause that [same letters] means that the number is the same. You can use the same letter if it is the same number."

The students had homework to return the next day. The homework featured the Tireless Ant context with 2 crumbs found in the container and drawings of the container for Days 1, 4, and 5 (Day 1 = 5 crumbs; Day 4 = 14, and Day 5 = 17). Of the 26 students, 1 student did not return the homework, 21 students resorted to formulas based on sub-total calculations (e.g., $a \times 3 = b + 2 = c$), 3 formulas conforming to the alphanumeric syntax (e.g., $\mathrm{J} \times 3 + 2 = R$), and 1 unclassified answer. The teacher started the lesson with a discussion of the students' homework answers. Three students volunteered to write their formulas on the blackboard. One formula was: <u>*number of the day*</u> × 3 = ___ + 2 = ___. The second formula was: $a \times 3 = b + 2 = c$; the third one was $J \times 3 + 2 = R$. The teacher took the opportunity to make a distinction between the formulas. The first two, she insisted, "separated" the calculations into two. The third formula did not. She insisted that it was not necessary to separate the calculations. Then the lesson continued with

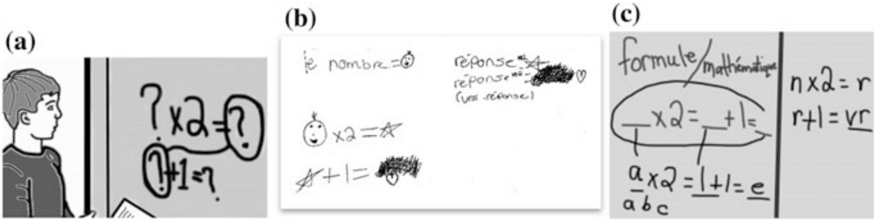

Fig. 1.6 Symbolic formulas in Grade 5 are still based on the calculation of sub-totals

another activity where the students were put in a position of formula interpretation. The activity included the formulas "$5 \times n + 2 = r$," "$3 \times n + 7 = r$," and a third question where the students had to produce their own formula. In each case they had to explain the meaning of the letters and coefficients within the context of days and crumbs. In general, the students were able to correctly identify the various terms of the formula. For instance, Miguel produced the formula $10 \times n + 5 = r$, and noted that 10 is the "added crumbs," n is "the number of the day," 5 is "first crumb," and r the "answer." An individual test took place two weeks later. By then, half of the 26 students were producing formulas conforming to the alphanumeric syntax, 7 students were producing formulas showing partial calculations and 5 students were producing other answers.

1.9 Grade Six

In Grade 6 the Tireless Ant activity did not include the drawings of the container and the crumbs/circles. Yet, the students were able to answer the questions quickly. Laura, for instance, said: "First, you have to take the number of the day and to multiply it by 2, because each day it [the ant] adds two crumbs. And you have to add 1, which is the crumb that the ant found in the container." The symbolic formula was easily reached too. It read: $(n \times 2) + 1 = m$. The appearance of brackets in the students' formulas was the result of a class discussion conducted by the teacher about the priority of operations.

During the activity concerning the interpretation of formulas, the teacher came to see Laura's group and challenged the students' interpretation. The formula under discussion was $5 \times n + 2 = r$. The students argued that the ant found two crumbs in the container and added 5 each day:

1. Teacher: Why not the ant started with 5 and added 2 each day?
2. Laura: Because times (pointing to the multiplication sign) means that [the ant] adds 5 each day, like 5 *times* the day… (she emphasizes the word "times").

The following day, the students explored the sequence shown in Fig. 1.7a. The students were at ease producing a symbolic formula for the general term of the sequence. Alex, for example, suggested $n \times 2 + 2 = r$.

The activity included the following formulas: "$N + N + 1 + 1 =$ ___" and "$2 \times N + 1 + 1 =$ __" (which were actually produced by students of another Grade 6 class). The teacher asked her students whether or not they thought that these formulas were correct and to explain. Referring to the first formula, Christiane answered: "Yes. N = number of the figure; ___ = number of rectangles in total; 1 = the rectangles added on the top, at the ends." Referring to the second formula she noted: "Yes. N = number of the figure; ___ = total of rectangles; 1 = the rectangles added on top, at the ends; 2 = two rows of rectangles." Another question

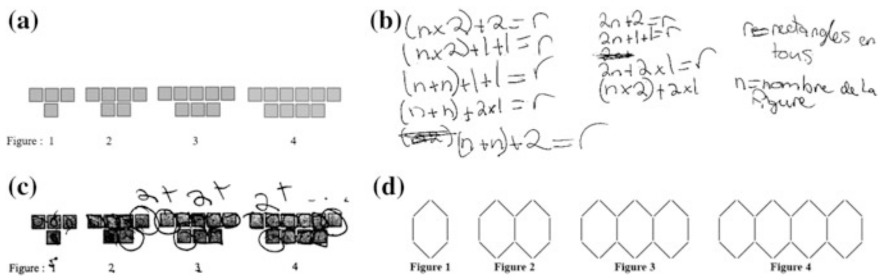

Fig. 1.7 Figural sequences and symbolic generalizations in Grade 6. **a** (left, top row) shows a figural sequence investigated in Grade 6. **b** (right, top row) shows Christiane's formulas. **c** (left, bottom row) shows traces of the students' perceptual and symbolic activity. **d** shows a sequence in the individual test

asked the students to produce as many formulas for the sequence as they could. Figure 1.7b summarizes Christiane's results.

As Fig. 1.7c suggests, the students have reached a very good coordination of perceptual and symbolic activity. Indeed, the students were able to interpret perceptually the given formulas "N + N + 1 + 1 = ___" and "2 × N + 1 + 1 = __". Figure 1.7c shows some traces of the students' perceptual activity: the two added rectangles were imagined at the right end of the rows; but also at the beginning of the rows, and also one at the beginning of the bottom row and the end of the top row. It is not the symbolic function that has evolved, but mathematical imagination and perceptual activity as well. The eye appears now as a theoretician (Radford 2010b).

Let us pause for a moment and discuss with more detail Fig. 1.7b. To produce the formulas in Fig. 1.7b, the students did not need to see the terms of the sequence, nor did they have to translate the meaning of the formulas and their components in natural language according to the context (i.e., the students did not need to refer to *n* as, e.g., "the number of rectangles on the bottom row"). Letters, constants, coefficients, and operations were uttered in a transliterated form only (e.g., "two *n* plus two equal *r*") or were not uttered at all. What this means is that, remarkably, for the first time in the students' mathematical experience, as we witnessed in the course of our longitudinal investigation, natural language was no longer leading thinking. At this precise point in the development of the students' algebraic thinking, abstract signs (what Peirce (1958) called *symbols,* i.e., abstract signs vis-à-vis the context) starting leading and words started following. In other terms, symbolic thinking has superseded verbal thinking!

To end this chapter, it might be important to say something about the results of an individual test in Grade 6. The test took place two days later. It included the sequence shown in Fig. 1.7d. The formulas with sub-totals disappeared completely. In the course of the years, with the help of the school Principal, we managed to keep most of the students in the same class. But in Grade 6 there were a few newcomers and two students moved to other schools. We lost Catherine and another student. Of the 31 students in our Grade 6 class, 21 students produced the expected alphanumeric formula—usually $n \times 5 + 1 = r$ or $(n - 1) \times 5 + 6 = r$. Ten students produced an incorrect formula for the problem. Of the 21 students who produced the expected alphanumeric formula, 19 were part of the cohort followed in this study and two students joined the cohort in Grade 5.

1.10 Concluding Remarks

In this chapter I presented the results of a longitudinal investigation on the emergence of algebraic symbolism in the context of sequence generalization. The investigation rests on a characterization of algebraic thinking based on its *analytic* nature. In sequence generalization, this idea means that the sought-after formula is not guessed but *deduced* from certain given data. In the course of the chapter I insisted that the formula does not necessarily have to be expressed through the alphanumeric symbolism. The formula can also be expressed through other kinds of semiotic systems.

The importance of distinguishing the semiotic systems through which the students produce their formulas is related to a dialectic materialist epistemological premise about cognition and signs, implicit in the Introduction, and that I can now state in full as follows. *The manner, depth, and intensity in which an object appears as an object of consciousness are consubstantial with the semiotic material that makes possible for such an object to become an object of consciousness and thought*. There are always limits to what can be thought and said within a semiotic system. For each semiotic system has its own *expressiveness*. In terms of Locke's and Condillac's philosophy of language and signs mentioned in the Introduction, what the dialectic materialist epistemological premise means is that language and semiotic systems in general are not merely the expressions of thought or mediators of it. As Vološinov notes, "It is not experience that organizes expression, but the other way around—*expression organizes experience*. Expression is what first gives experience its form and specificity of direction" (1973, p. 85; emphasis in the original). The alphanumeric symbolism and the Cartesian Graph symbolism, for instance, do not have the same expressiveness. There are inherent limits as to what can be said and thought within each one of them. Each one provides the students' experience of algebra with different form and direction.

Of course, this question about expressiveness is—reformulated at a more general level—the formidable problem that Vygotsky (1986) dealt with in the last chapter of *Thought and Language*. What our dialectic materialist epistemological premise means in the context of this chapter is that the conceptual deepness of the manner in which the variables and their relationships are noticed and the algebraic structure is revealed to the students is not the same in the various types of generalizations that we have discussed. In factual generalizations the formula is not expressed explicitly. It appears "in action," through concrete numbers and their operations. The variables and the relationship between the variables remain implicit. In contextual generalizations, by contrast, the formula is expressed at a more general level; the variables and their relationship become explicit and are referred to through contextual elements—spatial linguistic deictics (for example, “top” and “bottom”). While factual generalizations seem to go without difficulties in Grade 4, contextual generalizations were difficult to express. These difficulties reveal the students’ agony in coming to terms with a deeper level of algebraic structure consciousness. In Grade 5 things changed. The linguistic formulation of variables and their relationship became possible, the result being a deeper level of intelligibility. The algebraic formula entered the realm of *verbal thinking*. But it did not yet enter the realm of *symbolic thinking*. For this to happen, the teacher and the students had to continue working to achieve something that has far-reaching epistemological consequences. That is, as paradoxical as it may seem, the teacher and the students had to move to a conceptual realm where natural language ceases to be the main substance and organizer of thinking. Indeed, while natural language with its arsenal of conceptual possibilities offers the semiotic material to produce contextual generalizations, natural language has to recede into the background to yield space to a new cognitive form—symbolic thinking. The “deicticity” of contextual generalizations does not disappear: it becomes sublated into the new abstract signs of symbolic generalizations (see Fig. 1.6c).

Symbolic generalizations are, indeed, based on *symbols*—i.e., abstract signs vis-à-vis the context (see, e.g., Fig. 1.6b). The deictic nature of contextual generalizations may be formulated as an indexical and iconic form of signifying. The index and the icon signs (in Peirce’s sense) now have to recede for the symbol to appear. And Grade 6 was the moment in which this happened: it was the remarkable moment in which algebraic symbolic thinking emerged. The students had to overcome their tendency to think of the formula in terms of sub-total calculations (a symptom of the entrenched leading role that numbers *qua* concrete numbers had in the students’ thinking). Finally, this tendency disappeared and the focus shifted to variables, operations, and numbers reconceptualized at a higher level (as rates, for instance, as in the case of the multiplicative coefficient of linear formulas).

But we should not miss the point about the importance of the standard algebraic symbolism. The importance of the standard algebraic symbolism does not reside in its tremendous efficiency to carry out calculations only. It also resides in the possibilities it offers to reach new aesthetic modes of imagination and perception.

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Chapter 2
Implementing a Framework for Early Algebra

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Abstract In this chapter, we discuss the algebra framework that guides our work and how this framework was enacted in the design of a curricular approach for systematically developing elementary-aged students' algebraic thinking. We provide evidence that, using this approach, students in elementary grades can engage in sophisticated practices of algebraic thinking based on generalizing, representing, justifying, and reasoning with mathematical structure and relationships. Moreover, they can engage in these practices across a broad set of content areas involving generalized arithmetic; concepts associated with equivalence, expressions, equations, and inequalities; and functional thinking.

Keywords Algebraic thinking · Randomized study · Early algebra
Learning progressions · Qualitative methods · Curriculum

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2.1 Introduction

When tasked with the open question of measuring the impact of *early algebra*[1] on children's algebra-readiness for middle grades, our first challenge was to identify the "early algebra curriculum" from which impact could be measured. Essentially, such a curriculum as we envisioned it—that is, an instructional sequence that integrated core algebra concepts and practices across the elementary school mathematics curriculum through a research-based, multi-year approach—did not exist in curricular resources in the United States [US]. At best, we found that mainstream arithmetic curricula offered only a random treatment of "popular" algebraic concepts (e.g., a relational understanding of the equal sign, finding the value of a variable in a linear equation, finding a pattern in sequences of numbers), often buried in arithmetic content in ways that allowed one to potentially ignore or marginalize their treatment in instruction. This curricular challenge presented us with an obvious corollary: What is the *algebra* that we want young children to learn and that will suitably prepare them for a more formal study of algebra in the middle grades?

These challenges led us on a lengthy journey to apply a widely-acknowledged framework for algebra (Kaput 2008) as a conceptual basis for designing an early algebra curriculum for Grades 3–5. Such a curriculum would allow us to measure elementary grades students' potential for algebraic thinking as well as their readiness for algebra in later grades. In a separate line of work, we also began exploratory research that would allow us to back this approach down into the lower elementary grades (i.e., Grades K–2). We share part of this journey here on three fronts: (1) we characterize the algebra framework that has informed our approach; (2) we describe the curricular approach and its components, designed using this framework for Grades 3–5; and (3) we share evidence of the impact of this approach on children's algebraic thinking.

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[1]By *early algebra* we mean algebraic thinking in the elementary grades (i.e., Grades Kindergarten–5).

2.2 The Emergence of Early Algebra in the US

Research shows that, historically, algebra education in the US—an "arithmetic-then-algebra" approach in which an arithmetic curriculum in the elementary grades was followed by a formal treatment of algebra in secondary grades—was unsuccessful in terms of students' mathematical achievement (e.g., Stigler et al. 1999) and led to a widespread marginalization of students in school and society (Kaput 1999; Schoenfeld 1995). Algebra's resulting status as a gateway to academic and economic success (Moses and Cobb 2001) led to calls for identifying new approaches to algebra education. As part of this effort, scholars worked to develop new recommendations for school algebra instruction that would provide students with the kind of sustained experiences necessary for building informal notions about algebraic concepts and practices into more formal ways of mathematical thinking. Importantly, algebra education was re-framed as a longitudinal effort that would span Grades K–12 rather than one that began abruptly in high school (e.g., National Council of Teachers of Mathematics [NCTM] 2000; RAND Mathematics Study Panel 2003).

Recent US reform initiatives such as the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO] 2010) have reiterated the significant and increasing role algebra is now expected to play across school mathematics by outlining content standards and mathematical practices for algebraic thinking beginning at the start of formal schooling (i.e., kindergarten). While these efforts have strengthened the national discourse on the role of early algebra in school algebra reform, the development of a research-based approach to early algebra that would guide the systematic, long-term development and assessment of young children's algebraic thinking has been lacking. In this sense, we hope that the approach we share here might provide one route for clarifying and deepening the role of algebra in the elementary grades.

2.3 A Conceptual Framework for Early Algebra

The early algebra perspective that guides our work is based on Kaput's (2008) content analysis of algebra as a set of *core aspects* across several mathematical *content strands*. We discuss each of these here and how they are enacted in our work.

2.3.1 Core Aspects and the Algebraic Thinking Practices Derived from Them

Kaput (2008) proposes that algebraic thinking involves two core aspects: (a) making and expressing generalizations in increasingly formal and conventional symbol systems; and (b) acting on symbols within an organized symbolic system through an established syntax, where conventional symbol systems available for use in elementary grades are interpreted broadly to include "[variable] notation, graphs and number lines, tables, and natural language forms" (p. 12). While Kaput acknowledges differing views on whether and how acting on symbolizations such as variable notation should occur in elementary grades, he and others (e.g., Blanton et al. 2017a; Brizuela and Earnest 2008; Carraher et al. 2008) maintain that interactions with all of these symbol systems early on can actually deepen students' algebraic thinking. In our work, we also adopted this broad interpretation of symbol systems, along with the view that incorporating such diverse systems throughout children's algebraic work would be a potentially productive route to developing their algebraic thinking.

We derive four essential practices from Kaput's (2008) core aspects that define our early algebra conceptual framework: *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships (see also Blanton et al. 2011). We see the activities of *generalizing* and *representing* generalizations as the essence of Core Aspect (a). Furthermore, from Core Aspect (b), we take *justifying* generalizations and *reasoning with* established generalizations in novel situations as two principal ways of acting on conventional symbol systems, broadly interpreted. A critical component of these four practices is that they are centered around engagement with mathematical structure and relationships. For example, we take the view that the activity of justifying is not, in and of itself, algebraic, but it serves an algebraic purpose when the context is justifying generalized claims. In what follows, we elaborate on each of these four algebraic thinking practices as we interpret them in our work.

2.3.1.1 Generalizing

Generalizing is central to algebraic thinking (Cooper and Warren 2011; Kaput 2008) and the very heart of mathematical activity (Mason 1996). It has been characterized as a mental process by which one compresses multiple instances into a single, unitary form (Kaput et al. 2008). For example, in simple computational work, a child might notice after several instances in which she adds an even number and an odd number that the result is an odd number. In this, the child is starting to "compress" all of the instances of adding a specific even number and a specific odd number and getting an odd number as a result into the generalization that the sum of *any* even number and *any* odd number is odd. Engaging elementary-aged children in the activity of generalizing is vital because it strengthens their ability to filter mathematical information from common characteristics and to draw conclusions in the form of generalized claims.

2.3.1.2 Representing Generalizations

The activity of representing mathematical structure and relationships is as important as generalizing (Kaput et al. 2008). As a socially mediated process whereby one's thinking about symbol and referent is iteratively transformed (*ibid.*), the act of representing not only gives expression to the generalizations children notice in problem situations, but also shapes the very nature of their understanding of these concepts. As Morris (2009) notes, the practice of representing generalizations builds an understanding that an action applies to a broad class of objects, not just a particular instance, thereby reinforcing children's view of the generalized nature of a claim. In the example of evens and odds given earlier, children might represent what they notice in their own words as "the sum of an even number and an odd number is odd." They might represent generalizations in other ways, such as with variable notation. For example, a child might represent the Commutative Property of Addition as $a + b = b + a$, where, for the young child, a and b represent the counting numbers. Later, as students become more sophisticated, this number domain expands to include all real numbers.

2.3.1.3 Justifying Generalizations

In justifying generalizations, students develop mathematical arguments to defend or refute the validity of a proposed generalization. In elementary grades, the forms of arguments students make are often naïve empirical justifications. Research shows, however, that they can develop more sophisticated, general forms that are not based on reasoning with particular cases (Carpenter et al. 2003; Schifter 2009). For example, students might build "representation-based arguments" (Schifter 2009) where they use drawings or manipulatives to justify the arithmetic relationships they notice. In building an argument as to why the Commutative Property of Addition is reasonable,[2] a child might construct a snap-cube "train" of 3 red cubes followed by 4 blue cubes and visually demonstrate that the sum of the cubes (i.e., the length of the train) does not change when one flips the train around to become a 4-blue-cube, 3-red-cube "train." In a representation like this, the actual number of cubes is treated algebraically as a place-holder for any number of cubes. That is, the "3" and "4" become irrelevant in the more general justification the child is making.

[2]Technically, such properties are axioms and assumed to be true without proof. However, it is productive for children to think about why such properties are reasonable.

There are long-term dividends for engaging children in the practice of justifying the mathematical generalizations they make. For instance, Morris (2009) notes that the development of children's capacity to justify relationships about generalized quantities can help prepare children for a more formal study of proof in later grades. As such, justifying generalizations is an important act of algebraic reasoning.

2.3.1.4 Reasoning with Generalizations

Finally, algebraic thinking involves reasoning with generalizations as mathematical objects themselves. In this practice, children act on the generalizations they have noticed, represented, and justified to be true as objects of reasoning in new problem scenarios. For example, elsewhere we have observed young children building functional relationships that they represent with variable notation and with which they can reason as objects in solving new problem situations (Blanton et al. 2015a). Returning again to the example of evens and odds, a child might use previously noticed generalizations such as "the sum of an even number and an odd number is odd" to reason about the sum of three odd numbers. Cognitively, we see this type of reasoning as signifying an advanced point of concept formation in which the generalization has been reified in the child's thinking (Sfard 1991). Thus, cultivating this practice represents an important objective in learning to think algebraically.

2.3.2 Content Strands and Their Relation to Our Framework

Kaput (2008) further argued that Core Aspects (a) and (b) occur across three content strands:

1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics (p. 11).

Early algebra research has matured around several core areas relative to these content strands. Elsewhere (e.g., Blanton et al. 2015b), we have parsed these core areas, with three predominant areas being (1) generalized arithmetic; (2) equivalence, expressions, equations, and inequalities; and (3) functional thinking. We take generalized arithmetic to involve generalizing, representing, justifying, and reasoning with arithmetic relationships, including fundamental properties of operations (e.g., the Commutative Property of Multiplication) as well as other types of relationships on classes of numbers (e.g., relationships in operations on evens and odds). We take equivalence, expressions, equations, and inequalities to include developing a relational understanding of the equal sign and generalizing, representing, and reasoning with expressions, equations, and inequalities, including in their symbolic forms. Finally, we take functional thinking to include generalizing relationships between co-varying quantities and representing, justifying, and reasoning with these generalizations through natural language, variable notation, drawings, tables, and graphs.

Areas 1 and 2 align with Kaput's Strand 1, while Area 3 aligns with Strands 2 and 3. Although Kaput's content analysis—and our interpretation of it in our research—is not the only way to organize the content strands (or, our core areas) in which algebraic thinking practices occur, we do see this framework as reasonable and consistent with other perspectives (e.g., Carraher and Schliemann 2007; Cooper and Warren 2011).

2.4 Designing an Early Algebra Curricular Approach Using Our Algebra Framework

We expanded our algebra framework to establish an approach to teaching and learning early algebra that included an articulation of a curricular progression with associated learning goals, an instructional sequence to accomplish these goals, assessments to measure learning within the instructional sequence, and a characterization of students' ways of thinking as a result of their learning within the instructional sequence. This initial work, characterized in this chapter as Project LEAP,[3] focused on Grades 3–5. In particular, our approach built on the body of work concerning learning progressions and learning trajectories in educational research (Barrett and Battista 2014; Clements and Sarama 2004; Daro et al. 2011; Duncan and Hmelo-Silver 2009; Shin et al. 2009; Simon 1995), which uses an integrated approach to both supporting students' learning and characterizing learning in the context in which it is supported. In what follows, we briefly elaborate the theoretical foundations, methods, and design principles that guided our curricular approach to early algebra across Grades 3–5. Broadly, we followed a learning progressions approach to developing coherent curricular products (Battista 2011; Shin et al. 2009), instruction that targets students' development of understandings over a large span of time (Shwartz et al. 2008), and assessments to measure sophistication in student thinking over time (Battista 2011). In what follows, we describe the components of our learning progression.

[3]We use the term "LEAP" (Learning through an Early Algebra Progression) here in reference to our Grades 3–5 suite of projects that focused on understanding the impact of a systematic, multi-year approach to teaching and learning algebra in the elementary grades.

2.4.1 Curricular Progression

Our curricular progression elaborates finer grain sizes of the algebraic concepts and practices to be learned within each core area and at each grade level.[4] To construct this, we conducted a research synthesis and textbook analysis to specify (1) appropriate algebraic concepts and practices (e.g., a relational understanding of the equal sign; generalizing a functional relationship between two quantities) within our core areas, and (2) learning goals that characterized the depth of understanding that might reasonably be expected at each grade level and which could guide the design of learning activities for our instructional sequence. Finally, we sought external review of our proposed curricular progression to validate its consistency with empirical research and teaching and learning standards.

A guiding design principle for our curricular progression is to build sophistication in learning goals over time, starting from students' experiences and prior knowledge. Following Battista (2004) and as elaborated in Fonger et al. (in press), we balanced a dual lens on empirical research on students' understandings with an eye toward the canonical development of algebra over time in accordance with mathematical sophistication. This lens supported our specification of how we sequenced and ordered content across the grades. Our curricular progression served as a blueprint for designing an instructional sequence.

2.4.2 Instructional Sequence

Our instructional sequence is an ordered set of lessons across Grades 3–5[5] designed to build in complexity over time and to weave together the core areas (e.g., generalized arithmetic, functional thinking) and algebraic thinking practices (e.g., generalizing and representing generalizations) to support teaching and learning early algebra in an integrated way. Each grade level sequence consists of approximately 18 one-hour lessons that are intended to be taught along with the regular mathematics curriculum. While we follow a proposed sequence during implementation for our research purposes, there is flexibility with how teachers might incorporate lessons into their existing curriculum to accommodate their needs.

Using the curricular progression as a framework, we designed tasks or modified existing tasks from research that showed potential to facilitate students' construction of algebraic ideas (Clements and Sarama 2014), then built a core sequence of lessons using these tasks. We refined our instructional sequence through cycles of testing and revision. Moreover, we sequenced the introduction of core areas to

[4]We elaborate on this curricular approach in Fonger et al. (in press).

[5]Ultimately, our aim is to develop a Grades K–5 sequence. Our decision to focus initially on Grades 3–5 was guided largely by the more extensive early algebra research base available in upper elementary grades.

generally start from equivalence and a relational understanding of the equal sign, transition to generalized arithmetic and a study of fundamental properties of number and operation as well as other arithmetic generalizations, then progress to a study of generalized (indeterminate) quantities as a gateway for representing and reasoning with relationships between quantities through equations, inequalities, and functional relationships. Table 2.1 illustrates the lesson sequence and learning goals for Grade 3. Instructional sequences for Grades 4 and 5 were similar.

2.4.3 Assessments

We developed grade level assessments across Grades 3–7 to measure progress in the development of students' algebraic thinking in response to their participation in the Grades 3–5 instructional sequence and to monitor retention of that knowledge after the intervention (i.e., in Grades 6–7). Key algebraic concepts and practices identified in our curricular progression were used to design tasks that formed the basis for these grade-level, one-hour assessments. We designed assessment items to have multiple points of entry (e.g., students might use different strategies to solve a particular problem) and to include common items across several grades as a means to track growth over time. To strengthen the validity of our assessments, experts on teaching and learning algebra evaluated the extent to which the proposed assessment items aligned with algebraic concepts and practices in each of the core areas, and assessments were administered to elementary grades students and tested for psychometric soundness. The assessments have provided a critical means to measure effectiveness of our instructional sequence (see Sects. 2.5.1–2.5.3).

2.4.4 Student Thinking

We characterize students' thinking according to levels of sophistication, or qualitatively distinct ways of thinking, as evidenced in the strategies students use in written assessments and individual interviews. To strengthen the validity of our classification of student thinking, we accrued evidence of and distilled patterns in students' thinking over the span of several years (Stephens et al. in press). In our approach, the levels of sophistication observed in students' thinking is inseparable from the curricular and instructional context in which the learning was supported (see also Clements and Sarama 2004, 2014). In other words, the learning goals established in our curricular progression (and, subsequently, our instructional sequence) guide and support learning, while assessments measure that learning and levels of sophistication are the means by which we qualitatively characterize the nature of learning in that context over time.

Table 2.1 Overview of the instructional sequence for Grade 3

Lesson sequence and focus	Learning goals
Relational understanding of the equal sign (*Lessons 1–2*)	• Identify meaning of '=' as expressing a relationship between quantities • Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalence relationship (true/false number sentences) • Solve missing value problems by reasoning from the structural relationship in the equation (open number sentences)
Fundamental properties: additive identity, additive inverse, commutative property of addition, and multiplicative identity; Arithmetic relationships involving classes of numbers (e.g., evens and odds) (*Lessons 3–6*, *11*)	• Analyze information to develop a generalization about the arithmetic relationship • Represent the generalization in words • Develop a justification to support the generalization's truth; examine *representation-based arguments* (Schifter 2009) vis-à-vis empirical arguments • Identify values for which the generalization is true • Represent the generalization using variables • Examine the meaning of repeated variables or different variables in an equation representing a generalization • Examine values for which the generalization is true • Identify a generalization in use (e.g., in computational work)
Modeling problem situations with (linear) algebraic expressions (*Lesson 7*)	• Identify a variable to represent an unknown quantity • Informally examine the role of variable as a varying quantity • Represent a quantity as an algebraic expression using variables • Interpret an algebraic expression in context • Identify different ways to write an expression
Modeling and solving problem situations involving one-step, single variable linear equation (additive or multiplicative) (*Lessons 8–10*)	• Model a problem situation to produce a linear equation ($x + a = b$ or $ax = b$) • Identify different ways to write the representative equation • Analyze the structure of the equation to determine the value of the variable • Check the solution to an equation or determine if the solution is reasonable given the context of the problem • Informally examine the role of variable as an unknown, fixed quantity

(continued)

Table 2.1 (continued)

Lesson sequence and focus	Learning goals
Modeling problem situations involving linear functions of the form $y = x + b$, $y = mx$, or $y = mx + b$ with diverse representations (e.g., variables, words, graphs) and exploring function behavior (*Lessons 12–18*)	• Generate data and organize in a function table • Identify variables and their roles as varying quantities • Identify a recursive pattern, describe in words, and use to predict near data • Identify a covariational relationship and describe in words • Identify a function rule and describe in words and variables • Use a function rule to predict far function values • Examine the meaning of different variables in a function rule • Justify why a function rule accurately represents the problem data • Recognize that corresponding values in a function table must satisfy the function rule • Construct a coordinate graph to represent problem data • Given a value of the dependent variable and the function rule for a one-operation function, determine the value of the independent variable

2.5 Evidence of Growth in Students' Algebraic Thinking

It is reasonable to ask whether young children can successfully engage with a curricular approach such as that described here, that is, one that captures such a broad expanse of algebraic concepts and thinking practices across the elementary mathematics curriculum. This seems to be a tall order in an already crowded general mathematics curriculum, at least in the US. Our perspective, however, is that early algebra is not an "add-on" to existing school mathematics, but a means to help children think more deeply about that very content (Kaput and Blanton 2005). Early algebra has the potential to embed arithmetic concepts in rich algebraic tasks in ways that can deepen children's understanding of arithmetic concepts. In this sense, early algebra does not introduce a dichotomy in school mathematics (i.e., arithmetic *or* algebra), but is a means by which children—some of whom may already be struggling with arithmetic—can build deep mathematical knowledge with understanding. Our tasks are often designed to highlight this nexus between algebraic and arithmetic thinking by using arithmetic work as a springboard for noticing, representing, and reasoning with structure and relationships in number and operations. Moreover, we aim to facilitate the development of algebraic thinking—and mathematical understanding more broadly—through learning environments that rely on both small-group investigations of open-ended tasks where students represent their ideas in different ways (e.g., through drawings, written language, variable notation, and graphs) and rich classroom discourse that supports developing fluency with algebraic concepts and practices.

In this context, we examine next some of the evidence from several studies conducted by our project team that supports the viability of our approach. We look at evidence from two lines of research: quantitative studies conducted in Grades 3–5 (Project LEAP) as well as exploratory studies in Grades K–2 aimed at characterizing the cognitive foundations of children's algebraic thinking at the start of formal schooling. As described earlier, Project LEAP goals included the design of an instructional sequence for Grades 3–5, and we report here on studies addressing its effectiveness. We view the exploratory Grades K–2 work as prerequisite to the kind of systematic design and development that occurred in Project LEAP. Both serve our broader goal of developing a Grades K–5 instructional approach to early algebra education that has been rigorously tested for its ability to develop children's algebraic thinking and their readiness for a formal study of algebra in middle grades.

2.5.1 Project LEAP: Grade 3 Intervention

In Blanton et al. (2015b) we reported on our first quasi-experimental study designed to measure the effectiveness of the Grades 3–5 instructional sequence developed as part of Project LEAP.[6] We compared the algebra learning of third-grade students who were taught the Grade 3 sequence to students in a demographically and academically comparable control group. Approximately 100 students participated in the study. The Grade 3 sequence used in the intervention consisted of 19 one-hour lessons taught over the course of the school year by a member of our research team. Each lesson involved a preliminary small group activity that either reviewed previously taught concepts or previewed concepts addressed in the upcoming lesson. The remainder of the lesson focused on small group explorations in which students discussed a problem activity, collected and organized their data, looked for relationships, and represented the relationships through words, drawings, or variable notation. This was followed by whole-class discussions that revolved around teacher questioning designed to engage students in discussing their thinking about the generalizations they noticed, the nature of their representations, and why they viewed their observations as valid. Lessons focused on eliciting students' higher order thinking through both written and oral communication.

Control students were taught only their regular mathematics curriculum. All students were given our written, one-hour LEAP algebra assessment as a pre/post measure of shifts in their understanding of core algebraic concepts and practices.

[6]The LEAP Grades 3–5 instructional sequence and associated assessments are available upon request to Maria_Blanton@terc.edu.

From our analysis of student responses to the pre/post-assessment reported in Blanton et al. (2015b), we found that there were no significant differences between the two groups in terms of overall performance (percent correct) at pretest ($M = 18.22$, $SD = 12.36$ for the experimental group; $M = 14.99$, $SD = 10.58$ for the control group; $F = 2.01$, $p = 0.16$, $d = 0.28$). However, the experimental group showed significantly greater pre-to-post gains than the control group ($M = 65.51$, $SD = 21.01$ for the experimental group; $M = 21.97$, $SD = 15.37$ for the control group at post-assessment; $F = 143.6$, $p < 0.001$, $d = 2.37$). At the item level, the experimental group showed statistically significant pre-to-post gains for all but two of the pre/post-assessment's 19 questions. The control group did not show statistically significant pre-to-post gains on any of the assessment items. These results suggest that, overall, students as early as Grade 3 (approximately 9 years old) can successfully engage with core algebraic thinking concepts and practices over a broad expanse of algebraic ideas—as reflected in the algebra framework used in our approach—far beyond the occasional algebraic concept that they might otherwise see in their regular curriculum. At the same time, the business-as-usual curriculum control students received seemed to do little by way of developing students' algebraic understanding.

Moreover, as we reported in more detail in Blanton et al. (2015b), we also coded students' strategy use in their assessment responses so that we could more closely detail shifts in students' thinking. We found that experimental students exhibited more algebraic approaches to problem solving than did their control peers. This included that experimental students were more likely to interpret the equal sign relationally rather than operationally (Carpenter et al. 2003), correctly solve linear equations using strategies that invoked inverse operations, recognize varying quantities and represent operations on such quantities as algebraic expressions, recognize structural characteristics of equations (e.g., the Commutative Property of Addition) and develop arguments that invoked this structure, and recognize and represent with both words and variable notation relationships between two co-varying quantities.

2.5.2 Project LEAP: Grades 3–5 Intervention

Given the results of our Grade 3 study, we conducted a second quasi-experimental study with the goal to more extensively test the effectiveness of our Grades 3–5 instructional sequence.[7] In this sequel study, we compared the algebraic thinking of students who participated in a 3-year, longitudinal implementation of our Grades 3–5 instructional sequence to students in more traditional (arithmetic-focused) classrooms. Additionally, we followed these students into Grade 6 in a follow-up study

[7]See Blanton et al. (2017b) for a more detailed account of this study.

to assess retention of or shifts in their algebra knowledge (no intervention was provided in Grade 6).

Participants (n = 165) in the study were from two schools, one designated control and one designated experimental. One member of our project team taught the 3-year intervention in the designated experimental school, beginning with a Grade 3 cohort and continuing with this cohort for 3 years. Approximately 18 lessons were taught at each of Grades 3–5 as part of students' regular mathematics instruction. Students in both experimental and control schools were assessed at the beginning of Grade 3 (baseline data) and at the end of Grades 3, 4, and 5 using the one-hour, grade-level written algebra assessments developed in our curricular progression (see Sect. 2.4.3).

Students' performance (correctness) on common assessment items[8] was compared over time (Grades 3–5) and by group (experimental and control). Results of a two-factor, mixed-design ANOVA showed significant main effects for both experimental condition, $F(1, 144) = 137.03$, $p < 0.01$, $h^2 = 0.49$, and grade level, $F(3, 432) = 736.66$, $p < 0.01$, $h^2 = 0.78$, as well as a significant interaction between the two, $F(3, 432) = 70.29$, $p < 0.01$, $h^2 = 0.15$. Simple main effects tests revealed that there were no significant differences between experimental and control students at baseline (beginning of Grade 3), $F(1, 144) = 1.46$, $p = 0.23$. However, experimental students significantly outperformed control students at each subsequent time point: Grade 3 post-test, $F(1, 144) = 205.88$, $p < 0.01$; Grade 4, $F(1, 144) = 99.74$, $p < 0.01$; and Grade 5, $F(1, 144) = 103.28$, $p < 0.01$ (see Fig. 2.1).

We note that the intervention had the most impact at Grade 3, as indicated by the decreasing rate of performance of experimental students after Grade 3 (although experimental students' performance still improved year to year). We also note that by Grade 4, control students were being introduced to some of the algebraic concepts that were addressed in the intervention as part of their regular classroom instruction. As such, we think it is reasonable that there is a jump in their performance beginning in Grade 4. However, shifts in experimental students' overall performance (correctness) on the Grade 3 pre-assessment to the Grade 5 post-assessment from 22 to 84% offers perhaps even stronger evidence that elementary-aged students can successfully engage in a broad expanse of algebraic practices and concepts, as reflected in our algebra framework. Moreover, we suggest that the absence of a sustained, multi-year approach to fostering algebraic thinking leaves students significantly less prepared for algebra in middle grades, as indicated by control students' shifts on overall correctness from 20% (Grade 3 pre-assessment) to 61% (Grade 5 post-assessment). It is a positive result that there were shifts in control students' algebraic thinking by Grade 5 and, in our view, this reflects long-term efforts to integrate algebraic thinking into elementary grades. However, the difference in gains for the two groups shows that significant opportunities stand to be missed in current educational practice.

[8]Nine items were common across all Grades 3–5 assessments.

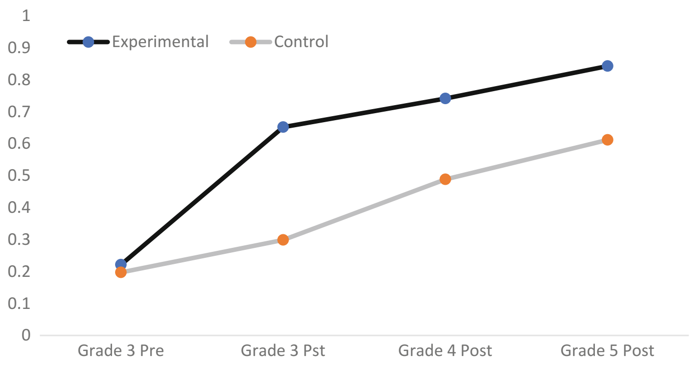

Fig. 2.1 Comparison of overall percent correct on Grades 3–5 common assessment items

To unpack these results further, we look here at students' performance (correctness) across 4 time-points—Grade 3 pre/post and Grades 4–5 post—on an item that captures how students were able to represent generalized quantities, an important transition point in the development of algebraic thinking. Although generalizing has rightfully received much attention as the heart of algebraic thinking (Cooper and Warren 2011; Mason 1996), Kaput (2008) argues for the equal importance of representing, or *symbolizing*, a generalization. On the following item, students were asked to represent and reason with generalized (varying) quantities[9]:

Piggy Bank Problem. *Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.*

a. *How would you represent the number of pennies Tim has?*
b. *How would you represent the total number of pennies Angela has?*
c. *Angela and Tim combine all of their pennies. How would you represent the number of pennies they have all together?*

Results (see Fig. 2.2) show that experimental students made greater gains in representing Tim's and Angela's numbers of pennies (parts a and b), as well as their combined number of pennies (part c), than did their control peers. We considered a correct response[10] to these items to be a letter to represent Tim's number of pennies (e.g., n), a related algebraic expression for Angela's number of pennies (e.g., $n + 8$), and a related expression such as $n + n + 8$ for the combined number of pennies. Experimental students correctly represented Tim's number of pennies with

[9]Adapted from Carraher et al. (2008).

[10]We recognize that a child might give a response such as n, m, and $n + m$, for parts a, b, and c, respectively. In a further analysis of strategy, we considered such responses. However, for overall correctness, we considered only the most stringent case in which students accounted for the fact that Angela and Tim had the same number of pennies in their banks in their representations.

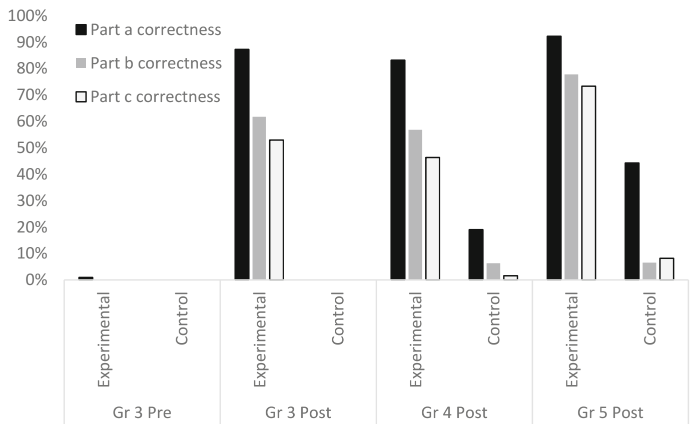

Fig. 2.2 Comparison of experimental and control performance (correctness) on the Piggy Bank Problem, parts a–c

variable notation (part a) at a rate of 1, 87, 83, and 92% across the 4 assessments, respectively. By contrast, only 0, 0, 19, and 44% of control students could do so. Students who could not correctly represent Tim's number of pennies with variable notation typically assigned this quantity a numerical value.

Similarly, experimental students made greater gains than control students in representing Angela's number of pennies as an algebraic expression (part b) across the 4 assessments (0, 62, 57, and 78% respectively). Meanwhile, only 0, 0, 6, and 7% of control students could correctly represent Angela's number of pennies across the 4 assessments. Finally, experimental students made greater gains in representing the combined number of pennies with an algebraic expression (part c) across the 4 assessments (0, 53, 46, and 73%, respectively) than did control students, whose overall percent correct was 0, 0, 2 and 8% across the 4 assessments, respectively.

We find these results to be compelling for various reasons. First, this is a particularly complex problem for young children because in an arithmetic-saturated experience, they have not learned to "see" and mathematize variable quantities in problem situations (see, e.g., Blanton et al. (2015a) for a treatment of progressions in young children's understanding of variable and variable notation). As such, even a simple task such as representing Tim's number of pennies is often beyond their perceptual field, as indicated by their action of assigning a numerical value to a varying quantity. In our view, a first step in understanding algebraic concepts such as those addressed in the Piggy Bank Problem is learning to perceive and represent a variable quantity (i.e., part a), after which students might notice and represent relationships between quantities (as in parts b and c).

Secondly, these results show that, unlike control students, experimental students were very successful at representing generalized quantities with variable notation. Moreover, experimental students were able to use variable notation in meaningful ways (e.g., they understood that the same letter was to be used to represent the number of pennies in each of Tim's and Angela's bank, since the number of coins was the same but unspecified). This calls into question the conventional wisdom that younger students are not "ready" for variable notation and should use those representational systems that are already available to them—particularly, natural language and drawings—to represent variable quantities, rather than variable notation (e.g., Nathan et al. 2002; Resnick 1982).

Finally, to test the claim that we set at the beginning of the study regarding whether participating in our instructional sequence would impact students' algebraic thinking in middle grades, we followed Grades 3–5 students into middle school and administered our Grade 6 algebra assessment (see Sect. 2.4.3) at the end of Grade 6. No intervention was given. We found that the experimental students (n = 46) significantly outperformed the control students (n = 34), with an overall correctness of 52% (experimental) versus 44% (control) on this assessment[11] one year after the early algebra intervention ended. In Isler et al. (2017), for example, we found that experimental students remained more successful in generalizing functional relationships and representing them in words and variables than did control students. Experimental students were able to correctly generalize and represent a functional relationship in words and variable notation at a rate of 48 and 65%, respectively, while control students were able to do so only at a rate of 26 and 41%, respectively. Such results suggest that when students experience a broad, sustained approach to early algebra instruction, they are better positioned for success in algebra in middle grades.

2.5.3 Project LEAP: Examining a Teacher-Led Grades 3–5 Intervention

Findings from our previous Project LEAP studies, summarized above, have led to a longitudinal, randomized study in 46 participating schools where we are currently following a Grade 3 cohort across Grades 3–5 as experimental students receive the intervention and control students receive their regular instruction. A key difference in this study was its experimental design (randomized) and the fact that teachers led the intervention as part of their regular classroom instruction. The utilization of classroom teachers to lead instruction is a core component of testing the efficacy of our intervention. It holds unique challenges that lie in the fidelity with which teachers might implement the sequence across different instructional settings, given their own varied professional experiences. To increase their fidelity of implementation, we provided all participating teachers with long-term professional development to strengthen their knowledge of algebraic concepts and practices, as well

[11]It should be noted that the analysis for Grade 6 data was for all items on the assessment (not just items common with the Grades 3–5 assessments) and that it included new, more difficult items.

as their understanding of students' thinking about these concepts and practices and how to craft classroom discourse that engaged students in dialogue around them.[12]

Results thus far show that, although there was no significant difference between experimental and control groups on the Grade 3 algebra assessment given at pre-test, experimental students significantly outperformed control students ($p < 0.001$) in overall performance on this assessment administered at post-test (see Fig. 2.3). In particular, participation in the intervention was associated with a 13% increase in post-test score compared to the control group, suggesting that the Grades 3–5 instructional sequence we designed using Kaput's (2008) conceptual framework shows potential to positively change the way children think algebraically in elementary grades and their potential for success in middle grades. We note, however, that improvements in overall performance for Grade 3 experimental students in this teacher-led study are not as robust as those for our previous interventions led by our research team (e.g., see Fig. 2.1). One possible explanation for this difference could be the diverse fidelity with which teachers implemented the intervention as opposed to the fidelity of implementation for a researcher with extensive knowledge of and instructional experience with the intervention.

2.5.4 Extending Our Work into Earlier Grades

Ultimately, early algebra is intended to be a focus of mathematics curriculum and instruction in the US across all of elementary grades, beginning in kindergarten (NGA & CCSSO 2010). A natural progression of research for us, then, is to consider how the conceptual framework we applied in our Grades 3–5 work might translate into earlier grades. We have initiated exploratory, qualitative studies on this, with the goal of understanding the genesis of algebraic thinking practices in children's thinking in Grades K–2. We provide a brief overview of some of our findings here.

In Blanton et al. (2015a), we provided evidence that Grade 1 (age 6) students participating in an 8-week classroom teaching experiment that focused on functional thinking could generalize, represent, and reason with (linear) functional relationships. As reported in that study, we developed a learning trajectory to describe first-grade participants' thinking about generalizing functional relationships, analyzing data from children's pre-, mid-, and post-instruction interviews. In particular, we identified eight different levels of thinking, ranging from *pre-structural* to *function-as-object*, exhibited by participants as they advanced through the teaching experiment.

At the pre-structural level, children could not describe or even implicitly use any kind of mathematical relationship in talking about function data. At the function-as-object level, some children had progressed to an ability to generalize

[12]For our analysis of teachers' fidelity of implementation, see Cassidy et al. (to appear).

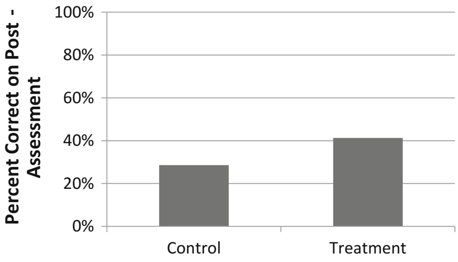

Fig. 2.3 Comparison of Grade 3 students' overall performance (correctness) on the Grade 3 algebra post-test in the teacher-led study

and represent a functional relationship with words and variable notation and reason with their symbolic rule as an object for exploring novel scenarios. For example, by the end of the teaching experiment some children were able to generalize a relationship between the number of cars in a train and the number of stops the train made, where it was assumed the train picked up two cars at each stop and the engine (the only "car" on the train before the first stop) was not counted.

One child represented this relationship as $R + R = V$ and described R as representing the number of stops and V as representing the number of cars. When asked how the relationship would change if the engine was counted, she noted that she would "just add 1" and represented this as $+1 + R + R = V$. In other words, she was able to reason with her first function rule as an object in order to solve the new problem and did not have to reconstruct a new function table and find a new relationship independently of her original one. In essence, at this advanced level students who were able to reason in this way no longer viewed the original rule as a *process* (Sfard and Linchevski 1994) of operating on numbers but instead were able to transform a function rule as an *object* (Cottrill et al. 1996; Gilmore and Inglis 2008). Moreover, children who exhibited thinking at this level understood boundaries concerning the generality of the relationship and conditions under which the generalization would not hold.

Elsewhere (Blanton et al. 2017a), we reported on how Grade 1 students participating in this study understood variable quantities and variable notation in the context of functional relationships. Again, as students progressed through the teaching experiment we found that their thinking advanced from what we characterized as a *pre-variable/pre-symbolic* level to a *letter as representing variable as mathematical object* level. We argue that at the most primitive level, students did not recognize a variable quantity in a mathematical situation and could not use or did not accept the use of any symbolic notation to represent such a quantity. As students progressed through the sequence, some were ultimately able to use variable notation to represent functional relationships and to reason with these symbolic rules.

In a related study, Brizuela et al. (2015) illustrate the variety of understandings about variable and variable notation held by Grade 1 children, including that (1) variable notation can signify a label or object; (2) variable notation can represent an indeterminate quantity; (3) quantitative relationships can be expressed through the ordinal relationships between letters in the alphabet; and (4) the inclusion of both letters and numbers in a single equation should be avoided. We also observed that these children were able to act on a mathematical expression that includes variable notation as a mathematical object. Our findings illustrate that given the opportunity, even very young children can use variable notation with understanding to express relationships between varying quantities. We argue that the early introduction of variable notation in children's mathematical experiences can offer them opportunities to develop familiarity and fluency with this convention. This raises an interesting question relative to prior research that has documented secondary school students' difficulties with variables and variable notation (e.g., Knuth et al. 2005; Küchemann 1981) and whether such difficulties might be ameliorated by a sustained introduction to variable and variable notation from the start of formal schooling.

2.6 Conclusion

Our goal here has been to describe the conceptual framework of our approach to early algebra, how we are enacting that framework through the design of a curricular approach to algebra instruction in the elementary grades, and a brief overview of some of our findings, reported in detail elsewhere, regarding the impact of this approach on children's algebraic thinking. Ultimately, our program of research aims to outline a curricular approach to teaching and learning algebra across Grades K–5 that can positively impact students' readiness for and success in algebra in middle and high school grades. Collectively, our studies contribute evidence to the perspective that elementary-aged children can engage in sophisticated practices of algebraic thinking—generalizing, representing, justifying, and reasoning with mathematical structure and relationships—across a broad set of core content areas involving generalized arithmetic; concepts associated with equivalence, expressions, equations, and inequalities; and functional thinking.

We have found that a research-based, comprehensive early algebra intervention across upper elementary grades (i.e., Grades 3–5) can statistically improve children's algebraic understanding and potentially improve their algebra-readiness for middle grades. Further, we have found that in lower elementary grades students exhibit a capacity for algebraic thinking beyond what we had originally hypothesized as possible. Our observations of children's algebraic thinking have been perhaps most striking in the early elementary grades (particularly, Grades K–1). Indeed, prior to our studies, we assumed that children in these early grades might have even more difficulty with the algebraic concepts with which adolescents so often struggle—for example, the object–quantity confusion associated with variables (McNeil et al. 2010) or the difficulty in shifting students' perspectives away from recursive thinking towards functional thinking (Cooper and Warren 2011). We have found instead that, in these early grades, children are far more able to think algebraically than we anticipated. In our view, providing sustained experiences, *from the start of formal schooling*, with the conceptual approach to early algebra described here holds promise for ameliorating the deeply held difficulties and lack of success that students have historically had with high school algebra in the US.

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Chapter 3
New Words and Concepts for Early Algebra Teaching: Sharing with Teachers Epistemological Issues in Early Algebra to Develop Students' Early Algebraic Thinking

Nicolina A. Malara and Giancarlo Navarra

Abstract We present the ArAl project, conceived as an integrated system of teacher education and classroom innovation, aiming at renewing the teaching of arithmetic in an early algebra perspective, guiding pupils towards the discovery of letters to express generalities. We focus on some theoretical key points (KP) and on the main language constructs (LC). Through excerpts of class-discussions, we show the incidence of KP and LC on the progressive construction and refinement of pupils' early algebraic thinking. Finally we discuss the difficulties met by teachers at the K–8 levels when they reflect upon their knowledge, beliefs, attitudes, and ways of relating with the pupils.

Keywords Early algebra · Algebraic thinking · Algebraic babbling
Metacognitive teaching · Noticing · Professional development of teachers
Linguistic approach to early algebra

3.1 Epistemological Roots of Early Algebra

Early algebra can be considered the result of a long process of teaching innovation, started in the 1960s after the modern mathematics movement, which rejected the conception of mathematics as an abstract, static, and isolated discipline in favor of a dynamic and evolutionary vision of it, rooted in the concrete world and open to

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interactions with different disciplines and contexts. Moreover, a methodological revolution shifted the attention from the passive learning of mathematical facts towards problem solving and mathematical discovery.

According to these new visions, there is a widespread awareness of the importance of investing in studies dedicated to the problems of teaching and learning, and of improving the culture of teachers to cope with the novelties. As far as the teaching of algebra is concerned, the first studies were diagnostic, looking for the most widespread, erroneous performance in students, for example, Küchemann's study under the heading of *Algebra* in Hart (1981). Particularly meaningful have been Kieran's surveys (1989, 1992), which clearly show how the difficulties with algebra are mainly due to the traditional teaching of arithmetic, teaching that completely disregards relational aspects and the control of meanings implied by calculation processes. Other studies have concentrated on projects of curricular innovation. The pioneering work in this area includes the English projects on the teaching of mathematics in Grades 6–10 (e.g., Bell et al. 1985; Harper 1987), which promoted an approach to algebra centered, from the very beginning, on problem solving, generalization, and modeling. In this context it is important to recall the new perspectives on algebra in the early grades that were proposed at ICME-5 (in Adelaide in 1984) inside the Working Group (WG) on algebra teaching and learning (Davis 1985).

In those years, scholars had different opinions regarding the relationship between arithmetic and algebra in education. Some scholars underlined the epistemological rupture between the two areas (e.g., Filloy and Rojano 1989; Herscovics and Linchevski 1994); others looked at them with a perspective of continuity, stressing their mutual synergies (e.g., Chevallard 1989, 1990). Nevertheless, there was general agreement on the need to set the foundations of the teaching of arithmetic within a relational perspective with respect to algebra. The main objectives were: (a) to overcome stereotypes such as the directional equal sign, or the lack of closure in arithmetic expressions; (b) to induce a structural vision of arithmetical expressions, detecting equivalences or similarities; and (c) to open the way to generalization and modeling for the genesis of the objects of algebra.

The acknowledgement of the importance of these issues was highlighted at the WG on algebra at ICME-7 (in Quebec City in 1992), where a specific area of teaching called pre-algebra was identified as a bridge between arithmetic and algebra. In that context, it was stressed that "*Within primary-school arithmetic there is ample opportunity for the development of algebraic thought*" and that "*letters could be used within children's arithmetic experience in order to facilitate their understanding of the meaning and significance of letters in later, formal algebra*" (Linchevski 1995, p. 114). At that time, moreover, the activities of syntactic transformation were no longer seen as isolated, but rather in their relationship with activities of representation and interpretation: Bell (1996) has spoken of the *essential algebraic cycle,* unifying in a whole the triad *representing, transforming, interpreting,* and their reciprocal interrelationships. Other scholars (e.g., Arcavi 1994; Arzarello et al. 1993; Boero 2001; Mason 1996) have taken into consideration also the metacognitive dimension, shifting the attention towards the control of the properties that legitimate the processes of syntactic transformation and the activation of anticipatory thinking. This control is seen as the ability to foresee (without carrying out syntactic transformations) new possible forms of an expression, by checking its meanings with reference to a given aim, or to hypothesize formal writings to be reached in order to achieve specific results. Arcavi (1994) summarizes all these aspects in the *symbol sense* construct. With reference to generalization, Mason (1996) maintains that pupils should be led to conquer the double awareness of *seeing the general in the particular* and *seeing the particular in the general* (p. 21) and, most of all, to become aware of the plurality of cases contained in a general statement.

In the second half of the 1990s there was a flourishing of studies on these aspects, mainly targeted at pupils aged 11–13. Some of the studies stood out for theorizing, within the framework of a linguistic vision of algebra, models of conceptual development of a socio-constructive type (e.g., Da Rocha Falcão 1995; Meira 1996; Radford 2000). In the USA there was widespread agreement on the idea that primary school syllabuses should be re-arranged in this perspective for the social needs of the 21st century (Kaput 1998).

In the international arena scholars have assumed different positions about the early usage of letters. Carraher and Schliemann (2007) have distinguished two different lines of thinking, one that focuses on "pre-algebra" as a transition between arithmetic and algebra and which postpones the use of letters until arithmetical learning has reached the upper grades, and the other, usually referred to as "early algebra," which can include the early introduction of letters to promote relational thinking and the coordination of different registers of representations. Thus, the label *Early Algebra* has been introduced to refer to the initiation towards (a) generalizations of relationships or properties through the observation of similarities in various numerical cases, (b) the verbal formulation in general terms of observed relationships, and (c) the symbolic translation of verbal sentences and the approach to algebraic reasoning through syntactically guided actions on the formulae obtained. With the evolution that has taken place with the various studies related to early algebra, that corpus has been legitimized as a specific subject area by the international scientific community, as documented by: (a) the many interventions devoted to this theme at international conferences since the 12th ICMI study "The future of the teaching and learning of algebra" (Chick et al. 2001); (b) the production of specific monographs (Cai et al. 2005; Cai and Knuth 2011; Kaput et al. 2008); and (c) the collective studies on Early Algebra, such as the research forum at PME 25 (Ainley et al. 2001), the *Early Algebra* Conference organized by David Carraher (in Evron, France, in 1998), and more recently the Early Algebra Topic Study Group at ICME-13 (in Hamburg in 2016). Several of these studies have also dealt with the problem of suitable teacher education with focused intervention on professional development (see, e.g., Blanton and Kaput 2003; Carpenter et al. 2003; Russell et al. 2011).

Our research places itself within this frame and has developed a project for Grades K-8 (5–14 years of age) called the *ArAl Project: paths in arithmetic to favor pre-algebraic thinking* (Malara and Navarra 2003). We used the expression ‘pre-algebraic thinking’ according to the original meaning of the term ‘pre-algebra’ (as discussed at the ICME-7 WG on algebra). In our view, pre-algebraic thinking concerns not only the development of relational arithmetic, but also the progressive *construction* of the algebraic language and the *development of the habits of mind* that will allow pupils to use algebraic language as a tool for thinking. Pre-algebraic thinking occurs in all the activities aimed at *building in pupils an attitude to look for regularities, relationships, and properties, and to express* them first in natural, and then in algebraic, language. In this way, pupils acquire the experiential ground–layer to activate the essential algebraic cycle. This view fits with the characterization of ‘*early algebraic thinking*’ given by Kieran et al. (2016, p. 10). Then, (*later*) *algebraic thinking* is characterized by students’ *achievement of a robust algebraic understanding* (Schoenfeld 2013) that allows them to deal with non-trivial tasks. Therefore, we can say that early algebra develops pre-algebraic thinking or that it promotes algebraic thinking. However, to avoid misunderstandings associated with the term *pre-algebra*, we will now speak of our approach as one focused on ‘early algebraic thinking’.

3.2 Early Algebra Within the ArAl Project: The Main Key Points

The ArAl project proposes a socio-constructive approach to early algebra and is structured as an integrated system of teacher education that merges early algebra teaching experiments with teachers’ educational processes based on teachers’ practice. The schools involved are spread throughout all regions of Italy. The activities are realized within the frame of institutional programs of teacher professional development. The practices of sharing and reflecting with teachers on the classroom transcripts are realized via web (e-mail, Skype, forum, etc.) and in apposite meetings in the schools or at the university.

The main idea on which the project is based is that the algebraic language can be learned in analogy with the learning modalities of natural language. In our view, the algebraic language should be built right from the earliest years of primary school, having pupils face pre-algebraic activities and the use of letters to codify relationships. The discussions of comparison of the short sentences they produce—and the constant, collective reflections on the meanings of each expression—progressively bring them to master the syntactical rules of the algebraic language. By analogy with a young child’s babbling when she starts to learn the natural language, we call *algebraic babbling* this process of construction/interpretation/refinement of ‘raw’ algebraic sentences. To better understand the meaning of this construct, it must be considered together with the other theoretical points of the project that we now present.

In the ArAl project, the image of early algebra is expressed through a set of key words and concepts that refer both to arithmetic and algebra, but the two disciplines are seen as evolving towards a different and original identity. We can consider the combined two disciplines as a meta-discipline, concerning not so much objects, processes, and properties of arithmetic and algebra, but rather the genesis of a unifying language, that is, a meta-language. In order to control the meta-disciplinary knowledge of early algebra, we bring the teachers to command the meaning of words and linguistic constructs that represent new conceptions of intertwining between arithmetic and algebra. Later we discuss some of these terms, that turn out to be fundamental in order to generate—both in teachers and in pupils—ways of seeing that are suitable to the development of algebraic thinking. In this sense we speak of 'epistemological issues in early algebra.'

In order to facilitate teachers' acculturation, we have conceived and shared with them a set of glossaries that concern: theoretical frame, mathematical topics, methodological-didactic and social issues, and linguistic issues related to the managing of discussions with the class. The theoretical constructs, made explicit by suitable keywords, become a cultural patrimony for teachers, who are about to carry out an authentic Copernican revolution in their being and acting in the class: they become aware that social interaction, argumentation, and verbalization are key elements in the construction of knowledge and that a stable acquisition of meanings happens through activities that emphasize metacognitive and metalinguistic aspects. We stress that the aim of the project is to prepare *metacognitive students*. In order to achieve it, we need to educate *metacognitive teachers*.

We present now the main key points (KP) underpinning our project:

- **KP.1**. *The socio-constructive aspects of knowledge*, which are typical of 'doing mathematics,' nurtured by collective activity in the class. The social construction of knowledge, that is, the shared construction of new meanings, is negotiated on the basis of the cultural tools and skills available to pupils and teachers; the contents of arithmetic and algebra are central, they emerge and condense through the teacher's orchestration of the individual contributions.

- **KP.2**. *The aspects of generalization and interplay between arithmetic and algebra*, with the shift of attention—in the teaching of arithmetic—from the procedural point of view to the relational one, the construction of arithmetical sentences—the recognition of those that are equivalent and their transformation through the basic arithmetical properties—as well as the approach to letters as a means to express in general terms observed numerical regularities.

- **KP.3**. *Identifying and making explicit the algebraic thinking often 'hidden' in concepts and representations of arithmetic*. The genesis of the generalizing language can be located in this 'unveiling'—when a pupil starts to describe a sentence like $4 \times 2 + 1 = 9$ no longer (not only) as the result of a procedural reading, 'I multiply 4 times 2, add 1 and get 9', but rather as the outcome of a relational reading, such as 'The sum of the double of 4 and 1 is equal to 9'. Pupils talk about mathematical language using natural language and do not focus on numbers, but rather on relations and the structure of the sentence.

- **KP.4**. *The central role of natural language as the main didactical mediator within the slow construction of syntactic and semantic aspects of algebraic language*. Verbalization, argumentation, discussion, and exchange promote understanding and critical review of ideas. We foster the relational point of view that brings pupils to elaborate complete and coherent argumentations, to compare them in their meanings, and to deal with the translation of verbal sentences expressing observed relationships into formal terms. To motivate pupils to face this task, we have conceived of a character—Brioshi—a Japanese virtual pen friend who loves to exchange mathematical questions by e-mail and to communicate through mathematical sentences (Malara and Navarra 2001). The mathematical language becomes a tool for communication. In this process letters are introduced to represent unknowns or variables, and their meanings are shared by pupils through collective discussions in class. In the production of representations in algebraic language, natural language also plays a meta-language role because it allows discussion at a meta-level of the meanings of the choices made by individual pupils and of the naive sentences they produce. The progressive mastering of the use of letters throughout the translation between verbal and formal sentences, which is sharply linked with interpretative aspects, characterizes the core process of algebraic babbling.

- **KP.5**. *The devolution*[1] *to pupils of the generation/interpretation of formulas and the progressive construction/refining of algebraic language*. This aspect represents an important moment of condensation in the evolution of algebraic babbling. The pupils, during a collective exploration of a thought-provoking task, are guided towards the detection of a rule and the individual phrasing of a verbal sentence that represents it. This is an important step that facilitates the pupil's assumption of the task to translate the sentence into algebraic language and, vice versa, to interpret a formula in verbal terms. In this way pupils become producers of genuine mathematical thought, overcoming the role of passive performers (examples are shown later).

- **KP.6**. *The metacognitive aspects.* We promote the shifting from concrete generative situations to the construction of concepts through reflective activities in class; for instance, this is done for the purpose of reifying the properties of the arithmetical operations, but also for becoming aware of their role in counting strategies. We promote the detection of similarities in figural patterns or arithmetical/algebraic sentences and the recognition of structural analogies, the identification and verbalization of the reasons underlying syntactic transformations of formulas, and the generation and interpretations of new formulas from the perspective of the development of formal reasoning. We also favor the interpretation of a given formula in different contexts, so that it can be conceived as an object representing all its possible interpretations in different words. We extend the attention to other languages (iconic, graphic, …), bringing pupils to face questions about the coordination of different types of representation.

[1]'*Devolution*' is a term introduced by Brousseau (1997) in his *Theory of Didactical Situations in Mathematics (Didactique des Mathématiques 1970–1990)*. It indicates a process between the teacher and her students where she, in presenting a problematic situation, brings them to assume the responsibility to deal with it. The devolution is fulfilled when the students actually accept the uncertainty implied in this assumption and they take on the commitment.

- **KP.7**. *Problematizing the activities.* In the ArAl project each activity is posed as an exploratory situation. We illustrate this issue through two episodes.

  First episode: The writing 42 + 15 = 11 + 42 emerges in a class; the teacher problematizes the situation asking the pupils to assess its correctness without calculating, reasoning on both sides of the equal sign and looking at relationships between the terms. These are some of the pupils' justifications drawn from the classroom transcripts: (A) "It is not true that they are equal and I explain it in two ways: $42 + 15 \neq 11 + 42$ and $42 + 15 > 11 + 42$"; (B) "42 is in both sides, 15 is bigger than 11, then 42 + 15 is bigger than 11 + 42"; (C) "I have written the sentence putting a question mark on the equal sign,[2] so Brioshi understands: 42 + (11 + 4) =?= 11 + 42 → 42 + 11 + 4 − 11 =?= 11 + 42 − 11 → 42 + 4 =? = 42 → 46 ≠ 42".

  Second episode: The core of the well-known 'pyramid of numbers' activity is the 'mini-pyramid', that is, two side-by-side bricks upon which there is another brick. On the visible side of each brick a number is written; the rule is that the sum of the numbers on the two bricks below is shown on the top brick. A 1st grade teacher proposes a mini-pyramid where there are: on the top brick 18, on the left brick 7 and on the right one a spot of ice cream that hides the number. She assigns the task: "Represent in algebraic language this situation so that Brioshi can find the number below the spot". In this formulation the unknown number is no longer the *result* to be found, but one of the terms used to express, in several ways, the relationships between them; that is, 7 + ✹ = 18; 18 = 7 + ✹; ✹ + 7 = 18; 18 = ✹ + 7; 18 − 7 = ✹; ✹ = 18 − 7; 18 − ✹ = 7; 7 = 18 − ✹. Then the coded sentences are interpreted in natural language and reflections on the relational view of the equal sign are made. In this way what we call *equations for fun*, solvable without particular formal transformations, are generated (we propose them at 1st and 2nd grade).

- **KP.8**. *The algebraic verbal problems, the construction of equations, and their naïve solutions*. The problematization of situations, like the ones described above, allows an early approach to verbal problems with one unknown and not immediately solvable. Algebraic verbal problems constitute the topic of the ArAl Unit 6[3]: *From scales to equations*, for pupils of 4th–6th grade. It presents a connected set of teaching episodes, each based on a problem on the use of a pair

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[2]For typing questions, here we have written "=?=". In our teaching experiments, the teachers put the question mark on top of the equal sign to stress that they are in front of a hypothetical equality; the pupils then have to express the reasons that support or refute it.

[3]The ArAl Units (at the moment there are 12 of them)—supported by the theoretical frame and the glossary—can be seen as models of teaching pathways for arithmetic in an algebraic perspective. They are structured in such a way as to make the teaching process transparent in relation to the problem situation being examined (methodological choices, activated class dynamics, key elements of the process, extensions, potential behavior of pupils, and difficulties they may encounter).

of scales, to favor the shift from manipulation of known and unknown weights to a reflection on the actions made. The representation of the relationships expressed by the text of the problem leads to the construction of the equation, and the reflection on the actions made leads to reifying the principles of equivalence of an equation and to formalizing the steps to solve it. More recently in 3rd grade we have carried out some teaching experiments involving additive verbal problems with an unknown datum that can be modeled by an equation such as $7 + 9 + a = 11 + 12$. For solving these problems, we adopt a strategy we call '*dynamic scenes,*' based on the use of a short video-clip. In the first scene, a visualization of the quantities at play and their relationships is carried out through different strips of paper. In the successive scenes, the manipulation of strips of paper, strategic cuts, and shifts of the strips are showed. Interpreting the meaning of the scenes, one arrives at the formalization of the problem and then discovers the additive cancellation rule and its role in determining the value of the unknown.

3.3 The Main ArAl Language Constructs

In early algebra, content knowledge is shown by identifying and progressively refining keywords or phrases and the relationships among them, with the aim to condense with increasing precision their crucial aspects, refining their clarity and consistency. In this perspective, some language constructs (LCs) of our theoretical framework have proven to be fundamental in order to generate among both teachers and pupils new ways to see arithmetic, suitable to generalization and algebraic representation. Now we summarize the main constructs and classify them according to their reciprocal links. Through some excerpts of classroom discussions we show how, from the beginning of primary school, the introduction of these LCs brings pupils to progressively reach habits of mind that promote algebraic thinking. We present three sets of LCs conceived respectively: (a) to foster the shift of attention from the result to the process; (b) to relate several representations of a natural number to a relational view of the equal sign; (c) to favor socio-constructive classroom practices that enhance the role of natural language.

3.3.1 Promoting the Shift of Attention from the Result to the Process

We analyze three dualities, sharply intertwined, that allow pupils to shift from the action plane to that of reflection: (LC.1) *Representing versus solving*, (LC.2) *Process versus product*, (LC.3) *Transparent versus opaque*.

- **LC.1**. *Representing versus solving*. A widespread belief among pupils, favored by the traditional teaching of arithmetic, is that the solution of a problem coincides with the identification of the result. This means that their attention is focused on operations. This prevents the exploration of mental paths that could generate algebraic thinking. In our case instead, pupils are slowly driven to shift from the cognitive level to the metacognitive one, where the solver interprets the structure of the problem and represents it through mathematical language. In this way the operational point of view is minimized in favor of the relational one.

- **LC.2**. *Process versus product*: The previous duality is strictly linked with one of the most important aspects of the epistemological gap between arithmetic and algebra: whilst arithmetic requires an immediate search of a solution, algebra postpones it and begins with a formal trans-positioning from natural language to a specific system of representation. When a pupil is guided to overcome his/her worry about the result (the product), he/she reaches a higher level of thinking, substituting the calculations with the observation of him/herself reasoning, controlling the structure of the problem (the process).

- **LC.3**. *Transparent versus opaque representation*. A representation in mathematical language consists of symbols that communicate meanings whose understanding depends both on the representation itself and on the ability of those who interpret them. Let us consider the so-called canonical form of a number, that is, the symbol related to its name (see later LC.4): we can say that it is poorer in meaning in comparison to other (non-canonical) representations of the number. For an extreme example: the non-canonical form $2^1 \times 3^3 \times 5^2$ provides more information on the divisors of 1350 than the canonical form. Regarding its divisibility, we can speak of the greater opacity of the writing 1350 versus the greater transparency of $2^1 \times 3^3 \times 5^2$. In general, transparency fosters the understanding of processes, that is, the ways in which a certain product is obtained; it highlights the strategies adopted, the possible mistakes, and the misconceptions underlying the solution to that particular problem.

  Example 1 (4th grade): It is part of an activity aimed at approaching the distributive property. The teacher presents the following situation (see Fig. 3.1): *Marina collects red and green marbles and places them inside the boxes as shown below. Represent the situation in mathematical language so as to find the number of marbles.*

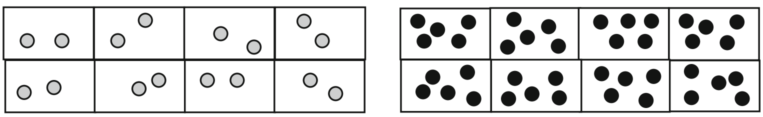

Fig. 3.1 Marina's boxes of marbles

The following pupils' translations are transcribed on the Interactive Whiteboard (IWB):

$$(a)\ 16\times 40=n;\quad (b)\ 2\times 4+5\times 4;\quad (c)\ n=5\times 8\quad n=2\times 8;$$
$$(d)\ 2\times 8=n\quad 5\times 8=n\quad n=2\times 8+5\times 8;\quad (e)\ 2\times 8+5\times 8=n;$$
$$(f)\ (2\times 8)+(5\times 8)=n;\quad (g)\ n=(2\times 8)+(5\times 8)$$

Melania: *The translation (a) is opaque for me.*
Teacher: *What do you mean?*
Bruno: *(a) is opaque because they have already found the number of marbles.*
Clara: *We didn't have to find this number, but to write a translation for Brioshi. They have already solved the problem.*
Bruno: *It's true, they didn't represent the situation. They found the product and not the process.*
Melania: [author of (b)] *I forgot something. I wrote 2 × 4 + 5 × 4 because I saw the separate lines of boxes. Now I realize that my representation is not complete, I must add × 2.* (She writes: 2 × 4 × 2 + 5 × 4 × 2 = *n*).
Franco: *Melania wrote (e), like me, (c) and (f) are equal because 4 × 2 is 8.*
Teacher: *Now can you choose a translation for Brioshi?*

Among the sentences written on the IWB, the pupils choose $n = 2 \times 8 + 5 \times 8$. Later the teacher proposes a new organization of the marbles putting in each box 2 reds and 5 greens, and asks the class to represent the new situation. Among the sentences, $n = (2 + 5) \times 8$ and $n = 7 \times 8$ show up.

Miriam: *What I have written, (2 + 5) × 8, is more transparent; Alessandro's writing (n = 7 × 8) is opaque. Opaque means it is not very clear; transparent means it is clear, you understand the process.*

Later in the class, the equality $2 \times 8 + 5 \times 8 = (2 + 5) \times 8$ is made explicit, favoring the recognition of the structural equality of the two sentences. In our project, activities like this are spread out; they allow the pupils to construct the experiential undertone for the reification of the distributive property.[4] (For more, see Malara and Navarra 2009).

This transcript shows how the class is familiar with mathematical discussion: the pupils express good argumentations and draw on important theoretical constructs (the dualities opaque/transparent and process/product). They are brought to compare representations and, reflecting on the employed symbols, they interpret the sentences and explain their differences.

[4]The ArAl Unit 11 is devoted to the construction of this property.

Example 2 (4th grade): The task is: *Translate the sentence* 3 × b × h *into natural language*. The class is comparing Lorenzo's proposal (*I multiply 3 by an unknown number, then I multiply it also by another unknown number*) with Rita's (*The triple of the product of two unknown numbers*).

Lorenzo: *Rita has explained* what 3 × b × h is, *whereas I have said* what you do.

Lorenzo evaluates the two translations focusing on the distinction between the operational and relational aspects related to the duality representing/solving. He recognizes in his sentence the operational point of view and in Rita's, the meaning of the sentence arising from a relational reading of it. The metacognitive control achieved by the pupil is high and it can be seen as a fruit of the type of teaching received.

3.3.2 Relating the Representations of a Number to a Relational View of the Equal Sign

- **LC.4**. *Canonical and non-canonical representation of a number*. Faced with the question, "Is 3 × (11 + 7) ÷ 9 a number?", students or teachers usually answer: "They are operations"; "It is an expression"; "They are calculations." In order to promote reflection on this aspect, we resort to the strategy of transcribing some information about a pupil—son of, friend of, desk mate of, etc. The class understands that those are different ways to describe the pupil: one is his/her name, whereas all the other representations expand the knowledge about him by adding information that the first name does not provide. The teacher then explains that, similarly, each number can be represented in different ways, through any expression equivalent to it. For example: 12 is his name, the so-called canonical form, all the other forms (3 × 4, (2 + 2) × 3, 36 ÷ 3, 10 + 2, 3 × 2 × 2, …) are non-canonical, each of which has a meaning in relation to the context and the underlying processes that generate them. This experience leads to the conclusion that 3 × (11 + 7) ÷ 9 is one of the many non-canonical forms of the number 6. The concept of canonical/non-canonical form also has crucial implications (for both pupils and teachers) in order to reflect on the meanings attributed to the equal sign. It becomes a kind of 'semantic ferry' towards generalization.

Example 3 (1st grade): The task is: *Given the number 8 + 4, the pupils have to choose equivalent forms from among the following representations: A. 7 + 2; B. 6 + 5 + 2; C. 8 + 3 + 1; D. 9 + 0 + 4; minimizing the calculations.*

Michele: *I know it! 8 plus 3 plus 1. You see that 3 plus 1 is the non-canonical form of 4.* (Others confirm.)
Teacher: *Have you understood what Michele and others observed? Nicola, what kind of comparison can you do here?*
Nicola: *They said that 8 plus 4 is the same as 8 plus 3 plus 1, because 8 remains equal and then, after 3, I put 1 and it still is 4* (he shows the numbers with his fingers).
Teacher: *But what a good pupil! Look now.* (She writes on the IWB: 39 + 4, 39 + 3 + 1, 39 + 3 + 2).

The pupils are amazed by such 'large' numbers; however, many are able to read them.

Teacher: *What number is equal to 39 plus 4?* (Many hands go up.) *Is it necessary to make calculations?*
Many: *No!*
Teacher: *You have to wise up!*
Alexandra: *The number is 39 plus 3 plus 1, because 39 is in all three expressions, 3 plus 1 is equal to the non-canonical form of 4, and 3 plus 2 is the non-canonical form of 5.*

This episode shows that since the beginning the pupils are educated to compare number sentences reflecting on the relationships between their addends. They learn to compare sentences without calculating their results. The number sentences 39 + 4, 39 + 3 + 1, 39 + 3 + 2 are proposed to favor the transfer between the two situations and to plant the seeds of relational thinking. These comparisons provide a foundation for the gradual, smooth transition to algebra (e.g., $a + 4$, $a + 3 + 1$, $a + 3 + 2$).

- **LC.5**. *Equal sign.* To reflect upon the meanings of the equal sign has crucial implications both for pupils and teacher. In $(6 + 11) - 2 = 15$, for example, both often 'see' *operations* on the left side and a *result* on the right side of the equal sign. The main idea is: 'I sum up 6 and 11, then take away 2 and get 15'. The usual teaching of arithmetic imprints in the pupils a meaning of the equal sign as a *directional operator*: it has a space-time characterization. In moving to algebra the pupils must learn to move around in a conceptual universe where they have to overcome the familiar space-time characterization: an equality like $2a - 6 = 2(a - 3)$ has a *relational* meaning; it states the equivalence between two representations of the same quantity. A consequence of the received imprinting is that the request 'Write down 14 plus 23' very often in primary school gets the answer '14 + 23 =' or '37'. The equal sign is viewed as an *indicator of a conclusion* and expresses the implicit belief that the conclusion will sooner or later be required by the teacher; '14 + 23' is viewed as an event waiting for its realization. The pupil is a victim of a lack (or rather a poverty) of control over meanings. In our approach this misconception is bypassed.

Example 4 (2nd grade):

Piero: *It is correct to say that 5 plus 6 'makes' 11, but you cannot say that 11 'makes' 5 plus 6. So it is better to say that 5 plus 6 'equals' 11, because then the contrary is also true.*

Piero's phrase expresses in a naïve, but convincing, way his relational view of the sign '='. Piero argues correctly, highlighting the differences in the meanings of the two sentences: the verbal term 'equals' has a *symmetric* connotation; the verbal term 'makes' does not, it has a *directional* connotation. Piero's reflection shows the metacognitive character of the teaching he receives.

3.3.3 Fostering Socio-constructive Classroom Practices Enhancing the Role of Natural Language

- **LC.6**. *Algebraic babbling*. In the learning of the native language, a child acquires little by little its meanings and the rules supporting it, gradually developing to its formal study in school, when he learns to read and reflect on the structural aspects of language. As we have already sketched, we believe that the mental models of algebraic thinking should be similarly organized from the early years of primary school, constructing algebraic language in a dense interlace with arithmetic, starting from its meanings. For this reason, we create an environment that stimulates the pupil's autonomous processing of encodings of verbal sentences into formulas alongside with their collective comparison within the class. The appropriation of the new language therefore occurs experimentally, and its rules mature gradually within a didactic contract that tolerates initial moments of syntactical inaccuracy. A key aspect in this frame is to help pupils understand the importance of respecting the rules of algebraic language. While students soon start interiorizing the importance of respecting the rules of natural language in order to facilitate communication, it is difficult to make them develop a similar awareness in relation to algebraic language. Therefore, it is necessary to help them understand that algebraic language, too, is a finite set of arbitrary symbols that can be combined according to specific functional rules to be respected (see points LC.8 and LC.9).

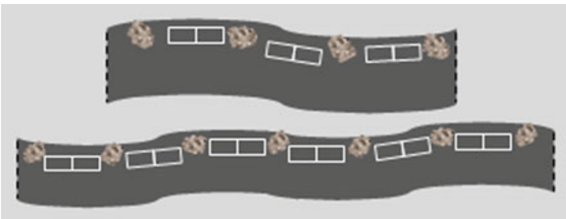

Fig. 3.2 Two examples of car parking

Example 5 (5th grade): The teacher poses a story problem concerning the council approval of several car parks that have to be built along the tree-lined roads of the town according to the rule ‘two cars between two trees’ (see Fig. 3.2): *Having the car parks to be made with the same pattern but different number of car spaces, the mayor asks that the approval be summarized in a formula which expresses, for all the car parks, the number of car spaces according to the number of trees.*
The pupils explore various numerical cases through drawings, collect their data, analyze them, and arrive at the following rule: ‘The number of car spaces can be found by multiplying by 2 the number of trees and subtracting 2’. The teacher begins a discussion to lead the pupils to reformulate the rule in a relational way.

Teacher: *Instead of saying ‘multiplying …’ how might we write the rule?*
Many: *It is equal to.*
Teacher: *Well, let us rewrite it. ‘The number of car spaces is equal to …’ to what? We cannot write ‘is equal multiplying’.*
Simone: *… is equal to ‘the number of trees multiplied by 2’.*
Laura: *… and then ‘taking away 2’.*
Teacher: (writing the sentence) *May we say the same thing using instead of ‘multiplied by 2’ a little word …?*
Giuseppe: *‘The number of car spaces is equal to the double of the number of trees minus 2’.*
Teacher: (writes the last sentence) *Now translate it to send it to the mayor.*

Each pupil translates the phrase. Translations are listed on the IWB (t = number of trees, cs = number of car spaces)

(a) $(2 \times t) - 2 = cs$ (b) $t \times 2 - 2$ (c) $(cs \times 2) - 2$ (d) $(2 \times t)(2 - 2)$

The teacher opens the discussion.

Mauro: *(c) is wrong, we must multiply the number of trees, not that of parking spaces.*
Renato: *I would remove (d), (2 − 2) is not involved.*

Two sentences remain: (a) $(2 \times t) - 2 = cs$ (b) $t \times 2 - 2$.

Teacher: *Do you think it’s really necessary to write* ‘cs’, *indicating what?*
Andrea: *I wanted to indicate the car spaces … the number of parking spaces.*
Teacher: *Do we have to use two letters* ‘cs’? *You used* t *to indicate the number of trees.*
Valentina: *We can use* c *alone!*
Teacher:

Do you agree that we use only c *to indicate the number of car spaces? Ok. Now: for you, which is the most correct, (a) or (b)? The sentence to be translated was: 'The number of car spaces is equal to the double of the number of trees minus 2'.*

Irene: *The sentence (a) is opposite.*

Andrea: *In (b) the number of car spaces is missing.*

Teacher: *Let us complete the sentence.* (She marks the sentence in parts with brackets and asks the pupils to translate each part into the corresponding symbol.)

This leads to the comparison between the verbal and formal sentence:
The number of parking spaces is equal to the double of the number of trees minus 2

p = 2× t - 2

$$p = 2\times t - 2$$

The most meaningful pupils' interventions that characterize the algebraic babbling are: (i) the answer given by many pupils, "it is equal to," which reflects a shift towards a relational wording of the rule that had started with the operational wording, "it can be found multiplying by 2"; (ii) Giuseppe's adding "the double of the number of trees," which is a conceptual and linguistic refinement in a relational sense of the expression "multiplying by 2"; (iii) the interpretation of the formulas (c) and (d) by Mauro and Renato and their accurate expression of the reasons why they are wrong; (iv) the teacher's question about the usage of '*cs*' that favors the pupils' understanding of the opportunity to reduce it to one letter; (v) Irene's intervention—where she compares the verbal sentence and its algebraic translation, observing the different positions of the subject—highlights her sensitivity towards the structural aspects of a sentence and the coordination of different representations; and (vi) Andrea's intervention underlines the sentence incompleteness because the subject does not appear.
For the refinement of the algebraic babbling, a key moment is the collective comparison of verbal and algebraic formulations of the rule where corresponding parts are underlined. This metacognitive activity allows the pupils to harmonize syntactic and semantic aspects.

- **LC.7**. *Argumentation*. A fundamental aspect in our approach to early algebra is the recognition of the potential role played by the relationship between argumentation and generalization in the social construction of knowledge. Only when argumentation becomes a shared cultural tool in the class can this relationship be made explicit and can students understand the role played by verbalization in the development of their capability to reflect on what they are saying. We could say that the power of argumentation is related to the fact that

Fig. 3.3 The first four pyramids

often those who start developing it are not completely aware of their ideas before they try to express them. As argumentation becomes a habit, the student understands its value and becomes aware of its role in comparing facts and in making their similarities gradually emerge.
Example 6 (6th grade): The class is working to find a rule that gives the number of black triangles according to the numerical position of a Fig. 3.3.

Ylenia: *On the line where the pyramids lie… for example, in the fourth pyramid the black triangles are four and the white are three… my pyramid of six floors has six black triangles and five white triangles on its base… The white (triangles) are always one less than the black ones. Maybe a pyramid with any number of floors has a number of black triangles on its base which is equal to the number of floors and as many white triangles as the black ones minus one.*

The teacher writes the following comment in her transcript: "Before her intervention, Ylenia wasn't aware of her conclusions but, as she was verbalizing, she started deducing and expressing the general rule".
This episode shows how pupils' implicit algebraic reasoning and generalizing emerge when argumentation and justification are central to teaching.

- **LC.8**. *Syntax and Semantics.* Control of the syntactical aspects of a new language occurs through its semantic control. In the traditional learning of mathematics, formulas are generally 'given' to pupils, thus losing their social value; it is necessary to lead them to understand that they are appropriating a new language that develops according to precise syntactical rules. As we sketched above, to highlight the value of mathematical language for communication, we invite teachers to propose an exchange of messages in formal language with either real or virtual correspondents (pupils, classes, teachers, Brioshi) engaged in the solution of the same problem. The collective comparison of formal sentences produced by the pupils and their interpretative analysis allow pupils to learn that algebraic language also has a syntax (which enables them to detect whether a sentence is correctly expressed or not) and a semantics (which enables them to detect whether it is true or false). So pupils acquire competencies in interpreting formulas and begin to conceive of the translation between these languages as the core of algebraic activity. Notwithstanding that natural language is systematically used in doing mathematics, it is necessary that pupils understand that algebraic language possesses a specific character, which creates an element of rupture with natural language. In our project the pupils are led to discuss these differences while becoming aware of the possible referents of mathematical terms and symbols. We offer two examples.

Example 7 (7th grade): Thomas represents the relationship between two variables as follows: $a = b + 1 \times 4$ and explains his writing.

Thomas: *The number of oranges,* a, *is quadruple the number of apples,* b, *plus 1.*

Katia: *It's not correct, because this would mean that the number of oranges is the number of apples plus 4. You have to put the brackets:* a = (b + 1) × 4.

Thomas and Katia exchange their views on their translations between natural and algebraic language, and on the semantic and syntactic aspects of mathematical writing. Katia intervenes at a metacognitive level and her argumentation is very articulated: she detects Thomas's syntactical mistake and correctly translates the verbal sentence. This episode shows how a metacognitive and socio-constructive teaching allows the pupils to assume an appropriate attitude for both early algebraic thinking and arguing.

Example 8 (5th grade): The class is given the task to represent in mathematical language the statement, *The double of the sum of 5 and its successive number*. As soon as the pupils' proposals are written on the IWB, Diana steps into justify her writing.

Diana: *Filippo has written* 2 × (5 + 6*), and it is correct. But I have written* 2 × (5 + 5 + 1) *because this way it is more evident that the number following* 5 *is bigger by a unit.*

Diana is explaining how her translation is clearer and more transparent because it considers the functional relationship between a number and its successor. Diana recognizes the syntactic correctness of Filippo's sentence, but considers it opaque: it does not make explicit the relationship between the addends. This episode shows that Diana has acquired either early algebra linguistic constructs or the attitude to make explicit all the relationships in play.

All these examples show that the teaching we promote generates not only an early algebraic thinking but also a wide range of linguistic, logical, and metacognitive abilities related to generalization, argumentation, and justification.

3.4 Teachers and Early Algebra: The Multicommented Transcripts Methodology

The early algebra approach requires a deep change of perspective in teachers. Their main difficulties concern the revision of their mathematical knowledge and beliefs that condition their teaching actions. They should learn to manage socio-constructive processes in the classroom, drawing on appropriate theoretical frameworks, comparing them to their own epistemology,[5] thus fruitfully and significantly enriching both culture and work in the classroom. The coordination of a mathematical discussion requires methodological skills that go beyond mere disciplinary competence. Teachers should foresee the development of classroom actions and form hypotheses about pupils' conceptual constructs and possible strategies to help them modify such constructs. From a social point of view, they should be able to create a good interactional environment, stimulating participation and mutual listening, avoiding judgment and leading the class to validate the arguments, and asking questions at a metacognitive level so that pupils can internalize the processes carried out.

In order to develop these skills—in tune with other scholars (e.g., Jaworski 2004; Mason 2002, 2008; Potari 2013; Schoenfeld 2013; Sowder 2007; Thames and Van Zoest 2013)—we enact educational processes with/for teachers that combine theoretical study, self-observation, and shared critical reflections on their practice. The glossaries allow the teachers to gradually attain a global vision of early algebra. As to teachers' ability to interpret signals when they are in the classroom, improvement can be obtained through the increasing awareness with which they learn to transform the recurring occasional observations and reflections into a personal methodology. The latter should result from the interlacing of observational skills, motivation to action, and knowledge of how it would be appropriate to intervene. Our aim is that teachers approaching early algebra become trainers of their own development, gradually sharing their experiences to generate a new *forma mentis*, that becomes a firm base for their autonomous development. Our hypothesis is that a fruitful exchange between theory and praxis can make teachers' competence evolve in two directions: first, in recognizing signals that their role is at stake either on the spot or in the organization of their theoretical tools; secondly, in processing the received signals so as to convert them into their own cultural patrimony.

Regarding the teaching methodology, we believe that observation and critical-reflective study of socio-constructive classroom processes are necessary conditions to foster teachers' development of awareness about the roles they must play in the class, the dynamics that characterize the mathematical collective construction, and the variables involved (Cusi and Malara 2015). In the perspective of constituting a community of inquiry, the teachers are organized into groups

[5]This term unifies a set of teacher's characteristics, such as knowledge, beliefs, orientations, goals, and ways of being inside the classroom.

according to the topics they are working on, and each group is coordinated by a researcher-mentor, who often goes into the classrooms to support the teachers' actions and frequently has face-to-face and web exchanges with them (Skype, e-mail). Periodically, work-sessions are held by the project leader and also joint meetings with all the teachers and mentors involved in the studies. A crucial aim of the teachers' educational process lies in leading them to perceive on which aspects they should concentrate and helping them understand how to intervene. Therefore we encourage the teachers involved in ArAl teaching experiments to observe their pupils' activity according to some key principles, as suggested by other scholars (e.g., Llinares et al. 2016): attending to childrens' utterances and strategies, interpreting them and appropriately deciding when to intervene to support them, but also observing the effectiveness of their actions (ways of listening, speaking, acting, reacting, …). One important method through which we try to promote these attitudes is the construction of what we call *Multicommented Transcripts (MTs)*, which develops in various steps: the teachers transcribe meaningful classroom episodes, send them by e-mail with their own comments to the mentor who makes his/her own comments, and then the mentor sends them back to the authors and other members of the team who can add further comments. So the MTs become important objects for the education of the teachers (Malara 2008), leading them to critically analyze their didactic interventions through the 'theoretical glasses' acquired in the project.

The comments in MTs highlight not only the positive aspects, but also often erroneous beliefs and behaviors. Frequently, they underline that teachers need to have better control of linguistic operative terms (*calculate, solve, find the result, it gives* ...), as well as that of algebraic terms (*connect*, *translate*, *represent*, *interpret*), the absence of which inhibit the development of a relational view in arithmetic. Furthermore, the comments give suggestions on how to guide the translation from natural to symbolic language, and vice versa. Particular care is devoted to helping teachers reflect on their actions in the face of 'false discussions,' that is, dialogues between teacher and only a few pupils where she rhetorically suggests the answers. Another weakness to be signaled would be closing the discussion through questions such as: "Is everything clear?" "Do you understand?" "Do you agree?", and not allowing pupils to re-examine the situation or not checking whether the conclusion has been effectively reached. Usually in the comments the project leader or mentors recommend that pupils be educated to argue thoroughly and coherently, with an appropriate use of language, underlining that the comprehension of mathematics occurs also through its collective and correct use. Moreover, they suggest that teachers promote a peer-dialogue interaction and limit their role as much as possible, stimulating questions in the classroom and drawing back during the answer phase: if the teacher is the constant pivot of the discussion, the social aspects of knowledge construction are weakened (for more on these aspects, see Cusi et al. 2011; Cusi and Malara 2015; Malara and Navarra 2011, 2016).

The fruits of this methodology, and in particular of the joint reflection on the MTs, strongly influence the development of the theoretical, methodological, and instrumental aspects of our project. In fact, meaningful excerpts of MTs become part of ArAl Units and are discussed in reports or transformed into learning objects

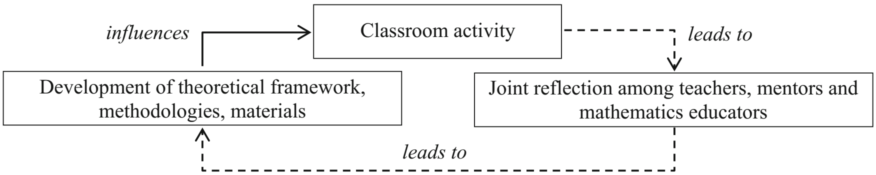

Fig. 3.4 The cycle of the mathematics education of the teachers

Fig. 3.5 The sequence of pearls

(Malara and Navarra in press). In this way they become educational tools for early algebra, offering teachers the possibility to develop the capability and sensitivity to act differently in the classroom (see Fig. 3.4).

In this cycle, the teachers learn to manage the socio-cognitive processes, comparing their epistemology about teaching arithmetic and algebra with the reference frames that they are offered and gradually internalizing the outcomes of the process, so as to consolidate them as a steady cultural patrimony about early algebra. Particularly effective are the occasions (meetings at school, university, the sharing via web of MTs among teachers) where collective debates develop on the classroom actions of teachers dealing with the same activity. Through these cross comparisons, the actions of one teacher can become a model of good practice for his/her colleagues. The following example offers a good model of a teacher's interactions while leading a discussion.

Example 9 (4th grade): Question: What is the color of the 27th pearl in this sequence (see Fig. 3.5)?

Students propose the following expressions:

(a) $9 \times 3 = 27$; (b) $6 \times 4 - 1 = 27$; (c) $27 \div 6 = 4\,\text{rest}\,3$ (d) $6 \times 4 + 3 = 27$

The teacher starts the discussion.

Samuele: *In my opinion Brioshi doesn't understand (a) because he does not know what are 9 and 3, and (b) because $6 \times 4 - 1$ is not equal to 27 but to 23.*

Francesco: *It's true, I got confused. I counted 3 white pearls and 3 black ones until I got to 27.*

Giovanni: *The module is formed by 6 pearls, 3 white and 3 black. You were wrong. The 27th pearl is the 3rd white of the 5th module because 27, which is the number we have to find, divided by 6, is equal to 4 with remainder 3. We need to look at the remainder to establish the 27th pearl.*

Mattia: *In my opinion they're both right, because the remainder 3 means that the 5th module started on. (d) is more correct than Giovanni's (c).*

Teacher: *Do you understand what Mattia said?*

Giuliana: *Emanuele (d) used the same numbers as Giovanni (c), but wrote a multiplication and an addition:* $6 \times 4 + 3 = 27$.

Teacher: *Who wants to explain better?*

Giovanni: *6 is the number of pearls of the module, 4 is the number of times that the module is repeated, and 3 is the number that I must add to 24, which is the product of 6 by 4 to get to 27.*

Marta: *I understand: 6 is the divisor, 4 is the quotient, 3 is the remainder and 27 is the dividend. Emanuele used the same numbers in the division as Giovanni.*

The pupils shift from the operative plane to the relational one and learn to appreciate the role of the Euclidean representation of the division. The episode shows how the effectiveness of the teacher's actions—paradoxically—consists in her marginality in the discussion: she limits herself to follow and carefully observe the development of the pupils' argumentations; she avoids giving indications or answers, but poses on the spot reflective questions. She induces *virtuous* behaviors in the pupils: they are encouraged to deepen or rephrase their argumentations and to intervene so as to clarify some claims of their classmates. Very often the MTs show instead how teachers have difficulty in promoting collective discussions with the result that the classroom interactions shrink into short fragmented dialogues between the teacher and the pupils.

In the long run, the MTs allow for ascertaining whether the classroom-leading strategies have changed, and how, during the training. Indications of the effectiveness of the training and of the teacher's professional growth are provided by the answers to the following questions: Does he/she modify his/her initial points of view or does he/she seem unaware of meaningful changes in his/her initial attitude? Is the teacher able to assume the appropriate roles in order to promote reflection on mathematical processes or objects? Does he/she foster linguistic interactions by encouraging verbalization, argumentation, and collective discussion? Does he/she negotiate and share with the pupils the ArAl theoretical framework? (on these aspects, see Malara and Navarra 2016).

Regarding the last point, we stress that a real and potentially effective sharing of the ArAl theoretical frame with the pupils occurs only if the teacher constantly communicates with the class by using the LCs. However, it is not sufficient to use such terms: the analysis of the MTs shows that teachers often forget to make sure that pupils also use them with a conscious control of the conceptual meanings that they condense. The consequence is ambiguity: the teacher uses them; the pupils seem to understand them, but actually don't use them during the discussions. The teacher doesn't notice it and goes along without checking whether the pupils acquire and use the terms with an authentically shared meaning.

The teacher should therefore not only get acquainted with the meaning of the LCs with the aim of stabilizing, conceptualizing, and mastering the meta-disciplinary knowledge of early algebra, but also improve and refine their use in a constant negotiation with the pupils. The acquisition of the key words of the discipline should be achieved through patient teacher-pupils cooperation, during which not only are the terms themselves important, but also the relationships that connect them.

In order to allow the terminology to settle in and improve, it is however necessary that the mathematical discussions be effective, that they promote communication and the sharing of meanings, and that they favor a thorough comparison among the sentences, expressed both in natural and in symbolic language. In other words: a stimulating discussion promotes the use of an advanced terminology, whereas an exchange of short sentences and word phrases generates poor terminology and syntactically mediocre or incomplete sentences, because the pupils delegate to the teacher the task of organizing the whole discourse. This completely blocks the functioning of the devolution, while the teacher gives up his role of guiding the pupils towards assuming their responsibility for the construction of their own knowledge. Let us consider an example. It concerns the interpretation of three formal sentences related to the same problematic situation. The pupils have to choose the correct one and justify the reasons for their choice.

Example 10 (5th grade):

In a pet store showcase there are 11 puppies. Some are visible, others not, because they are inside the house (see Fig. 3.6). *Which of the following phrases represents this situation correctly?*

$$\square\text{A. } d = 11 + 7 \quad \square\text{B. } 7 + d = 11 \quad \square\text{C. } 11 = d - 7$$

The pupils write their explanations, which are then copied on the IWB and discussed.

Besmala: *B tells you that the 7 puppies together with the ones inside are 11 overall.*

Martina: *I chose B because it is more transparent.*

Daniele: *B, because it summed all the dogs and gives a result of 11. A is wrong because the task wants us to find the suitable non-canonical form to find 11. C is not the answer, because if you do d minus 7 you cannot obtain 11, because we have to sum up all the dogs.*

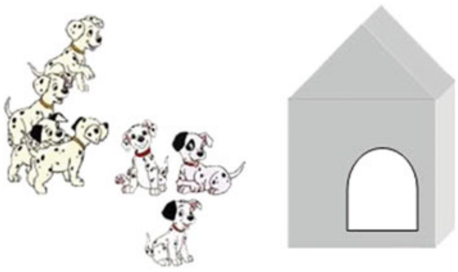

Fig. 3.6 The problem of the pet store

Clark: *B is correct, because 7 plus d, which is the small house, corresponds to 11.*

Sofia: *Representation B is the right one, because the puppies are 11 altogether, that is the sum of the visible ones and of the ones that are in the small house.*

The teacher considers the pupils' explanations, reported above, to be *correct* since they favor phrase B (the correct answer); so she doesn't analyze them in detail. This excerpt shows how a teacher can miss precious opportunities to test and consolidate important skills from an early algebra perspective. We stress the importance of taking care not only of formalization, but also of interpretation, based on a constant activity of relational reading of the formulas. The pupils resort to terms such as 'representation,' 'transparent,' and 'canonical form'; however, the teacher doesn't express the necessary sensitivity towards what their arguments express or their level of awareness in using such terms.

Besmala could be invited to reformulate her sentence in a relational sense, for example: "The sum between the number of the outside puppies and of the inside puppies is equal to their overall number." It would have then become evident that it is the translation of $7 + d = 11$.

It would be important to understand which meaning Martina gives to the concept of *transparent*: it seems closer to 'comprehensible,' that is, that the sentence can also be seen from a traditional point of view, based on operations (left of the equal sign) and result (on the right). Reflecting on the meaning of a term would allow for scrutinizing important conceptual aspects, as well as the meaning of the equal sign —thus favoring the shift from an *operational* to a *relational* perspective.

Not only does the teacher accept Daniele's and Sofia's statements about summing 'animals' and not 'numbers of animals,' but she doesn't even notice that Daniele's three justifications are cues to his operational viewpoint: "[B] 'gives a result' of 11," "the task wants us to find the suitable non-canonical form 'to find' 11," and "if you do C minus 7 you cannot 'obtain' 11". When Daniele speaks of 'canonical form,' he actually thinks of the operation that allows him to 'find 11'. Therefore, instead of fostering the pupils' relational view among the entities at play, the teacher allows (without noticing it) a hidden operational attitude.

Clark could be invited to reconsider his sentence; a frequent error of inexperienced pupils facing letters in algebra consists in associating them with the initial of the name of the object, not to the number of objects that it represents.

This analysis provides evidence that, even if a teacher actively takes part in a co-learning environment, it is not always easy for him/her to react appropriately to support/refine children's mathematical thinking. As documented in many of our research studies, the conquest of these capabilities requires a long time, an intentional self-monitoring on the part of the teachers, and a constant sharing of their own practices in the realm of teacher education programs focused on the critical analysis of such practice (see, for instance, Cusi et al. 2011; Cusi and Malara 2015; Malara and Navarra 2011, 2016).

3.5 Concluding Remarks

Leading 5- to 14-year-old pupils to approach early algebra essentially means leading them—through purposely-created problematic situations addressed in a socio-constructive way—towards a new language, with its semantics and syntax. Therefore, respecting its rules becomes essential for treating activities such as *translating, arguing, interpreting, predicting, and communicating* as mathematical activities. Carrying out calculations is still present, but is subordinated to 'higher' purposes: it is the groundwork for reasoning, argumentations, refutations, and corrections. As soon as the algebra that pupils deal with grows in complexity, they will be led to understand that the manipulation of symbolic forms is not self-referential, such understanding helping them mathematize, explore, reason, deduce, and achieve new knowledge.

What we have described shows educational aspects that we believe should be constantly developed in pupils, since these aspects support the growth of their algebraic thinking, promoting *metalinguistic* and *metacognitive* competencies, and consequently reflection on: (1) *language*, which promotes abilities to construct argumentations, to translate natural language into algebraic language, and to produce original thought; (2) the relationship between individual intuitions or productions and the social construction of shared knowledge; (3) passing from *concrete generative situations* to the conceptual condensation of the underlying mathematical facts and to the construction of the related concepts; and (4) some basic mathematical aspects, such as the evolution of counting strategies and the progressive recognition of the structural equivalence between sentences or, in the case of unknown and variable data, the generation of equations and functions.

Our report in this chapter should make clear that pupils can develop algebraic thinking as long as they are taught as metacognitive students. But in order to achieve this goal, it is necessary that teachers, in turn, learn to be metacognitive teachers. To promote metacognition in teachers we have conceived tools and enacted strategies involving them in a strict intertwining of reflections upon the knowledge in question (theory) and action in the classroom (practice). Our experience and our research studies have made us aware that changing teaching towards the perspective of early algebra requires a conversion of the teachers' professionalism: this is a slow process that must be supported through appropriate developmental programs.

Working in an early algebraic perspective means, for teachers, to become aware of the fact that arithmetic and algebra must be considered as interlaced disciplines right from the very beginning of primary school. In order to keep this perspective alive in classroom activity, teachers must improve their sensitivity to recognize the continuous micro-situations in which it is possible to contrast/compare the pupils' (and one's own) *operational* point of view to the *relational* one. We believe that on this basis pupils can experience, from the first school years, a conscious approach to algebraic language and thinking and, in general, a positive attitude towards mathematics. Our research shows that, in order to help pupils reach this goal and gradually and consciously build mathematical skills, it is necessary for pupils and teachers to share the specific terms of the theoretical frame for early algebra, using them constantly when they discuss, reflecting on their meaning, and letting their connections emerge.

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Chapter 4
Seeking, Using, and Expressing Structure in Numbers and Numerical Operations: A Fundamental Path to Developing Early Algebraic Thinking

Carolyn Kieran

Abstract The dominant focus on generalizing in the development of algebraic thinking has to a large extent obscured the process of seeing structure. While generalization-oriented activity remains highly important in algebra and early algebra, and in fact includes a structural component, equal attention needs to be paid to the complementary process of looking through mathematical objects and to decomposing and recomposing them in various structural ways. With the aim of instigating greater attention to structure and elaborating more widely on its meaning with respect to developing early algebraic thinking, this chapter explores the notion of structure and structural activity from various perspectives, and then presents a research-based example of 12-year-olds seeking structure within an activity involving factors, multiples, and divisors.

Keywords Structure · Early algebraic thinking · Properties · Structural equivalence · Number and numerical operations · Multiplication and division

4.1 Introduction

High school algebra involves working with generalized forms. The ability to see structure in these forms is crucial to being successful in algebraic transformational activity and to making sense of those transformations. While generalization has long been considered the heart of school algebra (e.g., Kaput 2008; Mason 1996), this focus on the process of generalizing has to a large extent obscured the process of seeing structure, even if generalizing does includes a structural component. While imbuing algebraic and early algebraic activity with generalization-oriented

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tasks remains a highly important aspect in developing students' algebraic thinking, equal attention needs to be paid to the complementary process of *looking through mathematical objects*, such as the expression $x^6 - 1$ or the number 989, and to decomposing and recomposing them in various structural ways (e.g., seeing that $x^6 - 1$ can be decomposed into $(x^3)^2 - 1$ or into $(x^2)^3 - 1$ and factored accordingly, or seeing that 989 can be decomposed into, for example, the structural expressions $9 \times 109 + 8$ or $9 \times 110 - 1$, or even $9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$). As the latter example suggests with its decomposition of 989 in terms of the *division algorithm* theorem, or according to place-value, as well as any number of other structural decompositions, attention to looking through mathematical objects at the primary and lower middle school levels means developing awareness of the possible and various ways of structuring number and the numerical operations of arithmetic. However, "students' experiences in learning arithmetic only rarely foster an appreciation of structure" (Arcavi et al. 2017, p. 53). Similarly, Mason (2016) has argued that looking at something structurally is an often-overlooked aspect of algebraic thinking. This chapter explores the notion of structure and structural activity from various mathematical, theoretical, and empirical perspectives, and then presents a research-based example of 12-year-old students seeking, using, and expressing structure within a selected domain of arithmetic activity, namely that involving multiplication, division, factors, multiples, and divisors.

4.2 Viewing Structure from Various Perspectives

Structure is without doubt one of the big ideas of mathematics (e.g., Kuntze et al. 2011; Mason et al. 2009) and is to be found everywhere in mathematics. A relevant example is drawn from Blanton and Kaput's (2004, p. 142) definition of algebraic thinking where they emphasize the foundational notion of structure: "[algebraic thinking is] a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationships." However, *structure* is often treated within the mathematics education community as if it were tantamount to an undefined term; it is further assumed that there is universal agreement on its meaning (Mason et al. 2010). That this may be problematic, in particular for mathematical teaching practice and research in early algebra with 5- to 12-year-olds, became obvious at the Early Algebra Topic Study Group at ICME-13 in Hamburg in July 2016 when one of the participants asked the others what they meant when they used the term *structure*. As participants attempted to express the notion of structure relative to the various content areas of early algebra, their responses suggested some uncertainty and a tendency to focus rather narrowly on the basic properties of arithmetic. With the aim of instigating greater attention to structure and elaborating more widely on its meaning with respect to the development of early algebraic thinking, this first section of this chapter examines and pulls together various perspectives on structure. It addresses structure and generalization, structure in numbers and numerical operations, and structure in figural patterns and functions.

4.2.1 Structure and Generalization

For Blanton et al. (2011), and in line with Kaput (2008), the essence of early algebra lies in generalizing mathematical ideas, representing and justifying generalizations in multiple ways, and reasoning with generalizations. They define generalizing as follows:

> Generalizing is the process by which we identify structure and relationships in mathematical situations. … It can refer to identifying relationships between quantities that vary in relation to each other. It can also mean lifting out and expressing arithmetic structure in operations on the basis of repeated, regular observations of how these operations behave. (p. 9)

This characterization of generalizing links it closely with the processes of "identifying, lifting out, and expressing arithmetic structure." In other words, generalizing in arithmetic involves identifying the structural.

And, conversely, the structural involves identifying the general, according to Mason et al. (2009):

> We take *mathematical structure* to mean the identification of general properties which are instantiated in particular situations as relationships between elements; these elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. Usually it is helpful to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced. … When a relationship is seen as instantiation of a property, the relation becomes (part of) a structure. (p. 10)

For Mason et al. (2009), attending to properties lies at the core of structural thinking, the latter of which they define as a disposition to use, explicate, and connect these properties in one's mathematical thinking. If a relationship between two or more objects is not seen as exemplifying some general property, then that relationship is not in itself related to structural thinking. Further, they assert that:

> Structural appreciation lies in the sense of generality, which in turn is based on basic properties of arithmetic such as commutativity, associativity, distributivity and the properties of the additive and multiplicative identities 0 and 1, together with the understanding that addition and subtraction are inverses of each other, as are multiplication and division. (p. 15)

In this intertwining of the structural and the general, one is led to ask whether discussion of the structural aspects of the various activities that are engaged in within early algebra and which aim at developing algebraic thinking in 5- to 12-year-olds could benefit from being expanded beyond the basic properties of arithmetic. What are some of the other structural properties that could be said to be included in activity involving numbers and numerical operations? Does the discussion need to be broadened even more when referring to structuring activity involving figural patterns—one of the most widespread approaches to developing algebraic thinking in early algebra? Is the notion of *structure* the same across the various content domains of early algebra (for these content domains, see Kieran et al. 2016, p. 10)? With these questions in mind, we move next to exploring the notion of structure in numbers and numerical operations, and follow this with exploring the notion of structure in figural-patterning and function-oriented activity.

4.2.2 Structure in Numbers and Numerical Operations

For basic notions on structure in mathematics and the activity of structuring in arithmetic, we turn first to Freudenthal (1991). He points out that the system of whole numbers constitutes an *order structure* where addition can be derived from the order in the structure, such that for each pair of numbers a third, its sum, can be assigned. The relations of this system are of the form $a + b = c$, which one calls an *addition structure*. In his book, *Didactical Phenomenology of Mathematical Structures,* Freudenthal (1983, pp. 112–113) describes the *multiplicative structure* of the natural numbers in terms that comprise more than the act of multiplying. It is the whole complex of relations $a \times b = c$, possibly also expressed as $c/b = a$, and complemented by $a \times b \times c = d$, $a \times b = d/c$, and all other relations one would like to consider in this context. It encompasses such properties as commutativity, associativity, distributivity, equivalence of $a \times b = c$ and $c/b = a$, and many more properties of this kind. But, according to Freudenthal, the structure of the natural numbers also allows for prescribing c in the relation $a \times b = c$ and asking for its splittings into two factors. Freudenthal asserts further that c can be split into its prime factors, with divisors and multiples being other means of structuring. As well, tying the order structure to the multiplicative structure yields the property that, given the product, increasing one factor means decreasing the other.

It is of interest that in Freudenthal's discussion of structure there is not just one all-encompassing *structure*. He refers, for example, to order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples, and so on. And these different but related structures have properties—in fact, *many* properties based on these structures, not simply the basic properties of arithmetic that are often referred to as the field properties. We notice too that Freudenthal also uses the phrasing, *means of structuring*, which puts forward the notion of alternative structurings that can be deduced from the basic structures. Freudenthal's perspective serves to broaden considerably the dimensions of any discussion related to characterizing structures and structuring activity within the mathematics of arithmetic and where the development of algebraic thinking is a goal.

Structures and structuring activity have naturally enough been a preoccupation of past research in school algebra and algebra learning (e.g., Kieran 1989, 2006a;

Warren et al. 2016). Certain aspects of this research are applicable to the present aim of enlarging the discussion of structures and structural activity in number and the numerical operations of arithmetic. In their research on structure, Hoch and Dreyfus (2004) define *algebraic structure* as follows:

> Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure. (p. 50)

One of the examples they provide is the expression $30x^2 - 28x + 6$ that students come to see as having a quadratic structure, which in turn allows it to be transformed into an equivalent factorized expression involving two linear terms. Their definition alerts us to the aspect of internal order, as well as to its possible structural decompositions. Warren (2003), in a paper on the role of arithmetic structure in the transition from arithmetic to algebra, and in line with earlier research of Morris (1999), similarly contends that knowledge of mathematical structure is knowledge about mathematical objects and the relationship between the objects and the properties of those objects. She states that:

> Mathematical structure is concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality). (Warren 2003, p. 123)

In addition to the field properties, we note that Warren includes mention of equivalence and equality properties as well as order properties.

From the research that has documented difficulties experienced by beginning algebra students with recognizing structure in algebraic expressions and equations, we obtain further insights for an enlarged perspective on structure (for overviews of this research, see Kieran 1992, 2007; for an alternative point of view on structure, see Kirshner 2001). Linchevski and Livneh (1999), who coined the phrase "structure sense," maintain that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. These researchers thereupon insist that instruction be designed to foster the development of structure sense by providing experience with equivalent structures of expressions ("equivalent structures of expressions" being sometimes referred to—in, e.g., Mason et al. 1985—as "equivalent expressions" or equivalent "forms") and with their decomposition and recomposition. Hoch and Dreyfus (2005, 2006) have also reported that very few of the secondary-level students they observed had a sense of algebraic structure, that is, very few could: "(i) recognize a familiar structure in its simplest form, (ii) deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form, and (iii) choose appropriate manipulations to make best use of structure" (2006, p. 306). Demby (1997) too found that algebra students were poor at identifying structure, in particular, the properties they use when they transform algebraic expressions—

despite having been taught how to use these properties. Thus, these algebra researchers suggest that improving attention to structure with younger students needs to go beyond focusing on the basic properties and should include experience with equivalence of compound and simple forms, that is, with equivalence expressed through decomposition, recomposition, and substitution, as well as with recognizing equivalence to familiar structures.

Some of these recommendations have been further unpacked in various proposals related to the development of algebraic thinking within arithmetic. For example, Ellemor-Collins and Wright (2009, p. 53) claim that *structuring numbers* means "organising numbers more formally: establishing regularities in numbers, relating numbers to other numbers, and constructing symmetries and patterns in numbers." For Subramaniam and Banerjee (2011, p. 91), "numerical expressions must be viewed not merely as encoding instructions to carry out a sequence of binary operations, but as revealing a particular operational composition of a number … how quantities or numbers combine." Slavit (1999) emphasizes the importance of being able to break an operation into its base components, of knowledge of operation facts, and of understanding the relationships between the operations. Asghari and Khosroshahi (2016, p. 1) argue that "mathematical thinking involving equality among young learners can comprise both an operational and a structural conception and that the operational conception has a side that is productively linked to the structural conception." Schwarzkopf (2015, p. 14, citing Winter 1982) advances the notion that "understanding an equality between two mathematical terms means understanding that the terms are different representations of the same mathematical object" (e.g., $5 + 4 = 2 + 7$)—a perspective on structure that is similar to the relational thinking approach to equalities promoted by Carpenter et al. (2003) and that includes attention to the role played by substitution in conceptions of equality (Jones et al. 2012).

In their research with elementary school children, Malara and Navarra (2016) point to the importance of expressing structural aspects of number in transparent, non-canonical ways, as illustrated by their example of 10-year-old students representing the sum of 5 and its successor: One student offered the expression "5 + 6", but a classmate argued that her own representation of "5 + 5 + 1" was clearer and more transparent because it expressed the functional relationship between a number and its successor. Similar transparency is stressed by Carraher et al. (2006), who have used the *N*-number line representation to help students focus on the structure of numbers and the relation between a number and its numerical neighbours.

The kind of "structural transparency" advocated by Malara and Navarra and by Carraher et al. is also emphasized by Arcavi et al. (2017), who argue that students' compulsion to calculate numerical answers can make it difficult for them to see patterns and mathematical structure. They describe an activity based on the well-known arithmetical sequence of triangular numbers arranged in dot formation. Arcavi et al. state that it is much easier to see the structure of the numerical sequence, and to generalize it, by observing the pattern in the uncalculated expressions 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, …, than by looking only at the total numbers of dots 1, 3, 6, 10, … for each member of the sequence. Expressing the total numbers of dots by means of their "unclosed" sums allows for seeing that the sequence is the sum of the natural numbers. Collis (1975), many years ago, brought attention to the importance of "acceptance of lack of closure" in the development of algebraic thinking (i.e., being able to accept, say, 8 + 4 as a number with the same legitimacy as its calculated value of 12). As pointed out by Mason et al. (2009), "by working on tasks which focus on the nature of the relation rather than on calculation, students' attention is drawn to structural aspects as properties which apply in many instances" (p. 15).

Other research-based suggestions related to developing elementary school students' structuring experiences with numbers and operations in arithmetic have included, to name just a few: activity with quasi-variables that brings out the additive inverse and additive identity properties (Fujii and Stephens 2001); the role of students' drawings to illustrate, for example, the doubling-halving property of multiplicative factors (Russell et al. 2011); multiplication-table tasks where students are encouraged to seek reasons why certain cells are the same (Neagoy 2015) and to articulate the structures underpinning the tables (Hewitt 1998; Mason et al. 2005); numerical tasks involving equivalence and compensation within addition and subtraction (as well as within multiplication and division), such as for example transforming 298 + 57 to 300 + 55 so as to make calculations easier (Baek 2008; Blanton et al. 2011; Britt and Irwin 2011); and the "three dice guessing" activity where students decompose by partition all the combinatorial possibilities from 3 to 18 (Wittmann 2016).

In the spirit of Freudenthal (1983, 1991), and as reflected in the research literature exemplified above, *structure* as it pertains to number and numerical operations at the elementary and early middle school levels encompasses many means of structuring—structuring according to factors, multiples, powers of 10, evens and odds, sums of 10, prime decomposition, and many more—such structurings often expressed in decomposed, uncalculated form. These structurings have properties, such as the basic properties of arithmetic, but also a multitude of other properties such as the successor property, the sum of consecutive odd numbers property, the sum of even and odd numbers property, equivalence and equality properties, and so on. To conclude this section, we would argue that the inclusion of such additional means of structuring and their properties within our discussions of structure related to number and numerical operations allows for a broader conceptualization of a fundamental aspect of early algebraic thinking and its development.

4.2.3 Structure in Figural Patterns and Functions

Number and numerical operations are not the only content included in early algebra research and teaching practice. Patterning and functions are also integral to this area of study. So what do we mean by *structure* in figural patterns and functions? As will be argued later in this section, the structure of number and numerical operations remains a central component even within these additional focus areas. One point that needs, however, to be brought out beforehand is that the term *structure*, in general, as it relates to figural patterns needs to be distinguished from the term *structure*, in general, as it relates to number and numerical operations. In activity that involves seeking structures in numbers and numerical operations, the structures are inherent to the numbers and numerical operations—a consequence of the axioms. Such is not the case with patterns. As pointed out by Carraher et al. (2008), pattern is not an acknowledged, much less well-defined, concept in mathematics. Patterns can be extended mathematically in any way that one wishes. There is no inherent structure to be uncovered and then generalized (Mason et al. 2010). Patterning involves the search for some regularity and the *imposing* of a certain structure. This imposing of structure affords some predictability to the pattern and so allows for generalizing beyond the provided set of examples of the pattern. Mason et al. (2009) remind us that many "mathematical-looking" tasks involve asking students to extend "patterns" and to predict the *n*th term; they emphasize that there must be some prior agreement or articulation of the actual underlying structure that generates the given sequence in order for a pattern to be considered a mathematical task—an articulation that researchers of figural patterning activity are generally careful to provide by means of the story context that accompanies the pattern (see, e.g., Moss et al. 2008, p. 157).

Once the actual framing of the underlying structure has been set out, the process of generalizing within patterning is considered to involve searching for some invariant property in a set of objects. In fact, distinguishing between what is invariant and what it is that is varying constitutes a crucial first stage in the activity of patterning (Kieran 2006b; Mason et al. 2005). According to Rivera (2013), and in line with Mason et al. (2009), the development of structural thinking within patterning activity involves the recognition of relationships of similarity and difference within a structure, followed by the perceiving of properties that characterize the objects being analyzed, and then by reasoning on the basis of the identified properties.

While many different types of patterns have been used in early algebraic activity, one of the most widely used types is that of the growing figural, or geometric, pattern. Based upon his extensive research on the development of algebraic thinking with 7- to 9-year-olds, Radford (2011) has argued that:

> Generally speaking, to extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one *spatial* and the other *numerical*; from the spatial structure emerges a sense of the figures' *spatial position*, whereas their numerosity emerges from a numerical structure. (p. 19)

In one of his studies, 7- and 8-year-olds were presented with the pattern shown in Fig. 4.1. Radford describes how one of the children, Carlos, when asked to draw the 5th term, very carefully produced the drawing shown in Fig. 4.2—one that did not conform to the two-row configuration of the given pattern. Carlos's geometric shape for Term 5 did not help him figure out its numerosity. On the other hand, other children who did attend to shape were still not attuned to numerosity, and vice versa.

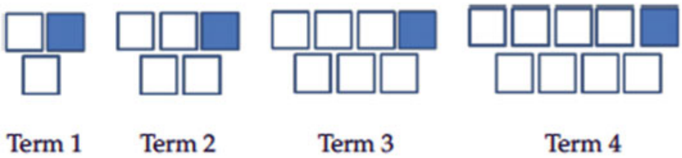

Fig. 4.1 The first four terms of a sequence given to the students in a Grade 2 class (Radford 2011)

Fig. 4.2 Carlos's drawing of the 5th term (Radford 2011)

Radford has emphasized that the linkage of spatial and numerical structures in figural patterns constitutes an important aspect of the development of algebraic thinking.

The day after the observations involving Carlos, the teacher discussed the row-wise geometric structure of the pattern with the children and thereby helped them to make links between the numerical and geometric structures. When she then asked the children about the number of squares in Term 25, one child, Mary, volunteered: "25 on the bottom, and 25 on the top plus 1." Radford (2012) explains the children's progress as follows:

> They became aware of the fact that the counting process can be based on a *relational idea*: to link the number of the figure to relevant parts of it (e.g., the squares on the bottom row). … The terms appear now not as a mere bunch of ordered rectangles but as something susceptible to being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. … This cultural transformation of the eye is not specific to Grade 2 students. It reappears in other parts of the students' developmental trajectory. It reappears, later on, when students deal with factorization, where discerning structural *syntactic forms* becomes a pivotal element in recognizing common factors or prototypical expressions. (pp. 216–217)

The point here is that the eye must learn to look for structural features in a variety of mathematical objects, structural features that involve decomposing and recomposing (see also Radford 2010). Malara and Navarra (2016) hinted at the structural transformation of the eye within purely numerical activity in their description of the young student who came to recognize that the number 6 could be decomposed into 5 + 1, thereby allowing one to see and to express the successor property of number. We shall return to this notion of the "structural eye" later on, near the end of the chapter.

Patterning is widely used in early algebra research studies that explore how children in the elementary grades come to think about and represent functional relationships, in particular, functions with a linear structure, but also exponential and quadratic structures (e.g., Cooper and Warren 2011; Rivera and Becker 2011). Rivera (2013), for example, describes in detail the structuring processes engaged in by various aged children across a variety of functional patterning tasks. There exists as well a substantial amount of research related to children's development of algebraic thinking in the context of functions that does not involve patterning tasks, but rather functional problem situations (e.g., Blanton et al. 2015; Carraher et al. 2006). Within this extensive body of research literature devoted to the theme of algebraic reasoning within patterning and functional activity is an understated aspect that pertains directly to our preceding discussion of structure in number and numerical operations. It concerns the explicit expression of sequences of operations.

An example of this aspect is drawn from the research of Moss and London McNab (2011), who aimed at developing 7- and 8-year-old students' awareness of linear functional relationships by means of both numeric and geometric (figural) patterns. The research method they employed involved first using geometric patterns (tile arrays) and then numeric (function machine) patterns and then moving back and forth between the two. They found that a bridging occurred between the two types of patterns that was enabled by the idea of a function rule: "It was the specific movement back and forth between the two representations, geometric and numeric, that ultimately supported the students to gain not only flexibility with, but also a structural sense of, two-part linear functions [i.e., $y = mx + b$]" (p. 296). In particular, they claim that it was the explicit expressing of the sequences of operations that corresponded to the functional structure of the numeric patterns that eventually came to be seen as a common thread in both the geometric and numeric patterns. As well, students became aware that expressions such as $15 \times 2 + 3$ were equivalent to $15 + 15 + 3$ by means of the parallel geometric and numeric structures and without necessarily calculating the totals for each. The emphasis on the role played by the explicit expressing of the sequences of operations in the Moss and London McNab study reminds us of the point made earlier by Arcavi et al. (2017) regarding the structure-developing role that can be played by observing the uncalculated expressions of the successive terms of a pattern sequence. Other types of activity involving explicit sequences of uncalculated expressions playing a similar role have been noted in research on "think of a number games" (e.g., Cedillo and Kieran 2003) and "tracking arithmetic" tasks (Mason 2017; Mason et al. 2005)—where students are encouraged to represent explicitly and in uncomputed form the operations that are applied to the thought-of numbers (sometimes represented as clouds) so as to more easily detect the properties being applied throughout and thereby explain the final results.

To recapitulate, the structures involved in figural patterns clearly include a numerical component (in addition to a spatial component) and, vice versa, the numerical aspects of the patterns are structural in nature. Recall Mary's response to the question posed to her in Radford's (2011) research regarding the number of squares in the 25th term of the pattern: "25 plus 25 plus 1"—a structural response that was expressed by a numerical decomposition, one that corresponded to her spatial decomposition of the figure. And decomposed numerical expressions constituted students' functional activity in the Moss and London McNab (2011) study. The point being made here is that structuring experiences involving decomposed numerical expressions are central not only to the content area of number and numerical operations, but also to the content areas of patterning and functions. We view such structuring experiences as fundamental to the development of early algebraic thinking.

4.3 Seeking, Using, and Expressing Structure in Numbers and Numerical Operations by 12-Year-Olds

This next section of the chapter presents a researched example of 12-year-old students engaged in structuring activity with numbers and numerical operations. The study (Kieran and Guzmán 2005) involved three classes of students from a Mexican private school in a task-based calculator environment and focused on the ways in which they sought, used, and expressed structures related to multiplication, division, factors, multiples, and divisors.

4.3.1 The "Five Steps to Zero" Problem: The Tasks and Learning Environment

Seeking and using structure involving multiplication and division has received less attention in the research literature than has addition and subtraction. Even less appears on the ways in which computing tools might be harnessed in the development of structural thinking in this area. In our study, we integrated a combined task-technique-theory perspective that was based on the so-called instrumental approach to tool use (Artigue 2002). Within this approach, mathematical concepts are considered to co-develop while the learner is perfecting his/her techniques with the tool. According to Lagrange (2000, p. 17, our translation and our emphasis): "The new instruments of mathematical work are of interest … because they permit students to *develop new techniques that constitute a bridge between tasks and theories.*" If techniques can constitute a bridge between tasks and the emergence of theoretical knowledge, then it is by looking at the techniques that students develop with the aid of their technological instruments, in response to certain tasks, that we obtain a window into the evolution of their structural awareness.

The tasks that we developed were based on the "Five Steps to Zero" problem (Williams and Stephens 1992; see Fig. 4.3—note that all whole numbers from 1 to 1000, with the exception of 851 and 853, which require six steps, can be brought down to zero in five or fewer steps). Successfully tackling this task situation, with the constraint of using only the whole numbers from 1 to 9 and only one operation per line, involves developing techniques for decomposing numbers (prime or

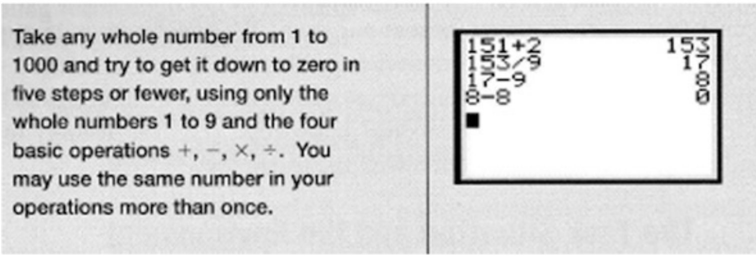

Fig. 4.3 The basic problem, "Five Steps to Zero" (adapted from Williams and Stephens 1992), accompanied by an example (151), displayed on the multiline screen of the calculator

composite) into other numbers in the same neighbourhood (not more than 9 away from the given number) that have divisors not larger than 9 so as to reach zero in five or fewer steps.

In the example illustrated in Fig. 4.3, the given number 151—a prime—was first converted into a non-prime number, followed by a test to see if the result was divisible by 9 (*divisible* being understood to mean *divisible without remainder*). As an alternative to the approach displayed in Fig. 4.3, a student might first subtract 1 from 151, and then divide 150 by divisors such as 2, 3, and 5, eventually arriving at 5, which can then be brought to zero by subtracting 5. Since students were encouraged to use as few steps as possible, this task provided fertile ground for learning, for example, the structural property that if a number has both a and b as divisors, then it is also divisible by $a \times b$, as long as $a \times b$ does not exceed 9.

Theoretical awareness of this property has been reported to be difficult for many students to develop. Past research has shown that, for example, just because students were able to find that $a \times b = ab$, they did not then state that ab is a multiple of b before first dividing ab by b (Vergnaud 1988). In another study involving older students, Zazkis and Campbell (1996, p. 542) asked: "Consider the number $M = 3^3 \times 5^2 \times 7$. Is M divisible by 7? Explain. Is M divisible by 5, 2, 9, 63, 11, 15? Explain." Students' understanding of divisibility and prime decomposition was found to be so poor that the researchers argued that developing a conceptual understanding of divisibility and factorization, which is essential in the development of conceptual understanding of the multiplicative structure of numbers, should be happening in the middle grades. We were thus interested in investigating how the tasks and tools that we designed to be used with the middle-grade students of our study would increase their structural awareness of factors, multiples, and divisibility.

We developed a set of ten activity sheets (see Fig. 4.4) involving tasks based on the "Five Steps to Zero" problem on which students worked over a period of one week (five classes of fifty minutes each). When students finished one activity sheet and handed it in, they were given the next one.

Several considerations were included in designing the tasks for the 10 activity sheets. Task #1 involved an even number (144) with several divisors below

1. Take the number 144. Write as many ways as you can for bringing 144 to zero, using as few steps as possible.
2. Take the number 151. Write as many ways as you can for bringing 151 to zero, using as few steps as possible.
3. Take the number 732. Write as many ways as you can for bringing 732 to zero, using as few steps as possible.
4. Describe your strategies for minimizing the number of steps.
5. Here is a solution proposed by a pupil for bringing 432 to zero: 432/2 = 216; 216/2 = 108; 108/2 = 54; 54/3 = 18; 18/3 = 6; 6 – 6 = 0. Show a way of bringing 432 to zero in fewer steps. Explain your strategy. Do you think it will always work? Why?
6. Here is a strategy proposed by a pupil for bringing 731 to zero: 731 + 7 = 738; 738/9 = 82; 82 – 1 = 81; 81/9 = 9; 9 – 9 = 0. Show a way of bringing 731 to zero with fewer steps. Explain your strategy.
7. The number 266 has as its divisors 2, 7, and 19. In other words, 266 = 2 × 7 × 19. What is the best strategy for bringing 266 to zero? Why? Explain why your strategy is the best.
8. Here is the strategy proposed by a pupil for bringing 499 to zero: 499 + 1 = 500; 500/5 = 100; 100/5 = 20; 20/5 = 4; 4 – 4 = 0. Show a way of bringing 499 to zero in fewer steps. Explain your strategy.
9. What do you consider to be the best strategies for bringing numbers down to zero?
10. Think of a number that your classmates would find difficult to bring to zero in five or fewer steps. Write down why you think it would be a hard number. Show the solution you found for your hard number.

Fig. 4.4 Tasks of the 10 activity sheets prepared to accompany the "Five Steps to Zero" problem

10—thus a fairly accessible number to start with. Note also that, by asking students to record on paper the solutions they were trying with their calculator, we would be able to trace how their strategies were evolving. Task #2 involved a prime number, which could be handled by means of the addition or subtraction of some number in order to have a composite that could be divided by numbers less than 10. Would students aim for an even number, a number ending in 5 or 0, or something else? Task #3 began with 732, which could be brought to zero in five steps if one began with a division by 3, 4, or 6; but a four-step solution required adjusting the given number so as to have a multiple of 9. After the first three open-ended tasks, the very important Task #4 asked students to describe in writing their techniques for minimizing the number of steps to be taken to reach zero. Task #5 presented a six-step approach for bringing 432 to zero. All of the divisors used in the given task were 2s and 3s. Would the students spontaneously think of combining some of the given divisors—for example, the first three 2s to yield a divisor of 8, and the last two 3s for a divisor of 9—so as to reduce the number of steps from six to three? Task #6, which was designed to provide some experience with multiples of 9, illustrated a five-step method for bringing 731 to zero that led off with the conversion of 731 to 738. Would students come to see that if 738 is divisible by 9, so too is $738 - 9$, and that this would save a step because the resulting quotient is immediately divisible by 9? Task #10, the final task, formed the basis for a follow-up competition in class.

The class was to be divided into three groups; each group settling on one of the "hard" numbers proposed and defended by its group members. Three rounds of competition involving two teams trying to solve the "hard" number of the third team were to take place. This competition and the individual justifications written down by each student as to their "hard" number were to provide further evidence of the evolution in their structuring activity during the week.

Each student was equipped with a graphing calculator. While the graphing capability of the calculator was not used, the larger screen of this kind of calculator made it possible for the students to record and observe at a glance all their steps toward zero—as opposed to the small one-line screen of a simple four-operation calculator. The calculator permitted students to carry out each of the basic operations in one step. Without having to keep track of all the intermediate moves that would normally capture their attention in a paper-and-pencil environment, they were free to focus on structural aspects. It is noted that the calculators that were used in the study did not provide the complete factorization of a number; thus, it was not possible to know instantly whether a given whole number was a multiple of some other number, or even whether or not it was prime—a constraint that was capitalized on for our study.

The teachers introduced the main task situation as follows. They began with the example of 360 and illustrated with the classroom view-screen (a room-size projection of the screen of the calculator that was hooked up to the view-screen device) that they could get down to zero in the following way: 360/2, 180/2, 90/3, 30/6, $5 - 5$. The teachers then requested volunteers to come forward to show how they might get to zero in fewer than five steps. After that, students were asked to suggest their own starting numbers, say, larger than 200, which other students came forward to solve. The students then began to work on the tasks of the activity sheets, either individually or in pairs, but each student filled in his/her own activity sheets.

Regularly during the week, individual students were invited to come forward to the classroom view-screen and to work out a task using that device. This allowed both the researchers and the classroom teacher to observe directly the nature of the approaches that the students were trying out in response to the problem tasks. It is important to note that the students whose work was being recorded on the view-screen had not already arrived at a solution beforehand, but were in fact allowing us to witness all the false starts, dead ends, and various structural relations they were trying to find. During the week that followed the classroom part of the study, four students representing a range of mathematical ability (as rated by the classroom teacher) from each of the participating classes were individually interviewed. A pre-test had also been given to the students prior to the study to inquire into their knowledge of divisors, multiples, and primes. The interviews, in conjunction with the pre-test results, gave the researchers the opportunity to explore at closer range the nature of the structural awarenesses that students had developed over the course of the previous week. In the sections that follow, we present and discuss samples of students' work that are representative of the ways in which the techniques of the three classes of students evolved.

4.3.2 The Emergence of Structurally-Oriented Techniques

The techniques that students used at the beginning of the week's activities tended to be based on simple criteria for divisibility, such as dividing by 5 if the number ended in 0 or 5, or dividing by 2 if the number was even. This is illustrated by the work of Marianne with the given number 151 from the second activity sheet (see Fig. 4.5). Her two recorded attempts suggested two different decompositions of 151: $5 \times 30 + 1$ and $5 \times 31 - 4$, both handled by means of inverse operations. She was clearly aiming at converting the given number 151 into a multiple of 5. We wonder whether she noticed the structural property that when two adjacent multiples of 5 (i.e., 150 and 155) are divided by 5, the two quotients that are obtained (i.e., 30 and 31) are consecutive numbers.

The initial techniques of Marianne evolved, just as they did for her classmates. On the third activity sheet with 732 (see Fig. 4.6), she showed a shift toward trying to find larger divisors.

On the fourth activity sheet, in describing the techniques that had emerged thus far for her, Marianne wrote (translated from Spanish):

> Divide by the largest divisor possible from 1 to 9; if there are no divisors, then add or subtract to obtain another number where the division is possible. After dividing, look at the result and test whether division is again possible. If not, repeat the previous procedure until arriving at a number less than 9 and finish the procedure with a subtraction.

Nicolas offers us another example of how pupils in this study were evolving from more basic techniques to that of trying to find the largest divisor possible. Having unsuccessfully tested whether 9 or 8 was a divisor of 930 (see lines 2 and 3 of Fig. 4.7), Nicolas's next efforts centered on finding another number in the neighbourhood of the given number 931 for which he could use large divisors

L1: 151 – 1	150	L1: 151 + 4	155
L2: 150/5	30	L2: 155/5	31
L3: 30/5	6	L3: 31 – 1	30
L4: 6 – 6	0	L4: 30/5	6
		L5: 6 – 6	0

Fig. 4.5 Two consecutive attempts by Marianne to bring 151 to zero (Note that line numbers have been added to make it easier to refer to specific lines of the screen display)

L1: 732/6	122	L1: 732/4	183
L2: 122/2	61	L2: 183 – 3	180
L3: 61 + 3	64	L3: 180/9	20
L4: 64/8	8	L4: 20/5	4
L5: 8 – 8	0	L5: 4 – 4	0

Fig. 4.6 Two attempts by Marianne with the given number 732

L1: 931 – 1	930	L26: 931 + 5	936
L2: 930/9	103.33	L27: 936/9	104
L3: 930/8	116.25	L28: 104/8	13
L4: 930/5	186	L29: 13 – 9	4
L5: 186/9	20.66	L30: 4 – 4	0
L6: 186/8	23.25		

Fig. 4.7 Various attempts by Nicolas to find suitable decompositions of 931

L1: 9 × 86	774	L10: 9 × 106	954
L4: 9 × 97	873	L11: 9 × 108	972
L6: 9 × 99	891	L12: 971 + 1	972
L7: 9 × 105	945	L13: 972/9	108
L8: 9 × 110	990	L14: 108/6	18
L9: 9 × 107	963	L15: 18/9	2
		L16: 2 – 2	0

Fig. 4.8 Marianne's shift to multiplication in her search for an appropriate structuring of 971

throughout. In attempting to find numbers in the vicinity of 931 that were divisible by 9, had Nicolas noticed the structural property that within every interval of 9 numbers there is exactly one number that is divisible by 9? His later work was to confirm that he had indeed discovered this property.

Toward the end of the week's activities, several students began to make structural breakthroughs. Their focus became more controlled in that instead of using successive trial and error with the divisors 9, 8, and 7, they started to search for techniques oriented around the use of the factor 9. The challenge of the activity had become that of finding a structural way to convert the given number into a multiple of 9 so as to arrive at zero in the fewest number of steps possible. Marianne, for example, wrote on her sixth activity sheet that she wanted "to subtract or add in order to arrive at a number divisible by 9; if you divide by the largest number, even if you do a subtraction or yet an addition, you will reach zero more rapidly."

With Marianne, it was not until the last day of the week, when she was using the classroom view-screen and was given the number 971 to bring down to zero, that we witnessed the structural technique that she had developed (see Fig. 4.8). Marianne began at once to search for a number in the vicinity of 971 by using the product of two factors, one of which was 9. She was in fact working in reverse, using multiplication rather than division, so as to try and arrive at a multiple of 9 in the neighbourhood of her starting number. Once she had found two multiples that were on either side of the starting number (see Lines 8 and 9 of Fig. 4.8), she successively refined her search until she reached a multiple of 9 that was within 9 units of 971 (see Line 11). The structure of multiplication with subtraction ($9 \times 108 - 1$) was then converted to addition with division (Lines 12 and 13).

A related technique that involved the structural interplay between dividing by 9 and using multiples of 9 emerged for another student, Mara, near the end of the week. While she was at the front of the class using the view-screen, a classmate suggested she try 731 as her initial number. After a few unsuccessful tries involving the search for neighbouring numbers that could be divided by 9, she seemed suddenly to notice that she could take the whole-number part of the quotient, which she rounded up to 82, and did a reverse multiplication (see Lines 16 and 17 of Fig. 4.9). The product told her immediately how much of a structural adjustment needed to be made to the initial number. We note that, had she truncated rather than rounding up the quotient, she would have saved a couple of steps in that 81 would have allowed an immediate subsequent division by 9. Another student, Pablo, had developed a similar technique (see Fig. 4.10). By multiplying 9 with 103 (Line 2), Pablo then inferred that the remainder on trial dividing 931 by 9 was 4, thereby leading to a structural decomposition of 931 as $9 \times 103 + 4$.

During the week following the classroom study, when individual interviews were held with some of the students, a revealing conversation took place with Nicolas. When asked what he would do if a given initial number was not divisible on the first step by a number between 2 and 9, he answered that he would add or subtract. So we continued by asking him how he figured out the amount that he needed to add or subtract, to which he responded that he had a certain "technique" (see Fig. 4.11 for the transcript of the relevant segment of the interview; I is the Interviewer and N is Nicolas).

Notice (Episode 36 of Fig. 4.11) that Nicolas is trying to control three factors at a time, all of them in the range of 2 to 9, in his search for a product in the neighbourhood of 431. On screen line L4, he enters the following multiple of 9: $9 \times 9 \times 5$, and sees that it yields 405. So, he then decides to adjust the second and third factors simultaneously. He decreases the 9 to 8 and increases the 5 to 6, entering $9 \times 8 \times 6$ into the calculator. This numerical expression produces the result 432, just 1 more than the given number (see screen line L5). He clearly realizes that $9 \times 8 \times 6 - 1 = 431$. This decomposition will allow him to bring 431

L1: 731 + 1	732	L16: 731/9	81.22
L2: 732/9	81.33	L17: 9 × 82	738
. . .		L18: 731 + 7	738
L10: 731 – 8	723	L19: 738/9	82
L11: 723/9	80.33	L20: 82/2	41

Fig. 4.9 Mara's shift from dividing to the reverse operation of multiplying the divisor by the rounded-up quotient

L1: 931/9	103.44	L3: 931 – 4	927
L2: 9 × 103	927	L4: 927/9	103

Fig. 4.10 Pablo's similar shift from dividing to multiplying in decomposing 931

32.	N:	Because (pause) well, I also have a "technique" that I use. First I do a multiplication, say, $9 \times 9 \times 3$ or something like that to arrive at another number, and I look at that number.
33.	I:	Let's see, repeat that for me one more time.
34.	N:	For example, if I have the number 571 and I multiply 9×9, it gives 81.
35.	I:	Let us say that I give you the number 431.
36.	N:	OK, so I go (and he picks up the calculator): L1: 9×9 81 L2: 81×3 243 L3: $9 \times 9 \times 4$ 324 L4: $9 \times 9 \times 5$ 405 So, like that, I arrive more quickly.
37.	I:	But I said 431. With this strategy that you have just described, how do you begin?
38.	N:	First, $9 \times 9 \times$ something, no? Until arriving close to the number. For example (he again picks up the calculator): L5: $9 \times 8 \times 6$ 432
39.	I:	Yes, I told you 431.
40.	N:	So, 431 plus 1, divided by 6, divided by 8, and so on.
41.	I:	Let's see.
42.	N:	(Nicolas picks up the calculator): L6: 431 + 1 432 L7: 432/6 72 L8: 72/8 9 L9: 9 – 9 0 And there it is!

Fig. 4.11 Segment of transcript from the interview with Nicolas where he describes his "technique" and illustrates it with the given number 431

to zero in four steps by means of inverse operations where each of the factors will be treated as divisors, except for the last one, which will be subtracted so as to arrive at zero. The complex of structural relations between multiplication and division and between addition and subtraction, as well as a structuring according to divisors and multiples, have all been expressed in Nicolas's mathematical work.

4.3.3 Analysis of the Evolution of Students' Structuring Activity

Some of the most powerful structural explorations that occurred during the week of activity on the "Five Steps to Zero" tasks involved the search for multiples of 9. Since students wanted to arrive at zero in the fewest number of steps possible, their initial techniques soon evolved into attempts to discover whether the given number was divisible by 9, or whether any numbers in the close vicinity (i.e., within 9 on either side of the given number) were. But how to find the right numbers in the close vicinity was the question. Furthermore, the students seemed unaware of the criterion for divisibility by 9 (i.e., sum the digits to see if the total is a multiple of 9) or how this test might be used to locate a multiple of 9 in the neighbourhood of the given number.

4.3.3.1 Variants of the *Division Algorithm*

The structural awareness that emerged for many students involved variants of the *division algorithm.* According to this theorem, any whole number can be decomposed and expressed as the product of two whole numbers plus remainder, that is, "For any $b > 0$ and a, there exist unique integers c and d with $0 \le d < b$ such that $a = b \times c + d$" (e.g., 989 = 9 × 109 + 8). Even though students were not taught this theorem, their work showed the different structural means by which they tried to obtain the value of c and thereby illustrated the ways in which they were beginning to think structurally about division with remainder—even if not always articulated explicitly. One variant of the division algorithm reflected in their work was the following: "For any $b > 0$ and a, there exist unique integers c and d with $0 \le d < b$ such that $a = b \times c - \mathrm{d}$" when the decomposition of the initial number a led to using the multiple of b on the higher side of a rather than on the lower side (e.g., 989 = 9 × 110 − 1).

But we also witnessed other structural "variants" of the division algorithm. For example, the techniques of Mara and Pablo evolved to take the form of carrying out a trial division by 9, followed by the multiplication of the truncated or rounded-up quotient with 9 in order to see how far the product was from the initial number—a structural approach that we named the "division algorithm invoking trial division" (e.g., 989/9 = 109.8888889, followed by 9 × 109 = 981). Since 989 can thereby be decomposed into 9 × 109 + 8, their approach for bringing 989 to zero involved using the inverses of addition and multiplication as in 989 − 8 = 981, 981/9 = 109, and so on. An implicit variant of this technique involved *looking at the size of the decimal portion of the quotient, without actually carrying out the related multiplication of 9 with the truncated or rounded-up quotient,* to provide a structural clue as to how close the given number was to a multiple of 9.

Another variant of the division algorithm was based on what we named the "division algorithm invoking trial multiplication." This approach, observed with Marianne, involved carrying out trial multiplications in order to find an appropriate value of c, as in, for example, the structural relation, $989 = 9 \times c + d$ (e.g., 9 × 106 = 954, 9 × 108 = 972, 9 × 109 = 981)—the latter multiplication clearly bringing the solver into the interval that is within 9 of the given number 989, thereby allowing for decomposing 989 as 9 × 109 + 8.

While the searches by Marianne for multiples of 9 always involved two factors, the technique that Nicolas came to develop involved a complete decomposition of the given initial number into three factors or more, accompanied where necessary by an addend-adjustment. His technique is one that we named "trial multiplication involving more than two factors" and synthesized as $a = b \times c \times d \times e \pm k$, where b, c, d, and e are whole number factors between 2 and 9, and k is a whole number addend or subtrahend such that $0 \leq k \leq 9$. In Nicolas's above work with the given number 431, he was able to analyze his trial sequence of operations, $9 \times 9 \times 5$, which had yielded 405, and adjust the expression in such a way that, by decreasing the second 9 to 8 and increasing the 5 to 6, the result ($9 \times 8 \times 6$) would be slightly larger than 405. The degree of control he showed not only in generating the multiple factors approach but also in changing the specific decomposition from $9 \times 9 \times 5$ to $9 \times 8 \times 6$ hinted at the structural eye that he was beginning to develop for number and numerical operations.

4.3.3.2 Developing a Structural Eye for Decomposing Number

Developing an eye for structure is surely a long process that needs to reinvent itself with every new type of mathematical object that is encountered. While the "Five Steps to Zero" activity lasted only a week, and there was no follow-up opportunity to see whether the ways in which students sought structure within that activity would carry through to their everyday mathematical work, the shifts in structural techniques that we observed suggest that the students had indeed begun to develop a structural eye for the multiplicative decomposition of number. This was highlighted in, for example, Mara's noticing that she could replace several trial divisions by just one, followed by multiplication of the rounded-up quotient by the divisor. From our observations, we conjecture that the shifts that emerged were partially motivated by students' lack of satisfaction with the initial trial-and-error methods they were using—dissatisfaction that pushed them to "develop a technique," to use Nicolas's words.

As was seen in the students' early trial divisions with the large divisors 8 and 9, the quotients usually contained decimals, which "got in the way" of reaching zero quickly. In the search for more effective means of tackling the tasks, Mara, as noted above, came to realize that she could "clean up" the decimal quotient and multiply it by the divisor to arrive immediately at a product that was in the required range of the given number. Marianne became aware that she could supplant her "messy" trial divisions with more focused multiplications involving two factors, one of which was 9. Nicolas came to develop a method that allowed him to arrive at a decomposition of the given number consisting of a complete set of factors plus any required additive adjustment. According to Subramaniam and Banerjee (2011), a refined, structural understanding of operational composition includes accurate judgments about relational and transformational aspects, such as judging how the contribution of one part of the expression will change if some of the numbers involved in the expression change. Such judgments were reflected in the compensations made by Nicolas in arriving at his complete decomposition of 431 (in Episodes 36–38 of Fig. 4.11) during the post-study interview.

One last point needs to be made with respect to students' beginning to develop a structural eye: It concerns the role of the tasks and the calculator. In Radford's

(2012) study with younger children, the teacher played a key role in helping them come to see the spatial structure of the pattern and to coordinate this with the numerical structure. In our study, the teachers played much more of an observational role. On the other hand, the inherent challenge of the "Five Steps to Zero" problem, as well as the wording of the task questions and the actual numbers used (Fig. 4.4), is likely to have contributed to encouraging the students to think more deeply about the structures that combine multiplication and division. Furthermore, had it not been for the presence of the calculator, the tasks that led to the development of students' structurally-based techniques, although doable, would surely have been less feasible. The calculators with their multiline screens permitted students to analyze successive results for possible indications of numerical structure. The classroom view-screen also enabled the sharing of newly discovered techniques. In sum, the nature of the tasks, the technological tool, students' collaborative work in pairs and in teams, as well as their own personal determination to find satisfying techniques to meet the challenges of the "Five Steps to Zero" problem, are considered to have all contributed to constituting an emergent, culturally-shared activity that underpinned the evolution of their "structural eye" and shaped the movement of their structural growth. But for such movement to further develop into what could be referred to as persistent structure-oriented practice in mathematical activity would require, in our opinion, "the mathematical work of the teacher in pressing students, provoking, supporting, pointing, and attending with care" (Bass and Ball 2003, p. vii).

4.4 Concluding Remarks

The aim of this chapter has been to instigate greater attention to *structure* and to elaborate more broadly on its meaning with respect to number and numerical operations in the development of early algebraic thinking. As characterized by Freudenthal (1983, 1991), *structure* encompasses the whole web of relations associated with the order, addition, and multiplication structures. These basic structures provide the foundation for multiple additional means of structuring. Furthermore, countless properties, in addition to the oft-cited basic properties of arithmetic, are generated by these structures. But, early algebra involves more than number and numerical operations; it also involves patterning activity and the development of functional thinking. Thus, other structures enter into play such as the spatial structures of figural patterns and various functional structures, namely linear, quadratic, and exponential. Nevertheless, a common aspect of all of these various approaches to early algebraic activity is the multitude of properties and means of structuration that are related to number and numerical operations.

To contribute to illustrating some of the ways in which we might elaborate more broadly on the meaning of structure within number and numerical operations and extend the discussion of the structural properties associated with arithmetical activity, a study (Kieran and Guzmán 2005) on the "Five Steps to Zero" problem was presented. It involved three classes of 12-year-olds who were observed as they generated multiple structural decompositions of the numbers they were given in their problem-solving activity. While students' structural decompositions of the given numbers were not unique, they all displayed an order structure, additive structure, multiplicative structure, and a structure combining the inverse relations between multiplication and division within the *division algorithm*; but their technical approaches also evolved to express structurings according to the related elements of factors, divisors, multiples, and remainder on dividing.

Structural properties that were explicitly indicated in the students' work included the following:

- If a number has both a and b as divisors, then it is also divisible by $a \times b$.
- When two adjacent multiples of a number n are divided by n, then the two quotients that are obtained are consecutive (e.g., "738 and 729 are two adjacent multiples of 9; when they are both divided by 9, the quotients are the consecutive numbers 82 and 81").
- Within every interval of n numbers, there is exactly one number divisible by n (e.g., "In the 9-number interval from 735 to 743 inclusive, there is exactly one number divisible by 9").
- If adding n to a number x yields a multiple of m, then so too will subtracting $m - n$ (e.g., "If adding 1 to 989 yields a multiple of 9, then so too will subtracting 8, due to the resulting difference of 9 between the two adjusted numbers 990 and 981").

Students' early solving strategies evolved in ways that entailed a genesis from the use of trial and error to more deliberate structuring according to the network of relations between multiplication and division. That the evolution occurred speaks powerfully for the initial use of trial-and-error methods that sparked the rise of stronger and more controlled techniques. The details that were provided of the ways in which students attempted to seek, use, and express alternative structures within the "Five Steps to Zero" problem contribute to better understanding how students of this age can come to extract certain multiplication/division structures within numerical activity.

The problem situation itself was pivotal to students' structural growth. But the "Five Steps to Zero" problem should not be considered only in the form in which it was used within this research study. This problem with the challenge of arriving at zero in a restricted number of steps has a generic quality to it. It could easily be adapted for younger students by changing the range of numbers, the range of operations, the number of steps, and even the permissible numbers to be used within the operations, such as even or odd numbers only, or just multiples of, say, 3, and so on. Researchers and practitioners could create variants of this problem, perhaps even for use without a calculator, as a means of developing students' awareness of structure for number and numerical operations at various grade levels in elementary school.

We consider the structuring activity described in this chapter to be a fundamental path to developing students' early algebraic thinking. As mentioned in the introductory remarks, high school algebra requires the ability to see structure within generalized forms. Algebra researchers have argued that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. They have thus offered several recommendations as to how instruction in arithmetic might be designed to foster the development of structure sense. A central suggestion has been that of providing experience with decomposition and recomposition of numerical expressions and with their structural equivalence. The activity involving the "Five Steps to Zero" problem has included exactly this type of experience—experience in ways of thinking that will be of value for the later structuring demands to be made in secondary school algebra (e.g., see Guzmán et al. 2010, for algebra students' work involving the structural relation that links factorizability, polynomial division with/without remainder, and cancelling terms in the simplification of expressions such as $(4x + 4y)/(x + y)$ and $(3x + 4y)/(x + y)$). In sum, we concur with Subramaniam and Banerjee (2011, p. 101), when they state that: "Numerical expressions emerge as a domain for reasoning and for developing an understanding of the structure of symbolic representation." More specifically, we contend that developing an understanding of the structure of number and numerical operations by means of various property-based, structural decompositions is vital to the emergence of early algebraic thinking. Indeed, to conclude we would argue that there is a dual face to activity that promotes early algebraic thinking: one face looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards "seeing through mathematical objects" and drawing out relevant structural decompositions.

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Chapter 5
Cultivating Early Algebraic Thinking

David W. Carraher and Analúcia D. Schliemann

Abstract This chapter describes a functions approach to early algebraic thinking developed in the context of classroom research with young students. We outline our approach, examine examples of students' reasoning in the classroom, and present interview and written assessment evidence of student learning. We also describe and evaluate a related program aimed at preparing teachers to promote algebraic thinking across the curriculum. Throughout, we attempt to identify conditions favorable to the cultivation of algebraic thinking in mathematics education.

Keywords Early algebra · Algebraic reasoning · Functions approach
Teachers' algebraic thinking

5.1 Introduction

Although young students may manifest rudimentary forms of algebraic thinking before they have been introduced to symbol-letter notation, there are good reasons for promoting algebraic thinking well before a first course in algebra. Fortunately, there are many opportunities for this, given the underlying algebraic nature of mathematics in the early grades. Here we examine how the basic operations of arithmetic can be approached from the standpoint of functional relations, facilitating the discovery of interconnections among standard topics and promoting students' formulation and representations of generalizations from early on. We illustrate, through examples taken from early algebra research, how we have engaged young students in various sorts of classroom activities designed to promote algebraic thinking. We also describe the key features of an in-service teacher education

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program that, with the contributions of mathematicians and physicists, was developed to build on early algebra research. The reader is cautioned that the present claims are still somewhat speculative and subject to modification over time as additional ideas are explored and new findings emerge in the field of early algebra. This is also to be expected given that principles for instruction cannot be justified solely through research, involving, as they do, value premises and goals (Hiebert 1999).

5.2 Algebra and Algebraic Thinking

Algebra is a branch of mathematics as well as a way of thinking (Gowers et al. 2008). In K–12 educational settings, it is often identified with a system of notation and procedures, heavily driven by syntax, for solving pure and applied problems. Most would agree that algebraic thinking emerged long before the advent of modern algebraic notation. Boyer (1968) breaks down the history of algebra into three stages: the rhetorical, syncopated, and symbolic phases. In the rhetorical stage of ancient Greece, statements and arguments were expressed verbally, occasionally accompanied by diagrams. Greek mathematicians before and including Diophantus were solving algebra problems and thinking algebraically without algebraic notation. It is useful to keep this in mind when attempting to define the minimal conditions for algebraic reasoning. There are ample opportunities for employing non-notational forms of representation, most notably, formulations in natural language, for students to express generalizations. This, however, does not imply that conventional algebraic notation should be withheld from instruction with young learners. We would claim, as do Blanton et al. (2017, also this volume), that algebraic notation has an important role to play in early instruction, especially in modeling extra-mathematical phenomena. Although there are dangers in premature mathematical formalism (Piaget 1964), it is also important to avoid belated formalism, including an overly delayed introduction of algebraic notation and other conventional forms of representation. Striking a balance is admittedly a challenge. However, we have found, and will attempt to exemplify that, in the early grades, students may easily learn to employ algebraic notation to express generalizations that they have reached by reflecting on the problem situation at hand. Gradually they begin to act directly on algebraic expressions in order to derive additional expressions. In this sense, algebraic notation initially tends to take on a “trailing,” rather than a leading, role.

We would suggest that “early algebra” refers to the algebraic knowledge, the algebraic thinking, and the (occasionally unconventional) representations and techniques of young students in solving problems that one would generally expect more advanced students to solve using modern algebraic notation.

As we have noted before (Carraher et al. 2000, 2005; Schliemann et al. 2007), there is no clear-cut break between early algebra and algebra. Early algebra, unlike what some have referred to as “pre-algebra,” is not to be viewed as a bridge students cross after they have studied arithmetic and before they study algebra. In principle, it can be developed and nurtured wherever there is arithmetic. This is because arithmetic is inherently algebraic. Were this algebraic character of arithmetic widely acknowledged and exploited in the early mathematics curriculum, perhaps the expression, *early algebra*, could be dispensed with. But, for the time being, at least in the United States, it can serve the useful purpose of highlighting important, often overlooked, facets of mathematics in the K–8 curriculum.

In the United States, algebra was long understood as entering the mathematics curriculum around adolescence, after, all being well, students had achieved a solid grounding in arithmetic. Earlier attempts to teach algebra would have been considered unpromising in view of the well-documented difficulties middle and high school students are known to have with algebra (see the extensive review by Kieran 2007). Studies have shown that, when students were introduced to algebra towards the end of middle school, many (a) displayed a limited interpretation of the equals sign (Booth 1988; Kieran 1981, 1985; Vergnaud 1985); (b) failed to understand the meaning of letters meant to serve as variables (Kieran 1985; Küchemann 1981; Vergnaud 1985); (c) refused to accept algebraic expressions as valid answers to problems (Sfard and Linchevski 1994); and (d) failed to solve equations with variables on each side of the equals sign (Filloy and Rojano 1989; Herscovics and Linchevski 1994).

Some researchers attributed such difficulties to constraints imposed by cognitive (under) development (Collis 1975; Küchemann 1981), concrete thinking (MacGregor 2001), or the inherently challenging nature of algebra. Herscovics and Linchevski (1994) claimed that many students were unable "to operate spontaneously with or on the unknown." Filloy and Rojano (1989) proposed that first-degree equations with a variable on both sides of the equals sign (e.g., $38x + 72 = 56x$) represent an historical and developmental divide separating arithmetical from algebraic thinking.

These views, however, were far from universal. Some mathematics educators (see, for example, Booth 1988; Kaput 1995, 1998; LaCampagne et al. 1995; Schoenfeld 1995) suspected that such difficulties tended to reflect shortcomings of early mathematics instruction, including the overly computational approach to arithmetic in elementary school.

Early on, Davis (1967, 1971–1972, 1985, 1989) proposed that preparation for algebra should begin in grades two or three; and he had intriguing film footage of young students working with activities such as "Guess My Rule" providing credence to his view. Beginning in the 1960s, Davydov and colleagues developed and undertook an innovative approach to early mathematics instruction centering initially, not on numbers, but instead on relations among unmeasured physical magnitudes. Their work and their impressive empirical results, made available to the English-speaking world somewhat later (Davydov 1991), gave rise to noteworthy initiatives (Dougherty 2008; Schmittau 2005; Schmittau and Morris 2004). Kaput (1995, 1998) argued that the weaving of algebra throughout the K–12 curriculum could lend coherence, depth, and power to school mathematics, and replace late, abrupt, isolated, and superficial high school algebra courses. The Algebra Initiative

Colloquium Working Group (Lacampagne et al. 1995; Schoenfeld 1995) proposed that algebra ought to pervade the curriculum instead of appearing in isolated courses in middle or high school.

At the turn of the century (see Kieran et al. 2016, for the historical background), the U.S. National Council of Teachers of Mathematics (2000) recommended that algebra be treated as a major strand interwoven throughout K–12 mathematics:

> Algebra is best learned as a set of concepts and techniques tied to the representation of quantitative relations and as a style of mathematical thinking for formalizing patterns, functions, and generalizations. Although many adults think that algebra is an area of mathematics more suited to middle school or high school students, *even young children can be encouraged to use algebraic reasoning as they study numbers and operations and as they investigate patterns and relations among sets of numbers*. In the Algebra Standard, the connections of algebra to number and everyday situations are extended in the later grade bands to include geometric ideas (NCTM 2000, p. 3, emphasis added).

This broadened concept of algebra to include certain ways of thinking, still a novel idea for mainstream American mathematics education, has the virtue of facilitating the discovery of connections among a variety of topics across the K–12 mathematics curriculum.

Kaput (2008) proposed that algebra and algebraic reasoning be thought of as being comprised of three strands:

1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics (p. 11).

Kaput's strands provide a useful point of departure for comparing and contrasting diverse broad approaches to early algebra (see reviews by Carraher and Schliemann 2007, 2010). "Generalized arithmetic" approaches focus on mathematical structures and the properties of number systems (e.g., Bastable and Schifter 2008; Carpenter et al. 2003). Other approaches (e.g., Davydov 1991; Dougherty 2008; Schmittau 2005) emphasize the use of mathematics to model and describe relations among quantities. Kieran et al. (2016) provide an overview of these approaches as well as somewhat similar approaches being pursued in Korea, China, and Singapore. Still other approaches (e.g., Blanton 2008; Carraher and Schliemann 2016; Moss and Beatty 2006) employ functions, first encountered in the very operations of arithmetic, as a means of integrating and deepening the study of myriad topics of the curriculum.

The three broad approaches to early algebra are really families of approaches. Ultimately any approach needs to confront many issues that might be addressed in diverse ways, with varying degrees of success. Take the case of functions, for example. How are functions to be integrated into specific topics of the curriculum so as to make sense mathematically, pedagogically, and in terms of learning and development? In mathematics, it may be legitimate to treat a function as a subset of the Cartesian product of two sets. But should students or teachers be introduced to functions in this manner, given that functions of greatest interest in K–12 mathematics are amenable to definition through a simple rule? What sorts of activities are particularly promising for introducing variables? What about other topics: fractions, decimals, etc.? What changes should be introduced into teacher education, both pre- and in-service, with the aim of promoting early algebraic thinking?

5.3 Functions and Algebraic Reasoning

We believe that a functions-based approach to algebraic reasoning can address many of the issues highlighted by both generalized arithmetic and modeling approaches.

The term "function" typically makes its first appearance in the eighth-grade curriculum, when students are 13 or 14 years old (see, e.g., The Common Core State Standards Initiative 2010, which mentions functions for the first time among the content standards for grade 8). There is a considerably broader view of functions, one that we subscribe to (Carraher et al. 2000, 2005), according to which the very operations of arithmetic, addition, subtraction, multiplication, and division, as well as a wide range of advanced concepts, are viewed as functions:

> One of the most basic activities of mathematics is to take a mathematical object and transform it into another one, sometimes of the same kind, and sometimes not. "The square root of" transforms numbers into numbers, as do "four plus," "two times," "the cosine of," and "the logarithm of." A nonnumerical example is "the center of gravity of," which transforms geometrical shapes…into points—meaning that if S stands for a shape, then "the center of gravity of S" stands for a point. A function is, roughly speaking, a mathematical transformation of this kind (Gowers et al. 2008, p. 10).

If "four plus" and "two times" can be regarded as natural language variants of the functions better known as $x + 4$ and $2x$, it is but a short step to realizing that all arithmetical operations and combinations of operations may be treated as functions.

We identify the onset of algebraic reasoning not with the solving of equations but, instead, with the formulation of and operation upon relations, particularly functional relations. Functions offer ample opportunities for bringing out the algebraic nature of arithmetic because: (a) the operations of arithmetic are themselves functions; (b) the concepts of domain and range (or co-domain), central to the definition of functions, support the introduction of variables as placeholders for arbitrary members of sets and the extension of the classes of number; (c) functions are amenable to multiple forms of representation (notably, written notation, graphs, tables, and formulations in natural language) that can be profitably employed in unison; and (d) equations and inequalities are naturally interpreted as the comparison of two functions.

Although functions implicitly flow throughout much of current early mathematics curricula, they are underexploited as resources for teaching and learning. As a simple example, let us imagine that a student solves a word problem by multiplying 9 by 3 and then subtracting 5, obtaining 22 as the answer. Both student and teacher might be inclined, rightly so, to view the response as a specific computation, $9 \times 3 - 5 = 22$, focusing their attention on the particular numbers at hand. However, with a bit of effort, the problem could be recontextualized as an instance of the function, $f(x) = 9 \times 3x - 5$, thereby representing an association between a *set* of possible inputs (the domain) and a *set* of possible outputs (the range or co-domain).

So, the introduction of functions in the early mathematics curriculum requires widening the focus of mathematical problem solving beyond operations involving fixed numbers (constants) to include variables and domains of numbers. This is no mean task, for it demands that problems be posed in ways that encourage students to seek out generalizations about numerical relations, be those relations axiomatic for a class of numbers (e.g., the field axioms) or associated with the constraints of a particular problem context.

Functions involving quantities pose challenges of their own, a quantity being a "property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference (BIPM et al. 2012, p. 2)." Quantities also include "counts" (number of entities) and dimensionless values such as the quotient of two lengths.

Although some quantities, for example, length, area, volume and speed, are standard objects of mathematics, developmental and mathematics education studies have shown that students' understanding of such concepts and their interrelations progresses according to characteristic paths and constraints (Piaget and Inhelder 1974), even with the benefits of instruction (e.g., Lehrer 2003). This is no less true of other quantities—weight, time, speed, unit-price, density, and so forth—that routinely appear in word problems (Liu et al. 2017; Smith et al. 1992). Whenever a student attempts to operate on quantities, beginning with the simple arithmetical operations of addition, subtraction, multiplication, and division, she is engaged in modeling (Greer 1997). Young students may find it challenging to determine what operations correspond to certain relations among quantities. For example, a student may initially find it odd to realize that the product of two different kinds of quantities (e.g., speed $\times$ time) yields a third kind of quantity (distance) (Schwartz 1988, 1996). No such transformations of kind occur for the multiplication of numbers. There are well-documented conceptual differences between operations on numbers and operations on quantities for the cases of addition, subtraction (Vergnaud 1982), and multiplication and division (Vergnaud 1983, 1988). Although one hopes that students will someday treat certain operations on quantities (joining amounts, offsetting one quantity value by another, comparing two values) as having clear counterparts in the addition and subtraction of numbers, at the onset of schooling students will often find it difficult to identify the operations on numbers that correspond to operations and actions involving quantities.

Shortly we will turn our attention to how mathematics problems may be framed so as to encourage generalizations and to address issues tied to relations among quantities in our own work. It is important to note that a growing number of studies are now available for similar analyses (Blanton et al. 2017, and this volume; Brizuela 2016; Brizuela et al. (2015); Ellis 2011; Moss and McNab 2011).

5.4 Algebraic Reasoning in the Classroom

We next describe students' classroom discussions and written work from a selection of lessons taken from three longitudinal studies we undertook with students from grades 3 to 5 (details of lessons and analyses are available at http://ase.tufts.edu/education/earlyalgebra/about.asp). The activities were designed to introduce (a) variables, (b) the number line as a resource for representing additive relations among quantities, (c) linear functions, (d) equations as a comparison between functions, (e) graphs in the plane, and (f) solving equations with variables on each side of the equal sign.

5.4.1 Variables in Relations

Students sometimes interpret a variable (typically, a letter in an algebraic expression) as representing a secret or mystery number, that is, a single value. This interpretation often happens to be valid for problems involving equations. However, to cultivate the idea that a letter can act as a placeholder for an arbitrary number or quantity value in a domain, it may be useful to engage students early on with problems involving unconstrained variables. When unconstrained, the ordered pair, $(n, f(n))$, would then represent *all* the ordered pairs in a function. This is the principal reason for considering relations, including functions, before introducing linear equations.

The Candy Boxes Problem in Fig. 5.1 (see Carraher et al. 2008b for a more detailed analysis) showed us that, under suitably supportive conditions, children as young as 8 years of age can acquire the idea of a variable as a placeholder for an arbitrary element of a set of numbers or quantity values. They can also express an additive relation over two sets. Such welcome advances tend not to occur spontaneously. They generally require scaffolding and discussion management by the teacher but, in our view, they are essential to students' future work on algebra.

On a first reading, the story appears to be about John and Mary and two fixed amounts. Not surprisingly, most students at first interpret the task as requiring them to guess how many candies there are in the boxes. If actual boxes of candy are used for the task, students heft or shake the boxes while listening carefully for any telltale sounds (students are sometimes miffed when they learn that the boxes are padded to prevent rattling). Some may question whether the instructor truly put equal numbers of candy in each box.

Students are asked to state, or to draw on paper, something that represents the number of candies for the two children. Eight- and nine-year-old students are inclined to assign specific values to each amount. For example, a student may make a drawing depicting a box with 7 candies to represent John's amount, and another box with 7 candies on which 3 additional candies rest, for Mary's amount. Other students will assign different numbers for John and for Mary.

John and Mary each have a box of candies.
The two boxes have exactly the same number of candies in them.
Mary has three extra candies on top of her box.
Draw or write something that shows how many candies John and Mary have.

Fig. 5.1 The Candy Boxes Problem

Occasionally, a young student will assign values that are inconsistent with the given information—for example, 9, 11. It is important that all children come to realize that, even though one may not know how many candies John has (he may have 9) nor how many candies Mary has (she may have 11), if we are to accept the problem description as accurate, it cannot be the case the John has 9 while, at the same time, Mary has 11. In trying to justify why such a case is invalid, students inevitably come to express some rule that has been violated, a rule such as "Mary has to have 3 more than John." After taking note of the range of predictions their classmates have made and eliminating invalid pairs, their attention gradually shifts to the valid pairs. In one classroom, once the invalid pairs were eliminated, a student expresses this shift in thinking: "Everybody had the right answer because everybody… has three more [for Mary]. Always."

As the discussion proceeds, some students may suggest a large number (e.g., one billion) for the amount in the box and increment that number by 3 for Mary's amount, as if to test the bounds of acceptable values. (What should the instructor adopt as the implicit domain? Should the instructor require that the suggested value for John's amount actually fit in the box? Or should any answer consistent with the problem wording be considered valid?) Others may draw question marks on each of two boxes and draw an additional three candies atop Mary's box.

So, gradually, with the help of the teacher, the focus shifts toward all ordered pairs consistent with the wording of the problem. That is, the teacher attempts to help students recognize the set of possibilities that are consistent with what we may describe as the relation, $(n, n + 3)$. Sooner or later, opportunities arise for discussing variables as placeholders for arbitrary values (although certainly not in these terms).

At some point, it becomes expedient to introduce a letter for representing unknown amounts. Although students may readily accept this suggestion, some first speculate that the letter the instructor suggests, n, likely stands for "nineteen" or "ninety" or some other number beginning with "n"). Others may suggest that n stands for 14 (n is the l4th letter in the alphabet). The instructor may continue the discussion and suggest, instead, that they adopt the convention that n (*or some arbitrary letter*) stands for "any number." Most students readily accept the suggestion. Even so, other issues present themselves. Some students may suggest that both John's and Mary's amounts be designated by the letter, n. It may still require a bit of discussion for students to realize that John's and Mary's amounts should not be represented by the same letter unless it were the case that they have the same amounts. Other students may propose that j represents John's amount and *m* represents Mary's amount. Although this is not incorrect, it does not capture the interdependence of the amounts.

The expressions, *j* and *j* + 3, *do* convey the interdependence of the amounts, but some students may find it unnatural to employ the letter *j* (which evokes "John") in Mary's amount, *j* + 3. In such a case, it is important to stress that *j* stands, not for John, but rather for the amount John happens to have.

A few more questions suggest some additional directions one might take after a first lesson on candy boxes.

- What should we use to represent the total amount of candies of John's and Mary's totals together? Is it possible that the total number of candies is 17? How about 16?
- If someone told us that there were 21 candies in the box, how many candies would each child have? What if there were $k-1$ candies?

With the next classroom example, we explore some of these additional issues about the use of variables for expressing additive relations, albeit in a different problem context.

5.4.2 Additive Relations in Diagrams and on the Number Line

The Heights Problem, described in Fig. 5.2, involves a simple additive (ternary) relation regarding the relative heights of three children. Two comparisons are given: the difference between Tom's and Maria's heights and the difference between Maria's and Leslie's heights. The question is to determine the difference in Leslie's and Tom's heights. Comparison problems are known to be more challenging for young children than "transformation" problems (see Vergnaud 1982, on additive structures). In earlier publications about this lesson (see Carraher et al. 2006, 2016), we focused on progress the pupils made in describing the general relation. Here we would like to call attention to some of the scaffolding provided by the instructor in helping the lesson move along profitably.

On one occasion, when we asked third graders to show the difference in heights of two students standing side-by-side in front of the classroom, a student tapped the top of the head of the shorter child and moved her hand horizontally to point to the shoulder of the taller one. When questioned about where the difference started and where it ended, she touched, simultaneously, the top of the shoulder and the top of the head of the taller child. After discussing the problem's statements, the instructor encouraged the students to focus on the differences between the protagonists' heights as lengths to represent the problem on individual worksheets.

Most of the students used drawings, vertical bars, or lines to show the three heights. Some of them assigned a height to Maria; others, like in the example of

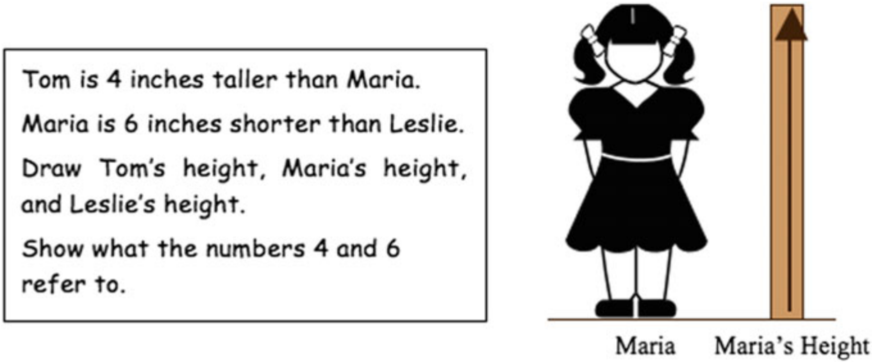


Fig. 5.2 The Heights Problem: Students were asked to show on paper the relative heights of the three children (reprinted with permission from the *Journal for Research in Mathematics Education*, copyright 2006, by the National Council of Teachers of Mathematics; all rights reserved)

Fig. 5.3 A student's representation of relative heights based on the premises that "Tom is 4 in. taller than Maria, and Maria is 6 in. shorter than Leslie"

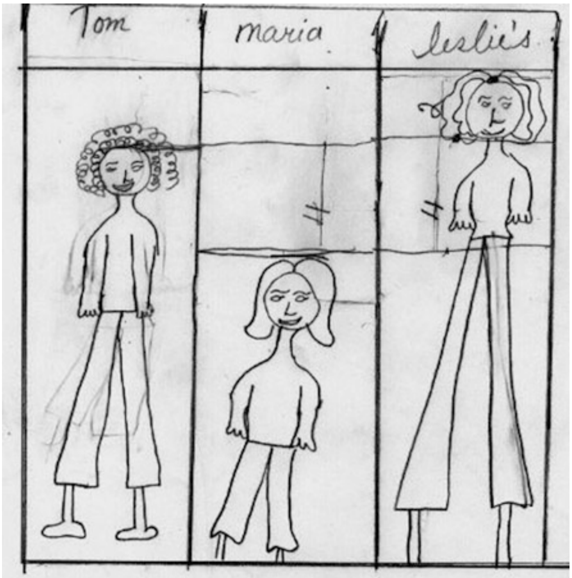


Fig. 5.3, merely indicated the differences between Maria's and Tom's heights and/ or those between Maria's and Leslie's and/or between Tom's and Leslie's.

On another occasion, a grade 4 class of students encountered the problem for the first time after they were already familiar with rudimentary letter-symbol usage. One of the students chose to represent the heights on an "N-number line" (see Fig. 5.4), similar to one they had been working with during previous lessons. On this line, the origin was labeled N. To the right were N + 1, N + 2, N + 3, and so forth; to the left were N − 1, N − 2, etc.

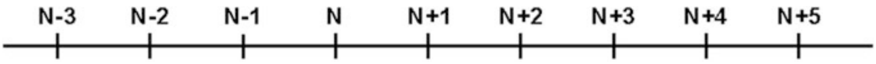


Fig. 5.4 The N-Number Line

In her representation of the Heights Problem (bottom of Fig. 5.5), she had placed Maria's name at N, Tom's name at N + 4, and Leslie's at N + 6. The instructor adopted her number line as a basis for a class discussion on the relative heights.

It is intriguing that this student realized that the N-Number Line would help to clarify the problem at hand. The other students voiced acceptance of her idea and represented Maria's, Tom's, and Leslie's heights at N, N + 4, and N + 6, respectively (middle line in Fig. 5.6). To explore whether the students realized that the placement of the first protagonist was arbitrary, the instructor asked the students whether Leslie's height might have been N. They were able to determine that Tom's height would then be $N - 2$ and Maria's would be $N - 6$ (bottom number line in Fig. 5.6). When the instructor proposed that Tom be placed at the value N, the students decided, after some discussion, to place Leslie at N + 2 and Maria at N − 4 (top number line in Fig. 5.6). When asked which of the three diagrams was correct, the students agreed that it did not matter; it all depended on the person whose height would be assigned the value N. It is evident that the number lines and algebraic notation being employed to help students structure the problem helped them articulate their thinking about the relations among the three heights.

5.4.3 Linear Functions in Tables and Algebraic Notation

Diagrams provide a useful way to examine the functions underlying figurate numbers (Weisstein 2017). Likewise, diagrams can be used to help students visualize and systematically explore discrete change in a sequence of geometric figures. They are often used in early grades for students to extend patterns. For example, triangular numbers (1, 3, 6, 10, 15, 21, …) can be conveniently modeled with diagrams of dots arranged in the shape of a right triangle, in which each successive triangle is formed by adding a new row of dots to the base of the triangle such that the new base has one more dot than the predecessor's base. It happens that triangular numbers correspond to the function, $f(n) = n(n + 1)/2$, where n refers to the step number.[1] Although young students can easily extend the sequence of figures, identifying the function is considerably more demanding.

[1]As others have noted (Moss and Beatty 2006), if one intends to eventually arrive at a closed form expression of the function, as opposed to a sequential listing, it is important to systematically keep track of the step number as well as the corresponding value of the function; this is easily achieved by using a table with a column for step numbers and another for the corresponding values of the function.

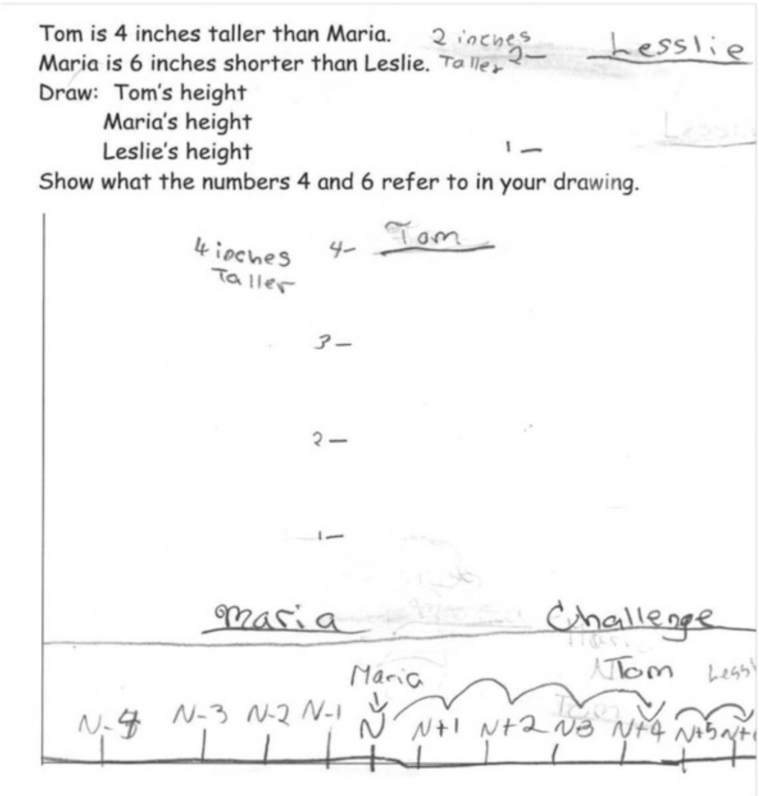


Fig. 5.5 A student's vertical drawing (vertical arrangement of numbered notches) showing that Tom was 4 in. taller than Maria (at the baseline) and Lesslie [sic] was 2 in. taller than Tom; she expresses the same relation in the lower part of her drawing using the N-Number Line (reprinted with permission from the *Journal for Research in Mathematics Education*, copyright 2006, by the National Council of Teachers of Mathematics; all rights reserved)

Fortunately, there are variants of figurate number problems that correspond to much simpler functions. In our studies, square-shaped dinner tables, successively adjoined end to end, provided a useful context for exploring the linear function, $g(n) = 2n + 2$, where n refers to the number of tables and $g(n)$ refers to the maximum seating, assuming that one person can sit at each free individual table edge in the configuration.

In the second half of third grade, students in one of the studies worked on the "Dinner Tables Problem" (see detailed analysis by Carraher et al. 2008a). Students first discussed and represented how many people could sit at the sides of separate square tables, under the constraint that only one person could sit at each side. Next, they were told that, in a restaurant, square tables were adjoined in a straight line, with one person sitting at each free edge of the tables. The instructor started by

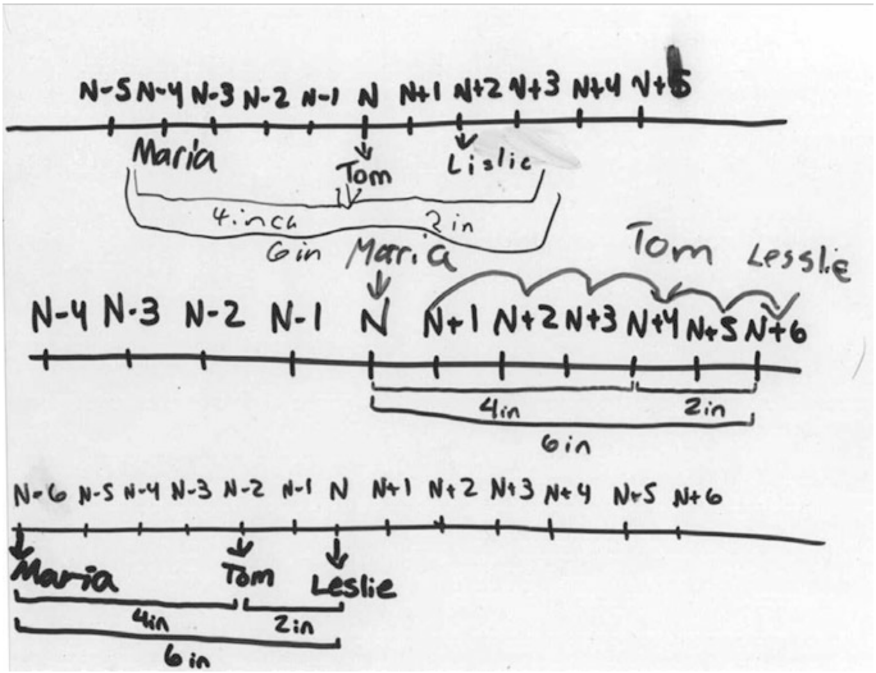


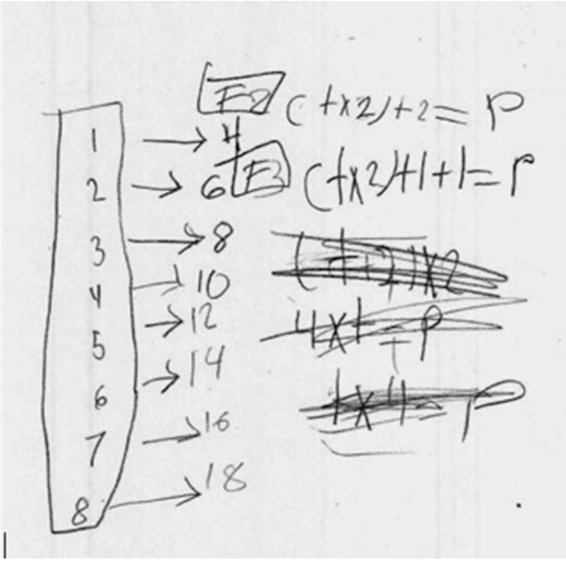
Fig. 5.6 Three "variable number lines" drawn by students and their teacher as they discussed the cases where Maria (middle number line), Leslie (bottom number line), and Tom (the upper number line) are successively assigned a height of N (reprinted with permission from the *Journal for Research in Mathematics Education*, copyright 2006, by the National Council of Teachers of Mathematics; all rights reserved)

drawing two, three, or more tables and, together with the students, counting the number of free edges of the tables and registering the results in a data table.

After exploring the pattern according to which seating capacity increased with the number of tables, the students were asked to find a rule that would allow them to determine how many people could sit at any number of tables, based on that number of tables and presuming that they did not have the time to go through all of the cases from 1 to n tables.

The students found several ways to parse the problem. And these variants were reflected both in the explanations and the formulas they came up with. As the instructor discussed and completed, on the blackboard, a data table emphasizing the relationships between possible number of tables and possible number of people, some students noted that the seating capacity could be represented by $n + n + 2$, reasoning that each n represented the number of people who could sit at each long edge of the line of tables and the 2 represented the people at the far ends of the line of tables. After examining the data in the table, others suggested the expression $2t + 2$ to represent two times the number of tables plus two seats at the ends (see Fig. 5.7). Those students who produced a valid algebraic rule did so only after

Fig. 5.7 A student's representation of the Dinner Tables Problem



reviewing the table of data. On one occasion, a student reasoned that it was as if each table could seat exactly two people, but, in his imagination, he incremented the number of tables by one to account for the two seats at the ends. A few students tried approaching the problem by working from the premise that n unattached tables offers $4n$ seats, from which the number of "blocked" edges would be subtracted; however, in no case did they find a way to succinctly express the number of blocked seats ($2n-2$).

5.4.4 An Equation as a Comparison of Functions

After being introduced to various representational forms of functions, from verbal statements to number lines, tables, algebra notation, and graphs (see details in Carraher et al. 2006, 2016; Schliemann and Carraher 2002, 2016), we found that fourth grade students, already familiar with functions and simplified functional notation, could come to understand and solve a problem as a comparison between two functions expressed in a data table and as two intersecting lines in a coordinate space (see details in Carraher et al. 2008b). Figure 5.8 shows the problem students were given.

Mike has \$8 in his hand and the rest of his money is in his wallet;
Robin has exactly 3 times as much money as Mike has in his wallet.
What can you say about the amounts of money Mike and Robin have?

Fig. 5.8 The Wallet Problem

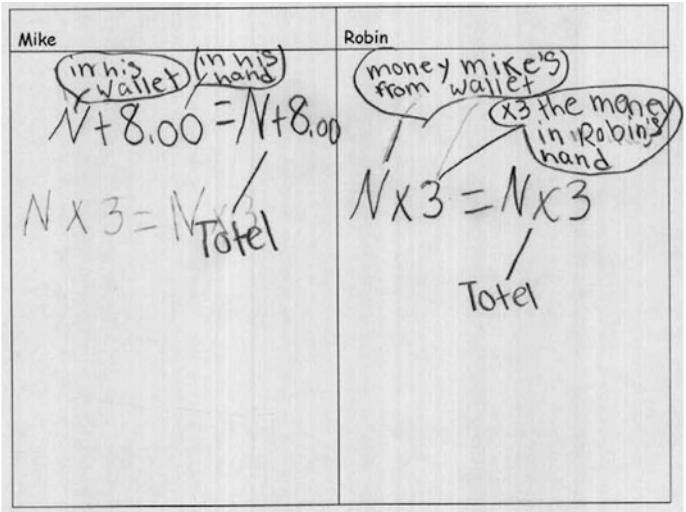


Fig. 5.9 A student's representation of the Wallet Problem (it is unclear why the student felt the need to provide (tautological) equations)

Note that, unlike typical algebra problems, this problem does not assert at the outset that Mike and Robin had the same amount of money. Nor does it ask students to find out how much money there is in the wallet or how much either of the boys had. Instead, students were simply asked to represent the problem, after a whole classroom discussion where they read the problem and were asked to consider who might have more money.

During the discussion, students invariably assumed that one of the two protagonists had more money than the other. Many interpreted one of the statements as meaning that Robin's amount was three times Mike's total amount; accordingly, they claimed that Robin had more money than Mike. Others noted that Mike already "began"[2] with more money, namely the $8 held in his hand. Some of those students claimed that Mike had more money.

Several minutes of whole-class discussion, moderated by the instructor, were needed for the students to realize that Robin only had three-times as much money as Mike had *in his wallet*, not three times Mike's total. The students were then asked to individually represent the situation in writing. Over 60% of them used algebraic notation to capture the functional relationships among the variables (see Fig. 5.9 for one example).

The instructor then suggested filling in a three-column table on the board, one for possible number of dollars in the wallet (w), another for Mike's amount ($w + 8$), and a third one for Robin's amount ($3w$). By examining the data in a table, the

[2]Such remarks reflect a student's proclivity to frame the co-variation among the variables in terms of a story unfolding over time.

students realized that, if there were $4 in the wallet, Mike and Robin would have the same amount of money; for amounts smaller than $4 Mike would have more money; and for amounts larger than $4 Robin would have more.

The instructor then showed the students a line graph in the Cartesian space (for Mike's amounts). The students recognized that the line represented Mike's total, noting that it "started at 8". They predicted that Robin's line would start at 0 and would be steeper than Mike's. They also recognized that the point where the two lines cross indicated that, when there was $4 in the wallet, Mike and Robin had the same amount. They also noted that Mike's total increased by $1 while Robin's increased by $3, for each additional dollar in the wallet.

For these students, w, the amount of money in the wallet, was a variable, not a single value. The two functions were equal for the case when there was $4 in the wallet. For all other values, one boy would have more money than the other.

These observations suggest to us that fairly young students can learn to understand and represent variables as placeholders for sets (rather than simply a single missing value). Furthermore, students appear able to grasp that algebraic expressions, $w + 8$ and $3w$, may represent functions that can be visually depicted and compared on the Cartesian plane. They also appear able to find solutions as well as conditions of inequality corresponding to the expressions, $3 \times w < w + 8$ and $3 \times w > w + 8$. This corresponds to the idea that the graphs represent both an equation and two inequalities. The students did not work with an equation in the traditional sense as a string of characters. We are using *equation* in the non-technical sense of a situation in which one amount is equal to another amount.

5.4.5 Solving an Equation-Like Word Problem Using Algebra

In one of the lessons (see details in Brizuela and Schliemann 2004), taught to fourth graders (9- and 10-year-olds), the instructor presented to the classroom known amounts of candies in transparent bags and unknown amounts of candies in two types of opaque containers, tubes and boxes. Bags, tubes, and boxes were displayed on two desks, in front of the classroom, so all students could see what each girl in the problem would have. Figure 5.10 shows how the situation was described.

Two students have the same amount of candies.

Briana has one box, two tubes, and seven loose candies.

Susan has one box, one tube, and 20 loose candies.

If each box has the same amount and each tube has the same amount, can you figure out how much each tube holds? What about each box?

Fig. 5.10 Description of the Tubes and Boxes problem situation

The situation corresponds to the equation $2x + y + 7 = x + y + 20$, although it is important to note that the students were not initially given the equation. Instead, they were shown a situation in which on one side of the desk there was Briana's candy, consisting of the content of two "tubes" (cylindrical containers), one closed box, and 7 "loose" candies in a transparent bag. On the other side of the desk there was Susan's candy, which consisted of the contents of one tube, one closed box, and 20 loose candies. The students were told that each tube contained the same number of candies, each box had the same number of candies (though not necessarily the same number as in a tube) and that Briana and Susan had the same total number of candies. The goal was to have the students determine how many candies were in each tube.

Early on, one of the students suggests that the tube has 13 candies and explains that, if there are 13 candies in Susan's tube, together with the bag, there would be the same amount of candies as on Briana's desk. Another one volunteers that the amount in the boxes does not matter. He explains that if Briana's tube has thirteen, then the 7 loose candies would make it 20, plus 13 more for her second tube, which would be 33. He explained that Susan had 13 in her tube plus the 20 loose candies, which is 33. He says that because of this, the box, no matter what number it is, will still make the two girls' total amounts equal. As another student asks the first student to, once more, explain his view, he further elaborates: "…in each tube it's thirteen chocolates and put together with the chocolates in the bag equal twenty. And then one tube with one tube will equal the same thing." Still another student jumps into the discussion and explains that she thinks that what Albert is "trying to say is that the candies that are in Susan's bag, are like sort of making up for another tube that Briana would have." She is referring to Susan's *additional* candies (20 − 7 = 13). This difference, favoring Susan, would need to be compensated by the candies in Briana's additional tube.

After the discussion, the students were asked to represent the situation and the solution to the problem in writing. Figure 5.11 illustrates the work of one of the students. She arranges drawings and algebraic notation to represent the situation in a two-column table; she cancels out one tube and one box from each side of the table, and represents what is left on one side (20 candies) as 7 + 13 and what is left on the other side as N + 7; and she cancels out 7 from each side to show that there are 13 candies in a tube.

In another example (Fig. 5.12), the student writes the equation 20 + N + J = 7 + 2N + J, and solves it explaining: "I broke 20 into 7 and 13. Then matched 7 and 7. Then broke 2N into N and N and matched them. Then 13 equals N."

A student demonstrated a similar solution on the board (similar to that shown in Fig. 5.12) and suggested that they "trash the boxes" (eliminate the boxes from consideration).

So, it appears that fourth grade students, after participating in our classes since third grade, could learn to meaningfully represent equations with variables on each side of the equals sign by acting on symbolic representations of the equation and enacting equal transformations on each side of the equation (Brizuela and

Fig. 5.11 A student's solution with drawings and algebra notation

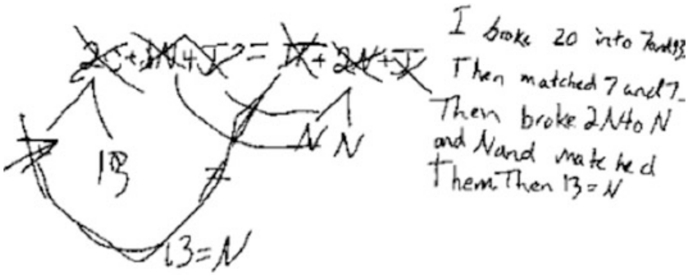


Fig. 5.12 A student's equation and solution to the problem

Schliemann 2004; Schliemann et al. 2013). This is perhaps surprising, in light of previous research results showing that older students (from 12 to 13 years of age) often have difficulties operating on unknowns, understanding that equivalent transformations on both sides of an equation do not alter its truth value, and solving equations where variables appear on both sides of the equal sign.

To be fair, it may well be the case that the first student's quick reasoning made the solution available to students who may not have been able to solve the problem on their own. Although this is quite possibly true, it also seems apparent that there are ways to frame problems in which the relations among quantities are meaningfully represented before algebraic notation is brought onto the scene.

5.5 Evaluating Student Learning

The classroom discussions and students' written productions across the many lessons they participated in along the years of each project show that algebraic reasoning and representations are within reach of many eight- to eleven-year-olds. The reader may, however, question whether this was the case for most of the students or just for a few of them. To address this question, as we describe next, we conducted individual interviews and collected data on written assessments by students who had participated in our lessons and by students in control groups of the same schools.

5.5.1 Interviews at the End of Fourth Grade

In the second of our studies, three to four weeks after the last early algebra class in grade 4, we evaluated students' progress in the intervention by individually interviewing them. Their responses were compared to the interview responses of a control group of 26 fifth graders from the same school. The following are some of our main findings (see Schliemann et al. 2003):

(a) 85% of the 4th graders in the intervention correctly stated that the equation 6 + 9 = 7 + 8 was true while 65% of the grader 5 control students did so.
(b) To represent that Mary had three times as much money as John, 70% of the intervention students represented John's amount as N and Mary's as N × 3. Among the controls only 29% of the children used variable notation.
(c) When asked which of three line graphs showed that "Mary has three times as much money as John", 78% of the students in the intervention group chose the correct line and 39% provided general justifications that took into account any possible pair of numbers. In the control group, only 46% of the students chose the correct line and only 25% justified their answers by considering any possible pair of numbers.
(d) The students were asked to represent in writing and to solve the following problem:

> Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?

To solve the problem, 56% of the students in the intervention group represented Harold's initial amount as a letter such as N, 49% represented Sally's amount as N × 4, 35% wrote N + 18 for Harold's amount after earning 18 more dollars, 17% wrote the full equation N + 18 = N × 4, and 27% correctly solved the problem. However, only 6% (four students) systematically used an algebraic method to simplify the equation. Among the controls, 23% of the students solved the problem, but not a single one used algebraic notation or equations to find the solution.

5.5.2 *Written Assessments at the End of Fifth Grade and Beyond*

At the end of the third three-year classroom intervention, in which 26 eight- to eleven-year-old children (third to fifth graders) participated in weekly early algebra lessons, we found (see Schliemann et al. 2012) that students' performance on a written assessment was significantly better than that of a control group for items related to the early algebra intervention, with the two groups performing equally well on items not related to the intervention lessons (Fig. 5.13).

In the following year, the school's regular teachers taught the third and fourth grade lessons to a new cohort of 24 students. Over the three years after the end of our intervention we followed up 20 students from the two cohorts. When they were at the end of seventh and eighth grade, these students were given a written assessment on questions related to the early algebra intervention as well as more advanced problems, typical of the middle and high school algebra curriculum in their schools. These problems were designed by the research team or taken from the NAEP (*National Assessment of Educational Progress*), MCAS (*Massachusetts Comprehensive Assessment System*), and TIMSS (*Trends in International Mathematics and Science Study*) assessments for middle and high school. Fifteen of the items were on solving linear equations, solving verbal problems by representing them as equations, solving the equations, and interpreting the results of their solutions. Five items included the graphical representation of non-linear functions in the four quadrants of the Cartesian space. The assessment was also given to 19 students from the same geographic area and grade levels.

Figure 5.14 compares the results of seventh and eighth grade control (n = 19) and experimental (n = 20) students for *Equations* and *Graphs* items, two years

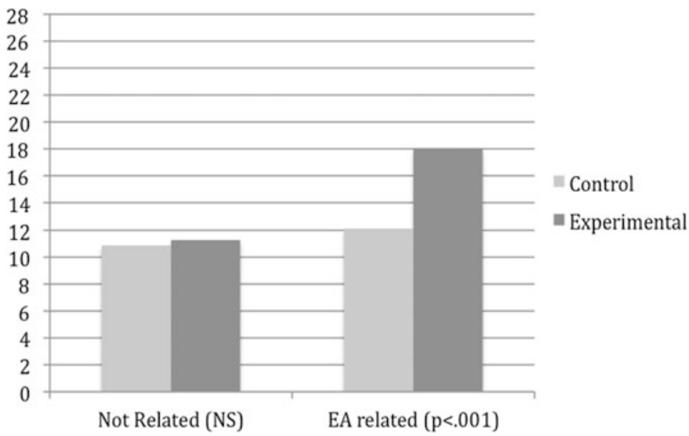


Fig. 5.13 Average number of correct answers by groups at the end of the intervention (fifth grade) for 28 items related and 22 not related to the intervention

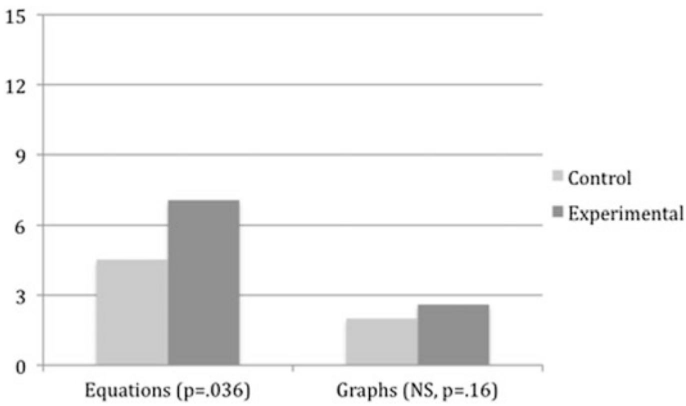


Fig. 5.14 Average number of correct answers by groups, two to three years after the intervention (i.e., at seventh and eighth grade), on 15 Equation items and 5 Graphs items

(seventh graders) and three years (eighth graders) after the end of the intervention. The treatment group performed better than the control group on both types of problems; the difference between the two groups was significant for *Equations*, but not for items on *Graphs*.

In addition, three years after the experimental group of students had concluded fourth and fifth grades and had completed seventh and eighth grades (they were then 12–14 years of age), our research group implemented a free one-week algebra Summer Camp. The camp was attended by six of the experimental group students and by 19 control students in the same grades. Assessment data were obtained for these students before and after the Summer Camp. Figures 5.15 and 5.16 show the two groups' results at the start and at the end of the summer camp, for equation and for graph items respectively (see Schliemann et al. 2012).

The early algebra students performed better than the control group on both sets of assessment items, before and after Summer Camp, and the difference between the two groups increased after participation in camp lessons. Differences were significant before and after camp for *Equations*. For *Graphs*, the difference between groups was not significant before, but became significant after the camp lessons. These long-term results, although preliminary and based on relatively few children, nonetheless suggest that students who had participated in the early algebra intervention were better able than their control peers to benefit from the camp lessons on topics of algebra that are part of the middle and high school curriculum.

5.5.3 In Summary

Our results, as well as those by Blanton et al. (2017, also this volume), who systematically documented the substantial and significant progress made by

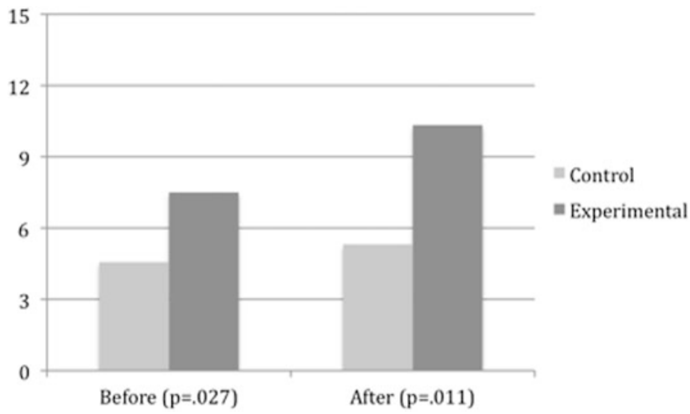


Fig. 5.15 Average number of correct answers by groups on 15 Equation items, before and after Summer Camp

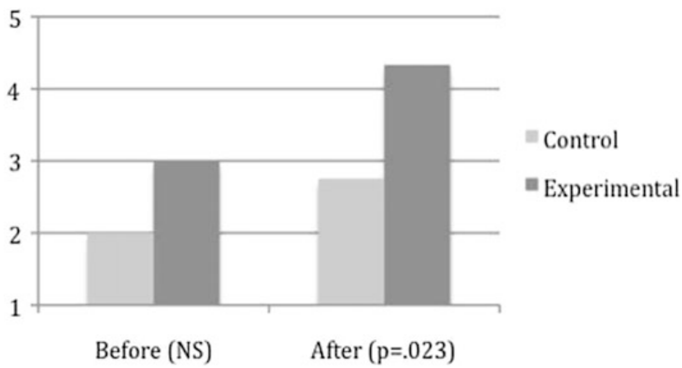


Fig. 5.16 Average number of correct answers by groups on 5 Graph items, before and after Summer Camp

students who participated in an early algebra intervention that also places strong emphasis on variables and functions from the early grades, support calls for changes in the curriculum. A question that frequently arises, however, regards the ability of teachers to implement such views in the teaching of the curriculum they are asked to follow. Next, we describe the results of a program that built upon our approach to algebra and functions in the early grades and aimed at preparing teachers in higher grades.

5.6 Preparing Teachers to Cultivate Algebraic Thinking

Several years ago, we developed, in close collaboration with mathematicians and physicists, and with input from target school districts, an intensive, three-semester long program of in-service teacher education constructed around the idea that functions and early algebraic thinking can play a central role in integrating otherwise isolated topics in the early mathematics curriculum, while providing ways to explore topics in the current mathematics curriculum more deeply. The interested reader will find a detailed description of the conception and structure of the program, known as The Poincaré Institute (https://sites.tufts.edu/poincare/), in Teixidor-i-Bigas et al. (2013) and Schliemann et al. (2017). In the three graduate-level courses offered by the program through the Tufts University departments of Mathematics and Education, teachers analyzed and implemented ideas and examples of classroom activities based on our previous classroom research and grounded in the mathematics of functions.

5.6.1 The Teacher Development Program

The program has offered, since 2011, three online and face-to face graduate level courses to four cohorts of 60 teachers each, from ten school districts in the Northeastern USA (six in Massachusetts, three in New Hampshire, and one in Maine). Teachers meet once a semester at the university campus and weekly in their schools. Instructors participate in online discussions, provide online feedback, and visit school districts once a month.

Teachers read and discuss written notes on mathematical content and solve and discuss math challenges in online groups of eight to ten teachers. Working in small groups of three to five teachers in their schools, they interview students about selected topics to understand their ways of thinking and plan, implement, evaluate, and improve classroom activities. They also analyze examples of videotaped lessons from our studies on algebraic reasoning in the early grades, in terms of mathematical concepts and representations, teaching strategies, and students' ideas and achievements. They discuss how course topics relate to the lessons they teach and consider how to modify prescribed curriculum activities based upon what they are learning in the program. Overall, teachers work an average of 10 h per week in course assignments.

The written notes, challenge questions, and course activities focus on the role of functions as a unifying concept throughout the K–12 curriculum, from the standpoint of mathematics and of mathematics teaching and learning. The courses emphasize multiple representations, including verbalizations, geometric tools (in particular, number lines and Cartesian graphs), function tables, and arithmetical-algebraic notation, as representations of the same abstract mathematical objects, to deepen the understanding of arithmetic operations, fractions, ratio, proportion, and the syntax of arithmetic and algebra. Teachers deal with equations and inequalities as comparisons between functions, solve equations using transformations of the plane, and are introduced to notions of change and invariance in terms of the behavior of functions.

The first course builds on the idea of numbers from representations of quantities to more abstract conceptions of numbers as mathematical objects. Teachers work on fractions and decimals, rational and irrational numbers, and the many ways numbers can be represented, for example, as points on the real line or an oriented segment. They then consider arithmetic operations as functions of a single or of two variables. The second course deals mostly with transformations of the line and of the plane as a way to understand functions and the solution of equations as the comparison between two functions. The third course compares linear to non-linear functions and takes a closer look at non-linear functions and rates of change. Questions regarding divisibility appear in the study of numbers, in the solution of Diophantine equations, and in the factorization of polynomials for solving polynomial equations.

The program devotes less time to the mechanics of algorithms and far more to understanding why these algorithms work and how mathematical concepts are interconnected. There is an emphasis on historic and typical middle school students' conceptions about the topics, on how mathematics is used to model science and real-life situations, and on how different representations can engage different types of learners and emphasize different properties of each model or tool.

5.6.2 The Impact of the Teacher Development Program on Teaching and Learning

We have encouraging evidence that the teacher development program contributes significantly to mathematics teaching and learning. Observations of classroom activities of 48 teachers in the first cohort attending the program by external evaluators (The Intercultural Center for Research in Education—INCRE), using the Reformed Teaching Observation Protocol (RTOP) measurements, found that, by the end of the program, teachers were increasingly (1) using and encouraging their students to use multiple representations (e.g., number lines, tables, graphs, and algebraic notation), (2) spending more time addressing their students' mathematical reasoning, and (3) analyzing mathematics problems from various perspectives. Changes in teachers' practice were even greater six months after this cohort of teachers had completed the Poincaré program, as found in additional in-class observations of 28 teachers. By then, the teachers' average RTOP score was nearly a full standard deviation above the average at the beginning of the program. These improvements were maintained one year later (INCRE 2014).

Course 1 was offered to the first cohort of teachers in the spring of 2011. At that time, student proficiency in mathematics (percentage of students at the Proficient and Advanced levels), according to the Massachusetts Comprehensive Assessment System test (MCAS) results, was below the levels for the state of Massachusetts, but very close to that of carefully matched districts. Three years later, the situation had significantly changed. Students in participating districts had narrowed the performance gap with regard to the state of Massachusetts. Furthermore, they had significantly outperformed the matched districts in terms of general mathematical proficiency. From the start to the end of teachers' participation in the program, changes for students in participating districts were significantly higher than those in matched districts, which showed practically no improvement. There was also a powerful "dosage" effect: districts' performance advantage over matched comparison districts increased in direct proportion to the percentage of teacher graduates in the district (Spearman's $r = 0.54$, $p = 0.007$). This provides additional evidence that performance differences between participating and matched districts were due to the program (see Schliemann et al. 2017; and short report available at https://sites.tufts.edu/poincare/research-and-impact/).

We are still in the early stages of understanding the mechanisms underlying the success of the Poincaré Institute in helping teachers gain expertise in advancing the mathematical proficiency of students. As the research unfolds, we hope to obtain a closer look at how functions can help provide teachers and students with tools for exploring mathematical ideas more deeply.

5.7 Closing Thoughts

We do not expect there to be universal agreement about the definition of "early algebraic thinking" because the idea entails theoretical issues (e.g., "What mental processes underlie students' understanding of functions and how do they develop?"), as well as value premises (e.g., "What are the most important sorts of mathematical competence to nurture among young students?"; "What features of algebra are most appropriate for young learners?"). This is not to say that a definition of early algebraic thinking is entirely arbitrary. Research about how children reason, particularly in special circumstances, where there is an effort for them to develop certain algebra-related competences, can help determine how promising an approach to early algebra may be.

Here we have taken the view that *algebraic thinking* refers to reasoning that expresses itself as statements or other representations denoting relations among sets of elements, typically numbers or quantities. We have found it useful to include, as representations of algebraic thinking, various forms (linguistic, tabular, graphical, diagrammatic, etc.) in addition to algebraic notation because algebraic thinking may be cultivated before algebraic notation is introduced. These alternative forms of algebraic expression are widely accepted as legitimate embodiments of relations (including functions), which, we have argued, play critical roles in algebraic reasoning insofar as they express associations between sets of numbers and quantity values. This inclusion of non-notational expressions of algebraic reasoning does not diminish the importance of algebraic notation itself, which, we have tried to argue, can lend itself favorably to algebraic problem solving among young students. Algebraic notation first offers itself as a means for succinctly expressing relations and generalizations that have been attained largely on the basis of semantics. Over time, algebraic notation plays an increasingly important role in structuring and guiding reasoning itself.

It is true that algebra entails axioms and conventions regarding the formulation and transformation of notational expressions. But, for most young students, these features, or perhaps the spirit with which such axioms and conventions are given, may not offer the best entryway into algebra. There is a vast body of research indicating that children's early mathematical understanding springs from their actions and reflections upon physical quantities. Actions of counting, partitioning, and joining sets of objects provide a major inroad to the set of natural numbers. Measuring and sharing continuous quantities pave the way to the rational numbers (or at least the non-negative rationals). Algebraic structures are indeed important, but one cannot reasonably argue that formal mathematical properties, such as the field axioms, are first accepted as arbitrary axioms. After all, they are confirmed through the child's experience. A young student acknowledges the associative, commutative, and distributive properties of the group of counting numbers for adding and multiplying not because she is willing to adopt them as axioms but rather because these properties actually match her own experience with collections of objects. The student eventually refrains from attributing commutativity to the operations of subtraction and division because the counterparts of these operations in the extra-mathematical world, for example, taking away and sharing, do not behave commutatively.

So, although, the student may someday come to understand algebra as a formal, self-enclosed language, algebra is born and largely raised, like arithmetic, in extra-mathematical situations. Arithmetic and algebra both entail modeling relations among quantities. Each of the tasks discussed here (Candy Boxes, Relative Heights, Dinner Tables, Wallet Problem, Tubes and Boxes) centered around the issue of describing relations among quantities.

Are quantities so different from numbers as to merit special consideration? We would respond with a qualified "yes." From the moment multiplication and division are introduced, students are asked to deal with situations for which they must represent relations involving three different kinds of quantities. For instance, students must make sense of the product of a speed and a time as a distance. This does not have a counterpart in the realm of pure number. Why does a volume divided by an area produce a length? What sort of function can take a volume as input and return a weight as output? If such cases fall within the realm of elementary mathematics (and not solely science), then mathematics, including arithmetic and algebra, entails the study of both number and quantity. This has significant implications for the education of prospective teachers of elementary mathematics.

Even so, numbers can have, and need to have, a life of their own. Students need to be able to operate on numbers without thinking about physical objects or their attributes. And algebraic expressions need not always be about quantities; they can and should represent numbers or even be taken as pure tokens that can be manipulated without a need for interpretation.

In the classroom episodes reviewed, there were some indications that, although the tasks were initially presented as "real-world problems," students increasingly focused, with encouragement from instructors, on features and constraints of the representations themselves. Word problems often require students to recast or reframe the problem in a way that makes it more general and therefore more amenable to expression through representations involving relations among sets of values. The Candy Boxes problem was initially presented as an exploration of specific amounts associated with the contents of boxes brought into the classroom. As the lesson progressed, the problem gradually became one of determining all ordered pairs of natural numbers (possibly within a limited domain of values) that qualified as members of the relation $(n, n + 3)$, the function $n \mapsto n + 3$ or its inverse, $n \mapsto n - 3$.

When a student chose to represent differences in heights as relative distances on the N-Number Line, the problem was transformed from one merely about the heights of three children to that of finding diverse, equivalent functions underlying the relation $(n, n + 4, n + 6)$. The Wallet problem, like the Candy Boxes problem, was interpreted initially as a simple comparison of two amounts of money. Each boy's amount needed to be re-conceptualized, not as a fixed value, but instead as a function of the amount of money in the wallet. Once several of the ordered pairs were found, a graph of each function could be drawn. On the graph, the solution manifested itself as the x-coordinate of the two lines' intersection; that corresponded to an amount of money in the wallet associated with an equal total amount for each boy. All values of x (the amount in the wallet) less than \$4 correspond to the inequality according to which Mike's amount $(x + 8)$ is greater than Robin's amount $(3x)$. All values of x greater than \$4 correspond to the inequality according to which Mike's amount is less than Robin's amount. So, the graphs express the equation, $x + 8 = 3x$, as well as two inequalities.

The Dinner Tables task was presented as a problem of finding a succinct rule according to which the seating capacity increased depending on the number of tables. Tables joined end to end offer seating places according to a "logic" that might be conceptualized in various ways, for example, as "two places at each table plus two end places $(2t + 2)$, or "three places at each end table plus two more for each table in the middle $(3 + 3 + 2(t - 2))$. Successful teachers tend to consider the degree to which an algebraic expression may match or mismatch a student's reasoning about the relations among quantities. They will also help students realize that different conceptualizations, and different algebraic expressions, may be equivalent in the sense that they validly describe the same relation.

The Tubes and Boxes problem corresponded to another comparison of functions, each of which initially seemed to depend on two variables, the number of candies in a box and in a tube. Equations with variables on each side are notoriously challenging for, and sometimes considered beyond the reach of, young students. But in the context of variable quantities in a meaningful situation, the students approached the problem by first eliminating one of the variables (the amount of candies in a box) and then used their intuitions about the operations that might preserve the equivalence and enable them to find a value for the remaining variable (the number of candies in a tube) that would yield the same total for each of two students. The claim that they solved the problem intuitively requires a proviso. The algebraic expressions did not simply register what students had concluded by their own intuitive methods. Once the students had at their disposal the algebraic notation on their individual written work and on the white board, they found it useful to make simplifications by acting directly on the notation. For example, they recommended partitioning the bag of 20 candies into 13 + 7, so as to allow them to remove 7 from each term. By modifying the equation accordingly, they registered a more streamlined representation of the problem. Already at this early stage in their understanding, the notation was helping them express what they knew about how the quantities were composed and to transform the equation into a simpler form.

In our earlier publications about early algebraic thinking, we placed considerable emphasis upon the achievements of the students, so much so that we may have understated the critical roles of the teachers. Although children may be capable of learning algebra from an early age, realizing this potential is not a simple matter of unleashing their capabilities. Algebra draws on ways of reasoning, kinds of problem situations, and systems of representation (notation, graphs, number line diagrams, certain ways of formulating relations in spoken language) that a child will generally not learn about, much less invent, on her own. The mathematics teacher and, to a lesser extent, the student's peers, play a vital role. The skeptic need merely imagine how much students would have learned had they been given written versions of the tasks and instructed to solve them on their own, without further discussion with and guidance from the instructor.

When algebra is approached from the perspective of relations, especially functional relations, the teachers may help improve students' performance in mathematics, as suggested by the results of our teacher development program. Whether this is due to the framing of mathematics, the nature of activities that come to the fore, or to the teachers' enhanced ability to attend to and nurture the thinking of students remains open to discussion. In any case, it would seem to demand long-term preparation of teachers to view mathematics and the teaching of mathematics from fresh perspectives.

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Part II
Learning to Think Algebraically in Primary and Lower Middle School

Chapter 6
Characteristics of Korean Students' Early Algebraic Thinking: A Generalized Arithmetic Perspective

JeongSuk Pang and JeongWon Kim

Abstract This chapter reports two studies that examined the early algebraic thinking of Korean students. Firstly, it deals with students' understanding of the equal sign, expressions, and equations as they progress through elementary school. Secondly, it investigates how third graders respond to diverse assessment items related to early algebraic thinking. The overall results show high percentages of correct answers. Whereas a majority of students showed a tendency to use computation, a detailed analysis of strategies used by students indicated some were capable of employing a structural approach. This chapter closes with discussions of the development of early algebraic thinking through the mathematics curriculum and the relationship between computational proficiency and algebraic thinking.

Keywords Early algebraic thinking · Equal sign · Expression · Equation
Variable

6.1 Introduction

Various studies in early algebra have been conducted on the nature, process, learning, and teaching of algebraic thinking (Kieran et al. 2016). Such studies demonstrate young students' algebraic thinking with the support of well-designed intervention programs promoting early algebraic thinking. This chapter reports two studies that examined the early algebraic thinking of Korean students. As early

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algebraic thinking has not yet been explicitly mentioned in the national mathematics curriculum of Korea, the results of the studies would be expected to reveal both the successes and the difficulties of algebraic thinking development under the current elementary mathematics curriculum. As such, this chapter is expected to contribute in two ways to the monograph: (a) As little has been known in international contexts of the algebraic thinking of Korean students, this chapter adds new and informative data to the field; and (b) by interpreting students' performance in relation to the current mathematics curriculum, this chapter urges intentional interest in improving the current mathematics curriculum to foster early algebraic thinking.

6.2 Background to the Study

6.2.1 A Generalized Arithmetic Perspective on Algebraic Thinking

Building on the identification of three strands of algebra by Kaput (2008), most of the early algebra research adopts a generalized arithmetic perspective with an emphasis on structures and relations arising in arithmetic, and a functional perspective with an emphasis on functions, co-variations, and changes. Note that the term *generalized arithmetic* has been used within the context of early algebra in a broad sense to include the properties and relations arising in arithmetical operations, without necessarily using letter-symbolic notations. As such, a generalized arithmetic perspective on content "not only includes number/quantity, operations, properties, equality, and related representations and diagrams, but also can include variables, expressions, and equations" (Kieran et al. 2016, p. 12).

As arithmetic has been regarded as the main context for early algebraic thinking (Carpenter et al. 2003; Kieran 2014), many studies have been conducted to probe children's understanding of the equal sign, expressions, and equations (Molina and Ambrose 2008; Stephens et al. 2013). It has been well documented that many students regard the equal sign as an operator to perform a calculation or as a signal to write down the answer that comes next (Kieran 1981).

The development of a *relational* understanding of the equal sign, which interprets the equal sign as a symbol to represent an equivalence relation between two expressions rather than as an operator, has been emphasized as fundamental to early algebraic thinking (Blanton et al. 2011; Knuth et al. 2006). Specifically, Matthews et al. (2012) developed a construct map for students' various conceptions of the equal sign in terms of four levels: (a) students at the *rigid operational level* are successful with typical equations having operations on the left side of the equal sign; (b) students at the *flexible operational level* are successful with atypical equations having operations on the right side of the equal sign or no operations; (c) students at the *basic relational level* are successful with equations having operations on both sides and accept a relational definition of the equal sign; and (d) students at the *comparative relational level* compare the expressions on the both sides of the equal sign and consistently generate a relational interpretation of the equal sign. The researchers designed a comprehensive set of tasks to assess students' understanding of the equal sign and ultimately of mathematical equality. The tasks were given to 224 students in Grades 2–6. Results showed that students had some difficulty when all operations were on the right side of the equal sign and experienced greater difficulty when operations were on both sides of the equal sign. Even students who successfully solved the items requiring a relational understanding of the equal sign tended to fail to generate a relational definition of the equal sign in words. An important finding of this study is that the children with an advanced understanding of the equal sign tended to solve difficult equations, which suggests a link between knowledge of the equal sign and algebraic thinking.

Byrd et al. (2015) focused on how a specific misconception of the equal sign may hinder students' learning of early algebra. The researchers differentiated the interpretations of the equal sign in three ways: (a) arithmetic-specific (e.g., "it means when you add something, you get the total"); (b) non-relational (e.g., "end of question", "a symbol to let you know the answer is next"); and (c) relational (e.g., "something is equivalent to something else"). Children who interpreted the equal sign in arithmetic-specific terms showed lower performance in solving early algebra items than those who defined the equal sign in a non-relational way but without using arithmetic-specific words. The negative effects of an arithmetic-specific view of the equal sign on early algebra learning occurred more for the fifth graders than for the third graders. This implies that an arithmetic-specific interpretation needs to be replaced by a relational or at least another non-relational view before students learn mathematical equivalence and its concomitant concepts in upper elementary grades.

An understanding of the different meanings of variable, coupled with a relational understanding of the equal sign, is fundamental in early algebra (Blanton et al. 2011; Usiskin 1988). A meaning of variable that is frequently used for lower graders at the elementary school level is that of a fixed but unknown number. However, it is not always easy for students to understand this prevalent meaning of variable, and seems to depend on the forms and structures in which it is used. For instance, according to Matthews et al. (2012), the items with letters as variables (e.g., $13 = n + 5$) proved more difficult than those with a similar format but without a letter variable (e.g., $8 = 6 + \square$). Note that students were able to easily solve equations with operations on the left side of the equal sign, but the use of variables rendered a dramatic increase in difficulty. In particular, equations with multiple instances of the unknowns on both sides of the equal sign such as $m + m + m = m + 12$ proved more difficult than the item asking for a relational definition of the equal sign. Students are expected to interpret algebraically the equations in which variables appear. Regarding the equation above, students need to realize that 'm' may be subtracted from each side, and that the simplified equation $m + m = 12$ or $2 \times m = 12$ may be divided by 2 on each side.

A variable can be used to express generalizations beyond specific numerical instances at the elementary school level (Blanton et al. 2011). For instance, while working with basic addition facts, young students can conjecture the commutative property of addition beyond particular number sentences. Young students are able to attend to general aspects by treating the specific numbers as *quasi-variables* (Fujii and Stephens 2008). Furthermore, such a property can be expressed through words (e.g., the sum of two numbers is the same regardless of the order of the numbers) or symbols (e.g., $\square + \triangle = \triangle + \square$ or $a + b = b + a$). The ability to use variables to represent a number in a generalized pattern is powerful for students to communicate their mathematical ideas succinctly (Brizuela et al. 2015).

A variable can also be employed to represent the relationship between two co-varying quantities (Blanton et al. 2011). However, children have difficulties in understanding or representing unknown quantities and tend to assign specific numerical values to solve a problem with unknown quantities (Carraher and Schliemann 2007). In conclusion, an understanding of the multiple meanings of variable and the ability to employ variables to express mathematical relationships or situations are significant in fostering students' algebraic thinking.

6.2.2 Development of Early Algebraic Thinking Through Instruction

Recent studies demonstrate that children are able to engage successfully with diverse aspects of essential algebraic ideas, and their ability can be enhanced through appropriate instruction with a well-developed curriculum. Recently some researchers have begun to compare students in an intervention program promoting early algebraic thinking with their counterparts being instructed with a typical arithmetic-based curriculum.

For instance, Britt and Irwin (2011) endeavored to promote students' algebraic thinking in arithmetic in their New Zealand Numeracy Project. Students with a new curriculum developed by the project were more successful than their counterparts with a conventional curriculum in solving all test items. These included not only simple compensation in addition but also complex equivalence with fractional values. The researchers emphasized that the newly developed curriculum led students to understand the underlying algebraic structure of arithmetic. By conducting a longitudinal study including students aged 12–14 the researchers demonstrated that sustained exposure to algebraic thinking in elementary school leads to more sophisticated generalization with the special symbols of algebra in intermediate school.

More recently, Blanton et al. (2015) demonstrated that, as early as grade 3, students are capable of developing algebraic thinking skills, when they are supported by appropriate tasks and teacher intervention that foster such thinking for a substantial period of time. The participants were third graders, 39 students from intervention groups and 67 counterparts from non-intervention groups. Whereas the former received specifically designed 19 one-hour early algebra lessons throughout the school year, the latter were taught by typical instruction. The study found that students in the intervention group statistically outperformed the non-intervention group in the post-test. Students in the intervention group were better in overcoming their misconceptions about the equal sign and noticing the underlying structure in equations, which helped them determine if the two sides of the equation had the same value without computation. More importantly, only students in the intervention group began to use an *unwind strategy* connected to inverse operations (e.g., to find the value of n in $3 \times n + 2 = 8$, students subtracted 2 from 8, and then divided 6 by 3 to yield 2), though they had not been formally taught this strategy.

Another noteworthy aspect of the Blanton et al. (2015) results was that as many as 74% of the students in the intervention group were able to model the problems that involved unknown quantities with variable notation, even though these students had assigned a specific numerical value to the unknown before participating in the program. The students were able to connect the variable notation across a series of problem situations and used it more frequently than words to represent the relationship between unknown quantities. This study showed that early and sustained exposure to algebraic thinking has a positive impact on students' use of variables.

6.3 Study 1: Students' Understanding of the Equal Sign, Expressions, and Equations

6.3.1 Overview

Given the importance of students' understanding of the equal sign as a basis for developing algebraic ideas, this section reports a study that examined such understanding (Kim et al. 2016). Assessment items from Matthews et al. (2012) were used. Because the items were developed on the basis of prior studies, this allowed for increased validity and reliability in examining students' comprehensive understanding of the equal sign, expressions, and equations. Students aged 7–12 years (i.e., from Grade 2 to Grade 6) were included to investigate whether their understanding of the equal sign, expressions, and equations improves as their grade levels increase following exposure to the current elementary mathematics curriculum.

6.3.2 Method

6.3.2.1 Participants

The participants for this study were from three elementary schools in two provinces. Overall academic abilities and socio-economic levels of students in the selected schools were considered as average in Korea. As this study investigated how students at different grades understand equivalence, we included students in Grades 2–6. Two classrooms for each grade in each of the selected schools were randomly chosen. A total of 695 students were included for the study: 135 second graders, 140 third graders, 140 fourth graders, 144 fifth graders, and 136 sixth graders.

6.3.2.2 Assessment Items

As already mentioned, assessment items from Matthews et al. (2012, pp. 345–347) were used. This instrument includes 27 items of three types: (a) *equation structure items,* such as deciding whether a given number sentence is true or false (e.g., 31 + 16 = 16 + 31 True/False/Don't Know), (b) *equal sign items,* such as asking students to write the meaning of the equal sign, and (c) *equation solving items,* such as finding the unknown number in a given equation (e.g., □ + 2 = 6 + 4). Among equation structure items, specifically *advanced relational reasoning items* are included, such as asking students to solve a given problem without direct computation (e.g., 17 + 12 = 29 is true. Is 17 + 12 + 8 = 29 + 8 true or false? How do you know?). There are also nine items that ask students to describe their answer or explain their solution process (e.g., What does the equal sign (=) mean? Can it mean anything else?). Some minor revisions of the original items were necessary. Specifically, for second graders, numbers less than 30 were used and letter variables were replaced by non-letter variables (e.g., "10 = △ + 6" in place of "10 = z + 6").

6.3.2.3 Data Collection and Analysis

The students in this study solved the assessment items in 40 min and all students' written responses were analyzed. Each item was scored either 0 for incorrect answer or 1 for correct answer. For the nine explanation items, three sub-categories were further used: (a) relational thinking, (b) computation, and (c) incomplete or incorrect explanation. "Relational thinking" here indicates that students explained their solution method by using the structure of the given equation or expression. To emphasize, we employed these sub-categories even for incorrect answers, because our purpose was to investigate the nature of students' understanding. Examples of student responses are included below in the results section.

After responses were coded according to correctness for all items and strategy use for the explanation items, they were analyzed quantitatively using the SPSS 12.0 program. Specifically, ANOVA and post hoc tests[1] were conducted to examine any significant differences for grades.

[1]An ANOVA test tells you whether you have an overall difference between your groups, but it does not tell you which specific groups differed—post hoc tests do.

6.3.3 Main Results

6.3.3.1 Students' Overall Performance

Figure 6.1 shows the results of students' performance for all items. The horizontal axis refers to items and the vertical axis refers to the percentages of correct answers from all grades. Note that Fig. 6.1 displays the percentages only for correctness, regardless of the strategies that students used. The results show that students were quite successful in almost all items (see the following sections for a detailed analysis of selected items). Using ANOVA, a significant difference for grade was found, $F(4, 688) = 125.838$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades. This implies that students' understanding of the equal sign, expressions, and equations improves as their grade levels increase until the fifth grade.

6.3.3.2 Students' Understanding of Equation Structure

Items from 1a to 2b ask students to decide if the given equation is true, whereas Items 3–8 ask for relational thinking. The percentages of correct answers for the latter were lower than those for the former. Generally speaking, the percentages of correct answers for equation structure items increased according to grade levels. Using ANOVA, a significant difference for grade was found, $F(4, 689) = 125.688$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades.

An analysis of the explanation that students wrote for Items 3–8 indicates that less than 35% of the students got the correct answer based on relational thinking. For instance, regarding Item 3 in Table 6.1, only 33.1% of the students justified the correct answer by relational thinking. For instance, some students wrote: "68 is larger than 67 by 1 and 85 is smaller than 86 by 1. So 67 + 86 is the same as 68 + 85." Others justified as follows: "67 + 86 is the same as 68 + 85, because 67 + 1 + 86 − 1 = 68 + 85. Here adding 1 and subtracting 1 makes the answer the

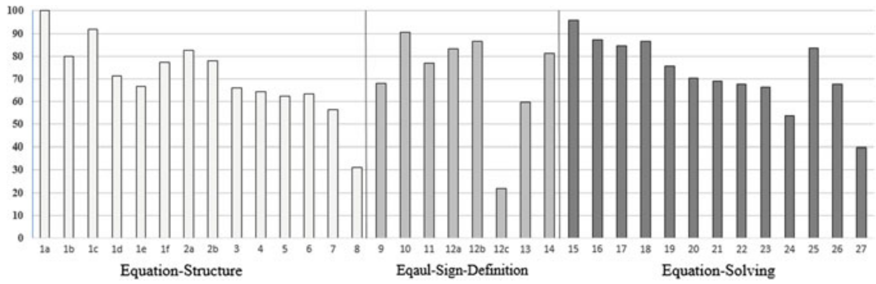

Fig. 6.1 Students' overall performance with respect to correctness

Table 6.1 Item 3 and students' responses

Item 3
Without adding 67 + 86, can you tell if the number sentence below is true or false?
67 + 86 = 68 + 85. How do you know? (Note: 7 + 4 = 8 + 3 for Grade 2)

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	88 (67.2)	58 (41.4)	34 (24.5)	21 (14.6)	25 (18.4)	226 (32.8)
	Relational	0 (0)	2 (1.4)	2 (1.4)	0 (0)	0 (0)	4 (0.6)
	Computational	1 (0.8)	0 (0)	3 (2.2)	0 (0)	0 (0)	4 (0.6)
Correct	Incomplete	13 (9.9)	19 (13.6)	14 (10.1)	24 (17.4)	24 (17.6)	94 (13.6)
	Relational	1 (0.8)	44 (31.4)	55 (39.6)	65 (45.1)	64 (47.1)	228 (33.1)
	Computational	28 (21.4)	17 (12.1)	31 (22.3)	34 (23.6)	23 (16.9)	133 (19.3)

same." Still others wrote: "67 + 86 = 68 − 1 + 85 + 1" and drew circles over '−1' and '+1.' The thinking of these students can be described, more formally, as using the associativity and commutativity properties of addition, $68 + 85 = (67 + 1) + (86 - 1) = 67 + 86 + (1 - 1) = 67 + 86$. However, even for upper graders, the percentage of those using relational thinking based on the algebraic structure of arithmetic was less than 50%. About 20% of the students in almost all grades used a computational strategy, even though the item explicitly states: "without adding 67 + 86." Some students got the correct answer, but gave incomplete or incorrect explanations, including "67 + 86 = 68 + 85 is true because it is the same" or "if you add each number, then 6 + 7 + 8 + 6 = 27 and 6 + 8 + 8 + 5 = 27, the number is the same." Regarding the incorrect answers, the most common response type was that of incomplete or incorrect explanations, such as "67 + 86 = 68 + 85 is false because the addends are different respectively." A rare example of an incorrect answer using relational thinking included, "67 < 68 and 86 > 85, so 67 + 86 = 68 + 85 is false."

Item 8 in Table 6.2 was the most difficult item among the equation structure items because it includes multiplication and the unknown number □. Note also that the new multiplier 8 is multiplied from the left in both sides of the first equation. Whereas the majority of students employed a computational strategy or gave an incomplete explanation of the strategy used to get the answer (e.g., "You know it if you just see it"), only 4.2% of the students were able to use relational thinking in their solution process (e.g., "There is the same 8 on both sides of $8 \times 2 \times \square = 8 \times 58$, so you simply divide by 8 only to get $2 \times \square = 58$").

6.3.3.3 Students' Understanding of the Equal Sign

Items 9 through 14 deal with the meaning of the equal sign. The percentages of correct answers to these items were high except for Item 12c. The percentages of correct answers for the equal sign items highly correlated with grade level and, in fact, a significant difference for grade was found using ANOVA, $F(4, 690) = 42.013$,

Table 6.2 Item 8 and students' responses

Item 8
Is the number that goes in the box the same number in the following two number sentences?
$2 \times \square = 58$, $8 \times 2 \times \square = 8 \times 58$ (Note: $2 + \square = 10$, $5 + 2 + \square = 5 + 10$ for Grade 2)
How do you know?

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	98 (74.8)	129 (92.1)	98 (70.5)	80 (55.6)	58 (42.6)	463 (67.2)
	Relational	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
	Computational	0 (0)	0 (0)	10 (7.2)	0 (0)	0 (0)	10 (1.5)
Correct	Incomplete	19 (14.5)	10 (7.1)	7 (5.0)	33 (22.9)	15 (11.0)	84 (12.2)
	Relational	3 (2.3)	0 (0)	1 (0.7)	4 (2.8)	21 (15.4)	29 (4.2)
	Computational	11 (8.4)	1 (0.7)	23 (16.5)	27 (18.8)	42 (30.9)	103 (14.9)

Table 6.3 Item 12 and students' responses

Item 12
Is this a good definition of the equal sign? Circle *True* or *False*
a. The equal sign means the same as. True False
b. The equal sign means add. True False
c. The equal sign means the answer to the problem. True False

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Correct	Item 12a	70 (53.4)	113 (80.7)	121 (87.1)	139 (96.5)	132 (97.1)	574 (83.3)
	Item 12b	97 (74.0)	128 (91.4)	124 (89.2)	114 (79.2)	133 (97.8)	595 (86.4)
	Item 12c	30 (22.9)	31 (22.1)	30 (21.6)	39 (27.1)	21 (15.4)	151 (21.9)

$p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades.

The most challenging Item 12 (see Table 6.3) asks students to determine whether the given definition of the equal sign is *true* or *false*. The percentages of correct answers for Items 12a and 12b were quite high from Grade 3 onward. However, only 21.9% of the students answered correctly for Item 12c. In other words, the majority of students have an understanding that the equal sign means "the same as", and that the equal sign does not mean "add". However, at the same time, many students thought that the equal sign means "the answer to the problem". Maybe students thought that it means to 'do' instead of to 'add,' as this might also mean subtract, multiply, or divide, but they seemed to agree with part c. More importantly, this non-relational thinking regarding the equal sign was persistent across all grade levels.

6.3.3.4 Students' Equation-Solving

Items 15–27 ask students to find the unknown number in the given equation. The percentages of correct answers for equation solving items were high except for Items 24 and 27. Again, the percentages of correct answers increased as grade levels rose. Once again, a significant difference for grade was found using ANOVA, $F(4, 689) = 95.288$, $p < 0.05$. Post hoc tests revealed a significant difference among grades except between the fifth and the sixth grades. This may be related to the elementary mathematics curriculum by which whole number operations are dealt with in multiple contexts up to the fourth grade but in the fifth and sixth grades mainly fraction and decimal operations are dealt with.

Whereas items 15–24 use the symbol variable □, the last three items use letter variables. Simply using a letter variable did not increase item difficulty, as shown in the results for Items 17 and 25 (see Table 6.4). However, multiple instances of a letter as a variable made it difficult for students, specifically second and third graders, to solve the given problem, as shown in the results for Item 26. In fact, the Korean elementary mathematics textbooks provide little opportunity for students to experience multiple occurrences of the variable.

Item 24 in Table 6.5 examines whether students use their understanding of the equal sign and equation structure to solve a given equation with relatively large numbers. Note that we changed the original equation, 43 + □ = 48 + 76, in Matthews et al. (2012) into 47 + □ = 48 + 76. This was done in order to see whether students were able to employ relational thinking for the close numbers 47 and 48, and to interpret the result for Item 24 in relation to the result for Item 3 (67 + 86 = 68 + 85) among the equation structure items. The results for Item 24 show that only 35.6% of the students solved the equation by relational thinking. Some students explicitly wrote their reasoning process (e.g., "The equal sign (=) here means the same as, therefore, the expressions are to be the same. Because 47 is less than 48 by 1, □ should be larger than 76 by 1, so the answer is 77."). Others used a computational strategy with an incorrect use of the equal sign (e.g., "47 + □ = 124 − 47 = 77, 48 + 76 = 124") or wrote an incomplete or incorrect

Table 6.4 Selected equation-solving items and students' responses

Item 17. Find the number that goes in the box. 8 = 6 + □
Item 25. Find the value of z. In other words, what value of z will make the following number sentence true? Circle your answer.
$10 = z + 6$ (Note: 10 = △ + 6 for Grade 2)
Item 26. Find the value of n. $n + n + n + 2 = 17$
(Note: ☆ + ☆ + ☆ + 2 = 17 for Grade 2)

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Correct	Item 17	79 (60.3)	116 (82.9)	118 (84.9)	139 (96.5)	130 (95.6)	582 (84.5)
	Item 25	80 (61.1)	111 (79.3)	120 (86.3)	137 (95.1)	128 (94.1)	576 (83.6)
	Item 26	40 (30.5)	79 (56.4)	111 (79.9)	119 (82.6)	118 (86.8)	467 (67.8)

Table 6.5 Item 24 and students' responses

Directions: *Find the number that goes in each box. You can try to find a shortcut so you don't have to do all the adding. Show your work and write your answer in the box.*
Item 24. 47 + ☐ = 48 + 76. How do you know? (Note: 7 + ☐ = 8 + 6 for Grade 2)

Response		Frequency (%)					
		2nd	3rd	4th	5th	6th	Total
Incorrect	Incomplete	110 (84.0)	69 (49.3)	56 (40.3)	41 (28.5)	29 (21.3)	305 (44.3)
	Relational	0 (0)	1 (0.7)	4 (2.9)	1 (0.7)	2 (1.5)	8 (1.2)
	Computational	0 (0)	1 (0.7)	5 (3.6)	1 (0.7)	0 (0)	7 (1.0)
Correct	Incomplete	6 (4.6)	20 (14.3)	8 (5.8)	23 (16.0)	17 (12.5)	73 (10.6)
	Relational	5 (3.8)	42 (30.0)	56 (40.3)	68 (47.2)	74 (54.4)	245 (35.6)
	Computational	10 (7.6)	7 (5.0)	10 (7.2)	10 (6.9)	14 (10.3)	51 (7.4)

explanation (e.g., "It seems that I can't explain it without adding the numbers."). We also found that the results for Item 24 were quite similar to the results for Item 3 (see Table 6.1), implying that students' understanding of the equation structure seems to influence their equation-solving abilities.

6.4 Study 2: Diverse Aspects of Early Algebraic Thinking in Third Graders

6.4.1 Overview

A multitude of studies have documented that elementary students can successfully develop essential algebraic ideas. This section reviews a study that examined third graders' early algebraic thinking (Pang and Choi 2016). Early algebra has not been explicitly addressed in the national elementary mathematics curriculum in Korea. We wondered how students not exposed to a specific intervention program or curriculum fostering such thinking processes would respond to the diverse assessment items related to early algebraic thinking.

In order to better understand our students' performance in the international context, we adapted Blanton et al. (2015)'s study for at least three reasons. Firstly, because Blanton et al. (2015) document the data from both the nonintervention group and the early algebra intervention group, we can locate our students' performance against both groups. Secondly, third graders participated in the study of Blanton et al. (2015). It is reasonable to examine the algebraic thinking of third-graders in our study, considering that it would be useful to examine the capability of these lower grade students with respect to engaging in early algebraic ideas. At the same time, these students have been sufficiently exposed to the elementary mathematics curriculum so as to enable the researchers to interpret their performance in relation to the curricular experience. Thirdly, the test items in Blanton et al. (2015) are sufficiently comprehensive in that they include *big ideas* in early algebra such as equivalence and equations, generalized arithmetic, functional thinking, and variables. The use of such items was expected to reveal multiple aspects of students' algebraic thinking that were developed while using the regular mathematics curriculum.

6.4.2 Methods

6.4.2.1 Participants

The third-grade students in this study were from seven elementary schools in four provinces. Overall academic abilities and socio-economic levels of students in the selected schools were considered as average in Korea. A written assessment was distributed to a total of 220 students. Unfortunately, 23 students did not answer the items asking for their explanation or justification. They were mostly in the one classroom in which the teacher did not emphasize the need to do so. After excluding the data from these 23 students, the data from the remaining 197 students were analyzed.

6.4.2.2 Assessment Items

The assessment items were from Blanton et al. (2015, pp. 83–86). One item among the original 11 items, dealing with proportional reasoning, was not included because it is not appropriate for Korean third grade students. Careful translation of the 10 items was conducted and a pilot test was administered in one third-grade classroom. A few revisions were necessary. Item 4, written in sentences, was changed into the form of a dialogue so as to make it more understandable, while keeping the meaning of the original item (see Sect. 6.4.3.3 for the detailed revision). A critical issue involving variable notation was raised. In Blanton et al. (2015), concepts associated with variables were integrated into the instruction and students were expected to be able to use letter variables to represent an unknown quantity in different problem contexts. However, in Korea, variable notation without letter symbols is used in the textbooks or workbooks for lower graders: For instance, (a) a variable as a fixed unknown number: as in $5 + \square = 7$ or $9 - \square = 5$ and (b) a variable as a tool for generalization: as in $\heartsuit + 0 = \heartsuit$, $0 + \heartsuit = \heartsuit$. Given this, in keeping with the original assessment items for comparison purposes, we developed supplementary items with the use of non-letter variable notation (see Sect. 6.4.3.5 for an example).

6.4.2.3 Data Collection and Analysis

The students in this study solved the assessment items in 40 min. A total of 197 students' written responses were analyzed for correctness for the items that have only one correct answer. The strategies employed by students were initially analyzed according to the coding scheme in Blanton et al. (2015). Whenever different strategies or responses emerged, new codes were assigned such as "correct answer with incomplete explanation" and "incorrect answer with an error to be noted". Criteria for determining correctness or strategy use are mentioned with the examples in the following results section.

In addition, unstructured interviews with nine students were conducted to investigate their reasoning processes in detail. For instance, the interviewees included students who answered correctly without explanation, students who used different strategies for similar assessment items, or students who used an apparently new strategy. The interviews were audiotaped and transcribed, which served to identify diverse aspects of those students' algebraic thinking.

6.4.3 Main Results

6.4.3.1 Comparison of Students' Overall Performance

Table 6.6 shows students' overall performance with the items rated in terms of percentage correct. Notice that four items (i.e., Items 3a, 3b, 4a, and 8b) were not included here because they were analyzed only for strategy use as in Blanton et al. (2015, p. 87). A cautionary note in reading Table 6.6 is that our main purpose was to better understand our students' overall performance in international contexts. Due to limited space, the results for some items are included in subsequent sections.

For most items, the Korean students performed as well as, or only slightly worse than, students in the Blanton et al. intervention group, and much higher than students in their non-intervention group. These items included figuring out a missing value in the equation (e.g., $7 + 3 = \square + 4$), evaluating an equivalence relationship (e.g., $12 + 3 = 15 + 4$ True/False), generalizing the commutative property of addition, and selecting a generalized algebraic expression on the basis of particular examples (e.g., $a - a = 0$ from $8 - 8 = 0$ to $12 - 12 = 0$). Regarding Items 5 and 9, the Korean students performed far better than students in the Blanton et al. intervention group, although they experienced substantial difficulties in Items 7 and 10. What follows is a detailed analysis of the strategies students employed on selected items.

Table 6.6 Comparison of students' overall performance by percentage of correct response

Item		Korean (N = 197)	Blanton et al. (2015)'s posttest correct	
			Non-intervention (N = 67)	Intervention (N = 39)
1	a	69.0	3.2	84.2
	b	70.0	1.6	84.2
2	a	74.1	31.7	86.8
	b	73.6	9.5	84.2
	c	75.1	14.3	89.5
4b		66.3	34.9	73.7
5		73.6	4.8	36.8
6		84.2	57.1	89.5
7	a	16.2	12.7	73.7
	b	15.2	7.9	63.2
	c	4.5	3.2	39.5
8a		84.7	49.2	89.5
9		85.7	28.6	52.6
10	a	76.1	52.4	86.8
	b	29.9	41.3	78.9
	c	47.7	7.9	31.6
	d	4.5	0.0	15.8
	e	47.2	41.3	55.3

6.4.3.2 Students' Understanding of Equivalence

The percentages of correct responses for the items addressing students' understanding of the equal sign, expressions, and equations were high. But when we analyzed their solution strategies, we found that the students solved the items by computation (coded as *computational* strategy) more often than noticing the underlying structure in the equation without computing (coded as *structural* strategy). For instance, whereas a majority of students responded correctly to Item 2b, 64.9% of them used a computational strategy and only 4.5% of the students used a structural strategy (see Table 6.7). The tendency of employing a computational strategy was lower for Item 2c, but still served as a main foundation for determining if the two sides of the equation had the same value. About 20% of the students consistently added the numbers to the left of the equal sign to get the solution (coded as *operational* strategy).

We wondered whether students were unable to use any structural strategy, even when they had a relational understanding of the equal sign. We interviewed some students who answered for Item 3 that the meaning of the symbol "=" in the number sentence 3 + 4 = 7 is "same as" or "equal," but who used only a computational strategy to find a missing value or to determine equivalence. As reflected in Episode 1, the student initially approached the item computationally, but was able to use a structural strategy when asked to solve it without computation.

Table 6.7 Item 2 and students' strategy use

Item 2 Circle True or False and explain your choice. b. 57 + 22 = 58 + 21. True False How do you know? c. 39 + 121 = 121 + 39. True False How do you know?		
Strategy	Example or explanation	Frequency (%)
Structural	Item 2b: True, because 58 is one more than 57 and 21 is one less than 22.	9 (4.5*)
	Item 2c: True, because 121 + 39 is 39 + 121 in reverse.	66 (33.5)
Computational	Item 2b: True, because 57 + 22 = 79 and 58 + 21 = 79.	128 (64.9)
	Item 2c: True, because 39 + 121 = 160 and 121 + 39 = 160.	71 (36.0)
Operational	Item 2b: False, because 57 + 22 = 79, not 58.	45 (22.8)
	Item 2c: False, because 39 + 121 is not 121.	41 (20.8)

*The sum of percentages does not reach 100 because the table includes main strategies

Episode 1 Emergent use of structural strategy versus tendency to compute.

Interviewer (I): (points to the student's written response *12 + 3 = 15, 15 + 4 = 19* for Item 2a: "12 + 3 = 15 + 4 True False How do you know?") *Do you necessarily need to compute 15 and 19 to solve this item?*
Student (S): *Probably.*
I: *Can it be done without computation?*
S: *I can solve it by comparison.*
I: *Okay. Why don't you compare?*
S: *If you compare 3 and 4, 3 is less. If you compare 12 and 15, 12 is less. So this part* (pointing to 12 + 3) is less.
I: *Okay. How about this* (pointing to Item 2b)?
S: *57 plus 22 is 79 and 58 plus 21 is also 79.*
I: *Without computation? How would you explain by comparison as you just did?*
S: *It becomes the same if I give 1 from 58, so I know it without calculation.*
I: *Right, very good! How about this* (pointing to Item 2c)? *Do you have to add these two numbers* (pointing to 121 and 39) *to find out 160?*
S: *I know right away because it just switches the positions of the numbers.*

In the episode above, the student went back to use a computational strategy for Item 2b, even after she had just solved Item 2a without computation. When asked to explain by comparison, however, the student was able to use a structural strategy (i.e., "give 1 from 58"). She then continued to use a structural strategy for Item 2c by justifying with a fundamental property of number and operations.

6.4.3.3 Students' Understanding of the Commutative Property of Addition

Item 2c in Table 6.7 examines students' understanding of equivalence, but it also reflects the commutative property of addition. As described, students' tendency of using a computational strategy for Item 2b was decreased for Item 2c. More significantly, only 3% of the students continued to use a computational strategy for Item 4a, whereas almost half of them used a structural strategy (see Table 6.8). This may be related to the slight but important difference between Item 2c and Item 4. Note that both items involve the commutative property of addition. However, Item 2c asks students to evaluate whether the given equation is true or false, and to justify their reasoning. In contrast, Item 4 does not include the equal sign and, more importantly, encourages students to reason without computation.

For Item 4b, 66.3% of the students answered that Yuna's thinking will work for all numbers. In fact, the majority of the students justified it by describing the commutative property of addition in words. It is not surprising because they had already learned it through their mathematics textbook. What is interesting here is that about 10% of the students justified their answer by writing another example (e.g., 1 + 2 = 2 + 1). We wondered whether the students were capable of thinking beyond particular instances to generalize the fundamental property. Episode 2 is an interview with a student who wrote a single instance for Item 4b.

Table 6.8 Item 4 and students' strategy use

<table>
<tr><td>Item 4 (original)
Marcy's teacher asks her to figure out "23 + 15." She adds the two numbers and gets 38. The teacher then asks her to figure out "15 + 23." Marcy already knows the answer.</td><td colspan="2">Item 4 (revised)
The following is the dialogue between Yuna and her teacher.
Teacher: Yuna, what is 23 + 15?
Yuna: If I add 23 and 15, I get 38.
Teacher: Then, what is 15 + 23?
Yuna: I already know it without computation!</td></tr>
<tr><td colspan="3">a. How does Yuna know?
b. Do you think this will work for all numbers? If so, how do you know?</td></tr>
<tr><td>Strategy</td><td>Example</td><td>Frequency (%)</td></tr>
<tr><td>Structural</td><td>Item 4a: It is the same as 23 + 15, because only the numbers are switched.</td><td>96 (48.7)</td></tr>
<tr><td>Computational</td><td>Item 4a: 23 + 15 = 38 and 15 + 23 = 38.</td><td>6 (3.0)</td></tr>
</table>

Episode 2 Generalization beyond particular instances regarding the commutative property of addition.

I: (reads Item 4a) *How did Yuna know?*
S: *Because only the positions of the numbers were switched.*
I: *Okay, you wrote 21 + 22. What does it mean?*
S: *21 + 22 is the same as 22 + 21.*
I: *Then, is it okay to use 3 and 4 instead of 21 and 22?*
S: *Yes!*
I: *Why?* (After no response from the student, interviewer continued) *Then, let's say that the first number is* □ *and the second number is* △*. Can we say* □ + △ = △ + □*?*
S: *Yes.*
I: *Why do you think so?*
S: *Because it just switches the position of the figures.*

As reflected in Episode 2, the student answered with the single instance of 21 + 22. But he knew that the numbers could be changed to other numbers, in fact, any numbers. In other words, he expressed the generalization in terms of specific numbers. Given the generalized representation of the symbols □ and △, the student was able to justify his thinking in words.

6.4.3.4 Students' Understanding of Equations

Item 9 examines how students solve a simple linear equation and justify their answer (see Table 6.9). Note that we changed the original equation $3 \times n + 2 = 8$ to $3 \times \square + 2 = 8$, because letter variables are not taught in Korea until Grade 6.

A noticeable result was that the percentage of correct answers for Korean students (i.e., 85.7%) was the highest for Item 9, which was much higher than for that of the Blanton et al. intervention group (i.e., 52.6%), as was seen in Table 6.6. To emphasize, mathematics textbooks in Korea do not deal with equations with two operations until Grade 3. More interestingly, most students used a different strategy (coded as *intuitive use of number facts*) than either the "Guess and Test" or "Unwind" strategies. According to Blanton et al. (2015, p. 57), the use of the Guess and Test strategy means that the student works through the equation in a forward manner, substitutes value(s) in for the variable and computes to see if the value is correct, and the Unwind strategy refers to the student working backward through constraints in the equation, inverting operations. Our students worked through the equation in a forward manner and seemed to notice the underlying structure of the given equation as a whole. Instead of substituting values for the variable (e.g., 3 and then 2, or arbitrarily initially choosing 2), our students started with the fact that (a certain number) +2 is 8, so the number must be 6. Then the question becomes easier because the original item turns into $3 \times \square = 6$. A noteworthy aspect is that students seem to be capable of seeing $3 \times \square$ as an object, which makes it easier for them to notice the structure of the equation. In this process, students could have

Table 6.9 Item 9 and students' strategy use

Item 9 Find the value of □ in the following equation. How did you get your answer? $3 \times \square + 2 = 8$		
Strategy	Example	Frequency (%)
Guess and test	$3 \times 3 + 2 = 11$, $3 \times 2 + 2 = 8$	5 (2.5)
Unwind	$8 - 2 = 6$, $6 \div 3 = 2$	2 (1.0)
Intuitive use of number facts	(A certain number) + 2 is 8, so the number must be 6. $3 \times \square = 6$, $3 \times 2 = 6$. $3 \times 2 = 6$, $6 + 2 = 8$	150 (76.1)

subtracted 2 from 8 to get 6 and then divided the 6 by 3 to get 2 (i.e., the unwind strategy). However, our students did not invert the operations, but employed familiar number facts in an intuitive manner (i.e., $3 \times 2 = 6$, $6 + 2 = 8$). In order to understand better students' thinking processes, we interviewed some of them who simply wrote "$3 \times 2 = 6$, $6 + 2 = 8$" for Item 9. Note that we gave them extra simple equations on the spot to trace their thinking.

Episode 3.1 Using 'equation sense' based on number facts.

I: *Could you explain how you solved this* (pointing to Item 9)?
S1: *3 times 2 is 6 and 6 plus 2 is 8.*
I: *Aha, how about this problem* (writing $7 \times \square + 3 = 24$)?
S1: *Some number less than 24, the product of 7 and a certain number should be less than 24 but at the same time close enough to 24. 7, 3, 21 and plus 3 is 24.*
I: *Okay, I will give you another problem with larger numbers.* (Writes $12 \times \square + 1 = 49$.)
S1: *12 times 4 is 48, here* (pointing to $12 \times \square$) *is 48 and plus 1 is 49.*
I: *Why did you think this* (pointing to $12 \times \square$) *is 48?*
S1: *Because 49 minus 1 is 48, then I thought 12 times what number makes 48.*

Episode 3.2 Noticing the structure in a linear equation

I: *How did you find 2 for Item 9* (pointing to $3 \times \underline{2} + 2 = 8$)?
S2: *I left 'plus 2' alone. 3 times a certain number, that product, that number plus 2 is to be 8. As 3 is there, I thought of the '3 times table'. 3, 2, 6 and 6 plus 2 is 8.*
I: *Okay, why don't you solve this problem* (writing $7 \times \square + 3 = 24$)?
S2: *3.*
I: *How did you know so quickly?*
S2: *This is the same. You just need to see a certain number plus 3. 7, 3, 21 and then plus 3 is 24.*
I: *Now, I will give you larger numbers.* (Writes $13 \times \square + 5 = 31$)
S2: *2!*
I: *Wow! How did you know the answer so quickly?*

S2: *It is the same, too. It says a certain number plus 5 is 31. 31 minus 5 is 26. So 13 times 2 is 26.*

As reflected in the episodes above, it was clear that the students were capable of noticing the structure of a given equation as a whole. On the basis of the understanding of the equal sign and expressions, S1 knew that the left side of the equation (i.e., $7 \times \square + 3$) should be 24 and, as there is '+3', "the product of 7 and a certain number should be less than 24 but at the same time close enough to 24." For another equation with larger numbers, S1 partially used an *unwind* strategy by inverting '+1' as '−1.' However, she still solved the equation in a forward manner; not by $48 \div 12 = 4$ but by $12 \times 4 = 48$, even though the number 12 is not in the common times table (the 2–9 times table). It was also obvious that S2 in Episode 3.2 was able to see $3 \times \square$ as an object, when he called it "that product, that number." His interpretation of numbers and expressions as objects was consistent. He explained both $7 \times \square$ in the equation $7 \times \square + 3 = 24$ and $13 \times \square$ in $13 \times \square + 5 = 31$ as 'a certain number.' S2 also partially used an *unwind* strategy when converting the addition of 5 to the subtraction of 5 (from 31) in $13 \times \square + 5 = 31$.

Given that both S1 and S2 partially used an *unwind* strategy, we might have coded their responses as the "Unwind" strategy. However, we chose not to do so in order to emphasize the noticing of the equation structure as a whole, rather than focusing on employing inverse operations step by step. Also worth noting is that students solved the equations very quickly in a forward manner through familiar number facts or "equation sense." In this respect, students might not use an inverse operation when they initially solve the equation, but merely mention it later to justify their answer.

6.4.3.5 Students' Understanding of Algebraic Expressions with Variable Notation

Item 7 examines students' understanding of algebraic expressions and, in particular, how they represent unknown quantities with variables (see Table 6.10). Note the original item was slightly altered because pennies are not used in daily life in Korea. 'Coins' were addressed instead of the specific coin penny, but kept the critical aspect of the item, that is to say, an indeterminate amount of coins.

Item 7 was the most challenging problem for our students. The percentage of correct answer was the lowest among the 10 assessment items (see Table 6.6) and, in fact, the percentage of "no response" answers was about 38%. The most frequent strategy students used was to assign a specific numerical value to the unknown quantity, although the item specifically says that the quantity is not known. Slightly less than 30% of the students assigned arbitrary numerical values to the unknowns of Items 7a, 7b, and 7c. In contrast, about 20% of the students assigned specific numerical values that were related to one another (see the strategy *value-related* in Table 6.10). Students used this strategy in a consistent way for Item 7a, 7b, and 7c.

Table 6.10 Item 7 and students' strategy use

Item 7
Hajun and Yejun have coins in their piggy banks, and the kinds of coins are the same. They know that their piggy banks each contain the same number of coins, but they don't know how many. Yejun also has 8 coins in his hand.
a. How would you represent the number of coins Hajun has?
b. How would you represent the number of coins Yejun has?
c. How would you represent the total number of coins they have?

Strategy	Example	Frequency (%)
Variable	Item 7a: □	25 (12.6)
Variable-related	Item 7b: □ + 8	23 (11.6)
	Item 7c: □ + □ + 8	9 (4.5)
Value-related*	Item 7a: 8	34 (17.2)
	Item 7b: 16	38 (19.2)
	Item 7c: 24	41 (20.8)

*This strategy code was given only to the responses in which a specific numerical value was assigned and interrelated with the unknowns of Items 7a, 7b, and 7c, such as in the provided example

Another strategy students used was to assign a non-letter variable (i.e., □) to the unknown quantity (coded as the *variable* strategy). What is important here is whether students were able to connect their representations in Items 7b and 7c to their representation of Item 7a (coded as the *variable-related* strategy). The majority of students related their representation in Item 7b to that in Item 7a, but did not relate their representation in 7c to those in Items 7a and 7b. In other words, the students who represented the number of coins Hajun has by □ (Item 7a) tended to keep their non-letter variable to represent the number of coins Yejun has by □ + 8 (Item 7b). However, they had difficulties in representing the combined number of coins Hajun and Yejun have as □ + □ + 8. Students instead assigned either a numerical value or □ which is assumed to simply represent that the combined number of coins is unknown.

In Korea, students are taught to represent the unknown number mostly by □ from the first grade. We wondered how students' responses would change if we provided them with variables. Against students' difficulties with Item 7c, we provided students with supplementary items in which both the number of coins Hajun has, and the number of coins Yejun has, are represented in the form of variables (see Table 6.11). We focused on those strategies in which students used a variable to represent the combined number of coins (i.e., the sum of responses related to 7a and 7b in the original items) and to flexibly operate with expressions involving such variable notation.

As shown in Table 6.11, when the specific variables were provided in the items, the percentage of correct answers increased in comparison to 4.5% for Item 7c, as was seen in Table 6.10. What is even more noticeable here is that it was easier for students to represent the unknown quantity as □ + △ than as □ + □ + 8. As the representation □ + □ + 8 includes two additions with the same symbol,

Table 6.11 Supplementary item 7 and students' strategy use

Supplementary Item 7 (S7). Hajun and Yejun have coins in their piggy banks, and the kinds of coins are the same. They know that their piggy banks each contain the same number of coins, but they don't know how many. Yejun also has 8 coins in his hand.		
a. The number of coins Hajun has is ☐ and the number of coins Yejun has is ☐ + 8. How would you describe the total number of coins Hajun and Yejun have?		
b. The number of coins Hajun has is ☐ and the number of coins Yejun has is △. How would you describe the total number of coins Hajun and Yejun have?		
Strategy	Example	Frequency (%)
Variable-related	Item S7a: ☐ + ☐ + 8	34 (17.2)
	Item S7b: ☐ + △	49 (24.8)

students seemed to attempt to compute further. Some students wrote it without the plus sign (i.e., ☐☐8) or put a rectangle to show the result of '☐ + ☐' (i.e., ▭ 8). Others wrote '☐ + ☐ + 8 = ?,' implying that it is not an object per se but something to be calculated.

6.5 Discussion and Implications

6.5.1 Development of Early Algebraic Thinking Through a Curriculum

Given the importance of early algebraic thinking, specific content domains aiming at fostering such thinking skills have emerged in various curricula, such as "patterning and algebra" (Ontario Ministry of Education 2005), "operations and algebraic thinking" (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010), and "number and algebra" (New Zealand Ministry of Education 2009). As aforementioned, the Korean national elementary mathematics curriculum does not include early algebra or algebraic thinking as a specific content domain. However, the two studies reported in this chapter indicate that students are capable of developing essential algebraic ideas from a generalized arithmetic perspective through the current curriculum. Specifically, a promising result was that except for a few items, our students' overall performance was similar to that of students in the intervention group in the Blanton et al. (2015) study. This means that new content areas are not necessarily needed in the current curriculum to induce early algebraic thinking and to make it accessible to students (McNeil et al. 2015). Early algebraic thinking can instead be fostered as a specific form of thinking while students learn typical content areas.

Another important result, as shown in Study 1, is that students' overall understanding of the equal sign, expressions, and equations evolves as their grade levels go up until the fifth grade. This tendency was consistent across different types of assessment items. Given the difficulties that lower graders such as Grades 2 and 3 experienced, however, specific pedagogical attention is needed. For instance, the equal sign is addressed in the first grade in Korea but only about half of the second graders in Study 1 understood that the equal sign means "the same as." More importantly, about 80% of the second graders had the misconception that the equal sign means "the answer to the problem." Note that this misconception persists even in upper grades. Such misconception must be related to curricular materials and instruction in which students see and use the equal sign (McNeil et al. 2015). According to Ki and Cheong (2008), our textbooks use equations mostly in a standard format (i.e., all operations are on the left side of the equation and the answer comes after the equal sign). Considering the importance of relational understanding of the equal sign as an essential idea for algebraic thinking (Blanton et al. 2011), diverse types of equations need to be utilized in curricular materials and instruction from the earliest grades.

The results of Study 2 also indicate our students' weaknesses in understanding algebraic expressions and representing the unknown quantities with variables. The students tended to assign a specific numerical value to the unknown quantity. Even the students who were able to use a non-letter variable had difficulty in connecting such a representation to other related contexts in a consistent way. Although variables have multiple meanings, they are frequently addressed in the current Korean curricular materials beginning in the first grade mainly as a fixed unknown quantity associated with missing-value problems (Pang et al. 2017). Variables to represent the relationships between varying quantities are addressed only from the fourth grade. Radford (2014, p. 260) postulates a framework for characterizing algebraic thinking in terms of three key notions: (a) *indeterminacy:* not-known numbers are involved in the given problem, (b) *denotation:* the indeterminate numbers are named or symbolized in various ways such as with gestures, words, or alphanumeric signs, and (c) *analyticity*: the indeterminate quantities are operated with as if they were known numbers. In order to increase our students' exposure to these key notions, improvement is needed in those parts of the current mathematics curriculum dealing with numbers and operations, in developing a relational understanding of equality, and in writing expressions or equations with variables to represent diverse problem contexts.

6.5.2 Computational Proficiency and Algebraic Thinking

Special attention in this chapter was given to a generalized arithmetic perspective in a broad sense so as to include equivalence, expressions, equations, and inequalities. A common and significant result of Studies 1 and 2 was the finding that our students tend to use a computational strategy in examining an equation structure or in finding an unknown number in an equation, even when the assessment items explicitly ask them not to use direct computation. Korean students are confident in computation; so it may be easy for them to calculate in solving a given problem or to use such computational ability when asked to justify their answers. On one hand, relational thinking or a structural approach over computation is desirable in dealing with mathematical equivalence. Our students need to be further instructed to notice the underlying structure of expressions or equations before jumping into calculation to get the correct answer. On the other hand, computational proficiency does not need to be discouraged in favor of early algebraic thinking. As shown in Episode 1, students with relational understanding of the equal sign are capable of using a structural strategy despite their tendency to compute. More interestingly, our students' computational proficiency seems to help them find the missing value in simple linear equations such as $3 \times \square + 2 = 8$, due to the way they think about such equations. On the basis of familiar number facts our students immediately noticed the structure of the given equation with two operations by regarding $3 \times \square$ as an object. As reflected in Episodes 3.1 and 3.2, the students were able to apply their algebraic reasoning about relationships to solve other equations with larger numbers.

To emphasize, arithmetic is a main context for early algebraic thinking. This study shows that students can be exposed to algebraic ideas as they develop the computational proficiency emphasized in arithmetic. The issue is then for teachers to elicit and foster students' early algebraic thinking through questioning with an emphasis on mathematical structure and relationships while they learn typical mathematical topics (e.g., Can you decide if the given equation is true or false without computation? Can you find the missing value in the equation without computation? What are the unknown numbers or quantities in the context and how can you represent them? Do you think this particular property of number and operations will work for all numbers?).

To conclude, this chapter is expected to provide information on Korean students' early algebraic thinking that develops by means of the current elementary mathematics curriculum. This chapter also shows that specific algebraic ideas need to be intentionally fostered in the curriculum from the earlier grades, because these ideas are not naturally developed in students as they progress through elementary school.

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Chapter 7
Function Tasks, Input, Output, and the Predictive Rule: How Some Singapore Primary Children Construct the Rule

Swee Fong Ng

Abstract Function-machine tasks are not part of the formal Singapore Primary Mathematics curriculum and hence not taught formally. The corpus of data shows that provision of the expressions input, output, and 'the rule is' aided primary children, particularly those in the upper primary grades, to construct the predictive rule underpinning function-machine tasks. Children's annotations showed that many were willing to write the literal form of *input* $\pm$ *a* = *output*, while others were open to the symmetric equivalence construct of the non-literal form of *output* = *input* $\pm$ *a*. Primary children's knowledge reflected the spiral structure of the Singapore Primary Mathematics curriculum, where number facts and processes are introduced in bite sizes. Children at all upper grades found implicit functions challenging.

Keywords Spiral structure of curriculum · Number facts · Algebraic thinking Function machines

7.1 Algebraic Thinking, Function, and Function Machines

Algebraic thinking "defies simple definition" (Driscoll 1999, p. 1). Radford (2001, p. 13) located the historical origins of algebraic thinking as emerging from "proportional thinking as a short, direct and alternative way of solving 'non-practical' problems." The introduction to the book *Algebraic Thinking: Grades K–12* sets out the theoretical discussion on what is algebraic thinking and how it differs from algebra—that there is "an algebraic *way of thinking"* (Moses 1999, p. 3, original emphasis). Such thinking incorporates forming "generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful

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symbol system, and exploring the concepts pattern and functions" (Van de Walle and Bay-Williams 2014, p. 276). Kaput (2008) suggested that there were three strands of algebraic thinking, all infusing the key ideas of generalization and symbolization. The first strand involves the study of structures in the number system, including those used in arithmetic, described by Usiskin (1988) as generalized arithmetic. Strand two explores the study of patterns, relations, and functions. The third strand seeks to study how best to capture the information or to model the situation symbolically. It is only normal for the human mind to organize the huge amount of information present in the environment by constructing meaningful relations with the various inputs and outputs and capturing the information symbolically (Fosnot and Jacob 2010). Therefore, to be able to organize information, Mason (1996) argued that it was necessary for the mind to detect what remains the same and what is changing and to construct an appropriate rule; this thereby reduces the demands on human attention so that the mind can function economically.

Function is the study of relationships. It is acknowledged that it is one of the most important topics in secondary school mathematics because it provides a means for "thinking quantitatively about real world phenomena and a context for studying relationships and change" (NCTM 2010, p. 7). Authors of the Comprehensive School Mathematics Program (1975) argued that functions should be used as a vehicle through which letters as variables and algebra are introduced. However, studies show that teachers and students alike are challenged by the concept and definition of function and its attending symbolism (Dubinsky and Harel 1992).

When functions are first introduced at the lower grades, the focus is on the covariation between the inputs and the outputs, and the rule underpinning a particular set of inputs and outputs. For example, when asked to look at this set of inputs and outputs (2, 7), (1, 6), (4, 9) (3, ?), (?, 10), and the question of "the rule is ___?", a first-grade child can fill in the missing numbers and state the rule as "+5". However, after the introduction of letters as variables, the child may complete the task and offer the general rule: $n + 5$. Function-machine tasks are instances of "seeing a generality through the particular and seeing the particular in the general" (Mason 1996, p. 65). That is, (2, 7) is a specific example of the rule $n + 5$. To scaffold the development of the concept of function, the *Navigating through Algebra* series (NCTM 2001) provides developmental activities where function is first introduced to kindergartners through very concrete means and eventually presented with formalized representations in grades 9–12. Willoughby (1999) introduced the concept of the function machine to kindergartners by getting them to guess how many sticks would be produced by a box that was designed to act as a function machine. There was a child hidden in the function-machine box who had been asked to use the rule of "adding 2" to produce the relevant outputs with specific inputs. By studying a number of inputs and the corresponding outputs, the kindergartners in Willoughby's class were able to predict the outputs after a number of examples; they were later able to figure out the rules defining the activity.

Willoughby continued to increase the complexity of the tasks. The children worked with composite functions and also with inputs and outputs using letters as variables. Willoughby concluded that "all children can understand abstract but important concepts, such as function, if the concepts are developed first from concrete activities and gradually abstracted over a long period of time" (p. 200). It is possible to provide such scaffolding because the early activities of function encourage children to look for relations that are located within simple arithmetic activities—activities that develop notions of algebraic thinking or algebraic reasoning. Schliemann et al. (2007) have provided empirical data to support all these theories of algebraic thinking. Their work has shown how the children who engaged with the mathematical tasks not only focused on computation but also were encouraged to think about relations and functional dependencies. The children were able to reason about the four binary operations as functions and work with mathematical generalizations before they learned formal algebraic notation.

7.2 The Singapore Situation

The Singapore primary mathematics curriculum places a heavy emphasis on (i) understanding of patterns, relations, and functions, and (ii) representing and analyzing mathematical situations and structures using algebraic symbols (Cai et al. 2005). Although the functional approach is developed across the primary mathematics curriculum, it is done on an ad hoc basis emphasizing the 'doing-undoing' process and in conjunction with the teaching of the four operations. For example, the teaching notes provided in the Primary Two teachers' guide (TG2A 1995) used the illustration in Fig. 7.1 to emphasize the 'doing-undoing' process to highlight the relationship between addition and subtraction and between multiplication and division. There was no emphasis on the construction of the forward or predictive rule that would encourage children to look for relationships between the input and output if they were given a set of items such as [(4, 7), (2, 5), (1, 4), (5, ?), (?, 12)]. In local textbooks and workbooks, function-machine tasks are offered as practice items (Ng 2004).

Fig. 7.1 Doing-undoing process showing the relationship between addition and subtraction and between multiplication and division

7.2.1 The Age and Individual Differences Study

In 2005, the longitudinal study *Age and Individual Differences in Mathematical Abilities: From Kindergarten to Secondary Schools* (henceforth Age and Individual Differences Study) examined the relationships amongst cognitive abilities, socio-motivational beliefs, and mathematical performance of Singapore children from kindergarten to secondary schools (see Lee et al. 2009, 2011, 2012, 2017). Although there is a wealth of information on children's learning and use of mathematical patterns (e.g., Nathan and Kim 2007; Orton and Orton 1999; Rittle-Johnson et al. 2013; Walkowiak 2014), there is a dearth of empirical studies on the relation between individual differences and knowledge of numbers, algebraic reasoning tasks, and solution of algebra word problems.

Children of the Age and Individual Differences Study were enrolled in primary schools that typically serve families from low to middle socioeconomic backgrounds. Children were tested annually for 4 years, once per year. At each time point, children were administered a battery of executive functioning tasks (see Lee et al. 2013), as well as the Knowledge of Numerical Operations test, which included items related to numerical and arithmetic proficiency. This test used the standardized Wechsler Individual Achievement Test-II (WIAT) (Wechsler 2001), which provided a measure of mathematics achievement that could be put on a common scale, with raw scores that could be compared across the grades. The questions were from a wide range of mathematics domains. With the WIAT, children from all grades progress as far in the test as their abilities allowed, which permits measures of individual differences in the questions attempted at each grade.

Children at each grade were tested with curriculum specific mathematics tasks. These included arithmetic word problems for lower primary grades (Primary One to Primary Three) and algebraic word problems for upper grades (Primary Four to Six). Completing number patterns and function-machine tasks were designed for children across all grades. Because this was a longitudinal study, all the curriculum mathematics tasks were grade specific, reflecting the mathematics content for each grade and the complexity of that particular grade.

One of the findings from the Age and Individual Differences Study that was significant for the research to be reported in this chapter was that the data related to the paper-and-pencil instrument that measured performance on the function-machine tasks for all grades showed that children at each grade had difficulties with such tasks. Given a set of inputs and their corresponding outputs, function-machine tasks require participants to provide the output for a given input, the input given the output, and to state the rule that undergirds that particular function machine. Some of the data for Primary One and Primary Five children are represented in Fig. 7.2. Each child was given an overall numerical score and the descriptive data showed that more participants were performing below the mean (Primary One: mean = 19.01, Std. Dev. = 10.94, N = 122; Primary Five: mean = 43.03, Std. Dev. = 20.58, N = 101).

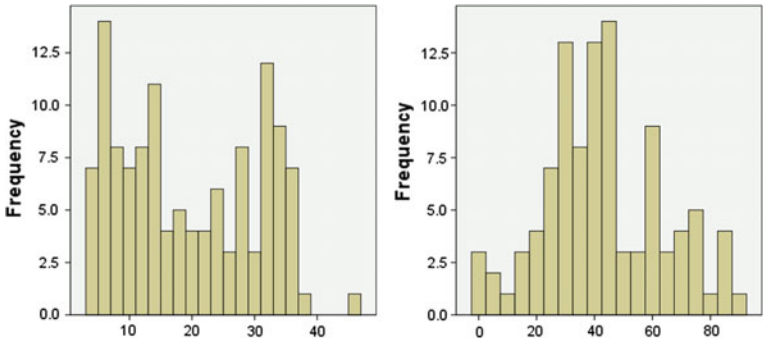

Fig. 7.2 Scores and their frequency for Primary One (left side) and Primary Five (right side) children on the function-machine tasks

The absence of qualitative data within the Age and Individual Differences Study meant that it was not possible to explain how the children (i) abstracted the rule that governed a given function, (ii) were convinced that their rule was correct, (iii) wrote the predictive rule, and (iv) worked out the input given the output. Furthermore it was not possible to explain why some children in the upper grades (i) succeeded with explicit functions involving an additive predictive rule (e.g., *input* ± *constant* = *output*) but not implicit functions (e.g., *input* ± *output* = *constant*) and why Primary Five children were challenged when the letter *n* was used as an input. Thus to gain insight into how these children engaged with the range of function-machine tasks, it was necessary to interview primary children at each level and listen to their explanations as they worked their way through the different function-machine tasks. As will be described shortly in Sect. 7.3, the resulting interview study reports on how primary children across the six primary grades constructed the predictive rule and what difficulties they encountered with such tasks. The empirical evidence gathered from the interviews is intended to have implications regarding the strategies teachers could try when they introduce such tasks to primary grade children.

7.2.2 The Spiral Nature of the Singapore Primary Mathematics Curriculum

Children's ability to engage with mathematical and algebraic type tasks does not happen in a vacuum. In mathematics, prior achievement is one of the best predictors of later achievement (Jordan et al. 2009) and is particularly important for mathematics where the learning of skills progresses in a hierarchical manner (Aunola et al. 2004). The work of Fuchs et al. (2006) highlights how lower mathematics skills could determine performance in mathematics at a higher level. Thus, the provision for a spiral structure in a primary mathematics curriculum can help to understand why children at each level might be more able to engage with function-machine tasks designed for that level. While the Singapore Primary Mathematics Curriculum does not include function-machine tasks, it does develop the concept of number at two levels: the structure of numbers and operations with numbers.

Structure and the nature of numbers: The Primary One syllabus partitions numbers up to one hundred into components of tens and ones. At each subsequent primary grade, the number size increases: one thousand at Primary Two and ten million at Primary Five—at each grade the number size increasing by a multiple of ten. The concept of fraction as an object is developed at Primary Four. The concept of decimals is also introduced and developed at Primary Four. The use of letters as variables is introduced only at Primary Six (Curriculum Planning & Development Division 2006, 2012).

Operations with numbers: By the end of Primary One, the intended curriculum recommends that children develop facility with addition up to 18. The two, three, four, five, and ten multiplication tables are formally introduced at Primary Two and those of six, seven, eight, and nine at Primary Three. Division involving the related multiplication tables is introduced at Primary Two and continued in Primary Three. Fraction as division, $a \div b = \frac{a}{b}$, is formally introduced at Primary Five. Likewise, multiplication involving fractions is developed at Primary Five, and Primary Six addresses division with proper fractions. Order of operations is introduced at Primary Five. With letters as variables in Primary Six, the objective is to work with notations of the form $a \pm 3$, $a \times 3 = 3a$, $a \div 3 = \frac{a}{3}$, including the simplification of linear expressions involving one unknown and evaluating simple linear expressions by substitution.

7.2.3 Structure of Items Underpinning the Design of the Function-Machine Tasks

Figure 7.3 provides the matrix that guided the construction of the function-machine tasks across the six grades within the Age and Individual Differences Study.

The function-machine tasks were constructed to reflect the progression of the primary syllabus and hence the items were increasingly more difficult. With the aim of controlling for the complexity of the tasks across the years, we drew upon Collis (1975) who theorized that the nature of task elements could have a marked effect on the facility of a given task. Items involving small numbers are easier than those constructed using larger numbers, which are easier than those using letters, in this order. This seemed reasonable because, when it comes to computation, even savants are affected by number size: “Great calculators struggle with great calculations like the rest of us” (Dehaene 2011, p. 151). As well, two reasons prompted us to include within the longitudinal study tasks involving implicit functions: (1) to increase the structural complexity of the function-machine tasks, and (2) to challenge the older children with tasks that are not structurally repetitive.

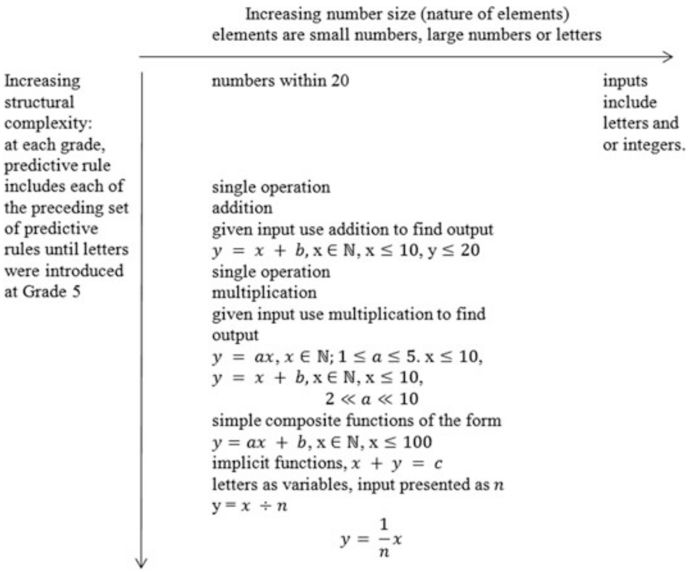

Fig. 7.3 The framework that guided that construction of the function machine tasks across the grades

7.2.4 The Function Tasks

In the Age and Individual Differences Study, in any one year, the function-machine instrument comprised three sub-parts, each part having five items. Part (a) was constructed using the more difficult items from the previous year; Part (b) items were representative of the core content in the present year's curriculum; Part (c) items were drawn from the easier items of the subsequent year's curriculum. The items were presented in this order. This three-part structure meant that it was possible to identify the state of the children's current knowledge. Those children who could solve all three parts indicated that they were performing beyond the current curriculum, while those who could complete Parts (a) and (b) were competent with the mathematical knowledge that was reflective of the curriculum for their grade level. For example, when children engaged with the function-machine tasks from Part (a), they may have identified the function rule $f(x) = x + a$, with task elements being particular examples of the general rule $f(x) = x + a$. However, when they moved to Part (b), they would have been confronted with elements representing the function rule $f(x) = ax$. The children could still apply the $f(x) = x + a$ rule, but when they tried to take into account all the (input, output) pairs, they might have realized that these (input, output) pairs are instances of particular examples of a different function rule, in this case $f(x) = ax$.

7.3 The Function-Machine Study

7.3.1 Participants

A total of 60 children were interviewed in the Function-Machine Study, ten children from each grade level. They attended the same student care-center that served to provide before- and after-school care to one neighborhood school, a non-selective school serving the local community. These children were considered a good sample because they were taught by different mathematics teachers. Hence their responses were more likely to reflect how these children engaged with the function-machine tasks and less so the pedagogy of the respective teachers. Furthermore there was less disruption to these children's curriculum time as they were interviewed before the start of or at the end of their school day. The participants were identified according to their levels and name code. Thus, P1S referred to a Primary One child with the code S.

7.3.2 Interviews

The interviews were not conducted in any chronological order starting with Primary One and working up to Primary Six. Rather the order was random, based on the advice of the administrator of the student care-center. It took a week to complete each grade level as the administrator's advice was to conduct the interviews two hours before the start of the afternoon session and two hours after the end of the morning session. Thus, it was possible to interview at most three children each day. All interviews were conducted in English, the medium of instruction in all Singapore schools.

In the Age and Individual Differences Study, the administrator had first read the general instructions in Fig. 7.4 before the administration of the tests proper, with all children receiving the same presentation. Hence, in the Function-Machine Study, the same set of protocols was applied for each child before commencing work on the function-machine tasks proper.

However, in the Function-Machine Study, because I wanted to learn how children made sense of the co-variation between the inputs and the outputs and whether they could state the relationship between them, the terms *input* and *output* were used from the outset of the interview. Figure 7.5 provides the details of the preliminary interactions with all the participants of the study. I feel it is necessary to provide this level of detail as a different set of interactions would result in different responses from the children.

After the introduction, the children were presented with the tasks. Figure 7.6 shows some examples of function-machine tasks, with the two on top exemplifying Primary One tasks and the two on the bottom, Primary Four (and above) tasks.

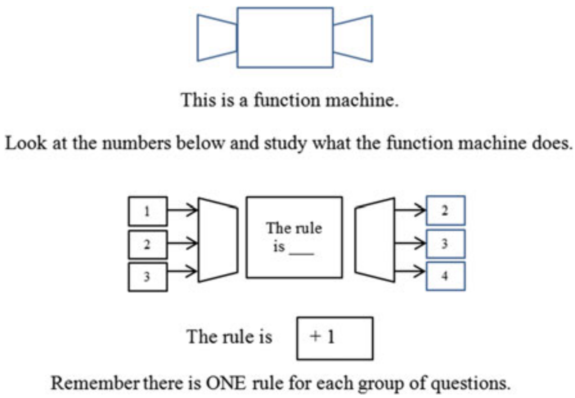

Fig. 7.4 All the children saw this function-machine presentation. The administrator in the main study read the instructions to the children as they were tested in their classes

Interviewer	Interviewer's intent and Interviewer Action (IA) and Child's Action (CA)
I pointed to the number 1 to the left of the function-machine. This number 1 is known as the input.	Intent: Gave the child a chance to engage with the input and output. The act of naming the numbers as input and output meant that the children had access to words that would enable them to identify the objects and talk about the activity of the function-machine. Otherwise, children would say "this one, that one," and it would not be clear which object they meant. IA: I then wrote the word input for the child. CA: The child repeats the word input. IA: Pointing to the number 2, the number 2 is the output. Writes the word output. CA: The child repeats the word output.
Can you name this number? Is it input or output?	Intent: This set of interactions provided children with the opportunity to test whether they could differentiate between the input-output relationships. IA: I pointed to the second input, in this case the number 2. CR: The pupil offered a response.
Is this number an input or output?	Intent: To provide further opportunities for these terms to become fixed in the minds of the children (Gladwell 2000) IA: I pointed to the number 3. CR: The pupil offered a response. IA: Is this number an input or output? I pointed to the number 4. IA: Is this number an input or output? I pointed to the last number in the sequence.
I then point to +1 in "the rule is"	Intent: To help children understand the demands of the task. IA: This machine uses the rule +1 to change the input into the output. I reiterated the point that there is only one rule for each group of questions. I point to the statement.

Fig. 7.5 Introductory interview protocol used with children

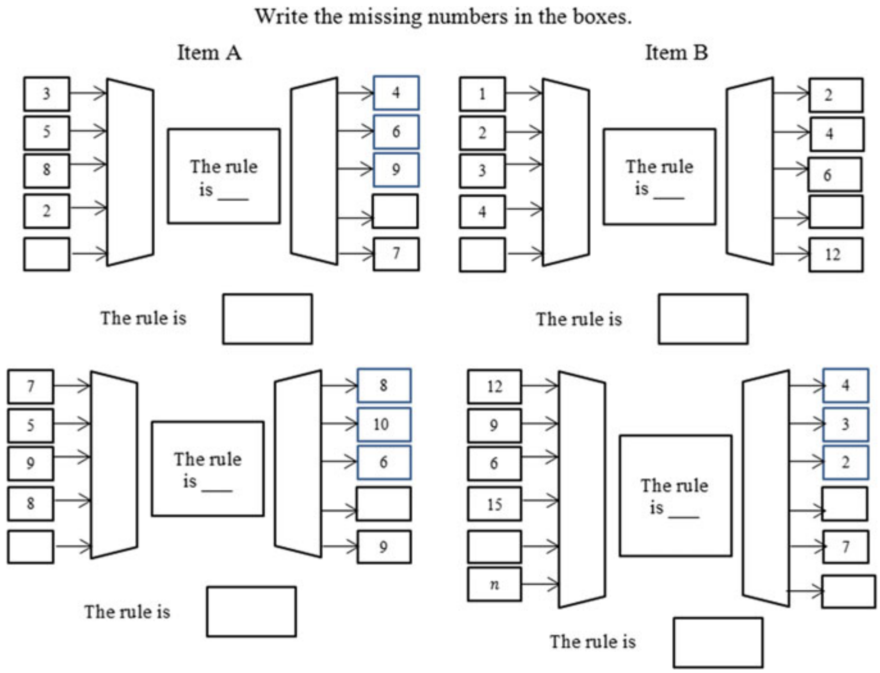

Fig. 7.6 Examples of variations of function-machine tasks presented to children across the grades. Top left simple explicit function of the form $f(x) = x + a$, $f(x) = ax$, at top right. Bottom left simple implicit function where *input* + *output* = *k*; *bottom right* $f(x) = \frac{a}{b}x$ with letter *n* as variable

Each of the children, after perusing a task, filled in the missing numbers. I used the following set of semi-structured questions when interviewing the children:

- What is the rule?
- How do you know the rule is correct?
- Write down the rule.
- How do you check the output for this input?
- How do you check the input with this output?
- Can you use the words input and output to show the relationship between the input and the output?
- Is there another way to write the rule for the machine? (if the child wrote *input* + *a* = *output*). This was an attempt to get the child to consider symmetric equivalence.
- Should a Primary Six child write $n \div 3$ as a predictive rule, the prompt "Is there another way to write this expression" was offered? This was a prompt to determine whether the child saw the equivalence between $n \div 3$ as the product of *n* and its reciprocal: $n \times \frac{1}{3}$.

There were fewer items in the lower grades. Thus, the duration of the interviews varied, from approximately 15 min for Primary One and an hour for Primary Six. However, if the children were seen to be struggling with a task, I would end the interview. As well, the interview was discontinued at any time a child expressed a wish to stop, but no child chose to do so. Note that all the annotations in the examples presented in the figures are the written work of the children themselves.

7.4 Challenges Faced by Children in Constructing the Predictive Rule

The performance of children with the function-machine tasks is discussed according to two levels: lower primary (Primary One to Three) and upper primary (Primary Four to Six). Because the lower primary mathematics syllabi focus on developing competency with operating with numbers, the children's engagement with the function-machine tasks reflected the ways in which lower primary grade children worked with the four binary operations. The upper primary grades expanded to include work with multiple operations, fractions, and variables.

7.4.1 Lower Primary Grades

7.4.1.1 Understanding the Demand of the Function Machine

Children in the lower primary grades (Grades 1–3) had difficulties understanding what they were expected to do with the function machine although the administration notes were read to them and I had demonstrated the workings of the function machine. Although the children seemed to understand that one machine had only one rule, some children understood this information differently. For example, for Q6, a Primary One question, the predictive rule should be *input plus* 1 = *output*. However P1S, who had a reasonably good command of her addition number facts, assumed the predictive rule was based only on the first input-output pair (Fig. 7.7, left side). She did not test the predictive rule with the other items in the set. Only when she was asked if the first predictive rule would apply to other items in the set (5, 6), (8, 9) did she realize that she had to test the first rule that came to her with the remaining items in the set. Once she knew what to do, she was able to apply this knowledge to find the output of the given input for each set of function-machine questions.

When asked what was the rule to change the input to the corresponding output, many Primary One children were able to say add 1 (for the case of Q6); however, they wrote 1 in the box "the rule is _____", excluding the operation sign from the

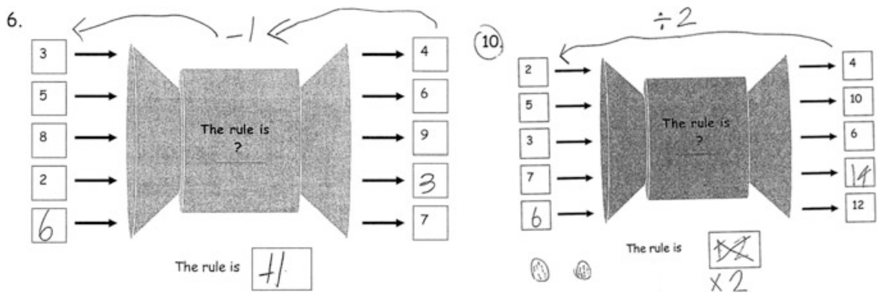

Fig. 7.7 The work of two students, P1S on the left, and P2A on the right. Arrows show how children indicated the relationship between the output and the input

rule. When asked how the machine would know what to do with the input number without the operation sign, the children promptly wrote the plus sign in front of the number 1. It was unnecessary to repeat this prompt for the remaining function questions. This is further evidence that these young children may not have understood the expectations of the function-machine questions even though an example had been provided to help them. Therefore in conducting such interviews with young children, it was important to ensure that children knew precisely what was expected of them.

Children at Primary Two were also struggling with interpreting the demands of the function-machine question. It was necessary to show P2A that there was a relationship between the input and the output by pointing to the input and the output and asking her to interpret the relationship between the two:

SFN (Interviewer): *The input is 3 and the output is 6* (*action first pointing to 3 and then to 6*) *Here the input is 5 and the output is 8.*
P2A took time to mull over the task and wrote down 3 without attending to the plus sign in the "The rule is ____" space.
SFN: *Look at the machine. What did the machine do to the input numbers?*
With this prompt, P2A wrote the addition sign in front of 3.

For every specific input, P2A used her fingers to check the accuracy of every output. It was noteworthy that, although initially she had difficulty interpreting the demands of the function task, once she had completed figuring out the rule for the addition function machines, she had no problems with multiplication either. For example, when she realized she had erroneously written an addition rule instead of a multiplication rule in "The rule is ____" box, she asked: "If write wrongly, then how?" She was advised that she could cancel the wrong rule and write the correct rule in the space provided, which she did (see Fig. 7.7, right side). These children's difficulties in understanding the expectations of the function machine may have contributed to their performing below the mean as presented in Fig. 7.2.

7.4.1.2 Competency with Number Facts Is Crucial to Identifying Predictive Rule

The ten Primary One children were able to use their knowledge of addition number facts to complete the function-machine task of the form $f(x) = x + a$. Although they were able to recall the number facts at will, they used their fingers to check the accuracy of the output by counting on their fingers. The children who used their fingers to check the accuracy of the answers often did not wish to show that they were using their fingers to support them in their counting. They hid their fingers under the table and moved their fingers to check the relationship between each input and output. They used the counting-on strategy, by counting on from the larger number, which they kept in mind, and counted-on using their fingers. Initially, the children did not use the relationship between addition and subtraction to help them find the input given the output. Their behavior suggested that they were incrementing from what they thought was the input and using the rule they had computed to confirm that the input was correct for the given output.

Most children knew that each function machine was governed by one rule. This was deduced when children moved from one function-machine task to another. P1YK moved seamlessly from addition to multiplication. In contrast, there were some children who, even though they could eventually find the multiplication rule, interpreted the first input–output pair as one of addition. However, the children who were confident with their number facts and the nature of the function machine behaved differently. They inhibited the rule that governed the previous function machine, studied the next function machine in its entirety, updated the information, and came up with the new function rule. P1YK was just such a pupil. He studied an entire function-machine task before quickly coming up with the predictive rule $\times 3$. He was extremely confident about his number facts. In fact, he refused to use arrows to show how he arrived at the input given the output. When asked how he was able to come up with the rules so quickly, P1YK explained that he was very good with numbers because each weekend his grandmother accompanied him to private mathematics classes where he was taught how to operate with numbers up to 400.

Those children who had not yet learned multiplication—an operation not taught until Primary Two—would initially apply an addition rule to the first input–output pair, but they stopped and said the rule (meaning addition) did not work for that question. Hence, they could not continue with the task because the function rule $f(x) = x + a$ could not be applied to all the input-output pairs where the rule was $f(x) = ax$.

The method the Primary Two children used to check their multiplication facts suggest that they knew that multiplication was repeated addition. For example, for Q10 (see Q10 displayed in Fig. 7.9), although those children were happy to write down the rule as $\times 2$, they tested their rule by using their fingers or tapping their pencil on the table, to ascertain that $input \times 2 = output$ for every input and its corresponding output. For example (2, 4) was reflected by tap, tap … tap, tap; (5, 10) was reflected by tap, tap, tap, tap, tap… tap, tap, tap, tap, tap, tap.

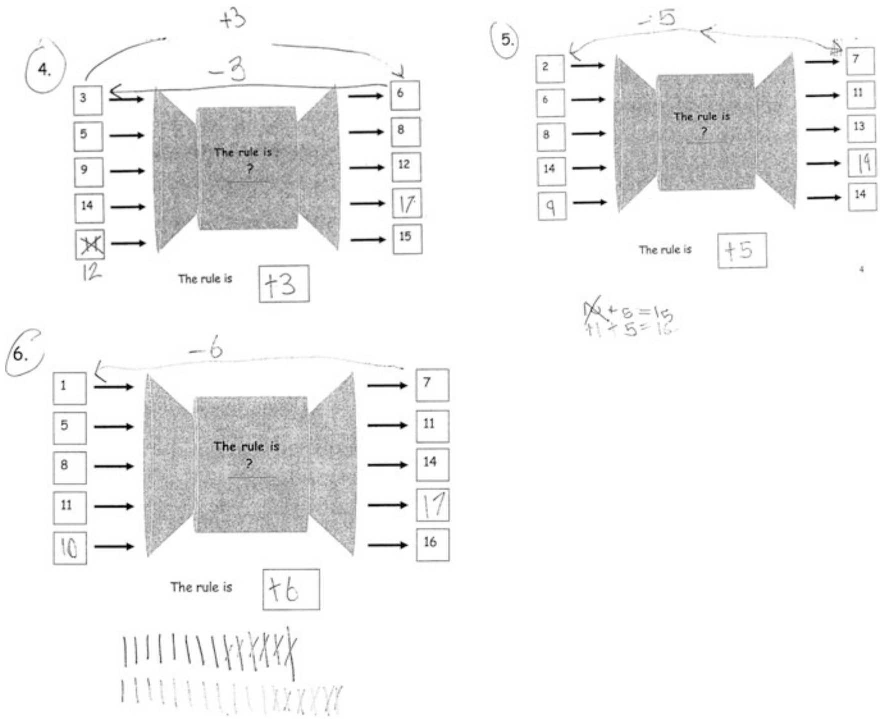

Fig. 7.8 P2A working with addition and different ways to figure out the input given the output

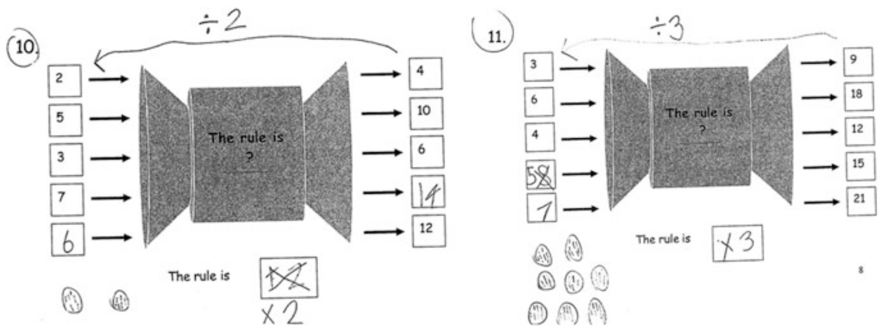

Fig. 7.9 P2A working with multiplication

Figuring the input given the output challenged children. P2A checked by counting down with her fingers (Q4), making a list (Q5), or finally using tally marks and crossing out +6 (Q6) (see Fig. 7.8).

To work out the input given the output of 12 (Fig. 7.9, left side), P2A drew rings and then made a tally mark in each ring, one at a time until she reached 12. Then she wrote 6 in the input space. When asked if $\times 2$ changes the input to the output and what she would write to move the output to the input, she wrote $\div 2$.

P2A adopted a guess and check strategy for Q11 (Fig. 7.9, right side). She had worked out that the rule was $\times 3$, but she then used an elimination strategy to work out the input for 15. Again she used tally marks to support her work. After she had written 8 in the input space, she drew two rings and she filled each of these rings with 8 marks. But when she remembered that the predictive rule was $\times 3$, she drew three rings and she put tally marks in each of them until she reached 15. Then she cancelled out 8 and wrote 5. She repeated the same process with the output 21. She drew 3 rings and distributed one tally mark to each ring until she reached 21 (one for you, one for you, one for you, …).

7.4.1.3 Relationships Between Operations: The Case of Addition and Subtraction

It is best not to assume that, if children constructed the predictive rule where they expressed the relationship between the input and the output as $input + a = output$, then they would construct the relationship of $output - a = input$. As discussed in the above section, to find the input given the output, one would hypothesize that these children would treat it as $output - a = input$. However, the behavior of these children suggested otherwise. They applied their knowledge of number bonds to find the input given the output. For example, in Q8 where the rule involved adding 3 to the input, and they had to supply the input for an output of 9, they asked, “what add 3 to give 9.” They saw it as: $? + 3 = 9$. Because $6 + 3 = 9$, hence, the input should be 6. To help children see that the relationship between the two operations—that is, subtraction is the inverse of addition, and that it was acceptable to use this relationship of $output - a = input$—it was necessary to ask children whether it was possible to use this relationship to arrive at the input. It was best not to assume that children saw the relationships immediately. It was necessary to draw their attention to the relationships between the output, the input, and the function rule. For example, it was necessary to test the “output–input” relationship by repeatedly asking the children: how would you go from 5 to 2. After they responded with five take away 3, the same question was repeated with the remaining outputs of 4 and 7. Four take away three is 1, seven take away 3 is 4. Then these children were asked if they could draw lines to show how to move from the output to the input, and only when they could do so was it possible to hesitate a guess that perhaps these children had constructed some relationship between addition and subtraction. At all times it was necessary to ask the children to point to each number in turn so that the researcher and the pupil were focused on the same numbers, be they inputs or outputs. This ensured that there was precision in perception and precision in expression. This form of focusing of attention ensured that there was synchronicity between the administrator of the test and the participant (Klein 1996).

7.4.1.4 Structural Equivalence

It is important for learners to know the relationships of equality, for example, the symmetric property of equality: "If $A = B$, then $B = A$" (Davydov 1962, p. 31). Such knowledge sets the foundation for further work, such as $A + e = B + e$ and $B + e = A + e$. However, structural relationships did not come naturally to children in this study. For instance, some of the Primary Two children were unable to provide the output for (7, ___) when the functional rule was $\times 3$, although they had learned the multiplication table for 3. This was because the multiplication table for 7 was taught only at Primary Three.

To ascertain whether these children were able to use language to state the general form of the predictive rule, Primary Three children were asked whether it was possible to state the relationship between the input and the output as: $input + a = output$. I found that it was necessary to mediate between the children and the tasks:

SFN: *Can we use words to show how the machine worked with the inputs and the outputs? Can we say* $input + 2 = output$? (Pointing to each input and its corresponding output).
Or can we write $output + 2 = input$?
I then wrote down these two relationships because it was difficult for the children to make sense of the words, as this was their first encounter with such terms. It was unfair to expect these children make sense of what was said to them. Also variations of the rule were offered to these children to assess whether they were ready to accept alternative ways of writing the forward rule.

The children did not reject these presentations, but would mull over the possibility of such relationships. To ensure that we both understood each other, I asked each pupil to write down the relationships. Of the 10 Primary Three, 6 preferred the *literal form*, that is, input + a = output. Figure 7.10 shows the work of one such child. The two examples on the left side of Fig. 7.10 show how this child prefers the literal form as it follows the left-to-right sequential order of the transformation of the input by the function machine into the output for the function $f(x) = x + b$. The example to the right shows the corresponding literal form for the function $f(x) = ax$.

Three of the ten Primary Three children preferred the non-literal form (with left and right sides of the rule switched) for the function $f(x) = ax$, that is, $output = input \times a$ (see Fig. 7.11).

Only one Primary Three child accepted both the literal and non-literal forms for the functions $f(x) = x + b$ and $f(x) = ax$ (see Fig. 7.12).

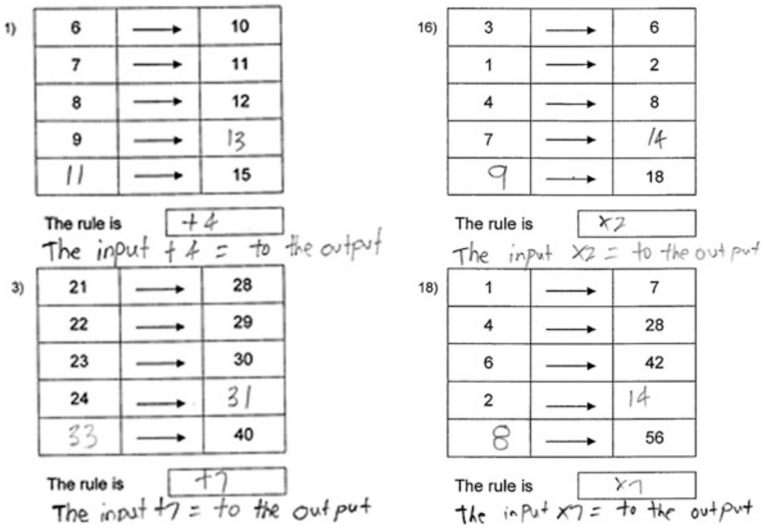

Fig. 7.10 On the left, the preferred literal order for the relationship for $f(x) = x + b$; on the right, for $f(x) = ax$ (by Primary Three children)

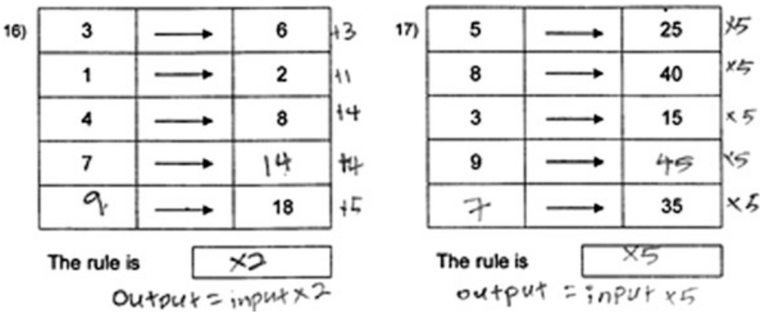

Fig. 7.11 A Primary Three child's written response for $f(x) = ax$, illustrating the non-literal form of the relationship

7.4.2 *Upper Primary Grades*

7.4.2.1 Structural Equivalence

Although the order of operations is not taught before Primary Five, Primary Four children were able to construct the predictive rule for 2-operation functions. Of those who could provide the predictive rule for such functions, four of the 10

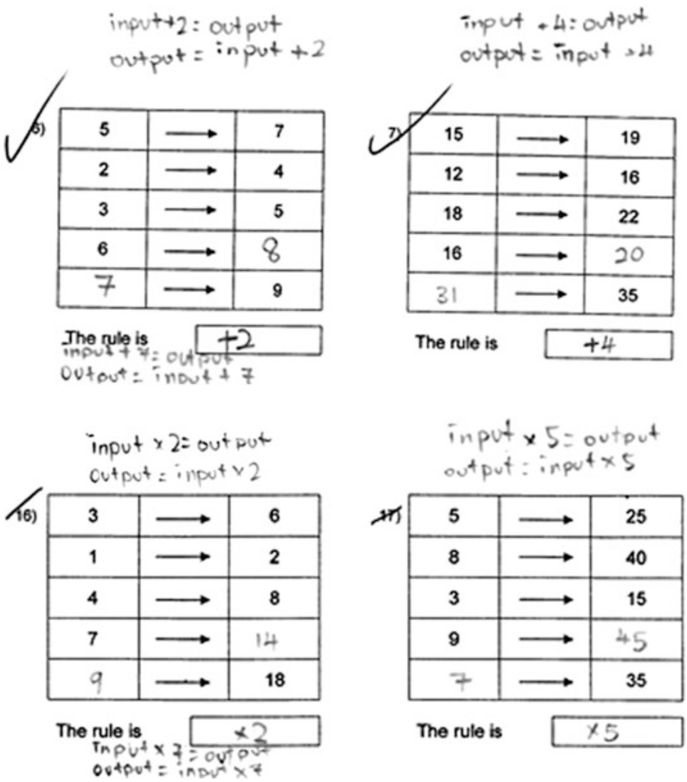

Fig. 7.12 One Primary Three child's acceptance of both the literal and non-literal forms of equivalence

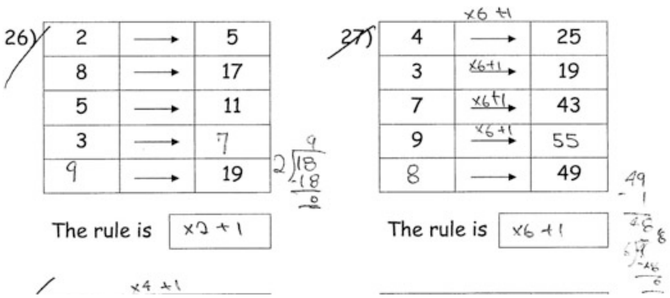

Fig. 7.13 Guess-and-check strategy used by a Primary Four child to construct the predictive rule for 2-operation functions

children applied a guess-and-check technique to help them work backwards to figure out the input for the given output. For Q26 (see Fig. 7.13), the thinking seems to be this: "18 divided by 2 is 9; if the output is 19, then the input must be

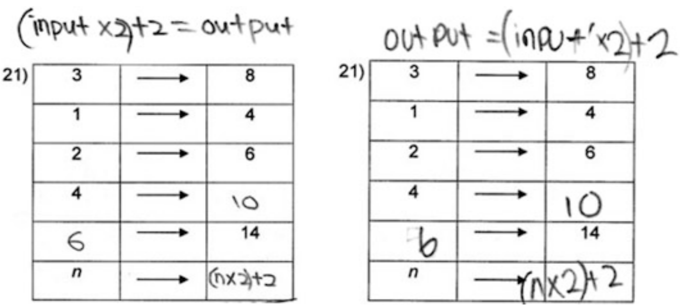

Fig. 7.14 The symmetric equivalences accepted by Primary Five children; note too the use of letters to represent the output

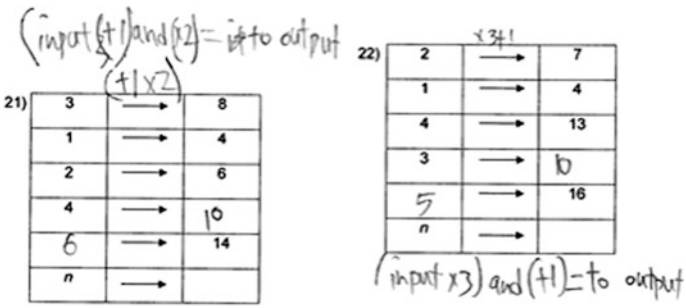

Fig. 7.15 Example of a Primary Five child's work where the rule, which is written in words, shows the use of the equal sign to announce the answer; note too that this child was unable to furnish an output for the input *n*

$2 \times 9 + 1$ as this gives a sum of 19." For Q27 (Fig. 7.13), knowledge of the 6-times table helped. The thinking was: "1 less than 49 is 48; when this is divided by 6, the quotient is 8." Hence, the input was 8. However, none of the Primary Four children wrote the expression: $input \times 2 + 1 = output$. Instead, they wrote the rule symbolically as $\times 2 + 1$ (as shown in Fig. 7.13).

Some Primary 5 children provided the predictive rule as: $(input \times a) + b = output$ (see Fig. 7.14, left side); others, the non-literal form: $output = (input \times a) + b$, (Fig. 7.14, right side).

Despite the noteworthy work involving structural equivalence, it was found that one Primary Five child treated the equal sign procedurally (cf., Kieran 1981, 1989), using it to announce the answer. His rule (see Fig. 7.15) could be read as: *input plus* 1 *and times* 2 *is equal to output*.

7.4.2.2 Checking the Accuracy of Their Predictive Rules

Children took their work seriously. Although their work suggests that they had command of their addition facts and multiplication tables, these children checked their work thoroughly, particularly the examples that involved multiplication and division. For example, although he was confident with his number facts, P5R insisted on checking the output obtained from a multiplicative rule by performing a long division with the multiplier of the predictive rule (see Fig. 7.16, upper half). Once he was sure of his multiplication-division correspondences, he completed the other function questions by recalling his number facts. Similarly, P5M (Fig. 7.16, lower half) meticulously checked the accuracy of his predictive rule with subsequent pairs of inputs and outputs: For a multiplicative rule (as in Q18), he divided a sample output by the multiplier of the predictive rule; for a predictive rule involving division (as in Q19), he checked his accuracy by multiplying a sample input by the reciprocal of the divisor of the predictive rule.

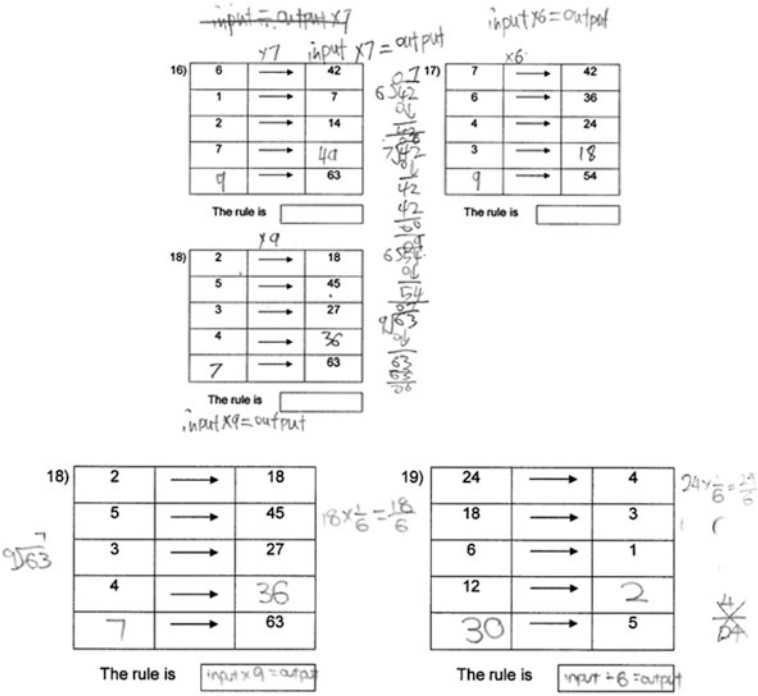

Fig. 7.16 The work of P5R, top set Q16–18, and P5M (bottom set Q18–19), showing how they checked the accuracy of their predictive rules

7.4.2.3 Working with Variables

Letters as variables are introduced formally at Primary Six. But as discussed in the design of the Function-Machine task, Part (c) items of the Primary Five Function-Machine instrument incorporated items where the letter n was used as an input. Such items were included in the instrument as a means to assess whether Primary Five children were ready to work with letters, at least as a generalized number (Küchemann 1981). All Primary Five children asked the same question or some variation of it: "what is n?" "I don't know what is n?" In response, I would say: "The letter n represents a number. It could be any number." It was interesting to note that those children who could work with 2-operation functions listened to my response, thought about it, and then wrote the output. Examples in Figs. 7.14 and 7.15 showed how some Primary Five children were able (and unable, respectively) to work with the letter n as an input. Perhaps those Primary Five children who were able to engage with the letter n as input had worked previously with the "model method" for solving algebraic type word problems, which may have helped them engage with n as an unknown unit. Readers are referred to Ng and Lee (2009) for a full discussion of the model method. The discussion in that paper may help readers understand why work with the model may be precursor to working with an unknown unit. However, Primary Six children did not have any problems with the letter n. They continued with each question and filled in the output as required for similar questions.

7.4.2.4 At the Top of the Spiral Curriculum

Only one of the ten Primary Six children could be said to have a good grasp of the various forms of the function machine. Child P6PQ was able to engage with symmetric equivalence for $f(x) = x + b$, $f(x) = ax$, and $f(x) = ax \pm b$; and with implicit functions, the reciprocal relationship where $x \div b = x \times \frac{1}{b}$, and letters as variables. Figures 7.17 and 7.18 provide some examples of P6PQ's work.

Compared to Q11 and Q12 (Fig. 7.17), Q13 and Q14 (Fig. 7.18) were more challenging as the predictive rule involved division, which could be represented by multiplication with the reciprocal of the divisor. Child P6PQ was still able to see the symmetric properties of equivalence. Furthermore, P6PQ saw multiplication as the inverse of division and applied that relationship to construct inverse rules that mapped the output to the input. Therefore, if $input \div 3 = output$, then $output \times 3 = input$. Furthermore, this pupil knew that division was related to multiplication with reciprocal: $input \div 3 = output$ was equivalent to $input \times \frac{1}{3} = output$. With 2-operation functions, Q21 and Q22, P6PQ was able to write the structural symmetric equivalences for these functions (see Fig. 7.19).

With 2-operation functions, P6HW (see Fig. 7.20) was more experimental with the predictive rule. For Q21, if $output = (input \times 2) + 1$, then $input = (output - 1) \div 2$. Although he had constructed the correct predictive rule for Q21,

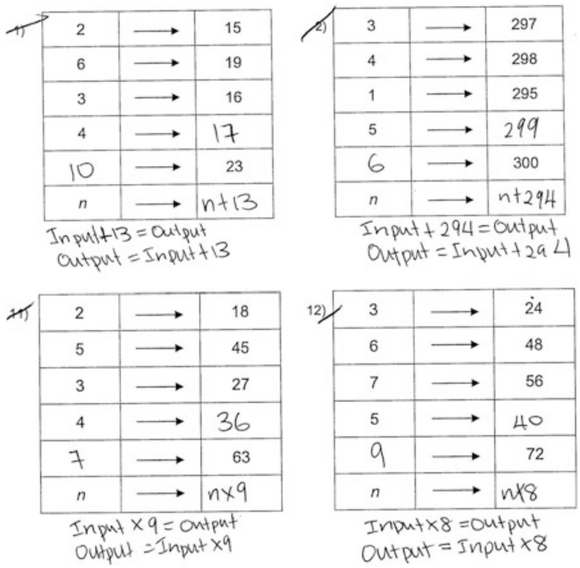

Fig. 7.17 With simple functions involving whole numbers, P6PQ was able to consider symmetric equivalence of both additive and multiplicative predictive rules

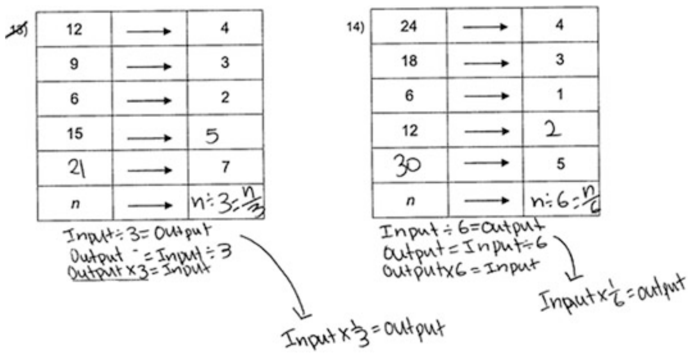

Fig. 7.18 P6PQ also demonstrated understanding of the reciprocal relationships between multiplication- and division-based predictive rules

the annotations for Q22 showed that P6HW did not assume the same structure for Q22. At first, P6HW tried out an additive structure ($2+5=7$; $1+3=4$; $4+9=13$—for the first three input-output pairs). At the same time P6HW kept a running tally in his head of the pattern to the addends he had calculated; and so, by the same reasoning, for the input of 3, add 7. P6HW had reordered the inputs so that they were not randomized as presented in Q22. However, because the difference for

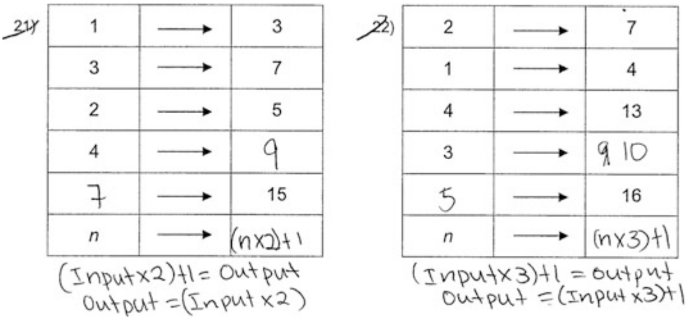

Fig. 7.19 P6PQ was able to write the structural symmetric equivalences for 2-operation functions

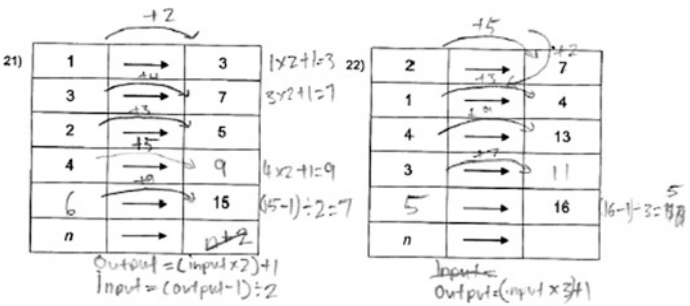

Fig. 7.20 Annotations for Q22 show how P6HW did not apply, by default, the 2-operation rule from Q21 to Q22

each input-output pair was not a constant, he decided against an additive rule and eventually wrote the correct predictive rule of $output = (input \times 3) + 1$. He was also able to find the correct input for the output 16 by carrying out the correct order of operations in reverse order. He was not, however, able to re-express his predictive rule for the case of the input being n.

7.5 Conclusion

The Function-Machine study was a component of the longitudinal Age and Individual Differences Study. The descriptive data from the Age and Individual Differences Study showed that more children were performing below the mean; however, it was not possible to provide explanations for this pattern within the data. Because the function-machine tasks were designed according to the curriculum, with each of the three parts having its specific role, it could be argued that children's work with the function-machine tasks was supported by the mathematics they were taught in the spiral curriculum. Fuchs et al. (2006) and Jordan et al. (2009) have argued that prior knowledge acquired at a lower level supports children's mathematical work at a higher level, and the evidence provided by the interview data from the Function-Machine study gave credence to support the extant literature. One could hypothesize that the spiral structure of the Singapore Mathematics Curriculum helped children expand their knowledge of the four binary operations, not unlike the work of Schliemann et al. (2007) and Willoughby (1999), and even explains why children were performing below the mean. If children had the mathematical knowledge relevant to Parts (a) and (b) of the task set, then they were likely to complete tasks in these parts because they could see how the (input, output) pairs were specific examples of a particular predictive rule. However, if they did not have the mathematical knowledge demanded by Part (c), they were unlikely to respond to items in Part (c).

The only way to access children's epistemology of the function-machine tasks was to talk with them and find out how they worked out the co-variation of the inputs and the outputs, how they constructed the predictive rule, and whether they would be comfortable in considering symmetric equivalences and using letters as variables in the predictive rule. Overall about eight out of ten children at each primary grade were able to complete the questions used in the study. These children had sound knowledge of their number facts for the four binary operations. Those who failed to do so were hampered by poor knowledge of number facts as they had to rely on their fingers to check their addition or multiplication facts. The corpus of data shows that these children's engagement with the function-machine tasks maps the mathematics of the spiral curriculum. If they have the necessary knowledge of number facts related to the four operations, then through a series of questions and answers those children in the lower primary were found to be able to cope with simple functions of the form $\mathrm{f}(x) = x \pm b, f(x) = ax$. Those in the upper grades were able to work with two-operation functions of the form $f(x) = ax + b$. Primary Six children were able to construct predictive rules involving the letter n for simple and 2-operation functions. However, one cannot be sure of the soundness of these children's understanding of letters as variables. Do they treat the letter n as a specific unknown or as a generalized number (Küchemann 1981)? The data suggest that those who could operate with letter n were treating them as generalized unknowns in that they could provide predictive rules appropriately for $f(x) = ax$ and $f(x) = ax \pm b$, where $\mathrm{a} \in \mathbb{R}$.

A few were willing to consider symmetric equivalences, that is, they were willing to accept the literal representation of the predictive rule of $input \pm a = output$ and its non-literal form of $output = input \pm a$, and likewise for other forms of the functions discussed. Where division and fractions were concerned, it was very rare to see Primary Six children come up with the equivalent form of the predictive rule $f(x) = x \div a$ as $f(x) = x \times \frac{1}{a}$.

The annotations provided by the children who participated in this study showed that they have never worked with such tasks in the formal curriculum. In fact the children's annotations showed them to be thoughtful and reflective learners in the way that they responded to my questions before they filled in the outputs, inputs, and the predictive rule. The lower primary children took about thirty minutes to complete the interviews and the upper primary children, at least an hour. The children in the upper primary grades needed more time as they had to consider a greater variety of functions tasks. Often children, after a prompt, would go quiet and remained so for a minute or so. I was left wondering whether I had lost them. But then they would complete the question, answering my prompt. I learned to respect these periods of silence without plying them with more questions.

Implicit functions proved to be challenging even to the competent Primary Six children (those cited in this study). Even with prompts to focus their attention on the inputs and the outputs, these children were challenged to see the relationship between the input, the output, that is, regardless of the inputs and the outputs, when the predictive rule involves the single operation of addition or subtraction, the sum (or difference) of each pair of inputs and outputs is a constant. Many prompts were needed before the more able children saw the relationship that *input* ± *output* = *constant.* But then even adults can be challenged by such tasks. This suggests that it is important to focus attention on explicit *and* implicit functions so that children cultivate flexibility in the way that they analyze relationships and not be fixated on explicit functions.

Words like inputs, outputs, and reciprocals were new terms to all these children. They are not found in any of the instructional materials. But the children's reactions to the introduction of the terms showed their pleasure in learning new words and using them appropriately. Rather than using generic and non-specific words like "this one" and "that one" to direct the attention of the interviewer, the use of specific terms seemed to give the children a sense that doing mathematics was more than getting the right answer. Using the right words helps.

In conclusion, according to the literature, to be engaged in algebraic thinking means to attend to relationships among the four binary operations, looking for and generalizing relationships inherent within the elements presented in the mathematical tasks, be they simple or complex. Findings from the current study show that, with an appropriate mathematical diet comprising varied function tasks, children are able to engage in algebraic thinking involving seeing the general in the particular and articulating and using appropriate symbolism to capture the general rule.

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Chapter 8
Algebraic Understanding of Equalities in Primary Classes

Ralph Schwarzkopf, Marcus Nührenbörger and Carolin Mayer

Abstract Learning of mathematics in primary classes should not be reduced to learning algorithms and routines or procedures. It is more important to foster children's thinking and reasoning of meaningful relations between objects and operations. In addition, from our point of view, the algebraic concepts should be more relevant than the algebraic symbols. In this sense, the conceptual understanding of equalities seems to be an essential basis for a flexible and sophisticated understanding of equations. Hence, discussing and explaining equalities should play a prominent role in the teaching and learning of mathematics from the beginning of primary school. In our design study we develop learning opportunities for primary school children that involve comparing terms and tasks with a view toward the underlying mathematical structures. In this chapter we discuss our theoretical background and some results of our video-based qualitative analysis of learning situations in the area of reasoning about equalities.

Keywords Algebraic thinking · Equalities · Equations · Collective argumentation

8.1 Introduction: Equations in Primary Classes

Typical arithmetic curricula in primary schools offer potential opportunities to engage in algebraic thinking. So arithmetical knowledge in primary classes already includes abilities of conversion that ultimately harbor algebraic potential, but without relying on formal algebraic tools such as elaborated representations and

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terms (Nührenbörger and Schwarzkopf 2015; Nührenbörger 2015). With regard to equations, first equations in the sense of "early algebra concepts" (e.g. 3 + 7 = 5 + 5 or 3 + ? = 10) exist to promote algebraic ways of thinking in primary schools (Kieran 2011). But in contrast to the more formal algebra in secondary education, in everyday primary school the equal sign mostly appears in a non-algebraic sense: it assigns a result to the operations, that is, it seems to be essentially a demand to compute than a sign for symmetric equality between two terms. It is hence not that surprising that children develop a perspective for the equals sign as a calculation-operator (see Steinweg 2013 for a summary; Cai and Knuth 2011; Kieran 1981; Seo and Ginsburg 2003). Within this perspective, the understanding of equations is fixed to everyday experiences with calculation problems: 3 + 7 = 10 means that a calculation on the left side of the equal sign leads to a calculation-result on the right side of the sign. This "calculation-result interpretation" (Winter 1982) of the equal sign includes that notations like 10 = 3 + 7 are not accepted and that equations like 4 = 4 seem to be senseless because the computation is on the wrong side or missing at all (Kieran 1981). It is not very surprising that children, within this somehow empirical understanding of equations, complete equations like 17 + 23 = __ + 20 in the following way: 17 + 23 = 40 + 20 = 60 (Carpenter et al. 2003; Falkner et al. 1999).

There is no doubt that children have to overcome this dominating calculation-result interpretation of the equal sign on their way to the algebra of secondary education (Knuth et al. 2006). To reach this goal, one can foster a broader understanding of the equal sign by confronting children with equations of different forms, for example, with missing numbers on the left side of the equal sign (… + 12 = 35), with the whole calculation on the right side (27 = 13 + …) or with calculations on both sides (19 + 48 = 20 + …). But, from our point of view, the concepts are more relevant than the signs. In other words, the teaching and learning of mathematics in primary school should first work on the development of *equality*-concepts before focusing on the formal use and properties of their symbolic representations in the sense of *equations*. Before we discuss this approach in more detail, we give an example involving children from the 2nd class of school (7 years old) who argue about equalities without using the signs of equations.

8.2 Theoretical Background

8.2.1 Discussing Equalities: An Example

Jessica and Maria are noting pairs of calculation tasks on a strip of paper as a team: One of them writes down a problem, and then the partner notes another pair of terms with the same result. The children have already found two pairs of terms ("8 + 7" and "7 + 8" as well as "9 + 6" and "6 + 9"), implicitly following the commutative law. Now Jessica proposes the next term "8 + 6", but Maria, instead of using the commutative law, writes down “9 + 5”. This serves to disrupt the cooperative activity and gives rise to a dispute between the two children:

Jessica Why put a nine there, this isn’t even a nine and a five.
Maria Look (..) you add one here, ok? And that what you have added here is taken away there, see? It’s the same problem, only that it doesn’t look the same. But you get the same result in the end. (..) Get it?
Jessica But eight is nine here and…er… six five. That is plus and that is minus.
Maria No, it’s like, you take away one from the six and you count this one in addition to the eight. Understand?

Jessica is surprised by Maria’s proposal. She has expected the term 6 + 8, obviously still having in mind something like the commutative law. Maria, then, explains how to construct the new task from the already existing one: Adding one to the first number and taking away “what you have added” from the second number leads to “the same problem”, although the numbers change. Speaking from an algebraic point of view, Maria explains the new kind of equality with the associative law. In German speaking countries this understanding of the associative law is called “constancy principle of the sum”.

In her reaction, from the expert’s point of view, Jessica gives a short but corresponding interpretation of the same idea: One plus and one minus (see Fig. 8.1). Nevertheless it is unclear whether Jessica understands that it is the same sum as she may just be indicating the operation done to each addend (i.e., add 1 and subtract 1).

But, surprisingly, Maria does not agree. In her second explanation, she changes the order of the operations: First of all, you have to take one from the six, only afterwards you can give this one to the eight—this in fact seems to be a different understanding of the same equality (see Fig. 8.2). As well it is possible that she demonstrates her flexibility in presenting her understanding in a different way to see if that would help Jessica to understand her point.

As we can see here, the second graders’ discussion about their understandings of the associative law surely serves as a very fruitful learning opportunity in the sense of developing their early algebraic understanding. The notation of symbols in the

Fig. 8.1 Jessica: $8 + 6 = 8 + (1 - 1) + 6 = 9 + 5$

Fig. 8.2 Maria: $8 + (1 + 5) = (8 + 1) + 5$

sense of equations does not seem to be necessary for this learning opportunity. On the contrary: If the girls had written the tasks in the form of second graders' equations (8 + 6 = 14, 9 + 5 = 14), they would probably not have had any discussion to compare the terms. In this sense the equal sign sometimes avoids the emergence of fruitful learning opportunities on the way to algebraic understanding.

The discussion develops because one of the girls feels misunderstood by the other one—Jessica's interpretation of Maria's idea seems to irritate Maria so that she feels the necessity to sharpen her explanation. In other words: The special circumstance of the interaction process—namely a disagreement between the participants—seems to foster the need to discuss the equality in a more sophisticated way. This kind of disagreement (we call it a productive irritation—see Sect. 2.4) and its necessity to be negotiated within a collective argumentation plays an important role in our theoretical approach within our long-term design-science project PEnDEL (i.e., practice-oriented development projects in discussion with educators and teachers). One of the project's main goals is to understand and to design substantial learning opportunities that help children on their way from arithmetic to algebra. In the following, we present some core ideas of our project.

8.2.2 The Aim of Our Research

Within our PEnDEL project we started a variety of different design experiments, such as whole class instruction, group work, and peer interviews (Cobb et al. 2003; Nührenbörger et al. 2016; Wittmann 1995). The experiments were planned on the basis of already designed mathematical problems, mainly taken from "mathe 2000" (Wittmann 2001b), a famous German project that set up the understanding of mathematics education as a *design science* (Wittmann 1995). The outcomes of mathe 2000 are "substantial learning environments," that is, series of mathematical problems with the following characteristics:

1. They represent fundamental objectives, contents, and principles of mathematical learning at a particular level.
2. They are based on fundamental mathematical content, processes, and procedures beyond this level and contain a wealth of mathematical problems ("exercises").
3. They can be flexibly tailored to the specific conditions of a particular class.
4. They integrate mathematical, psychological, and educational aspects of mathematics teaching and learning and therefore provide a rich field for empirical research. (Wittmann 2001a, p. 2)

All of our experiments are videotaped and analyzed within a qualitative research paradigm, based on the theoretical framework of the interactionist-epistemological approach (Steinbring 2005; Yackel and Cobb 1996). Especially, we understand mathematical knowledge as a social construction, being constituted and differentiated by the negotiation of meaning within the mathematics classroom interaction

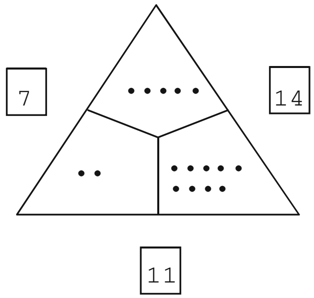

Fig. 8.3 Arithmogons: solution to the arithmogon whose given outside squares consist of the numbers 7, 11, and 14 (Wittmann 2001b, p. 193)

(Voigt 1994). Following the tradition of interpretative instructional research in mathematics education, we generate our theory by analyses within video-based case studies (Bauersfeld et al. 1998).

Our goal is to strengthen the content-related concept of equality in primary school. Hence, the equal sign and its algebraic correct and formal usage do not play a leading role within our learning experiments. In some of the learning experiments, such as those involving arithmogons (Wittmann 2001b), the equal sign does not even appear in the tasks. In the arithmogon illustrated in Fig. 8.3, one possible task would be for the designer to provide only the numbers in the outside squares and to ask the learner to find numbers for the inner fields such that the sum of every two inner fields corresponds to the number in the square affixed to that side of the triangle.

But nevertheless, the learning environments focus on equality. In this sense, in the research project we analyze the ways of (algebraic) reasoning and understanding of arithmetic equalities, without the latter needing to be formally established in the form of equations (Nührenbörger 2015; Nührenbörger and Schwarzkopf 2015). The underlying assumption of the project is that an overly close linking of an equality concept to the equal sign will, on the basis of teaching routines, seduce the children into calculating the terms immediately, without examining the underlying content-related equalities.

8.2.3 The Balance Between Empirical and Relational Knowledge

According to the theoretical approaches of Steinbring (2005), however, the learning of mathematics in general and the learning of algebra especially oscillate between two epistemological poles. At one pole, we have empirical knowledge. On this level, equations are symbolical representations of calculations. The equation 49–4 = 45 means the fact that subtracting 4 from 49 will lead to the number 45, that is, the equation relates a calculation of (normally) two numbers to one result; one equation consists of three objects. From this point of view, the equations 81–36 = 45, but also $9 \cdot 5 = 45$ and $15 \cdot 3 = 45$ symbolize two more calculations with the same result[1]: They are different calculations with the same result. Equations, within an empirical point of view, only exist in the form "operation = result", that is, the typical equations at this level can be read as "number sentences".

At the other pole, mathematics consists of relational knowledge: The focus is no longer on the numbers, the calculation, or the result, but on the structure of the numbers, the comparison of terms, regularly (but not always necessarily) written with variables. As an example, the underlying content of the given equations above is the difference between square numbers and their structure can be expressed as follows:

$$a^2 - b^2 = (a + b) \cdot (a - b) \quad \text{for all integers}\, a, b.$$

It is clear that pure relational expressions like this equation are not typically accessible to children in primary level. But, it is also clear that algebraically substantial learning opportunities can not only deal with calculations in the sense of empirical situatedness. The children would only learn to calculate faster, that is, they would increase their factual knowledge about calculation results and routines.

So, how can children come to a more algebraic view on equations? We now discuss this question first in general on the basis of square number differences, having in mind that the extent of this problem would be too challenging for primary level children. We then concentrate our discussion on the primary level. For this, we limit our problem to the special case $b = 1$ in a special form, namely that of equalities involving the terms a^2 and $(a + 1) \cdot (a - 1) + 1$.

According to Steinbring (2005), it is important that new structural insight is linked with empirical findings. Hence, a fruitful learning opportunity can only arise if a balance is provided between empirically situated and general relational interpretations. Within this balance, we stress a conceptual understanding of equality, more or less independent from formal rules in the sense of equations: An equality is a relation between two *different* representations of *one* mathematical object. In conclusion, the children should learn that two empirically different looking objects can represent the same mathematical object in a structural sense (Winter 1982). In the example above, the terms $7^2 - 2^2$ and $9 \cdot 5$ represent the same object, because one can transform one representation to gain the other:

$$7^2 - 2^2 = (7 + 2) \cdot (7 - 2) = 9 \cdot 5.$$

Of course, an equality means more than only the fact that two terms lead to the same calculation result. For example, although the results of the calculation $15 \cdot 3$ and $7^2 - 2^2$ are the same, the equation $15 \cdot 3 = 7^2 - 2^2$ does not represent an equality according to the given definition, because the terms do not represent the same mathematical object. They are only equal in an empirical sense. Instead,

[1]Note that the raised dot is used here to signify the operation of multiplication.

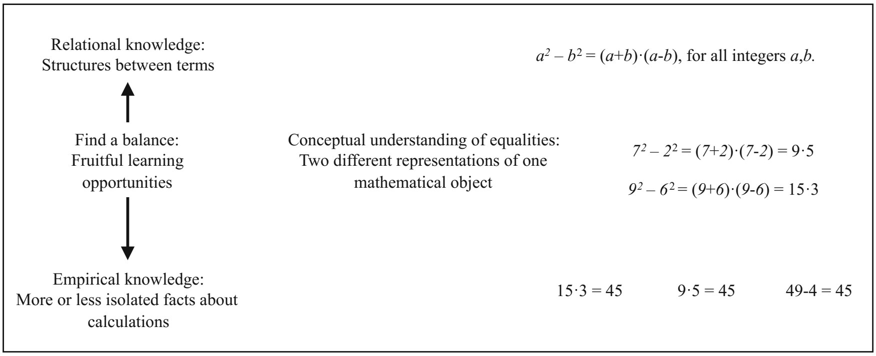

Fig. 8.4 Balance between empirical and relational knowledge

within our context of differences between square numbers, the concerning equality would be represented by $15 \cdot 3 = (9 + 6) \cdot (9 - 6) = 9^2 - 6^2$.

Consequently, equations like $4 = 4$ seem to be senseless within this understanding of equality because already the representations on both sides of the equation are equal: 4 does not only equal 4, but it is the same. From this point of view, it is neither surprising nor wrong when children reject equations of the type $a = a$ (Kieran 1981, p. 319; or Steinweg 2013, p. 73).

Summarizing our point of view (see Fig. 8.4), mathematical knowledge develops in the oscillation between two poles: on the one side is the empirical situatedness, that is, the factual knowledge about numbers, calculations, and calculating-routines. And on the other side is the relational generality, where the focus is only on structures between objects, represented in the form of mathematical symbols. Of course, the way from empirical knowledge to relational knowledge is a very complex and long term learning process, analyzed by many mathematics educators. For example, Hefendehl–Hebeker (1998) analyzes “the shift of view” from arithmetic to algebra, Krummheuer (1995) reconstructs “modulations of framings”, while van den Heuvel-Panhuizen (2003) discusses the “miracle of learning.”

8.2.4 *Collective Argumentation*

In line with Steinbring (2005) and Miller (1986, 2002), we understand the “learning is a balance between empirical and structural knowledge” as constituting what we refer to as “fundamental learning processes.” These processes are very hard to realize because they require that children reorganize their knowledge in order to re-interpret already familiar learning topics from a new, more structural perspective. For example, children would not only have to memorize addition facts up to 20; they are also intended to acquire a structural understanding of addition, using equalities that are based on algebraic laws in order to build up sophisticated calculation strategies like 7 + 9 = 6 + 10 or 7 + 9 = 7 + 10 − 1.

According to Miller (1986) and to our own observations (Nührenbörger and Schwarzkopf 2015), children often do not feel the need to think about mathematical structures in a sufficiently intensive manner in the sense of fundamental learning processes. For example, given a series of tasks that follow a pattern like

$$30 + 20 = \qquad 31 + 19 = \qquad 32 + 18 = \qquad 33 + 17 = \qquad \ldots$$

many children describe the underlying pattern in a quite empirical way, for example: Increasing the first number by one (two, three, … any number) and at the same time decreasing the second number by one (two, three, … the same number) causes the result to remain constant. From our point of view, it is very important that on the way to algebraic thinking children are able to see and describe patterns like this, but:

> Structural thinking is much more than seeing a pattern, such as 'when one number increases by three the other goes down by three'. (Mason et al. 2009, p. 23)

So, how can we encourage children to think in a more intensive manner about arithmetic terms in order to initiate steps of fundamental learning processes? In our experiments we confront children with somewhat productive irritations, that is, we try to disrupt children's expectations. Knowing the rule "+1−1 = 0" by routine, the children in our studies analyze pairs of multiplicative terms like these:

5·5 4·6	6·6 5·7	7·7 6·8

When we asked the children what they would expect regarding the results of the pairs, most of them answered in the same way as within the addition series: 5 · 5 must be the same as 4 · 6 because increasing one number by one (two, three, … any number) and at the same time decreasing the second number by one (two, three, … the same number) causes the calculation-result to remain constant. Afterwards, the children found out by calculating that their expectation was not true—although the difference between the results is quite small. This phenomenon was nearly always very surprising for the children—knowledge that always seemed to be true in the first grade was no longer safe.

This irritation was always very productive, because for the children it seemed to be clear that their observation stood in contradiction to their arithmetic rule. So they would have to think about it in a more intensive way to understand the inequality or, better, to understand the equality between 5 · 5 and 4 · 6 + 1. Children can gain access to a structural understanding of this equality when they can model the multiplication with fields of dots, as is illustrated, for example, in Fig. 8.5.

From a theoretical point of view, fundamental learning processes can only be realized within "collective argumentations", that is, the social process whereby

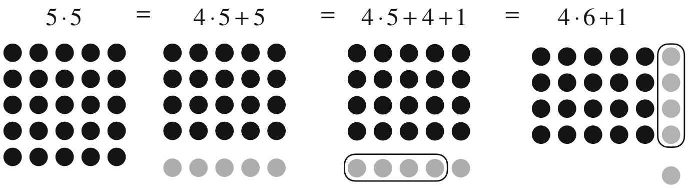

Fig. 8.5 Multiplication fields for a structural understanding of $5 \cdot 5 = 4 \cdot 6 + 1$

children find reasons for an observation that contradicts the expectations that have been built up by their past experiences and routines. According to Miller (1986), these social conditions are necessary to enable children to think about their factual knowledge in a more structural way, that is, to start the processes of fundamental learning. Hence, in our experiments we confront the children with problems and phenomena that make it somehow impossible for them to use their previous routine so that they have to solve the problem in an argumentative way (Miller 1986; Schwarzkopf 2000, 2003).

In summary, to enforce the emergence of substantial learning opportunities in the balance between empirical and structural knowledge, we develop tasks that initiate mathematical needs for collective argumentation on equalities. Our intention is to confront the children with a “productive irritation” (Nührenbörger and Schwarzkopf 2015), that is, with confronting them with equalities that go against their experience-based expectations. This approach is based on Piaget’s (1985) work on cognitive conflicts. The tasks or problems, for example, provide phenomena that were not expected by the children so that they have to reflect on the given structures and see the need to reinterpret the experienced mathematics behind the problem. In this sense, a productive irritation is based on a deviation from acquired expectations that has to be cleared up by the participants.

8.3 The Learning Environment “Computing Chains”

8.3.1 Didactical Background of the Learning Environments

In the following we present an example of a substantial learning environment designed to foster an algebraic understanding of arithmetic equalities. Substantial learning environments in general are rich in content and based on subject-related didactical considerations. They refrain from artificial (teaching) set-ups, but concentrate on mathematical patterns and structures (Wittmann 2001a). Substantial learning environments have a special design in common, trying to offer complex algebraic structures to young children: Objects and operations are presented with the help of special geometrical forms or pictorial figures as squares, triangles or arrows, which serve as support for a structural reading of the given problems.

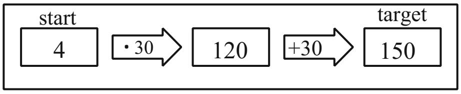

Fig. 8.6 Computing chains

Numbers and operations are used and combined in a way that allows numerous fundamental mathematical activities for exploring complex relational networks. As we have referred to several substantial learning environments in previous articles (e.g., Nührenbörger and Schwarzkopf 2015), this chapter concentrates on "computing chains" (taken from the German school book "Das Zahlenbuch" ["The number book"]).

Computing chains (see Fig. 8.6) consist of a series of operations (noted in the arrows) that transform a "start number" s (noted at the left of the chain) successively into a "target number" t (noted at the right).

To become familiar with the presentation of the tasks, the children start by simply calculating some target numbers. The given example represents the structure $t = s \cdot 30 + 30$. Thus, the children use their empirical knowledge as they calculate a given task and note the result in the form of the target number. After a few such exercises, the children can become aware of the structure beyond the chain, for example, by analyzing the equality behind the equation $(s + 1) \cdot 30 = s \cdot 30 + 30$. Here, we can speak of relational knowledge in that the focus is no longer on the results of calculations but on the structure of numbers and the comparison of equal terms. Of course, children would use the arithmetical language to describe and understand this phenomenon: Instead of multiplying by 30 and then adding 30 again, you can also add 1 to the start number and multiply the result by 30 to get the target number. As described earlier, in this way children can develop equality concepts before using the symbolic representations in the form of equations. The substantial learning environment of computing chains using boxes and arrows to present the objects and operations in the problem allows a structural analysis of the given equality. Hence, computing chains provide opportunities to discuss equalities on the basis of arithmetic operations. In this problem context, arithmetical operations can become the central algebraic objects in discussing equalities—this is an essential aspect of substantial learning opportunities that are aimed at helping children think in an algebraic way (Steinweg 2013, p. 123).

In the next section we discuss two examples of learning opportunities in the context of computing chains. In our design study, pairs of 4th graders (10 years old) were confronted with series of tasks that were intended to initiate a collective argumentation around some productive irritation. Our focus of analysis will be on the question of whether we can see an algebraic core within the children's discussion of the problematic equalities: Do they discuss only empirical aspects of the equalities or can we discern indications of more structural arguments?

8.3.2 *Example 1: Comparing Computing Chains*

In the first example (Fig. 8.7), the boys Jens and Noah computed the following working sheets—Jens took the left and Noah the right one.

As you can see, each of them discovered a pattern in his series of tasks and continued with two appropriate computing chains: The start numbers increase from row to row by 1 and the operators in the arrows stay constant, which causes the target number to rise by 30. Obviously, the relation depends on the distributive law: The start number and the increase of 1 are distributed to the operator "· 30", so that an increase of 30 appears in the sequence of target numbers—adding or subtracting 30 in the second step has no effect on this pattern.

After the phase of individual work the learners compared their columns of computing chains. With regard to a cooperative way of learning they had the chance to discover mathematical phenomena on their own, discuss these findings, and make more discoveries together.

Especially, it is often surprising for children that different computing chains lead to the same target numbers—because of the wide-spread routines in mathematics lessons, this phenomenon seems to be irritating for many children.

In the example shown in Fig. 8.7, every set of two neighbored chains shows this equality. The episode starts when the interviewer asks the children to compare their columns of computing chains:

Interviewer: Let me push them together into the middle (*pushes the problem sheets to the middle*). If you compare them now, do you notice anything?

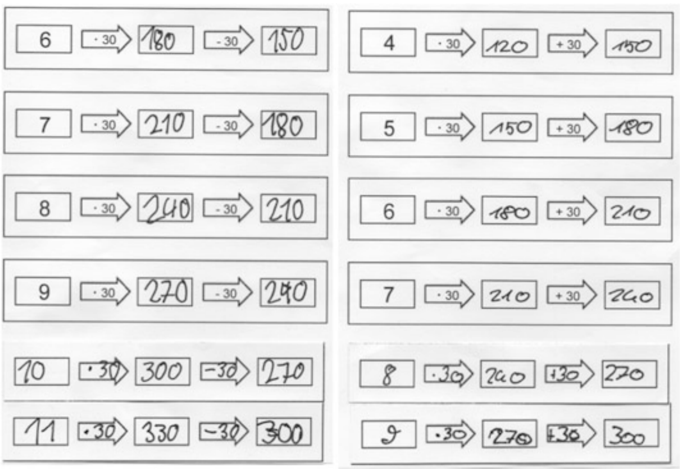

Fig. 8.7 Jens' and Noah's work sheets

Jens: I see. 4 times, 5 times. Oh. You have lower numbers. You have 4 times, 5 times, 6 times, 7 times, 8 times (points to Noah's start numbers)

Noah: Because you have minus here (points to Jens' second arrow numbers), you must have higher ones so that the result stays the same

Jens: Yes.

When having a look at both blocks of chains, Jens makes a new discovery. He no longer compares the vertical differences between the numbers, but focuses on the problems in the horizontal neighborhood. He seems to be surprised about the discovery, that the start numbers of Noah's chains are lower than his numbers. Due to this discovery, and maybe also to his own astonishment, Noah feels motivated to start some reasoning. The boys seem to be irritated in a productive way because they start to examine the detected phenomenon in a more intensive way, especially starting a collective argumentation involving reasoning about its structure. Noah points to Jens' second operator, a subtrahend, and highlights the higher start numbers of Jens' chains in comparison with the constant target number.

Noah compares the problems in a qualitative way. He does not specify his justification numerically, but has a qualitative idea of balancing amounts so that the total at the end is the same. If one has got more than the other at the beginning, then one has to take something away, so that it is the same at the end.

After having discovered the differences of start numbers in combination with the equality of target numbers and having started to justify this phenomenon, the interviewer asks for further justification.

Interviewer: What do you mean?

Noah: Well (.)

Jens: I get two times more, well more, and so that it's the same for both of us in the end, I must sub, subtract 30 and he needs to make 30 plus […].

Jens develops Noah's comparison to a quantitative comparison considering the numerical values. As Noah does before, Jens relates the starting numbers of the problems, the target numbers as well as the second operators, to each other and furthermore articulates these relationships. Jens "gets two times more" at the beginning and as "it's the same for both of [them] in the end," Jens has to "subtract 30" and Noah has to "make 30 plus." Algebraically speaking, Jens sets up the following equation: $(x \cdot 30 + 2 \cdot 30) - 30 = x \cdot 30 + 30$.

In his argumentation, Jens stresses the start number as well as the subsequent operation. In the first phase of the episode he describes: "You have 4 times, 5 times, 6 times, 7 times, 8 times." In the second phase he relates his beginning of the problems to the one of Noah and states that he "gets two times more." Jens seems to regard the whole terms $4 \cdot 30$, $5 \cdot 30$, $6 \cdot 30$ etc. in Noah's chains and the terms $6 \cdot 30$, $7 \cdot 30$, $8 \cdot 30$ etc. in his chains, and not only the particular start numbers. If he were only looking at the start numbers, he would have talked about "two more" instead of "two times more." He therefore interprets each term at the beginning, consisting of an operation and an object, as one whole object. Also he relates the second operator −30 and +30 to the beginning, strengthening the assumption that Jens looks at the whole objects $4 \cdot 30$, $5 \cdot 30$ etc. and not only at the start numbers, as they could not explain the different operations in the opposite direction. To sum up, Jens does not refer to the start numbers alone in his argumentation, he does not compare $x \cdot 30$ and $(x + 2) \cdot 30$, but relates to the equality between the terms $x \cdot 30 + 30$ and $(x \cdot 30 + 2 \cdot 30) - 30$.

After Jens stated this equality relation between the two columns of computing chains, he justified it.

Jens: […] because we have I think (.) yes, have 60 more in the middle, me 60 more than Noah, and I must then go minus 30, then Noah is only 30 away from me, from my number and he goes 30 plus, then we are at the same number.

At the end of the argumentation, Jens focuses on the middle numbers of the computing chains. Hence, for him, these middle numbers seem to be important to secure the reasons for the equality between the target numbers. He remarks that the middle numbers of his chains are "60 more" than the corresponding ones of Noah. From our point of view, this empirical observation is a link within the argumentation to the factual knowledge of the children: The difference between the middle numbers is 60, so adding 30 to the lower one and subtracting 30 from the higher one equalizes the difference (see Fig. 8.8).

In summary, Jens first specifies the equality relation between the two columns of target numbers by comparing the relations between start numbers and the second operators. He then relates these comparisons to each other with the help of the middle number: The empirical fact that the differences between the middle numbers are always 60 seems to be essential for Jens to make sure that his thoughts on the structures are correct. Algebraically speaking, the argument sets up the following relation:

$$(x \cdot 30 + 2 \cdot 30) - 30 = (x \cdot 30 + 60) - 30 = x \cdot 30 + (60 - 30) = x \cdot 30 + 30.$$

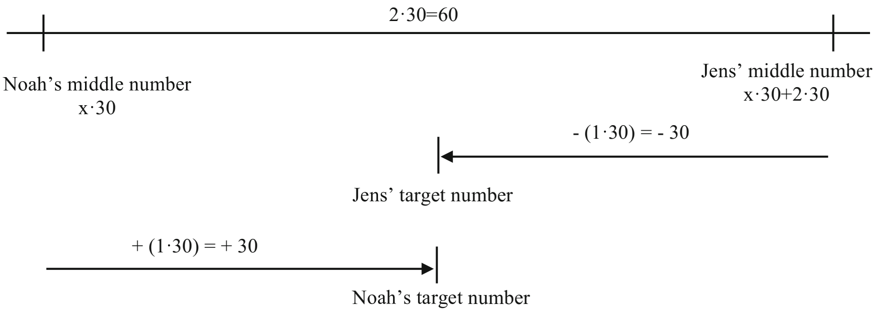

Fig. 8.8 Jens' argument on the equality

Hence, Jens argues that both computing chains are different representations of the same mathematical object, namely ($x \cdot 30 + 60 - 30$).

8.3.3 *Example 2: Completing Computing Chains*

In the second example (see Fig. 8.9) the children have to figure out the second operator of computing chains. The intention of the blackened middle numbers is to push the children to compare objects and operations in order to find the second operator, without computing the middle number. A help for this problem is the first row of the column: Having in mind that 320 is the result of $8 \cdot 40$ the children could think that in the second row one has to add $5 \cdot 40$ in the second arrow because of the distributive law: $3 \cdot 40 + 5 \cdot 40 = 8 \cdot 40$. To understand the underlying equality of this equation, the children would have to see that the terms $3 \cdot 40 + 5 \cdot 40$ and $8 \cdot 40$ are different representations of the mathematical object $(3 + 5) \cdot 40$.

Nevertheless, of course, the children might compute the middle number mentally and find the solution in a more empirical manner: $3 \cdot 40 = 120$ and because 320–120 = 200 the operation in the second arrow must be "+200". This possibility to link the problems to the empirical pole of mathematical knowledge is always given in our learning environments: We do not want to demand too much of the children by confronting them with somehow pure algebra problems that belong to secondary level. Hopefully, children would evolve from this empirical point of view to a more structural one, observing the relations between the rows of the columns, especially the dependence of the second operator on the start number.

However, at the beginning of the following episode, the pair of 4th graders Nils and Dilay, who are working together, have already found the start number for the

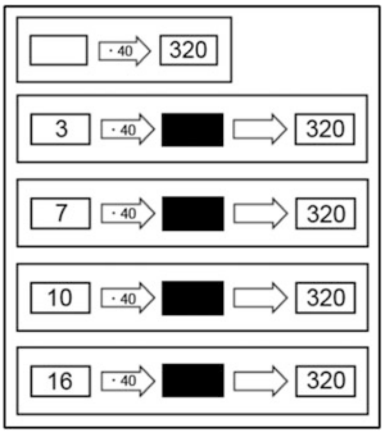

Fig. 8.9 Figuring out the second operator of the computing chain

short computing chain (Fig. 8.10). Dilay first looks at the longer chain with the start number 7 and finds the second operator for this computing chain:

Dilay: I've got it (writes down +40). Here it is plus 40.
Interviewer: Why?
Dilay: Because here it is eight times 40 (*points to the short computing chain*) and here it is seven times 40. And when we do not calculate this (*covers the middle number with her fingers*), then one just has to do 40 again because only then it is 320.
Interviewer: Yes, well done.

Dilay finds the correct solution, +40, writes it down and then gives reasons for her solution. She points to the short computing chain and compares the belonging term $8 \cdot 40$ with the beginning of the second chain, $7 \cdot 40$. She concludes that the missing operator must be +40. Because she stresses the start numbers of both chains, she might have noticed the difference of 1 between them and reasons that "one just has to write 40 again" with regard to the distributive structure of the two problems: $8 \cdot 40 = (7 + 1) \cdot 40 = 7 \cdot 40 + 1 \cdot 40$.

Of course we do not know whether Dilay had already calculated the middle number before she offered her solution. Maybe she calculated the middle number, determined the difference of 40 between the target and the middle number and noticed afterwards, that "one just has to do 40 again," that is, she found the solution with an empirical point of view. But, the important aspect is that she characterized her argument in more structural terms: Dilay emphasized that her argument was valid without referring to the result of the middle number: "And when we do not calculate this." However, the argument of Dilay initiated a statement from Nils:

Nils: Is it correct at all? (*looks at the computing chains*) It is, isn't it?
Interviewer: Nils, do you know what Dilay just meant?
Nils: Hm. Yes. I think, there is one more (…) (*points to the start number of the short chain*). For this there must be one more time forty. (*points to the second arrow of the long chain*), because that are seven and eight, that would be 320 too, but because seven (..) because it uhm uhm is one less, there has to be one more of the forties in addition (.) that's why it is plus forty.

At the beginning, Nils wonders if Dilay's solution is correct. He agrees quickly, but still does not seem to be completely satisfied. After the interviewer asks him whether he understood Dilay's explanation, he starts to give reasons for Dilay's solution by himself. He examines the short chain and points out that there is "one more". Maybe he compares the start numbers of both chains and states that eight is one more than seven. Eventually, Nils not only compares the start numbers but, as Jens did in the above mentioned example as well, looks at them in connection with the first operator "times 40". He might then have realized that the short computing chain has one more "times forty" than the beginning of the long chain. When looking at Nils' further argumentation, this interpretation can be assumed.

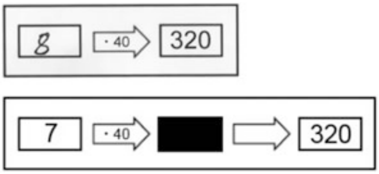

Fig. 8.10 Discussed part of the task "Figuring out the second operator"

He regards the second operator of the long chain and states: "there has to be one more of the forties in addition." Nils seems to balance the lower amount in the beginning of the long chain with the second operator, so that the result is the same as the one of the short chain. Algebraically speaking, he uses the distributive law to distribute 8 times 40 to 7 times 40 and 1 times 40: $8 \cdot 40 = 7 \cdot 40 + 1 \cdot 40$. Nils compares the numbers in the two chains, focuses on their differences, and transforms one term into the other: $8 \cdot 40 = (7+1) \cdot 40 = 7 \cdot 40 + 1 \cdot 40$.

Hence, Dilay and Nils develop an argument that justifies the equality of the terms $8 \cdot 40$ and $7 \cdot 40 + 1 \cdot 40$ by referring to the common mathematical object $(7 + 1) \cdot 40$. Although we do not know whether Dilay found the solution empirically, she did state her argument with reasons for the solution that were not based on the empirical finding of the middle number.

8.4 Closing Remarks

In this chapter we tried to highlight two aspects related to the learning of algebra within primary levels of schooling. First, we argued for concentrating on the concepts of equality rather than stressing the use of the equal sign in a formally correct way. From our point of view, this conceptual understanding is essential to bridge the gap between the empirical knowledge about calculation tasks and the pure relational knowledge about the structures of algebraic terms. It is clear that this kind of conceptual understanding can only be realized in learning settings of adequate complexity in the sense of substantial learning environments. But, and this is the second main focus of our chapter, the processes of fundamental learning that involve the interplay between empirical and relational knowledge can only take place when children are engaged in collective argumentation—only processes of argumentation enable children to overcome their empirical understanding of mathematics. We tried therefore to stress the importance of productive irritations for the emergence of fruitful learning opportunities on children's pathway to algebraic concepts—exemplified by a focus on equalities between arithmetical terms. Irritations arise in situations where previously held mathematical views, approaches, notions, or expectations fail in the socially-specific necessities of interaction. They become productive for the initiation of fundamental learning processes if children develop options for bridging the gap between expectation and disappointment by collective argumentation. These social processes are the source and motivation for children on their way from an empirical use of calculating rules to more algebraic, structural understandings of equalities. This in turn can lead, later on, to an adequate understanding of algebraic equations and to a conceptual foundation for the use of algebraic formalism in general.

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Chapter 9
Observations of Structure Within Shape Patterns

Aisling Twohill

Abstract Constructing general terms for shape patterns supports children in reasoning algebraically about relationships between quantities and relative rates of change. This chapter describes a research project wherein I investigated the strategies children attending Irish schools used when asked to solve shape-patterning tasks. The research instrument was a task-based group interview, and the children's interactions shed light on a number of catalysts for the broadening of their observations of the pattern structure. Such catalysts included peer interactions, concrete materials, and teacher prompts. In this chapter I draw attention to children's observations of structure, and seek to trace the thinking of children whose observations broadened from an initially narrow or limited perspective.

Keywords Shape patterns · Generalization · Structure · Task-based interview

9.1 Introduction

Blanton et al. (2011) have pinpointed four key practices of algebraic thinking: (a) generalizing, (b) representing, (c) justifying, and (d) reasoning with generalizations. They emphasize that these practices must focus upon structures and relationships. As well, Blanton et al. (2015) have identified functional thinking as an appropriate content domain within which children may apply key practices of algebraic thinking. Functional thinking embodies an approach that sees functions as descriptions of relationships about how the values of some quantities depend in some way upon the values of other quantities (Chazan 1996). In shape patterns, children are asked to discover or explore a function that relates the number of elements of some component of a term, to the position of that term in the pattern.

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For example, in the pattern of fences depicted in Fig. 9.1, the number of posts in each fence is a function of the position number (x), where the specific function is $f(x) = 3x + 1$.

Blanton et al. (2015) highlight the role of functional thinking in young children's algebraic thinking by stating that functional thinking includes generalizations of co-varying quantities and their relationship, representations of these relationships, and reasoning with the relationships in order to predict functional behavior. Concurrent with the definition of functional thinking above, many of these constituent skills of functional thinking focus on quantities and the relationship between quantities. For children to explore these relationships, and in order to define and apply functions, their thinking can be supported by broad and multi-faceted observations of structure within patterns. Such observations may include figural aspects of terms along with numerical quantities of components. In this regard, Rivera and Becker (2011) have stressed that sole attention to the numerical aspects of a pattern indicates that children are only superficially grasping the commonality within the structure of the pattern. Examples of figural observations of Fig. 9.1 pattern are depicted in Fig. 9.2.

The focus of the research described in this chapter is children's observations of structure within patterns. Sixteen children, with an average age of 9.6 years, participated in group, task-based interviews where they sought to collaboratively construct general terms from shape patterns. Children were expected to apply functional thinking to the context of shape patterns, as they were prompted to reason about and represent co-varying quantities and their relationships. In my analysis of the children's discussions, constructions, drawings, and gestures, I explored children's tendencies to attend to relationships between terms, or relationships between a term and its associated position in a sequence. Research suggests that when children begin to explore the structure of patterns, their natural tendency inclines towards a *recursive approach*, that is, the examination of the mathematical relationship between consecutive terms in a sequence (Lannin 2004; Rivera and Becker 2011). When children are learning to interpret patterns it may be

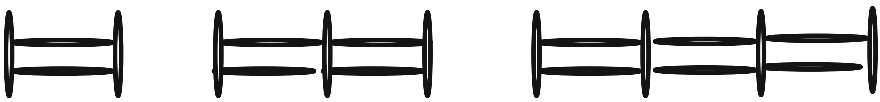

Fig. 9.1 A pattern of fences, wherein the number of posts in each fence is a function of the position number of the fence

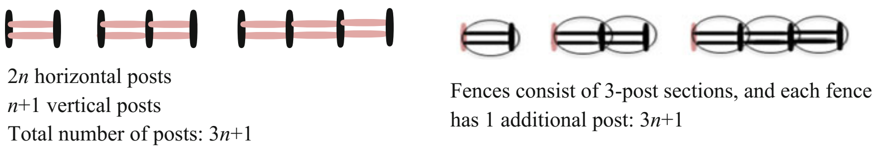

Fig. 9.2 Possible figural observations of the Fences pattern

necessary for a teacher to encourage the child to consider an *explicit approach*, whereby the child identifies a rule for the relationship between a term and its position in the pattern (Lannin et al. 2006). Taking into consideration the challenges inherent in solving problems when children attend only to numerical aspects of the pattern structure, rather than also incorporating figural aspects, my investigation of the children's observations of structure incorporated an exploration of their inclinations to discern both figural and numerical aspects of the shape patterns presented to them (Twohill 2017).

The analysis and findings in this chapter are drawn from a research study involving primary school children in Ireland.[1] Within the Irish Primary School Mathematics Curriculum (PSMC), recursive reasoning is favored in the presentation of learning objectives and associated activities, while explicit reasoning is not outlined as an approach to pattern solving (Government of Ireland 1999). As I will outline later in this chapter, the omission of functional thinking, and specifically shape patterning, from the Irish PSMC indicates that it is highly improbable that most children attending Irish primary schools will have engaged with activities designed to develop functional thinking. In my research I aimed to investigate what strategies children attending Irish primary schools would adopt when asked to construct general terms for shape patterns, with and without real life contexts. In this chapter I focus on the children's observations of relationships within the structure of the patterns presented to them.

9.2 Generalization

9.2.1 Generalization from Shape Patterns

A core aspect of algebraic thinking is the construction of a general case for a pattern, or for a scenario that contains a generalizable phenomenon (Kaput et al. 2008). Within formal algebra, students are required to apply a mastery of generalization skills, to understand what it is to generalize, and to manipulate expressions of generalizations while accepting their generalizability. Traditional curricula prevalent in many countries delay children's engagement with generalization until secondary school; thus, many students fail to successfully develop the skills and habits of mind involved in abstract thinking (e.g., Arcavi 2008; Kilpatrick and Izsák 2008; Mason 2008).

[1]In Ireland children attend primary school for eight years, typically from the age of four or five. The classes are referred to as Junior Infants, Senior Infants, and 1st through to 6th class. As such, Senior Infants would be equivalent to 1st grade, 1st class to 2nd grade, etc. The children who participated in this study were attending 4th class, and for ease of reference I refer to their grade as 5th grade throughout this chapter. In the study of Nic Mhuirí (2014) the children attended 6th class, and in this chapter their grade is referred to as 7th grade.

Generalization of a phenomenon involves the analysis of visible instances of the phenomenon, and the application of conclusions to cases that are not observable. As such, the capacity to generalize supports us in utilizing the structure of the world around us so that we can conjecture and predict. In mathematics, young children are often encouraged to generalize properties of computation, such as the addition or multiplication of zero, so that they may build on what they have observed with some numbers, in order to perform mental computations with other numbers. Indeed, Mason asserts in this volume that, in order to make sense of much of mathematics, children must generalize. Mason highlights evidence pointing to children's early ability to generalize, and also describes typical instances of young children working from the abstract to the particular, thereby applying generalizations to specific instances. In exploring children's construction and application of generalization, many research studies have focused on shape patterns, for example, Radford (2011), Rivera and Becker (2011), and Warren and Cooper (2008). Lannin et al. (2006) suggest that generalizing through patterning activities may create a bridge between students' knowledge of arithmetic and their understanding of symbolic representations.

Central to an understanding of children's early algebraic thinking are concepts of "the particular" and "the general" (Mason and Pimm 1984). Mason and Pimm identified three ways of talking about numbers: as specific numbers, generic numbers, and the general case. Specific numbers are identified without the use of symbols, or when symbols are used, there is no ambiguity about the value associated with the symbol. The general case is an expression of something that is true for all numbers, or in the solution of patterns, an expression that defines all terms of the pattern. A generic example is an example where a specific number plays the role of the general number, "but one presented in such a way as to bring out its intended role as the carrier of the general" (Mason and Pimm 1984, p. 287). For example, when solving for patterns, children may refer to specific far terms, such as the 100th term, in order to express their generalization; but their use of a numbered term is intended to describe the general case (Mason and Pimm 1984; Radford 2010).

Resonating with Mason and Pimm's trichotomy of specific, generic, and general cases, Radford (2001) asserted that children's algebraic generalizations may be factual, contextual or symbolic. Factual generalizations involve instantiating a general structure to specific terms, whereby children do not express a generalization as applicable to all terms, but apply an "operational scheme" which allows them to calculate a value for particular terms. Contextual generalizations, by comparison, involve the consideration of non-specific terms. While contextual generalizations are not completely abstract, or general to all terms, they indicate a distancing from the specific, whereby children may refer to "the next term" or to a generic term. Symbolic generalizations involve the abstract expression of disembodied mathematical objects, wherein children express the algebraic concepts with no reference to their method of calculation, or to any specific term. Radford (2001, p. 88) has emphasized the complex developments required in children's thinking in order for them to engage in this level of abstract expression. For children to express their thinking symbolically, without reference to specific or situated instances, requires a different perspective on the mathematical objects involved. This higher order perspective must be accompanied by a "layer of discourse" appropriate to the description of the mathematical objects—in this case general terms for patterns, without alluding to any specificities, either in specific terms or in pointing towards thought processes involved in constructing the generality.

9.2.2 Generalization in the Irish Primary School Mathematics Curriculum

Mason (2009) stated that "young children are able to generalize, because without it they could not function in the world and certainly could not grasp language" (p. 159). When children enter primary schools, they bring with them nascent skills in generalization. It is the role of educators to nurture and develop such skills so that, by the time children leave primary school, they possess the competences necessary for engagement in abstract symbol manipulation. In this section I describe the content of the Irish PSMC as it pertains to patterning, and I highlight what I perceive as missing elements which, if present, could support teachers in fostering existing algebraic thinking skills as identified by Mason (2009).

The Irish PSMC is organized under the five Strands of Number, Algebra, Shape and Space, Data, and Measures. Content is prescribed under each Strand from when children commence school at four or five years of age. The Algebra Strand of the PSMC contains patterning activity, but this is largely limited to repeating patterns in the first two years of primary school, and numerical sequences thereafter. Also, as there is no specific guidance within the curriculum as to the purpose of, or intended pedagogical approach related to, patterning activities, the algebraization of any patterning activity is very much at the discretion of the classroom teacher, and thus vulnerable to inconsistencies. Clements and Sarama (2009) caution that teachers need to be aware of the role of repeating sequential patterns and of where they fit into children's observations of structure within patterns. For example, Papic (2007) emphasized that when teaching patterning to young children, teachers must remain cognizant of focusing children's attention on the commonality, or unit of repeat. Such detail is not outlined in the PSMC. Furthermore, in the presentation of the curriculum no attention is paid to consideration of far terms for patterns. Children may not, therefore, consider terms beyond their perceptual range, and may not be facilitated to consider "indeterminate quantities conceived of in analytic ways" (Radford 2011, p. 310).

Kaput (1998) suggests that for students to develop algebraic reasoning skills with which they can unlock many sophisticated areas of mathematics, it is important that they engage with algebraic processes over time and that this engagement is purposeful. Within the PSMC, instruction in algebraic thinking commences early, and patterning serves to support computation throughout primary school. However, generalization or identifying a far term in a pattern are not mentioned. While there is content relating to the formulation of rules for patterns, all the pattern examples provided lend themselves most readily to recursive solutions. Furthermore strong tendencies towards drilling children in recursive approaches are evident in textbooks popular in Irish primary schools (Twohill 2013). Eivers et al. (2010) found in an examination of teaching approaches in Irish classrooms that textbook use was pervasive, and suggested that teachers were possibly overly reliant on textbooks, indicating that many children's exposure to patterning may be limited to recursive approaches with no mention of generalization.

In assessments of the attainment of Irish children in algebraic thinking, these gaps within the primary school curriculum appear to be evident. In 2015, the International Association for the Evaluation of Educational Achievement (IEA) conducted an iteration of Trends in International Mathematics and Science Study (TIMSS). TIMSS measured the mathematics and science skills of children in many countries at both 4th grade and 8th grade. In Ireland more than 9000 children participated, and both of the Irish cohorts achieved a scale score significantly greater than the overall mean for all countries (Clerkin et al. 2016). The performance of the 8th grade cohort on the Algebra content domain, however, indicates that Irish students may have experienced greater challenges with items assessing Algebra than with those assessing Number, and Data and Chance. The Irish cohort achieved a mean score of 501 for Algebra, significantly lower than the mean score for Number (544), Data and Chance (534), and the overall scale score for mathematics (523).

Along with curricular content and Irish students' performance on assessments that evaluate their algebraic reasoning, it is pertinent to highlight the typical situation in Irish classrooms in terms of student agency. In this chapter I will refer to how interactions within interview groups supported children's thinking. In this section, therefore, I draw attention to the relevant experience the children may have had in developing the necessary skills inherent in using 'exploratory talk' during discussions (Mercer and Littleton 2007). Without opportunities to develop shared practices such as justifying answers, and building upon each other's suggestions, children may collaborate at very superficial levels when working together on mathematical tasks. Nic Mhuirí (2014) researched trends in mathematical talk occurring in Irish 6th grade classrooms and found that the incidence of exploratory talk was extremely low. Supporting the findings of Nic Mhuirí, Eivers et al. (2010) also cited low levels of group discussion during mathematics lessons, where a teaching approach typified by being textbook based and centered on the teacher predominated in 7th grade.

9.2.3 Observations of Structure

Strømskag (2015) defines a shape pattern as a sequence of terms, composed of 'constituent parts,' where some or all elements of such parts may be increasing, or decreasing, in quantity in systematic ways. While a limited number of terms of a shape pattern may be presented for consideration, the pattern is perceivable as extending until infinity. In order to construct a general term for a shape pattern, children must "grasp a regularity" in the structure of the terms presented, and generalize this regularity to terms beyond their perceptual field (Radford 2010, p. 6). Similarly, Mulligan and Mitchelmore (2009) present "structure" as the definition of a pattern, which is most often expressed as a generalization, being a "numerical, spatial, or logical relationship, which is always true in a certain domain" (p. 34).

In seeking to construct a generalization for a pattern, children may adopt a variety of approaches. Lannin et al. (2006) identified an explicit approach as establishing a relationship between a term and its position in the pattern. In contrast, a recursive approach involves comparing consecutive terms in order to identify a relationship, which is then used to construct subsequent or preceding terms. A 'whole-object' strategy entails identifying a term of the sequence as a unit, and constructing other terms by generating multiples of the unit. To gain insight into a greater range of patterning, and a structural understanding beyond the most basic repeating patterns, children may benefit from opportunities to consider an explicit approach, and some children may require intervention to do so (Lannin et al. 2006; Rivera and Becker 2011). Thus, teaching activities and materials should avoid overusing sequences that foster a recursive approach. Students need recourse to both explicit and recursive methods of solving patterns, and their thinking should be developed to include an ability to determine which method is appropriate in a particular situation (Lannin 2004). Watson et al. (2013) concur by emphasizing that explicit and recursive thinking should not be considered as hierarchical but complementary, and that children will be supported in developing robust, flexible reasoning skills when facilitated in engaging with both.

9.3 Methods

In seeking to explore the strategies children use, I presented patterning tasks to four focus groups, each of which had four participants. The children were attending 5th grade in Irish primary schools and had a mean age of 9.6 years at the time of the interviews. The children were not schooled in any way in preparation for the interviews, and no teaching intervention took place prior to the children's participation. Of the four interviews (one interview per group), three were video recorded, and one was audio recorded. To facilitate strong student agency during the interviews, I encouraged the children to share and explain their ideas with each other and to explore differences between their approaches without verification from me (Howe et al. 2007). For ease of reference, Table 9.1 identifies the four groups of children and their chosen pseudonyms, which I used throughout the research study.

Table 9.1 The four groups of children who participated in the task-based interviews

Group name	Participants
Group 1	Grace, Ciaran, Fiona, and Daniel
Group 2	Arina, Alex, Jay, and Cherry
Group 3	Christopher, Danny, Lily Rose, and Jane
Group 4	Emily, Wyatt, Luigi, and Orla

Task-specific interview questions were presented on a worksheet, and I asked further questions to probe children's thinking and to encourage children to compare and justify their responses. The written interview questions were designed to motivate children to firstly explore the patterns and to proceed to the construction of general terms. The verbal interview questions and prompts were designed to best facilitate the children in engaging with the patterns and in articulating their thinking in response to the tasks.

Goldin (2000) advises that "by analyzing verbal and nonverbal behavior or interactions, the researcher hopes to make inferences about the mathematical thinking, learning, or problem-solving of the subjects" (p. 518). In seeking to explore children's mathematical constructions, I was conscious throughout that my inferences from children's comments were approximations of their true meaning. As Van Manen (1990) attests, "a *good* phenomenological description is an *adequate* elucidation of some aspect of the lifeworld" (p. 27, emphasis added). While I sought to unpick as best I could how and why children thought about the mathematical tasks, I posit that it is not possible to feel a sense of completion, or closure, in relation to the children's thinking, but rather that interpretation is ongoing (Postelnicu and Postelnicu 2013).

As discussed above, shape patterning and explicit thinking are not formally explored in most primary school mathematics classrooms in Ireland and many of the children would experience the tasks as novel. I felt, therefore, that children's interaction with the tasks would be best explored if the children were supported to work within their ZPD (Vygotsky 1978). Working alone with an unfamiliar adult researcher, children may be inclined to respond in ways they believe are expected (Ginsburg 1997), and this may limit their autonomy to take risks with novel tasks, or to tease ideas out. In comparison, working within groups, children have opportunities to create "personal mathematical insights" (Goos 2004, p. 263).

The patterns presented to the children are outlined in Fig. 9.3. As seen in Fig. 9.3, Pattern 1 is a Beams pattern used in previous research by Radford (2010). Pattern 2 is a modified version of an X pattern. Similar patterns have been used by Rivera and Becker (2011) and by Warren and Cooper (2008). The difference between this pattern and those used in previous research is that the pattern grows in an asymmetrical fashion. Participants are required therefore to consider both the rate of change, the element of the pattern that is growing, and the direction of growth. For example, even-numbered terms are relatively simple to describe, as expressed by Jane: "the same way in term 10, it was like half of 10 was the amount of tiles on the legs." Odd-numbered terms are more challenging, and during this

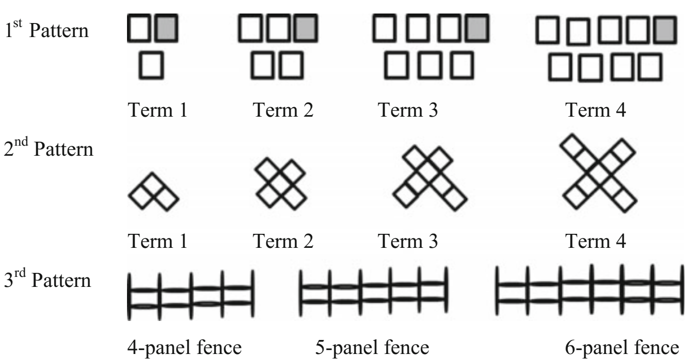

Fig. 9.3 Patterns presented to the children

research project, some children struggled to find the number of tiles on each leg for the 75th term, due to difficulty in halving 75. The thinking required in order to realize that 75 must be reduced by one, halved, and then a final adjustment of 2 diamonds added to the lower legs involves considerable algebraic reasoning, which was beyond the capabilities of some children during their interviews. Grace was reasonably successful in this, when she articulated: "Because ehm, you could just find eh, the nearest number to, eh, but like eh, you could choose the doubles that equal 74, and then you just add one more."

To begin thinking about each pattern the children were asked to describe what they observed of the pattern terms presented. I then asked them to extend to the subsequent two terms, for Patterns 1 and 2, and to a preceding and a subsequent term for Pattern 3. Children were asked to construct the subsequent and preceding terms using tiles or matchsticks, and to draw them. Some children constructed their own terms whereas some collaborated with a fellow group member. I anticipated that manipulation of concrete materials might support the children in constructing understanding, particularly in a context where they are encouraged to interact with their peers, learning from and questioning each other (Bruner 1966; Dunphy et al. 2015). In the introduction to each interview I urged children to discuss their thinking, to question each other, and to share their ideas.

Following the children's discussions and constructions of next terms, I asked the children to construct near and far terms, where far terms were sufficiently large as to play "the role of generalized number" (Stacey 1989, p. 150). To conclude work with each pattern, the children were asked to verbally describe a general term. I did not ask children to express their generalization using symbols, but rather I took the position that abstract symbolism is not a necessary component of algebraic thinking (Radford 2010). The use of variables in generating rules from patterns is semantically challenging, as children must see a single variable as simultaneously fulfilling the roles of "dynamic general descriptor" of terms in relation to their position and as a generic number in an expression (Radford 2000). Variable use is not present in the PSMC in Ireland before 5th grade, and therefore would be entirely novel to most, if not all, of the children who participated in this research. In planning the interviews, I did not consider it appropriate therefore to expect children participating in this study to work with the position number of terms in a pattern in both ordinal and cardinal roles (Radford 2000).

9.4 Findings and Discussion

9.4.1 Overall Findings

When coding the children's utterances I deemed a comment to indicate an explicit approach if it identified, described, queried, or drew upon a relationship between terms and their position in the pattern. Comments that spoke of relationships between consecutive terms were deemed indicative of a recursive approach. Other approaches identified during transcription and coding were approaches using 'counting' of elements and 'whole-object approaches.' Children who used a whole-object approach treated a term, or the constituent part of a term, as an object and constructed a subsequent term by generating multiples of this object. Some whole-object approaches involved a 'final adjustment' wherein the child adjusted the construction to cater for contextual or numerical aspects of the pattern. Table 9.2 contains examples of children's comments and their corresponding codes, and Table 9.3 presents the number of comments coded as recursive, explicit, whole-object, or counting, broken down by pattern.

Table 9.2 Examples of children's comments and the corresponding codes

Child	Pattern	Comment	Code
Cherry	3	I matched the number in the middle and then, like, on the outside they add one on	Explicit
Wyatt	1	I think the 86th will have one more than 86 that's 87 on top and then take away and it's 86 on the bottom	Explicit
Alex	3	To make the 3-panel fence you would have to take away 3 posts from the 4-panel	Recursive
Ciaran	2	That's like, it's going up in twos each time, look cos these grow then these grow then these grow so it'd probably be like 16, at the other end	Recursive
Daniel	2	Ok, 1, 2, 3, 4, 5, 6, 7. 1, 2, 3, 4, 5, 6, 7, 8, 9. So term 3 is 7 and then that's 9	Counting
Emily	1	Because if you add the… On term 6 if you add… If you double it, it would equal to… if you doubled the top number 6 and 6 it would make 12	Whole-object
Emily	1	I think it could be since this one is 6, you double the top and that's 12, it could be 12 on the top row, then since it's 5 on the bottom, it could be 11 on the bottom. 12 on the top and 11 on the bottom	Final adjustment

Table 9.3 Tally of coded comments per pattern

Total	Counting	Whole object	Final adjustment	Recursive	Explicit
Pattern 1	18	4	2	28	64
Pattern 2	15	1	0	63	77
Pattern 3	16	4	1	14	39
Total	49	9	3	105	180

It is important to emphasize while presenting these statistical counts that they are an approximation of the proportions of time and focus given by children to the various approaches. Many factors impacted my production of these counts, which could alter the true proportion of focus given by the children. For example, it was not possible for me to hear every comment uttered. Some children may have repeated themselves at different times within the interviews and I may have included this as two comments. Also, all coding depended upon my interpretation of the children's comments. As such, these counts must be seen as the best information available at the moment, but not as an exact representation of the situation.

Patterns observed in the numbers of coded comments presented in Table 9.3 suggest that children expressed more ideas of an explicit nature, than of any other, when describing, extending, and generalizing from the patterns presented to them (180 comments in total). It may be surprising that an explicit approach seemed to dominate the groups' thinking about the patterns presented to them, considering previous research in the area and the approach to patterning within the PSMC, as discussed earlier in this chapter (Government of Ireland 1999; Lannin 2004). Twelve of the 16 children demonstrated explicit thinking at some stage during their engagement with the patterns, while the remaining four children did not apply explicit thinking in order to construct terms. It is pertinent to highlight that the high figure for the counting approach relates predominantly to children's comments during the initial description and extension elements of their engagement with the patterns. As identified by Barbosa (2011), counting is a natural and appropriate approach to take to one's exploration of the structure of a pattern.

While there seem to be indications from research that recursive thinking is likely to be the intuitive approach of many children, I found the inclination of children to demonstrate explicit thinking to be of particular interest. Adopting a hermeneutic phenomenological stance, I analyzed contributing factors to children's inclination to think explicitly. In this section I outline the impact on the children's thinking of (a) the concrete materials; (b) my contributions as facilitator; and (c) the children's interactions with each other within the interview groups.

9.4.2 Concrete Manipulatives and the Physical Construction of Terms

Six of the sixteen children seemed to have spontaneously explored connections between terms and their positions, before observing their peers making this connection. For example, in response to Pattern 1, when asked to construct the 12th term, Ciaran and Grace (who were both participants in Group 1) demonstrated spontaneous explicit thinking independently of each other, in saying:

Grace it's obviously going to have 13, because term 5 had 6 squares on the top and 5 on the bottom, so term 12, so it's gonna be really easy

Ciaran how many tiles are needed for term 12? 1, 2, 3, 4…12; 12 multiplied 2 is 24, plus one is 25

Grace identified each term as consisting of a top row and a bottom row, containing $n + 1$, and n tiles, respectively, as presented in Fig. 9.4. Ciaran identified terms as containing t diagonal pairs, with an additional tile on the top right corner, as presented in Fig. 9.5.

Grace and Ciaran made these comments following their physical construction of pattern terms using tiles. Preceding their construction of the terms, both children had tended strongly towards recursive thinking where comments such as "each time you're adding 2" were typical along with references to the total number of tiles as always being an odd number. A focus on 'numerical' aspects of the terms also seemed to dominate their group's interactions before construction of the terms. However, following their use of the manipulatives, both children succeeded in drawing on 'figural' aspects in constructing factual generalizations (Radford 2010; Rivera and Becker 2011). I thought therefore that the pathway followed by Grace and Ciaran in their thinking about this pattern merited further analysis.

Fig. 9.4 Grace's construction of Term 12 of Pattern 1, presented as two rows of tiles, where the bottom row contains n tiles, and the top row contains $n + 1$ tiles

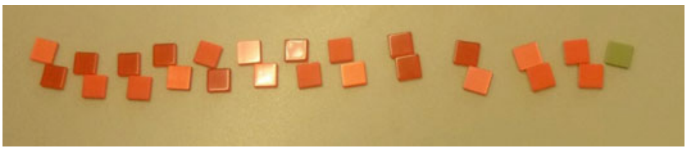

Fig. 9.5 Ciaran's construction of Term 12 of Pattern 1, presented as n pairs of tiles presented diagonally, with one additional shaded tile

Warren and Cooper (2008) found that the use of manipulatives combined with number card identifiers for terms, supported children in shifting their focus from relationships between subsequent terms, to relationships between terms and their position. In exploring the strategies Ciaran and Grace adopted, I sought to consider whether the use of manipulatives played a role in broadening the children's focus in this way. I examined, therefore, excerpts of the group's conversation during the period of the interview when their perspectives appeared to shift. The following transcript is the initial reaction of the children when they are presented with four terms of Pattern 1 and asked to describe what they see:

Daniel [reads question] Oh look, it's going up in twos.
Grace I already said that.
Ciaran Yeah, but you see, [points], 3.
Daniel 3, and then 2, and then another 2.
Ciaran Yeah, but look, each time it gets bigger.

Within this transcript, Grace's first comment of "I already said that," referred to a previous comment, which was inaudible, and immediately preceded Daniel's reading of the question. While Grace and Daniel immediately referred to the difference between subsequent terms, Ciaran seems to prefer to construct his thinking independently, as he says "but you see" and points to the three tiles comprising the 1st term in the pattern. Having counted the tiles in term 1, he then observes that each subsequent term is "bigger" without specifying size or the quantity of tiles. Interpreting this comment, I would suggest that Ciaran was not listening closely to his peers at this point, as he seems to be trying to convince them that the terms are growing in size, when Daniel and Grace have both already acknowledged this aspect. Unfortunately Grace and Fiona tended to whisper to each other during this section of the interview, and much of their discussion was inaudible, even though I placed a recorder on the desk right in front of them. In Table 9.4, I present the children's discussion following from the transcript above, and I focus on Ciaran's comments as he refines his thinking and begins to identify aspects of the structure of the pattern, such as the difference between consecutive terms.

Ciaran's initial three comments during this discussion focused on the total number of tiles for terms in the pattern, and he attributed the property of being odd or even to the totals rather than mentioning relationships between terms, or between terms and their positions. Daniel, in contrast, focused strongly on the rate of growth between terms in the pattern, repeatedly referring to "going up", and 2 as the rate of change between consecutive terms. Daniel and Ciaran faced each other during this exchange, while Grace and Fiona held a parallel discussion.

After Daniel's comment "3 and then another 2, and then another 2", Ciaran's attention seemed to shift, whereby he used the term "more", as opposed to "bigger," which he had used in the earlier extract, in reference to the difference between consecutive terms. This may indicate that Ciaran is beginning to refine his sense of the quantity of tiles that are added to construct each new term (as well, perhaps, of the physical configuration of pairs of light tiles that increase by 2 each time, in addition to the darker tile at the end of the pattern that does not increase in number

Table 9.4 Discussion among Grace, Ciaran, and Daniel where the progress of Ciaran's thinking is isolated for examination

	Comment made	Description
Daniel	1, 2, 3, then, 1, 2, 3, 4, 5, that's going up	
Ciaran	[Counts, pointing with finger]	
Ciaran	They're all even	Mistakenly observes that the total quantity of tiles in each term is an even number
Fiona and Grace	[Whispering inaudibly]	
Daniel	Yeah, they're always going up in twos	
Ciaran	Yeah, but, Daniel, look, can't half a 3, can't half a 5, can't half 7, can't half 9. And then they're all odd	Notices that the totals are not divisible by 2, so his previous observation of evenness is corrected: the quantity of tiles within every term is an odd number
Daniel	They're all odd, and these are evens	
Grace	They're all odd, 3 s are odd, 5 s are odd, 7 s are odd, so they're all odd	
Ciaran	No these are odd. See look can't half a 3, can't half a 5, can't half a 7, can't half a 9, can you?	Repeats the observation that the quantity of tiles within every term is an odd number
Daniel	No, and also 2s, get it? 3 and then another 2, and then another 2	
Ciaran	Yeah so, they're all going up, each time there's more	A reference to "more" rather than "bigger"
Daniel	Number 1 they're all odd and number 2 there's adding 2	
Ciaran	Yeah each time you're adding 2, but you're not adding that, the darker one, but they're all even, I mean they're all odd	A recursive description of the pattern, drawing a connection with the sequence of odd numbers

—as hinted at by his last comment in Table 9.4). In his comments from this point, he began to incorporate a comparison of consecutive terms for similarities and differences "each time you're adding 2, […] they're all odd." Tracking Daniel's thinking in a similar way, we can see that his initial observation was of the difference between successive terms, and he progressed from here to describe both the odd-numbered total of tiles in each term and the difference between terms. Daniel and Ciaran seemed to draw each other's attention to two salient details, the total number of tiles for each term and the rate of growth between terms. In the following paragraphs, I analyze subsequent extracts from the children's discussions, which offer further insight into their thinking about Pattern 1. During their conversations, the children's observations of the structure of the pattern develop to include both recursive and explicit approaches.

Within this group, an overly zealous consideration of whether total numbers of elements were odd or even at times dominated the children's discussions of this pattern and subsequent patterns. Frobisher and Threlfall (1999) suggest that children may notice aspects such as whether the total number of elements of a figure is odd or even, rather than the aspects we expect them to notice such as the rate of change or relative rates of change. Aside from the odd or even aspect, the children seemed to be focusing strongly on recursive and numerical aspects of the pattern, whereby the term number had not yet been mentioned. Also little reference was made to the size or shape of terms, other than when Ciaran observed that "each time it gets bigger." Shortly after this exchange, and before construction of the terms using manipulatives, Grace suggested that "the number on the bottom, that eh, the number on the bottom, eh, each one on the bottom is less and each one on the top is more." While this comment does not indicate any relationship between the number of tiles and their position in the pattern, Grace did separate the two rows of the terms in a figural manner in order to investigate the structure of the pattern.

After the children commenced physical construction of the 5th term of this pattern, the following exchange took place:

Ciaran Will I help you on 6 cos I'm done 5 now?
Daniel I'm done 6.
Ciaran No, you have to make it like this. You have to go diagonally see like this.
Daniel Like a pattern.
Grace It goes like that, it goes like that, it doesn't go straight.
Ciaran Yeah, you have six of them going diagonally, and then you need to put a red one in the corner, which is the weird one.
Daniel Yeah.

Daniel erred in his construction of the 5th term and, as shown in this transcript, Ciaran tried to explain the structure of the 6th term to him by describing 6 diagonal pairs, with an additional top tile, as portrayed in Fig. 9.6.

During this section of the interview Grace and Fiona spoke to each other, at times almost inaudibly, and Daniel and Ciaran continued to work together. Ciaran again referred to the tiles "kinda going diagonally", and Daniel added "oh yeah, because that one, that's sticking out, that's sticking out, so, and they're not like all in a line." When I asked Daniel and Ciaran which terms they had constructed, Daniel identified his construction as the 6th in the pattern, but Ciaran corrected him, saying "No, I'll do 6, look, you need 6 of these going diagonally look, look, you

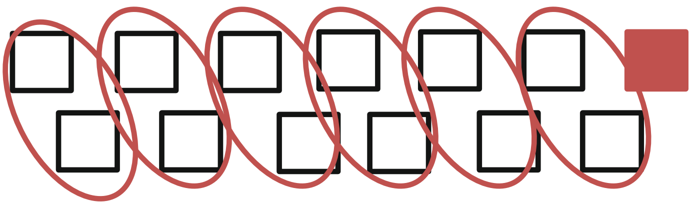

Fig. 9.6 A diagram based upon Ciaran's description of the 6th term of Pattern 1

need like this but do the 3 more, you see" and Grace joined in describing her construction of the 6th term as "you know you do 7 squares, and 6 at the bottom."

Both Grace and Ciaran at this point are describing the terms explicitly—they do not refer to the relationship with previous terms, and the quantities of tiles they describe are related to the position number of the terms. I next asked the children to describe the 12th term, as a near generalization of the pattern. At this point Grace stated "it's obviously going to have 13, because term 5 had 6 squares on the top and 5 on the bottom, so term 12, so it's gonna be really easy," as highlighted at the beginning of this section. Ciaran did not offer such a confident description at the start, but he did construct the 12th term perfectly, during which the following exchange occurred with Daniel:

Ciaran	1, 2, 3, 4, 5, 6, 7, 8, 9, we need 4
Daniel	We need 3 more
Ciaran	No, do you remember we're making 12

Ciaran required 13 tiles for the top row of the term, indicating possibly that his thinking had broadened from the sets of pairs of tiles to a view of the terms as consisting of two rows, where the top row of a term contains $n + 1$ tiles.

Ciaran and Grace are referring to particular terms in this situation, and have not constructed abstract generalizations, but they appear to have applied an understanding of the commonality of this pattern, as they worked with "particular instances of the variable" (Radford 2011). While a variety of factors may contribute to the children's thinking, the change in how Ciaran and Grace think about the pattern after constructing the terms with the concrete manipulatives is noteworthy. Perhaps the physical act of construction supported them in isolating elements of the pattern, and in considering the relationship between, on the one hand, quantities of elements in constituent parts of their constructed terms and, on the other hand, the relevant position number in the pattern.

9.4.3 Facilitator Prompts and Questions

During two of the interviews, I felt that children's discussions indicated that their perspective was focusing on the relationship between consecutive terms, or how the patterns were growing, to the exclusion of any other aspect of the structure. Therefore I encouraged the children to consider a relationship between terms and their position in the patterns. For example, during their discussion of Pattern 2, Group 1 students were making slow progress and tending to use vague language. I felt at that point that a prompt would be appropriate in focusing their attention, and supporting them in engaging with the task. I asked three times whether the children could see any connection between the term number and the quantity of tiles on the legs of the x-shape. The children at this time seemed to be struggling to associate a number with terms or to perceive the rates of change of constituent parts of the pattern terms, and my prompts were not supportive of their thinking, as seen in the following transcript:

AT Can you see any connection between the term number and the length of the legs? Or between the term number and the term? If we look at term 1, the number is 1. Look at the term, term 2 the no is 2, look at the term, term 3, term 4. Ok, Grace can you explain to us where… why you decided 21?

Grace Because like, em, you see the 6 that had em 3 on all each sides and then one diamond in the middle. So, em two threes would be six, and then two sixes would be twelve, and then eh twelve add one that would be like 13, and then the 10, it would be.. Like each time whenever you add something like 7, that would be just 2 more, so it would be eh 15, and then 8 that would be 17, and then the 9 that would be the 19, so the 10 has to be the 21.

Daniel Oh yeah.

AT Ok, now em, I'm going to ask again for you to look back and see can you see any connection between the term and the term number.

Daniel That, they're longer.

As highlighted by Radford (2000), children need to perceive the same number as fulfilling both ordinal and cardinal roles simultaneously when engaging with patterning problems. I posit that during this section of the interview, the children were not succeeding in this dual perception. Daniel's response to my question may indicate a persistent focus on the relationship between consecutive terms, but Grace's response was more complex in nature. She began by considering the 3 tiles on each side of the 6th term, whereby she combined four groups of three in order to find the total quantity of tiles on the legs, to which she added the one central tile for the total number of tiles. From this 6th term however, she seemed to progress recursively in order to find a total number of tiles for the 10th term. The children's recursive approach did not seem at this point to be supporting them in thinking about bigger terms, but neither did my promptings to consider an explicit approach. Most of the children continued to struggle with this pattern, with only Grace succeeding in constructing the near generalization outlined above. Later in the conversation I also drew the children's attention to figural aspects of the patterning terms, as I felt that their focus was dominated by the numerical aspects. After receiving this prompt, Ciaran proceeded to construct a far generalization.

Group 2's discussion of Pattern 2 also seemed to be dominated by recursive reasoning, to the exclusion of any other perspective. I prompted the children by asking whether anyone could see a connection between terms and their corresponding term numbers in Pattern 1, but none of the children replied to my question, and they continued to discuss the terms recursively.

Aside from prompting the children, each of the worksheets on which the patterns were presented included a final question "Can you see a connection between the term number and the term?" When designing the interview schedule, I was aware that it was possible that this question at the end of children's discussions about Pattern 1 could have prompted some children to broaden their thinking to include an explicit approach to Patterns 2 and 3. I deemed the question appropriate therefore as a means of exploring the children's observations of the structure of the pattern. As indicated by the analysis in this chapter, not all children adopted an explicit approach in seeking to solve Pattern 2 or Pattern 3, and many did not respond to this question when it was presented to them. As an example, in interviewing Group 2, I pressed them to consider whether there was a connection between a term number and the number of elements within the associated term. Our discussion related to Pattern 1 is presented in the following transcript:

Arina [Reading the question at the end of the worksheet] So, can you see a connection between the term number and the term?
Arina It's just… the connection… term one, sorry.
Alex Well I can't see any connection, I can't see any connection between the term, but I think I can see a connection between the numbers.
Arina Yeah, the numbers, term one.
Alex Oh yeah, like all of them, when term one is going on to term two it skips, it skips four and then it just goes on to this.
AT What do you think, Jay? Do you see a connection between the term and the term number?
Jay Erm… no.
AT Cherry?
Cherry I think… because like term one is like, there's like one on the bottom.
[A child visits the interview room, and interrupts the interview with a message inviting the children to visit a junior class to teach them mathematics games, when their interview has concluded. The children respond with smiles and a chorus of "yes!"]
AT Now, Cherry, you were saying what connection you saw between the term and the term number?
Cherry Like the term number has one and then the bottom one is one.
AT Great and is that true for every term?
Cherry Yeah.

In this excerpt from the children's discussion, Arina and Jay proffer no opinion on the question, and Alex demonstrates recursive thinking in describing the sequence of the total quantities of tiles in consecutive terms. Cherry does make a link between the term number and the quantity of elements in one constituent part of the terms, that is, the bottom row. It is surprising that none of the other children built upon this observation, as from my viewpoint as facilitator I expected it to be enlightening for others. Unfortunately, I feel that the message delivered by the child who visited the interview room drew children's attention away from the mathematics, as they were obviously looking forward to their visit to the junior class. Also, Cherry did not indicate an inclination to think explicitly when she was presented with Patterns 2 and 3. I would hypothesize that my question "is that true for all terms?" was possibly too closed and summative in nature. A more open question such as "do you observe anything similar happening elsewhere in the pattern?" may have facilitated children in examining the structure of the pattern, in a manner that the summative question did not.

9.4.4 Group Interactions

In this section I will discuss the impact of verbal interactions between children on the relationships children explored within the structure of the patterns.

In Group 3, Lily Rose seemed to begin by reasoning recursively when describing and extending all the patterns. For example, she stated: "I just added one more each on the top so all of them have three on them now," in explaining her drawing of the 6th term of Pattern 2. Following their construction of the 5th term and drawing of the 6th, I asked Lily Rose's group to consider the 10th term in this pattern. Lily Rose's answers focused strongly on figural elements of the terms and were quite vague, as in "it would be about that long" and "I think it would be this long and this wide." Her use of the 'hedges' "about" and "I think" may indicate some ambiguity in her thinking, and a desire on her part to present her ideas as proposals rather than conclusions (Rowland 2007). Such ambiguity, or unwillingness to present an idea as fully formed, may be expected in this context, given the novelty of the tasks; but it may also indicate that she has engaged with the alternative thinking of her group members and is beginning to question, or rethink, her approach. During the group discussion, the children's opinions varied; and so I asked them to discuss the differences between their ideas. The children discussed the distribution of tiles on the legs of the x-shape, and the presence of "some middle square." During this discussion Lily Rose suggested that the 10th term would have five tiles on each leg but neglected the central tile. It was not clear whether she was using explicit thinking, or building from the 6th term to the 10th, in that she followed "because half of 5 is 10 [sic] and then you give one each every time" with "it would have 5 going up and 5 going down; I think you might have 5 going across and 5 going left and right."

After further discussion about the 10th term, I asked the group to construct a far (75th) term for this pattern. At this point, Lily Rose demonstrated explicit thinking, as she sought without hesitation to halve 75, and adhered to the asymmetric nature of the pattern by stating, "you will have to give one side, one extra ones. Like down at the bottom two, they might have one extra than the top two." Again Lily Rose used the plausibility shield "might" in presenting her construction. While she was quick to suggest halving 75 as a strategy for identifying the number of tiles required for each leg of the x-shape, her language indicated that she remained tentative about the structure of the general term. Discussion of this pattern with her peers, along with attention to the tasks presented, seem to have supported Lily Rose in fine-tuning her thinking from her early vague statements to a very specific explicit strategy for describing the 75th term, with attention to the quantities involved and their positioning.

9.5 Conclusion

In considering the algebraic thinking demonstrated by the children who participated in this study, a question arose as to whether it would be reasonable to anticipate that children would identify explicit relationships in the structure of the patterns. Radford (2011) presents algebra as a cultural construct and states that children's skill development will benefit from facilitation by the children's educational environment. Similarly, Lannin (2004) contends that most children will intuitively reason recursively, and Lannin et al. (2006) recommend that alternative perspectives on the structure of patterns may require intervention. As discussed in a previous section of this chapter, it is improbable that the children had experienced any instruction or facilitation in observing structure within shape patterns prior to their engagement with the patterns in the group interviews. However, Mason (2008) contends that children demonstrate a facility in thinking algebraically from early childhood, including specializing and generalizing. Indeed, many of the children who participated in this research demonstrated a willingness to use explicit thinking in solving the patterning tasks presented. In the task-based group setting where children were encouraged to talk and physically construct terms with manipulatives, some of the children seem to have demonstrated thinking that Radford (2011) and Lannin (2004) have suggested may not be intuitive.

Two important elements of the children's engagement with the novel tasks presented were the interactions within the groups and the use of concrete materials in constructing pattern terms. Both catalysts of peer interactions and use of manipulatives were more supportive of the children's progress than prompts or leading questions from me as task facilitator. The progress demonstrated by Grace, Ciaran and Lily Rose may indicate that their thinking was supported by their interaction with the concrete materials and interactions with their peers. A further element that supported the thinking of some children was observation of figural, as well as numerical, elements of pattern structure (Twohill 2017). This study is too small to present definitive findings in that regard, but would support a rationale for further investigation in the Irish context.

Within this volume (see Chap. 14), John Mason emphasizes the need for research to inform teacher actions, and for such actions to broaden children's observations of structure. The analysis presented here is consonant with Mason's position, and demonstrates how facilitating children in observing structure in multiple ways supports their success in generalizing shape patterns. The value inherent in children constructing their understanding is largely accepted in mathematics education research, but teaching approaches underpinned by transmission persist in many classrooms in Ireland (Dooley 2011; Nic Mhuirí 2013). The success achieved by some children in this research, on novel high-order tasks, demonstrates the potential of discovery methods, and points to the benefits of specific catalysts to support children's observations of structure. A next phase of this research will involve student teachers in implementing discovery-based approaches with shape patterns in task-based group work in order to explore their sense of self-efficacy in this setting.

Equally, English (2011) warns that teachers and policy makers should not underestimate children's ability to take on and work with new ways of thinking. English states that children "have access to a range of powerful ideas and processes and can use these effectively to solve many of the mathematical problems they meet in daily life" (p. 491). In the forthcoming redesign of the Irish Primary Mathematics Curriculum, it will be important to reflect the broad range of thinking strategies of which children are capable, as evidenced in this research.

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Chapter 10
Generalizing Fractional Structures: A Critical Precursor to Algebraic Thinking

Catherine Pearn and Max Stephens

Abstract Our research focuses on how students find an unknown whole, when given a known fractional part of the whole, and its equivalent quantity. This chapter will show how Year 5 and Year 6 students, who have yet to meet formal algebraic notation, create algebraic meaning and syntax through their solutions of these fraction problems. Some students rely on diagrammatic representations using different mixes of multiplicative and additive strategies. Other students use fully multiplicative approaches to find the whole. Some students' solutions show how they use "best available" symbols to move beyond arithmetic calculation and show evidence of algebraic thinking, especially when students are able to treat particular numerical and fractional values as quasi-variables. This chapter sets out to identify those precursors of algebraic thinking that allow students to move beyond particular fraction values to generalize their solutions.

Keywords Fractions · Generalization · Algebraic thinking

10.1 Introduction

The National Mathematics Advisory Panel (NMAP 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fraction problems are central goals of students' mathematical development and are the critical foundations for algebra learning. The links between fractional knowledge and readiness for algebra have been highlighted by many researchers, such as Wu (2001), Jacobs et al. (2007), Empson et al. (2011), and Siegler et al. (2012). According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a

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C. Kieran (ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds*, ICME-13 Monographs,
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dynamic understanding of algebra" (p. 17). Many researchers argue that the basis for algebra rests on a clear understanding of equivalence and rational number concepts (Lamon 1999; Wu 2001).

Siegler et al. (2012) used longitudinal data from both the United States and United Kingdom to argue that "elementary school students' knowledge of fractions and division uniquely predict those students' knowledge of algebra and overall mathematics achievement in high school, five or six years later" when all other factors such as whole number arithmetic, intelligence, working memory, and family background were controlled (p. 2).

Three distinct aspects of algebraic thinking identified by Kieran (1981), Jacobs et al. (2007), Mason et al. (2009), and Stephens and Ribeiro (2012) are important for this study. They are students' understanding of equivalence, transformation using equivalence, and the use of generalizable methods. The importance of these three key ideas underpins our research.

10.1.1 The Australian Curriculum Context

According to the rationale given for the *Australian Curriculum: Mathematics* (ACARA 2016) the mathematics curriculum:

> … focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

The *Australian Curriculum: Mathematics* (ACARA 2016) presents fractions as a clearly important topic across Years 5–8. However, we notice that the focus at Year 6 is on finding fractional parts of a known whole and at no stage directs the attention of teachers to finding the whole when given a known fractional part. Some might think that this could be addressed in Year 7 when students are asked to solve problems involving addition and subtraction. But, this appears to exclude multiplicative solutions to fraction problems especially those involving an unknown whole. Table 10.1 implies that the link between fractions and algebra is limited to number patterns and sequences involving fractions. In Year 7 students are expected to be introduced to the concept of variables and to use letters to represent numbers. But any bridge between fractional knowledge and algebraic thinking is left unstated or at best implicit. Drawing attention to the importance of linking fractional competence, in particular, understanding, using, and generalizing fractional structure, is the focus of this chapter.

Table 10.1 Content descriptors from *Australian Curriculum: Mathematics* (ACARA 2016)

Year	Fractions and decimals	Patterns and algebra
5	Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103)	Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107)
6	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)	Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)
7	Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153)	Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)
8	Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)	Simplify algebraic expressions involving the four operations (ACMNA192)

10.2 Previous Related Research

The current study builds on the research of Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015), who investigated students' quantitative reasoning with fractions and algebraic reasoning in writing and solving equations. Their research sample involved 18 students in total from middle school and senior high school. Lee (2012) focused on two seventh grade students, Lee and Hackenberg (2014) wrote about one of the two students, and Hackenberg and Lee (2015) focused on 12 of the larger group of 18 middle and high school students. In their research, students were given a fractional relationship between two collections of objects. The actual size of the collections was initially given but in later questions was unstated and so needed to be represented by "unknowns". Students were first asked to "Draw a picture of the situation", and were then asked to write an algebraic equation to represent this relationship. An example of the first kind of problem used by Lee and Hackenberg (2014) is the *Tanya-David Money Problem*:

> *Tanya has $84, which is $^4/_7$ of David's money. Can you draw a picture of this situation? How much money does David have?*

In this fraction task the fraction $^4/_7$ represents an amount of $84. A typical solution to this problem might require students to represent Tanya's money by a rectangular shape consisting of four equal parts, and David's money by a rectangular shape divided into seven equal parts. With an appropriate pictorial representation, this problem could be solved, without writing an algebraic equation, relying on the fact that the $84 can be split into four equal parts consisting of $21 and so the seven equal parts would be presented by $21 × 7 ($147). But in the algebra tasks the corresponding numerical quantity is unstated. For example, in the *Sam-Theo CD Problem* students were told:

> *Theo has a stack of CDs some number of cm tall. Sam's stack is two-fifths of that height. Can you draw a picture of this situation? Can you write an equation for how tall the height of Sam's stack is?*

Lee and Hackenberg (2014) presented this second problem as an instance of reciprocal reasoning. Having drawn a picture of a rectangle that was partitioned into five equal parts, Willa, the student who was the focus of this study, then drew another rectangle that spanned two of those five parts. Willa used t to represent the height of Theo's stack and s to represent the height of Sam's stack. Willa then wrote a correct equation $s = (^2/_5)\ t$. A correct pictorial representation of the fractional relationship allowed Willa to show the relationship between Sam's smaller CD stack and Theo's larger CD stack regardless of the specific number of CDs making up the respective collections. Willa was also able to write $t = (^5/_2)\ s$, explaining that she "got 10 pieces by taking Sam's two-part stack height five times and then divided by two to find Theo's five-part stack height". We call this "reverse fractional thinking". It is interesting to note that Willa obtained the second "reciprocal" relationship by manipulating her pictorial representations, as distinct from transposing the first equation algebraically. The research by Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015) shows that fractional knowledge is closely related to establishing algebra knowledge in the domains of writing and solving linear equations. They concluded: "Teaching fractions and equation writing together can create synergy in developing students' fractional knowledge and algebra ideas" (Lee 2012, p. 9). Willa and the other students interviewed in their study needed a clear understanding of fractional structures in order to write correct algebraic equations in those cases where the physical amounts being represented by a fraction were left unstated.

The focus of the Lee and Hackenberg research was on whether students were able to write *specific algebraic equations using appropriate algebraic notation*, namely, being able to write and solve an algebraic equation to represent a multiplicative relationship between two unknown quantities. Our study involves students in the final two years of elementary school who have met fractions but are not yet expected to use and write algebraic notation. While it would be reasonable to invite these younger students to attempt to solve problems similar to the *Tanya-David Money Problem*, it would be less reasonable to present students with problems like the *Sam-Theo CD Problem*, unless they can be led into these more abstract problems by careful scaffolding. With younger students, we are keenly interested in the methods they use to approach problems such as the *Tanya-David Money Problem*, including contexts where they are provided with diagrammatic representations of the quantities involved and other contexts where no diagram is supplied.

Our research question is focused on identifying indicators of generalized fraction thinking as students attempt to find an unknown whole when presented with a known quantity representing a known fraction of the whole. Like Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015), our study will use specific fractional quantities. Initially, like Lee and Hackenberg (2014), we will attach specific quantities to the known fractional part as our younger students are asked to find the whole. However, it will be necessary to follow Lee and Hackenberg's later tasks, such as the *Sam-Theo CD Problem* and introduce progressive degrees of generality into our questions if we are to establish strong evidence of students' being able to generalize fractional structures even if they do not express this in written algebraic equations involving appropriately designated unknowns. That is a major challenge for our research. How we deal with these challenges will become evident subsequently in this chapter in the development of our research instruments, especially in our design and use of the interview protocol.

10.3 The Pilot Study

Eighteen Year 6 students (11–12 years old) from a metropolitan school in Melbourne were chosen as they were deemed by their teachers to be highly successful in mathematics. They were assessed using a paper and pencil test including three reverse fraction tasks (Pearn and Stephens 2016).

The three reverse fraction tasks (Fig. 10.1) specifically require students to find a whole collection when given a part of a collection and its fractional relationship to the whole. Similar tasks were used in the research by Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015). Elements of algebraic thinking may be demonstrated in these fractional tasks where students can use equivalence, transformation, and generalizable methods to solve them.

Each fraction question (Fig. 10.1) was scored out of three. Zero was given when the question was not attempted or the answer incorrect. One mark was given for a correct response with no explanation or if there was some evidence of a correct diagram which the student did not take further (starting point). Two marks were given for a correct answer with limited explanation and three marks were given for a correct answer with a mathematically complete explanation.

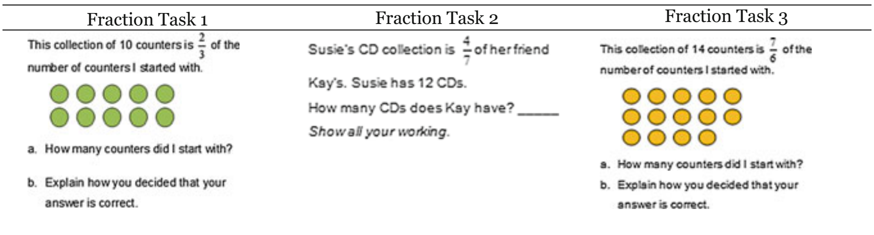


Fig. 10.1 The three fraction tasks

In this pilot phase of the study, three research questions needed to be addressed:

1. What solution strategies were used by students who successfully solved the three given fraction tasks?
2. What key features are shown in the methods used by these students?
3. Can the methods used be interpreted as providing evidence of algebraic thinking and generalization?

10.3.1 Results

Of the eighteen students, eleven obtained correct answers to the three questions; and seven of these provided clear explanations. Four students got two questions correct and three students got only one correct. Multiplicative methods were used by fourteen of the eighteen students in at least one question. Two students who solved only the first problem used additive methods, either arguing that since two-thirds is equivalent to 10 a further one-third is needed to make a whole; or having found one-third to be equivalent to 5, two successive thirds needed to be added to make 15.

The remainder of this section examines the strategies used in the solution of the three reverse fraction tasks by the seven students who provided three correct solutions together with mathematically complete explanations. Our focus is to examine closely the strategies these students used to solve the three reverse fraction tasks and to look for evidence that might anticipate algebraic thinking.

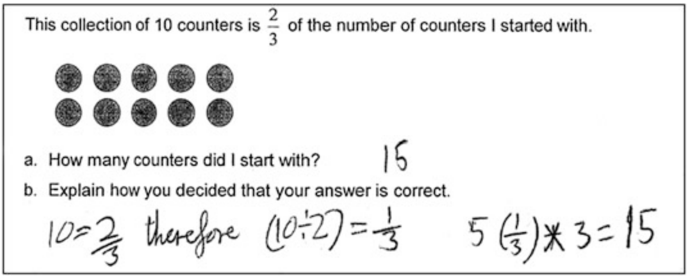
10.3.1.1 Fraction Task 1

In response to Fraction Task 1 all three students shown here initially made the connection between the number of objects and the given fractional part and divided by the numerator of the given fraction to find the number of objects in the unit fractional part. They then calculated the number of objects in the whole by multiplying by the denominator.

For Fraction Task 1 Student 3 (Fig. 10.2) made the connection between the number of objects and the given fractional part and started with an equivalence statement between ten and two-thirds. This student found the number of objects in the unit fractional part by dividing by the numerator of the given fraction and calculated the number of objects in the whole by multiplying by the denominator.

Student 3 used abbreviated symbolic/number notation for all three fraction tasks; such as 10 equals two-thirds as a shorthand way for saying that ten counters is two-thirds of the group.

Student 6 (Fig. 10.3) started with an equivalence statement between two-thirds and ten. Like Student 3, the symbolism may seem idiosyncratic but the

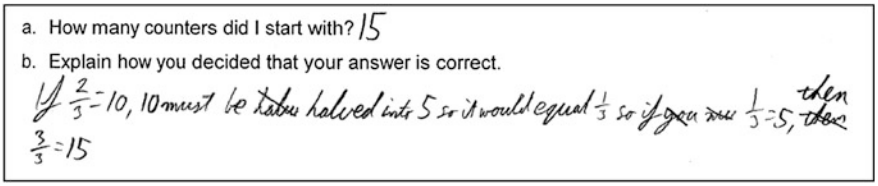

This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.

a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.

$10 = \frac{2}{3}$ therefore $(10 \div 2) = \frac{1}{3}$ $5 (\frac{1}{3}) * 3 = 15$

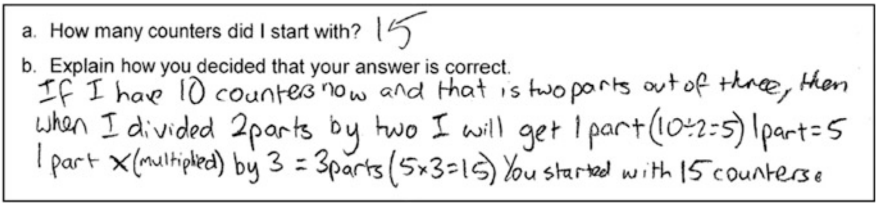
Fig. 10.2 Student 3 response


a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.

If $\frac{2}{3} = 10$, 10 must be taken halved into 5 so it would equal $\frac{1}{3}$ so if you see $\frac{1}{3} = 5$, then $\frac{3}{3} = 15$

Fig. 10.3 Student 6 response to Fraction Task 1


a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.

If I have 10 counters now and that is two parts out of three, then when I divided 2 parts by two I will get 1 part (10÷2=5) 1 part = 5 1 part × (multiplied) by 3 = 3 parts (5×3=15) You started with 15 counters.

Fig. 10.4 Student 5 response

mathematical meaning is clear as this student moved from two-thirds to one-third to three-thirds, referring to the equivalent number of counters.

In responses to Fraction Task 2 (not included) and Task 3 (see Fig. 10.8), Student 6 created an initial equivalence statement by writing ${}^{4}/_{7} = 12$ and ${}^{7}/_{6} = 14$ respectively. In both cases Student 6 first found the relevant unit fraction and scaled it up to find the whole. Recording is again idiosyncratic, for example, ${}^{1}/_{6} = 2 \times 6 = 12$ where Student 6 compressed two operations into one symbolic statement. These two equivalent expressions, $10 = \frac{2}{3}$ by Student 3 and $\frac{2}{3} = 10$ by Student 6, appear to be mathematical objects that can be operated on to obtain the unknown whole. Student 5 (Fig. 10.4) used reverse thinking, which started with a detailed written response.

10.3.1.2 Fraction Task 2

In the work samples shown here, both students initially made the connection between the number of objects and the given fractional part and divided by the numerator of the given fraction to find the number of objects in the unit fractional part. They then calculated the number of objects in the whole by multiplying by the denominator.

Student 2 (Fig. 10.5) initially made the connection between the number of objects and the given fractional part and dividing by the numerator of the given fraction found the number of objects in the unit fractional part. She then calculated the number of objects in the whole by multiplying by the denominator. This student used abbreviated symbolic/number notation for all three fraction tasks.

Student 1 (Fig. 10.6) also used the equal sign in a way that some might deem incorrect. However, the 4 written after the 12 refers to the numerator of the fraction, hence the division by 4 to obtain 3. Multiplying by 7 is needed to transform one-seventh to a whole. The size of the fraction, although unstated, guided this student to a correct solution that was generalized to the other two fractional tasks. Some researchers, such as Mason (2017), refer to this as 'tracking arithmetic,' where changes to the fractional quantities are tracked onto their equivalent numerical representations using parallel operations.

The combination of these two methods anticipates very closely how one needs to solve $\frac{4}{7}x = 12$.

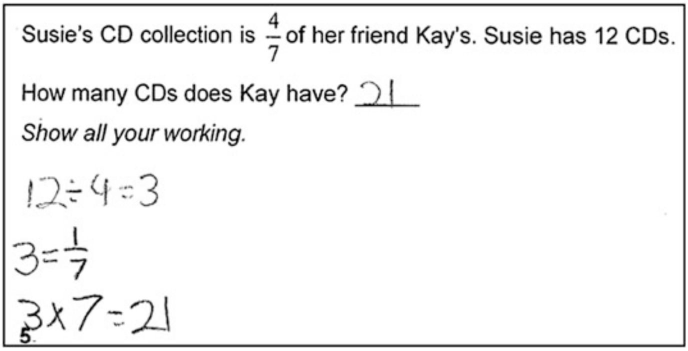


Fig. 10.5 Student 2 response

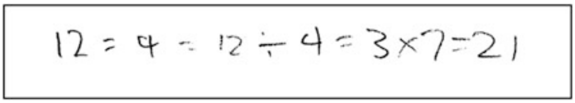


Fig. 10.6 Student 1 response to Fraction Task 2

10.3.1.3 Fraction Task 3

The three students shown here initially connected the number of objects and the given fraction, then found the number of objects equivalent to the unit fraction. They then calculated the number of objects in the whole by multiplying by the denominator. Student 5 (Fig. 10.7) incorrectly wrote 7/6 ÷ 1/7 = 1/6 (instead of multiplying by one-seventh) but correctly operated on 14.

Student 6 gave a more abbreviated response for this task (Fig. 10.8) than for Fraction Task 1 (Fig. 10.3). Here the 14 (counters) was related by an arrow to the 7 (numerator) and ÷2 the implied relationship. That is, since the number of counters is double the numerator, Student 6 divided 14 by 2 and concluded that, since $\frac{1}{6} = 2$, then "×6" gives 12.

Student 4 used symbolic/number notation for all three fraction tasks. Figure 10.9 shows this student's response to Fraction Task 3.

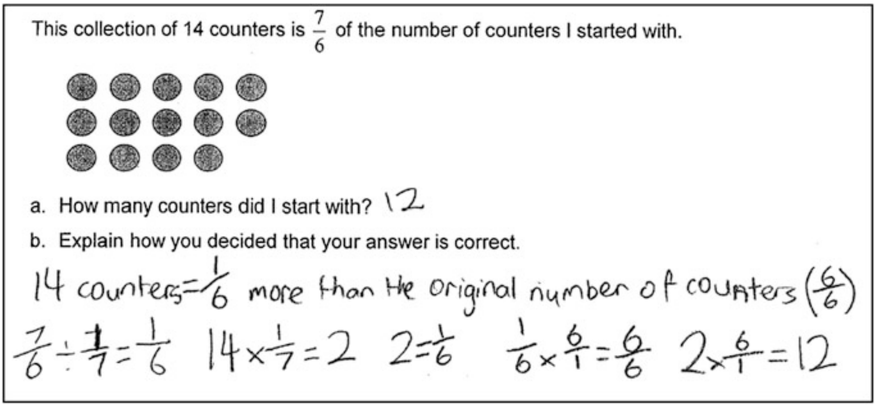


Fig. 10.7 Student 5 response

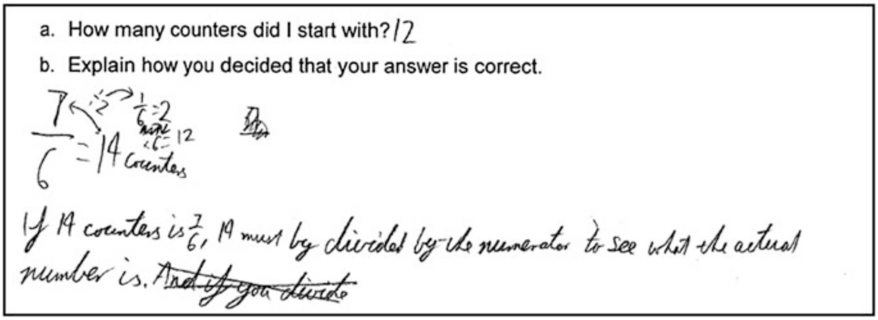


Fig. 10.8 Student 6 response

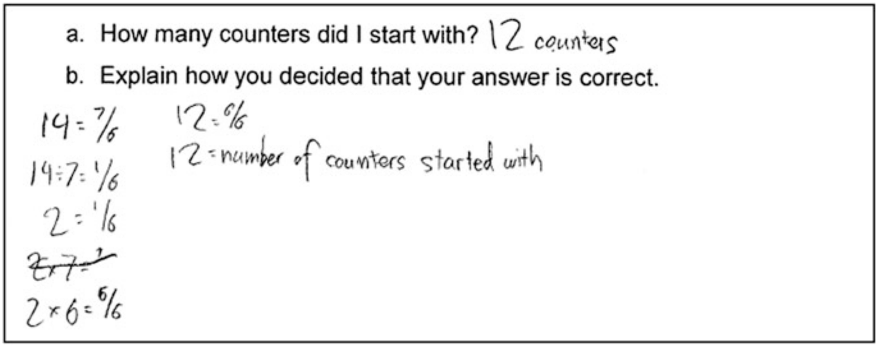


Fig. 10.9 Student 4 response

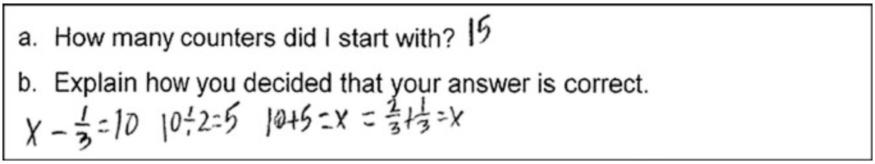


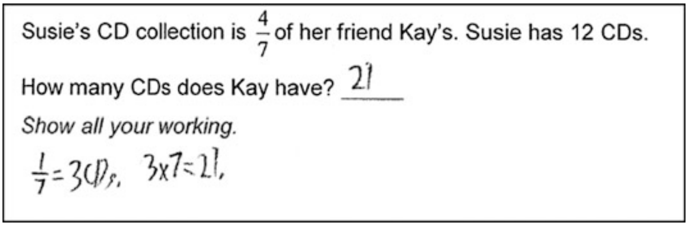
Fig. 10.10 Student 7 response to Fraction Task 1

In the examples shown above (Figs. 10.2, 10.3, 10.4, 10.5, 10.6, 10.7, 10.8 and 10.9) all students initially made the connection between the number of objects and the given fractional part, and then divided by the numerator of the given fraction to find the number of objects in the unit fractional part. They then calculated the number of objects in the whole by multiplying by the denominator. Treating the fraction and the objects represented by the fraction separately *parallels* what is needed to solve simple algebraic equations.

10.3.1.4 Use of Mixed Methods by Student 7

Student 7 was the only student from this group who used different solution methods for the three fraction tasks. In Fraction Task 1, Student 7 used a mixed method rather than a totally multiplicative one (see Fig. 10.10), by calculating what one-third was, then added on one-third to the original two-thirds to find three-thirds. This student was comfortable with using 'x' to denote the whole, but this use of an unknown did not involve any symbolic manipulation.

Figure 10.11 shows a shorthand response by Student 7 for Fraction Task 2, finding the unit fraction and then multiplying by the denominator (7) to calculate the whole: This fully multiplicative response is very similar to the way Student 2 responded to the same task in Fig. 10.5.

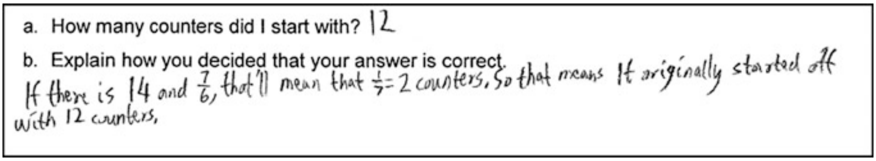

Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.

How many CDs does Kay have? 21

Show all your working.

$\frac{1}{7}$ = 3CDs, 3x7=21,

Fig. 10.11 Student 7 response to Fraction Task 2


a. How many counters did I start with? 12

b. Explain how you decided that your answer is correct.

If there is 14 and $\frac{7}{6}$, that'll mean that $\frac{1}{7}$=2 counters, So that means It originally started off with 12 counters,

Fig. 10.12 Student 7 response to Fraction Task 3

For Fraction Task 3 (Fig. 10.12) this student incorrectly wrote $^1/_7$ instead of $^1/_6$ but self-corrected, multiplying by 6 to reach the correct answer.

These Year 6 students know that they must move from any given fraction to its unit fraction and then scale up to one-whole. Some may do this additively but scaling up multiplicatively is the ultimate goal for transitioning to algebra. At the same time, they "track" the same mathematical operations on the number of objects represented by the given fraction, using "best available" symbols to record their chains of reasoning.

10.3.2 Algebraic? What Some Experts Say

As part of our pilot study, we asked an international group of 15 mathematicians, mathematics educators, and practicing teachers to contrast responses of these seven students with a sample of responses from other students who had obtained the correct answers by using additive or pictorial methods. We asked the respondents what if anything, in their opinion, "anticipated algebra" in the group discussed above, bearing in mind that none of the students has yet been introduced to formal algebraic notation. We left these experts to give their own meaning to "anticipating algebra". Three responses referring to what Students 1 and 6 had written are quoted here. One Professor of Mathematics expressed concern: "These students may have difficulty with algebra because they do not understand the meaning of the equals sign". By contrast, a Professor of Mathematics Education, referring to Fig. 10.3 said that "$\frac{2}{3}$ = 10 is treated like $\frac{2}{3}\,x = 10$". She went on to say that "this latter algebraic expression is solved by dividing by 2 and multiplying by 3, in exactly the same way as Student 6 had done in Fig. 10.3. A third Professor of Mathematics, outside Australia, commented that Students 1 and 6 were:

> *'tracking arithmetic': not allowing parameters (numbers) in the problem to be calculated, so that at the end their role and influence can be tracked, and generalization consists of treating each original number as a place holder, so treating it algebraically.*

Another Professor of Mathematics Education drew attention to "students' use of the property (rule) of equality: multiplying or dividing same number to both sides" as evidence of algebraic thinking. Another mathematics education researcher gave a more extended response:

> *As Blanton and Kaput (2005) argue: "algebraic reasoning is characterized by generalizing from a set of particular instances and expressing them in increasingly formal and age-appropriate ways" (p. 413). While limited by the arithmetic nature of the three reverse fraction problems, it is clear that these students are generalizing and formalizing more effectively than other students who rely on purely pictorial methods such as circling rows in Task 1 or counting by pairs in Task 3. These methods are least likely to be successful in Task 2.*

Seven practicing teachers who were consulted generally supported the view that the kind of thinking shown in these three examples was algebraic or could be seen as anticipating algebra, specifically through students' use of equivalence and generalizability. While there was no uniform agreement about what exactly anticipated algebra in the work samples presented to this group, the majority of the teacher respondents drew attention to features that indicated a definite shift away from arithmetical/calculation thinking.

10.3.3 Conclusions from the Pilot Study

These responses call for a stronger conceptual framework for the terms "anticipated algebra" or "anticipating algebra". The three fraction tasks used in this pilot study required students to find an "unknown" whole when presented with a given collection that is a subset representing a given fraction of the whole. While some students made explicit use of the diagrams provided in the first and third questions, no student felt a need to create a diagram for the second question. Where students consistently used multiplicative methods, usually by first finding the unit fraction and its equivalent quantity and then scaling up to find the unknown whole, there was stronger evidence of generalizability and algebraic reasoning across the three questions. However, it should be noted that no student employed explicit symbolism in the form of an unknown. Nor did any student propose a solution strategy whereby the unknown whole could be found by explicitly dividing the known quantity by its equivalent fraction. Nor did any student propose multiplying first by the denominator and then dividing by the numerator. Typical multiplicative solutions where the given quantity was first divided by the numerator of its equivalent fraction, to find its unit fraction equivalent, and then having this quantity scaled up by multiplying by the denominator may be perceived as mathematically equivalent to dividing by a given fraction. But these are essentially two-step strategies, quite different from a one-step "divide the quantity by the given fraction" which is more clearly generalizable.

In answering the three research questions that were to be addressed in this pilot study, the following tentative conclusions can be drawn:

1. Students who successfully solved these problems typically moved from the given fraction to its related unit fraction and its corresponding quantity. Some then moved additively to find the whole. The majority of the eighteen students were able to find the whole multiplicatively by scaling up. However, "scaling up" can imply scaling up by multiplication only or it can also include scaling up by repeated addition. Both approaches are acceptable when the given fraction is tied to a specific quantity. But as the Lee and Hackenberg (2014) study shows, repeated addition is conceptually difficult to apply when the quantity associated with a fractional part is unspecified.
2. Many successful solutions employed a shorthand way of relating the known fraction and its related quantity. Being able to connect the known fraction and its related quantity in this way allowed students to operate simultaneously on both to find the whole. Most students chose to do this multiplicatively although some successfully employed additive methods.
3. Those experts cited above who see evidence of algebraic thinking appear to point in particular to where students establish implicitly or explicitly an equivalence relationship between a given fraction and its related quantity, and are then able to operate on both components of this relationship to find the whole fraction and its equivalent quantity. As well, multiplicative methods based on equivalence appear to be generalizable. However, this rests on an assumption that students who have used this method will use it when presented with other fractions and with other related quantities or where no quantity is specified. That appears to be likely for those students who consistently use an equivalence-based multiplicative solution strategy across the three questions. Being able to use this method for *any* fraction and *any* quantity would be clearly algebraic and generalizable. But it is difficult to draw this inference confidently from the current examples without some stronger evidence of students' generalized thinking.

In the next phase of the study, we will seek to demonstrate whether and how students who rely implicitly or explicitly on an equivalence relationship between given fractions and their related quantities can find an unknown whole *regardless* of the particular fractions or quantities used. This reasoning might be expressed symbolically or it might be equally well described using mathematical terms that convey a similar generalizable meaning. In other words, are students able to treat the different given fractions as *quasi-variables* (Fujii and Stephens 2001)? In other words, are students able to recognize that the operations that they apply to a given fraction are repeatable and generalizable across other instances?

10.4 Follow up Study 2

The findings from the pilot study shaped the second phase of the study in which a written questionnaire containing the same three reverse fraction tasks were given to 46 Year 5 and Year 6 students in a different school. Unlike the pilot study where only students who were deemed to be highly successful participated, all Year 5 and Year 6 students in the second school were given the opportunity to participate. Working across a broader range of abilities was intended to allow a clearer identification and classification of solution strategies used by students at these year levels. It would also allow the researchers to examine whether particular solution strategies offered greater potential for generalized algebraic thinking. Most importantly, the second phase of the study was intended to gather more explicit and robust evidence of algebraic thinking and generalization. Based on an analysis of students' written responses, a sub-sample of the 46 students participated in a one-to-one interview utilizing the same fractions as in the original assessment (see Fig. 10.1) but where the quantities were changed or unspecified, and no diagrams provided (see Sect. 10.4.2).

10.4.1 Completing the Paper and Pencil Assessment Protocols

At the end of the 2016 school year, 46 students from Years 5 and 6 (11 and 12 year olds) from an inner city Melbourne primary school completed a paper and pencil test: the Fraction Screening Test (Pearn and Stephens 2015). Three days later, 17 students were selected and interviewed to gain further insights into the strategies they used to solve the three reverse fraction tasks (Fig. 10.1), which were part of the Fraction Screening Test. The selection process is described in Sect. 10.4.3.

The students' previous written responses to the three reverse fraction tasks from the Fraction Screening Test were analyzed to determine the types of strategies students used. These responses were then classified according to six categories: Incomplete, Visual Methods, Additive/Subtractive Methods, Mixed Methods, Multiplicative Methods, and Advanced Multiplicative Methods. As discussed below these classifications were applied to students' responses to the set of three tasks overall, as distinct from classifying responses task-by-task.

In Table 10.2 *Incomplete* refers to students whose written responses were incomplete or who did not attempt any or all of the three reverse fraction tasks in Fig. 10.1. *Visual* refers to students who showed explicit partitioning of the diagrams shown in Fig. 10.1 for reverse Fraction Tasks 1 and 3 before using additive or subtractive strategies involving the number of objects, but *in the absence of* any fractional notation or any equivalence relationships. Some "visual" students attempted to create a diagram for reverse Fraction Task 2 (Fig. 10.1) to represent the 21 CDs before attempting to solve the task.

Table 10.2 Methods used for the three reverse fraction tasks (n = 46)

	Incomplete	Visual	Additive/subtractive	Mixed	Multiplicative	Advanced multiplicative
Year 5 (n = 26)	6	3	4	10	3	0
Year 6 (n = 20)	6	1	5	4	2	2
Total (n = 46)	12	4	9	14	5	2

Additive methods refers to students who used additive or subtractive methods with explicit use of fractional notation and equivalence relationships, sometimes without explicit partitioning of the given diagram, or creating a new diagram, to find the whole. These students could find the number of objects needed to represent the unit fraction and then added or subtracted the appropriate number of objects needed to make the whole. However, to be classified as additive there was no evidence of any multiplicative strategy apart from finding the unit fraction and its equivalent quantity.

Mixed methods refers to students who, having found the unit fraction and its equivalent quantity, used multiplicative strategies to solve at least one task while still using additive/subtractive strategies to solve at least one other task. This "scaling up" or "scaling down" would involve (repeated) addition or subtraction of the quantity corresponding to the unit fraction from the given number of objects—thus, mixed methods involved only minimal multiplicative reasoning.

Multiplicative methods refers to students who consistently used multiplicative reasoning to solve at least two questions successfully. Generally these students found the quantity represented by the unit fraction and then scaled up or down to find the whole. Multiplicative methods would always, and only, involve scaling up by means of multiplication from the quantity corresponding to the unit fraction.

Advanced multiplicative methods describes students who consistently used either algebraic notation to find the whole, or used a one-step method to find the whole by dividing the given quantity by the known fraction.

10.4.2 Developing an Interview Protocol

The interview protocol included reverse fraction tasks similar to those shown in Fig. 10.1 but with progressive levels of abstraction, starting from particular instances and becoming progressively more generalized. Our goal in including Questions 4b, 5b, and 6b (Fig. 10.13) where the quantity was unspecified was to assist those students who had used multiplicative thinking in their written solutions for the three reverse fraction tasks to move forward to more robust generalizations. A second goal was to investigate whether those students who had correctly used

Original tasks	Change of number 1	Change of number 2, and Unknown number
1. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.	1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain your thinking.	4. a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?
2. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? ____ *Show all your working.*	2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? ____ b. Explain your thinking.	5. a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay's CDs, how would you find the number of CDs Kay has? b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had?
3. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.	3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with a. How many counters did I start with? b. Explain your thinking.	6. a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with? b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

Fig. 10.13 Six interview tasks linked to the three original written tasks

Think about the tasks you have just done.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

Can you tell me what you would do? Please write your explanation in your own words.

Fig. 10.14 Question 7 from the interview

visual methods or a mix of multiplicative and/or additive methods could be induced to adopt more consistent multiplicative and generalizable strategies as they completed the interview tasks.

Some might argue that students who use a mixed method involving division (to find the quantity equivalent to the unit fraction) and then apply addition (or subtraction) of the required number of unit quantities (to reach the whole) are also using a generalizable method, even if somewhat more difficult to describe than the use of division followed by multiplication. However, as the interview results show, while adding on does work when specific fractions are used, this method becomes quite difficult when the quantity is not known, and was seen to be not feasible or generalizable when students were presented with 'any fraction' (Fig. 10.14).

If students were able to complete these six questions, they were presented with Question 7 (Fig. 10.14), which required that they come up with a completely generalizable method.

The interview protocol was intended to respond to questions raised by Carolyn Kieran (personal communication, July 2016, at ICME-13) who encouraged the researchers to vary the quantities, and also include non-specific quantities associated with each of the given fractions. This suggestion is supported by the research of Marton et al. (2004), which shows how numbers can be varied in order to foster a generalizable pattern. Stronger evidence of algebraic thinking would be apparent if students consistently used multiplicative methods, or progressed to using multiplicative methods, in responding to reverse fraction questions where the quantities were changed but the fractions remained the same (Questions 4a, 5a, 6a) as in the first three interview tasks (Questions 1–3, Fig. 10.13).

Where the original three fractions from Fig. 10.1 were used with different known quantities (see middle column of Fig. 10.13), we were interested to see whether students' solution strategies in this Follow Up Study 2 replicated strategies that other students had used in the earlier Pilot Study or whether the interview questions induced them to change from additive/subtractive methods to generalizable multiplicative methods. In particular, we needed to ascertain whether those students who had relied on additive or subtractive methods, with or without a diagram, were able to use multiplicative methods once the diagrams were no longer provided.

Subsequent interview questions (4b, 5b, 6b) were designed to make additive and/or subtractive strategies less attractive and less easy to use. In these questions students were asked how to find an unknown whole if they had *any number of counters* which represented a specific fraction of the whole. Finally students were given the general question about *any quantity* with *any fraction* and asked how they would find the whole (Fig. 10.14). This important question type was not included by Lee and Hackenberg in their research. Our aim was to ascertain whether students could generalize their solution method, for example, by dividing the unknown quantity by the numerator and then multiplying by the denominator. This question also provided an opportunity for more confident multiplicative thinkers to use algebraic notation to represent the unknown quantity and its accompanying fraction.

10.4.3 The Interview

Students to be interviewed were chosen from the 32 students who successfully solved at least two of the three reverse fraction tasks regardless of the methods used. Initially 19 students were interviewed but two interviews were terminated as students lost interest or went 'off-task'. In the final sample shown in Table 10.3 there were seven girls and 10 boys with nine students in Year 5 and eight in Year 6.

Referring back to Table 10.2, students were selected as follows for the interview. Three of four students described as using only visual strategies were interviewed; five of nine students who used additive strategies; four of 14 students who

Table 10.3 The sample of students interviewed (n = 17)

Year level	Boys	Girls	Total
Year 5	5	4	9
Year 6	5	3	8
Total	10	7	17

Table 10.4 Methods used by interview group to solve the original three reverse fraction tasks (n = 17)

Year level	Visual	Additive	Mixed	Multiplicative	Advanced multiplicative	Total
5	2	3	2	2	0	9
6	1	2	2	1	2	8
Total	3	5	4	3	2	17

Table 10.5 The interview scoring framework

Level	Description
0	Not able to successfully complete any questions
1	Completed some or all of Questions 1–3 with known fraction and given quantity
2	Completed all questions with known fractions and a given quantity (Questions 1–3, 4a, 5a, and 6a). Relied on additive methods to solve Questions 4b, 5b, 6b. Could not give a generalizable response to Question 7
3	Completed Questions 1–6 using multiplicative and/or mixed methods. Gave an appropriate non-symbolic generalizable response to Question 7
4	Completed Questions 1–6 using consistent multiplicative methods. Used suitable algebraic notation to give a generalizable response to Question 7

used a mix of multiplicative and additive methods; three of five students who used only multiplicative methods; and two students who used advanced multiplicative strategies. The subgroups of students' methods are shown in Table 10.4.

10.4.3.1 Administration of the Interview

A one-to-one structured interview was conducted at the school with four interviewers each using up to seven questions designed by the researchers (see Figs. 10.13 and 10.14). Students, for example, who were unable to answer Questions 4b, 5b, or 6b were not pressed to answer Question 7. The record of interview consisted of a three-page document that included the questions and space for students to record their answers and to explain their thinking. Students were encouraged to think about, and articulate, their response before writing anything on paper. At the start of the interview students were shown a copy of their responses to the three original reverse fraction tasks. This was then left on the table for students to refer to, if required.

Each interview took approximately 15 min per student. Students were free to correct their written responses to interview questions. Interviewers encouraged students to first verbalize and then write down their thinking. Students were able to leave the interview at any point. The written records from the interviews were analyzed by two researchers independently, using the scoring framework shown in Table 10.5.

10.4.4 Results of Interviews

No student was scored at Level 0, a result that is unsurprising given that all students selected for the interview had successfully answered two of the three reverse fraction tasks from the Fraction Screening Test. Two of the three students who used visual methods for the original three reverse fraction tasks were able to use additive methods for the interview with no diagram, and were deemed to be at Level 2. A third student who had used visual methods was scored at Level 1. The results of the interviews are summarized in Table 10.6.

Two students using additive methods in the written test were also at Level 2 in the interview and one was placed at Level 1. Two students who had used additive methods in the written test converted to fully multiplicative methods and deemed to be at Level 3. Four students who used mixed methods in the written test successfully solved all interview questions and solved Question 7 multiplicatively. These four students were all scored at Level 3 for the interview.

All three students who had used multiplicative methods in the written test were scored at Level 3 in the interview. These students continued to apply generalizable multiplicative methods throughout the interview. Two students who used advanced multiplicative methods for the written test answered all questions multiplicatively in the interview and responded to Question 7 using appropriate symbolic notation and were deemed to be at Level 4. The numbers boxed in Table 10.6 identify 11 of the 17 students, including two who had used only additive methods on the written test, who when interviewed scored at either Level 3 or Level 4, thus demonstrating clear evidence of being able to generalize a procedure that is independent of a particular fraction or quantity. At Level 3 this was typically expressed as "divide by the numerator and multiply by the denominator". Table 10.6 also shows five students who consistently used multiplicative methods or advanced multiplicative methods on the reverse fraction tasks, all achieving Level 3 or 4 on the interview.

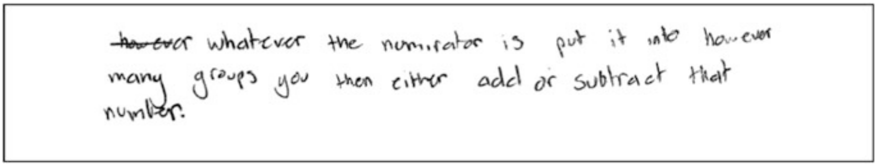
Table 10.6 Evidence of generalizing fraction structures as a result of the interview

Reverse fraction tasks (Fig. 10.1)		Interview score (Table 10.5)			
Written test methods	Number	Level 1	Level 2	Level 3	Level 4
Visual	3	1	2	–	–
Additive	5	1	2	2	–
Mixed	4	–	–	4	–
Multiplicative	3	–	–	3	–
Advanced multiplicative	2	–	–	–	2

10.4.4.1 Illustrative Examples

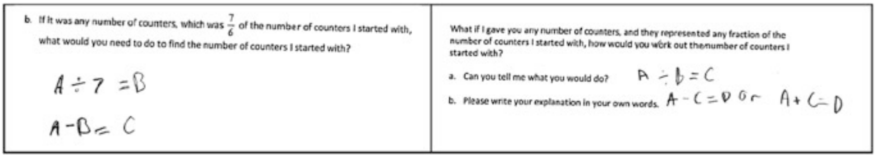
Students scoring at Level 2 or below were restricted to using additive methods when it was no longer appropriate or useful. This is illustrated in Fig. 10.15 by Student G's attempted solution to Question 7 which only appears to be generalizable since it depends on knowing how many times the quantity equivalent to the unit fraction needs to be added or subtracted. It appears that Student G has determined that the number of times the quantity corresponding to the unit fraction needs to be added to the given quantity can be obtained by subtracting the numerator from the denominator (with a subtractive compensation when the numerator is larger than the denominator). Student G's response to Question 7 has the appearance of generalizability and demonstrates a deep perception of the problem when the fraction and the quantity are both unknown, but this method cannot be written as an algebraic solution. This encapsulates the fundamental limitation of the additive method employed by Student G. The same limitations are evident in Student E's response to Question 7 as shown in Fig. 10.16.

Student E (Fig. 10.16) uses algebraic notation in what looks like a generalized solution but this solution only works for specific cases where the given fraction is one part more (or less) than the whole. While this student confidently solved all



~~however~~ whatever the numerator is put it into however many groups you then either add or subtract that number.

Fig. 10.15 Student G's additive response to Question 7



b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

A ÷ 7 = B

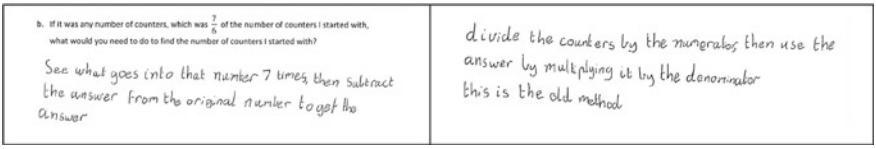
A − B = C

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out thenumber of counters I started with?

a. Can you tell me what you would do? A ÷ b = C

b. Please write your explanation in your own words. A − C = D or A + C = D

Fig. 10.16 Student E's responses to Question 6b and 7



b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

See what goes into that number 7 times, then subtract the answer from the original number to get the answer

divide the counters by the numerator then use the answer by multiplying it by the denominator this is the old method

Fig. 10.17 Student K's response to Question 6b followed by a generalized response to Question 7

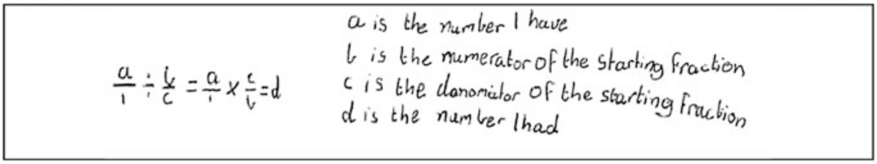


Fig. 10.18 Student J's symbolic multiplicative response to Question 7

preceding interview tasks, she was unable to give an appropriate multiplicative response to Question 7. Her overall response was scored at Level 2.

Student K (Fig. 10.17) used a method similar to the one used by Student E to answer Question 6b (Fig. 10.16). But when presented with Question 7, Student K suddenly realized that the preceding method would no longer work, and then gave a fully generalizable answer to Question 7: "Divide the (number of) counters by the numerator, then use the answer by multiplying it by the denominator". When Student K writes "This is the old method," she is referring to the fully multiplicative method that she had used in her written responses. As a result Student K was scored at Level 3 for the interview.

Student J (Fig. 10.18) consistently solved all previous interview questions by dividing whatever quantity was given by its equivalent fraction. In Question 7 Student J, who represented all quantities and fractions algebraically in a fully generalized solution, scored at Level 4. Only one other student gave a comparable symbolic response and was scored at Level 4.

10.4.4.2 Summarizing the Results

The interview achieved its stated purposes.

1. Students who relied on visual partitioning methods to solve the reverse fraction problems were least likely to be able to generalize their solution strategies. For these students, each question was a new problem that had to be considered on its own terms.
2. A necessary precursor to being able to generalize a solution was to recognize, implicitly or explicitly, an equivalence relationship between the given fraction and its related quantity. This allowed students to find the quantity related to the unit fraction that could then be scaled up to a whole, additively or multiplicatively.
3. Additive methods were less easily generalized, even using an equivalence relationship.
4. Multiplicative methods were clear precursors to generalization: typically dividing by the numerator to find the quantity equivalent to the unit fraction and then multiplying by the denominator. (They are also generalizable by dividing by a given fraction; or by first multiplying by the denominator to obtain a whole number equivalent and then scaling down to find the unit equivalent.)

5. Generalizable methods provided evidence of algebraic thinking when students could describe what needed to be done if a given fraction was related to *any quantity*.
6. Fully generalizable methods demonstrated algebraic thinking when students could describe in non-symbolic terms how to find the whole, given *any* fraction and *any* quantity.
7. Some students demonstrated clear algebraic thinking by using symbols such as *a*/*b* and *c* to represent any given fraction and any given quantity respectively in order to generalize their solutions.

10.5 Discussion and Conclusion

The nine students, who used either mixed methods or multiplicative methods (including advanced multiplicative) to solve the three reverse fraction tasks from the Fraction Screening Test, showed clear evidence in the interview that they were able to deal with variations in both fractions and corresponding quantities and to generalize their methods. One student, having successfully solved the first three questions, explained subsequent solutions as: “same as I did before” and later gave an explicit symbolic response for Question 7.

The interview questions as shown in Figs. 10.13 and 10.14 allowed these nine students to treat variations in the given fractions as ‘quasi-variables’; that is, recognizing that the same multiplicative operations applied regardless of the fraction. In responding to Question 7, which posed the problem in its most generalized form, students typically referred to dividing by the numerator and multiplying by the denominator. Two students who had used additive methods to solve specific fraction problems recognized that additive methods were not appropriate when dealing with unknown quantities and were able to shift to fully generalizable multiplicative thinking.

These 11 students appear well positioned for formal algebra as expected in Year 7, as given in the Australian Curriculum: Mathematics (ACARA 2016) where they will be introduced to “the concept of variables as a way of representing numbers using letters (ACMNA175)”. The two students in this group who were scored at Level 4 showed even stronger evidence of being able to create and “simplify algebraic expressions involving the four operations (ACMNA192)” as recommended for Year 8 (Table 10.1).

Students who are dependent on visual methods or additive methods are likely to experience difficulty in adopting a multiplicative approach and describing a rule as implied in ACMNA133 (Table 10.1). These students are most likely to experience difficulty in transitioning to algebra. It needs to be noted that 14 of the 46 students who completed the paper and pencil tests were unable to complete more than one of the reverse fraction tasks in the initial test. Six of these were Year 6 students about to transition into secondary school. These students appear to be at risk in subsequent years when meeting linear equations involving rational coefficients and in relation to proportional reasoning.

By careful scaffolding of the tasks, the interview also assisted some students who had relied previously on additive methods for the original three reverse fraction tasks to convert to multiplicative and generalizable methods. These students were persuaded to use multiplicative methods when solving questions that presented either an unknown quantity or, in the case of Question 7, with both an unknown fraction and unknown quantity.

The current study has limitations relating to its sample size. The pencil and paper questionnaire was given to students in two classes in the same school and from these classes seventeen students were subsequently interviewed. While this is a broader sample than what was used in the pilot study where only Year 6 students who were identified by their teachers as mathematically capable were involved, there is a clear need to replicate the study with larger samples of students. This study expanded the key ideas from Lee and Hackenberg whose research was confined to 18 secondary aged students. Our study shows that the key algebraic ideas investigated by Lee and Hackenberg are also important for elementary age students and need to be systematically addressed in the elementary years prior to moving into the secondary school system. Three important overarching algebraic ideas appear to be common across both the Lee and Hackenberg research and our own. These are equivalence, transformation using equivalence, and the use of generalizable methods. These three ideas are used to identify algebraic thinking even when younger students are not able to use symbolic notation.

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Chapter 11
First Encounter with Variables by First and Third Grade Spanish Students

Marta Molina, Rebecca Ambrose and Aurora del Rio

Abstract This chapter presents findings from a teaching experiment on the initial understandings that primary Spanish students demonstrated when they were first introduced to the use of letters to stand for an indeterminate varying quantity in a functional relationship. We provide a detailed account of our task design and class activity to show how understanding of variable notation for functional relationships was cultivated. We discuss the degree to which results from previous studies generalize to the Spanish context. Our results, similar to those of previous studies, support the introduction of variables in elementary grades.

Keywords Algebraic symbolism · Functional thinking · Primary school
Variables

11.1 Introduction

Within the curricular innovation proposal and research line known as early algebra, researchers have begun to establish that young children are capable of developing an understanding of letters as variables when they are provided with opportunities to participate in classroom activities and discussions about functions, variables, and their representations (Blanton et al. 2015; Carraher et al. 2008). It remains to be seen the extent to which the findings generalize to different contexts and languages. Our research represents an effort to replicate the success of other projects in a Spanish setting. We built on the work of Brizuela et al. (2015b) by exploring six-

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and eight-year-old children's initial understanding of letters when used to describe functional relationships between co-varying quantities. Updated curriculum documents in Spain for primary education (Spanish Ministry of Education and Science 2014) explicitly include as an aim "describing and analyzing change situations, finding patterns, regularities and mathematical laws in numeric, geometric, and functional contexts" (p. 19388). Research-based evidence from the Spanish context is needed to guide the introduction of these new early algebra guidelines in classrooms. Considering that the early algebra proposal has already begun to have an effect in many curricula, information about students' first reactions and ideas related to algebraic elements is of use to guide the implementation of future curricular proposals. In this case we focus on variable as well as variable notation and its use in algebraic expressions together with numbers and operation signs.

11.2 Theoretical Framework

We share with other authors in this volume a belief that arithmetic and algebra can and should be integrated starting early in children's educational careers. We chose to use functional relationships, as a starting point, in part because teachers can easily adapt story problems in their existing curricula to go beyond "isolated experiences in computation" (Blanton and Kaput 2011, p. 7) to become algebraic problems by asking children to think of a variety of cases and then making generalizations about those cases. Blanton and Kaput (2011) reported that the teachers they worked with readily incorporated functional thinking tasks into their teaching when their "algebra sense" was awakened. Moreover, several researchers have found that young students can engage in thinking about functions. Following up on these studies, in the study here reported, we elected to launch our work in algebra by exploring functions.

Since functions are used to describe patterns and relationships that hold for sets of numbers, students need to go beyond thinking of specific pairings to thinking more generally. To do so requires conceptualizing an *indeterminate quantity,* a placeholder of a sort for any number that might be in the set. Radford (2011) argued that reasoning about *indeterminate quantities* is critical to algebraic thinking and pointed out that mathematicians thought about *indeterminate quantities* long before they had a notation for them. Radford provided examples of second-grade students alluding to the idea of indeterminate quantities when they were discussing a growing pattern that had gotten so big that they could no longer handle the calculations associated with it. He noted that, while the students did not have a symbol or even a word for the indeterminate quantity, they could point to it. He emphasized that the concept of *indeterminate quantity* was independent from the notation used to represent it.

Nevertheless, we need some notation to represent the idea of an indeterminate quantity in symbolic form. Our modern mathematical convention is to use a letter from the alphabet to represent indeterminate quantities. We suspect that mathematicians chose to borrow the symbols of our alphabet system for their work in algebra because these symbols were familiar and at hand (particularly in printing houses). As adults, they could easily refer to the mathematics context to recognize the irrelevance of the sound associated with the symbol when interpreting its meaning. We assume that mathematicians were not considering that children might not as easily let go of the letter-sound correspondence when encountering letters in an algebraic context. As will be evident in the data we present, representing the idea of indeterminate quantity with a letter confounded students' interpretations of the notation. Our work with the children has led us to better appreciate Radford's (2011) point about the importance of distinguishing between the concept of indeterminate quantities and the symbols used to represent them.

Given that mathematicians have chosen to use alphabetic symbols for variables, most mathematics educators do the same when initially engaging children in algebraic reasoning. Research on students' interpretation of letters as variables (Knuth et al. 2005; Küchemann 1981; Usiskin 1988) distinguishes static conceptions of variable as objects (i.e., a label for an object or an object itself) or as specific unknowns (fixed value), from dynamic conceptions as generalized numbers (representing multiple values, one at a time) or indeterminate varying quantities (representing, at once, a range of numbers). To obtain the symbolic representation of a functional relationship in a problem-solving context, a variable can be temporarily used as an unknown. However, to define or work with the function, the variable must be understood as an indeterminate quantity that can take on many values (the domain of the function).

In keeping with others who employ an emergent perspective (see, e.g., Cobb and Yackel 1996) we believed we should use the students' initial understandings as the starting point of instruction as we moved towards the conventional meaning of variables, so that students could construct their own understandings of variables and variable notation as they completed tasks and interacted in the classroom. We believed we needed to provide students with opportunities to grapple with algebraic notation and build on students' incipient ideas as we enculturated them into conventional uses of variables in mathematical expressions. By introducing letters to represent indeterminate quantities, we wanted to create a "semantic space" to be filled with meaning (Sfard 2000). The introduction of new symbols gives students the opportunity to negotiate meanings for the mathematical objects they represent even when they still know very little about them (e.g., just that they represent a quantity). We share Sfard's assumption of a dynamic interplay between symbolizing and sense making. From this view, the introduction of a new symbol is a decisive step in the creation of a new object.

11.3 Connections to Previous Research

Several researchers have reported what happens when students first begin using letters in algebraic expressions. Radford (2014) described how a fourth-grade student proposed to use a letter when being asked to simplify the "message problem." After writing the formula "1 + 1 + 2 × ____ = ____" on the board, the teacher asked the students to fill in the gaps. One student suggested using a letter, so the formula became $1 + 1 + 2 \times n = n$. Another student disagreed stating that both "n's" did not have the same value. During a different activity, this same student proposed to use the formula $N + N + 1 = R$, explaining that R is the answer (in French, 'la réponse' is the answer). In both cases, letters appeared spontaneously, without prompting from the teacher, and the students gave them different meanings.

Brizuela et al. (2015a) explored the first steps of a six-year-old girl making sense of letters in mathematics in the context of a teaching experiment. They illustrated the diversity of her understanding related to variables and variable notation, and how her ideas evolved within and across each of her interviews. In the first interview the child accepted the researcher's suggestion to use letters in order to shorten the headers of a function table with the first letter of each word. When she was asked to use another different letter she argued that she had never used a letter in mathematics but she had used shapes, making an association between letters and shapes to represent unknown quantities. She assigned the letter a fixed quantity based on its position in the alphabet. Later on in the interview she used letters to represent a fixed quantity with an arbitrary value, and also to represent quantities with finite variation. In the second and the third interviews she utilized the idea of a letter representing a varying unknown and variables as mathematical objects. Their findings contribute to evidence that variable notation is within the reach of lower primary-grade students and of how the initial understanding of letters can evolve within the context of working on tasks about co-varying relationships.

In Brizuela et al.'s (2015b) teaching experiment, other first graders could also overcome their initial tendency to want to fix a specific value to quantities in functional relationships, with individual children expressing different understandings of variable notation within one interview. Children were able to take a critical first step in developing fluency with variable notation by coming to accept that quantities represented by variable notation were indeterminate. From the gathered evidence, and in line with Sfard's (2000) position previously mentioned in this chapter, Brizuela et al. (2015a) argued that conceptual understanding does not need to precede the introduction of symbolic notation, rather acquiring variable notation can be considered much like any language acquisition which evolves over time as students gain fluency. Students take up the use of algebraic notation at different rates with some incorporating it into their "personal repertoires" (Carraher et al. 2006, p. 109) more readily than others. For that reason, Carraher and colleagues recommend undertaking "systematic teaching experiments and research" (p. 111).

Blanton et al.'s (2015) analysis of third graders' (8- to 9-year-olds) assessment answers showed that students did not initially know what to do with variables to represent a functional relationship. This finding indicates that while children are capable of appropriating the use of variables to express functional relationships, this knowledge is not part of the informal knowledge that they bring into the classroom with them. It needs to be cultivated through carefully crafted tasks and artfully orchestrated discussions in which the teacher steers children toward conventional uses of variables, while at the same time honors each child's emergent understanding of the notation (Radford 2011).

In this chapter, we expand on Brizuela et al.'s (2015a) work by showing how Spanish students responded to tasks similar to those used in previous research. In addition, we compare the work of six-year-olds and eight-year-olds to show the initial understandings children of different ages demonstrated when first introduced to the notation of using a letter standing for an indeterminate quantity to represent a functional relationship. We provide a detailed account of task design and class activity to show how understanding of variable notation for functional relationships was cultivated.

11.4 Methodology

Our research approach is a teaching experiment (Molina et al. 2011) in which we designed and implemented generalization tasks that involved linear functional relationships. The data reported here belong to a larger project aiming to explore and characterize elementary Spanish students' functional thinking. We worked with three groups of students in the same school in the south of Spain, one group from each of Grade 1 ($n = 30$), Grade 3 ($n = 27$), and Grade 5 ($n = 25$). Here we focus on the first two groups. In each of the grade levels we implemented four to five sessions (i.e., lessons) of one to one-and-a-half hours each. One of the researchers taught the function lessons while the regular classroom teacher was also present in the classroom. All instruction was in Spanish, the children's native language. These students had no prior experience with generalization or with function tasks. They also had never used symbols or shapes to represent variables or unknowns in school.

All tasks involved linear functions of one variable with natural numbers presented through a familiar context (see Table 11.1). We provided students with a context where they could express their reasoning and strategies and use multiple representations in a meaningful way. We did not provide them with any introductory explanation; rather we presented them with a set of questions to consider so

Table 11.1 Contexts and functions considered in the first sessions in Grades 1 and 3

Grade	Context	Function
1	At an animal shelter, every dog wears a collar with its name on it	$F(x) = x$
3	Maria and Raul are two siblings that live in La Zubia. Maria is the older sister. We know that Maria is 5 years older than Raul	$F(x) = x + 5$

as to elicit their thinking. The first questions asked students to explore the functional relationship by considering particular cases. We intentionally chose non-consecutive cases to avoid leading the students towards using recursive thinking where the attention is directed to only one of the sets of numbers connected by the function. Next, we included questions that directed students' attention beyond particular cases towards the relationship that connected both variables mentioned in the context. In this way we anticipated that students would follow the inductive reasoning process (Cañadas and Castro 2007).

We considered two types of relationships connecting the independent and dependent variables: correspondence and covariation (Smith 2008). The first one, correspondence relationship, refers to the rule that allows describing the dependent variable in terms of the independent one. The second one, covariation relationship, refers to how changes in the independent variable affect the dependent variable. Most questions referred to the correspondence relationship (i.e., the relationship of the dependent variable with the independent one), but some referred to the covariation relationship. In some later questions, we asked students for the inverse correspondence relationship: the relationship of the independent variable with the dependent one.

In some questions, we asked the students to make use of different representations, such as tables, verbal language, and letters, so that they could have multiple opportunities to explore the functional relationship and express their thinking. Letters were introduced to represent indeterminate quantities. For example, in the first session with the first graders, sentences such as "We need Z collars for Z dogs" were presented to the students to determine its validity. No specific explanation was given to them about the meaning of letters until the end of the first session. In proposing the tasks, we intentionally chose to use a letter (Z) that does not belong to the Roman numeration system, nor one that was the initial letter of any of the quantities mentioned in the context of the problem. We did this hoping to move students away from the interpretation of the letter as a label. In the third grade, the letter Z was introduced during the discussion group session, instead of in the written task where students were asked to initially choose their own letters. We also hoped that using Z as the variable would interfere with children's consideration of the ordinal position of the letter in the alphabet, as significant to the interpretation of the variable, since there are no letters after it.

Tables were used to organize the data and help students to later represent the general relationship (the function) either with letters or with words. For example, in the first session in third grade we asked the students to make a table to organize all the information they had about the ages of both siblings and later we gave them an incomplete 3×9 table with the headings "Raul's age", "operations to compute Maria's age" and "Maria's age".

Students in first grade tended to work individually on the paper and pencil tasks even though they were sitting in bigger groups of around 10 students. Students in third grade sat in groups of 3–4 students and freely decided whether to work individually, in pairs, or in groups. We presented the context to the students and read most of the questions aloud so that reading difficulties would not interfere with students' participation in the tasks. After most children completed two to three questions, either individually or in small groups, the researcher orchestrated whole-group discussions so that their answers and ideas could be shared. The aim was not getting a correct answer, but rather sharing different verbalizations of the relationships that connect the two variables and the ways in which children represented them in writing.

The classroom sessions were video-recorded: one stable camera was used to capture whole class activity and another roving camera operated by a researcher was used to capture the activity of individual students and brief one-on-one conversations between students and the researcher. Students' written responses were also collected.

11.4.1 Description of the First Sessions of Both Grades

As both groups of students had no previous experience working with functions, for the first session we chose the easiest function for first grade, the identity function, and a slightly more difficult function for third grade, including just one addition (see Table 11.1). The contexts described in Table 11.1 were carefully chosen to match both types of functions.

In the case of first grade we chose the idea of providing dogs with collars. We made clear to the students that no dog could be left without a collar and no dogs could get more than one collar. The shelter as the context for the dogs could allow students to consider large quantities of dogs (e.g., 100, 1000, a "thousand of millions"). In this context, the idea of variable could emerge because the number of dogs in a shelter is a varying quantity.

We started the session presenting the context to the students using stickers of dogs and collars and gluing a collar on each dog. Orally several particular cases of numbers of dogs and numbers of collars were considered. Afterwards, we presented the students with true/false (T/F) sentences related to the functional relationship: first including small numbers, then big numbers, and finally variable notation (see Table 11.2). Letters were not introduced before the T/F sentences because we wanted to explore students' natural reactions to them. As mentioned earlier in the chapter, symbols were introduced to provoke meaning making and negotiation. Therefore, the meanings given to the letters by the students would depend on their own understanding and how the discussion would develop.

The T/F format was chosen to make the tasks approachable to the first-grade students' writing level. Students were asked to decide whether the statements were true or false. They had several minutes to write their answers to the questions and then we began a whole-group discussion during which students had to give their answer and explain why they thought that each sentence was correct or not. After considering some sentences with numbers, we asked them to consider others with letters. In the group discussion we focused on the sentences, "We need Z collars for Z dogs," and later, "If we have N dogs, we need Z collars." With these two

Table 11.2 True/false sentences proposed to first-grade students

Examples of questions	Description
If we have three dogs, we need two collars	Focused on the correspondence relationship. Considering small or big numbers
In the shelter there are one hundred dogs. We need one hundred collars	
We need Z collars for Z dogs	Focused on the correspondence relationship. Using letters to represent indeterminate quantities
Let's imagine that in the shelter there are N dogs. Then we need Z collars	
On Friday two more dogs arrive. Then we need two more collars	Focused on the covariation relationship
Let's imagine that on Friday many new dogs arrive to the shelter and we have twice as many dogs as on Thursday. Then on Friday we need twice as many collars as on Thursday	
We have seven collars. Then we can put collars on seven dogs	Focused on the inverse correspondence relationship. Considering small or big numbers

sentences we wanted to provoke general verbalizations of the functional relationship as well as invite students to give meaning to the letters. To end the final discussion, we told the students that letters could represent any number.

In the first session in third grade, we asked the children to consider a situation about the ages of two siblings (see Table 11.1). The context of ages was chosen because we thought it was an accessible context for children to think about varying quantities with a constant difference. Many of the children in the class had siblings and frequently thought about how their ages would change as they grew up. This context has the limitation that it doesn't make sense to think of numbers much bigger than one hundred but it was considered sufficient for the purpose of the task. The idea of variable could also emerge as our ages are constantly changing as we grow older.

First, we asked students for the age of Maria given particular ages for Raul and required them to verbally express the relationship between both ages. Secondly we requested that they provide the information in a table. We provided them with a 3×9 table with the headings: Raul's age (1st column), Maria's age (3rd column), and the operations that allowed them to get the latter from the former (2nd column). We hoped that this organization of the data would help them give meaning to our next request: to represent Raul's age with a letter in the last row of the table, and to indicate how to use the letter to work out Maria's age. We made this request on paper and clarified it orally to each student who expressed confusion about what to do. As in first grade, we intentionally avoided specific explanations about the use or meaning of letters until the end of the session.

In the whole-group discussion, we first asked students for possible values for Raul's age, Maria's age, and the operations they used to find out Maria's age, while recording the information in a table on the blackboard. Then, we asked the students for letters to represent Raul's age and chose letter Z to work with, in order to avoid letters that either appear in the Roman numerals or are initials of the names of the children in the story. However, students decided to use other letters when sharing their thinking and letter Z was not adopted, except for one case appearing at the end of the discussion. Asking the students which operation to perform on Raul's age to find Maria's age led to the verbalization of multiple meanings assigned to the letter. We clarify that we used a letter because we did not know Raul's age, but that it could represent any possible age.

11.4.2 Data Analysis

We analyzed students' written work and their verbal contributions to class discussion. We employed Blanton et al.'s (2015) scheme to code students' use of variable to describe functional relationships:

(a) letter-related (variables are used to relate one quantity to another),
(b) letter-new (letters are assigned to each quantity without representing the relationship between them) and
(c) value (a specific numerical value is assigned to the unknown).

We created a combined category, "letter related and value", as some third-grade students evidenced different uses of letters in their answers. Within both letter-related categories, we found answers that suggest different understandings of variable as we comment on below. We also added the category "reject letters" because some responses evidenced rejection of using letters in favor of particular quantities.

11.5 Results

We begin by separately presenting the results from each group of students. Then we compare them. The results correspond to the number of students attending the first session in each group: 29 and 24 respectively (one first grader and three third graders did not attend class that day).

11.5.1 First Graders' Understanding of Variables

In Table 11.3 we show the options chosen by the first-grade students for the written task in those sentences where letters appeared. Most of the students chose the

Table 11.3 Answers of first-grade students to the written task (correct answer in bold) (n = 29)

Sentence	True	False	Empty
We need Z collars for Z dogs	**26**	3	0
If we have N dogs we need N collars	**19**	7	3
If we have N dogs we need Z collars	5	**21**	3
If we have P dogs we need C collars	4	**22**	3

Table 11.4 Coding of first-grade students' explanations during the whole-group discussion (n = 13; one student gave two explanations, classified as letter related and value respectively)

Coding	Number of students	Examples of students' responses	
Letter related	6	Student's explanation	Sentence
		They are the same letters	We need Z collars for Z dogs
		They are different letters	If we have N dogs we need Z collars
Value	5	Z is zero and there is no collar and no dog	We need Z collars for Z dogs
		Z has 28 as value, since it is the last letter in the alphabet	
		N is the first letter of none, and Z is the first of zero	If we have N dogs we need Z collars
		The value of N is 14, because it is his position in the alphabet, and the value of Z is 27	
Rejects the use of letters	3	Z is not a number	We need Z collars for Z dogs
		This doesn't exist	If we have N dogs we need Z collars

correct answer in each of the four sentences, and 13 of them showed consistency in their answers by choosing the four sentences correctly.

During the whole-group discussion 13 out of 29 students volunteered to explain their thinking to the rest of the class. All of them had given some answer in the sentences including variables in the written task. In Table 11.4 we gather some of their verbal explanations. Most of the students' explanations evidenced acceptance of the use of letters, but five gave it a particular value, using the alphabet-position of each letter or relating it to the first letter of a particular word (e.g., Z is from zero, even though in Spanish zero is written as "cero," but they sound alike). Six students compared the letters without assigning a value to them, just arguing if they were "equals" or not. For example, a student said, "I don't know the meaning of this letter, but Z and Z are equals and then it is the same number of dogs and collars."

These children were using the letter as a representation of an indeterminate quantity.

In attending to the consistency between the explanation in the whole-group discussion and the written task, we realized that only one student gave an inconsistent explanation: in the written task she chose that "We need Z collars for Z dogs" is true, while during the whole-group discussion, she rejected the use of letters arguing that the phrase was false, because Z wasn't a number.

11.5.2 Third-Graders' Understanding of Variables

In Table 11.5 we show our coding of third-grade students' initial response in completing the written task and we include some comments made during the discussion.[1]

To use letters to represent functions, it is not only necessary to interpret the letters as an indeterminate varying quantity but also to dare to mix them with the arithmetic symbols that students have studied and previously represented only with numbers. We believe this to be an advance in thinking. Eleven of the 24 students were able to represent the relationship between both ages by incorporating numbers with letters in expression. Four different ways were detected. First some students wrote expressions combining letters and numbers to express the relationship detected in a way similar to that normally used by mathematicians (e.g., see Sara in Table 11.5). Second and third, others used the positions of the letters in the alphabet to express the relationship, either by choosing letters five positions away to represent the ages of each sibling (e.g., see Noelia in Table 11.5), or by using letters with two meanings at the same time: an indeterminate quantity and a specific value (e.g., Salma in Table 11.5). This latter use of letters might be the result of rejecting the combination of letters and numbers in the same sentences as detected by Brizuela et al. (2015a). Finally, two students (Sergio and Clara in Table 11.5) even combined several of these ways of expressing the relationship: they represented the additive relationship existing between the two variables using a letter plus 5 in the column of the table reserved for the needed operations to get Maria's age; and then wrote another letter, five places further in the alphabet, in the column of the table for Maria's age. One student (e.g., see Miguel in Table 11.5) used the words "adding five" in the second column rather than symbols to express the relationship between the children's ages. He did so in all the rows of the table.

Three students (Clara, Nuria, Pedro in Table 11.5), using the letter to relate one quantity to another, wrote a number sentence including letters, numbers, and operation relating them (e.g., $C + 5 = H$; $G + 5 = 12$). Four students wrote number sentences (with or without letters) in the second column, showing some resistance to leaving an algebraic or arithmetic expression open.

[1]Names of students used in this chapter are pseudonyms.

Table 11.5 Coding of third-grade students' written response in completing the table ($n = 24$)

Coding	Number of students	Examples of students' responses				
		Student	Raul's age	Operation to calculate Maria's age	Maria's age	Comments using information from the discussion
Letter related	6	Sara	R	R + 5	No answer	
		Sergio	A	A + 5	F	F is five positions later than A in the alphabet
		Clara	C	C + 5 = H	H	
		Noelia	R	No answer	W	W is five positions later than R in the alphabet
		Miguel	R	"Adding five"	No answer	
Letter related and value	5	Carmen	A	A + 5	65	
		Salma	R	R + E	24	E was used because it is the fifth letter in the alphabet
		Nuria	R	G + 5 = 12	No answer	
		Pedro	L	L + 5 = 140	140	
Letter New	4	Maite	R	C[a]	M	I wrote here an R of Raul, and here a C of five[a]
Value	1		A	21 + 5 = 26	No answer	
No answer	8					

[a]C is the first letter for the number 5 in Spanish (cinco)

Five students (those whose responses were classified under "letter related and value"), even though they were able to represent the relationship between both quantities, felt the need to assign particular values to both ages (see Table 11.5).

Regarding the answers classified in the "letter new" category, the students tended to choose the initial of the name of the child in the story to represent his/her age. Related to this meaning, one of the students used the letter C (initial of "cinco" in Spanish) to represent "adding five."

Analysis of the transcript from the whole-group discussion showed students articulating each type of response illustrated in Table 11.5. During the discussion of their responses, children demonstrated a willingness to use letters in place of numbers, and they showed emergent understanding of quantities as indeterminate. The following interaction (translated from the Spanish original) illustrates this:

Carmen: *I'll do A plus 5*
Researcher: *A plus 5, and then, what would be the age of Maria?*
Carmen: *For example, 47*
Researcher: *47, why?*
Carmen: *Because if A is equals to 42, I add 5 to it.*

While this child is speaking of a particular case of Maria's and Raul's ages, she specifies it as "an example" and uses the conditional "if" to discuss the case. While she is not yet articulating an understanding that A could stand for any number, she does seem to recognize that Maria's age depends on Raul's age and will vary. We concluded that she was developing "a positive disposition toward variables" (Brizuela et al. 2015a, p. 59) and that, in expressing herself in this way, she was potentially bringing her classmates along with her.

Other students while understanding letters as indeterminate, turned to the order of the letters in the alphabet to be able to use letters to express the functional relationship (see Pedro's explanation below). In his written work, Pedro was able to use letters within a number sentence to express the relationship between both students' ages, but assigned a specific value to Maria's age. However, here in this extract from the discussion, he provides a general representation of Maria's age using a letter. Interestingly, as did other students, he felt the need to capture in this abstract representation the additive relationship between the two quantities by making use of the order of the letters in the alphabet.

Pedro: *Z plus…, no, if it were A plus 5 equals E*
Researcher: *A plus 5 equals E. Let's see. Explain it. How do you get to that answer?*
Pedro: *Between A and E there are 5*
Researcher: *There are 5, Where? In the alphabet?*
Pedro: *Yes*
Researcher: *Ah, ok. That is why you chose A to express Raul's age and the E… what does it mean?*
Pedro: *The age of Maria.*

Other students provided evidence of a static interpretation of letters, as objects or labels. This is initially the case with Maite, as can be detected in the following extract from the discussion. Later, she shows a change in her interpretation of the letter as she is requested to use a letter that cannot be considered as the initial of Raul's name. In addition, we observe that initially she is not worried about the order of the letters in the alphabet, but when Raul's age is represented with the letter Z she begins to consider order, an idea that had been introduced by other students within the discussion.

Maite: *I wrote here an R of Raul (she points to the first column), and here a C of five*[2] (She points to the second column)
Researcher: *A C of five, here. And why did you write a C of five?*
Maite: *Because everything was adding five.*
Researcher: *Ok and if R is Raul's age, how would you write Maria's age?*
Maite: *M*
Researcher: *Why?*
Maite: *Because it is the initial.*
Researcher: *And what does the R have to do with the M? Which relationship is there between the R and the M that you chose?*
Maite: *They are the initials.*
[a bit later in the discussion, after bringing students' attention back to the case of using Z to represent Raul's age]
Maite: *The age you have to write down is D, because Z plus 5 would be D.*

Those students who completed the third column of the table did not write an expression in the form of "$n + 5$." These children may have felt that they had to put something new in the third column. In this case, some students gave Maria's age a specific value (see Table 11.5) or chose another letter to represent her age. In the following extract this is the case.

Noelia: *For example, letter C, C plus…*
Researcher: *If I write a C here, you are saying* (she writes in the first column)
Noelia: *The other C plus five and now in the other one, H,*
Researcher: *H? And this H, what does it mean?*
Noelia: *Because C plus 5 is the H…. H is the age of Maria.*

We wondered if Noelia and other children like her felt that Maria's age had to be a single symbol rather than an expression with three different symbols. This reaction could be expected as students even in secondary school evidence need of closure when working with algebraic and arithmetic expressions. They tend to conceive of expressions whose numeric value is not shown as incomplete and tend to finish them (Tirosh et al. 1998).

11.5.3 Comparison of First- and Third-Grade Results

As could be expected, in both groups there were students that rejected the use of letters. Also in both groups some students showed acceptance of the use of letters and some were able to argue about the function without having to talk about

[2] We remind the reader that five in Spanish is "cinco."

specific values. Language competence of first-grade students makes it difficult to analyze the meaning they ascribed to variables in responses classified as "letter related." The student might be thinking of the letters as an unknown or as an indeterminate varying quantity, or even as an object. In the case of third graders, if we attend to reactions during the group discussion, we notice that understanding of letters as an indeterminate quantity appears in several cases. For example, Maite and Pedro argued that they still have to add five, no matter which letter they chose for Raul's age. On the other side, we can notice a preliminary approximation to the idea of variable in Carmen. She assigned the value 60 to the age of Raul in the written paper, while during the discussion she argued that Raul's age could be 42. The use of the letter as an object was clearly evidenced by some third-grade students' responses, those that used the initial of the names of the children in the story, as could be detected in Maite's initial answers during the discussion.

Tasks proposed in both grade levels made different demands in relation to the use of letters as a consequence of the different functional relationships involved. In the case of the first graders, in order to judge the veracity of the T/F statements, students had to give meaning to given sentences expressing the functional relationship by using letters to represent indeterminate quantities. In the case of the third graders, the task requested that students represent operations on the indeterminate quantity, and therefore give meaning to operating with letters and numbers. In that sense, the task had a higher cognitive demand. Some of the uses of symbols proposed by the students avoided this difficulty. Yet other students used sentences combining numbers and symbols to describe the function. As a result, an interesting and diverse use of letters to represent the functional relationship was evidenced, which gave richness to the later discussion. The more open nature of the third-grade task, apart from the age difference, was key to allowing this wider variety of meanings to appear.

11.6 Discussion

In our concluding discussion, we wish to raise a few points, as well as relate our results to the findings of some others who have done research in the area of early algebra. We address here the relevance of lower primary-grade students' use of letters, the question of which symbols to use to represent the idea of indeterminate quantity, and how the design of the tasks and the discussions about them played a role in promoting students' sense-making of letters in the functional contexts that were considered.

11.6.1 Relating Our Findings to the Results of Others

The variety of responses to students' first exposure to using variable notation is consistent with data presented by Carraher et al. (2008), which has shown that a few children appropriate variable notation readily, while others shy away from it, and some attempt to employ variables but do so in unconventional ways. This finding indicates that, in the classroom setting, educators can expect to have some "early adopters" and these early adopters can model for their classmates ways to use algebraic notation to express mathematical relationships. While the teacher will have to initiate the idea (Lobato et al. 2005) of using letters to represent indeterminate quantities, subsequent interactions can rely on the "early adopters" to propagate the use of variable notation among their classmates. In this case, this would be the students who are at the point of formulating the idea asked for by the teacher. The range of adoption of variable notation indicates the need for teachers to balance their initiation of this new notation with opportunities for students to use it to express their ideas and to make sure that, as they adopt it, they can communicate effectively with others about their thinking.

Unlike the six-year-old girl in Brizuela et al.'s (2015a) study, neither the first nor the third graders had previous experience representing unknown quantities either with letters or shapes. In addition they had not yet participated in any of the lessons of the teaching experiment where the research was framed. Nevertheless, important similarities are identified in the results of both studies: students assigned to the letter a fixed quantity based on its position in the alphabet; other students used letters to represent a fixed quantity with an arbitrary value; and others, to represent indeterminate quantities. Both Brizuela et al.'s and our findings evidence that variable notation is within the reach of lower primary-grade students. Our results provide interesting examples of how students might accommodate their thinking about the position of the letters in the alphabet to use letters to represent functional relationships between indeterminate quantities.

11.6.2 The Relevance of Lower Primary-Grade Students' Using Letters

Letters, which can be assigned many meanings, are an essential component of mathematics language. Within the context of algebra, letters are used to represent unknowns, indeterminate varying quantities, generalized numbers, as well as parameters (Usiskin 1988). Part of the competency that students are expected to develop in the study of school algebra is understanding and using this diversity of meanings given to letters, as well as being able to use letters together with operation signs and numbers to express mathematical general sentences in a compact and precise way (Arcavi 2005).

Research on secondary students' understanding of variables and variable notation points to this topic as a content where students have limited understanding (Fernández-Millán and Molina 2016; Molina et al. 2017). The distinction among unknowns, indeterminate varying quantities, and parameters is not understood by most secondary students; most tend to interpret letters as replacers of objects or words (Furinghetti and Paola 1994; Küchemann 1981). In secondary school, students can also show different understandings of the letter in a particular algebraic expression (Filloy et al. 2008). For example they interpret the letter as unknown as well as variable in the following equation $x + x/4 = 6 + x/4$. Therefore, traditional approaches to teaching algebra are not succeeding in the development of an appropriate understanding of letters as used in algebraic contexts. This fact supports the pertinence of exploring both the possibility and the ways of initiating students into the use of letters much earlier, as a means of giving them more opportunities and time to develop such understanding.

Within the specific context of functions, which has herein been considered, letters are tools to lead student thinking to a more general conception of the relationship existing between two variables, in particular, to recognizing the varying nature of variables. Using a story context has the potential of helping students make sense of the math; however, they also carry the risk of leading students to think about particular cases/values. In the situations we considered, the children could think about animal shelters or siblings they know, and give answers inspired by those examples. More research is required to identify rich contexts or differently designed tasks that give rise to functional relationships where the varying nature of the variable concept is more perceptible.

11.6.3 Deciding Which Symbols to Use for the Idea of "Indeterminate Quantity"

Our results, along with those of Brizuela and her colleagues (Brizuela et al. 2015a, b), raise the question of which notation to use to represent indeterminate quantities. As noted above, many of our students, like those in Brizuela and her colleagues' research, inferred that the position of the letter in the alphabet was significant, treating the alphabet as having a one-to-one mapping with numbers. In this way, their knowledge of the alphabet may have been interfering with their understanding of the indeterminate quantities we were asking them to consider. Students in first grade have only recently learned the order of the letters in the alphabet and are just beginning to sort out the symbol-sound correspondence that makes the alphabet such a powerful tool. When seeing the same symbol in a mathematics context, they are likely to evoke their existing associations for the symbol and think about the meaning that it holds when they are reading and writing. Interestingly, this type of response has also been found in studies with secondary students (MacGregor and Stacey 1997).

Some have argued that mathematics is its own language and, if we adopt this assumption, then when learners become adept at mathematical notation they become bilingual, being fluent in expressing ideas in two symbol systems—spoken words and mathematical symbols. In a way, young children learning algebraic reasoning could be thought of as acquiring three symbol systems: (1) their native spoken language and how to use words to express ideas (L1); (2) the written system of their language and how to use written symbols to represent words (L2); and (3) the written system of mathematics which has its own symbol system that represents ideas (L3). This line of reasoning led us to consult the language acquisition research to think further about introduction of the use of symbols for indeterminate quantities. When trying to make meaning of new words in one language, learners can make inferences within that language based on context cues, or think of associations in other languages making inter-language cues. Both of these processes are quite natural (Ender 2014). When children are making meaning for letters in mathematics by referring to their position in the alphabet, we can think of them as using inter-language cues. Our challenge is to support them in recognizing the differences between L2 and L3, so they do not get distracted by what language teachers would call "false cognates."

11.6.4 How the Task Design and Discussion Orchestration Promoted Students' Willingness and Ability to Employ Variables to Express Their Mathematical Ideas in Both Groups

The familiarity of the context, together with the simplicity of the functions, were elements that supported students in beginning to make meaning for variables. Requiring students to use the letter to describe a linear function provoked the emergence of a wider variety of meanings for the variables than did the identity task. Even though the function considered in the third-grade task was also quite simple, it required the students to use letters in combination with numbers and operation signs. Therefore, it led students to apply the notation and meanings learnt in arithmetic to create a new mathematical object, that is, an algebraic expression. In the first-grade task, as the demand of representing the functional relations was not included, but rather the representation was given to be judged, it was very difficult for the researchers to distinguish between the students' answers to determine if they were thinking of the letter as an indeterminate quantity or as an object.

Using a table to help third-grade students consider various cases involving known quantities before asking them to think about indeterminate quantities seemed to foster understanding. In our opinion, the introduction of the letter in the context of a table helped the students to understand what we were asking them to do, because the table supported them in seeing the pattern of the relationship between the two quantities. In the case of first graders, as they are not so familiar with the table representation, we used T/F sentences. In the different sentences we provided, they could also perceive a pattern that helped them give meaning to letters in that the linguistic structure of the sentences was very similar. We need to remember that for both groups of students, using letters in mathematics was a new endeavor. In addition, as previously mentioned, the consideration of a letter different from the initial of the children's names in the story was a key element of the discussion in third grade that helped some students to move away from their interpretation of the letter as an object/label.

To conclude, our data indicate that using letters to represent indeterminate quantities holds promise as a viable starting point for early algebra instruction. Furthermore, we recommend introducing letters in mathematics from the first grade so as to gradually break students' rejection of their use. As was seen, some students' answers will help their classmates to develop meaning for the new symbols.

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Part III
Teaching for the Development of Early Algebraic Thinking

Chapter 12
Making Implicit Algebraic Thinking Explicit: Exploiting National Characteristics of German Approaches

Anna S. Steinweg, Kathrin Akinwunmi and Denise Lenz

Abstract German mathematics teaching-units in primary school lack explicit algebra learning environments. Then again, many national characteristics of teachers' attitudes and beliefs, everyday school life in mathematics classes, and deep-seated approaches that expect children to communicate and argue about mathematical findings, provide favorable prerequisites for algebra. Moreover, the contents taught have the potential to address algebraic thinking if approached from a new perspective. Yet, teachers and children are mostly unaware of the algebraic potential of certain tasks. This chapter includes three studies with a special explicit focus on possible key ideas, children's abilities, and challenges offered by tasks. These evaluated ideas illustrate in interweaving perspectives feasible approaches that enable teachers to integrate algebraic thinking into their classroom culture. Moreover, the implicitly given opportunities revealed by the special focus of each study are hoped to lead to a sensible acceptance of algebraic thinking in primary math classes and its curriculum.

Keywords Awareness · Generalization · Properties · Relational thinking
Key ideas · Patterns · Structures

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12.1 Introduction

Algebraic thinking is an important branch of mathematics from the start (Brownell et al. 2014). Working on numbers and operations is twofold (Müller and Wittmann 1984). On the one hand, numbers are regarded as digits or strings in a place value system. This allows carrying out algorithms and calculating discrete solutions. On the other hand, numbers and operations form an algebraic structure with special properties. This allows thinking about patterns, terms, and equations as special objects (e.g., Kieran 1981; Sfard 1991; Tall et al. 2001). Both of these perspectives on numbers and operations are crucial for a substantial mathematical education from the very start. Yet, the latter is almost neglected in daily school life in Germany. The authors of this chapter wonder why the various implicit possibilities for implementing and supporting algebra in primary mathematics are largely unknown to teachers and not taken up in textbooks and syllabi.

In the next section three different aspects are identified and described which may pave the way for a sensible and deliberate teaching and learning of algebra in German primary schools. This analysis of the special situation in Germany therefore frames the offered approaches. In particular, the issues and opportunities raised by the national characteristics are focused on. Afterwards in Sects. 12.3–12.5, the interwoven perspectives underpinning three research studies, and their respective theoretical frames, methods, and results, are outlined in detail. The topics of these studies are at the core of recent research that Kieran et al. (2016) identify as "a focus on mathematical relations, patterns, and arithmetical structures" (p. 10). The perspectives evince optional ways of implementing algebra and algebraic thinking in daily school life.

12.2 Issues and Opportunities for Supporting Early Algebra in German Primary Mathematics

The situation of German primary mathematics concerning early algebra is ambivalent. On the one hand, algebraic topics have no tradition and still no explicit place in German primary school curricula and teaching-units, unlike in some other countries (e.g., NCTM 2000). German primary curricula and standards mention algebra in a very limited way, if at all. Also there are only a few national research studies on algebraic thinking in the lower grades (e.g., Akinwunmi 2012; Gerhard 2013; Lenz 2016; Nührenbörger and Schwarzkopf 2016) and the 'Early Algebra' movement has not (yet) spread in Germany. At the same time, we also face the same didactical problems concerning this topic that are mentioned internationally (Malle 1993), for example, weak conceptions of variables (Franke and Wynands 1991; Specht 2009). On the other hand, when we take a closer look at the German primary mathematics classroom, we can identify many characteristics that offer opportunities to support algebraic thinking.

Astonishingly, primary classroom interaction and teachers' attitudes towards teaching and learning mathematics are very coherent throughout the country in spite of the fact that there are different curricula in the federal states, but all based on a common national standard (KMK 2004). Primary maths in German classrooms is very up-to-date concerning many fruitful teaching and learning principles (Krauthausen and Scherer 2007; Radatz et al. 1996; Schütte 2008; Steinweg 2014a). At least regarding the following three themes, a common ground can be identified.

- *Teacher attitudes and practices*: Teaching is no longer understood as passing on knowledge but as supporting individual and constructive life-long-processes. For instance, it is common to support children's individual solving strategies in terms of framing learning as discovery (Winter 1991). This goes hand-in-hand with seeing each child as an individual being with individual needs, abilities, and experiences (Bauersfeld 1983). With this in mind teachers also take into account prior knowledge, and misconceptions or mistakes are treated in classroom interaction as learning opportunities. Individuals' reactions and solutions to tasks are valued and integrated into the classroom discussion and interaction (e.g., Gallin 2012; Kühnel 1916/1966; Selter 1998).
- *Tasks and problem posing*: The teaching units offered are mainly substantial learning environments (Wittmann 1998), closely linked to mathematics as the science of patterns. Tasks are embedded in learning environments and are therefore related to each other. So-called operative variations (Wittmann 1985) of arithmetical tasks build up patterns and offer opportunities to discover mathematical relations among them. Most German textbooks and worksheets offer tasks with mathematically sound patterns to be spotted and to be described.
- *Expectations on learners*: The expected reactions in classroom interaction differ from just giving numerical answers. The German national standards (KMK 2004) expect children of all ages to communicate and to argue mathematically. This includes commenting on solutions, describing one's own solving processes, detecting patterns, defending different approaches, explaining certain patterns, and so forth. Primary teachers are very much aware of the importance of these so-called process competencies and try to support them in the classroom (Walther et al. 2008).

In summary, German norms around daily classroom interaction and common beliefs about teaching and learning reveal bright opportunities for early algebra. Yet, the algebraic potential of patterns and structures is fairly unknown to both teachers and children. Algebraic thinking is mostly understood as a content of secondary school, being very abstract and possessing no links to primary maths. This might originate from a lack of knowledge about the nature of algebraic thinking as well as a traditional view of algebra. Kaput et al. (2008b, p. xviii) call this belief the "algebra-as-we-were-taught-it, [which] follows arithmetic-as-we-were-taught-it." In an already overfilled curriculum it is understandable that primary school teachers might have some reservations about this alleged new additional content. As a consequence, teachers do not promote algebraic thinking explicitly in primary maths class.

Because of this situation, we believe that it is of high importance to promote researchers' and teachers' awareness concerning algebraic thinking and to make the implicit algebraic thinking that is already present in German classrooms explicit. The objective is to encourage teachers to integrate algebraic thinking into their classroom. Moreover, the aim is to enable teachers to become aware of their already addressing algebraic thinking in their maths class. This might finally lead to an acceptance of algebraic thinking in the primary math class and its curriculum.

In this chapter we present three different research studies, which build on three perspectives regarding the integration of algebraic thinking in primary maths classrooms. Each study, which is grounded in the national characteristics of German primary mathematics teaching, has a separate focus:

1. A focus on topics and contents of primary school mathematics that contain opportunities for promoting algebraic thinking. Section 12.3 clarifies the nature of algebraic thinking by describing four essential key ideas.
2. A focus on the algebraic competencies of primary school children. Section 12.4 provides insight into the potential of children's generalization processes with respect to the development of algebraic thinking.
3. A focus on the design of tasks and problem posing. Section 12.5 describes the challenges of task design and how tasks might promote algebraic thinking even in very young children.

12.3 Study I: All Eyes on Key Ideas

One perspective concerning the integration of algebraic thinking in the primary maths classroom takes as its starting point the mathematical topics and contents taught in primary school. In spite of the fact that algebra is not mentioned explicitly in curricula and syllabi, many topics are related to algebraic ideas and content fields. From this perspective, algebra is not a new content to add but a content field to be identified within already taught topics.

12.3.1 Theoretical Framework

The theoretical framework of this study includes two different aspects: an analytical and a constructive part. The analysis given in the first subsection tries to differentiate between the terms *pattern* and *structure* from a mathematical point of view. The constructive part, which is presented in the second subsection, suggests an option for categorizing topics into key ideas.

12.3.1.1 Patterns and Structures

Often mathematics itself is described as the science of patterns (Devlin 1997). In this view, all mathematical theories arise from patterns spotted. Even axioms characterize patterns to build on. Not surprisingly, teaching and learning about patterns and structures is no special topic but fundamental for all mathematics lessons. “Mathematics ‘makes sense’ because its patterns allow us to generalize our understanding from one situation to another” (Brownell et al. 2014, p. 84).

Becoming aware of patterns allows us to see sense in mathematics and to appreciate its beauty. This awareness is at least twofold. On the one hand, seeking patterns can be classified as meta-cognitive; on the other hand, there is a cognitive component of awareness that is characterized by “knowledge of structure” (Mulligan and Mitchelmore 2009, p. 38).

Patterns can be described as “any predictable regularity, usually involving numerical, spatial or logical relationships” (Mulligan and Mitchelmore 2009, p. 34). Constructing a pattern of numbers or shapes by making up a rule or a certain operative variation (Wittmann 1985) of a given number or task is a very creative process. If, for instance, the pattern of a number sequence is creatively made up, the regularity then is fixed and can be used, continued, and described (Steinweg 2001).

In this research study *structure* is understood as mathematical structure and not as a category system to describe the individual pattern awareness of children (on different uses of the term structure c.f. Rivera 2013; also Kieran in this volume). Mason et al. (2009) recommend “to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced” (p. 10). They point out the difference between the spotting of (singular) relations and the use of the given example as such for the general structure with certain properties:

> Recognising a relationship amongst two or more objects is not in itself structural or relational thinking, which, for us, involves making use of relationships as instantiations of properties. Awareness of the use of properties lies at the core of structural thinking. We define structural thinking as a disposition to use, explicate and connect these properties in one’s mathematical thinking. (Mason et al. 2009, pp. 10–11)

Hence, detecting structures, in contrast to patterns, requires mathematical knowledge about objects and operations. The relation between mathematical objects is essentially determined by mathematical structures (Wittmann and Müller 2007). Awareness of structures often suffers from the fact that structures are mentioned only briefly and only formulated in ‘rules’ in mathematics lessons. Unfortunately, these condensed statements are not an appropriate tool to become aware of the logical structures and properties of mathematical objects and relations, which are fundamental for mathematics.

Sufficient knowledge of mathematical structures is crucial for both teachers and children. Only well trained teachers are able to understand the mathematical structures and to make them accessible for children (Chick and Harris 2007; Devlin

1997). One approach for obtaining access to mathematical structures lies in explicit learning environments that enable children to explore, use, describe, and even prove patterns originating from underlying structures (Steinweg 2014b).

12.3.1.2 Algebraic Key Ideas

The main issue, worked on in this study, is to become aware of and to appreciate algebraic topics in primary class interaction. Hence, the most important question is which mathematical ideas are key, when it comes to algebraic thinking. The international research discourses provides several possibilities concerning the framing of algebra in primary school mathematics, for example, NCTM (2000). Besides standards and curricula various research projects outline different approaches or major ideas of algebraic thinking. Kaput (2008, p. 11), for example, identifies three strands of algebra, which are generalized arithmetic, functional thinking, and the application of modeling languages.

In the German context outlined above further detailing of algebraic content has not yet occurred. The initial step has to build on the existing terms used in syllabi and standards in order to receive broad acceptance and to make an impact on daily school lessons and mathematics textbooks. This possible link is the content area 'patterns and structures', which is given in the national standards (KMK 2004). Mathematics in primary school offers various opportunities to become aware of algebra as a mathematical background, that is, mathematical structures. In order to encourage sensitivity to important learning opportunities, the common topics are re-structured according to key ideas of algebraic thinking (Steinweg 2017).

(1) Patterns (& Structures)
(2) Property Structures
(3) Equivalence Structures
(4) Functional Structures

The first idea is briefly described above (also see Sect. 12.4). The second lies in the properties of numbers (e.g., parity, divisibility) and operations (e.g., commutativity, associativity, and distributivity). Examples of this key idea are presented below. The third key idea holds learning opportunities in evaluating, preserving, or construing equivalence in given correct or incorrect equations by assessing terms, and so on. The main issue here is to overcome the urge to solve equations but to focus on the relation of given numbers, sums, differences, products, or quotients (Steinweg 2006).

> Inviting children to find 'quick ways' to do arithmetic calculations such as adding the same to both numbers to reach an easier calculation (47–38 = 49–40) and the many variants, can be an entry into appreciating structure. (Mason et al. 2009, p. 14)

The last key idea involves learning environments on functional structures, (i.e., mainly proportional) relations, and co-variation aspects, for example, 'number and partner number' (Steinweg 2003)—also used in the study described in Sect. 12.4.

12.3.2 Methodology

The research design follows a constructive approach against the background of mathematics education as a design science (Wittmann 1995). In the research project (Steinweg 2013) learning environments that are suitable for the key ideas outlined above are designed and evaluated. Learning environments provide—besides some implementation ideas—in particular, tangible examples of common tasks in order to uncover their algebraic potential. Each learning environment includes various tasks in a booklet to be handed out to the children along with further mathematical background and educational information for teachers in a teacher's guide. The teachers participated in an introductory meeting in which the tasks and possible teaching arrangements—given in the guidelines—were discussed. They committed themselves to implement all of the 'extra' tasks in the booklet among the usual textbook tasks in daily classroom work over a period of 10 months. The frequency, intensity, and depth of the use of the learning environments were to be decided freely by the teachers. There was no specific focus on the child-teacher-interaction while working on the tasks—with the exception of some mathematics lessons randomly visited by the researcher. The research therefore focused on the question:

> *Does the implementation of 'new' tasks structured by key ideas* via *learning environments show any effects on children's algebraic competencies?*

Six German primary school classes with 144 children from 2nd to 4th grade (on average 7- to 9-year-olds) participated in the project. Additionally, two children per class took part in video-recorded interviews throughout the project period.

In the results presented here, we focus on distributivity as one element of the key idea 'property structures'. The main challenge is to see equations and expressions in a meta-perspective way. For instance, in the expression $2 \times 8 + 5 \times 8$ children have to spot the specific 'internal semantic' (Kieran 2006, p. 32). Only if the common factor is identified as an important component in the products can the 'variable' factors be summed up. For a start the two products have to be regarded as objects in a sum and then the two different factors can be added to create a new product (7×8), which is equal to the sum of two products. Of course, it is always possible to take a procedural perspective and to calculate expressions to determine the specific result (product, sum, etc.). This arithmetical perspective is very much supported in primary mathematics. The change in perception of expressions and equations is therefore crucial and challenging.

12.3.3 Research Results on the Example of Distributivity

Tasks can be designed in such a way as to take advantage of the natural urge to calculate (Fig. 12.1); for example, summing up multiplication table results yields a new sequence that can be identified in the 3rd line of the table as consisting of the

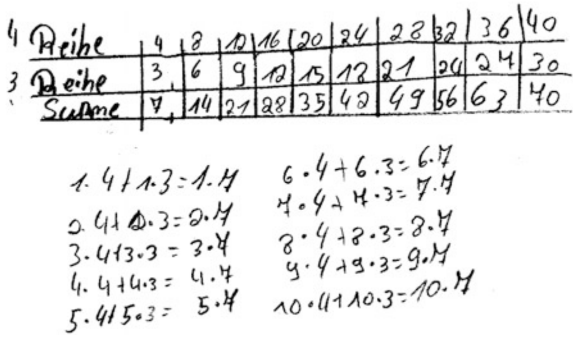


Fig. 12.1 Exploring distributive structures in symbolic representations

sum of the addends (cf. table with addends and sums in Fig. 12.1). This may, at first, be a surprising result for the children. If other examples are tested and in a next step the addends are rediscovered as products, as given in the lower part of Fig. 12.1, the underlying general idea can become more and more clear.

Besides tasks in symbolic representations, rectangle areas as a representation of multiplication (length by width) are used as well in the tasks of the given booklet. Such rectangles are provided by the teachers as representations on worksheets or 'actively' made up by the children by cutting out sections of grid paper. If rectangles are accepted as multiplication representations, manipulating these rectangles by cutting and re-interpreting the two part-rectangles as multiplications can be the next step to explore and understand distributivity (Fig. 12.2).

As the main research question aims to evaluate the effects of the implementation of the learning environments, results of a pre- and post-test are of interest. The results of the test item $10 \times 5 - 4 \times 5 = \_\_\_ \times \_\_\_$ (corresponding to distributivity) are herein documented by way of example (see Table 12.1).

Most likely, the children participating in the project had already experienced derive-and-combine-strategies for solving multiplication tasks in class. This approach to the multiplication tables, which is used in German mathematics in primary school, is somewhat peculiar. There is no longer 'doing tables,' but working on core tasks (e.g., doubles, times 5, times 10) and the use of derive-and-combine-strategies to solve other multiplications. Only core tasks

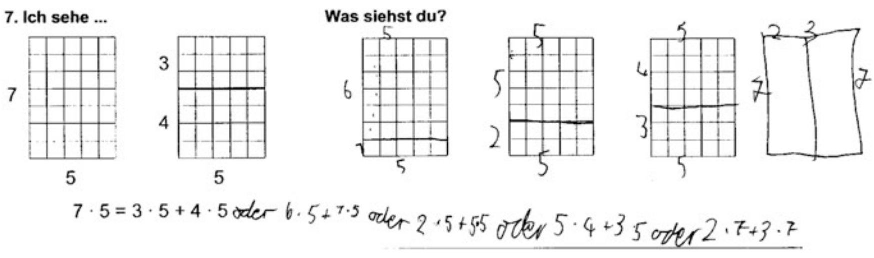


Fig. 12.2 Exploring distributive structures by interpreting rectangles as multiplications

Table 12.1 Results for solving $10 \times 5 - 4 \times 5 =$ ___ $\times$ ___

Category	Pre-test (n = 135) (%)	Post-test (n = 133) (%)
'Proceptual'	1.5	32
Procedural	32.5	35
No answer given	66	33

should be known by heart as facts (sometimes known as 'helping facts' in the Anglo-Saxon literature). For example, in order to solve 7×8 the children are encouraged to combine the known facts 2×8 and 5×8. This combination is possible because of distributivity. Even so, the task item was found to be quite hard to handle for the participating children in the pre-test (Table 12.1).

Prior to the project two-thirds of the children had no idea what to fill in the blanks. Only in very few cases were children able to combine the two given multiplications referred to in Table 12.1 into 6×5 and thereby make use of the structure (what we are naming the 'proceptual' or algebraic perspective). After participating in the project, one-third of the children were able to give this answer. Another third of them responded with a result such as 3×10, which is fitting because of the equivalent result 30 (the "procedural" or arithmetical perspective). Despite the fact that these results are still far from being satisfactory, the increase in the numbers of children using an algebraic perspective is considerable.

12.3.4 Discussion

The project gives an initial indication that it is possible to foster algebraic thinking by providing sound learning environments. The challenges offered to the children support effects on understanding and increased performance on algebraic tasks. Yet, the impact of learning environments alone is not enough to support all children. Teachers' instructions and interaction in classroom discussions as well as the specific role of representations have to be focused on in further studies.

As mentioned above, the project provided no binding specifications to teachers regarding how to focus on distributivity, but offered different opportunities to explore this mathematical structure via the 'new' tasks in learning environments. As a "good balance between skill and insight, between acting and thinking, is […] crucial" (Drijvers et al. 2011, p. 22), further effort should focus on exploring the differences between procedural and structural/conceptual work on tasks.

The developed *key ideas* may function as *bridges and guiding principles* between arithmetical and algebraic topics. If common arithmetical strategies—like derive-and-combine—are seen from a different angle, they actually are algebraic. This has to be made more explicit to both teachers and children. From a meta-perspective view the procedures performed are determined by mathematical structure and the properties of operations, that is, by algebra. Last but not least, this 'new' perspective and awareness implies "better understanding of rules and procedures" (Banerjee and Subramaniam 2012, p. 364).

12.4 Study II: All Eyes on Children's Algebraic Thinking

In this section, we explore how to exploit the potential of German primary mathematics classroom culture by making explicit the algebraic character of children's daily mathematical communication and reasoning. We believe that algebraic thinking is already taking place in the present maths lessons implicitly due to the national characteristics described earlier. It is then necessary to clarify the nature of algebraic thinking and to support its recognition in students' actions and communications. We illustrate this by focusing on the *generalization of patterns*—one of the most important parts of algebraic thinking (Kieran et al. 2016).

12.4.1 Theoretical Framework

"Patterns and Structures" are fundamental content in German maths classes—starting from the primary school level or even earlier. To discover, to describe, and to reason about patterns are essential activities according to the national primary mathematics standards (KMK 2004). Working on patterns and structures holds great opportunities for algebraic thinking as it can evoke children's generalization processes, which are considered essential for algebraic thinking (Kaput 2008; Mason et al. 2005). Unfortunately, teachers are mostly unaware that such opportunities exist.

Generalizing mathematical patterns is one of the main approaches to algebra and also to the introduction of variables. Mason and Pimm (1984) describe *generalizing* as "seeing the general in the particular." The concepts of variables as *general numbers* (or indeterminates, see Freudenthal 1973) and as *varying numbers* (variables in functional relations, see Freudenthal 1983) are powerful tools for generalization. Thus, the use of variables enables students to communicate, to reason, to explore, and to solve problems on a general level (Malle 1993). Variables can therefore be introduced as meaningful and necessary signs in the context of generalization.

Generalizing is an important part of any mathematics classroom in which the focus is laid on patterns and structures—thus also in German primary maths class. Patterns and structures have to be constructed actively by the learners by interpreting the given mathematical signs (Steinbring 2005). In order to achieve this, the students' challenge is to see something general in the particular (Steinbring 2005). Whenever learners communicate about mathematics, including when they talk about regularities, structures, and relations, they find inevitable the need to generalize. But before they are introduced to algebraic symbols and conventional signs for their generalizations, they face the problem of trying to say something general without having the necessary tools, such as variables. They are thus compelled to find their own fitting signs that can represent their explored mathematical patterns and structures. In the last decade, research has focused on the competencies of young children in the field of the emergence of algebraic thinking. Studies have revealed promising findings. Young learners are able to generalize and reason about patterns, number relations, and arithmetic laws (e.g., Bastable and Schifter 2008; Cooper and Warren 2011; Schliemann et al. 2007). Radford (2003) claims the importance of natural language as well as non-symbolic forms of generalization (e.g., by means of gestures).

In order to exploit the algebraic character of primary school students' communication of patterns and structures, a study was conducted to explore their individual generalization processes. The study focused on the following research questions:

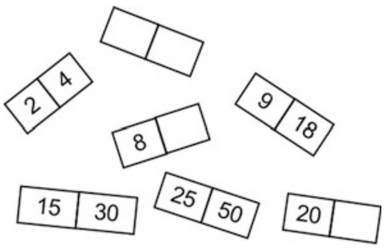
> *How and with which linguistic resources do primary school students generalize mathematical patterns? How do students develop variable concepts by generalizing mathematical patterns?*

12.4.2 Methodology

The presented interview study (Akinwunmi 2012) investigated the generalizing processes of primary school children. Thirty participating fourth graders (approximately 9–10 years) were engaged in three different task formats that included tasks that are known to focus on the exploration of patterns:

(1) "Think of a number" (e.g., Mason et al. 1985; Sawyer 1964): Children explore and explain the structure that lies in the following task. "Think of a number. Add 4. Add 8. Subtract the number you thought of. Subtract 2. The result is 10."
(2) "partner numbers" (Steinweg 2003): Children explore and describe the relationship of pairs of numbers (Fig. 12.3) and fill in some missing values.
(3) "growing patterns" (e.g., Orton 1999, see example on the next page).

Fig. 12.3 Example for "partner numbers"



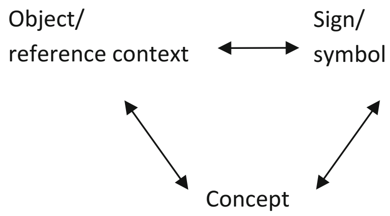


Fig. 12.4 The epistemological triangle (Steinbring 2005, p. 22)

The interviews focused on individuals' oral and written descriptions and explanations of patterns. The students were chosen from three different schools and included a heterogeneous range of achievements in mathematics according to their teachers. All interviews, each of an approximate duration of 45 min, were conducted by one of the authors; they were videotaped and transcribed.

The data were analyzed by a group of researchers by means of the epistemological triangle (Fig. 12.4) based on Steinbring's (2005) theory of the construction of new mathematical knowledge in classroom interaction.

The epistemological triangle can be used to reconstruct the referential mediation between mathematical signs/symbols and the reference contexts that serve for the interpretation of the signs. Steinbring (2005) describes the interdependence among the three entities by explaining that, "the referential mediation is steered by conceptual mathematical knowledge and at the same time, conceptual mathematical knowledge emerges in the referential mediation" (p. 179).

The study presented here reconstructs the development of variable concepts by observing the construction of knowledge by means of new referential mediations between mathematical signs and reference contexts.

12.4.3 Research Results

The presentation of the results is divided into two subsections. The first subsection (12.4.3.1) gives an insight into the analysis of an exemplary generalization process from an epistemological oriented perspective by revealing students' development of the variable concept. The second subsection (12.4.3.2) gives an overview of the children's forms of generalization and presents their linguistic tools from a semiotic perspective.

12.4.3.1 Epistemologically-Oriented Analysis of the Generalization Process

The epistemologically-oriented analysis of children's individual generalization processes is illustrated below with reference to an interview sequence with Lars, a student who worked on the growing pattern of "The L-Numbers" (Fig. 12.5).

First, Lars was asked to continue the pattern and to calculate the required number of squares for L1 to L10, L20, L100, and for other figures. His calculations showed that he split the figures into two sections: the vertical squares (including the linking square), which total one more than the term number of the sequence, and the horizontal squares which equal the term number. Using variables, his strategy could be described by the explicit formula $(n + 1) + n$. When requested to explain his strategy for calculating the required number of squares for any term of the sequence, his first statement involved citing an example: "I calculate, for example, 5 + 4. That's how I get the result." When asked to write down a description of his strategy, he drew a figure (Fig. 12.6) and explained it as illustrated below.

Lars' description of 'The L-Numbers':

Lars:	So this is five *(points to the five vertical squares)* and this is five (*points to the five horizontal squares including the linking square in the left corner*). So and ehm (*adds the two arrows and the plus sign to his figure*).
Interviewer:	Great. Now can you explain to me in detail, what exactly you mean (*points to the arrows*)?
Lars:	This downwards (*moves his pen alongside the vertical squares*) plus this (*moves his pen alongside the horizontal squares*), I calculate.
Interviewer:	Ah, ok. Good.

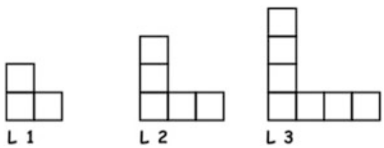


Fig. 12.5 Growing pattern of "The L-numbers"

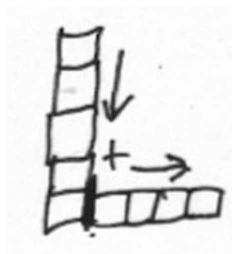


Fig. 12.6 Lars' description of the pattern

Lars: And ehm, this here (*points to the linking square in the corner of the 'L'*) belongs to the downwards. That's why I put a thicker line there (*retraces the line between the vertical and the horizontal squares*).

In this interview scene Lars used a drawing of the fourth figure of the sequence to describe the structure that he saw in the growing pattern. To this concrete figure he added a vertical and a horizontal arrow and a plus sign. With his explanation of the arrows "this downwards plus this, I calculate," he pointed out that these expressions represent the summands of the addition of the two parts of the 'L'-figure. The words "downwards" and "this" can therefore be construed as word variables as they referred to the varying number of squares in the two parts of Lars pattern. They enable Lars to describe the general structure of the 'L-numbers' beyond his first example, 5 + 4. He took these signs from a geometrical context as they initially indicated a direction within the given figure. In this new context they now referred to the varying amount of needed squares for one part of the figure. Thus Lars constructed a new referential mediation between the used words and arrows and the general structure of "The L-Numbers" (Fig. 12.7).

It is this mediation that is characteristic of the concept of variable as a general or varying number in the way that it includes the semiotic nature of the relation between one unifying symbolic object and its referring to multiple instances. Creating that symbolic object lies at the very heart of generalization (Kaput et al. 2008a, p. 20). It is important to note that this object is not necessarily a symbolic variable in the form of a letter. A 'variable' here appears in the form of words, signs, or symbols. We agree with Radford (2011, p. 311) that algebra can be considered as a "particular way of thinking that, instead of being characterized by alphanumeric signs, is rather characterized by the specific manner in which it attends to the objects of discourse." Therefore, we can say that the process of generalizing mathematical patterns is fostering students' concept of variable by naturally establishing this kind of mediation.

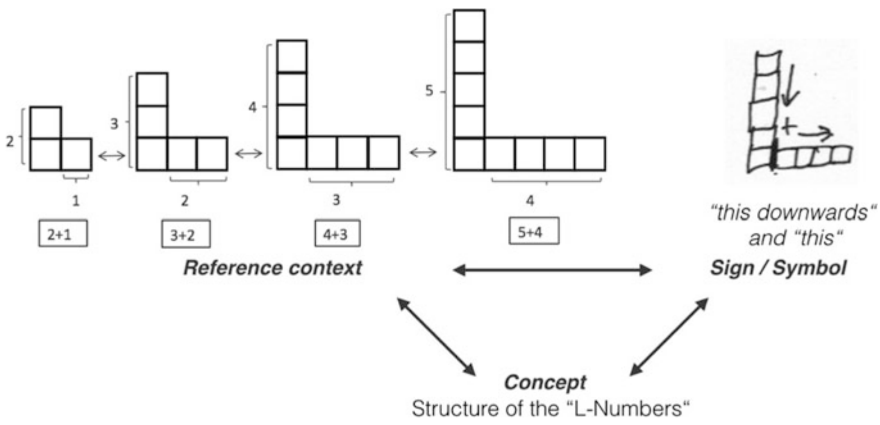


Fig. 12.7 Interpreting Lars' drawing and his explanation with the epistemological triangle

12.4.3.2 Children's Linguistic Forms of Generalization

Across the study's different task formats and the various individual reactions, different linguistic forms of generalization could be identified that served the students as tools for generalization (Table 12.2).

We found that children mixed and combined the above forms even within one description. Although the first four forms of generalizing are limited in terms of creating generally valid statements because the description does not apply to all objects in the pattern, they nevertheless illustrate the general character of the pattern. As such they also present possibilities for learners to express statements that can be understood as "general" in the classroom discussion.

12.4.4 Discussion

The analysis of the interviews of which we could just present a brief insight above shows that when asked to explore and especially requested to describe mathematical patterns and structures, children felt the necessity to generalize in order to be able to communicate about their discoveries in the way that Lars did. The individual use of signs or symbols with variable character originated from the motivation to refer to a mathematical structure in general and to describe it beyond the visible objects of the pattern. Learners spontaneously used signs or symbols drawn from other contexts. In the new context of generalization, they served as variables (as Lars' adverb of direction "downwards", the deictic expression "this," as well as the arrows in the described interview scene). In the individual process of generalizing, children constructed new referential mediations between the used signs and symbols and the general structure. Thus, this mediation shaped the concept of variable. The linguistic forms of generalization occurred in the

Table 12.2 Children's linguistic forms of generalization

Forms of generalization	Description of the category	Illustration by means of the term x^2
Stating one example	Students use one example and explicitly indicate this as an example	"For example it's three times three."
Listing several examples	Students list several examples and sometimes refer to a continuation	"It's one times one, two times two, three times three and so on."
Quasi-variables	Students use concrete numbers combined with a generalizing expression	"I always calculate three times three."
Conditional sentences	Students phrase conditional sentences	"If it is three, then I calculate three times three."
Variables	Students use words or signs with variable character	"You have to calculate the number times the same number." "?·?"

interaction while working on mathematical patterns and structures and took on the role of variables in the context of generalizations. They enabled the learners to describe mathematical patterns and structures in general and therefore served for the propaedeutic development of variables as general or as varying numbers.

We believe that it is important for teachers to be able to identify and support students' attempts to generalize as they themselves are a key to children's algebraic thinking and to the development of the variable concept. Teachers have to understand that these generalization processes take place in primary math class when students communicate about mathematical patterns and structures. The linguistic differentiation among forms of generalization presented above can help to open teachers' eyes and ears to generalization processes that occur in classroom communication and therefore aid in nurturing the awareness of algebraic thinking.

12.5 Study III: All Eyes on Tasks

This section proposes to expand the scope of common tasks used in maths lessons. Dealing with variables and establishing relationships as important aspects of algebraic thinking were addressed by a task design appropriate for children from 5 years on. In addition to addressing different aspects of variables, the tasks also promoted relational thinking. An interview study tried to find out which relations children describe between known and unknown quantities, represented as marbles and boxes.

12.5.1 Theoretical Framework

Algebra focuses not only on procedures, which are directly operable, but also and very importantly on the concepts that are represented in equations as relations between numbers, objects, or variables (e.g., Steinweg 2013). Relational thinking especially describes this way of thinking and therefore is an important part of algebraic thinking. Relational thinking refers to the recognition and use of relationships among numbers, sets, and relations. It enriches the learning of arithmetic and can be a foundation for smoothing the transition to algebra (e.g., Carpenter et al. 2005).

Another important part of algebra and the emergence of algebraic thinking is that of variables. While "variables" can be hard to define, several authors mention different aspects of variables with the aim of clarifying the field. At least, three different kinds of variables can be defined: *Unknowns* describe a specific, but undetermined number, whose value can be evaluated. For instance, in the equation $25 + x = 30$, x can be determined (e.g., Freudenthal 1973; Usiskin 1988). *Variables* describe a range of unspecified values and a relationship between two sets of values (Küchemann 1981). In this way, variables appear in statements about functional relationships. *General numbers* describe indeterminate numbers which appear in generalizations, such as descriptions of properties of a set as in $a + b = b + a$ (e.g., Freudenthal 1973). Some research on children's understanding of variables also emphasizes *quasi-variables*, which by the use of examples expresses a basic structure in general terms (e.g., Fujii and Stephens 2001). Related to this work on variables is the substantial empirical research on relational thinking about numbers and operations in symbolic equations (e.g., Carpenter et al. 2005; Steinweg 2013; Stephens and Wang 2008). We note however that, in some of the tasks used in these latter studies, namely 28 + 32 = 27 + __, students still have the opportunity to calculate and, thus, there may be no need to use relational thinking.

In contrast, Stephens and Wang (2008) used the following task to investigate 6th and 7th-graders' relational thinking:

$$18 + \underset{\text{Box A}}{\square} = 20 + \underset{\text{Box B}}{\square}$$

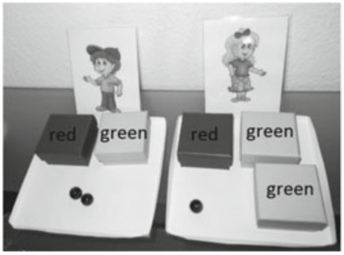
Students had to put numbers in the boxes named A and B to make the sentence correct. Tasks with more than one unknown quantity seemed to have the potential to push students to use and show relational thinking, instead of using computational methods to find a solution. Examples of tasks with more than one unknown and which are represented with concrete objects can also be found in Affolter et al. (2003) and Schliemann et al. (2007). In these studies, boxes containing rods and marbles were used in order to provide access to variables in equation situations. To encourage student's relational thinking, concrete materials would seem to have an advantage, especially for younger students, even as young as kindergarteners. However, studies on relational thinking with concrete material and unknown quantities are few. Therefore the following study addressed the question:

How do young children describe relations between known and unknown quantities that are represented with concrete materials?

12.5.2 Methodology

To find out what competences children already have for dealing with unknown quantities, clinical interviews (e.g., Selter and Spiegel 1997) with 82 children aged 5–10 years (kindergarten and elementary school) were conducted and videotaped. The underlying concept was to "translate" different kinds of equations with variables into a representation that young children could handle. Therefore known and unknown quantities were represented with concrete materials in the form of marbles and boxes. The boxes represented unknown quantities because their content was unknown. To make the task accessible, the following story was told: "Here you see two children. They are playing with marbles. Some marbles are packed up in different colored boxes and some marbles are separate. Boxes with the same color always contain the same amount of marbles."

Task of type B



How many marbles have to be in the green box, so that both children have the same amount of marbles?

Following task of type C

How many marbles have to be in the lime-green box, so that both children have the same amount of marbles?

Fig. 12.8 Tasks and interview questions (Lenz 2016, p. 176) (The labels designating the colors were subsequently added to accommodate black-and-white publishing constraints.)

The main study included 12 tasks on four levels of difficulty (Lenz 2016). In this section of the chapter two tasks are chosen as examples because the transition between tasks of type B and C (shown in Fig. 12.8) was found to be especially interesting. In tasks of type B children can answer with a concrete number. For instance, there is one marble in every green box (see Fig. 12.8). The green box represents an aspect of variables that can be identified as an unknown. The content of the red boxes is unknown and their determination is not necessary for the solution of the task. In contrast, the boxes in task C (see Fig. 12.8) take on another role. They can be seen as variables that represent a functional dependency. Since the amount of marbles in both boxes is unknown, no concrete number can be given. It can only be said that the lime-green box contains one marble more than the other box. This describes the relationship between the two unknown quantities.

12.5.3 Research Results

To gain insight into students' work, the following transcript shows how the 4th-grader Rick (11 years old) dealt with the consecutive tasks of type B and C.

Interviewer: How many marbles have to be in the green box, so that both children have the same amount of marbles? (task B, Fig. 12.8)

Rick: One.

Interviewer: And how did you get that?

Rick: Because…one plus one (*points one after another to the girl's green boxes*) plus this one marble (*points to the girl's marble*) are three. And here (*points to the boy's green box*) is also one marble in, plus the two loose marbles. And that's then the same (*points one after another to the girl's red box and the boy's red box*).

Rick gave the correct answer immediately and justified the number of marbles in the green boxes. He also pointed to the two red boxes and named them as "the same" without having to know the number of marbles contained.

Interviewer: How many marbles have to be in the lime-green box, so that both children have the same amount of marbles? (task C, Fig. 12.8)

Rick: Two marbles.

Interviewer: And how did you get that?

Rick: Because I think there is one marble in (*points to the girl's box*), plus the loose marble are two marbles. Then there (*points to the boy's box)* just can be two marbles, because you have to get the same result.

… [The interviewer gives different examples of amounts of marbles for the different boxes. Rick gives the corresponding number of marbles of the other box.]

Interviewer: Can you say in general, how to indicate the number of marbles in the boy's box?

Rick: You have to, uh, here is any number of marbles inside (*points to the girl's box*) plus the one marble (*points to the girl's single marble*), then there must not be as many as in this box (*points to the boy's box*), but one more in there.

Rick was confronted with a task in which both unknowns depended on each other. In order to give an answer to the interviewer, he mentioned discrete values for both boxes. In response to the interviewer's further questions, he was able to state a general relationship: he described the amount of marbles in the girl's box as "any number," which can be interpreted as a general number.

The responses of the other children covered a broad spectrum. We evaluated their various responses in two ways—according to the nature of the relationship that they expressed between the two quantities and according to the way in which they were handling the unknowns. The categories that were used in the evaluations were partly based on the distinctions described in the theoretical framework above and partly on other distinctions that emerged from the children's responses.

12.5.3.1 A First Evaluation: Relationship Between the Quantities

Regarding the answers to the task of type C, some children directly described a relationship between the two quantities in the boxes, as was the case with the 4th-grader Luca: "In the green box is always one marble more than in the orange box." Other children referred to the dependency between the amounts of marbles in the boxes, as did the 4th-grader Kathy: "It depends on how many marbles are in the orange box." Here, Kathy did not specify the relationship between the amounts of marbles in the boxes, but did have a sense of the dependency. Other children neither described a relation nor referred to the dependency between the amounts of marbles in the boxes. The 2nd-grader Lena (8 years old) mentioned specific numbers for the amounts of marbles in both boxes: “In the green box are three marbles and in the orange box are two marbles.” Other children wanted to shake the boxes to hear how many marbles were inside.

12.5.3.2 A Second Evaluation: Handling the Unknowns

The children’s answers were also classified according to how they treated the unknowns. In some cases, the amounts of marbles in the boxes were seen as *general numbers*, that is, the amount of marbles in one box was considered a generalized indeterminate number in relation to the amount of marbles in the other box. As noted above, Luca said: “In the green box is always one marble more, than in the orange box.” Here the amount of marbles is undetermined; it is always one marble more—no matter how many are actually in it.

In other cases, the amounts of marbles in the boxes were seen as *quasi-variables*: the children recognized the relationship between the amounts of marbles in the two boxes, but rather than stating a general description they mentioned specific numbers. The six-year old kindergartener Adam said: “…if there are eight or nine marbles in the orange box, then I take one marble more, that’s nine or ten marbles for the green box.”

For others, the amounts of marbles in the boxes were seen as *variables* where the amounts of marbles in the boxes depended on each other. As mentioned above, the 4th-grader Kathy explained: “It depends on how many marbles are in the orange box.” Further requests showed that she could handle the variation of numbers as a functional relationship, even if she did not specify it in terms of a static relationship.

The amount of marbles in the boxes was seen by others as an *absolute number* in that they referred to a specific number of marbles in the box, partially without taking the two related boxes into consideration. Clara from kindergarten (6 years old) answered: “Four…because the box is so small, there just fit four marbles in.”

Lastly, the amount of marbles in the boxes was seen as an *undeterminable*: Children said that the amount of marbles could not be defined. Axel (a 2nd grader) said: “I’m not a clairvoyant”; Rob (another 2nd grader) said: “I have to open the box.”

12.5.4 Discussion

The task design shows how algebraic thinking can be built on a concrete level. The boxes as representations for unknowns offer a possibility to get in touch with variables at an early stage. Relational thinking can be stimulated at this early stage by leaving the numerical values ambiguous. The tasks look simple at first glance and are visually very similar. However, they allow the construction from simple to mathematically complex contexts. They are therefore suitable for working from kindergarten to the secondary level and for addressing different aspects of variables while promoting relational thinking at the same time. In particular, the difference between the task types B and C marks a special breaking-point in the use of variables. Their roles change from an unknown that can be determined to a variable whose value cannot be known but can be described as a relation. Hence, tasks of type C strengthen the use of relational thinking since relationships between the sets have to be established. Different approaches to the solution of the tasks can also be made clear by operating on the tangible material (boxes and marbles). For example, in task B, both red boxes can be removed in order to clarify their irrelevance for the solution of the task. In later grades, it is possible to transfer the underlying structures to the formal level. Placeholders, symbols, or letters can replace the real boxes. Thus, with regard to the variables as well as with regard to the establishment of relationships, different changes in the levels of representation can take place.

12.6 Conclusions

This chapter aimed to explore how existing characteristics of German mathematics teaching could serve as opportunities to promote early algebraic thinking. Though a national perspective, it may serve as a framework for many other countries facing comparable issues and obstacles on the way to supporting algebraic thinking. The common aim of our research community is to provide fruitful learning environments and therefore learning opportunities for children regarding algebraic themes.

The above outlined ideas aim to overcome apparent stumbling blocks that cannot be attributed to children but to the given framing of mathematics lessons. Children are very capable of generating sound and viable reactions to algebraic challenges. Hence, we tried to emphasize three evaluated and promising approaches for supporting children's algebraic competencies. The common denominator of the three viewpoints that were presented lies in the existing implicit opportunities that have to be made explicit. This includes creating sensitivity to the algebraic potential of the mathematical content already taught, encountering children's abilities, and paying attention to the nature of the challenges created when designing tasks. If teachers, researchers, and curricula developers are aware of the potential of already daily used tasks (Sect. 12.3), the rich scope of children's abilities (Sect. 12.4), and the great effect of minor changes in problem posing (Sect. 12.5), then children will benefit sufficiently.

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Chapter 13
Early Algebra as Analysis of Structure: A Focus on Operations

Deborah Schifter

Abstract This chapter presents examples from classrooms to illustrate how work within two branches of early algebra—functions and generalized arithmetic—can provide a context for highlighting the operations as distinct objects. The examples emphasize two major themes: the role of representations in the study of structures associated with the operations and teachers' actions that draw students' attention to those structures.

Keywords Functions · Generalized arithmetic · Operations · Structure
Representations

13.1 Introduction

In extant practice, the teaching of calculation in the elementary and middle grades tends to focus on procedures for producing correct results. Although students realize there is a different procedure for each operation, the distinction among operations may fall into the background. Confusion about the operations frequently results in consistent procedural errors. Indeed, common errors in subtraction or multiplication can be interpreted as an application of structural properties that apply only to addition.

For example, consider such errors as these:

- $35 - 16 = 21$ Decompose the numbers into tens and ones; subtract the tens $(30 - 10)$ and subtract the ones $(6 - 5)$; add the results $(20 + 1)$. (The correct answer is 19.)
- $35 \times 16 = 330$ Decompose the numbers into tens and ones; multiply the tens (30×10) and multiply the ones (5×6); add the results $(300 + 30)$. (The correct answer is 560.)

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The basic approach behind these errors is related to a strategy that works for addition: To add 35 + 16, decompose the numbers into tens and ones; add the tens (30 + 10) and add the ones (6 + 5). The sum of the results provides the correct answer, 51. Students who make the errors illustrated above may be thinking of their correct addition strategy as the way *numbers* work, rather than how *addition* works. For that reason, despite a teacher's corrections, students often continue to apply the incorrect procedure. The error is likely to persist unless its underlying foundation is examined.

Early algebra, with its emphasis on recognizing and expressing mathematical *structure* (Kieran et al. 2017), has the potential of rooting out such errors. In the context of this chapter, mathematical structure refers to those behaviors, characteristics, or properties that remain constant across specific instances. Particularly relevant for the argument made here is that each operation has a unique set of structures. For example, switching the order of terms in an addition expression does not change the value of the expression, but switching the terms of a subtraction expression does (unless the value of the expression is 0). For another example, given an addition expression, when one addend is increased by some amount, the value of the expression increases by that same amount; but given a multiplication expression, when one factor is increased by some amount, the value of the expression increases by the value of the other factor times that same amount.

A focus on the behavior of addition, subtraction, multiplication, and division helps students come to see an operation not exclusively as a process or algorithm, but also as a mathematical object in its own right (Kieran 1989; Sfard 1991; Slavit 1999). As the operations become salient, seen as objects with a set of characteristics unique to each, students are positioned to recognize and apply their distinct structures.

Key to this work is representations—diagrams, physical objects, or story contexts—that embody relationships among quantities defined by the operations. For example, addition may be represented as the joining of two sets and subtraction as comparison or removal. An arrangement of equal groups or an array can represent multiplication or division.

The linking of spatial and numerical representations of structure, as well as story contexts that embody the structure, is a feature of early algebra that is gaining recognition among researchers. In the language of Radford (2011), "The awareness of these structures and their coordination entail a complex relationship between speech, forms of visualization and imagination, gesture, and activity on signs (e.g., number and proto-algebraic notations)" (p. 23). Warren and Cooper (2009) hypothesize that "abstraction is facilitated by comparing different representations of the same mental model to identify commonalities that encompass the kernel of the mental model" (p. 90). Moss and London McNab (2011) theorize that "the merging of the numerical and the visual provides the students with a new set of powerful insights that can underpin not only the early learning of a new mathematical domain but subsequent learning as well" (p. 280). As illustrated in the classroom examples below, images of the operations become thinking tools for students that they can call upon to reason about arithmetic symbols.

Even if curriculum materials focus on the structure of the operations, in order to use them effectively, teachers need to understand the underlying mathematics and maintain an orientation and intention consistent with those of the curriculum developers (Stein et al. 2007). It is the teacher who, in response to what students say and do, poses a question or underscores an idea or who offers suggestions to help students consider and develop new options. The examples below provide an opportunity to examine teacher moves that draw students' attention to key issues.

The work presented here draws from three research and development projects[1] co-led by the author with Susan Jo Russell and Virginia Bastable. In each project, the researchers provided professional development for twelve to forty teacher-collaborators who enacted early algebra lessons in their classrooms. Teachers audio-recorded a subset of their lessons and wrote narratives, based on their recordings, of what happened. A smaller subset of the lessons was video recorded. The classroom lessons described below are taken from the video recordings and teachers' written narratives. For purposes of readability, lessons from different classrooms are presented as a single composite class.

13.2 Analyzing Structure in the Domain of Functions

The study of functions in the elementary grades most often focuses on linear functions. Because linear functions involve a multiplicative component and an additive component, they provide students with the opportunity to consider the difference between multiplicative and additive structures.

In a fourth-grade classroom, the teacher, Ms. Bergeron,[2,3] gave the class the following context: *Sam is collecting pennies in a Penny Jar. There are 3 pennies at the start, and he adds 4 pennies each round.*[4] Students were to find the number of

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[2]Ms. Bergeron is a composite of teachers whose lessons were video recorded. Students' names are pseudonyms.

[3]Ms. Bergeron was among 40 teachers who field tested the second edition of the curriculum *Investigations in Number, Data, and Space* (Russell et al. 2008) and provided cases for the professional development curriculum module, *Patterns, Functions, and Change* (Schifter et al. 2017). Field-test teachers met with curriculum writers for professional development for one week during the summer prior to field testing, two full days during the school year, and three-hour monthly after-school meetings. Many of the teachers had worked with the curriculum writers in previous projects that focused on deepening teachers' understanding of mathematics content and attending to student thinking.

[4]The lessons described here were developed by the curriculum writers and currently appear in TERC (2017). The video recorded lessons appear in Schifter et al. (2017).

pennies in the jar after 1 round, 2 rounds, 3 rounds, etc. They represented the two variables in a table (see Table 13.1) and found the number of pennies in the jar for an increasing number of rounds either by counting or adding.

The objective of the lesson was not only that students find the number of pennies in the jar at any given time. Rather, the Penny Jar context was a pretext for students to engage with mathematical structure.

13.2.1 Penny Jar Episode 1

The image of pennies collecting in a jar (see Fig. 13.1) does not bring out the distinct multiplicative and additive structures in the function, but the image in Fig. 13.2 does. The additive component, the 3 pennies at the start, is illustrated by a row of white circles. The multiplicative component, 4 pennies added each round, is illustrated by the gray array.

After her class explored several Penny Jar contexts, Ms. Bergeron first presented the image as shown on the left in Fig. 13.2 and then added rows to illustrate adding groups of pennies to the jar until there were six rows of gray circles, as shown in Fig. 13.3.

Table 13.1 Representation of the two variables in the initial Penny Jar context

Number of rounds	Total number of pennies
Start with	3
1	7
2	11
3	15
4	19

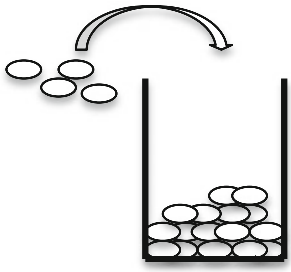

Fig. 13.1 Pennies collecting in a jar

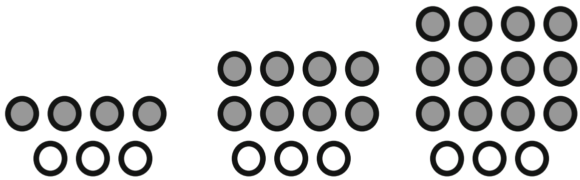

Fig. 13.2 The initial Penny Jar context: 3 pennies at the start and 4 pennies added each round

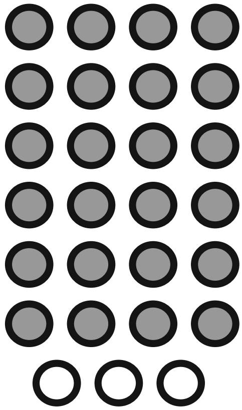

Fig. 13.3 The penny jar after the 6th round of adding pennies

Ms. Bergeron:	*We've got 6 gray rows here. Without actually putting the pennies in, can you imagine what more rounds would look like? On the 10th round, what would it look like? How many rows would we have and how many columns?*
Zoe:	*I think it will be 10 rows at the end of the 10th round and 4 columns at the end of the 10th round.*
Ms. Bergeron:	*If there are 10 rows, how many pennies did she put in the jar?*
Sally:	*40*
Ms. Bergeron:	*Where did the 40 come from?*
Sally:	*If there are 10 rows and you put in 4 each time, 10 times 4 is 40.*
Ms. Bergeron:	*And how many pennies are in the jar?*
Sally:	*43.*
Ms. Bergeron:	*And where did the 3 come from?*
Sally:	*The start number, which is 3.*

By showing the students six rows in the array and asking them to imagine ten, students needed to manipulate the image in their minds, thus building a portable and flexible tool. The image of the start number as a row of white circles and the pennies added each round as an array of gray circles was not only a representation of a small set of instances, but could stand for *all* instances of the Penny Jar, no matter how many rounds. That is, in their minds, students could picture an array with any number of rows, standing for any number of rounds.

In order to help students establish the representation as a meaningful tool, Ms. Bergeron continued to ask questions, with different students responding, checking to make sure they not only were seeing its additive and multiplicative components, but also were making correspondences among the image of circles, the components of a table, and the context of the Penny Jar.

Later, the class worked on function rules. Each student was given a penny jar situation with different parameters, and their task was to figure out what rule could be used to find the number of pennies in the jar for any round. When the class came together to discuss what they had done, Viktor, who had been given the situation, *4 pennies to start, add 9 each time*, presented his rule: "The round number multiplied by 9, plus 4, is the number of pennies in the penny jar."

In specifying the function rule as it relates to the context, students have to sort out what distinguishes addition from multiplication. Precisely because 9 is *added* over and over again, the rule is to *multiply* by 9.

When the teacher asked if students had other ways of writing the rule, Brenda raised her hand. Instead of stating the specific situation that had been given to her, Brenda offered a rule that covered all of the Penny Jar situations: "You multiply the number of rounds by the number you add each time and then you add the start number."

Ms. Bergeron recorded students' rules in a combination of words and symbols, as shown in Fig. 13.4.

Although it's not a big leap from here to write the function rules with algebraic notation—which, in fact, the class would get to—that was not the main goal. Instead, the major goal was that students more deeply understand the structure underlying the Penny Jar context. The class continued to offer different ways of formulating a rule, and checked those rules against the context with specific numbers.

In formulating a rule, students recognized what was common across specific instances. In this way, students' work in early algebra is metacognitive (Cusi et al. 2011). As Malara and Navarra (2003) describe it, students "substitute the act of calculating with looking at oneself while calculating" (p. 230). Students reflected on the actions they took to find the number of pennies in the jar at a given round and found a way to generalize for any round.

In this classroom episode, students generalized at different levels of abstraction. Viktor specified the common structure for a particular Penny Jar context (start with 4 and add 9 each round), whereas Brenda identified the common structure across all Penny Jar contexts. Up until now, the class relied on the image of the Penny Jar in their exploration.

Another level of abstraction is to see common structures across contexts, which the class would pursue in coming lessons. Students explored sets of buildings in which the number of windows (including skylights) was a function of the number of floors. Buildings were represented as cube towers in which each visible square unit contained a window. Figure 13.5 shows the first three towers for buildings that have six windows on each floor and two skylights.

Students created tables and function rules for this new context, which also mapped onto the array representation used for the Penny Jar. Students came to see

- (round # x 9) + 4 = total number of pennies
- multiply the number of rounds by the number you add each time and add the start number

Fig. 13.4 Ms. Bergeron's recording of a student's rule for the Penny Jar situation

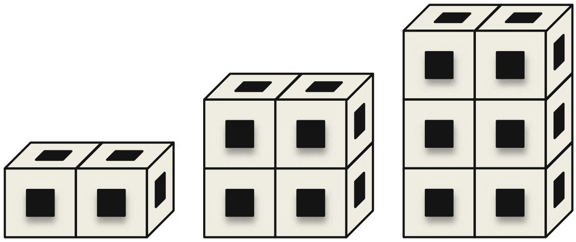

Fig. 13.5 The first three towers of the Building Towers context

Table 13.2 Representation of the two variables in the later Penny Jar context

Number of rounds	Total number of pennies
Start with	4
1	9
2	14
3	19
4	24
5	29
6	34
7	39

that the skylights played the same role as the start number in the Penny Jar and the number of windows on each floor corresponded to the number of pennies added each round.

13.2.2 Penny Jar Episode 2

In another Penny Jar lesson, students were given problems to draw out the distinction between proportional and linear relationships. Given the situation, *4 pennies to start and 5 added each round*, students created a table (see Table 13.2) for the first 7 rounds. Then they were asked, how many pennies after 14 rounds?

Many students think that since 14 is double 7, the number of pennies in the 14th round must be double the number of pennies in the 7th round, that is 78 pennies. This answer, of course, is incorrect. After the 14th round, there are 74 pennies in the jar. This doubling error, or more generally the assumption of proportionality, is common (MacGregor and Stacey 1993; Orton and Orton 1994; Scanlon 1996).

When Ms. Bergeron asked the class about the number of pennies after 14 rounds, one student, Joyce, offered another idea. She didn't assume proportionality—she knew she had to pay attention to the additive component—so she suggested that you double the value for 7 rounds and then add the starting number. Doubling 39 and adding 4, her answer was 82.

There are different moves a teacher might make when an error arises. Frequently teachers will simply indicate the answer is wrong or suggest a different method to arrive at the correct answer. In this case, Ms. Bergeron recognized Joyce's error as an opportunity for the class to think about structure. She opened up class discussion, inviting students to comment on Joyce's strategy or offer their own.

Sheenah said, "You would use the start number only once. You would probably have to do 39 and then minus off 4 and do another 39." Sheenah recognized that Joyce's strategy included the start number too many times. By subtracting $39 - 4$, Sheenah found the number of pennies added in rounds 8 through 14. She added that to the number of pennies in the jar after 7 rounds, and got the correct answer of 74.

Maria said, "When you double 39, you double the start number. So after you double, you need to subtract 4." Maria, too, recognized the problem with Joyce's method, and explained why it was necessary to subtract 4 rather than add.

The context of the problem and associated representations played a key role in allowing students to think through what was incorrect about Joyce's calculation and how to correct it. The story of a jar with a fixed number of pennies to start and a given number added each round provided an image that students could hold onto, as well as language that supported fluent communication. In their explanations, Sheenah and Maria talked about needing to include the "start number" only once in the calculation.

13.3 Analyzing Structure in the Context of Number Systems

The second area of early algebra involves students noticing, articulating, and justifying structural properties of the operations. In the context of this chapter, the term, *structural property*, refers not only to the commutative, associative, distributive, inverse, and identity laws of addition and multiplication, but also structures that can be derived from those laws, as well as structures particular to subtraction and division.

The structural properties students examine are often implicit in calculation strategies they frequently use. For example, when asked to solve $39 + 15$, a student might say, "I gave 1 from the 15 to the 39, and that gives me $40 + 14$. That's easy to solve." Implicit in this move is the property, *In an addition expression, when 1 is subtracted from one addend and added to the other, the sum is unchanged.* Or a child just learning math facts might say, "$6 + 6 = 12$ so $6 + 7$ must be 13, because it's 1 more." Implicit here is the property, *Given an addition expression, when one addend increases by 1, the sum increases by 1.* However, even when such computational strategies have been discussed in class, not all students make sense of them, and those students who use the strategies fluently are likely not to have thought through when and why they work.

In order to help students focus on such structures, some projects (e.g., Carpenter et al. 2003), taking up ideas from Davis (1964), engage students in determining whether a given number sentence is true or false. For example, students might be given such number sentences as

$$57 + 89 = 56 + 90$$
or
$$129 + 58 = 129 + 59$$

One can determine whether these number sentences are true by calculating:

- 57 + 89 = 146 and 56 + 90 = 146; therefore the first equation is true;
- 129 + 58 = 187 and 129 + 59 = 188; therefore the second equation is false.

However, students are encouraged to apply *relational* thinking, that is, to analyze structure to answer the question. The first number sentence is true because one addend decreased by 1 and the other increased by 1. The second number sentence is false because one addend remained fixed while the other addend increased.

In the classroom examples below, the teacher uses another approach to explore structural properties: Students are given sequences of related expressions and discuss what they notice.

13.3.1 Episode 1: Equivalent Addition Expressions

Third-grade teacher Lauren Fried presented the sequence of expressions shown below.[5,6]

$$14 + 1$$
$$13 + 2$$
$$12 + 3$$
$$11 + 4$$

The numbers were purposefully low in order to allow students to look for relationships without getting lost in numbers at the edge of their comprehension. The property at play was that, given any addition expression, if one addend decreases by 1 and the other increases by 1, the sum remains the same.

[5]The classroom interactions described here were taken from video recordings and teachers' written narratives based on audio recordings. Ms. Fried is a composite character composed of several teachers who taught the same lesson sequence, which appears in Russell et al. (2017, pp. 88–111). The same resource also includes video and more detailed textual descriptions of the classroom lessons (pp. 45–75).

[6]The teachers who comprise Ms. Fried are among 21 teachers who participated in a professional development project. In monthly after-school sessions and three full-day meetings each year, participants investigated mathematics, shared written narratives of their classrooms, and discussed video of their lessons. Nine teachers participated for four years; 12 teachers for two. Some teachers had participated in professional development with the researchers prior to this project.

Students started with all kinds of observations—"There's a 4 in the first one and the last one, and there's a 3 in the second and third, but they're in different places"—but eventually the class discussed the change from one expression to the next. Students specified that as they went down the list, one addend increased by 1 and the other addend decreased by 1, and the totals were all 15.

After checking their ideas out with two addends that total 24, students were asked to write a general claim, working individually or in pairs. Some of the stronger statements included the following.

- *If you take away a number and add one number it will equal the same answer.*
- *If you have two numbers and have a sum and want to have the same sum with a smaller number then you have to bring the first addend down and bring the second addend up.*
- *If you take away one from one number and add one to another number it should equal the same answer.*

Malara and Navarra (2003) coined the term "algebraic babbling," in recognition of the fact that stating a generalization is difficult and most likely new to students. Analogous to the way children learn natural language, students learn to communicate in mathematical language by starting from the intention to state an idea and through collective discussion, verbalization, and argumentation, gradually become proficient in syntax.

Ms. Fried shared these statements with her students and then challenged them to be more precise. She asked, "What operation are you discussing?" "What does the word *number* refer to?" "What about the word *one*?" In response to each challenge, students suggested edits until they co-constructed a "class conjecture": *If you have two addends, and you take away 1 from an addend and add 1 to the other addend, it should equal the same sum.*

The goal in this process is not to formulate the most concise or precise statement possible. After all, the teacher could present a conjecture to the students, and they might even understand it. Rather, the goal is to have students learn to communicate about mathematics, which means they must take on the task of putting their ideas into their own words.

Once the class had formulated a conjecture, the next task was to use manipulatives, pictures, diagrams, or story contexts to illustrate the relationships in the claim. One pair of students came up with the context of girls at the beach. *There are 20 girls sitting on the sand and 4 girls in the water, 24 girls in all. If one girl from the beach goes into the water, there are 19 girls on the beach and 5 in the water, but the same 24 girls are still there.*

Prior to the whole-class discussion, the teacher cut out 24 "girls" which she taped to the board for students to illustrate the girls' movements (see Fig. 13.6).

Working with representations like the one shown in Fig. 13.6, students not only recognize a numerical pattern, but they see *why* the pattern holds. The single action of moving an object from one group to another represents both decreasing one addend by 1 and increasing the other addend by 1. Because no objects were removed and nothing else was included, the total remains the same.

Students also realized that they could move more than one object and the same principle holds. Any number of girls could move from the sand to the water. As long as no girls leave the beach and no additional girls arrive, the total number of girls remains fixed.

It is essential that students not only become clear about the representation, but also that they see how it corresponds to their conjecture and the arithmetic equations. Just as Ms. Bergeron asked her students to specify the meaning of the representation of circles in terms of the Penny Jar, here, too, Ms. Fried asked such questions as, "Where in the representation do you see the two addends?" "What in the representation shows that the two numbers are added?" "Where do you see the sum?"

This same idea can be shown with a variety of contexts and manipulatives, and students used their representations to make a general argument. One child, Melody, presented as her two addends a long stick of red cubes (shown as light gray in Fig. 13.7) and a long stick of blue cubes (shown as dark gray in Fig. 13.7), both sticks longer than shown here. She said, "We don't know how many cubes are on the stick." That is, her stacks of cubes could represent any two addends. "And if we take this many"—she removed some red cubes—"and put it onto there"—adding them to the blue stick—"it would be the same thing. The red one got smaller and the blue one got bigger, and it's the same."

Ms. Fried asked questions to clarify with Melody and the class exactly what she meant and how the representation related to the conjecture. Even though there were necessarily specific numbers of red and blue cubes, Melody was clear that this was irrelevant. She hadn't counted them; they could represent any number. And when

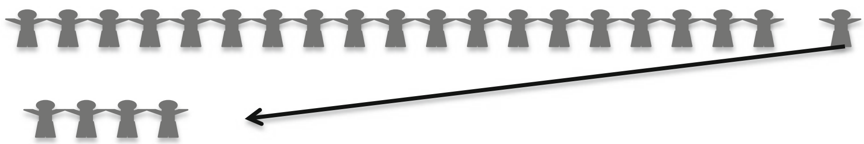

Fig. 13.6 Teacher's representation of the girls at the beach situation

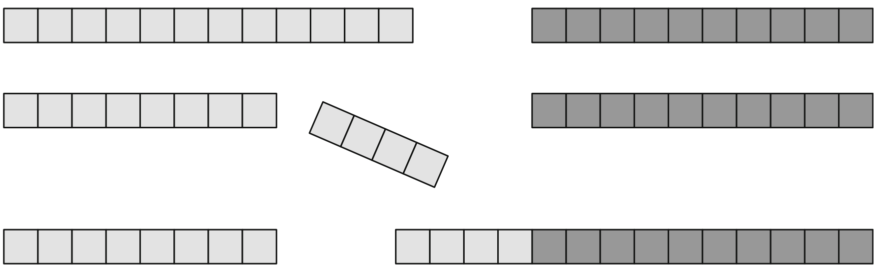

Fig. 13.7 A student's representation of alternative decomposition of the same sum

she broke off some red cubes, it could have been any number up to the number of red cubes in her stack. The act of moving some cubes from one stack to the other, conserving the total number of cubes, demonstrated the conjecture, which now read, *If you have two addends, and you take away some amount from an addend and add the same amount to the other addend, it should equal the same sum.* At the elementary grades, arguments like this one constitute proof.

However, even though the class conjecture stated, "If you have two *addends*…" and even though the representations show two quantities that are joined, the fact that they were *adding* still fell into the background. Many students think of a property of addition as about *numbers* rather than particular to *the operation*. For this reason, it is essential for students to continue their exploration by examining another structural property, analogous to this one, but for a different operation.

13.3.2 Episode 2: Equivalent Subtraction Expressions

Looking at a sequence of equations as shown below, Ms. Fried's students were surprised to realize that their rule of adding 1 to one number and subtracting 1 from the other didn't work for subtraction; that is, it didn't produce equal differences. But this launched an exploration of what does work for subtraction.

$$\begin{array}{ll} 17+5=22 & 17-5=22 \\ 18+4=22 & 18-4=14 \\ 19+3=22 & 19-3=16 \\ 20+2=22 & 20-2=18 \end{array}$$

Ms. Fried's class went through a similar process of examining sequences of subtraction expressions and working to formulate a class conjecture. They agreed on this statement: *If you're doing subtraction, you can make both numbers go up by 1 or both numbers go down by 1, and you get the same difference.*

Students next devised representations, using cubes, number lines, pictures, and/or story contexts, to explain why this generalization is true. One pair came up with a story about a boy holding helium balloons. If he holds more than 12 balloons, he'll float away. He starts with 15 balloons and quickly has to pop 3. If he starts with one more balloon, he has to pop one more balloon. Clearly, the story doesn't have to be realistic. Its purpose is to provide an image of subtraction that allows students to reason about how the variables are related.

Latesha: *Every time you add another balloon, it has to be popped, so both numbers keep going up by 1, but he's always still got 12.*

Ms. Fried: *Here's my question. This picture shows the amounts changing by 1 each time. Can we use the balloon context to show adding an amount that's not 1?*

The teacher asked students to turn and talk about this. After a few minutes in pairs, the teacher brought the group together. The discussion started with specific numbers.

Ezra: *If he has 17, he has to pop 5. But it doesn't just have to go up by 1. Give him 100 more; then he has 117, so he has to pop 100 more.*
Ms. Fried: *So how many does he have to pop?*
Isaac: *He has to pop 105. 117 take away 105 is the same as 17 take away 5.*
Roberta: *It could be 1017 and 1005.*
Maria: *He'd have to pop 1005. He'd have to do it really fast.*

Then students started using general language.

James: *The more balloons you give him, the more he has to pop. That's why both numbers keep going up.*
Sage: *And they have to go up by the same amount. If you higher the first number, you have to higher the second number, or you won't get back to the same answer.*

As a result of this discussion, the class revised their conjecture. Their first conjecture about subtraction dealt with increasing or decreasing the terms by 1. Now it read, *If you're doing subtraction, you can change the first number and the second number by the same amount and you get the same difference. You can add an amount or subtract an amount, but you have to do the same thing to both numbers or you won't get the same answer.*

At the start of their exploration of subtraction, after having proved a claim for addition, the class had expected the same claim to work for both operations. They quickly saw that it didn't work, and recognized the limits of their first claim. This brought the operations to the fore in students' minds. The contrast between addition and subtraction helps students come to see an operation not exclusively as a process or algorithm—a set of instructions to *do* something—but also as a mathematical object in its own right, each operation with a different set of structures.

Toward the end of their exploration of the subtraction claim, these students reflected on the work they had done. In the midst of this discussion, one student said, "When we got the idea of seeing if our addition rule works for subtraction, I was like, of course it works. And then it was like uh-oh, it doesn't work, and I lost all hope." She concluded, "I'm happy we found a very close but different rule."

Another student said, "I wonder what happens with multiplication."

13.4 Comments About the Classroom Examples

This chapter has presented classroom episodes that illustrate students and teachers working on two different types of early algebra tasks: work with functions and work with generalizations in the context of arithmetic. Both of these examples demonstrate early algebra as a way of thinking: Students develop the habit of looking for structure, and articulate, test, and prove rules or conjectures for an infinite class. That is, students think analytically about indeterminate numbers.

In these episodes, the teacher used the tasks to make explicit how structures differ for each operation. In their work with linear functions, students developed images to represent the additive component and the multiplicative component of the function. In their work with generalized arithmetic, students investigated two conjectures that contrast the behavior of addition and subtraction.

Key to the students' work in these episodes is the linking of story contexts with spatial and/or numerical representations of structure. Here, the teacher's role is essential. In both classrooms, the teacher continually challenged the students to identify correspondences across representations. About the Penny Jar: "What is this row of circles in terms of the Penny Jar story?" "Why do we multiply the number of rows by 9?" "Where do the rows appear in the table?" About girls on the beach: "Where in the representation do we see the two addends?" "Where is the sum?" "How do we see one addend increasing by 1 and the other decreasing by 1?" About the balloon story: "The more balloons you give him, the more he has to pop. How does that help us think about our conjecture?" "How do you see that idea in the equations?"

Having made connections across representations, students were able to explain their reasoning in terms of story contexts and images. In Ms. Bergeron's class, students talked about the "start number" from the Penny Jar context to explain why one calculation strategy was incorrect and why another approach would result in the correct answer. In Ms. Fried's class, students defended their conjectures with stacks of cubes and stories about girls at the beach or a boy holding balloons. Relying on such representations, students—both those who were explaining and those following the reasoning of classmates—used their own powers of reasoning rather than remembered symbol patterns.

13.5 Impact on Student Learning: Focus on Operations Versus Focus on Numbers

From the projects in which the classroom examples were drawn, two additional data sources (beyond the classroom interactions documented in video and written narratives) provide evidence of the impact on student learning of lessons that focus on the behavior of the operations. One source is a set of interviews with students of collaborating teachers. The second source is a written assessment of students whose teachers participated in an online professional development course.

13.5.1 Interview Data

As part of a teaching experiment with twelve teachers of grades 2–5, teachers implemented two instructional sequences to explore contrasting generalizations such as that illustrated in the case of Ms. Fried. The sequences, written by the researchers, consisted of twenty to twenty-five 15-minute lessons. The twelve teachers met with the researchers at monthly after-school 3-hour sessions and four full-day sessions. Nine additional teachers who had already been working in the project for two years and collaborated with the researchers to design the sequences also attended these meetings. The sessions were opportunities to work on mathematics together, share narratives in which teachers reported on classroom interactions, and view video of participants' lessons.

From each of twelve classrooms, one-on-one interviews were conducted at the beginning and end of the school year with three students representing the range of learners, characterized in terms of strong, average, or weak in grade-level computation. In one strand of the interview, students were given pairs of subtraction problems (for example, $10 - 3 = 7$; $10 - 4 = ?$) that illustrate a structural property not explored in the instructional sequences: *Given a subtraction expression, if the second term (the subtrahend) increases by 1, the difference decreases by 1.* Students were asked to describe what they noticed, come up with other pairs of problems that illustrate the same feature, state a conjecture, and use a representation to explain why the conjecture must be true.

In the analysis of interview data (Higgins, in preparation), one of the dimensions that distinguished students' conjectures was "salience of the operation: the degree to which students attend to the behavior of the operation versus focus almost exclusively on the numbers when drawing generalizations and articulating conjectures." Some students articulated generalizations that were fundamentally about the operation: "When you have the same numbers, once you subtract more, you'll have less. And if you subtract less, you will have more." Other students showed no evidence of attention to the operation: "The numbers in the middle, you just add 1. Then the answer you take away 1. The first numbers are the same."

In interviews conducted at the beginning of the year, lack of salience of the operation was found for close to half the students (n = 36). After having worked on lesson sequences that explored a different set of generalizations about the operations, the percentages improved. At each grade level, more students explicitly referenced the operation or talked about what they were noticing in operation-specific terms. The operation was no longer just part of the background, but became something that students realized they needed to attend to when articulating what they were noticing.

13.5.2 Written Assessment Data

Based on what the research team learned with collaborating teachers, the team designed a year-long on-line professional development course (Russell et al. 2012b) with the following goals: to help teachers understand and look for structural properties implicit in students' work in number and operations, bring students' attention to such properties, and support students to articulate, represent, and create mathematical explanations of the properties. The book, *Connecting Arithmetic to Algebra* (Russell et al. 2012a), formed the basis of the course. Participants engaged in three main kinds of activity: discussing chapters from the course text, doing mathematics activities designed for adult learners, and writing *student thinking assignments* in which they analyzed their efforts to engage their students with course ideas. Teachers posted and responded to weekly assignments and participated in six synchronous webinars across the year.

The teachers were not given lessons or tasks to take to their students. Rather, the nine *student thinking assignments* given over the course of a school year suggested questions teachers would pose to their class. For example, the first assignment is shown in Fig. 13.8. Researchers wrote written responses to each teacher's *student thinking assignments.*

The first year the course was offered, pre- and post-course assessment data were collected from 600 students of 36 participating teachers and, as comparison, 240 students from 16 non-participating teachers in the same school systems (Russell et al. 2016). Assessment items included those in which students were asked to explain why they think two expressions are equal. Student responses were coded for the type of explanation they provided: (a) no explanation; (b) a computational explanation; or (c) a relational explanation, that is, an explanation that refers to mathematical structure. For instance, to explain why $9 - 5$ and $10 - 6$ are equal,

Choose *one* of the following questions to work on with your own students:

Is this number sentence true? $2 + 5 = 3 + 4$

Is this number sentence true? $19 + 6 = 20 + 5$

Is this number sentence true? $3 \times 7 = 7 \times 3$

How do you know they are equal? 3×16 and 6×8

How do you know they are equal? $34 + 27$ and $31 + 30$

How do you know they are equal? 50×10 and 5×100

As you plan, think about how you will help your students learn to have mathematical discussions. Record the class discussion (audio or video) to have a record after the class session is over. When you listen to the recording, note student responses that particularly intrigue, surprise, or please you. Choose a passage to describe in detail in writing, including actual student dialogue. Write about two or three of the students' responses. Your posting should be about two pages in length.

Fig. 13.8 The first *student thinking assignment* given to the teachers a student could carry out both computations, showing that each expression equals 4, or the student could give a relational explanation: e.g., “Since 9 is 1 less than 10 and 5 is 1 less than 6, the difference is the same.” In the post-intervention assessment, students of teachers in the Participant Group provided significantly more relational explanations than in the Comparison Group.

The assessment also asked students in grades 3–5 to write a story problem for a given multiplication expression. In the posttest, 74% of the Participant Group (n = 475) produced a correct story, but only 48% of the Comparison Group (n = 180). Students of grades K to 2 were asked to write a story problem for a subtraction expression. Although the Participant Group (n = 128) showed significant progress from pretest to posttest—from 28% correct to 74% correct—the difference with the Comparison Group (n = 60) was not significant.

13.5.3 In Summary

Data from both sources suggest that lessons in which teachers draw students' attention to the distinct structures of each operation help to make the operations a salient object in students' mathematical experience. The interview data demonstrate how, when students notice patterns across calculation problems, they recognize the pattern as related to the structure of a given operation. The written assessment data reveal that, once students have an opportunity to explore and represent properties of the operations, they have a better understanding of contexts that are modeled by the operations and rely on structural properties to explain the equivalence of arithmetic expressions.

However, within the interview data, even after the intervention, there were still students at each grade level that produced conjectures in which the operation was invisible. In the post-intervention written assessments, there were students who continued to rely on computation to prove the equivalence of two expressions and students who could not create a story problem for a given arithmetic expression. To make the operations salient objects in *all* students' mathematical experience requires persistent effort.

13.6 Conclusion

With the considerable attention in the elementary and middle grades given to calculation procedures for addition, subtraction, multiplication, and division with different categories of numbers, it might *seem* that students have extensive experience with the operations. However, while students might remember calculation procedures, little attention is given to *structural properties* that distinguish each operation from the others. A consequence of such absence is the lack of salience of the operations in students' minds. The operations are interpreted as instructions to perform a set of steps rather than as objects, each with its own set of characteristics and properties.

The absence of understanding structure carries into students' study of algebra. For example, consider such common errors as these:

- $(3 + 5)^2 \neq 3^2 + 5^2$ but we see algebra students write $(a + b)^2 = a^2 + b^2$
- $2 \times (3 \times 5) \neq (2 \times 3) \times (2 \times 5)$ but we see algebra students write $2(ab) = (2a)(2b)$

If students learn to call upon a variety of representations to reason about the structure of the operations in the elementary grades, the tools they develop will help them analyze such errors and support their fluency in algebra.

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Chapter 14
How Early Is Too Early for Thinking Algebraically?

John Mason

Abstract My answer is that it is never too early. In order to learn arithmetic it is necessary to think algebraically, although not necessarily using symbols. Some evidence for algebraic thinking amongst young children is given, followed by suggestions as to why such thinking has not always been promoted and developed. Specific pedagogic actions are outlined that focus on the expression of generality as the core of algebraic thinking, including examples of task-contexts that invoke reasoning both with and without using numbers. Finally, it is proposed that the critical feature for promoting algebraic thinking is not the tasks given to learners, but rather the opportunities noticed by teachers for calling upon learners' powers to express and manipulate generalities, and that this is enriched when teachers engage in similar tasks at their own level, so as to sensitize themselves to pedagogic opportunities when working with learners.

Keywords Expressing generality · Algebraic thinking · Teachers' noticing
Children's powers

14.1 Introduction

My answer to the title question is that it is never too early for sensitively directed generalization and abstraction, which is in total agreement with Hewitt (1998) and with Gattegno (1970). Indeed in some sense it is impossible to be too early, although of course it is always too early for insensitive instruction. This is confirmed by many authors (e.g., Cooper and Warren 2011, p. 207). I develop the proposal that the use of material objects for learners to manipulate is only a special case of the general proposition that to appreciate and comprehend something it is useful to have mediators, to make use of confidently manipulable objects (which

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may be diagrams, symbols, or material objects) in order to recognize, detect, and locate structural relationships. Such relationships are not confined to any particular example but are instances of general properties shared by all examples. The whole point of algebra is to permit the manipulation of referents rather than of material objects themselves, in order to gain insight into the general. Algebra cannot be 'too early' if it emerges out of growing awareness of and familiarity with expressions of generality.

My approach follows Maslow (1971): I am interested in what is possible, happy that others research what is currently the case. The first section (Sect. 14.2) summarizes some of what is possible based on the research of but a few of the many that could have been used. The second section (Sect. 14.3) indicates why these and other reports of children's mathematical behavior ought not to be any surprise, using the discourse of natural human powers and the education of awareness, again referring to only a selected few of the many possible authors. The third section (Sect. 14.4) suggests why the possible is not always actualized. The fourth section (Sect. 14.5) indicates some ways of working sensitively with children in order to enable them to display their powers in a mathematical context. The fifth section (Sect. 14.6) offers some tasks pitched at the adult level but intended to provide experience roughly parallel to what children might be experiencing, because this is how teachers can maintain and enrich their sensitivity to their learners. The final section is by way of a concluding summary.

14.2 A Taste of What Is Possible

This section rehearses some of the possibilities for exploiting the early algebraic thinking of young children. In each case, whether repeating patterns, symmetry, or counting, the children are called upon to become aware of, and to articulate, generality. Arithmetic without generality is a purely clerical activity; arithmetic which calls upon children to become aware of generality is mathematics.

14.2.1 Repeating Patterns

When kindergarten teachers asked how they might develop the mathematical thinking of young children using activities they were already using with children, Papic (2007) suggested making repeating patterns. She reports use of a wide range of tasks, some used for assessing what children can do with patterns: copying towers of colored blocks with a repeating pattern and determining missing elements that are shielded from the child (Papic 2013). The important thing is that very young children can develop a sense of repetition and indeed even more complex patterns, if suitably challenged. The pictures in Fig. 14.1 are but instances of all of

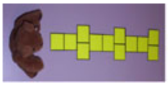
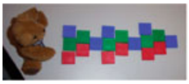
Achievement	*Examples*	
Copying and extending a repeating pattern Simple repeating patterns Complex repeating patterns	 model	 copy: age 4.1
Generating their own repeating pattern	 age 5	 age 5.4

Fig. 14.1 Children copying and constructing repeating patterns (Papic 2013) (reprinted with permission from author)

the children in the study displaying the power to copy repeating patterns, and to create their own. All they need is to have their attention directed to this possibility.

Chen (2015) reports on children detecting repeating patterns despite distracting aspects of the context. A significant feature of Confucian heritage teaching of mathematics is to vary the context or setting in which the concept is embedded as seen in Fig. 14.2. Here Chen shows how children can take into account complexity, such as changing directions, as well as identifying missing elements in a repeating pattern.

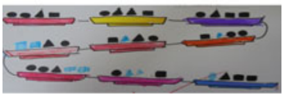
Complexifying the context	 Notice the rising and falling aspect, which has to be ignored.
Recognizing missing elements in repeating patterns	 age 6  age 6 Each line of three boats should be the same. Notice the change in direction.

Fig. 14.2 Complex contexts for detecting repetition (Chen 2015) (reprinted with permission from author)

Articulating and expressing generality about repeating patterns		Child T11: "See this is a pattern of 'I' for Isaac, three green across, three blue up and three yellow across … two times".
	 Even more sophisticated articulation when the camels were joined into a circle	Child T11: "Big purple, little purple, little yellow, big green. It's three times."

Fig. 14.3 Articulating observed repetitions (Papic 2013) (reprinted with permission from author)

An important feature of education at every age is to work on expressing what is imagined using material objects, diagrams, words, and symbols. The next examples illustrate ways in which the materials can be changed but the essential idea remains the same. In Fig. 14.3 there are but two of many instances in which children constructed their own complicated pattern using different materials and in slightly different settings (e.g., the circularity of the camels).

Papic and Mulligan (2007) developed a sequence of discernible phases or stages of emergent pattern recognition and use. While recognizing that mathematics education research usually takes the direction of increasingly fine discernment, I myself am more interested in alerting teachers to possibilities, and letting them watch and listen to children responding to stimulating tasks without trying to classify the complexity of those responses. What is important for me is extending and challenging, rather than gauging and evaluating, under pressure from institutions seeking evidence of progress. As will emerge later, developing a sensitivity to notice, accompanied by possible actions arising because of what is noticed, is more important to me than a series of labels for describing, distinguishing, and even worse, comparing learners' behavior.

14.2.2 Symmetry

Bornstein and Stiles-Davis (1984) tested children aged 4–6 on discriminating between symmetric and asymmetric pictures, finding a range of achievement in which sensitivity to vertical axes of symmetry dominated horizontal axes, with oblique axes following rather later. They noticed that many of the patterns constructed by children had strong similarities with mirror-symmetric decorations on stone tools and pots of early humans. Bornstein and Stiles-Davis were also looking

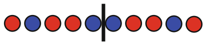
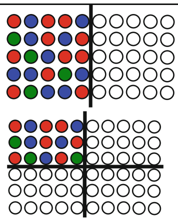
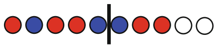
Creating patterns with reflective symmetry	
using one axis of symmetry; using two axes of symmetry	
Recognizing missing elements in symmetrical patterns	
Articulating what is missing in symmetric patterns with justification	E.g.: "this one has to be the same as that one" extending to "to make it symmetrical" to "because they are the same distance from the mirror line"

Fig. 14.4 Reasoning using symmetry

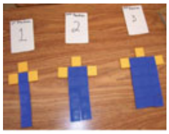
Predicting the number of elements required on a growing pattern	 age 9

Fig. 14.5 Predicting (Moss and Beatty 2006) (reprinted with permission from authors)

for a developmental sequence, but without the children having been encouraged previously to work at symmetry it is hard to chart development. Figure 14.4 displays different task structures for stimulating learners to make use of symmetry in their reasoning.

14.2.3 Counting

Moss and Beatty (2006, p. 449) have shown that promoting conjecturing and predicting (Fig. 14.5), using the language of "I have a theory" with 9-year-olds, the children quickly picked up the practice and integrated conjecturing, and presenting evidence for their conjectures, into their functioning (educating their awareness). They soon moved from displaying instances of their 'theories' on a grid (as a point graph) to 'inventing' negative numbers so as to extend their graphs to the left of 0. Their pattern sequences extended to quadratic relationships as they continued to propose and negotiate multiple expressions of generality.

Blanton and Kaput (2011, p. 9) reinforce the observation that children's development of algebraic thinking has traditionally been constricted by inappropriate assumptions about what is possible: "… the genesis of these ideas [functional analysis] appear[s] at grades earlier than typically expected."

Cai et al. (2011) report on how learners are supported in developing algebraic thinking in China and Singapore, suggesting that this accounts for superiority in algebra later as compared with educational systems in which not only formal algebra, but expressing generality and development of algebraic thinking are not introduced until high school, if at all.

Cooper and Warren (2011, p. 197) summarize their own and other people's research by saying that "results have … shown that students can generalize relationships between different materials within repeating patterns across many repeats." They go on to suggest that the use of tables can help students generalize not only in the younger years, but later in school, for example when working with equivalent fractions.

My claim is stronger still: learners need to generalize simply in order to engage with arithmetic seen merely as the manipulation of numerals (symbols standing for numbers). As they meet more and more complex mathematics, more and more sophisticated mathematical ideas, the need to 'see the general through the particular' and also 'to see the particular in the general' grows in complexity and sophistication correspondingly.

For me, as for most mathematicians, arithmetic is much much more. It is the study of actions on numbers in order to study and make use of numbers themselves. This is what makes arithmetic part of mathematics. In order to make sense of mathematics, in order to appreciate and comprehend the topics that learners meet, it is necessary to use and develop one's natural powers to imagine and to express, to specialize and to generalize, to conjecture and to convince. These can be strengthened by acknowledging and calling upon them at the earliest opportunities. For example, Hewitt (1998) shows that arithmetic competence and proficiency depend on a sense of, an awareness of, generality without which learners are doomed to tackling each new task ab initio. Competence and proficiency involve the recognition of particular tasks or situations as instances of a class of similar situations, which constitute a person's example space (Watson and Mason 2002, 2005). Learners who have not been called upon to express generality are at a major disadvantage as they move through school. As Caleb Gattegno famously said (private communication 1978) "the real problem with teaching mathematics in school is what to do after age 12, when they have learned the entire [current] school curriculum" (or words to that effect).

Mason et al. (1985) made a case for multiple roots of algebraic thinking, with correspondingly multiple routes into explicit algebraic thinking, and showed how this could be done, drawing on materials and ideas stretching back to Chinese manuscripts, Egyptian papyry, and Babylonian tablets, as well as projects in the UK in the 1960s and 1970s. This was extended in Mason with Graham and Johnston-Wilder (2005) to a program of tasks designed to promote and reinforce the expressing of generality. School algebra is seen as a language for expressing generality, which, because of its succinct use of compact symbols, is readily manipulable, and so can be used to solve equations. The source of those equations is the expression of necessary relationships between quantities with as-yet-unknown values. This goes back to Newton (1707) who famously shifted the focus of mathematicians from setting up systems of equations that express necessary relationships to the solving of equations. Newton thought that the translation of problematic situations into algebra was essentially trivial, though colleagues such as Ward (1713), who were perhaps more experienced in teaching novices, disagreed (Mason 1999). Evidence seems to be largely on the side of Ward rather than Newton, as attempts are made in each generation to simplify, perhaps even automate, the modeling of simple situations with algebraic symbols. The use of Singapore bar diagrams (Fong 1993; Lee et al. 2013) is but one recent attempt in this direction.

It is worth noting that languages that are both expressive and readily manipulable are emerging from algebra into the domain of computer technology. LOGO, Boxer, TuneTalks, computer algebra systems, and more recent offshoots, such as TouchCounts, provide people of different ages and maturity with expressive and manipulable languages. All of these invoke algebraic thinking and algebraic awarenesses.

14.3 Why What Is Possible Is No Surprise

No one expects young children to internalize or memorize all two- and three-digit additions and subtractions, much less multiplications and divisions. Instead, children are expected to internalize procedures that are sequences of actions for achieving these tasks. Yet telling someone how to perform these involves complex action sequences with choices being made according to the particular numbers (see last section for ways to re-experience this for yourself). However, for procedures such as long multiplication or division or adding and subtracting fractions to be effective, it is necessary not simply to memorize a sequence of steps, which are prone to getting mixed up or be forgotten, but rather to appreciate what the steps enable to be done next. This is educating awareness, in the sense of Gattegno (1970, 1987), in parallel with training of behavior (Mason 2008a).

It is a plausible conjecture that Egyptian and Babylonian scribes did not attempt to express methods of solving problems in general because of the effort required in using cuneiform on moist mud tablets and because of the expense of preparing papyry. Much more sensibly, they show worked examples, with assertions such as “thus is it done” (Gillings 1972/1982) to indicate the potential generality. Newton (1707; see Whiteside 1972, p. 135) comments specifically: “However, so that I may develop an intimacy with this method of reducing problems of this sort to an equation and make it clear—and since skills are more easily learned by example than precept—I have thought it right to append the solutions of the following problems ….” The use of worked examples has been an integral component of teaching mathematics through the ages, despite its weaknesses (Chi and Bassok 1989; Renkl 2002).

Any child who can walk and talk has already exhibited massive use of natural powers, including generalizing and abstracting. Indeed, Davydov (1990) and followers see humans as moving naturally from the abstract to instantiation in the particular, rather than the other way round (see, e.g., Schmittau 2004; Dougherty 2008). Goutard (1958) independently expressed it as: “In all fields of education, and especially with young children, we must start with indefinite situations (for such is the reality in which they live).” One possible reason is that language is fundamentally general. In order to be specific and particular, it requires additional work with adjectives and adverbs and-or direct use of deictics and pronouns, or even the addition of pointing and touching. The Russian claim is that children quite naturally learn to perform the reverse process of seeing the particular as an instantiation of something more general. It was Gattegno’s realization of the complexity and powerfulness of this move by young children, from the abstract indefiniteness of language to internalizing it for their own use, by themselves, which led him to develop his *Science of Education*, including *The Silent Way* (Gattegno 1970, 1973, 1975/1988, 1987; see also Young and Messum 2011).

Children appear to comprehend quite readily the use of schematic diagrams such as bar-diagrams for depicting relationships in word problems, without requiring correspondence between marked length and actual length (Golomb 1992; Lee et al. 2013). The issue in interpreting any diagram is recognizing what relationships are being indicated schematically. The same applies to geometrical diagrams as well. Of particular importance is recognition of what can change in the diagram and still the relevant relationships will be maintained, in other words, to appreciate the scope of generality of the properties being instantiated as relationships.

Expressing generality succinctly and precisely is made possible by the use of referents, and the expression of properties using those symbols. This applies differently to algebra and to geometry. Among other things, the symbols help the reader to ignore attributes that are not relevant to the mathematical relationships. Young children show themselves to be adept at this, using colored counters to stand for imagined objects, such as cars in a car park or children in a classroom, as well as using letters to stand for qualities of objects, such as their size. The power of succinct expressions lies in the manipulative possibilities, as mentioned earlier.

Where kindergarten teachers, indeed teachers of children of every age make use of and develop children’s powers to imagine and to express what they are imagining, using gestures, diagrams, words, and symbols, children are being prepared to think mathematically, to engage with explicit not just implicit algebraic reasoning. I know of at least one child who burst into speech only when he became frustrated that his parents were not correctly and quickly interpreting his pointing. Algebra as a language makes it possible to refer rather than point, to avoid ambiguity in the use of prepositions such as *this* and *that*, and to deal with many, usually infinitely many, cases all at once. This is truly powerful thinking. Indeed Gattegno (1984, p. 20) proposed that a lesson is mathematical only when it is “shot through with infinity.”

Gattegno (1975/1988) also suggested that “I made my brain” and that that is what every child who can walk and talk has managed to achieve. I think he meant that the brain develops not simply through experience of the material world, but as a result of processing that experience; that learning involves ‘educating your awareness,’ which means becoming sensitized to notice phenomena, and accumulating actions that can be enacted with a minimum of attention so that attention is available to provide overall direction. One manifestation of this is Jerome Bruner’s notion that scaffolding involves the teacher upholding attention foci when the learner’s attention is fully absorbed by some detail (Bruner 1996, pp. 75–76).

The only real constraint on learners is what they impose on themselves. However, imagined constraints are strongly influenced by the milieu: teachers, parents, institutions. Dweck (2000) has provided massive evidence that children’s discourse, often picked up from adults around them, can be altered so as to shift from a negative, psychologically blocking stance towards a positive, psychologically opening stance. It can be roughly summarized as helping children turn “I won’t …” and “I can’t …” into “I can and I will … try harder and differently” (Open University 1982; see also Boaler 2010).

14.4 Why the Possible Is Not Always Actualized

Teaching is a caring profession that depends on the relationships between teacher and learners, and between teacher and mathematics in order to engender a productive disposition between learners and mathematics (Andrews 2016). Very often one or other of these ‘cares’ is stressed with the other disappearing into the background. For example, concentration on social organization of groups of learners without also attending to the mathematics being discussed stresses one without building on the other. It leads to unhelpful pedagogic actions such as ‘dumbing down’ (Henningsen and Stein 1997; Stein 1987; Stein et al. 1996).

Just as many textbook authors, educators, and teachers see multiplication as repeated addition when in fact repeated addition is only an instance of multiplication, so textbook authors, educators, and teachers often seem to mistake the use of tasks involving pattern generation and expressions of generality for algebraic thinking, when it is only an instance of generalization. As I have said on many occasions “A lesson without the opportunity for learners to generalise mathematically, is NOT a mathematics lesson” (Mason et al. 2005, p. 1). In the absence of other stimuli, children will generalize from a few lessons that they find tedious, that “mathematics is not for me.” It is surely much better to provoke them to use their powers mathematically! To learn to think mathematically, to appreciate and comprehend the mathematical enterprise involves actively generalizing and abstracting at every turn. The only reason children do not succeed is because their powers have been allowed to atrophy or because they have been induced to leave them at the classroom door, usually because the child has interpreted the teachers’ actions as not requiring the use of those powers. This mis-impression is likely whenever the teacher tries to do the learning for the children, by doing the specializing and the generalizing, the imagining and the expressing, under the misapprehension that 'they are not yet ready to do that for themselves' when in fact they have been doing it since birth if not before.

14.4.1 Falling into Habits

It is all too easy to fall into habits of how tasks are offered. "Habit forming can be habit forming" as the Zen master says (Shigematsu 1981, no. 341). Many, such as Thorndike (1922, p. 194) have tried to exploit it, following the adage that 'practice makes perfect'. But reinforcing mechanicality at best trains behavior, which is inflexible and which blocks creative responses to situations. One example of habit forming is always offering learners the particular and expecting them to generalize, rather than sometimes offering a partial generality, or a very general statement, so that learners can make use of and develop their power to specialize as well as to generalize. Another example is providing learners with the first few terms of a sequence and asking for successive terms. This makes two significant mistakes. First, it habituates learners into reasoning forward, often inductively, and directs their attention away from looking at something structurally, that is, seeking out relationships which are instances of general properties (Kieran this volume). Second, any sequence can be extended mathematically in any way you like. Before expressing a generality, it is essential to have some underlying general structure to express (Mason et al. 1982/2010). Thus when learners are offered a sequence of pictures, whether geometrical or otherwise, there is no generality to express until there is a statement of how the picture sequence is formed and extended. Alternatively, a sequence consisting of a repeating pattern can be presented to learners, with the proviso that the block that generates the sequence appears at least twice. Then (and only then, Mason 2014) can you be sure that the sequence can be extended uniquely.

For example, if you are told only that a block that generates the sequence by being repeated has occurred at least once, then the sequence *AABAABA* can be extended in two ways:

AAB.AAB.AAB.AAB … but also *AABAABA.AABAABA.AABAABA* … .

An interesting exercise is to look for sequences that can be extended in at least n ways for different values of n. Notice that that remark has shifted attention to 'two ways' being seen as a potential parameter, and this is precisely what variation theory exploits (Marton 2015), and the type of thinking that needs to be promoted if learners are going to appreciate what mathematical thinking is like and what the enterprise of mathematics is about.

14.5 Sensitively Promoting Generalization and Abstraction

The notion of sensitivity is being emphasized because it is all too easy to try to push children, telling them things that they are not likely to be able to relate to their own experience. Of course there is nothing wrong in telling people things. The mistake is to assume that they have internalized what has been said together with the speaker's way of perceiving. Indeed, it is a mistake even to assume that they have made appropriate sense of what has been said. In other words, when something is expressed, there are two actions expected of the hearer: to comprehend what is said, and to appreciate what is said by experiencing the situation so that what is said is experienced as an expression of what is perceived. This often fails because the teacher and the learners are attending to different things. Even when they are attending to the same thing, they may be attending in different ways (Mason 2003).

Teaching by listening turns out to be far more effective. Although it lies behind various reform movements, trying to engineer other people's teaching to match some imagined ideal proves to be ineffective (as the vast education experiment of the last 3000 years amply demonstrates: cycles of attempts at reform rarely move the center of gravity of practice away from 'filling and drilling', that is, telling and then testing). Instead of asking a question and then waiting for an answer, judging its appropriateness against the thought which prompted the question (Love and Mason 1992), teaching by listening involves putting learners in situations where they naturally ask questions (Davis 1996; Meyer 2013), and where the mere fact of a respected 'other' being present influences learners' behavior. Care for both learners and mathematics then enables you to respect the mathematical thinking, acknowledging (praising if relevant) specific actions, perhaps even labeling those actions so that they can be referred to and drawn upon in the future (a part of scaffolding mentioned earlier).

Recognizing that children have already demonstrated and used astounding natural powers just to be able to walk and talk, and drawing upon, invoking, and evoking these could be a central feature of teaching. That this has not become standard practice is for me a sign of failure for my generation of mathematics educators and teacher supporters.

The *transposition didactique*, recognized and labeled by Chevallard (1985), captures the shift from an expert becoming aware of something, to converting that awareness into a sequence of actions for learners to carry out. What almost always happens is that expert awareness (expert experience) is converted or transposed into training in behavior: the instructions the expert gives in order to try to reproduce their experience in and for the learner. It takes a great deal of care for both the learner and the mathematics in order to propose tasks that open up possibilities for learners rather than closing them down.

14.5.1 Multiple Expressions for the Same Thing

Algebraic manipulation ought, in my view, to be a trivial matter. It can arise perfectly naturally when several learners each express the same generality, but differently (as usually happens when generalizing picture sequences, but also in other situations). If several different looking expressions appear, it is natural to assume that there is some way to get from one to the other without using the original source: in other words, by manipulation of the expressions themselves. Having started in this vein with secondary students, and drawing on various previous projects in the UK, we at the Open University, like many others, soon realized that it was also available to primary children (Mason 1990). Appreciation of the 'rules' for manipulating algebraic symbols is both informed and reinforced by reflection on the properties of arithmetic operations (emerging as the axioms of algebra, and then of arithmetic). This is but one more example of how the rules of algebra, as well as being the generalization of experience of arithmetic, can arise perfectly naturally in children's awareness when they are offered tasks that bring the need for manipulation to the surface. In the light of the extensive evidence of children's natural powers, it is a reasonable conjecture that teachers who have run into difficulty teaching algebra may not have drawn upon children's powers appropriately.

14.5.2 Worked Examples and Tracking Arithmetic

Seeing someone work through an example using a particular technique can be very instructive especially if the learner then tries a similar example for herself. But I suggest that it is only effective when the learner has some sense of what is particular and what is general; what are parameters in the examples, and what is structural. In other words, it depends on learners attending to the same things as the teacher, and in the same way. For example, in $C = 2\pi r$, both the 2 and the π are structural, while the r is a parameter; in the example of a circle with radius 2, the circumference is $2\pi \times 2$, where one of the 2's is structural and the other is not. Furthermore, it really helps if the learner develops an inner incantation or patter that guides their actions, for the issue is usually not so much what to do next, but how you know what to do next. This has been verified by researchers looking at what is effective about worked examples (e.g., Chi and Bassok 1989). Having an appreciation and comprehension of the procedure and why it works is also a contributing factor, because training behavior (memorizing a procedure step by step) is inflexible and by itself, at best, instrumental (Skemp 1976). Trained behavior is of limited use without educated awareness, which means developing a repertoire of actions that can then come-to-mind (actually, come-to-action along with associations coming to intellect and positive disposition coming to emotion).

Tracking arithmetic is one way to move quickly from specific examples to recognizably algebraic symbols (Mason 2016; Mason et al. 2005, p. 21; Mason et al. 2007). You use a particular number (as in Think of a Number games) but refuse to permit that number to be absorbed into any calculation. You can then track its route through the various steps, replace it with a cloud standing for some as-yet-unknown or unspecified number, and then eventually move to using a symbol.

In a similar manner, if you can check whether an answer is correct, you can track that number, then replace it by a cloud or symbol to produce algebraic constraints (equations or inequalities), which you can then set about resolving. Mary Boole pointed out that what is needed is to 'acknowledge the fact of [y]our ignorance,' to denote that by a symbol, and then to express what you do know using that symbol (Tahta 1972). As Davydov and followers have amply demonstrated (Davydov 1990; Dougherty 2008; Schmittau 2004), using letters for as-yet-unspecified quantities is no barrier when handled sensitively and appropriately, even for young children. It has long been recognized that expressions often start as verbal phrases or clauses, migrating to succinct short forms, and then to single letters when learners are confident in expressing generality. This is the origin of 'rhetorical algebra' as an intermediary between arithmetic and algebra. Indeed Küchemann (1981) delineated six different modes. If as-yet-unspecifieds (numbers in the mind of someone not present) are symbolized by, with, and for young children, there might be no need for long transitions into algebraic manipulation.

For learners who have not yet been stimulated to think generally rather than always in particular, it may not even occur to them that the context in which a mathematical task appears can be altered. Getting learners to make alterations to the context or setting of a task for themselves contributes to their sense of power and control over a task-type, rather than feeling at the mercy of whatever might appear on an examination.

14.5.3 Reflection

Pólya (1962) suggested that reflecting back on what had happened during work on a problem is one of four phases of effective mathematical thinking. But as Jim Wilson (personal communication 1984) pointed out, "it's a phase more honored in the breach." Although it is always tempting to rush on to the next task either to escape lack of success, or to capitalize on success, making use of released energy of satisfaction or frustration in order to reflect and form images of future choices can be a much more worthwhile investment of time and energy.

One important role for teachers is to have what Bruner called 'consciousness for two,' namely to be aware of actions being enacted while learners are deeply in the flow of an action (Bruner 1986, pp. 75–76). Teachers can use this awareness (actually it is an awareness of an awareness of an awareness: Mason 1998) to prompt learners to withdraw from the action momentarily in order to contemplate that action, its effectiveness now, and its potential effectiveness in the future, or what seems to be blocking progress. This is part of the "discipline of noticing" (Mason 2002a).

There are numerous pedagogic strategies that can be used to bring back to mind effective actions so that they are more likely to come-to-action again in the future when needed. For example, asking yourself:

What was effective?
What is it about the task that made that particular approach effective?
What other similar tasks could be resolved using the same method?
What other choices of parameters would give the same answer?
What is the set of possible answers to tasks like this?
What would make a task similar to this one?
Are there any parameter values that will not work, or constraints on such parameters?

This last is effectively opening up the 'question space' (Sangwin 2006), which in some sense corresponds to the 'example space' (Watson and Mason 2002) being explored. In addition to a range of central and peripheral examples of some concept or procedure, the example space includes construction techniques for altering an example, and when appropriate, a sense of what boundary examples there might be. The question space is structured around the various constraints that make questions do-able.

14.5.4 Reasoning About Numbers

Reflecting on the relationships between arithmetic operations, and expressing these, produces the axioms of algebra, and justifies the claim appearing in most textbooks since the 15th century, that algebra is (can usefully be seen as) 'arithmetic with letters'. Unfortunately this rather off-hand approach has put off many learners, who have been mystified by what the letters are doing and what they are for. My long standing conjecture is that it is because the origins of algebraic expressions as expressions of generality is often circumvented that algebra is taught as if it is about the manipulation of meaningless symbols. Many adults report that algebra was a significant watershed for them because no one explained what it was about. This is a pity, because generalized arithmetic is one of the several roots of, and routes into, algebra (Mason et al. 1985). For learners experienced in expressing generality, and using more and more succinct formulations, it soon becomes obvious that in many cases there are different-looking expressions of general relationships that must always give the same answer. This suggests that it should be possible to get from one expression to other equivalent expressions without having to go back to the source situation. In this way the 'rules of algebra,' which are simply the expression of properties of arithmetic operations, can be experienced and internalized, then built upon by a succession of increasingly complex tasks.

Arithmetic seen as part of mathematics must refer to the study of properties of numbers, including relationships such as associativity, and distributivity of multiplication over addition. Merely performing addition, subtraction, multiplication, and division as algorithms is not mathematics. But making use of properties so that multiplying 49 by 51 is re-constructed as $(50 - 1)\ (50 + 1)$ or recognizing that $87 + 54 = 84 + 57$ without doing any calculation contributes to and is part of algebraic thinking. Thinking algebraically is appreciating the generality, recognizing an instance of a general property.

14.5.5 Reasoning Without Counting

Children struggling with arithmetic, for whatever reason, can often display superior powers of reasoning when the objects are not numbers. For example, reasoning about game strategy, such as Secret Places (Mason et al. 2012) can be done with and by young children, as O'Brien (2006) demonstrated. Reasoning about symmetry is also accessible to young children. For example, in Fig. 14.6, what colors must the blank circles have so that all three vertical lines are mirror lines for the pattern? Reasoning about magic squares (see next section) without actually having a particular one at hand is also possible, though it requires careful introduction so that children appreciate what the coloring means (Mason et al. 2012).

Introducing the mathematical implications of prepositions such as between, ahead, behind, to the left of, and to the right of can be developed through reasoning that, despite not making much if any reference to numbers, enriches and extends experience of mathematical reasoning, which seems to me to be algebraic in nature. For example, a sequence of statements of the form '*A* is ahead of *B*' and '*X* is behind *Y*' is revealed and, as each one appears, decisions have to be made as to whether there is sufficient information accumulated to place all of the 'people' in order, or whether the new information contradicts what is already known. Variants include statements of the form '*A* is between *B* and *C*,' and even quantitative statements such as '*A* is 3 ahead of B.' Forays into two dimensions are also possible. The point is to invoke children's natural power to consider possibilities without having them physically manifested in front of them, so that they develop control over their mental imagery, and gain familiarity with manipulating what is not actually present.

Fig. 14.6 Parallel mirror symmetry reasoning

14.6 Parallel Tasks for Adults

In order to experience something of what it may be like for learners to engage with new ideas and new procedures, it is useful as a teacher to put oneself through similar experiences on a regular basis. One way to do this is to take on learning something new, whether mathematical or not, as for example in Fig. 14.7. Catching how you use your attention can give you a taste of what it might be like for children doing similar tasks, including the amount of effort required. Of course using an actual pegboard is easier for children, but an adult can do it without.

Using unusual lines of symmetry can sensitize you to what it might be like for children working with a single mirror line (see Fig. 14.8). Variations, including parallel mirrors, are of course possible. Note also the task structure: deduction and encountering the mathematical theme of *freedom & constraint* (Mason and Johnston-Wilder 2004a, b).

What is required to be able to work from images rather than needing to use physical manifestations? Jerome Bruner (1966) uses this idea as the basis for his three modes of (re)presentation, and describes in detail how to encourage learners to move from physical material to diagrams and images, and thence to symbols. This was translated into a trio of frameworks for informing pedagogical design and choices in the moment (Mason 2002b; Mason and Johnston-Wilder 2004a, b; Open University 1982). Examples are presented in Figs. 14.9 and 14.10.

Expressing the generality (see Fig. 14.9) that gives the position of the one hundredth (ultimately one would like the *n*th) cell, and then articulating how to find

The three pictures show a magic square whose numbers have been obscured. Nevertheless it is the case that the sum of the numbers under the cross-hatching must be equal to the sum of the numbers under the diagonal shading. Justify this by showing how both of the shadings are made up of overlapping full lines in the magic square.

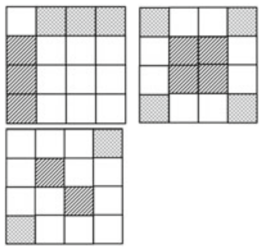
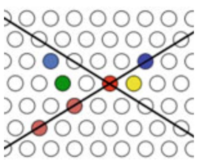


Fig. 14.7 Magic square reasoning



Taking both solid lines as mirrors, for which of the empty cells is the color already determined, and for how many further cells can you freely choose the color while maintaining the symmetry?

Fig. 14.8 Even more complex reasoning using symmetry

Two repeating patterns are shown, with at least two copies of the generating pattern visible.

What color is the 100th cell in each case? In what position is the 100th shaded square?

Fig. 14.9 Predicting repeated patterns

Use the first diagram to explain why the ‘up and down’ sum 1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25 and generalize.

Use the second and third diagrams to provide an alternative way of seeing (Mason 2008b).

Extend or modify the diagrams to display other sums, such as the sum of the numbers leaving a remainder of one when divided by 3.

Fig. 14.10 Pebble pattern counting

If you draw a single square, then develop a sequence of pictures by adding, at each stage, all the squares touching exactly one vertex of one previous square, you get a sequence of pictures. How many squares will appear in the *n*th picture?

Pictures 2, 3 and 4

Fig. 14.11 Complex generalities to express

the position of the 100th (ultimately *n*th) cell of a given color brings many adults up against a shift in thinking, typical of switching from ‘doing’ to undoing.’

Developing flexibility in interpreting diagrams can contribute to flexibility when creating diagrams, and in seeing algebraic expressions in different ways. Expressing generality can be tougher than first appears, as in the task illustrated in Fig. 14.11, taken from Mason (1988, p. 29). It is worthwhile extending the sequence, by first articulating for yourself how the sequence could be generated. A Pólya-inspired approach might be to specialize (Pólya 1945, 1962): work first on the pictures that appear at stages which are a power of two, before trying to work on the intermediate stages.

A more focused approach with higher likelihood of experiencing something relevant, is to ‘tie one’s hands behind one’s back’ so to speak, to force yourself to re-construct the steps and reasoning behind familiar procedures. See, for example, Fig. 14.12. Most people experience cognitive dissonance, to put it mildly. They find that they have to go back to first principles. Because the usual format is altered, internalized actions are inappropriate and have to be re-thought and re-formed. The result is that it is possible to experience uncertainty, to notice attention movements that might parallel the efforts of a child learning to do column addition. If this task doesn’t succeed, try writing the dictated numbers with the digits in columns.

Mirror Dictation

Get someone to read out a sequence of five or six numbers, some with four digits, some with three. Everyone else writes them down in a column, EXCEPT that the units digits are on the left. Everyone then adds their numbers so as to get the correct answer, even though it has the units digit on the left.

Now write down two four digit numbers from right to left, and subtract the smaller from the larger.

Now select a four-digit number and a three-digit number, and write them down from right to left so as to perform a long multiplication.

Now select a five-digit number and a two-digit number, and write them down from right to left so as to perform a long division.

Fig. 14.12 Mirror dictation task

Even three-column addition or subtraction can be a challenge on a spreadsheet. Yet this is what young children are expected to internalize. They are also expected to appreciate and comprehend what they are doing and why it works! Try to construct a spreadsheet to perform long division (one digit per cell). The restrictions imposed by the minimal tools available in a spreadsheet highlight the complexity of instructing a machine to perform a complex task. This may provide a taste of what it is like when learners are being instructed in carrying out arithmetic procedures. One difference is that teachers, experienced with arithmetic, have something to fall back on, to aid in the reconstruction, whereas learners may not, unless some work has been done first concerning the essence, the essential awarenesses that lead to the arithmetic algorithms.

14.7 Conclusions

It is well known that assumptions turn into expectations, and that expectations have a significant influence on behavior: your own expectations are framed by the expectations of those around you, both subtly and overtly expressed. The abiding question is how the assumptions and expectations of parents and institutions can be opened up to the enormous powers that children have displayed when they learn to walk and talk, and to call upon these powers when being with children.

Algebra, or rather the use of natural powers which, when expressed in words and symbols is recognizable as algebra, is called upon from birth if not before. To learn arithmetic, that is, to gain facility with numbers, is to think algebraically, even if not explicitly. There is a long tradition of invoking and evoking children's use of their own powers, and making opportunities to promote the development and refinement of those powers, but this tradition is constantly being submerged in the mistaken desire that children perform arithmetic. Performance follows and is part of appreciating and comprehending, but appreciating and comprehending requires more than simple training of behavior. Behavior is what can be trained, but its flexibility is only possible when awareness has simultaneously been educated. As Gattegno (1970, 1987) put it, "Only awareness is educable," and the Upanishads (Radhakrishnan 1953, p. 623) effectively extend that to include "Only behavior is trainable," which in turn depend on "Only emotion is harnessable," further augmented by "Only attention is directible" (Mason and Metz 2017).

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Chapter 15
Cycles of Generalizing Activities in the Classroom

Susanne Strachota, Eric Knuth and Maria Blanton

Abstract This study considers classroom situations in which students and the teacher co-contribute to promoting generalization. It specifically focuses on the ways in which students and a teacher in one classroom engage in generalizing arithmetic. Generalized arithmetic is an important route into early algebra (Kaput in Algebra in the Early Grades. Routledge, New York, 2008); its potential as a way to deepen students' understandings of concepts of school arithmetic makes it an important focus of early algebra research. In the analysis we identified generalizations around properties of arithmetic and the actions that promoted these types of generalizations, and then considered the relationship between these actions. Analysis revealed that generalizations became platforms for further generalization.

Keywords Early algebra · Teachers promoting generalization · Student learning Generalizing · Generalization · Generalized arithmetic

15.1 Introduction

For many students, algebra continues to be a gatekeeper to future academic and employment opportunities. In response, initiatives (e.g., *Common Core State Standards Initiative* 2010; National Council of Teachers of Mathematics [NCTM]

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2000, 2006) and conferences [e.g., the *Algebra Initiative Colloquium* (Lacampagne et al. 1995), the *Nature and Role of Algebra in the K–14 Curriculum Conference* (National Research Council 1998), and the *Mathematics Learning Committee* (National Research Council 2001)] have re-conceptualized algebra, suggesting that algebraic thinking should not only play a prominent role in elementary grade mathematics, but should also be treated as a longitudinal K–12 strand of mathematical thinking. This shift in perspective has led to increased attention being placed on early algebra—the introduction of algebraic thinking in the elementary grades.

Traditionally, for most students the study of algebra has been limited to one or two courses in secondary school, often with little to no explicit interweaving with other mathematics topics. In contrast, early algebra initiatives reframe algebra as a way of thinking about mathematics that extends throughout elementary grade levels and topics. In 2000, NCTM stated their position when they advocated that algebra be taught as a way of thinking and problem solving, beginning in elementary school and extending throughout mathematics, and not as an isolated course in high school. In response to the question "What is algebra as a strand of a school mathematics curriculum for all students?" NCTM explains in their position statement that:

> All students should have access to algebra in a pre-K-12 mathematics curriculum, including opportunities to generalize, model, and analyze situations that are purely mathematical and ones that arise in real-world phenomena. Algebraic ideas need to evolve across grades as a way of thinking and valuing structure with integrated sets of concepts, procedures, and applications. (NCTM no date)

15.1.1 Introduction to Generalization

As expressed by NCTM as well as mathematics education scholars, generalizing is at the core of algebra (Cooper and Warren 2011; Kieran 2007; Mason 1996). Yet, students struggle to generalize (English and Warren 1995; Lee and Wheeler 1987; Stacey 1989), often making weak generalizations, and rarely justifying their generalizations (Breiteig and Grevholm 2006; Knuth el al. 2002; Koedinger 1998; Usiskin 1987).

Supporting generalizing in the mathematics classroom requires a better understanding of the source of students' generalizing, that is, understanding the instructional mechanisms that encourage students' generalizing—the focus of the study reported here. In particular, the study reported here seeks to identify the processes that prompt and substantiate mathematical generalization, with the aim of understanding the actions in the classroom that initiate, refine, and sustain generalization.

One way to support algebraic thinking in elementary mathematics classrooms is through generalized arithmetic (Kaput 2008). *Generalized arithmetic* “involves looking at arithmetic expressions in a new way, in terms of their form rather than their value when computed” (Kaput et al. 2008, p. 12). Some early algebra research focuses on generalizing numbers and operations more broadly, exploring the potential of building on arithmetic by recognizing and articulating mathematical structure and relationships (e.g., Blanton and Kaput 2005; Carpenter et al. 2003; Davis 1985; Kaput 2008). Other research, such as that of Carraher et al. (2008), hones in on how specific elementary mathematics topics, such as arithmetic properties, can serve as a means or platform for fostering algebraic reasoning. In this study, we explore generalizing arithmetic more broadly, but use a specific topic, the Commutative Property of Addition, to illuminate the findings.

Children have a natural inclination to notice and discuss regularities and patterns in the number system—this is the foundation for constructing, testing, and justifying generalizations (Schifter et al. 2008). The field offers substantial research that describes algebraic ways of thinking in elementary grades and supports the feasibility of early algebra by demonstrating students’ capabilities (e.g., Mason 2008). However, the processes underlying the development of algebraic reasoning “are not well understood” (Ellis 2011, p. 309). A next step in early algebra research, then, is to understand ways in which students’ engagement in such activities is both prompted and supported.

Ellis (2011) defines the activity that supports generalization, or *generalizing-promoting activity*, as consisting of actions and interactions, working “in concert” with one another to promote generalization (p. 310), and she goes on to note that generalizing is “influenced by—and influences—the interrelated actions of students, teachers, problems, representations, and artifacts” (p. 337). Our research drew upon the work of Ellis (2007, 2011) to identify activity related to generalizing. Our goal is to demonstrate how the instruction provided during an early algebra intervention can influence the ways in which students generalize about arithmetic. We show that “algebrafied” instruction supports students’ in developing and justifying generalizations, and describe “algebrafied” instruction in a way that may inform practice (Blanton and Kaput 2003; Kaput and Blanton 2001). We share one approach to supporting algebraic reasoning in elementary mathematics and outline this approach in a framework. We view the framework as a potential resource for pre-service teacher preparation and in-service professional development on early algebra instruction, as well as for lesson planning for teachers who aim to incorporate algebraic reasoning in elementary mathematics.

15.1.2 Introduction to the Study

In this chapter, we present a two-part study that is a part of a larger research project concerned with the fundamental question of how to support students in elementary grades to be prepared for middle grades algebra and beyond (viz., Blanton et al.

2015). The study, from which the data described in this chapter were drawn, involves a quasi-experimental comparison of Grades 3–5 (ages 8–11) student performance from two classroom contexts: an intervention classroom (implementing an intervention based on an *Early Algebra Learning Progression* [EALP]) and a traditional elementary mathematics classroom. The EALP consists of a curricular framework, with learning goals, an instructional sequence of lessons to achieve those goals, assessments, and levels of sophistication that characterize students' understandings over time (Fonger et al. 2015, in press). Drawing upon Kaput's (2008) analysis of algebra in terms of content strands and thinking practices, we designed the EALP, which then guided the development and implementation of our longitudinal early algebra intervention, as well as the development of accompanying assessments.

The study we report on here had a two-part approach. The first part of the study focuses on analyzing intervention and control student responses to EALP assessment items to demonstrate differences in their understandings of the Commutative Property of Addition. Based on the results from this analysis, we argue that traditional arithmetic approaches to properties do not provide sufficient opportunities for students to engage in algebraic thinking practices, and that the critical difference between the early algebra approach and a traditional approach to elementary mathematics is *the nature of instruction*. Therefore, the second part of the study focuses on exploring the instruction that led to the differences observed in the first part of the study. Figure 15.1 shows the research questions for both parts of the study.

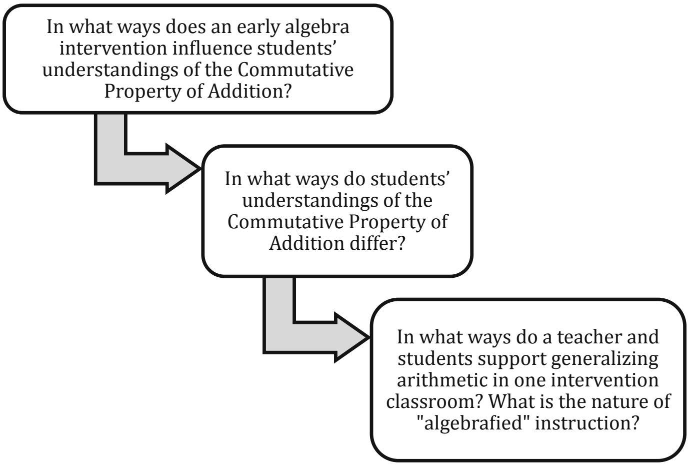

Fig. 15.1 Research questions

15.2 Conceptual Framework

In this section, we define three constructs that are central to the research reported here: algebraic activity, generalized arithmetic, and generalizing. Additionally, we elaborate on some of the research in the area of generalization in order to situate our study within the extant literature.

15.2.1 Algebraic Activity

Consistent with NCTM and other early algebra initiatives, we conceptualize algebraic thinking as a mental activity. We also conceptualize algebra as a tool for representing and reasoning about generality. These ideas align closely with Kaput's (2008) core aspects of algebra. Kaput describes algebra in two ways: (1) Algebra as symbolizing generalizations of patterns and constraints in a consistent, structured way and (2) algebra as a conventional symbol system that can syntactically organize and represent reasoning and actions on generalizations. In Kaput's first core aspect he conceptualizes algebra as a mental activity or a way that one might reason, whereas in the second core aspect he conceptualizes algebra as a mathematical system or something that one might reason about. Here we synthesize both aspects of algebra. That is, we investigate students' algebraic thinking and the ways that students interact with the symbol system of algebra to represent their thinking.

15.2.2 Generalized Arithmetic

Although historically, elementary mathematics has tended to focus mainly on arithmetic and computational fluency, we view generalized arithmetic as a means for developing students' algebraic thinking. As mentioned above, *Generalized arithmetic* "involves looking at arithmetic expressions in a new way, in terms of their form rather than their value when computed" (Kaput et al. 2008, p. 12). We view arithmetic properties as one of the more accessible ways in which teachers might foster algebraic thinking. Thus, the examples we provide in this study are situated in classroom contexts of generalizing about arithmetic properties.

Russell et al. (2011) underscore the central role generalizing plays in mathematical activity: "At its core, mathematics is about observing examples to find regularities, noticing structure and relationships, forming conjectures about the observations, and then proving and concluding general statements" (p. 2). Likewise, Kaput (1999) views algebraic thinking as the process of generalizing mathematical ideas from a set of particular instances, justifying those generalizations through discourse, and then expressing them in age-appropriate formal ways.

15.2.3 Generalizing

The process of generalizing, the activities that support generalizing, and the results of generalizing are distributed, social activities constructed from an interplay and integration of individual reasoning, collective reasoning, discourse, and tools (Ellis 2007, 2011; Jurow 2004; Latour 1987). From this perspective, generalizations are shared ideas that occur in verbal and written form, and generalizing is a social and generative process situated within a context. Generalizing, specifically, occurs when a person or group of people engage in one of three actions: (a) relating common characteristics across particular instances, (b) searching for examples beyond the original situation or idea, or (c) extending to broader or new situations from particular situations (Ellis 2011, p. 311). See Table 15.1 for a complete description of each generalizing action. These generalizing actions served to underpin our lesson design, as well as contribute to our subsequent analyses, which will be described later.

15.3 Study Part 1: Assessment Performance

In this section, we share the results from the comparison of the performance on the EALP assessments of students who received an early algebra intervention to students who did not receive an early algebra intervention. In particular, we focus here on ways in which the early algebra intervention influences students' understandings of the Commutative Property of Addition, comparing performance of the intervention and control groups.

Table 15.1 Generalizing actions (adapted from Ellis 2007)

Generalizing actions	
Relating	*Relating Situations*: forming an association between two or more situations *Relating Objects*: forming an association of similarity between two or more present objects
Searching	*Searching for the Same Relationship*: repeating an action to determine a stable relationship between two or more objects *Searching for the Same Procedure*: repeating a procedure to test whether it remains valid for all cases *Searching for the Same Pattern*: repeating an action to check whether a pattern remains stable across all cases *Searching for the Same Solution or Result*: repeating an action to determine if the outcome of the action is identical every time
Extending	*Expanding the Range of Applicability*: applying a phenomenon to cases beyond the case from which it originated *Removing Particulars*: removing contextual details to develop a global case *Operating*: operating on an object to generate new cases *Continuing*: repeating an existing pattern to generate new cases

15.3.1 Setting and Participants

The initial sample included a Grade 3 (ages 8–9) cohort of 103 intervention students and 67 comparison students from two different elementary schools in the same district in the North Eastern part of the United States. In Grade 4 (ages 9–10), 105 of those intervention students and 67 of those comparison students completed the assessment. Due to attrition, 90 intervention and 61 comparison students completed the Grade 5 (ages 10–11) assessment. The two elementary schools followed the same curriculum and pace, and the student populations in each school were similar in terms of socioeconomic status.

15.3.2 The Comparison Group

The students of the comparison classes were taught mathematics in a regular classroom setting using the Pearson Education (2014) *enVisionMATH* curriculum. A few practices that are highly emphasized throughout the *enVision* text are modeling from a story problem, writing and completing a number sentence, writing to explain, acting ideas out, drawing ideas, talking about ideas, talking with a partner, thinking about how to solve the problem, making sense of the problem, modeling thinking aloud, and interacting in small groups. There is also an emphasis on the use of vocabulary for each topic and using tools such as connecting cubes or a number chart.

The *enVision* curriculum differs from the early algebra curriculum because it is not explicitly focused on fostering students' engagement with algebraic thinking practices. While the *enVision* curriculum includes algebra content (there are exactly six "algebra sections in Grade 3), none of the "algebra" sections or other sections address content that is typically described as fundamental ways of thinking about early algebra. For example, a relational understanding of the equal sign [i.e., understanding that the equal sign indicates an equivalence relation rather than a direction to compute (see Carpenter et al. 2003)] is typically considered essential to engaging in algebraic thinking. And, there is no explicit focus on the use of the equal sign or student misconceptions about the equal sign throughout the topics. Relatedly, with the exception of a few sections that address properties, addition sentences often appear in the following form "7 + 7 = ?".

The Commutative Property of Addition is introduced and reintroduced several times throughout the Grade 3 text. Some questions focused on the Commutative Property of Addition suggest that students refer to a number line to explain how the property works. Other questions ask: if students know 3 + 4 = 7, how might the property help them solve 4 + 3? One section includes missing-number sentences that represent the Commutative Property of Addition (e.g., __ + 8 = 8 + 2). In all, the treatment of this property is exceedingly minimal in comparison to the related lessons designed for the early algebra intervention curriculum.

15.3.3 The Early Algebra Intervention

The Grades 3–5 early algebra intervention was taught by a teacher-researcher as part of students' regular classroom math instruction (the regular classroom teacher was present during some of these lessons). Each lesson introduced new concepts and revisited ideas that were addressed in previous lessons; so within and over the course of each grade the lessons built on each other. Each grade-level intervention consisted of about 20 lessons per grade constituting about 10% of the mathematics instruction for each year. Because students received the intervention during their regularly scheduled mathematics instruction, the total amount of time spent on mathematics remained unchanged.

The development and implementation of our early algebra intervention was based on Kaput's (2008) analysis of algebra in terms of content strands and thinking practices. Therefore our intervention was designed to support students' engagement with the four algebraic thinking practices of *generalizing*, *representing*, *justifying,* and *reasoning with* mathematical structure and relationships (Blanton et al. 2011) across three content areas: *generalized arithmetic*; *equivalence, expressions, equations, and inequalities*; and *functional thinking.* Here we focus primarily on students' engagement with the algebraic thinking practices in the area of generalized arithmetic. However, the content areas of algebra are interrelated; thus, *equivalence, expressions, equations, and inequalities* are also addressed.

To offer some insight on the nature of the intervention lessons and the ways in which they engage students in the aforementioned algebraic thinking practices, we share an excerpt from one lesson, which later appears in a transcript provided in the findings section. Table 15.2 outlines part of a Grade 3 intervention lesson on the Commutative Property of Addition. The lesson objectives were the following:

- Generalize and represent fundamental properties by observing structure in computational work, describe these properties in words and variables, and understand for what values they hold true.
- Begin to develop an understanding of the limitations of empirical arguments and the power of representation-based reasoning when justifying conjectures.
- Understand the use of different variables to represent fundamental properties.

We evaluated students' algebraic understandings using a one-hour written assessment at the beginning and end of Grade 3, at the end of Grade 4, and at the end of Grade 5. The study spanned 3 years, so that we could track students' learning over time. We compared the performance of students who received the

Table 15.2 Excerpt from Grade 3 intervention lesson

Grade 3 Lesson 4: Exploring fundamental properties (commutative property of addition)	
Student task	Teaching tips
A. Which of the following equations are true? Use numbers, pictures, or words to explain your reasoning 17 + 5 = 5 + 17 20 + 15 = 15 + 20 148 + 93 = 93 + 148	*Pose to students and have a few students share responses and justifications. Encourage students to respond to each other's contributions*
B. What numbers or values make the following number sentences true? 4 + 6 = ___ + 6 25 + 10 = ____ + 25 ___ + 237 = 237 + 395 38 + ___ = ____ + 38	*Pose to students and have a few students share responses and justifications. Encourage students to respond to each other's contributions. Use these problems to listen for and reinforce a relational view of the equal sign*
C. What do you notice about these problems? (What can you say about the order in which you add two numbers?) Describe your conjecture in words	*Encourage students to share conjectures and respond to each other's conjectures. Discuss whether students' conjectures are mathematically the same or different. Use students' descriptions of "any number" to discuss the use of letters to represent two arbitrary numbers in preparation for part D*
D. Represent your conjecture using variables	*Ask students to share their representations and make sure to include a discussion of using the same vs. different variables (e.g.,* $a + a = a + a$ *vs.* $a + b = b + a$*) so that students understand why it is important to represent this conjecture using different variables. Discuss why two different conjectures using different letters (e.g.,* $a + b = b + a$ *and* $m + n = n + m$*) are equivalent. Ask students to describe what their letters represent*
E. Can you express your conjecture a different way using the same variables?	*Using the commutative property, students should note that* $a + b$ *could also be represented as* $b + a$
F. For what numbers is your conjecture true? Is it true for all numbers? Use numbers, pictures, or words to explain your thinking	*Again, encourage students to share their justifications and respond to each other. Pay attention to how students use numbers: as specific examples or as generic examples to illustrate a general case? Press students to explain: "How do you know it is <u>always</u> true?" Place their conjecture about the commutative property in a prominent place so that students can refer back to them*

intervention to that of students who received only regular instruction. The students who received only regular instruction are referred to as the comparison group.

15.3.4 Data Collection

In the analysis, we focus on one assessment item (see Fig. 15.2), which evaluated students' ability to generalize, represent, and justify the Commutative Property of Addition, a topic taught to both the intervention and comparison students.

15.3.5 Data Analysis

Assessments were coded based on correctness, as well as on the strategies students used. Inter-rater reliability scores were computed and at least 80% agreement was achieved between the coders. When coders disagreed, they discussed codes until agreement was obtained. To determine significance, frequency Tables (2×2) of the intervention versus comparison groups by item correctness were created for all parts of the task (a, b, and c) and were statistically examined using Chi-square tests for all assessments.

Part a was coded as correct if the student named the Commutative Property of Addition, stated the property in words, or explained that the same numbers were used in the two additions. Part b was coded as correct if the student symbolically represented the property by writing an equation using letter variables. If the student represented the property in other ways, such as by writing an equation using variables *and* numbers or writing an expression instead of an equation, the response was coded as incorrect.

Part c was coded as correct if students justified their responses by stating that the same numbers are used or explained that the numbers are switched/flipped/swapped or by naming the property. Figures 15.3 and 15.4 illustrate the coding scheme by means of examples from two students' work. On Part c in Fig. 15.4, technically the

> **Marcy's teacher asks her to solve "23 + 15." She adds the two numbers and gets 38. The teacher then asks her to solve "15 + 23." Marcy already knows the answer without adding.**
>
> **a) How do you think Marcy knew the answer without adding again?**
> **b) Write an equation using variables (letters) to represent the idea that you can add two numbers in any order and get the same result.**
> **c) Will Marcy's idea always work? Explain why.**

Fig. 15.2 Assessment item from the Grade 3 pretest and the Grades 3, 4, and 5 posttests

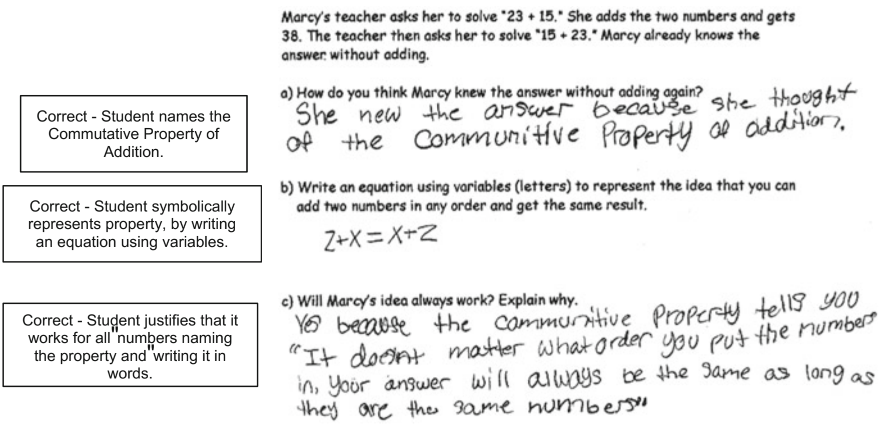

Fig. 15.3 An intervention student's response from Grade 4 posttest

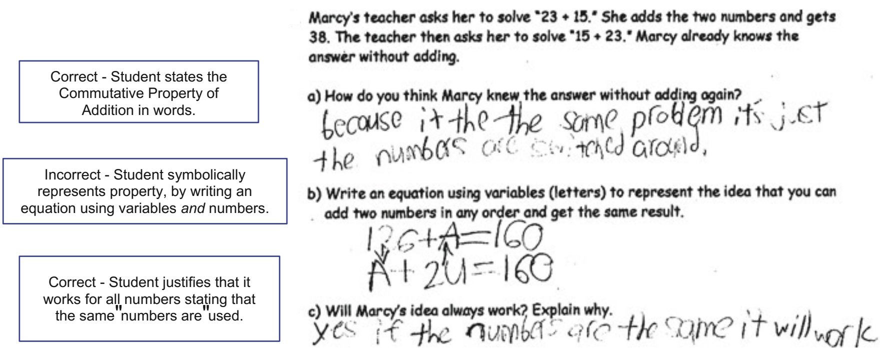

Fig. 15.4 Another intervention student's response from Grade 4 posttest

operations have to be the same and either addition or multiplication, but based on our conversations with students we infer that they assume the operation does not change. While we do not know that students are thinking that the operation remains the same, we believe that giving them the benefit of the doubt is reasonable.

15.3.6 Results: Part 1

At the Grade 3 pretest, results of the written assessments showed no significant differences between the intervention and comparison students. At the Grades 3, 4, and 5 posttests, the differences between the intervention and comparison groups were not significant in Part a; however when students were asked to generalize and justify their reasoning in Parts b and c, the differences in the performance of the groups were significant for each posttest. That is, the students who participated in the early algebra intervention were more successful in representing a generalization symbolically and providing a justification for Marcy's reasoning. This suggests that the intervention helped students develop and represent generalizations about arithmetic properties.

Table 15.3 summarizes the results. At the Grade 3 pretest, 36% of comparison students and 47% of intervention students noticed the structure of the Commutative Property of Addition in Part a, 1% of comparison students and 1% of intervention students were able to write an equation using variables to represent the Commutative Property of Addition in Part b, and 24% of comparison students and 22% of intervention students identified structure in equations when providing a justification for Marcy's thinking about the Commutative Property of Addition in Part c. The differences between the results of the comparison and intervention groups were not statistically significant for any part of the task at the Grade 3 pretest.

At the Grade 3 posttest, 70% of comparison students and 81% of intervention students were able to notice the structure of the Commutative Property of Addition in Part a, 2% of comparison students and 51% of intervention students were able to write an equation using variables to represent the Commutative Property of Addition in Part b, and 30% of comparison students and 65% of intervention students provided a structure-based argument to justify Marcy's thinking about the Commutative Property of Addition in Part c. The differences between the results of the comparison and intervention groups on Parts b and c were statistically significant with a chi-square value <0.05.

At the Grade 4 posttest, 83% of comparison students and 85% of intervention students were able to notice the structure of the Commutative Property of Addition in Part a, 32% of comparison students and 50% of intervention students were able to write an equation using variables to represent the Commutative Property of Addition in Part b, and 54% of comparison students and 78% of intervention students provided a structure-based argument to justify Marcy's thinking about the Commutative Property of Addition in Part c. The differences between the results of the comparison and intervention groups on Parts b and c were statistically significant with a chi-square value <0.05.

Table 15.3 Percentage correct by item part by testing session

	Intervention				Comparison			
	3 pre	3 post	4	5	3 pre	3 post	4	5
Part a	0.466	0.814	0.853	0.889	0.358	0.697	0.825	0.918
Part b	0.01	0.51^{x}	0.495^{y}	0.889^{z}	0.015	0.015	0.317	0.327
Part c	0.223	0.647^{x}	0.878^{z}	0.889^{z}	0.239	0.303	0.54	0.672

Note Superscripts x, y, and z denote a chi-square p-value < 0.05 for Grade 3 post, Grade 4, and Grade 5, assessments respectively

At the Grade 5 posttest, 92% of comparison students and 89% of intervention students were able to notice the structure of the Commutative Property of Addition in Part a, 33% of comparison students and 89% of intervention students were able to write an equation using variables to represent the Commutative Property of Addition in Part b, and 67% of comparison students and 88% of intervention students provided a structure-based argument to justify Marcy's thinking about the Commutative Property of Addition in Part c. The differences between the results of the comparison and intervention groups on Parts b and c were statistically significant with a chi-square value <0.05.

Although this study does not explore potential learning progressions in algebra, it is interesting to note the consistency of percentage correct on Part b between the Grade 3 posttest and the Grade 4 posttest contrasted with the percentage correct in Grade 5 for the intervention group. This trend may provide insight as to how these students progress when generalizing and symbolizing. Of the intervention group, 51% of students at the Grade 3 posttest and 50% of students at the Grade 4 posttest were able to write a correct response for Part b. This means they were able to write an equation using variables to represent the Commutative Property of Addition. Alternatively, if students wrote an equation using variables *and* numbers (see Fig. 15.4 for an example) or wrote an expression (e.g., $a + b$ and $b + a$), Part b was coded as incorrect. The percent of correct responses provided by the intervention students on Part b remained the same between Grades 3 and 4 but increased significantly in Grade 5. The increase in Grade 5 may indicate that these students are more comfortable using variables and numbers or expressions before using equations with variables to represent arithmetic properties.

15.3.7 Discussion: Part 1

An analysis of the curriculum used in the students' regular math instruction—when the intervention group was not experiencing the research-based lessons—confirmed that both the intervention and comparison groups studied the Commutative Property of Addition. Furthermore, both groups were successful in recognizing and referring to the Commutative Property of Addition. The comparison group, however, was not as successful as the intervention group when prompted to represent the property and justify the generalization.

These findings suggest that arithmetic properties can serve as useful contexts to engage students in developing and symbolizing generalizations, but the inclusion of properties in early grade mathematics instruction will not initiate students to engage in these practices. That is, simply teaching arithmetic properties does not foster generalizing. Rather, the combination of topics that can serve as a springboard for algebraic reasoning, such as arithmetic properties, and instruction that supports students' in developing and symbolizing generalizations, may be key to fostering algebraic reasoning in elementary grades. That is, the intervention students' success in representing the property and justifying why the property holds true for all numbers is a result of instruction that promoted generalizing arithmetic. This raises the question of how instruction served to promote generalizing arithmetic. This question lies at the core of the second part of this study, which seeks to identify the processes that prompted and substantiated students' generalizing in this particular classroom. The results of the first part motivate the second part of the study, which investigates the nature of instruction in the context of the early algebra intervention.

15.4 Study Part 2: Classroom Practices

Part 2 focuses on classroom situations in which students and their teacher co-contribute to promoting generalizing and representing and justifying generalizations. Based on the results of Part 1, we argue that the way in which properties were taught in the intervention effectively supported students in reasoning algebraically about the Commutative Property of Addition. In response to the findings above, the aim of this study is to analyze "algebrafied" instruction (Blanton and Kaput 2005; Kaput and Blanton 2001), specifically by attending to the ways in which a teacher's and students' actions support generalizing in one intervention classroom. What is the nature of "algebrafied" instruction? Or more specifically, in what ways do a teacher and students support generalizing arithmetic in one intervention classroom? This is the research question for Part 2 of this study.

15.4.1 Setting and Participants

The sample consists of the participants in the Grade 3 intervention classroom reported on in Part 1 of this study. Data for the second phase of analysis were the videotaped intervention lessons.

15.4.2 Data Analysis

Understanding the processes and interactions that foster generalization is critical to supporting students' generalizing (Ellis 2011). We integrated into our analysis Ellis's seven types of generalizing-promoting actions (see Table 15.4) as a framework to draw attention to the actions that contributed to generalization. The categories of generalizing-promoting actions are not mutually exclusive; furthermore, we view generalizing-promoting actions and generalizing actions (see Table 15.1) not as mutually exclusive, but rather as interactive and interchangeable actions. Then, in a later phase of our analysis, consistent with Ellis's (2011) conceptualization of generalizing as a situated and dynamic process, we consider the interaction between generalizing-promoting actions and the generalizations that were the result of generalizing-promoting actions and generalizing actions.

Using Ellis's (2007) taxonomy of generalizing actions (Table 15.1) and the framework of generalizing-promoting activity (Table 15.4), we analyzed classroom video and transcripts to make sense of how students engage in the act of generalizing. After watching the video recoded lessons, we read the associated transcript and used Ellis's categorizations to identify and label instances of generalization.

Generalizing actions (*GA*) represent the mental actions of relating ideas, searching for similar ideas or extending ideas towards generality; they are inferred from a person's activity or talk. For instance, relating might occur if students relate two or more models that they have constructed with unifix cubes. Searching might occur if students repeatedly compute arithmetic expressions, reversing the order of the numbers each time, in search of a pattern or relationship. Extending might occur if students identify a pattern when repeatedly computing arithmetic expressions and extend the idea to all cases by replacing the particular numbers with variables.

Generalizations (G), that is, final statements of generalization, which Ellis (2007) refers to as *reflection generalizations*, include for instance, identifying a general pattern when looking at multiple expressions (e.g., 3 + 5, 5 + 3, 2 + 6, 6 + 2…). These can be characterized according to identification or statement, definition, and influence. Identification or statement involves determining and articulating "a general pattern, property, rule, or strategy" (Ellis 2007, p. 245).

Table 15.4 Generalizing-promoting actions (adapted from Ellis 2011)

Generalizing-promoting actions
Publicly generalizing This may involve—(a) creating an association between two or more problems, objects, situations, or representations; (b) identifying an element of similarity across cases; or (c) expanding a pattern, idea, or relationship to reach beyond the case at hand
Encouraging generalizing This may involve—(a) prompting the formation of an association between two or more entities; (b) prompting the search for a pattern or relationship; (c) prompting the expansion beyond the case at hand; or (d) prompting the creation of a verbal or algebraic description of a pattern or rule
Encouraging sharing of a generalization or idea This may occur as formal or informal requests for sharing broadly or restating ideas
Publicly sharing a generalization or idea This may occur as revoicing another member's generalization or publicly validating or rejecting another member's generalization
Encouraging justification or clarification This may involve asking members to clarify a generalization, describe its origins, or explain why it makes sense
Building on an idea or a generalization This may take include refining an idea or using it to create a new idea, rule, or representation
Focusing attention on mathematical relationships This may entail directing others' attention to specific mathematical features of a problem or activity

Definition involves recognizing and describing the common characteristic of a class, pattern, or other situation. Influence involves applying a previously constructed reflection generalization to a new situation.

Generalizing-promoting actions (*GP*) are activity or talk that seems to support the process of constructing or refining a generalization (Ellis 2011), for instance, prompting students to identify the similar structure of two or more equations. These include publicly generalizing, encouraging generalizing, encouraging sharing of a generalization or idea, encouraging justification or clarification, building on an idea or a generalization, and focusing attention on mathematical relationships.

The Grade 3 transcript excerpt below (Fig. 15.5) provides an example of the initial analysis. In the next iteration, we will re-analyze the instances of promoting generalization and consider the responses that were made to generalizing actions and their resulting generalizations so as to illuminate the relationship between these actions. "*GA*" indicates a generalizing action. "*G*" indicates a generalization (we bolded the specific part that was coded as a generalization). "*GP*" indicates a generalizing-promoting action (we italicized the specific part that was coded as a generalizing-promoting action).

In lines 3 and 4 of Fig. 15.5, the teacher is using the task to encourage generalizing. She asks students "What goes in the blank?" to prompt the formation of a relationship between the expressions. Tammie responds, and the teacher encourages justification by asking "why?". Tammie justifies her claim, and in lines 8 and 9 the teacher encourages another student to revoice Tammie's idea. This action is an example of the generalizing-promoting action of publicly sharing a generalization or idea and encouraging justification or clarification. Sam responds to the teacher's request by introducing a new situation in line 10. By introducing a new situation, Sam is engaging in the generalizing-promoting action of encouraging relating and focusing attention on mathematical relationships. The teacher and Sam clarify his idea in lines 13 through 17. Then the teacher states a generalization, "We have to have the same amount on both sides" in line 19. The teacher continues by describing the equation and makes another generalization by stating that the sums are equivalent, "Because it's the same numbers…" in line 22. Following the generalization, she encourages relating and searching by introducing a new idea ($4 + 6 = \_\_ + 4$) in lines 25 and 26. She requests justification—a generalizing-promoting action—in line 29, and Morgan responds with a generalization.

15.4.3 Results: Study 2

By analyzing the data according to the taxonomy (Ellis 2007) and framework (Ellis 2011), we identified the actions of promoting-generalizing, generalizing, and a new construct: *responding to generalizing*. As a result, we conducted a second round of coding, in which we considered the relationships between promoting-generalizing actions generalizing actions, and generalizations. In the first subsection, we will

Teacher: There was some really great 3rd grade thinking that was going on. And I want to talk about it whole-group, because I think it's really worth sharing. Let's look at this first one, which was probably the easiest for everybody. *"4 + 6 = ___ + 6." What goes in the blank?* [*GP/GA*]

Tammie: *4.*

Teacher: *Why?* [*GP*]

Tammie: *Because that's the only way it could be even.* [*GP*]

Teacher: Okay. that's the only way that it could be even. *Somebody want to explain that a little differently?* [*GP*]

Sam: *Um, it's the only way it could be even because it says 4 + 6, so the only other way to do it would be to change the number that it's af-, like, change 6 to a 7, and change 4 to a 3.* [*GP/GA*]

Teacher: Okay, so, if there was a 7 over here, we could have put a different number in the blank. But because they already put a 6 here, this was the only way to do it. Do you want to add to that, bud? [GP]

Sam: *Um, 4 + 6 and then there's a 6 on the other side, because you couldn't put a 3 there because then it would be a whole different number.* [*GP*]

Teacher: Good. That equal sign tells us that the amounts have to be the same, right? ***We have to have the same amount on both sides*** [*G/GP*]. So when we're dealing with the same number, like 6, our only option is to put this other number here. 4 + 6 = 4 + 6. Do you have to add 4 + 6 to figure out if what you have on the left hand side? No! ***Because it's the same numbers, right?*** [*G/GP*] There's not a lot of math here. We don't have to know our facts for this problem. We probably do in 3rd grade, but we don't have to, because we know 4 + 6 is the same as 4 + 6. Could I do this? *Would this be okay? 4 + 6 = __ + 4? Could I put a 6 here?* [*GP*]

Students: Yes!

Teacher: Oh, how come? How come that works? [GP]

Morgan: ***Because it could be, like, a turnaround fact.*** [*G/GP*]

Teacher: Very good, Morgan! This is a turnaround fact, commutative property. [GP] And we'll talk about this at some point, you'll talk about this in your 3rd grade classroom, 4 + 6 and 6 + 4, same exact thing, just turned around like Morgan said. [GP] Turnaround fact. Good job.

Fig. 15.5 Grade 3 transcript excerpt from first round of analysis explain the motivation behind a second round of coding. In the following subsections we will describe the actions that we identified in the second analysis, what we observed about these actions, and demonstrate how they relate to the transcript excerpts that support our claims.

15.4.3.1 Motivation for Round 2 Analysis

In the first round of analysis, some actions that we identified fit into multiple categories of Ellis's generalizing-promoting actions (2011) and generalizing actions (2007). Furthermore, we noticed that the actions within an episode and across episodes occurred in interaction with each other. And lastly, we noticed that certain types of actions emerged from the same kind of action or prompted the same kind of action. In particular, we observed how the teacher responded to generalizations.

As before, the transcript below (see Fig. 15.6) was coded using Ellis's taxonomy (2007) and framework (2011). In this episode, the teacher began the lesson by asking students to individually build any length "train" using two different colors of unifix cubes. She then used as an example one student's, Mikel's, train, which had a length of 7 and was composed of 4 red cubes and 3 yellow cubes, and asked if anyone had a train that was similar (a term that may have been initially quite vague for some students) to Mikel's train.

In line 1 of Fig. 15.6, the teacher encourages relating. Next, in lines 4 and 5 Tommy relates the two trains by creating an association ("switcheroo") between them and thereby makes a generalization. Then in lines 7 and 8, the teacher focuses students' attention on the mathematical relationship by asking if the trains have the same number of blocks. After students respond, she encourages them to extend beyond the case at hand by representing the idea in a new way, as a written expression. This action prompts Tommy to relate the new idea (the expressions) to the original idea (the trains). Thus, the teacher's action in lines 14 and 16 functions both as encouraging extending and encouraging relating. To close, Tommy generalizes when he identifies a common characteristic of the trains and the expressions—"they're switcheroo."

After reflecting on our analysis, we felt as though some actions were not accurately captured by our coding scheme. For example, we mentioned that lines 14 and 16 function as a generalizing-promoting action in two different ways. We also observed that many of these actions in this episode, but also across episodes, were related, even inseparable, but we were not accounting for these relationships. For example, we noticed that lines 4–5, 6–8, 10–12 and 19–20 were closely connected. In particular, lines 4–5 contain a generalization and lines 6–8 are a response to that generalization.

We actually began drawing arrows between these actions to indicate the relationship. Lastly, we felt as though some moments that proved to be important moments later, for example lines 10–12, were not characterized as significant according to our analysis. As a result, when we revisited the data we had three goals. First, we looked at related codes to see if we could conflate previous

1	Teacher:	Ahh. Maddie has a train similar to this. *Maddie, what does your train say?* [*GP*]
2	Maddie:	4 yellow and 3 red.
3	Teacher:	3 reds. So, *do these trains match each other? What do you think, buddy?* [*GP*]
4 5	Tommy:	***No, because they, because they, because they switcheroo, because the yellow's 4 and the red is 3 and then the 4 is red and then the yellow is 3.*** [*G/GP*]
6 7 8	Teacher:	Good, you noticed something that's happening, but can I ask you something? You said, no these don't match. *Do they have the same amount of blocks in each of the trains?* [*GP*]
9	Students:	Yes.
10 11 12	Teacher:	Yes. So they're good partners. But let's write what you just said. You said they're switcheroo. So, I think it was Maddie, but I just switched them in my head. Maddie, is that yours, 4 yellow?
13	Maddie:	Yeah.
14	Teacher:	*How would we write an, how would we write an expression for this train?* [*GP*]
15	Maddie:	4 + 3
16	Teacher:	Good. And let's stop there. And Mikel, *what could we say for your train?* [*GP*]
17	Mikel:	3 + 4
18	Teacher:	3 + 4. And what did you say about these expressions, buddy? [GP]
19 20	Tommy:	***They, they're, they're, they just switcheroo because 4 is on the, because red is 4 and the 3 is yellow and the 3 is 4…***[*G/GP*]

Fig. 15.6 Grade 3 transcript excerpt motivating the need for interrelatedness analysis

categories of activity into a new category. Second, because we observed that actions within and across episodes were sometimes related, we reconceptualized episodes as situations that could become objects for relating themselves. This process enabled us to code actions that seemed insignificant in the moment, but later played an important role in developing or refining a generalization. Lastly, because we noticed that certain actions were connected to other actions we focused on the interrelatedness of activity.

Although we do not elaborate on this finding, we feel as though it is worth noting that many generalizing-promoting and generalizing actions observed in this classroom began with or involved relating particulars—either situations or objects (such as equations). Interestingly, this finding is consistent with Ellis (2007). In her 2007 paper on generalizing actions and generalizations, Ellis categorizes students' final statements of generalization and refers to these statements as reflection generalizations. In the teaching episodes we analyzed, the goal of instruction was to generalize the Commutative Property of Addition. This generalization corresponds to Ellis's reflection generalization of identifying sameness or a general principle. Moreover, Ellis argues that the reflection generalization of making a statement of sameness often emerges from the generalizing action of relating (Ellis 2007, p. 245), which was prevalent in this classroom.

15.4.3.2 Round 2 Analysis

After our first round of analysis we revisited the data in a second round of analysis that focused on the interrelatedness of activity. Through the iterative analysis, we identified a continuous and dynamic cycle of generalizing-promoting and generalizing actions. By dynamic we mean that there are many ways in which the students and teacher may navigate through the sequence. Each cycle builds on the previous cycle because generalizations build on previous generalizations. Furthermore, as the sequence is continuous it increases in sophistication and has a generative quality. Through the iterations of analysis, we found that (1) generalization becomes a platform for further generalization, (2) generalizing-promoting activity can become the object of further generalizing-promoting activity, and (3) generalizing is generative in nature.

(1) *Generalization becomes a platform for further generalization.*

For instance, in the following classroom episode Kyle's generalization is the basis for another generalization. Note that the interaction described below is occurring in a whole class discussion and the actions identified in brackets refer to the new framework of interacting generalizing-promoting actions (see Fig. 15.7), which is introduced in the following section.

Teacher: Let's think about this. If you have 3 + 2 like we had, 3 + 2, and 2 + 3, so you have 3 + …you're adding 3 + 2 on this side and you have the same numbers on this side, what happens? [*action a*]

Kyle: Um, it's a true equation. [*action e*]

Teacher: It's a true equation. That's good. Why is it a true equation? [*action a*]

Kyle: Because there's a, there's the same amount on the equal sign, and if there's the same amount on the equal sign it's true. [*action g*]

Teacher: All right. I think I know what you're saying. Tell me if this is it. "Then it is a true equation because you have the same amount on both sides." [*action c1*] Right?

Kyle: Mmhmm.

Kyle generalizes when he says "if there's the same amount on the equal sign it's true." Later when students represent the Commutative Property of Addition using variables—another generalization—Kyle's generalization becomes an object of generalization itself because students are able to relate Kyle's generalization to

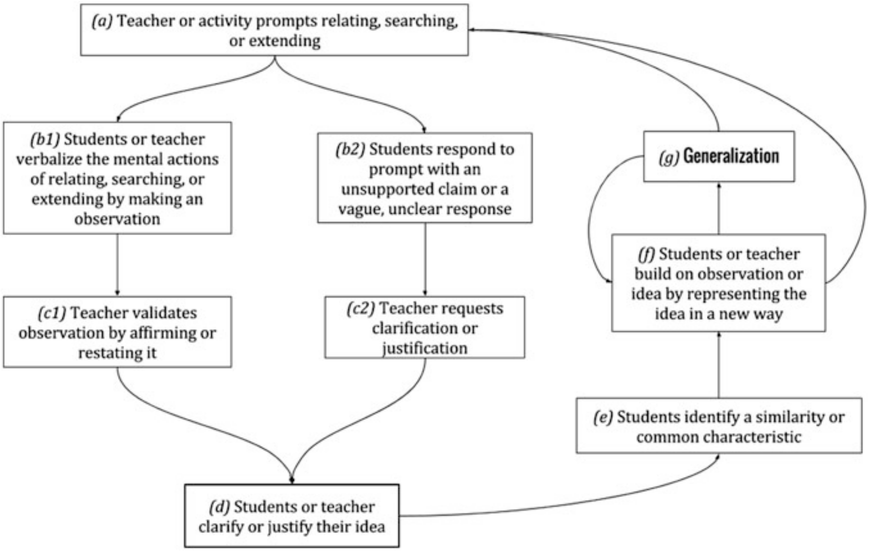

Fig. 15.7 The framework of interacting generalizing-promoting actions

other generalizations or equations or instances that represent the Commutative Property of Addition to construct a formal mathematical representation using variables (e.g., $a + b = b + a$). This is evident in the previous transcript (Fig. 15.6), which actually occurred after Kyle's generalization. For example, Tommy creates an association, which he refers to as "switcheroo," by building on Kyle's idea. Specifically, he connects the particular situations for which Kyle generalized about with the particular situations, which were constructed using the unifix cubes.

The episode with Kyle is also an example of relating particulars. As noted, according to Ellis (2011), one way to promote generalization is to encourage relating particulars. In the classroom episode above, students relate particular instances to build a generalization (Fig. 15.7, action *a*); that generalization becomes an object for relating and thus constructing future generalizations (action *g* leads to action *a*).

Through the iterative analysis, we identified a cycle of action in this particular classroom; in this cycle are actions that promote generalizing, then generalizing occurs, and the generalizing itself becomes an action that promotes further generalizing. The cycle is a framework of interacting that illustrates actions and types of talk observed in this classroom. While some actions emerge from a certain action or prompt the same action, the framework is not absolutely linear. In no cases did we observe all actions (*a*–*e* in Fig. 15.7) in the order of the framework.

(2) *Generalizing-promoting activity can become the object of further generalizing-promoting activity.*

Generalization often prompts further generalization. In some instances, it was evident that students were engaging in generalizing-promoting actions, but the teaching episode did not result in a generalization; instead the episode produced an idea used for relating, which prompted future generalization. The following transcript was used previously to demonstrate the first round of coding. We use the same transcript again to demonstrate our point and to provide an example of how the second round of analysis enhanced our understanding of students' generalizing.

As previously explained, the teacher began this lesson by asking students to individually build any length "train" using two different colors of unifix cubes. She then used one student's, Mikel's, train, which had a length of 7 and was composed of 4 red cubes and 3 yellow cubes, as an example. She prompted relating by asking if anyone had a train that was similar to Mikel's train (action *a*).

Teacher: Ahh. Maddie has a train similar to this. Maddie, what does your train say? [*action a*]

Maddie: 4 yellow and 3 red. [*action b1*]

Teacher: 3 reds [*action c1*]. So, do these trains match each other? What do you think, buddy? [*action d*]

Tommy: No, because they, because they, because they switcheroo, because the yellow's 4 and the red is 3 and then the 4 is red and then the yellow is 3. [*action e*]

Teacher: Good, you noticed something that's happening [*action c1*], but can I ask you something? You said, no these don't match. Do they have the same amount of blocks in each of the trains? [*action d*]

Students: Yes.

Teacher: Yes. So they're good partners. But let's write what you just said. You said they're switcheroo. So, I think it was Maddie, but I just switched them in my head. Maddie, is that yours, 4 yellow? [*action a*]

Maddie: Yeah.

Teacher: How would we write an, how would we write an expression for this train? [*action f*]

Maddie: 4 + 3 [*action b1*]

Teacher: Good. And let's stop there. And Mikel, what could we say for your train? [*action a*]

Mikel: 3 + 4 [*action b1*]

Teacher: 3 + 4. And what did you say about these expressions, buddy? [*action a*]

Tommy: They, they're, they're, they just switcheroo because 4 is on the, because red is 4 and the 3 is yellow and the 3 is 4…[*action e*]

Tommy relates the two trains and makes an observation when he describes the trains as "switcheroo." In response, the teacher validates Tommy's observation and helps to clarify his claim (actions *c1* and *d*). She then encourages the class to build on the observations by representing them in a new way, as a written expression (action *f*), and prompts Tommy to relate the new idea (the expressions) to the original idea (the trains) (action *a*). Tommy identifies a common characteristic of the trains and the expressions—"they're switcheroo" (action *e*).

As noted, this transcript was previously presented to demonstrate our data analysis process. By applying the new framework, the framework of interacting generalizing-promoting actions (Fig. 15.7), to this transcript we are able to highlight potential implications for instruction. In this instance, the new framework suggests that a productive next step would be for the teacher or an activity to prompt relating, searching, or extending by building on Tommy's generalization (action *a*). For example, the teacher might recommend that students search for other instances of "switcheroo" by encouraging them to test particular instances with other operations. Then she might suggest that students relate these particular instances, aiming to extend the application of the concept of "switcheroo" to multiplication. Not surprisingly, since the framework emerged from these data, we observe the teacher taking this approach.

(3) *Generalizing is generative in nature.*

The teacher uses the conversation about "switcheroo" to prompt extending (action *a*) when she asks students to describe what they noticed by making a conjecture, or a "math statement" (action *f*). A student makes a generalization by stating the Commutative Property of Addition in words when she says, "If you are adding two numbers and you have the same numbers on the other side of the equal sign, then it is a true equation because you have the same amount on both sides." The teacher writes the statement on the board and pushes students to clarify the meaning of the statement (action *c1*). The teacher continues to build on the generalization by asking students to think about representing the generalization in a different way (action *g* leads to action *f*). Without being prompted to use variable notation, one student, Una, surprises the teachers (the intervention teacher and the classroom teacher who happened to be present) by suggesting that letters replace the specific numbers to represent that the conjecture works for all numbers.

Teacher: How could we represent that this will work for all numbers all of the time [*action a and f because the action is stemming from g*]? So for any number that this conjecture that we wrote is true, and our conjecture was: "If you have two numbers on the left-hand side of the equal sign and you're adding them, and the same two numbers on the other side but in a different order, then it'll be a balanced equation because you have the same amount on both sides," right? Una, do you think you could already do it today?

Una: Yes. $e + a = a + e$. [teacher writes $e + a = a + e$] [*action g*]

Teacher: Okay. So. Ms. O [the classroom teacher] just got blown away over there.

Ms. O: I almost fell off this little seat.

Teacher: She almost fell off her chair. Almost down. Okay, let's look at this. Let's look at what Una wrote. "$e + a = a + e$." Una, talk to me about this. [*action c2*]

Una: Um, the *e* stands for one number and the *a* stands for the other number and the *a* and the *e* on the other side stand for the same number except I changed it to $a + e$ instead of leaving it $e + a$. [*action d*]

Teacher: Okay. So talk to me about this. Why did you choose to use two different variables? I agree with you. Why'd you do that? [*action c2*]

Una: Because if I did the same, because if I did um, one, both of, one letter on both sides it would be the same number and it wouldn't be a turnaround fact, but if I did two letters it would be different, it would stand for different numbers, so it would be a turnaround fact. [*action d*]

Teacher: Good. So what you're saying is you chose an *e* and an *a* because you wanted to represent two different values, and you wrote them in a different order, $e + a$ and $a + e$, because you wanted to show that you turned them around. Una, slap me five. Nice job. Do you guys agree with what Una wrote?

Students: Yes.

Teacher: Yep. Could Una use different letters if she wanted to? [*action f*]

Students: Yes.

Teacher: Yep. She could do different combinations of letters.

The teacher responds to Una's suggestion by requesting clarification (action *c2*). After Una clarifies her reasoning (action *d*), the teacher prompts students to build on Una's idea by discussing the possibility of representing Una's idea a different way in the same form (action *f*). Each generalizing-promoting and generalizing action in these episodes contributed to Una's final statement in which she represented an arithmetic property with variable notation. Furthermore, Una's generalization can serve as a platform for future generalization.

15.4.3.3 Discussion: Part 2

The series of teaching episodes presented here is but one example of the pattern identified in this classroom. In sum, we have presented a framework that represents how generalizing-promoting actions and generalizations occurred in relation to each other in this classroom. We observed that generalizing-promoting actions and generalizations interact in a generative cycle, and the cycle is generative because it is continuous and each action builds on a previous action. Thus, actions in the cycle increase in sophistication and generalizing becomes a platform for further generalizing.

In the first round of analysis in Part 2 we learned that some actions could be characterized as contributing to generalizing in multiple ways and that actions interacted within an episode and across episodes. In the second round of analysis, we observed a cycle of actions that promote generalizing, and learned that generalizing itself becomes an action that promotes further generalizing. The first round of analysis supported Ellis's (2007) taxonomy of generalizing actions and her framework for generalizing-promoting activity (2011). Then by attending to the interrelatedness of actions, we built on Ellis's work by adding the action of responding to generalizations.

15.5 Conclusion

The results of this study motivate future research questions. For instance, prompting relating, searching, or extending was identified frequently in this classroom. Sometimes prompting occurred in response to a generalization; this was a novel finding. We consider prompting a critical action because it initiates a cycle of generalizing actions. Thus, research on prompting relating, searching, and extending has a strong potential to influence the practice of teaching. Questions that researchers might explore are: In what ways do students prompt relating, searching, or extending? Are certain types of prompting more mathematically productive? How are certain types of prompting related to the generalization they promote? What is the nature of tasks that prompt relating, searching, or extending?

We argue that the results of this study have a high-impact potential to inform instruction. We suggest that the framework of interacting generalizing-promoting actions be viewed as a starting point for research on supporting mathematically productive generalizing in the elementary classroom. Moving forward, we hope that researchers will test, scrutinize, and refine the framework and propose that a tested and refined version of this framework could have direct implications for educators. For instance, such a framework might be applicable to designing and planning lessons that promote generalizing. However, we recognize that engaging in actions that are consistent with this framework does not guarantee generalizing will occur. We also recognize that there are many ways to support generalizing, and that this is only one approach.

We hope that future research explores this avenue and aims to uncover other ways to support generalizing because research in this area is critical to advancements in mathematics education. Education, in general, is "aimed at helping students develop robust understandings that will generalize to decision making and problem solving in other situations, both inside and outside the classroom" (Lobato 2006, p. 431). To this end, generalization is at the core of research on learning. Furthermore, the process of generalizing a set of particular instances, and justifying and formalizing the generalization, are fundamental to mathematics (Kaput 1999). Understanding how students generalize "will help us understand how students enter into the specialized disciplinary discourse of mathematics" (Jurow 2004, p. 280). Thus, we view research on mathematical generalization as relevant to all ages of students and levels of mathematics and consider research on generalization especially productive. Such research contributes to knowing how students learn the authentic thinking practices of mathematics by moving "away from the predominant preoccupation with numerical calculations," and placing the "focal emphasis on typical and important ways of mathematical thinking" (Dörfler 2008, p. 159).

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Chapter 16
Scaffolding Teacher Practice to Develop Early Algebraic Reasoning

Jodie Hunter, Glenda Anthony and David Burghes

Abstract In recent years there has been an increased emphasis on algebraic reasoning in primary classrooms. This includes introducing students to the mathematical practices of making conjectures, justifying, and generalizing. Drawing on the findings from a classroom-based case study, this chapter provides an exemplar of how professional development can lead to shifts in teacher practice to develop a 'conjecturing atmosphere' in the classroom. The findings affirm the important role of the teacher in introducing student-related mathematical practices. Careful task design and enactment, teacher questioning, and noticing and responding to student reasoning were all key elements in facilitating students to make conjectures, justify, and generalize.

Keywords Algebraic reasoning · Primary school · Professional development
Teacher change

16.1 Introduction

Important changes have been proposed for mathematics classrooms of the 21st century in order to meet the needs of a "knowledge society." A key aspect of proposed changes is greater emphasis on the teaching and learning of early algebra

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in primary classrooms (Blanton et al. 2016; Carpenter et al. 2003). Mason (2008) identifies algebraic reasoning as an essential type of thinking for "participation in a democratic society" (p. 79). The design and enactment of tasks along with specific pedagogical actions are important factors to develop early algebraic reasoning in the classroom. When existing curriculum materials are used, teachers need to recognize and adapt materials to exemplify opportunities and enact planned tasks in such a way that algebraic reasoning occurs (Blanton and Kaput 2005). Alternatively, teachers may purposefully design tasks to elicit aspects of algebraic reasoning. In addition to developing tasks, teachers need to recognize spontaneous opportunities for algebraic reasoning during enactment across a range of tasks. This can present challenges for primary teachers who have often not had experience in using pedagogical actions that facilitate algebraic reasoning (Blanton and Kaput 2005)

Creating a classroom culture that focuses on justification and generalization is not an easy task for a number of reasons. Firstly, there are considerable challenges related to the difficulties that students may encounter in both constructing and justifying generalizations. These challenges are related to a lack of understanding of generality along with difficulties with mathematical language and symbolism, and a lack of problem-solving skills necessary to construct an argument (e.g., Bieda et al. 2006; Callingham et al. 2004; Chick 2009). Secondly, many teachers themselves may not have had experience in constructing and justifying generalizations or promoting these practices in their own classrooms. In classrooms where frequent viable algebraic reasoning opportunities occur, students make purposeful conjectures, construct mathematical arguments, justify ideas, use age-appropriate proof, and generalize their ideas (Bastable and Schifter 2008; Blanton and Kaput 2005). Mason (2008) terms this the development of a 'conjecturing atmosphere' whereupon an ongoing expectation is that generalizations will be expressed and treated as conjectures, and then justified or disproved. Traditionally in mathematics classrooms, the space for students to generalize has been constrained because as Mason (2008) maintains it is often the teacher who provides the examples, cases, and methods during lessons.

Within the field of early algebra, previous research studies have typically focused on either professional development for teachers (e.g., Blanton and Kaput 2008; Franke et al. 2008; Jacobs et al. 2007) or shifts in the classroom practices and the consequent student understanding as early algebraic reasoning is introduced (e.g., Bastable and Schifter 2008; Blanton and Kaput 2005). The purpose of this chapter is to bridge these two areas of research by linking the professional development experiences of one teacher with the shifts in her classroom practice. Drawn from a larger study involving qualitative case studies of six teachers, we provide an account of this teacher's journey in shifting her practice to optimize algebraic reasoning opportunities over a school year. Specifically, we focus on this teacher to explore the following two key research questions:

(1) How can professional development scaffold teachers to identify and use opportunities to engage students in making conjectures, justification, and generalization?

(2) What pedagogical strategies and teacher actions support students in making conjectures, justification, and generalization in primary classrooms?

Before presenting the research findings we provide a summary of the research literature that informed the initial design of the professional development focus. In particular, we review what is known about effective professional development and the teacher's role in developing opportunities for students to engage in algebraic reasoning.

16.2 Literature Review

16.2.1 Effective Professional Development for Teachers in Early Algebra

Teachers take a critical role in reforming classroom practice by the use of pedagogical actions that facilitate algebraic reasoning and through the development of norms that support classroom and mathematical practice. However, many teachers themselves have had little experience with the classroom practices and rich, connected types of algebra that support integration of algebraic reasoning into primary classrooms (Blanton and Kaput 2005; Franke et al. 2008). Consequently, teachers require lengthy and in-depth support across many components of their daily practice. It is important to cultivate professional development experiences that both re-conceptualize algebraic reasoning for teachers and also support them to engage with the types of algebraic reasoning experiences that are relevant to primary school students (Franke et al. 2008). Effective professional learning is situated in practice. Ghousseini and Sleep (2011) note that "learning in and from practice requires being able to see, hear, and understand the many details of classrooms (e.g., the content, the students, and the work of the teacher) and use this knowledge to analyze and improve one's own teaching" (p. 148). These researchers maintain that effective practice-based professional development uses representations (activities, exemplars, and tasks) that engage learners (the teachers) with a particular representation of practice, for example, developing algebraic reasoning, and then supports the teachers as learners to become a deliberate user of this practice.

The findings of research studies that have investigated effective professional development (e.g., Back et al. 2009; Earley and Porritt 2009) and those studies that have specifically investigated teacher development programs focused on algebraic reasoning (e.g., Blanton and Kaput 2003; Franke et al. 2008; Koellner et al. 2011; Schifter et al. 2008a) provide us with insight into the important elements of effective practice-based professional development. These studies highlight the key indicators of effective continuing professional development to support the development of algebraic reasoning as: (1) opportunities to develop learning communities; (2) a focus on student learning and understanding; and (3) the facilitation of reflection on teaching practice.

16.2.1.1 Opportunities to Develop Learning Communities

A key aspect of effective professional development is offering teachers the opportunity and space to collaborate and build a strong professional community (Back et al. 2009; Earley and Porritt 2009; Franke et al. 2008; Koellner et al. 2011). For improved learning for students to occur, teachers need to have opportunities and reasons to question their practice through intentional inquiry (Ghousseini and Sleep 2011; Jaworski 2008). The use of inquiry as a tool within a community can enable teachers and educators to explore key questions and issues in practice. A community of inquiry develops when the use of inquiry shifts from being a tool to becoming a 'way of being' through which the participants in a community develop their practice (Jaworski 2008). Examples of how a community of inquiry can be developed are evident in research studies (e.g., Blanton and Kaput 2008; Franke et al. 2008; Jacobs et al. 2007; Koellner et al. 2011) that report on the principles of effective professional development in early algebra. Common features of these successful case studies are the facilitation of reflection on mathematical understandings, student thinking, and instructional practices.

16.2.1.2 A Focus on Student Learning and Understanding

Teacher knowledge is recognized as key to effective teaching, both within early algebra and in a wider context (Askew et al. 1997; Blanton and Kaput 2008; Shulman 1987). Solving and analyzing tasks and predicting student responses during professional development offers teachers the opportunity to develop pedagogical and specialized content knowledge required for teaching. For example, Franke et al. (2008) report on an aspect of their professional development where teachers worked collaboratively to create a written conjecture about commutativity, edit, and justify it. By working through the process of justification, teachers were both able to consider the arguments that students may use but also reflect on their own proof schemes. The use of tasks such as these during professional development also offers opportunities for teachers to understand how to develop specific aspects of algebraic reasoning.

16.2.1.3 The Facilitation of Reflection on Teaching Practice

A key element of professional learning that facilitates change such as the cultivation of early algebraic reasoning in the classroom is teacher reflection on practice (Back et al. 2009; Ghousseini and Sleep 2011; Jacobs et al. 2010; Schön 1987). The first essential step in developing the capacity to reflect on practice is that of noticing. As Jacobs et al. (2010) note, "noticing is a common act of teaching" (p. 169). However, to develop expertise within a profession, it is important to learn to notice relevant phenomena in a particular way (Jacobs et al. 2010; Schön 1987). This process of learning to notice is only developed through engaging with the act of noticing. It requires both knowledge of the relevant aspects to notice within a situation and the ability to be aware of them and respond appropriately while engaged in the act of teaching (Franke et al. 2008; Jacobs et al. 2010). Within the context of early algebra, professional development activities need to facilitate teachers to develop their understanding of the pedagogies they are using, the mathematical content involved, and the ways in which students make sense of algebra. Furthermore, teachers also need to develop a disposition of inquiry and reflect how their own practices are aligned or are in contrast with the ideas of pedagogies from research into early algebra, which they are seeking to adopt (Ghousseini and Sleep 2011).

16.2.2 Teachers' Roles in Developing Opportunities for Algebraic Reasoning

Within the classroom, in order to develop a conjecturing atmosphere, the teacher takes an important role by both planning opportunities for generalization and drawing on spontaneous opportunities during task enactment or discussion. Blanton (2008) proposes a model that characterizes the five components of building a generalization in the classroom. These include facilitating students to (1) explore a mathematical situation; (2) develop a conjecture or mathematical statement; (3) test the conjecture; (4) revise the conjecture if it is not true; and (5) develop the conjecture into a generalization if there is sufficient evidence to show it is true. Research studies (e.g., Carpenter et al. 2003, 2005; Fosnot and Jacob 2010; Schifter et al. 2008b) have explored how tasks may be purposely designed to elicit conjectures and generalizations from students. For example, Carpenter et al. (2005) describe how true and false number sentences can be used to facilitate student exploration of the properties of zero. The students were initially provided with a false number sentence (e.g., $78 - 49 = 78$), which provoked an in-depth discussion of the property of zero in relation to subtraction. The teacher introduced number sentences involving addition and subtraction with large numbers (e.g., $78956 - 0 = 78956$ and $0 + 5869 = 5869$) to press the students to articulate generalizations about the properties of zero in addition and subtraction.

As an alternative to purposefully designed algebraic tasks, other researchers have noted the value of supporting opportunities for algebraic reasoning across a wider range of mathematical tasks and activity. That is, by carefully monitoring student observations and questions during small group work, teachers can identify student conjectures for exploration and draw on this as a spontaneous opportunity. For example, Schifter et al. (2008b) report on a classroom episode where young students were working to generate ways to make 10. As the teacher observed the students working, she noted that many of them were utilizing the commutative principle, which they had informally termed 'turn arounds.' Noting that some students were making statements such as, *turn arounds always work,* prompted the teacher to use this to lead a discussion and probe understanding of additive commutativity.

Research has identified other pedagogical actions that are also important during task enactment and large group discussion to facilitate student engagement in both mathematical practices and algebraic reasoning. For example, when launching a task to the class, teacher questioning can be used to specifically focus student attention on the patterns and relationships within the task (Smith and Thompson 2008). Following the development of conjectures such as described above, students then need opportunities to engage in testing whether they are true. This requires the teacher to position students to agree or disagree based on mathematical arguments and to facilitate students to use concrete materials and representations to develop their arguments. For example, Bastable and Schifter (2008) provide an example of a teacher positioning students and facilitating them to build arrays using unifix cubes to convince their classmates of the generality of the commutative principle for multiplication. Further examples are provided in research studies (e.g., Carpenter et al. 2003; Schifter 2009; Schifter et al. 2008b) that demonstrate how students can justify conjectures about operations involving odd and even numbers. In these studies, students represented even and odd numbers using blocks, cubes, or drawings. They showed the meaning of the operation (addition) through joining the sets that represented all even or odd numbers and were able to use the structure of the representation to show the claim would always work. Through engaging students in this type of process, they begin to privilege reasoning based on representations (Schifter 2009).

16.3 Methodology

The data used within this chapter are drawn from a larger study (Hunter 2013) involving a year-long professional development and classroom-based intervention focused on developing early algebraic reasoning in a mathematical inquiry community. An aim of the larger study was to examine the key pedagogical strategies and classroom and mathematical practices that teachers can use to facilitate early algebraic reasoning in the classroom.

Participants in the study included two groups of primary teachers, one group of three teachers from England and the other group of three teachers from the Channel Islands. The schools in the study used the Mathematics Enhancement Programme (MEP) curriculum material. The MEP curriculum material was developed in order to improve mathematics teaching and learning in the United Kingdom by drawing on findings from the Kassel project (Burghes 2004). It includes resources such as lesson plans, workbooks, and online inter-active resources. Many of the tasks in the curriculum material have implicit opportunities to facilitate students to engage in algebraic reasoning due to their structural basis.

The focus in this chapter is on one teacher and her 25 Year-Three students (7- to 8-year-olds) from a semi-rural school in the Channel Islands. The teacher was an experienced teacher who was interested in strengthening her ability to develop early algebraic reasoning within her classroom. This teacher was selected as an exemplary case study. The changes that she undertook and shifts in practice observed were similar to half of the teachers who participated within this study (three out of six participants).

The study was split into four phases, the first, which was in the year prior to the beginning of the professional development, and the remaining three phases, which corresponded with the three terms over the school year during which the professional development in the form of cluster workshops was undertaken. These comprised of a mix of half-day or full-day workshops when the participating teachers were released from their classroom teaching. There was one full-day workshop in term one, two full-day workshops in term two, and one full-day and two half-day workshops in term four. The focus of professional learning during the workshops consisted of four separate but related components: (1) understanding of algebraic concepts, (2) task development, modification, and enactment, (3) classroom practices, and (4) mathematical practices.

Data gathering included classroom observations prior to and during the year-long professional development, video records of professional development meetings, audio recorded interviews, post lesson observations, detailed field notes of 23 lessons with each teacher, and classroom artifacts. The results reported in this chapter draw on the data from the professional development meetings related to developing classroom and mathematical practices to provide a context for the changes in the classroom. Other key sources of data include the classroom observations and audio-recorded teacher interviews.

Data from the classroom observations were analyzed and used to inform and shape the overall framework of teacher actions that emerged within the teacher over the three phases of professional development—a framework that aimed at supporting the development of early algebraic reasoning in her students. Ongoing data analysis supported the revision of the professional development. For example, from analysis of the classroom observations and discussions during study group meetings, it was observed that the teachers needed professional development in facilitating students to generate and explore conjectures. In response a task was designed to enable the teachers to explore possible conjectures that students would make and how these could be justified.

Retrospective data analysis used the QSR International (2012) *NVivo 10* qualitative software program. This included multi-levels of coding using both parent and child nodes. The initial codes were developed from a variety of sources including research literature, the initial viewing of the video records, and the observational and reflective field notes. Repeated viewing of the videos and re-reading of the transcripts and field notes confirmed or refuted the initial hypotheses and codes and other hypotheses and codes were developed as necessary.

Two central aspects that constituted the development of the framework of teacher actions and which are described shortly are the following: (1) teacher actions to develop and modify tasks and enact them in ways that facilitate algebraic reasoning and (2) teacher actions to develop mathematical practices in students that support the development of their algebraic reasoning. Both types of teacher actions are summarized in a table within each of the three sections that focus on the professional-development-related phases of the study and illustrate the changes in the teacher actions over the course of the study.

16.4 Findings and Discussion

Within this section examples are provided of how the teacher shifted from, on the one hand, task enactment and lessons that were conducted in a teacher directed, procedural way, devoid of opportunities for students to engage in algebraic reasoning to, on the other hand, enacting tasks and lessons in a way that supported her students' engagement in algebraic reasoning. The findings from the four phases of the study begin with a snapshot of the classroom prior to the professional development. Data from the professional development sessions relevant to the development of mathematical practices are presented at the beginning of each section. This is followed by an explanation of the changing teacher actions and an analysis of classroom episodes to illustrate the shifts in the teacher's practice toward integrating algebraic reasoning into the everyday mathematics lessons.

16.4.1 Phase One: Prior to the Professional Development

Prior to the professional development, observations of five lessons suggested that the teacher did not draw on opportunities to develop or enact tasks in a way that facilitated the students' engagement in algebraic reasoning. The tasks used in the lessons were taken directly from the curriculum with no modification and enacted in a teacher directed, procedural way. Tasks that had the potential to develop algebraic reasoning were enacted in ways that focused on computation rather than on the relationships within the task. Opportunities to explicitly identify or examine the properties of numbers and operations were not drawn upon. For example, in one lesson the students constructed two alternative solutions that implicitly drew on the commutative property. However, after recording these on the whiteboard and asking the students to describe what they had noticed, the teacher then stepped into offer a brief explanation of the commutative property herself:

Otto: It's the other way around…it's, it's the same but it's just changed around
Teacher: And that's one of the really important things in multiplication, isn't it? It doesn't matter if we do two times five or five times two.

During the observations prior to the professional development, there was no evidence of explicit identification or examination of the properties of numbers or operations during lessons. This meant that, for students, the properties remained implicit and they were not provided with opportunities to develop deep generalized understanding as advocated by many researchers (e.g., Carpenter et al. 2003; Schifter et al. 2008b).

16.4.2 Phase Two

16.4.2.1 Learning to Recognize Algebraic Reasoning Opportunities

In the first phase of professional development, there was an emphasis on developing the teacher's understanding of the links between arithmetic and algebra. This included extending pedagogical content knowledge to include the expected progression of student learning and potential misconceptions related to early algebra. Key activities were used during the professional development to achieve this. For example, research articles (e.g., Blanton and Kaput 2003; Carpenter et al. 2000, 2003; Kazemi 1998) were used as multi-purpose tools. The range of articles and excerpts that were used focused on extending teacher understanding of early algebra, providing models of classrooms that would support early algebraic reasoning, and promoting reflection on current practice. Discussions held after reading each article required the teachers to respond to questions such as "what did you find interesting?" or "are there any ideas that you could bring to your classroom after reading that?" The articles also served the purpose of developing links between research and classroom practice.

The teacher in this case study used the research articles to reflect on her own practice. For example, when presented with an adapted framework from Hunter (2007), which detailed classroom and mathematical practices linked to the development of algebraic reasoning, she described how she would use this to analyze her own practice: *I'd like to take this away and highlight the things I think I do all the time and then look at the things I never do; but then to also go and have a look because what you think you do, is it really what you do? And look at my philosophy in terms of my practice.* She also used excerpts from research material to develop her understanding of the areas within the primary mathematics curriculum that had links to early algebra and noted how she could use the material while planning mathematics lessons to select and/or modify tasks to focus on early algebraic concepts.

Further activities in the initial professional development meeting included predicting and analyzing student responses to algebraic tasks, which could be used in the classroom. For example, one activity included predicting potential responses to an open-number-sentence task (e.g., $8 + 6 = \_\_ + 5$). The teacher's initial response drew on a previously taught computational strategy: *My class would look at the left hand side and the right hand side and then would say we will start with the left hand side because we could work that out and they would put the 14 above there and then write the 14 above that and work it out.* Following extended discussion with the other teachers in the study group and further pressing from the researcher, she identified a possible misconception related to the equal sign: *they might add them all.* She was also able to identify a potential relational strategy (indicating each part of the equation): *So six and the five and then make that nine.* In later phases of the study, the teacher used activities from the workshops such as these to develop new tasks and modify existing MEP tasks to use during her mathematics lessons.

The second part of the activity involved an analysis of student assessment data from task-based interviews where students had responded to similarly structured tasks. Often the student responses demonstrated misconceptions of the equals sign. The teacher displayed a strong interest in the second part of the activity focused on analyzing her students' responses. She engaged in a prolonged analytical discussion with the researcher and her colleagues to understand what the varying responses indicated about her students' reasoning. At the end of the meeting the teacher requested copies of the student responses so she could analyze these further. Initially the teacher showed limited insight into student thinking. She needed opportunities to develop a framework to make sense of students' algebraic reasoning. Teacher knowledge of expected student progressions and potential misconceptions are important factors in developing classrooms with a focus on early algebra (Franke et al. 2008). As will be illustrated in Sect. 16.4.3.1, the teacher in this study used this developing understanding in later lessons to modify existing MEP tasks to specifically address student misconceptions of the equals sign.

The selection, design, and enactment of tasks were a key focus during the professional development meetings in all phases of the study. In the first part of the professional development, the group of teachers and the researcher worked together to examine the MEP curriculum material for links to early algebra content. The teachers then highlighted these as focal parts of their mathematics lessons and discussed ways in which the tasks could be enacted to focus on early algebra. Additionally, the researcher highlighted specific tasks from the MEP curriculum material to provide ways for the teachers to investigate how the existing tasks could be modified and further developed.

16.4.2.2 Early Changes to Task Design and Enactment

Following the initial professional development, the teacher undertook specific actions to shift her task design and enactment and to introduce mathematical practices that support the development of students' algebraic reasoning. An overview of these actions is shown in Table 16.1.

The teacher began intentionally developing and trialing ways to adapt her planning to integrate algebraic reasoning into her lessons. She examined MEP lesson plans and rather than asking students to complete the whole task, she presented them with those parts of the task that focused their attention on an algebraic concept. For example, in one lesson the teacher began by using a task involving an array and two equations (e.g., $3 \times \_\_ = 6$, $6 \div \_\_ = 2$) to focus student attention on the general relationship between multiplication and division:

Teacher: (records $3 \times 2 = 6$ and $6 \div 3 = 2$) Let's have a look at those, did anyone notice anything? Three times two equals six and six divided by three equals two. With your partner, what do you notice about those please?

After the students talked with their partners, she asked a student to say what he noticed:

Tristan: They're just the other way around… because the three is in the middle and the six is at the beginning and at the end.

The teacher then directed the students to examine related equations where the position of the numerals has changed. However this, and the following teacher questioning, moved the focus to specific equations limiting the opportunities for students to further explore the relationship between multiplication and division:

Teacher: So it's the same digits. Would it work if I put them in any order? If I did this *(writes* $2 \div 3 = 6$ *on the board)* two divided by three equals six because I've got the same numbers. Just talk that one through with your partner or what about this one, three divided by six equals two, is that true? Or six divided by three equals two *(writes the different equations on the board)* Are any of those true?

This was followed by further whole class discussion involving individual students using magnetic counters to model whether each equation was true. By asking the students to use magnetic counters to solve each equation, their attention was shifted specifically to calculating answers and thus the focus on the inverse relationship was lost. In this case concrete material was introduced as a tool to solve the task rather than as a means of developing an argument and proving or justifying. The lesson concluded with the teacher writing the equation $a \times b = c$ and then stating a conjecture:

Teacher: I have this theory that for every pair of factors and a product I can make two multiplications and two divisions let's see if that's right. With your partner at Planet X can you see if you can come up with equations for that?

Table 16.1 Developing framework of teacher actions in Phase Two

Teacher actions to develop and modify tasks and enact them in ways that facilitate algebraic reasoning	Implement tasks as problem-solving opportunities
	Emphasize student effort to approach and complete cognitively challenging tasks
	Extend or enact tasks to include opportunities for generalization
	Interrogate tasks for opportunities to highlight structure and relationships
Teacher actions to develop mathematical practices that support the development of algebraic reasoning	Require students to explain their reasoning

Overall, this lesson illustrates how opportunities for the students to develop and explore their own conjectures and prove and justify their reasoning were missed by the teacher telling her students the conjecture that she had developed and then guiding them towards generating equations to match the conjecture.

At this stage of the study, although the teacher had begun to plan for algebraic reasoning, there were still limited opportunities for engagement with mathematical practices associated with algebraic reasoning. For example, key mathematical practices, such as making conjectures and developing generalizations, justifications, and proofs (Bastable and Schifter 2008; Mason 2008; Selling 2016), were not established within the classroom during this phase. The teachers' practice of seeking examples and cases was promising, but her propensity to offer conjectures potentially reduced student opportunity to generalize.

16.4.3 Phase Three

16.4.3.1 A Focus on Mathematical Practices that Support Algebraic Reasoning

Throughout the third phase of the study, a focus was maintained on task development and how this related to developing student reasoning in specific areas of early algebra. In one professional development meeting, the teacher described how she reviewed class data from the student task-based interviews that had been a focus in a previous meeting. She noted that many students gave incorrect responses related to an operational understanding of the equal sign. In response to this, she began intentionally adapting tasks from the MEP material to focus student understanding on the equal sign. Opportunities were given during the professional development meetings for the teachers to watch and share video clips of how they were embedding practices aligned with early algebra into their mathematics lessons. On this occasion, the teacher shared a video clip from her classroom with the group in which a task (e.g., $36 - 6 = \_\_ + 20$ and $24 + 4 = \_\_ - 2$) was developed and adapted to focus student understanding on the equal sign.

During the professional development meetings in the third phase, another key focus was on the development of mathematical practices. To investigate how conjectures could be built upon in the classroom, one activity included the development and justification of conjectures. The teachers were first asked to brainstorm conjectures (both correct and incorrect) that they had heard their students make during lessons. The teacher was able to readily provide a range of conjectures that she had begun to notice students making during mathematics lessons. These involved conjectures about identity elements: *if you multiply by zero you will always end up with zero;* about odd and even numbers: *odd plus even always makes odd;* and even some incorrect conjectures: *if you multiply by ten, you add a zero to the number you are multiplying.*

The subsequent activity began by asking the teachers to reflect on different types of age-appropriate proof and justification. The initial discussion focused on different types of justification strategies used by young students, which have been illustrated in various research studies (e.g., Carpenter et al. 2003; Knuth et al. 2009; Schifter 2009). These included: an appeal to authority, justification by example, and generalizable arguments using representations. Following this, the teachers were asked to work together to develop a verbal explanation along with a physical representation to justify conjectures about odd and even numbers. This led to the teacher noting the value of using physical representations as tools to facilitate students' understanding of the structure and properties of numbers: *those ones are very powerful (points to a diagram of multi-link cubes in rows of twos to represent even numbers) because we counted in twos, so two, four, six, eight; so with the twos you can just keep going and how to make it odd, you can just put one in.*

In a focus group discussion in a following meeting, the teacher volunteered information about how she was embedding the suggested new and innovative practices: *Duncan made a statement about adding odd and even numbers. I have always got them to explore that just by finding lots of examples to sort of support it, whereas this time we actually proved it because we got two little piles of two unifix [cubes] and little piles of one; and realized visually that if you were adding an even number to an even number, you will always get an even number because you will not get any of the individuals, so that was taking it on board to proof because you could visually see it.*

16.4.3.2 Shifts in Task Design and Enactment

The teacher continued to undertake specific actions to shift her task design and enactment and to introduce mathematical practices. Table 16.2 presents an overview of the actions she undertook in the third phase of the study.

In the classroom, the teacher drew on a case study (Carpenter et al. 2003) presented during a professional development meeting to introduce the mathematical practices of generalization, justification, and proof. In a purposefully planned investigation of zero, student attention was drawn to a number sentence that had been constructed to involve the target number of 20 (e.g., 20 + 0 = 20) and they were asked to discuss what they noticed. The teacher then facilitated the students to develop a conjecture and find examples that illustrated the conjecture. Following this, she pressed them beyond the use of examples as justification by requiring that they prove their conjectures using a range of concrete materials (e.g., using counters, drawings). She then asked them to symbolize their generalization. Similar to the finding of Carpenter et al. (2003), this context provided the teacher with a rich area to scaffold students to develop and investigate conjectures and generalizations. It also provided the students with an initial opportunity to use concrete materials and representations as a means to develop an argument and establish a general claim.

Table 16.2 Further development of framework of teacher actions in Phase Three

Teacher actions to develop and modify tasks and enact them in ways that facilitate algebraic reasoning	Adapt tasks to highlight structure and relationships. This may include changing numbers or extending to multiple solutions
	Structure tasks to address potential misconceptions
	Use enabling prompts to facilitate all students to access tasks
	Implement tasks by focusing attention on patterns and structure
	Recognize and use spontaneous opportunities for algebraic reasoning during task enactment
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to develop mathematical explanations which refer to the task and its context
	Facilitate students to use representations to develop understanding of algebraic concepts
	Ask students to develop connections between tasks and representations
	Provide opportunities for students to formulate conjectures and generalizations in natural language. Leads students in examining and refining these conjectures and generalizations
	Listen for conjectures during discussions. Facilitate students to examine these
	Require students to use different representations to develop the clarity of their explanation

Although the teacher had now begun to adapt her task design to include opportunities to engage students in mathematical practices, at this point in the study her interventions did not extend to drawing on spontaneous opportunities during task enactment. For example, when the students referred to odd and even numbers or other patterns they had noticed, she heard them but did not let these ideas develop further. In one lesson, a student listened to two solution strategies and noted that the solution had drawn on the commutative law. Later the teacher commented on this:

Teacher: I was really impressed that they retained things from last term. You know Julio was like 'oh that's the commutative law.'

Although she had noticed the statement, she had not used the opportunity to engage students spontaneously in further investigation or discussion.

It was during the latter half of the third phase that the teacher began to recognize and draw upon spontaneous opportunities, including student-generated conjectures about the patterns they noticed. For example, in one lesson a student conjectured that dividing by four was the same as finding one quarter of a set. After recording this initial conjecture, the teacher then asked her students to work in pairs to investigate the conjecture by exploring what happened when a set of 12 counters was divided by four. The teacher then used this opportunity to facilitate students to explore the relationship between different fractional numbers and division. The whole class discussion began with a student agreeing with a conjecture Paul had made:

Jasia: (records 1/3 of 12 = 4 and 12 ÷ 3 = 4) It is because one third is three (points to the denominator) and there is three (points to ÷ 3) here and you have divided them all by the same; so the same as 12 and 12 divided by three equals four.

The teacher revoiced the explanation and then asked the students to generalize the conjecture:

Teacher: Paul wanted to know, his idea was: is dividing by four the same as finding one quarter. This time we've divided by three. Is that the same as finding one third? Jasia agreed with that and coming back to Paul's idea, dividing by four is the same as finding one quarter. Can anyone think what dividing by *n* would be the same as?

After further discussion the teacher returned to Paul who had made the original conjecture:

Paul: Finding one *n*th.

As this phase of the study progressed, the teacher continued to recognize opportunities within the curricular material that could be extended through task design to facilitate mathematical practices. This was then extended to noticing and using spontaneous opportunities within enacted tasks. This indicates that she was now beginning to use pedagogical moves described by Smith and Thompson (2008) and Blanton and Kaput (2005) to develop algebraic reasoning. The teacher initiated a growing expectation that students would offer conjectures to be tested and then developed in generalizations. The teacher herself noted this shift in her practice in a follow-up interview after the lesson: *Now I make more conscious decisions about which bits to go with and which bits to say I'll come back to you… Like today, Paul's idea was that dividing by four is the same as finding one quarter, so I thought well I'll get the counters out and see if we can make that link…So that's been the shift because we wouldn't have done that in the past. We'd have talked about it and then I would have said 'dividing by four is the same as finding one quarter.* These actions meant that she was beginning to facilitate a 'conjecturing atmosphere' in her classroom, such as described by Mason (2008) and others.

16.4.4 Phase Four

16.4.4.1 Role Played by Collaboration, Discussion, and Reflection

During the last phase of the professional development research project, the teacher identified the opportunities for collaboration and discussion within the study group as an important element for her development. She identified the reflective discussion questions and subsequent conversations about embedding algebraic reasoning and creating a certain classroom culture as a significant driver of changes in her classroom practice. In the final meeting she linked changes in practice to: *the amount of time we've had working together as a three, because it's been sustained over the year and it's been backed up with the reading and the on-going discussion between the three of us for the year.*

In the final set of professional development meetings, the teacher engaged in reflective discussions about the changes in both her own understanding of algebra and her classroom practice. She described her previous understanding of algebra as: *the missing number and shoving in an X here.* She related to the group that her school experiences of mathematics had focused on computation and procedures. She spoke of encountering algebra at teachers college: *I got to the first maths tutorial in the first week and it was that little problem, you know how many moves to get those three people and they all have to swap places. My maths lady said 'why does that work? Show me with algebra' and I was like 'oh my god I'm on the wrong course'.* In contrast, the teacher stated that she now viewed algebra as much bigger than she previously perceived and related algebra to generalization: *it's the way they think and it's the way they take something and can take it to a wider context.*

Within this broadening perspective it was clear that the teacher's perception of algebra had expanded and she now viewed facilitating algebra in the classroom as encompassing more than just content. Her own lack of experience with algebra during schooling was her motivation to ensure her students engaged in mathematical practices related to algebra; her increasing expectation that students would explain and justify was a personal response: *if he doesn't learn to explain and justify, he will be like me in his first tutorial and think nobody has ever asked me to justify that before.*

She acknowledged the significant shifts in the way her students engaged in making conjectures and generalizing and the key role she took in developing these practices: *they talk more mathematically, they come up with conjectures; but if they weren't asked the same sort of questions, if the language of conjecture and generalization suddenly stops, then that's going to filter away from them.* The teacher was also able to identify how the pedagogical content knowledge she was developing related to early algebra. In particular, she recognized the growth in her understanding of relational reasoning. At the same time, she knew her growth was ongoing because she told the group that, while confident with addition type problems, she still found using relational reasoning to solve number sentences involving subtraction challenging. To solve these she had to either draw a picture or think with concrete materials.

During professional development meetings in this final phase, the teacher was able to use her developing pedagogical content knowledge to critique the structure of tasks planned by the group to develop algebraic reasoning with their students. For example, a task was designed and used by the group with the aim of developing student understanding of the equal sign by asking students to record equations involving the target number 27 (e.g., $27 = 10 + 10 + 7 = 40 - 13 = 27 + 5 - 5$). Following this, in another task students were asked to solve true or false number sentences. While solving the true or false number sentences, it was evident that some students still did not view the equal sign as representing equivalence. After the lesson the teacher critiqued the tasks, telling the group: *I think maybe because we historically present children with a lot of things with the answer just being one box, that sort of one where they had to look maybe provoked that thinking a little bit more. You know at the beginning where they said something, something equals and then the next child does, equals, I don't know … when I look at it now I think it is a fantastic activity and a fantastic assessment… but maybe they are just seeing and the next one, and the next one, and now it's my turn and they don't actually see the equal sign, whereas this question here and that one here in particular really made them think about the idea of balance.*

16.4.4.2 Classroom Context: Development of a 'Conjecturing Atmosphere'

Table 16.3 shows the teacher actions that were introduced during the fourth and final phase. It is important to note that these actions built upon those previously established and outlined in Tables 16.1 and 16.2.

The teacher now designed tasks and carefully considered how to enact them in a way that exemplified opportunities for students to engage in mathematical practices. She described herself thinking as she planned about how to: *draw out the commutative law from this one, or this could be a great discussion point for, like the other week when we were doing timesing by one, or dividing by zero, get them to come out with conjectures.* During lessons the teacher maintained the expectation that conjectures would be expressed and proved while facilitating a consistent expectation for generalization. She used questioning such as: *would it work for different numbers?* Or: *can I change that into something that would work for any number?*

The teacher also consistently engaged her students in building generalizations in the classroom. She achieved this by noting the conjectures that students made and then facilitating the whole class to investigate these. This involved testing and revising the conjecture and developing it into a generalization. A new expectation that was developed was that students would justify their conjectures using concrete materials. For example, a student made a conjecture about dividing by one: *It's just like you're getting one group and dividing it by one group so you have already*

Table 16.3 Further development of framework of teacher actions in Phase Four

Teacher actions to develop and modify tasks and enact them in ways that facilitate algebraic reasoning	Recognize and use links to algebra in tasks across mathematical areas
	Implement tasks as open-ended problems
	Anticipate student responses that could provide opportunities for algebra
	Recognize and use spontaneous opportunities for algebraic reasoning from student responses
Teacher actions to develop mathematical practices that support the development of algebraic reasoning	Lead explicit discussion about mathematical practices
	Listen for implicit use of number or operational properties. Use these as a platform for students to make conjectures and generalize
	Facilitate students to represent conjectures and generalizations in number sentences using symbols
	Ask students to consider if the rule or solution strategy they have used will work for other numbers. Consider if they can use the same process for a more general case
	Promote use of concrete forms of justification
	Require students to translate between different representations

done it. If you've got a number and you divide it by one, it ends up that number. The teacher then asked the student to demonstrate and justify this idea using materials: *Show what you mean with counters on the board.*

As the teacher's students gained more experience in justification, they more readily drew on material to prove their reasoning. For example, using a task involving the distributive property (e.g., Write the correct sign for 9×14 __ $9 \times 7 + 9 \times 7$), the teacher facilitated the students to draw on representations to justify their reasoning. Building on previous work that investigated how relational reasoning could be used to solve tasks involving the distributive property, many students began to generalize the distributive property to solve the tasks. The teacher asked a student to share her explanation (see Fig. 16.1): *Seven add seven is 14 [notates an arrow from each seven and writes 14 underneath] and there is a 14 there [indicates 14 on the left-hand side] and they are both times nine so you have got nine times 14 and nine times 14.*

The teacher then asked the students to work in pairs using Misty's reasoning to prove whether $9 \times 6 = 9 \times 3 + 9 \times 3$. A student began by building an array to represent 9×6. Misty then developed this further (see Fig. 16.2): *Because there is three there [indicates splitting the six rows into three by drawing a line]. There is three rows there and three rows there and that is just the same as those [points to $9 \times 3 + 9 \times 3$ in the equation] and then it is times nine [points across the rows].*

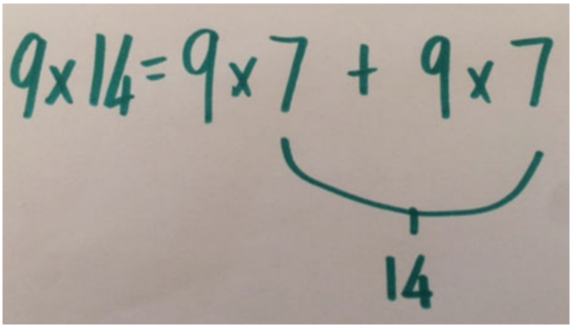

Fig. 16.1 Student representation of the equation

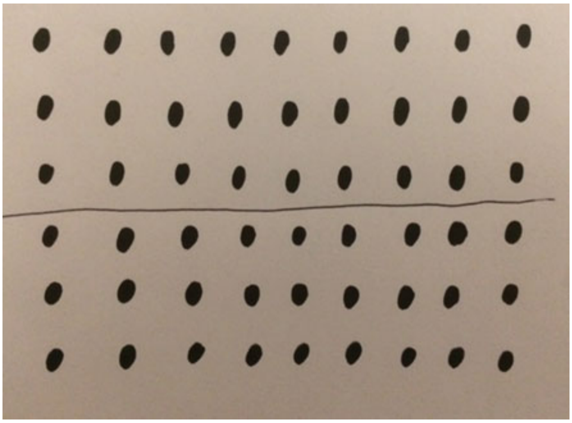

Fig. 16.2 Student justification of reasoning

Another key shift in the final phase was that the teacher now consistently drew upon students' responses to spontaneously and seamlessly integrate algebraic reasoning into the lessons. For example, in one lesson the teacher had asked the students to think about an efficient method to solve 26 − 8 = ___. A student responded by saying: *you could break it down to six and two.* The teacher first clarified the response: *Break what down to six and two* and after the student responded, then pressed for justification: *Why have you chosen to break eight down into six and two? Why not one and seven or four and four?* After student discussion, the teacher then asked the student to apply and generalize the strategy to similar equations: *Let's just go off track a bit, if you were doing 34 take away seven, with your partner can you just talk about how Millie and the other children would tackle that?*

The consistent focus on the mathematical practices of justification, generalization, and proof led to students drawing on previously examined conjectures and generalizations in their explanations. For example, the teacher asked the students to investigate what numbers would have a remainder of one when you divided them by two. A student drew on her understanding of odd and even numbers from previous discussions: *fifteen because if you divided it into twos and it is an odd* *number so you have one left over.* Similarly, in a later lesson many students used a generalization from an earlier lesson, that anything multiplied by zero was zero, to argue that $6 \times 0 + 5 = 7 + 4$ was a false number sentence.

Together with shifts in the way that the tasks were implemented and the focus on algebraic reasoning, there were resulting shifts in how the students engaged in the classroom mathematics activity. Along with the teacher, the students had also developed their own algebra ears and eyes and more readily drew upon algebraic reasoning, using structural aspects and patterns to solve tasks. By carefully monitoring and using student reasoning, the teacher developed a 'conjecturing atmosphere' as advocated by a number of researchers (e.g., Bastable and Schifter 2008; Blanton 2008; Mason 2008). Using a similar model to the one that Blanton (2008) describes, the teacher engaged her students in building and using generalizations by noting and exploring conjectures.

The teacher introduced an expectation that conjectures would be justified with concrete material. In this way, the students began to use representations to develop reasoned, general arguments. Schifter (2009) outlines specific criteria for student justification of general claims through the use of representation-based proofs. Examples from the current study meet Schifter's criteria that the meaning of the operation (e.g., multiplication) be represented in the manipulative and in the structure that are both involved and that the proof show that the claim (e.g., distributive nature of multiplication) would work for all cases. A number of researchers (e.g., Carpenter et al. 2003; Schifter 2009) argue that facilitating students to use concrete material to justify conjectures and explanations enhances students' work with proof in later years.

16.5 Conclusion and Implications

This chapter illustrates how professional development can scaffold teachers to identify and use opportunities to engage students in making conjectures, justifications, and generalizations. The findings illustrate how the use of specifically designed professional development activities, reflection on and in practice, and the collegiality of the professional development group supported the teacher to develop a classroom context in which a 'conjecturing atmosphere' (Mason 2008) was developed.

Research studies that focus on effective professional development (e.g., Earley and Porritt 2009; Franke et al. 2008) note the importance of teachers having time and space to collaborate. This was also important in the development of the teacher's understanding of early algebra and how to integrate this into mathematics lessons. The teacher's actions within the group sessions illustrated that she viewed change as an on-going collaborative process as she sought feedback from both the researcher and other study group members. The teacher identified the collegiality established through working collaboratively over a long period of time as a significant factor in the changes in her practice along with the activities that promoted reflection on practice.

In the initial phase of the professional development program it was important to develop teacher awareness of potential student reasoning related to early algebra. The teacher's knowledge of the expected progression in student understanding and potential common misconceptions related to areas of early algebra was developed in a number of ways. Research articles were used as an early stimulus to develop her personal understanding of early algebra and student pathways in understanding key concepts. Another important component of teacher development in this area was the opportunity to actively engage in analyzing student work related to early algebra concepts. Similar to studies by Stephens et al. (2004) and Franke et al. (2008), the use of data from assessment tasks was a useful way to increase teacher understanding of students' algebraic reasoning and misconceptions. The anticipation of student responses to tasks and analysis of student responses during mathematics lessons were other ways in which understanding of student reasoning was developed.

Deliberate planning for algebraic opportunities was another important factor in the teacher's development and there were notable shifts in the way algebra was integrated into the mathematics lessons over the duration of the professional development. Of key importance was for the teacher to recognize the inherent algebraic structure of number and operations. In the later part of the study, professional development activities that facilitated engagement in mathematical practices such as generalization and justification led to the teacher widening her conceptualization of early algebra. The teacher was then able to use opportunities to develop her understanding of these practices further during her lessons.

While the influence of the external expert and community of learners was significant for this teacher, her sustained active inquiry into her own practice was a key feature in warranting her efforts to change practice. Inquiry into her practice resulted in a new appreciation of the value of physical representations as tools to facilitate student understanding of the structure and properties of numbers and to develop forms of proof. In the classroom, shifts in practice were evident as the teacher provided opportunities and support for students to engage in these mathematical practices. Furthermore, she developed an appreciation for student capacity to engage in early algebraic reasoning and mathematical practices.

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Chapter 17
Preservice Teachers' Knowledge to Teach Functional Thinking

Sharon McAuliffe and Cornelis Vermeulen

Abstract This chapter reflects on a study of preservice teachers' knowledge to teach functional thinking as a strand of early algebra, and its implications for teacher education. While there is a growing body of research related to young children's algebraic thinking, much of the research within teacher education has focused on the assessment and development of teachers' and students' early algebra and less on the preparation of teachers to teach early algebra. The results of this study, based on the written reflections and video-recorded lessons of preservice teachers, highlight issues related to the ways in which knowledge is used when teaching functional thinking in the early grades.

Keywords Mathematical knowledge for teaching · Early algebra · Preservice teacher education · Functional thinking

17.1 Introduction

Early algebra (EA) is a relatively new topic within the South African primary school curriculum and is intended to conceptually prepare primary school students for the future study of formal algebra. Students in the early grades are expected to work with both geometric and numeric patterns by copying, extending, and describing patterns, and to create their own patterns. The study of numeric and geometric patterns in the later grades develops the concepts of variable, relationships, and functions. Understanding functional relationships is intended to enable students to eventually describe the rules generating the patterns (Department of Basic Education 2011).

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High school students' problems with algebra are a well-recognized concern across different countries and have resulted in a plethora of studies focused on understanding the nature of the problem and the search for solutions. There is growing evidence to support the exposure of children in the earlier grades to algebraic thinking and to help build connections between arithmetic and algebra in a meaningful way. The intention is to develop children's arithmetic thinking with a focus on structure rather than continue the limiting concept of arithmetic as a computational tool only. Blanton (2008) describes EA as a way of making important ideas of mathematics, particularly algebra, accessible and relevant to children. EA provides depth, meaning, and coherence to children's mathematical understanding and creates the opportunity to generalize relationships and properties in mathematics.

Building a practice to develop children's algebraic thinking requires a significant process of change for preservice teachers (PSTs), who are often schooled in traditional arithmetic ways of doing mathematics. PSTs in South Africa enter teacher education with a diversity of school experiences of learning algebra, which usually involves a high degree of procedural understanding. While it is important that PSTs have the opportunity to develop both conceptual and procedural understanding of the mathematics they will teach, it is equally important for them to be able to transform and use this knowledge to make it accessible for learners (Rowland et al. 2009). Preservice teacher education, through providing courses that are effective in building knowledge for teaching mathematics, is a critical element in the development of mathematics teachers. Fey et al. (2007, p. 27) asked the question: "How can prospective teachers be given a good start on developing essential knowledge of algebra for teaching?" and found there was very little available in the mathematics education literature at that time. Coincidentally, Hohensee (2017, p. 232) raised the same issue a number of years later—this time for the case of early algebra—and found that only four studies had been carried out to investigate PSTs' preparation to teach early algebra. This chapter therefore attempts to contribute to this under-represented aspect in the mathematics education research literature.

The purpose of this chapter is to investigate PSTs illustrations of knowledge for teaching functional thinking during their teaching practicum and to better understand the knowledge PSTs need to teach functional thinking, using the findings from a study undertaken with PSTs in South Africa (Mc Auliffe 2013). Data for this chapter are taken from PSTs' written reflections and their video-recorded lessons related to teaching a "pattern generalization and functional thinking" (Wilkie 2016, p. 246) lesson using a variety of different approaches. These data provide an understanding of the issues related to teaching functional thinking as a strand of EA and their implications for teacher education.

17.2 Literature Review and Theoretical Framework

This section outlines two aspects of the theoretical framework underpinning the research, that is, teacher knowledge and early algebra. The framework that is derived from research on these topics guided the design and implementation of the EA course for PSTs and the analysis of the PSTs' written reflections and video-recorded lessons.

17.2.1 Teacher Knowledge

Much has been written about the importance of teachers' mathematical knowledge but there is no agreement on the content and structure of that knowledge, nor a single accepted framework for describing mathematical knowledge needed for teaching (Clay et al. 2010). Nevertheless, it is accepted that the work of Shulman (1986, 1987) and colleagues "initiated a new wave of thinking about teacher knowledge by suggesting that content should matter in teaching" (Goulding and Petrou 2008, p. 1). Previous research recognized the role of subject matter knowledge and pedagogical knowledge, but paid little attention to a special body of knowledge for teaching. Rather than seeing teacher education from the perspective of either content or pedagogy, Shulman (1987) proposed to consider the relationship between the two knowledge bases as the intersection of content and pedagogy and introduced the notion of pedagogical content knowledge (PCK). He described PCK as the "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding," and defined it as:

> the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. (Shulman 1987, p. 8)

There are many different iterations of teacher knowledge frameworks scattered across mathematics education, focusing on different aspects of teacher knowledge and emanating from the work of Shulman, such as Ma's (1999) profound understanding of fundamental mathematics, Chick and Harris's (2007) particular focus on pedagogical content knowledge, and the Baumert et al. (2010) COACTIV (Cognitive Activation in the Mathematics Classroom) project.

Holmes (2012) provides an extensive list of frameworks, categorized into content knowledge and content knowledge for teaching. The latter includes Ball et al.'s (2008) "Mathematical Knowledge for Teaching" framework, from which the study herein reported draws. Ball and colleagues have extended the work of Shulman on teacher knowledge through practice-based research of teachers' work in the classroom. Over the past decade, they have developed a model of teacher knowledge: Mathematical Knowledge for Teaching (MKT), which elaborates in more detail "the fundamentals of subject matter knowledge for teaching" by delineating sub-domains of knowledge, and by measuring and validating knowledge of these domains (Ball et al. 2008, p. 402). MKT is recognized as a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, for example, engineering and carpentry, and constitutes the mathematical knowledge, skills, and sensibilities required for teaching mathematics. Ball et al. (2009) define MKT as mathematical knowledge *entailed by teaching*, in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to learners. MKT is based on the premise that teachers need to know mathematics and know how to use mathematics in the work of teaching; it comprises the domains of knowledge shown in Fig. 17.1.

MKT includes both subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is further divided into common content knowledge (CCK), specialized content knowledge (SCK) and horizon knowledge. *Common content knowledge* is the mathematical knowledge and skill used in settings other than teaching. It involves correctly solving mathematics problems, recognizing incorrect answers, and using mathematical terms and notation correctly, which is not exclusive to the work of teachers. *Specialized content knowledge,* however, is the mathematical knowledge and skill unique to teaching and involves teachers in doing a kind of mathematical work that others do not. It implies unique

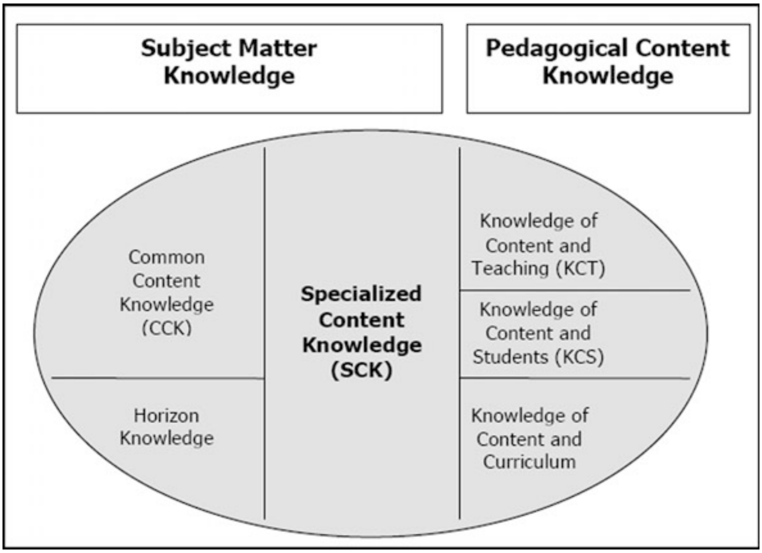


Fig. 17.1 Model of mathematical knowledge for teaching (Ball et al. 2008, reprinted with permission)

mathematical understanding and reasoning for teaching and requires knowledge beyond that being taught to learners. *Horizon knowledge*, a provisional inclusion in the model, is an awareness of how mathematical topics are relayed over the span of mathematics included in the curriculum. This is about teachers' understanding what the mathematics looks like in grades above and below the grade they are instructing. The category of pedagogical content knowledge in this model is different from that of Shulman and entails *knowledge of content and teaching (KCT), knowledge of content and students (KCS), and knowledge of content and curriculum.* It is thus knowledge that combines knowing about content and knowing about teaching, learners, and the curriculum.

This research focuses on the specific category of SCK, which involves unpacking mathematics to make it accessible and understandable for learners and requires the teacher to have a "deep and connected understanding of mathematics and relationships among ideas" (Bair and Rich 2011, p. 295). Ball et al. (2004) describe SCK in terms of eight tasks teachers engage in when teaching mathematics. These were later condensed into six by Kazima et al. (2008) and entail the following: *defining*—attempts to provide a mathematical definition; *explanations*—teachers explain an idea or procedure; *representations*—teachers represent ideas and in various ways; *working with students' ideas*—teachers engage with both expected and unexpected students' mathematical ideas; *restructuring tasks*—teachers change set tasks by scaling them either up or down; and *questioning*—teachers ask questions to move the lesson on. We return to these characteristic tasks of SCK in the Discussion section (Sect. 17.5).

17.2.2 Early Algebra with a Focus on Functional Thinking

Bridging the gap from arithmetic to algebraic thinking has been identified as problematic for students (Herscovics and Linchevski 1994). However, there is a growing body of research, based largely on the influential work of researchers such as Lins and Kaput (2004), Kieran (2004), Carpenter et al. (2005), Carraher et al. (2008), and Kaput and Blanton (2008), which suggests that the development of algebraic reasoning in the earlier grades could help alleviate the difficulties students have in learning algebra in high school. Linking arithmetic and algebraic thinking in the early grades could help develop the necessary skills for future success in algebra by creating the opportunity for students to "foster a particular kind of generality" in their thinking (Lins and Kaput 2004, p. 47). If students and teachers were to regularly spend the first six years of primary school simultaneously developing arithmetic and algebraic reasoning, the study of algebra later would become a "natural and non-threatening extension of the mathematics of primary school" (Cai and Moyer 2008, p. 3).

While there is acknowledgment of the eclectic and diverse views of early algebra (EA) , the work of the pioneer researcher, Kaput (2008), has helped define the field in more specific terms. Blanton and Kaput (2005) offer a definition of EA as a process in which learners generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways. They later use the term "early algebra" to refer to algebraic thinking in the elementary grades, which is designed to help children "see and describe mathematical structures and relationships for which they can construct meaning" (Blanton 2008, p. 6). EA is not an attempt to introduce symbol manipulation earlier to younger children, but rather an attempt to reform and update the teaching of arithmetic in a way that stresses its algebraic character. It requires understanding of how the arithmetic concepts and skills can be better aligned with the concepts and skills needed in algebra so that learning and instruction is more consistent with the kinds of knowledge needed in the learning of formal algebra (Carpenter et al. 2005). Early algebra is not to be seen as additional work to the current curriculum requirements. It is not a topic to be taught after children acquire arithmetic skills and procedures, but is developed in parallel with the development of arithmetic knowledge. It is about developing a way of thinking and reasoning that benefits all aspects of mathematics. Kieran (2004, p. 149, emphasis added) defines algebraic thinking as follows:

> Algebraic thinking in the early grades involves the *development of ways of thinking within activities* for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, *such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting.*

Kaput (2008) identifies two core aspects of algebra: (i) generalizing and expressing generalizations in increasingly systematic ways using conventional symbol systems and (ii) acting syntactically on symbols within organized symbol systems. These core aspects run through each of three algebraic strands: *Generalized arithmetic* (algebra as the study of structures and systems in arithmetic and including quantitative reasoning); *functions* (algebra as the study of functions, relations, and joint variation); and *modeling* (algebra as the application of a cluster of modeling languages both in and out of mathematics). *Functions*, and concomitantly functional thinking, provide a context for developing ways of thinking algebraically within pattern activities by creating opportunities for students to study change, to analyze relationships, to notice structure, to generalize, to problem-solve, to model, to justify, to prove, and to predict (Kieran 2004).

According to Wilkie (2016, p. 245), a key aspect of early algebra is exploring the functional relationship between two variables: noticing and generalizing the relationship, and expressing it mathematically—whether in words or in symbols. Smith (2008, p. 143) has defined functional thinking as a type of "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances." It involves ideas of change and representation through tables, flow diagrams/function machines, and graphs. Generalizing with a view toward the idea of a function means recognizing regularity through elementary patterning.

There are many reasons given in the literature for a functional approach to learning algebra through the use of pattern activities. Barbosa et al. (2009, p. 2) highlight the usefulness of patterns to help build a "more positive and meaningful image of mathematics," as well as develop crucial skills related to "problem solving and algebraic thinking." Mason et al. (2005) suggest that manipulating familiar objects can inspire confidence and is the beginning of noticing structure. The structure eventually emerges in the form of a generalization or expression. Trying out a case involving familiar objects and moving from the simple case towards analyzing a collection of particular cases helps build early generalizing. Samson and Schafer (2007) report that results from their research highlight the benefits of working with pattern activities that include opportunities to engage with algebraic thinking processes as a precursor to formal algebra. Pattern activities provide a special opportunity for teachers to develop a particular kind of generality in students' thinking, that is, an immersion in the "culture of algebra" (Lins and Kaput 2004, p. 60). Such activities provide students with a set of experiences that enables them to see mathematics—sometimes called the science of patterns—as something they can make sense of, and provides them with the habits of mind that will support the use of the specific mathematical tools that they will encounter later (Schoenfeld 2008). Pattern tasks that lead to generalization are important in making the transition from arithmetic thinking to algebraic thinking in that they provide a useful introduction to the concept of variable and future work with symbols (Michael et al. 2006).

17.3 Methodology

This section outlines the methodology followed in this study, and highlights the design and the data collection and analysis process.

17.3.1 Design

This was a qualitative case study involving 26 third-year Bachelor of Education PSTs, enrolled in an Early Algebra course. There were nine Foundation Phase (FP) (Grades 1–3) PSTs and 17 Intermediate/Senior Phase (ISP) (Grades 4–7) PSTs working in small, mixed-ability groups to encourage discussion and interaction across the phases. The EA course ran for one academic year, spanning 24 weeks, with contact sessions three times a week, and included eight weeks of teaching practicum in schools. It was designed to address both CCK (common content knowledge) and SCK (specialized content knowledge) of EA using an integrated, dynamic, and student-centered approach. The EA course focused on two of the Kaput (2008) content strands for algebraic thinking: algebra as the study of structures and relations arising in arithmetic, and algebra as the study of functions.

The coursework consisted of working with algebraic problems related to these two strands through the mathematical activity of generalizing. This involved generalizing as a process of identifying structure and relationships in mathematical situations, such as children recognizing that the order of adding two numbers does not matter (Blanton et al. 2011). It moved beyond "generalizing and expressing generalizations" and involved "extending one's thinking beyond producing a generalization, to reasoning with generalizations as objects in themselves" (Blanton et al. 2011, p. 9). Groups were expected to work collaboratively in solving problems and were encouraged to experiment, analyze, generalize, justify, and communicate their thinking to one another and the whole class. This process was mediated by a researcher who guided discussion and highlighted key features of the mathematics and the related thinking processes. The CCK was integrated with SCK through two different routes.

The first involved reading and discussing journal articles related to EA, addressing both theory and empirical work related to EA and children. Each group took a turn to lead the discussion on a particular article and to be interviewed by the researcher about key aspects of the article. This was followed by a whole class discussion and an opportunity to reflect using journaling. The journaling was encouraged to help the PSTs identify some of the key points of discussion and to reflect upon the deeper issues related to the teaching and learning of EA.

The second element of the coursework involved researching and planning an EA lesson to be taught during a future teaching practicum. Students had to share their planned lesson with their group for feedback and critique. The majority of the PSTs (22 out of 26) selected to teach a lesson focusing on functional thinking using growing patterns, which followed the prescribed national curriculum: Foundation Phase (6- to 9-year-olds) students are expected to be able to copy, extend and describe patterns using physical objects and to create and describe their own patterns, while the Intermediate/Senior Phase (9- to 13-year-olds) are expected to extend pattern tasks to include finding input and output values and to practice thinking about and describing functional relationships between numbers (Department of Basic Education 2011).

Interestingly, some of the Foundation Phase PSTs chose to extend the pattern tasks to encourage younger students to look for functional relationships focusing on two or more varying quantities and to describe "mathematical relationships for which they could describe meaning" (Blanton 2008, p. 6). Eight out of the 22 pattern and functional thinking lessons were in the Foundation Phase and 14 in the Intermediate & Senior Phase. The pattern lessons were based on different contexts and involved using real-life contexts and fictional characters such as animal/dinosaur/caterpillar eyes and tails, creating different clothing combinations, using string and cuts, tables and people to be seated, and geometric patterns using matches. All lessons focused on "functional thinking through a process in which arithmetic tasks are transformed into opportunities for generalizing mathematical patterns and relationships by varying a single task parameter," such as the number of people in a group (Blanton and Kaput 2005, p. 10). All pattern tasks involved linear relationships requiring one and two-step functional rules, that is, $y = \mathrm{a}x$ and $y = \mathrm{a}x + \mathrm{b}$.

17.3.2 Data Collection and Analysis

There were various components to the data collection of the original study (Mc Auliffe 2013): firstly, the EA lessons were video recorded and followed by a post-lesson interview with each PST to discuss and critically reflect on the teaching of the lesson. The PSTs used notes from these interviews to prepare written lesson reflections on their practice, which were later submitted electronically. Secondly, PSTs were given a copy of their video-recorded lesson and asked to complete a questionnaire related to their use of SCK to teach the patterns and functional thinking lesson. Lastly, two focus groups of PSTs were interviewed and asked to reflect on the development of their knowledge for teaching early algebra content in the EA course, on the teaching approach used in the EA course, and on the experience of the teaching practicum. Here, we report on data collected from PSTs' written lesson reflections and video-recorded lessons only and focus on four different aspects of their SCK for teaching early algebra, namely: (i) representations, (ii) working with students' responses, (iii) restructuring tasks, and (iv) questioning. These aspects were selected as they (a) indicate how PSTs attempt to use their knowledge to make functional ideas accessible to learners and (b) illustrate the issues PSTs face when teaching functional thinking lessons.

A content analysis approach was followed in which qualitative data from the reflections and video recordings were studied and coded either as CCK or SCK, using ATLAS.ti (2011). The descriptors for each code were guided by the relevant literature and based on the theoretical framework of the study. The coded data were then analyzed further in an attempt to extract illustrations of knowledge for teaching functional thinking with a focus on specialized content knowledge for teaching functional thinking.

17.4 Results

This section, which begins with a few additional remarks situating our study with respect to SCK, will then focus on data drawn from the activity of two of the participating PSTs, one whose practicum teaching involved a Grade 2 class and the other a Grade 7 class. The presentation of results will include, for each of the two PSTs, a brief description of the pattern generalization lesson, an excerpt selected from the PST's lesson reflection, and a transcript from part of the video-recorded lesson.

17.4.1 Specialized Content Knowledge for Teaching Functional Thinking

Wilkie's (2016) research on the professional learning of practicing teachers to develop their students' functional thinking was designed to provide teachers with an understanding of pattern generalization. She offers a detailed elaboration of the types of knowledge needed for teaching functional thinking based on the Ball et al. (2008) model, that is, specialized content knowledge (SCK), knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of curriculum (KC). SCK in Wilkie's (2016) research focuses on two different approaches to pattern generalization, namely recursive and explicit, and includes some explanation of the different terms used for these approaches, such as near and far generalization, co-variation and correspondence, and finding local and relational rules. In her research, KCS focuses on the knowledge of the processes students go through in using these just-mentioned two approaches, common issues and errors, the possible learning progression pathways, and assessing written work. KCT involves knowing more about the types of representation and questions that help students to develop functional thinking and to be used in teaching, as well as working with errors and knowledge of progression. KC refers to the teachers' knowledge of national curriculum requirements for teaching functional thinking, including content descriptions and activities. The results from our study are also linked to the Ball et al. (2008) model, but with a predominant focus on SCK, as the mathematical knowledge and skills unique to the teaching of functional thinking sometimes overlap with KCS and KCT, as recognized by Wilkie (2016). The following examples were chosen from a Grade 2 and a Grade 7 class to give coverage across two different phases of schooling in South Africa. They reveal interesting and different issues for PSTs in acquiring SCK for teaching functional thinking.

17.4.2 Keri—Grade 2

17.4.2.1 Brief Summary of the Pattern Task

The focus of this lesson is on doubling and halving and Keri uses the opportunity to help students understand the interconnected relationship between the inverse operations. She takes the task further using the context of cutting strings of wool to introduce the possibility of different representations of the same concept to the Grade 2 learners. Keri introduces the lesson using different whole-class clapping activities. This is followed by counting-forward and -backward activities, such as counting in 2s from 32 upwards and from 67 backwards. Individual students are then asked to add different double numbers: 2 + 2; 3 + 3; 5 + 5; after which they double whole numbers such as 7, 4, 9 and halve numbers such as 6, 8, 10, and 12.

Number of cuts	1	2	4	8	16	64
Total number of pieces of wool	2	4	8	16	32	128

Fig. 17.2 Function table displaying number of cuts and total pieces of wool

The class of 30 students is then divided into three different ability groups: top, middle, and bottom. Keri takes the top group onto the mat while the rest of the class returns to their desks to complete mathematics worksheets. She demonstrates and explains the "Cutting Wool" pattern problem to the group, which involves taking a piece of wool, folding it in half and making one cut. Keri then asks the students: "how many pieces of string do you have?" and records the result in a table (see Fig. 17.2). She then takes the two pieces left after the cut, folds each in half and asks a student to make one cut in each and to count the number of pieces of string. She again records the results in the table. The cutting activity is repeated a number of times with the remaining pieces of string.

17.4.2.2 Keri's Reflections on the Lesson

Her lesson reflection provides a summary of her lesson, activities, and sequencing of actions, including the various representations used to develop the mathematical concept of doubling and halving:

> I really liked the introduction of the lesson, the body percussion, as it introduced patterns and beats to the Grade 2s in a fun and exciting way. The exposition of the lesson, cutting wool, was practical and made it easier for the Grade 2s to see the pattern being dealt with. The table used in the exposition also further helped the Grade 2s grasp the pattern. During the exposition I also introduced algebraic symbols by getting the Grade 2s to see a shorter way of writing the information in the table, i.e., instead of writing out the whole words *cuts* and *pieces* they could just write a "c" and a "p."

There is mention later in her reflection of a focus on algebraic thinking and, while this is not elaborated upon, she does make use of a data table (function table), different representations, and varies the input values. It is not clear from the transcript whether she expects students to generalize nor is there any mention of how she establishes a linkage between the table-representation and the previous numerical activities. She plans different activities to draw attention to the task of working with multiples of 2 to emphasize the quasi-variable aspects of these numbers, but it is not used in a general way to represent many numbers—as would be the case with a variable (Fujii and Stephens 2008).

> When it came to the exposition, I feel that I lost the focus of algebraic thinking a bit as my lesson moved more towards the doubling and halving concept. This was due to the fact that I took the data table away too soon. It was clear to see that the Grade 2s were grasping the doubling and halving concepts, so I should have moved onto and focused more on the algebraic thinking aspect. I could have kept the data table there for longer and gotten the

> Grade 2s to see the variety of different relationships between the information filled out on the data table, i.e., counting in twos, doubling, halving, multiplying by two, etc., thereby exposing them to different ways of looking at and describing patterns they saw. I could have exposed the top group a bit more too. I could have extended their algebraic thinking by giving them different amounts and getting them to predict the answer, as this would have challenged their thinking more.

Keri appears to want to use the data table to focus more on the algebraic thinking part of the lesson by transferring the data collected from the cuts and pieces of wool into the tabular form and helping students to see different relationships. Her reflection shows awareness of the connections between different representations of the doubling and halving, but she also identifies some difficulty in moving between different forms of arithmetic and algebraic reasoning. There is reported evidence provided by Moss and McNab (2011, p. 278) who have worked with Grade 2 students and who found that the students were able to generate a rule for growing pattern activities, but needed "targeted specific pedagogical support" to move from patterns to generalization. They highlight the literature reporting that the move from "perceiving patterns to finding useful rules and algebraic representations is difficult." While Keri's reflection highlights connections and the search for patterns, there is no mention of the purpose of the pattern activity to move from the simple particular case of "seeing what is going on," in order to generalize for a collection of particular cases (Mason et al. 2005, p. 23). Patterning activity that does not involve the work of noticing the "underlying structure to generalize across specific instances" falls short of engaging students in algebraic thinking (Kieran et al. 2016, p. 17).

17.4.2.3 Video Extract from the Lesson

The following video extract is taken from a later part of the lesson in which Keri has moved from the numeric oral activities to the functional thinking task and uses this to show students how the same concept can be represented in different ways. She uses the context of the Cutting Wool task to ask students questions so as to elicit particular responses. Here she reveals more about her knowledge for teaching functional thinking:

Teacher: How many times did you cut your piece of wool?
Student 1: Two times.
Teacher: Two times and how many pieces of wool are there now?
Student 1: Four.
Teacher: Four pieces of wool and we go back to our data table and we fill in our information. What must I write by the number of cuts?
Student 1: Two.
Teacher: And what must I write by the pieces of wool?
Student 1: Four.

Teacher: Okay, so I write two and four, right. So I have two cuts and I have four. Now who can tell me what happened to the piece of string? Does it look the same as it did when we started? What's happened to that piece of string?
Student 2: It's getting less.
Teacher: Is it getting less or is it getting more?
Student 2: It's getting smaller.
Teacher: It's getting smaller, yes. That piece of wool is getting smaller. Right, now if I now have four pieces of wool. How many pieces did I have before I cut it twice?
Student 3: Two.
Teacher: We had two pieces of wool, yes.

She continues the lesson making a different number of cuts and we pick up the discussion again below:

Teacher: If I made eight cuts how many pieces of string would there be?
Student 4: Sixteen.
Teacher: Yes, and if I made sixteen cuts how many pieces of string would there be?
Student 4: Thirty-two.
Teacher: The last one, if I made thirty-two cuts how many pieces of string would there be?
Student 5: Sixty-four.
Teacher: Okay, I've got a nice big number. Do you want to try this big one. Okay let's see if you can do this one. If you had sixty-four cuts—how many pieces of string?
Student 6: One hundred and twenty-eight.
Teacher: And how did you know it was one hundred and twenty-eight?
Student 6: I doubled the sixty and I doubled the four.

The questions posed to the students relate specifically to varying input values. There is an instance later in which she gives the total number of pieces of wool and asks for the number of cuts and her students answer successfully.

In summary, Keri demonstrates knowledge of different ways of working with multiples of 2 through the use of clapping, counting forwards and backwards in 2 s, and doubling and halving activities. She includes the Cutting Wool task (contextual pattern activity) to extend the students' understanding of the number pattern and introduces input and output values related to the function $y = 2x$ with the help of the students' responses. The Grade 2 students are able to recognize the correspondence between the number of cuts and pieces of wool, but Keri does not extend the questioning to help the students to move from specialization towards generalization. She misses the opportunity to highlight "the variable on which the pattern or sequence depends" and keeps "its structure as a function well-hidden" (Kaput 2008, p. 7). It would have been helpful to expand the students' algebraic thinking by extending the task to include more questions related to higher input values and eventually moving to generalizing the pattern.

17.4.3 Bryn—Grade 7

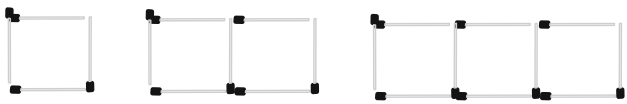
17.4.3.1 Brief Summary of the Pattern Task

Bryn is working at a higher grade within primary school and is keen to connect different representations of functional relationships. There is much in the literature to support this approach and to help students "to think about functions in diverse and legitimate ways" (Wilkie 2016, p. 253). He has previously developed the concept of function over a series of lessons and is currently working with geometric growing patterns to analyze and generalize for different linear functions. This lesson starts with a square match pattern and involves the addition of three matches to create the next term in the sequence. The students help the teacher to extend the pattern and the relevant data is captured in a function table (Fig. 17.3).

17.4.3.2 Bryn's Reflections on the Lesson

Bryn's lesson reflection below shows his interest in helping students make connections to the function activities of previous lessons and he builds on this knowledge to generalize for a different function. The students have been previously working with direct proportional problems and he uses this knowledge to link to the geometric pattern activities.

> I decided to go back to some previous work done about three lessons ago where I put the 3 and 8 times table in a flow diagram, and learners were challenged to give the *n*th term for both. They then realized that the 3 and 8 times table can be written as $3n$ and $8n$ respectively. This knowledge I used when getting them to find the pattern formula $y = 3x + 1$. Once they had completed the table and noticed the constant difference of +3, I ask them where else they find it? Their reply was the 3 times table which could be written $y = 3x$. I then got them to compare the output of $y = 3x$ to their pattern and they were easily able to see that the difference is +1 every time and they were then able to construct the formula $y = 3x + 1$. This was not in my lesson plan but I'm glad I used it because it gives the



Number of squares	1	2	3	4	5	6
Total number of matches	4	7	10			

Fig. 17.3 Function table of number of squares and matches

> learners a better understanding to constructing formulas and helps them see the relationship the constant difference and the formula. This works well and I will definitely use it in the future. I then explained the co-ordinate graph to them and together with them we constructed the graph for $y = 3x + 1$. Learners easily understood the graph. They were now able to compare the pattern across a range of representations and that was the objective of the lesson.

This reflection reminds us of the critical role of the teacher in providing the opportunity for students to engage in functional thinking by "creating activities, describing variable quantities, posing appropriate questions, and engaging in classroom discourse" (Smith 2008, p. 145). Bryn links the work from a previous pattern lesson to help students generate a function rule for the given geometric pattern. He helps students construct the rule/formula by encouraging them to compare the outputs of the $y = 3x$ function and the outputs of the new function $y = 3x + 1$.

Students recognize the difference between the outputs of the two functions as "+1" and are able to generate a function rule for the square match pattern above, which enables them to calculate the output value given a correspondent input value (Barbosa and Vale 2015). Bryn works hard with the students to connect the outputs of the different functions to move towards a generalization for the square match activity. However, there is no mention of or link to the structure of the geometric pattern: students are not requested to analyze the pattern formed by the matches in terms of what remains constant and what varies. Thus, while the pattern can be linked to prior learning, this approach might have limitations when students are required to generalize different and unrelated functions.

17.4.3.3 Video Extract from the Lesson

The following is a video extract from a later part of Bryn's lesson and focuses on representing and comparing functions using tables and graphs. He starts by teaching students how to plot the $y = 2x + 1$ graph using the input and output values from the function table and then gives the students the opportunity to create their own graph of the $y = 3x + 1$ function.

Teacher: What I want you to do is, first take one of these. So the line paper I gave you guys now, you get it with blocks right. So what I want you to do on the page, just look here quickly. I want you to draw my table that I drew there okay. So I want you to draw at least 20 blocks this way and 20 blocks that way. So y- and x-axis must have at least 20 blocks on each. Okay guys, just stop there quickly. I don't think there will be enough time for everyone to draw their own one. I will just show you quickly on the board. Okay so what line is this? Where are these points from? What is the equation for that line there, those points there? (pointing at the table of values).

Student: $2x + 1$

Teacher: Remember that table there. Okay, so this table is $y = 2x + 1$. It corresponds with this picture and the table corresponds with the co-ordinates graph. This one with the co-ordinate graph okay. So let's do the one that we tried previously, the one that was… What was the previous one that we did? y equals to?

Student: $3x + 1$.

From the planning and execution of the lesson, it seems that Bryn wants students to understand how the function concept can be represented in different ways starting with the picture of the geometric pattern, drawing the table of the input and output values, and then using this to draw the graph of the function. However, there is little opportunity for students at this stage to discuss what is happening in the transformation between the different representations of the function or to understand the relevance of such information. Bryn tries to initiate such a discussion towards the end of the lesson when analyzing his graphs of the two linear functions, but runs out of time to develop the content further.

Teacher: Okay so which one is adding 2? (pointing to Fig. 17.4). Okay so he (student) says that, in this one here I'm adding 2 and in this one here I'm adding 3. So do you see a resemblance between the two?

Teacher: So the more I add here, the steeper my graph is (pointing at Fig. 17.4). So this number here will affect what is called the gradient of the equation, the gradient of the graph. So what I want you guys to do firstly, plot these two in two different colors for me.

Bryn tries to make use of his knowledge of functions to help students "see and describe mathematical structures and relationships for which they can construct meaning" (Blanton 2008, p. 6). He assists students to generalize mathematical ideas and to express them in formal and age-appropriate ways, but fails to establish those

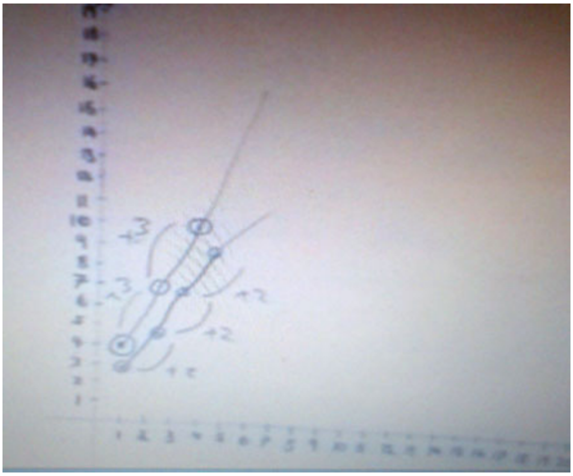


Fig. 17.4 Preservice teacher's illustration of linear graphs

generalizations through the discourse of argumentation (Blanton and Kaput 2005). There is strong teacher-guided instruction and prompting of answers, which reduce the opportunity for students to construct their own understanding of the function concept. According to Schmidt (2008), curriculum coherence involves the logical sequencing of a topic to reflect the inherent structure of the discipline (mathematics) and needs to focus on the interconnections between ideas within concepts. There is evidence of planning in terms of sequencing of tasks, but there is insufficient engagement of students in connecting the tasks and understanding the mathematical relevance.

Overall, Bryn's written and verbal illustrations demonstrate his knowledge for teaching functions as a series of tasks related to analyzing geometric patterns, drawing a table of the data, finding the function rule, and sketching the graph of the function. However, it is important for students to make effective and purposeful use of symbols in ways that are inherently sensible and meaningful (Schoenfeld 2008). It is not clear if Bryn's students understand the links between the different activities or appreciate functions as a way of thinking within activities such as "analyzing relationships between quantities, noticing structures, studying change, and generalizing" (Kieran 2004, p. 149). Students need to have experience of problem solving and modeling, but they also need to be encouraged to justify, prove, and predict. It is essential for teachers to select "tasks that encourage students to reason flexibly" and to allow students "to model a variety of situations, connect important mathematical ideas, and build a basic understanding of algebraic concepts," which will hopefully help to increase their success in algebra at the higher levels (Lannin 2004, p. 223).

17.5 Discussion and Implications

The purpose of this chapter is to investigate PSTs illustrations of knowledge for teaching functional thinking during their teaching practicum, to better understand the knowledge PSTs need to teach functional thinking, and to identify the implications for teacher education. The verbal and written illustrations (reflections and video transcripts) highlight the complexity of the SCK needed for teaching functional thinking and the challenges PSTs experience in teaching the topic.

Keri makes use of her SCK for teaching functional thinking to plan activities to help students recognize the link between arithmetic and algebraic thinking, albeit tentative. She is able to select activities that have the potential to build functional thinking and she plans to try to connect different representations of the same function $y = 2x$. She uses the function table to help students recognize the link between the number of cuts and the number of pieces of string. Unfortunately, the activity does not evolve into generalizing relationships, and the opportunity to make the arithmetic activity algebraic through deliberate generalization is lost (Kaput 2008). Keri experiences some issues in relation to connections and coherence in how the mathematics is managed in instruction. There is evidence of the development of her SCK for teaching functional thinking in her planning to work with multiples of two using different representations, but she does not do enough to help her students to analyze the function behavior, to generalize the relationships between co-varying quantities, or to express these relationships in words (Blanton et al. 2011). She works with activities in highly separated ways without enough emphasis on the relationships between them (Venkat and Naidoo 2012). This highlights the importance of investigating different ways in which PSTs use their knowledge for teaching, work with, and connect tasks in the context of the classroom to develop algebraic reasoning.

There are different aspects to Bryn's SCK for teaching functional thinking at the Grade 7 level. Students are challenged to work with geometric patterns, analyze relationships, make predictions, justify their thinking, and construct function rules. Bryn has high expectations for his students and he wants them to develop a deep understanding of the inter-connectedness of the different representations of a function. He extends their understanding beyond the requirements of the curriculum to include and compare gradients of different but related functions. Bryn demonstrates SCK for teaching functional thinking through his use of different representations of functions, working with students' responses to help generate a function rule, and questioning and probing student thinking about the nature of the sketched graphs. However, there is little connection between the structure of the geometric pattern and the function rule, and students are not encouraged to study the function by noticing the relationship between the number of squares and matches (Blanton 2008). Students are not required to justify their thinking or responses, and they do not establish generalizations through the discourse of argumentation (Blanton and Kaput 2005). As with Keri, these are issues related to connections and coherence in planning and presenting functional thinking lessons, as well as challenges in working with student responses, task design, and questioning. This highlights the need for preservice education programs to create greater opportunities for students to become more fluent and flexible in designing and sequencing functional thinking lessons for a range of students. The role and actions of the teacher are crucial in developing algebraic thinking through designing activities that challenge students, providing coherent explanations, and asking probing and stimulating questions requiring students to reflect on their observations (Kieran et al. 2016).

The MKT model offers a useful framework to categorize and describe different types of knowledge needed for teaching and is used in this context to understand PSTs' knowledge for teaching functional thinking. There are some challenges in terms of the classification of knowledge as illustrations appear to overlap SCK, KCS, and KCT and the boundaries between knowledge domains are sometimes blurred. Ball et al. (2009) have acknowledged this challenge and suggest that it may be useful to see the knowledge categories as inter-related rather than as separate and unique domains of knowledge. Notwithstanding this difficulty, the model is useful in helping to focus more specifically on the tasks required for teaching functional thinking and how PSTs interpret these tasks in practice.

This research helps us to gain a better understanding of the SCK needed for teaching and the implications for teacher education programs. Firstly, it is well documented that for students the shift from arithmetic thinking to algebraic thinking is difficult—as is evident in Keri's lesson. It is important for PSTs to have the knowledge and understanding of the differences between these two ways of thinking. They should encourage their students to reason flexibly, to make and justify generalizations, to recognize the limitations of justification by example, and to move towards general arguments (Lannin 2004; Stephens et al. 2015). Secondly, PSTs need to have knowledge of patterns, both the strategies and the representations that can be used to help students to generalize, and they need to be able to transform this knowledge into forms that are pedagogically powerful (Shulman 1987; Yesildere and Akkoç 2010). Bryn demonstrates knowledge of different representations of functions, such as geometric patterns, tables, graphs, and equations, and attempts to render this knowledge understandable to his students. It is important for PSTs to be able to transform what they know and to make their knowledge accessible to students through the selection of "appropriate forms of representations," giving "clear explanations of concepts," and using "questioning to assess and develop students' knowledge and understanding" of functional thinking (Rowland et al. 2009, p. 36). This requires opportunities within teacher education courses for PSTs to experiment in teaching functional thinking and to reflect in and on the process. Thirdly, it is apparent from this study that PSTs could benefit from knowledge of learning trajectories for functional thinking. Blanton et al. (2015, p. 514) use trajectories to describe "children's thinking about generalizing algebraic relationships in functions" and identify three essential features of learning trajectories (see also Clements and Sarama 2004): learning goals, instructional activities, and developmental progression. If Keri and Bryn had a better understanding of where their students were coming from and going to, it would likely have impacted on the selection of their lesson goals, their task design, as well as their questioning of the students. In all probability, it could have produced quite different results. Including learning trajectories in teacher education courses means that PSTs would be able to draw upon this knowledge as a framework to develop their students' functional thinking and to plan and design instruction that involves different levels of sophistication in generalizing functional relationships.

17.6 Conclusion

The results of this study are useful in helping us understand the development of knowledge for teaching functional thinking and how this might inform EA teacher education courses. While much progress has been made in developing a framework for mathematical knowledge for teaching, there is still work needed to determine "what kinds of learning opportunities effectively help PSTs to develop such knowledge" within different mathematics topics (Thanheiser et al. 2010, p. 3). The illustrations provided by the PSTs begin to elucidate their engagement in the development of their own SCK and their understanding of functional thinking, facilitated by opportunities to discuss and reflect on the theory and practice in the EA course. Through guided support and reflection, the teaching practicum can become an important contributing factor to develop knowledge for teaching functional thinking, thereby merging theory and practice. The EA course gives the PSTs opportunity to gain knowledge of practice through participating in, and reflecting on, practice, thereby helping to create a reflective disposition (Matos et al. 2009). This research contributes to a more "reflective mathematics education culture" in which teacher educators and PSTs begin to engage with the conceptualization of knowledge needed for teaching EA and hopefully other topics thereafter (Gellert et al. 2009, p. 54).

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Conclusions and Looking Ahead

Carolyn Kieran

A reading of the 17 chapters in this volume discloses the rich and varied perspectives according to which the chapter authors theorize about and examine early algebraic thinking from the 5-year-old in kindergarten to the 12-year-old in lower middle school, within the content area that has come to be referred to as *Early Algebra*. As pointed out in the Introduction to the volume, one cannot help but be struck by its international dimension, with authors from five of the seven continents. Early algebra as a bona fide field of research and practice has truly emerged as a worldwide movement over the past several years.

From its beginnings in the 1980s, when early efforts involved mostly 13- and 14-year-olds and centered on ways of bridging arithmetic and algebra, to its reorientation for the primary school level in the decades that followed, the historical evolution of the movement that sparked its global adoption is briefly presented in Malara and Navarra's chapter (Chap. 3). The downward push into the primary grades did much more than expand the emphasis from what was considered important within the earlier perspective, a perspective that encompassed developing meaning for variables, the equal sign, and alphanumeric representations of expressions and equations. The new movement adapted those earlier areas of attention for the younger student, but even more importantly, it articulated a focus that included structural aspects of number, operations, patterns, and functions, as well as the mathematical processes of conjecturing, justifying, and generalizing structural relationships and properties. This perspective on early algebra, quite different from its 1980s predecessors, was groundbreaking. In the process of its evolution, researchers developed a diversity of theoretical frameworks, with some favoring an early initiation into the use of algebraic symbols and others not. While many of the volume chapters reflect this diversity, there is also remarkable uniformity on certain issues, such as the pivotal role of the teacher in early algebra instruction.

In this concluding section of the volume, four cross-cutting themes that reflect the simultaneous diversity and uniformity that permeate the volume are discussed

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and interwoven with some suggestions for further research that can assist in moving this emerging field even further ahead:

(1) Varied theoretical perspectives on what constitutes early algebraic thinking;
(2) Improving attention to structure with primary and lower middle school students;
(3) The multiple facets of students' early algebraic thinking; and
(4) Pivotal roles of the teacher and the curriculum.

Theme 1
Varied Theoretical Perspectives on What Constitutes Early Algebraic Thinking

As can be discerned from reading this volume, there is no single theoretical perspective or unique definition as to what constitutes early algebraic thinking and how its development can be appropriately investigated or promoted. While little disagreement exists as to the importance of the process of generalization as a central component of algebraic thinking, the various ways in which this process is conceptualized and integrated within exemplars of the various theoretical frameworks offer a range of insights for current and future work in this area.

For Mason (Chap. 14), a pioneer in promoting the notion that the expression of generality is the core of algebraic thinking—but not necessarily requiring alphanumeric notation—the process of noticing patterns and generalizing starts at a very early age as a child begins to perceive similarities and differences in the world around her. For Blanton, who early on collaborated with another of early algebra's pioneers, Jim Kaput[†], and for the colleagues with whom she currently conducts research in this area (Chap. 2), early algebraic thinking comprises generalizing, representing, justifying, and reasoning with mathematical structure and relationships, usually involving recourse to alphanumeric notation. While both of these theoretical perspectives revolve around the idea of generalization, Radford (Chap. 1) maintains that one of them conflates algebraic thinking and generalization and the other conflates algebraic thinking with alphanumeric symbol use. He argues further that notations are neither a necessary nor a sufficient condition for algebraic thinking and that generalization is a common attribute of human thinking and cannot consequently capture the specificity of algebraic thinking. In order for the reasoning underpinning the generalizing process to be considered truly algebraic, he proposes that it must "(i) resort to indeterminate quantities and idiosyncratic or specific culturally and historically evolved modes of representing/symbolizing these indeterminate quantities and their operations, and (ii) deal with indeterminate quantities in an *analytical* manner."

Another dimension that distinguishes theoretical frameworks is the nature of the content strands that are the main focus. While the Blanton et al. (Chap. 2) framework includes both functional and generalized arithmetic content strands, the perspective developed by Carraher and Schliemann (Chap. 5) identifies the onset of algebraic thinking with the formulation of and operation upon relations, particularly functional relations. Within their framework, the basic operations of arithmetic are approached from the standpoint of functions. Students from the 3rd grade onward employ algebraic notation, as well as alternative forms of algebraic expression (e.g., linguistic, tabular, graphical, diagrammatic, etc.), to express generalizations of problem situations.

A framework within the generalized arithmetic content strand, and one that also favors guiding young students toward the use of letters in expressing generality, is that of Malara and Navarra (Chap. 3). The epistemological roots of their framework, which are strongly linked to a linguistic approach, make it quite unique among early algebra frameworks. In their framework, the two disciplines of arithmetic and algebra are seen as a combined meta-discipline with a singular unifying language. For Malara and Navarra, developing algebraic thinking involves building in students an attitude of looking for regularities, relationships, and properties, of reflecting on them metacognitively, and of expressing them first in natural, and then in algebraic, language—with "algebraic babbling" characterizing the initial expressions of early algebraic thinking and progressing to the use of fully-fledged algebraic language, which is viewed as a tool for thinking.

While different points of view have been expressed within these theoretical frameworks, in particular with respect to generalization and the introduction of alphanumeric symbols, as well as the nature of the content strands that are considered the central focus, their application to the learning and teaching of algebraic thinking—presented under Themes 3 and 4 below—yields an incredibly rich portrait of the early algebra practices of which our young students are capable and the teacher actions that support such learning.

Theme 2
Improving Attention to Structure with Primary and Lower Middle School Students

In contrast to the dominant focus on generalizing in the development of algebraic thinking, Kieran (Chap. 4) argues that such a focus has to a large extent obscured the process of seeing structure. While generalization-oriented activity remains highly important in algebra and early algebra, and in fact includes a structural component, she proposes that equal attention needs to be paid to the complementary process of *looking through mathematical objects* and to decomposing and recomposing them in various structural ways. She suggests, in the spirit of Freudenthal, that *structure* as it pertains to number and numerical operations at the primary and lower middle school levels encompasses not just the order, additive, and multiplicative structures, but also structurings according to factors, multiples, powers of

10, evens and odds, sums of 10, prime decomposition, and many more—such structurings often expressed in decomposed, uncalculated form. These structurings have properties, such as the basic properties of arithmetic, but also a multitude of other properties such as the successor property, the sum of consecutive odd numbers property, the sum of even and odd numbers property, equivalence and equality properties, and so on. Indeed, she argues that there is a dual face to activity that promotes early algebraic thinking: one face looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards "seeing through mathematical objects" and drawing out relevant structural decompositions.

Related to this stance, Mason (Chap. 14) suggests that, rather than always offering learners the particular and expecting them to generalize, consideration should be given to offering a partial generality, or a very general statement, so that "learners can make use of and develop their power to specialize as well as to generalize." He cautions that the routine practice of providing students with the first few terms of a sequence and asking for successive terms habituates them into reasoning forward, often inductively, and directs their attention away from "looking at something structurally, that is, seeking out relationships that are instances of general properties."

The notion that structure is a key idea of early algebra is also elaborated in the chapter by Steinweg et al. (Chap. 12). Based on their premise that early algebra is not a new content to add, but a content field to be identified within already taught topics, they have drawn out from the common topics of the existing primary mathematics curriculum in Germany four key ideas of algebraic thinking: (i) patterns (and structures), (ii) property structures, (iii) equivalence structures, and (iv) functional structures. The first idea contrasts pattern awareness with detecting structure, which requires mathematical knowledge about objects and operations. The second lies in the properties of numbers (e.g., parity, divisibility) and operations (e.g., commutativity, associativity, and distributivity). The third key idea relates to evaluating, preserving, or construing equivalence and focusing on the relations of given numbers, sums, differences, products, or quotients. The fourth key idea involves learning environments on functional structures, relations, and co-variation aspects.

Schifter's chapter (Chap. 13) is another one where structure is emphasized, with *structure* being defined as referring to those behaviors, characteristics, or properties that remain constant across specific instances. Relevant to the content of her chapter is that each operation has a unique set of structures. She argues that a focus on the behavior of addition, subtraction, multiplication, and division helps students come to see an operation not exclusively as a process or algorithm, but also as a mathematical object in its own right. Her research shows that "to make the operations salient objects in *all* young students' mathematical experience requires persistent effort on the part of the teacher with the use of tasks that make explicit how structures differ for each operation."

Theme 3
The Multiple Facets of Students' Early Algebraic Thinking

The two main content strands of early algebra are functions and generalized arithmetic. While these content strands provide the principal orientations of the learning environments within which students' early algebraic thinking is developed, the relative emphases given in the various studies to aspects such as symbolizing, representing, generalizing, particularizing, justifying, seeking structure, and verbalizing, as well as the range of grade levels, topics, and novelty of certain tasks, disclose a much more nuanced and multi-faceted picture of early algebraic thinking among 5- to 12-year-olds than would be suggested by the identification of these two strands alone.

With respect to the functional content strand, the research by Ng (Chap. 7) highlights an activity that has received comparatively less attention in studies of early algebraic thinking, that of the function machine. While function-machine tasks are not part of the current Singapore primary mathematics curriculum, the latter does emphasize (i) understanding of patterns, relations, and functions, and (ii) representing and analyzing mathematical situations and structures using algebraic symbols (from Grade 6 onward). In an earlier Singaporean paper-and-pencil assessment study covering a range of topics, students across the primary grades were found to have difficulties with the function-machine tasks, not really knowing how to interpret them. Ng (Chap. 7) wished to investigate further this phenomenon by interviewing students at each primary grade. After initiating students to the demands of the function-machine tasks, she found that those who had sound knowledge of their number facts for the four binary operations—supported by a variety of counting methods and conceptual knowledge of the operations—could use this knowledge to complete the function-machine questions used in her study. As will be seen below to be also the case for students taught with the Korean curriculum (Chap. 6), numerical proficiency—in combination with the contributions of a curriculum characterized by certain key features from early algebra—can play a potential role in the development of algebraic thinking.

Other research with a functional thread is presented in the chapter by Molina et al. (Chap. 11), who conducted a teaching experiment to uncover the initial understandings of first and third grade students when first introduced to the use of letters to stand for an indeterminate varying quantity in a functional relationship. The tasks they designed supported students in beginning to make meaning for variables. Their results provide further evidence that alphanumeric notation for variables is within the reach of lower primary-grade students, an argument that has been made consistently by Blanton et al. (Chap. 2) and by Carraher and Schliemann (Chap. 5), based on studies conducted within their longitudinal programs of research. Molina et al. also point out that, while the teacher will have to initiate the idea of using letters to represent indeterminate quantities, subsequent interactions can rely on the "early adopters" to propagate the use of variable notation among their classmates.

Drawing from a six-year longitudinal program of research, Radford (Chap. 1) takes us on a journey through the transition from non-symbolic to symbolic forms of algebraic thinking in pattern-generalizing activity by students as they moved from Grade 2 to Grade 6. While problems of increased difficulty were presented to the students in each successive grade, a core problem remained invariant each year: *The Tireless Ant*. By reference to this problem, and by focusing on those grades where the transition from the non-symbolic to symbolic forms of algebraic thinking actually occurred, Radford begins by describing the nature of the Grade 4 students' genuinely algebraic generalizations—even if they were not using alphanumeric symbols. The teacher's well-designed questioning helped move some of the students from a *factual* to a *contextual* form of generalization, but the challenges that other students were experiencing were evident from the lack of linguistic clarity in expressing the relationship between the variables. While the class made substantial progress toward the production of alphanumeric formulas, students' difficulties with such formulas centered not on the numbers themselves but on the operations, that is, with dealing with the operations in an unclosed form, which hampered them from moving more fully toward a *symbolic* form of generalization. Radford continues with the story of the evolution that occurred in Grade 5 and finally in Grade 6 when students began to think with algebraic symbols rather than translate language into symbols.

Also with a functional theme, but without algebraic notation, is the research reported by Twohill (Chap. 9), which involved 9- and 10-year-olds collaboratively constructing general terms for shape patterns. While shape patterning can be approached with or without an explicit connection to functions, neither shape patterning nor functional thinking is included in the Irish primary mathematics curriculum. The students in the Twohill study were encouraged to use *explicit* rather than *recursive* means for generating additional members of the pattern sequences, which most were able to do without too much difficulty. According to Twohill, however, it was not her prompts that supported the students' progress as much as it was the use of concrete materials in constructing the pattern terms and the students' interactions among themselves. Several other chapters in this volume point to the importance of the use of manipulable objects (which may be diagrams, symbols, or material objects) and a variety of representations in order to recognize, use, and express structural relationships, as well as to the role of students' interactions with fellow students, and with their teachers, in the development of early algebraic thinking.

Moving to the content strand of generalized arithmetic, the research reported by Schwarzkopf et al. (Chap. 8) deals with algebraic understanding of equalities. In one of the examples they present involving pairs of numbers, two 2nd grade students justify the associative law by means of what the authors refer to as the *constancy principle of the sum*: "Adding one to the first number and taking away 'what you have added' from the second number leads to 'the same problem'." The authors note that this nice example of early algebraic thinking occurred without any written equal sign or equation notation, but in a context of collective argumentation. They provide additional examples involving collective argumentation and with a focus not on the numbers, the calculation, or the result, but on the construction of arithmetical operations as mathematical objects.

The development of algebraic thinking described in the chapter by Kieran (Chap. 4) involved seeking, using, and expressing structure in activity involving multiplication, division, multiples, and divisors. Three classes of 12-year-olds, who worked throughout one week on tasks related to the *Five Steps to Zero* problem, were observed as they generated multiple structural decompositions of the numbers they were given in their problem-solving activity. The structural awareness that emerged involved variants of the *division algorithm* and the development of a structural eye for the multiplicative decomposition of number. A related structural eye was developed in the students who participated in the research carried out by Malara and Navarra (Chap. 3). These researchers cite the example of a 5th grade class that was given the task to represent in mathematical language the statement: *The double of the sum of 5 and its successive number*. As soon as the pupils' proposals were written on the board, a student, Diana, stepped forward to justify her writing and to critique what was written by another student: "*Filippo has written* $2 \times (5 + 6)$*, and it is correct. But I have written* $2 \times (5 + 5 + 1)$ *because this way it is more evident that the number following* 5 *is bigger by a unit*." Malara and Navarra point out that Diana was explaining how her translation was clearer and more transparent because it considered the relationship between a number and its successor.

Another recent study within the strand of generalized arithmetic is the research on generalizing fractional structures reported by Pearn and Stephens (Chap. 10). Noting the importance of fractional understanding in algebraic equation solving, they point out that, while the Australian primary mathematics curriculum does include a focus in 6th grade on finding fractional parts of a known whole, at no stage does it direct the attention of teachers to finding the whole when given a known fractional part. In their main study involving Grades 5 and 6 students, and by means of a paper-and-pencil assessment and subsequent interviews, Pearn and Stephens found that students who were able to find an unknown whole regardless of the particular fractions or quantities used were also able to express their reasoning with mathematical terms that conveyed generalizable meaning. The researchers argue for three important overarching ideas in identifying algebraic thinking, even when younger students are not able to use symbolic notation: equivalence, transformation using equivalence, and the use of generalizable methods.

We conclude this theme with some findings related to the longer-term benefits of the primary-school development of algebraic thinking. Carraher and Schliemann (Chap. 5) followed up a group of students from their 3rd and 4th grade intervention studies and assessed these students when they were in 7th and 8th grades on a variety of topics, including solving linear equations, solving verbal problems by representing them as equations, solving the equations, interpreting the results of their solutions, and representing graphically non-linear functions. The assessment was also given to a group of students from the same geographic area and grade levels. The early algebra students performed better than the control group on all of the problem types. Similar results were found when they carried out an additional study involving a one-week algebra summer camp for 12- and 14-year-olds, half of whom had participated in their earlier intervention program at primary school. Once again, the early algebra students performed better than the control group on the assessment items, before and after summer camp, and the difference between the two groups increased after participation in camp lessons.

Theme 4
Pivotal Roles of the Teacher and the Curriculum

In the concluding remarks of their chapter, Carraher and Schliemann (Chap. 5) draw attention to the crucial role of the teacher in the development of students' early algebraic thinking:

> In our earlier publications about early algebraic thinking, we placed considerable emphasis upon the achievements of the students, so much so that we may have understated the critical roles of the teachers. Although children may be capable of learning algebra from an early age, realizing this potential is not a simple matter of unleashing their capabilities. Algebra draws on ways of reasoning, kinds of problem situations, and systems of representation (notation, graphs, number line diagrams, certain ways of formulating relations in spoken language) that a child will generally not learn about, much less invent, on her own. The mathematics teacher and, to a lesser extent, the student's peers, play a vital role. The skeptic need merely imagine how much students would have learned had they been given written versions of the tasks and instructed to solve them on their own, without further discussion with and guidance from the instructor. (Carraher and Schliemann, Chap. 5)

The "further discussion with and guidance from the instructor" is emphasized in considerable detail in, among others, Chap. 3 (Malarra and Navarra), Chap. 13 (Schifter), Chap. 14 (Mason), and Chap. 15 (Strachota et al.). But a study reported by Pang and Kim (Chap. 6) suggests that an appropriate curriculum is also essential for the cultivation of early algebraic thinking.

In Korea, the primary mathematics curriculum, which is rooted in a generalized arithmetic perspective, does not include mention of early algebraic thinking; however, attention to the equal sign and the use of variable notation without letter symbols is found in the textbooks as early as the first grade, with alphanumeric variables being introduced to represent the relationships between varying quantities from the 4th grade onward. Pang and Kim (Chap. 6) report two studies that assessed students' early algebraic thinking within the existing curriculum, one study (Grades 2–6) dealing with the equal sign, expressions, and equations, and the other study (Grade 3) with a variety of early algebra topics that were included in one of Blanton et al.'s (Chap. 2) assessments (see the Blanton et al. 2015b study cited in Chap. 2), topics such as equivalence, functional thinking, variables, and generalized arithmetic. In both Korean studies, the students scored very well, with their computational proficiency often serving as a basis for their relational understanding.

An additional finding of great interest is drawn from the second Pang and Kim study that showed that the Grade 3 Korean students performed just about as well as, or slightly better than, the Grade 3 (USA) intervention group in the Blanton et al. study and much better than the Grade 3 (USA) non-intervention group that had not experienced the curricular progression designed by the Blanton team (see Table 6.6 in Chap. 6). The nearly equivalent performance on the Blanton et al. assessment by the Grade 3 Korean students and by the Grade 3 intervention group of the Blanton et al. study suggests that it is quite feasible for national curricula to embed features that allow these curricula to compare well with the curricular progressions designed and developed within longitudinal programs of research with respect to the cultivation of early algebraic thinking. Pang and Kim argue that "new content areas are not necessarily needed to induce early algebraic thinking and to make it accessible to students; early algebraic thinking can instead be fostered as a specific form of thinking while students learn typical content areas." However, one would likely not be wrong in inferring from Pang and Kim's findings that it is not just any existing curriculum that can support such learning; certain key features would seem to be necessary, as well as competent teachers who can interpret the, sometimes hidden, intent of that curriculum. It is perhaps a sign of the relative newness of early algebra as a field of study that the worldwide emergence of research in this area has yet to be reflected in the equally worldwide creation of national curricula that promote early algebraic thinking.

An alternative point of view is expressed by Steinweg et al. (Chap. 12), who propose that, despite the lack of any early algebraic tradition in German primary school mathematics, the cultural characteristics of teachers' attitudes and beliefs about teaching, the nature of everyday school life in mathematics classes, and teachers' pedagogical approaches that include the expectation that students communicate and argue about mathematical findings, provide favorable prerequisites for developing early algebraic thinking. They suggest that the contents of the existing curriculum have the potential to address algebraic thinking if approached from a new perspective. The existing curriculum, which is structured around content and process standards, includes for the 4th grade (when primary schooling ends, except for Berlin) the following content: number and operation, space and shape, patterns and structures, magnitude and measurement, and data, frequency and probability; and the following processes: problem solving, communicating, arguing, modeling, and representing. To support their proposal, Steinweg et al. focus on three aspects that they consider would help develop teachers' awareness of the algebraic nature of what is already being taught: the structural elements of primary school mathematics, the existing ability of primary level students to think algebraically, and the manner in which tasks can be designed so as to promote algebraic thinking even in very young learners. The next step, which elaborates how teachers might become aware of the early algebraic perspective that Steinweg et al. propose, remains, however, to be developed.

Mason (Chap. 14) states that the critical feature for promoting algebraic thinking is the opportunities noticed by teachers for calling upon learners' powers to express and manipulate generalities. He proposes that teaching by listening is far more effective than trying to push students and then assuming that they have made appropriate sense of what has been said. According to Mason, teaching by listening involves putting learners in situations where they naturally ask questions, but it also involves developing possible pedagogical actions arising from what is noticed. As brought out by Strachota et al. (Chap. 15), who focused on the generalization-promoting actions of a Grade 3 teacher, the ways in which a teacher *responds to students' generalizations* are crucial for generalizations to become platforms for further generalizations. For example, when one of the students in her 3rd grade class stated a generalization, the teacher encouraged others to revoice the generalization. This public sharing of the generalization led to the teacher requesting clarification and justification. Another student subsequently introduced a new, but related, idea, followed by further questioning on the part of the teacher. This continued the generalization cycle, which yielded additional generalizations, clarifications, and justifications.

However, the action of "responding to generalizations" seems to be one that can be difficult for teachers to learn. Mc Auliffe and Vermeulen (Chap. 17) reported a study on the preparation of preservice teachers to teach early algebra. The results, based on the written reflections and video-recorded lessons of the practicum experience of two preservice teachers (Grades 2 and 7), highlight the issues involved in applying specialized content knowledge to teach functional thinking. Despite their coursework addressing both theory and empirical research related to early algebra, both of the case-study preservice teachers experienced difficulties: the Grade 2 teacher in bringing the students to the point of generalizing the relationships of the problem situation, and the Grade 7 teacher in bringing the students to connect the structure of the figural pattern and the function rule, and to justify their thinking and establish generalizations through the discourse of argumentation. By means of the two case studies, Mc Auliffe and Vermeulen draw out the challenges faced by preservice teachers in learning to ask probing and stimulating questions. They point to the need in preservice teacher education for greater emphasis on generalizing and on how to apply such knowledge in practice. They suggest that further research in this area could help to determine the kinds of learning opportunities that would effectively help preservice teachers to develop the knowledge they need to teach early algebra.

Some suggestions in this regard can be gathered from research in early algebra professional development. While it can be argued that professional development offers different, and perhaps more, opportunities than does preservice education for adapting one's teaching to the cultivation of early algebraic thinking, the issue of asking probing questions has been a focus of several studies. For example, Malara and Navarra (Chap. 3) have conceived and shared with teachers a set of glossaries that concern the theoretical frame, the mathematical topics, the methodological-didactic and social issues, and the linguistic demands related to the managing of discussions with the class. They also make use of what they call *Multicommented Transcripts (MTs)*, wherein teachers transcribe meaningful classroom episodes and critically analyze their didactic interventions, in particular their manner of questioning and of responding to students' verbalizations, before sending the transcripts off to the course mentors who add their own comments. Schifter (Chap. 13) describes the tool of the *Student Thinking Assignments* given over the course of a school year that suggests questions that teachers could pose to their class. However, the issues involved in shifting one's teaching practice toward the development of early algebraic thinking are considerably more complex and clearly go beyond expanding the ability to ask probing questions and to listen and react appropriately to students' responses, as is seen in Chap. 16.

The chapter by Hunter et al. (Chap. 16) describes the year-long evolution that took place within a 3rd grade teacher as she broadened her own perspective on algebra, coming to view the facilitation of early algebraic thinking as more than just dealing with content, rather as also engaging students in mathematical practices related to conjecturing, justifying, and generalizing. While the influences of the external professional-development expert and the community of teacher learners in her group were significant for this teacher, her sustained active inquiry into her own practice and the development of her own personal framework of teacher actions were key features in warranting her efforts to change practice.

Summing Up

This concluding section of the volume has attempted to highlight a sampling of the central contributions offered within its 17 chapters. While a careful reading of each chapter offers much more nuanced detail with respect to the development of early algebraic thinking than could be encapsulated above, it is nevertheless hoped that the synthesis has been able to capture some of the most recent and exciting features of the newly emerging and rapidly evolving field of research and practice in early algebra. Its main features can be summed up as follows:

- A view of early algebra that includes not just an adapted version of the time-honored initial algebraic topics of creating meaning for the equal sign and equality, variables, and expressions and equations, but also (and especially important) a focus on relations, patterns, and structures in numbers, operations, and functions, and on the mathematical processes of seeking structure, conjecturing, generalizing, and justifying.
- An identification of early algebraic thinking as reasoning that expresses itself as statements or other representations denoting structural relations among numbers, quantities, operations, and patterns.
- An increased emphasis on structuring activity in early algebra, including detecting structure within numbers, numerical operations, patterns, and functions (i.e., *seeing through mathematical objects*), with attention to the properties of numbers (e.g., parity, divisibility) and operations (e.g., commutativity, associativity, and distributivity), as well as to equivalence expressed through decomposition, recomposition, and substitution—an emphasis suggesting that there is a dual face to activity that promotes early algebraic thinking: one face

looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards "seeing through mathematical objects" and drawing out relevant structural decompositions.

- Characterizing the diverse ways in which structure is expressed by students who are developing algebraic thinking in the different content areas of early algebra.
- The need for creating, at the national levels, mathematics curricula for Grades 1–6 that include key features of early algebra and that can thereby contribute to the development of early algebraic thinking.
- The critical role played by teachers in cultivating early algebraic thinking by means of actions that include, among others, careful task design and enactment, the prompting of structuring and generalizing activity, and listening, noticing, questioning, and responding appropriately to students' efforts in such activity.
- The need in preservice teacher education for greater emphasis on generalizing and on how to apply such knowledge in practice.
- The elaboration of key measures within professional development programs regarding the pedagogical actions that facilitate early algebraic thinking.
- Identification of the possible role played by the interaction between numerical proficiency and early algebraic thinking in students.
- The introduction of alphanumeric symbols as early as the 1st grade of primary schooling versus their introduction later in the 3rd, 4th, or 6th grades.
- The importance of the use of manipulable objects (which may be diagrams, symbols, or material objects) and various representations in order to recognize, use, and express structural relationships.
- The role of student interactions with both fellow students and teachers in the development of algebraic thinking.
- The longer-term benefits of developing early algebraic thinking in primary school.
- Provision by researchers of adequate information on the nature of the curricular framework and the supporting teaching actions when reporting their studies on the development of early algebraic thinking.

Future research addressing the above issues, as well as the additional suggested questions formulated within the body of each individual chapter of this volume, will serve to enhance further our understanding of how to support the development of early algebraic thinking in 5- to 12-year-old students around the world. The present volume contributes to this growing field of knowledge with the evidence it presents regarding the teaching and learning of algebraic thinking within early algebra from various theoretical perspectives and in different contexts and cultures.

Author Index

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Subject Index

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