Chapter 4
Pre-service Elementary Teachers' Generation of Multiple Representations to Word Problems Involving Proportions

Ryan D. Fox

Abstract How well can pre-service elementary teachers answer word problems? Furthermore, can they represent the same answer in multiple ways? To answer these questions, I conducted a study with four pre-service elementary teachers answering word problems that incorporate proportional reasoning to investigate the strengths and opportunities for growth. I found three pre-service elementary teachers generated different representations of the correct solution: writing proportions to solve by cross-multiplication, drawing pictures to solve by repeated addition, and creating tables to solve by percents. One pre-service elementary teacher did struggle to produce effective strategies to solve some of the presented word problems.

Keywords Pre-service elementary teachers · Mathematical knowledge for teaching

For the past 30 years, education researchers have used various terms to identify and explicate a specialized knowledge for teaching mathematics: starting with Shulman's (1986) Pedagogical Content Knowledge and continuing with Mathematical Knowledge for Teaching (Ball et al. 2008), Profound Understanding of Fundamental Mathematics (Ma 1999), and Mathematics for Teaching (Davis and Simmt 2006). For pre-service elementary teachers (PSETs), the search for such a specialized knowledge base feels more elusive and challenging. Mathematics teacher educators deal with the intertwined issues of PSETs' lack of confidence regarding mathematics and length of time away from mathematics classes (Goulding et al. 2002). Without proper content knowledge and confidence in the subject, teaching mathematics well to elementary students becomes a problem of worldwide significance (Vula and Kingji-Kastrati, this volume; Shaughnessy and Boerst, this volume; Lin and Hsu, this volume; Pilous et al., this volume). Yet, mathematics teacher educators persist. To support the mathematical development of PSETs,

R.D. Fox (✉)
Belmont University, 1900 Belmont Blvd, Nashville, TN 37212, USA
e-mail: ryan.fox@belmont.edu

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mathematics teacher educators engage in programs of teaching and research to address this important issue. This chapter discusses one small step toward this goal.

The research question guiding this study is: how can a pre-service elementary teacher generate multiple representations of a solution to apply her knowledge of proportional reasoning to a sequence of contextual problems? This study addresses the research question by examining the quantity and quality of representations the PSET generates. Three of the four participants provided responses of similar quality and quantity. The fourth participant struggled to produce the same quantity and quality.

4.1 Relevant Literature

As an activity for students, generating multiple solution paths to a single question is consistent with procedures with connections, a high-level cognitive demand task within Stein et al.'s (2000) framework. In their explanations to procedures with connections tasks, Stein and colleagues suggest multiple representations use "visual diagrams, manipulatives, [and] symbols…[to make] connections among multiple representations…to develop meaning" (p. 16). However, in previous works, researchers have shown PSETs struggle to demonstrate this desirable activity. Depaepe et al. (2015) showed PSETs struggled to answer questions correctly involving fractions. Because of their content struggles, PSETs could not access the appropriate pedagogical content knowledge to support students' different representations to solutions to test questions.

Researchers have shown the progress PSETs made to develop stronger mathematical knowledge for teaching. Baek et al. (2017) found PSETs could generate many representations, particularly using pictures, to word problems that involved fractions. However, not all PSETs could answer questions correctly. Although PSETs made progress in performing a valuable activity, they often struggled with challenging mathematical content: coordinating multiple units within a single question. Stohlmann et al. (2015) started with PSETs who did not possess conceptual understanding of topics from the elementary curriculum. After a course focusing on multiple representations, PSETs changed their beliefs on teaching to include an emphasis on conceptual understanding and meaning making.

Turner and Rowland's (2011) work on the Knowledge Quartet can describe the nature of a specialized knowledge of teaching elementary mathematics. As Turner and Rowland mentioned, "[while] we believe certain kinds of knowledge to be desirable for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher…*ought* to know" (p. 197, emphasis in original). Their work is an extension of the work of Rowland et al. (2005). The earlier study examined the application of university students' mathematical knowledge developed from their teacher preparation program to their clinical experiences at the end of their preparation program. The four categories they elaborated were foundations, transformations, connections, and contingencies. Turner and Rowland (2011) defined foundations as pre-service teachers' "knowledge [and] understanding… in preparation (intentionally or otherwise) for their role in the classroom" (p. 200). This component of the quartet is the only one not defined in terms of the practice of teaching. Instead, foundational knowledge is a collection of networks of information a teacher develops before his or her own teaching career begins. Foundational knowledge is generally knowledge PSETs acquired before they begin their teacher preparation program. In order to develop multiple solution paths, PSETs need access to a single solution path. Answering a question correctly from the elementary mathematics curriculum would comprise part of a PSET's foundational knowledge.

As PSETs begin their teacher preparation program, they develop the next component of the Knowledge Quartet, transformation. Rowland et al. (2005) describe transformation knowledge as a "focus on knowledge-in-action as *demonstrated* in planning to teach and in the act of teaching itself" (p. 261, emphasis in original). Transformational knowledge could be developed through the generation of multiple representations to a solution. As school students, PSETs solved many word problems that involve setting up a proportion with an unknown quantity and determining the value of the unknown. Such work would be classified as foundational knowledge. Transformational knowledge could involve explaining other connections between quantities in the proportion or providing illustrations to encourage students to visualize the quantities involved in the proportion.

In my previous works (Fox 2012, 2013), I examined how two pre-service elementary teachers solved word problems involving proportions. The two participants, Stephanie and Hope, had two contrasting approaches in the solutions to the problems. My original intention in selecting participants was to have Stephanie and Hope serve as opposing ends of performance on these word problems, with other participants fitting somewhere in between. Stephanie took a rather consistent approach to solving the problems (Fox 2012). In her desire to be as "clear" as possible, Stephanie repeated a three-step algorithm as a solution to each question: re-writing of the scale from the problem, a sub-division of the number line segment into the appropriate number of parts, and adding the wholes and parts to get to the final correct answer. Hope's approaches were less consistent than Stephanie's (Fox 2013). Hope wanted to find an approach that could answer all word problems. However, when the numbers involved in the problems changed—from whole numbers to fractions to mixed numbers—Hope's attempts did not transition well. She did develop an algorithm to getting the right answer in working with mixed numbers. She could not reproduce the algorithm when reflecting on her work in a later interview. In this chapter, I want to outline the work of two other participants and find connections across participants.

4.2 Method

This study involves the same method as outlined in previous reports (Fox 2012, 2013). All four participants in this study are PSETs. All four participants were in their second year of their undergraduate careers when they participated in the study. All had completed the university's one required mathematics content course required for their teacher preparation programs. None of the participants had taken a mathematics methods (pedagogy) course.

I interviewed each participant four times. The four-interview sequence was a modification of an interview sequence suggested by Seidman (2006). In detailing a three-interview sequence that could be applied to all social sciences, Seidman (2006) mentioned key features of each interview:

> The first interview established the context of the participants' experience. The second allows participants to reconstruct the details of their experience within the context in which it occurs. And the third encourages the participants to reflect on the meaning their experience holds for them. (p. 17)

Four participants completed my four-interview sequence. In Interview 1, I asked about the participant's background in mathematics and desire to teach elementary school. I concluded the interview with five word problems. After each word problem was a question for reflection. I asked follow-up questions to probe for additional information from the participant's reflection. I asked the participants ten word problems each in Interviews 2 and 3. Together, these interviews extend Seidman's middle interview into two separate interviews: each interview involved different details of mathematical content. In Interview 4, each participant reflected on her experiences in the Interviews 1, 2, and 3. I posed no word problems in Interview 4. Because the focus of this chapter is on the participants' mathematics, data from Interview 4 is not included in this chapter.

The four interviews in this study satisfy Wilson's (2013) definition of semi-structured interviews: "[t]he semi-structured interview…allows some standardization of questions and also the freedom to explore and add new questions as unexpected topics emerge" (p. 41). I asked each participant the same word problems. When having the participant explain her response, I would break away from the interview guide to explore why the participant wrote down what she did. The participant's responses to my reflection question led to additional questions that I could not foresee asking before the interview.

In this study I used two categories of word problems, which are described in Table 4.1. During Interview 1, I asked three Road Map and two Floor Plan word problems. For all five word problems, the number of miles and the number of feet are whole numbers. Across the twenty word problems in Interviews 2 and 3, I asked ten Road Map and ten Floor Plan word problems. Five Road Map word problems contain miles represented as a fraction between 0 and 1. Five Floor Plan word problems contained feet represented as a fraction between 0 and 1. Five Road Map word problems contained a number of miles written as a mixed number; five Floor Plan word problems contained a number of feet written as a mixed number.

Table 4.1 Word problems given to each participant throughout the first three interviews

Category	Stem of word problem	Total number of questions asked
Road Map	Let's say that I am looking at a map and the map has printed on it, "1 inch = ___ miles". How far apart are two towns if they are ___ inches apart on the map? How do you know your answer is right?	13
Floor Plan	If I had a drawing of a floor plan of a house, and the plan has printed on it, "1 inch = ___ feet". How wide is a [room] if the [room] is ___ inches wide on the plan? How do know your answer is right?	12

The word problems posed in this study would satisfy Crespo's (2003) introduction regarding mathematical tasks: "Even the most routine of mathematical activities can be constructed into a worthwhile mathematical experience when posed in such a way as to engage students in mathematical inquiry" (p. 244).

I recorded all interviews using audio and video recording devices. The video recording devices captured the written responses of the participant; the audio recording devices captured the discussion between each participant and me. During the interviews, I took field notes to capture my initial impressions of the participants' responses and to assist with later analyses.

To provide additional analysis of the participants' work, I developed ternary diagrams to map the performance of each of the four participants. I reviewed each of the participant's written responses to the 25 word problems. I coded a participant's response to each word problem using one of three codes: without a correct response, a correct response with a new representation, or a correct response with a previously used representation. I placed a participant's distribution of codes on the same ternary diagram to determine if any differences existed in the rate of codes applied across the four participants.

4.3 Results

In this section, I begin by providing a summary of results from Fox (2012, 2013) for Stephanie and Hope. I then provide more detailed results from two other participants, Brooklynne and Arielle. I selected the third and fourth participants as representatives of most PSETs' performance in the same mathematics course.

4.3.1 Stephanie

Stephanie's answers (Fox 2012), revealed the same process used for responses to word problems with proper fractions and mixed numbers in Interviews 2 and 3.

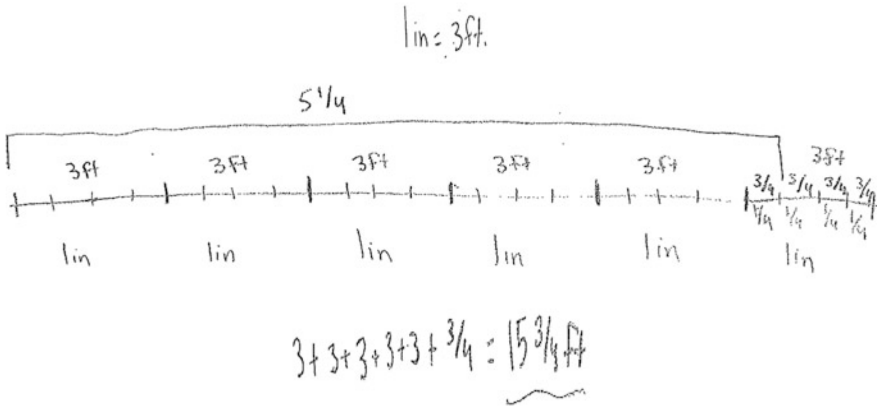

Fig. 4.1 Stephanie's representations of solutions (Fox 2012)

Each response always included a re-writing of the scale, the use of a number line to represent both the number of inches and the number of corresponding number of feet or miles, depending if the word problem was from either the Road Map or Floor Plan category. When using fractions, Stephanie represented both the numerator and denominator as number of pieces on the number line. An example of Stephanie's work on a Floor Plan word problem from Interview 3 can be found in Fig. 4.1.

4.3.2 Hope

Two themes in Hope's work are struggle and success (Fox 2013). In her written work, Hope did not provide correct final answers to six of the 25 word problems: one during Interview 1 and five during Interview 2. During Interview 3, Hope found success by repeating one algorithm that worked for word problems with mixed numbers. An example of Hope's success can be seen in Fig. 4.2, a Floor Plan question with 1 in. = 3 ft and the room being 6 2/5 in. wide.

4.3.3 Brooklynne

Brooklynne provided correct answers to her written responses for all 25 questions. In Interview 1, Brooklynne correctly answered the word problems, and explained answers as if she were thinking of her future students. Brooklynne wrote a paragraph explanation for each reflection. An example from Interview 1 can be seen in Fig. 4.3.

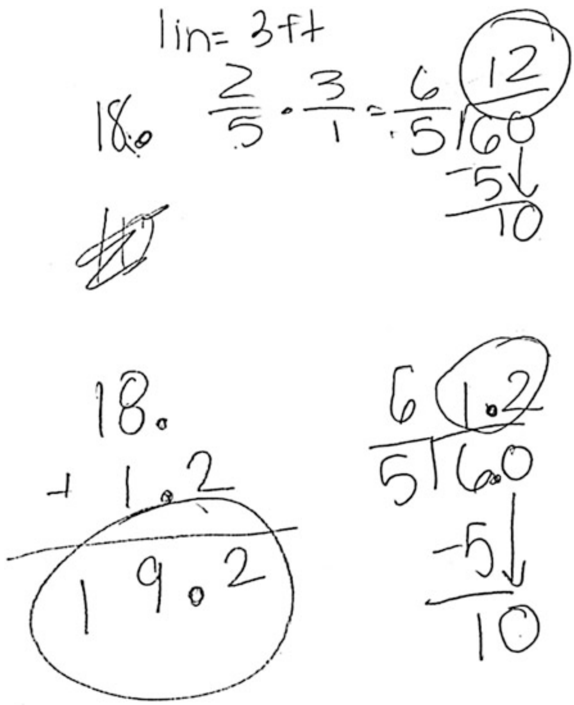

Fig. 4.2 Hope's successful repetition of an algorithm to answer problems in Interview 3

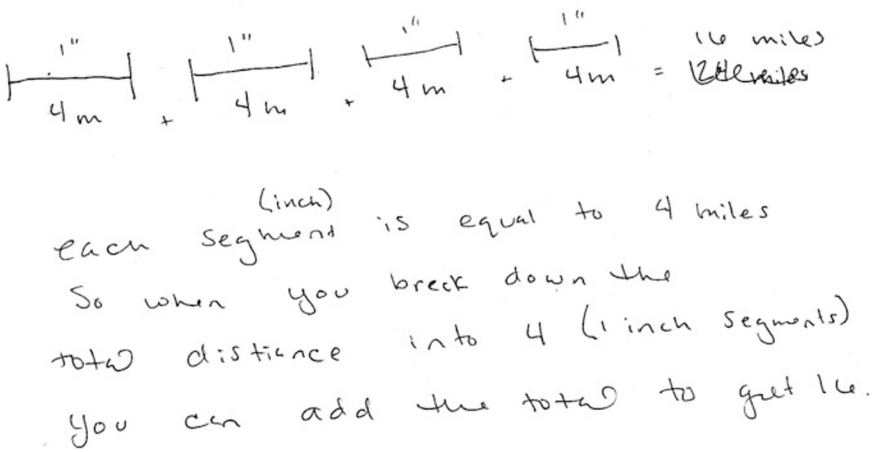

Fig. 4.3 Brooklynne's written reflection to a Road Map word problem in Interview 1

However, Brooklynne did not provide the same picture and written explanation strategy to all word problems in this interview. Her initial solution to the first word problem (a Road Map question filling in the blanks with 1 in. = 1 mile and 6 in. apart) was to provide a solution as if she was explaining the solution to a peer: set up a proportion and use cross-multiplication to determine the unknown value. Brooklynne acknowledged drawing pictures to represent the solution could be a more desirable alternate to the solution for younger students than the one she had:

> I think you could easily draw this out on a board….So you can draw one inch equals one mile…and then you can add them altogether and say six miles.

In Interview 2, Brooklynne represented whole-number multiplication as the repeated addition of whole numbers. In a Road Map word problem, Brooklynne correctly identified how using the scale, 1 in. = 4 miles, can be used to find the distance between two towns if they are only one-fifth of an inch apart: divide one unit by five to represent one fifth of an inch, and then do likewise for each picture representing the four miles. Brooklynne drew rectangles to represent the solution involving word problems from the Floor Plan category, as seen in Fig. 4.4. When I probed to ask why she drew two-dimensional pictures for floor plans, instead of straight-line distances between cities on a map, Brooklynne said:

> I think because this [question] is saying it's wide. It's like…a two-dimensional measurement, around….Like, wide, or length, or something like that.

In Interview 3, Brooklynne used different strategies to facilitate the computations in her solutions. In Fig. 4.5, Brooklynne used repeated addition to multiply four and three quarters by four. When determining the sum of three quarters four times, Brooklynne added two groups of partial sums to get three. She added that answer to four fours to get the final answer of nineteen.

In a reflection to a different Road Map word problem, Brooklynne said:

> I did the same thing for this one, but, I just used point two five instead of, um, point five. Because it was a fourth instead of a half. And then I just did the same thing where I added them all up, and they equaled one. And then I added all threes to equal twelve. And then I got thirteen.

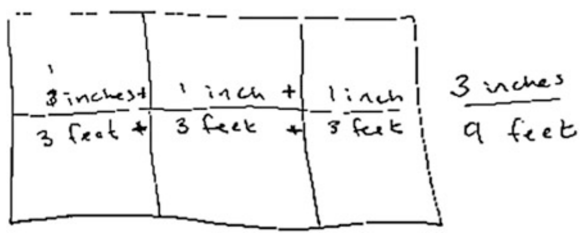

Fig. 4.4 One example of Brooklynne's rectangles to represent a Floor Map word problem

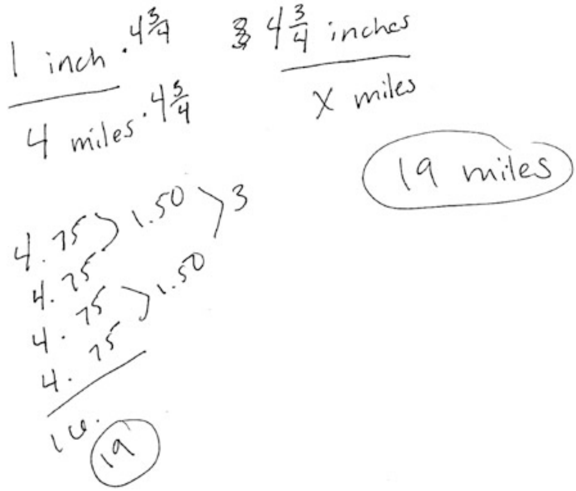

Fig. 4.5 A collection of Brooklynne's solutions and representations

4.3.4 *Arielle*

Across the first three interviews, Arielle provided the correct answers to all 25 word problems. Figure 4.6 includes examples of Arielle's representations for the final answer: proportions, repetitions of the scale value, and a table of values. In the figure, the proportions is for a Road Map word problem in Interview 1. The repetition of the scale value is for a Floor Plan word problem in Interview 3. The table of values is for a Road Map word problem in Interview 3. Arielle referenced her use of three different approaches for one solution to a Road Map question in Interview 2:

> First I drew out….So, half of one is a half. So half of four is two. So that's how I kind of saw it right away. But I still drew the picture. And I still wrote out what I thought down there when I just saw at first. Like, one divided by a half equals a half. Four divided by a half is two. And then I did the cross multiply. So I kind of checked it three times.

For the final word problem in Interview 3 (a Floor Plan question with 1 in. = 3 ft and the room being 6 2/5 in. wide), Arielle noticed that, for the same scale of floor plans, a larger number of inches in the floor plan corresponded to a larger number of feet on a floor plan. The previous Floor Plan word problem had the same scale, but used a drawing that was 2 2/5 in. wide. Arielle reflected on her use of number sense in her reflection to this word problem:

> Six and two fifths is at least double two and two fifths. So, I knew it had to be at least greater than sixteen.

Arielle saw patterns from previous questions that would help her in answering later questions. Additionally, Arielle noticed my convenient choice of numbers to get whole number answers. During Interview 3, Arielle commented:

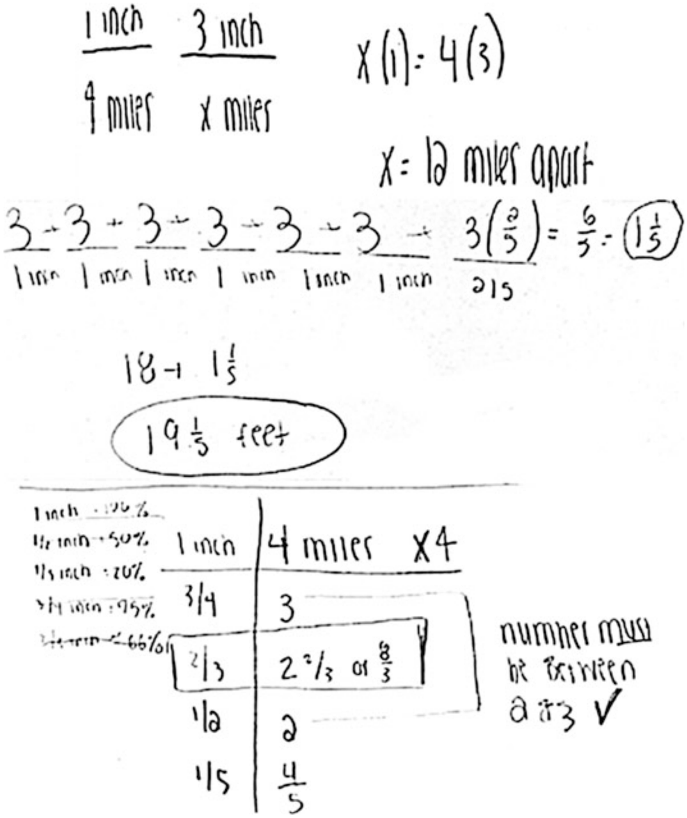

Fig. 4.6 Collection of Arielle's representations of solutions

> So, what's three fourths of four? So. I got the twelve over four. And I divided that. And It's three miles. And at first I was looking at it like why is it coming out so evenly? And then I went back and looked and I was like, wait. It's four and four. Like three fourths. One. Two. Three. Four. I should have just one, two, three miles.

4.4 Discussion

In this section, I will discuss the work of the four participants together and the potential for future work. Because these PSETs did not have much interaction with the topics of fraction multiplication and proportions recently, I believe some participants would struggle generating a single—let alone generating multiple—representations of the solution to the problems. Hope's work seemed to fit that belief. Brooklynne's work showed more correct answers and more representations of those correct answers than what Hope's work showed. Stephanie's work involved more representations than Brooklynne's work. Arielle's work involved more representations than any other participant.

4.4.1 Examining the Work of All Four Participants

For all participants, the knowledge employed to answer all questions is consistent with the codes found in the foundations component of the Knowledge Quartet (Turner and Rowland 2011): "overt subject knowledge" and "reliance on procedures" (p. 200). Three of the participants also demonstrated an activity found in the transformations component: "choice of representation" (p. 200). Arielle's reference connecting the numbers used in one word problem to numbers used in prior word problems could be consistent with Turner and Rowland's code for connections, "decisions about sequencing" (p. 201). As other researchers in this volume (e.g., Lajoie, this volume) determined, the contingency component of the Knowledge Quartet depends on the strength of the other three components of the Quartet.

Stephanie and Arielle provided three representations to a single solution. They differed in how they used the three representations. In most solutions, Arielle used the same strategy as Stephanie of representing the length as repetitions of the given scale. Stephanie's three steps followed the following sequence: re-write the scale, draw the picture to scale, then add up the corresponding values. Arielle's three steps seemed to inform each other: the picture, the chart, and the proportion all provide different contexts toward the same answer. In Interview 2, Arielle reflected on helping her younger brother with mathematics homework. She provided an interesting insight into how her preparation to become a teacher presented itself when helping her brother:

> And I explain things to him [in] so many different ways. And he sometimes gets frustrated because I'm talking to him like he is in second grade. But I am not doing it on purpose. It's just kind of like the classes I am taking.

Stephanie's approach is to create the one best explanation that would support as many students as possible in a single explanation. Arielle wants to support the mathematical development of as many students as possible by presenting different approaches and encouraging students to use the one approach that they would want to implement. Both PSETs exhibit components of a transformational knowledge: behavior "directed towards a pupil (or a group of pupils)…which follows from deliberation and judgement informed by foundation knowledge" (Turner and Rowland 2011, p. 201).

Table 4.2 provides the distribution of word problems that involved correct representations using new representations, correct responses using previously used representations, or without a correct representation. Because each participant attempted all 25 problems, the sum of each row will be 25.

Table 4.2 Distribution of responses by participant across three categories

Participant	Number of questions without a correct response	Number of questions with a correct response but a previously used representation	Number of questions with a correct response and a new representation
Brooklynne	0	16	9
Stephanie	0	17	8
Hope	6	14	5
Arielle	0	14	11

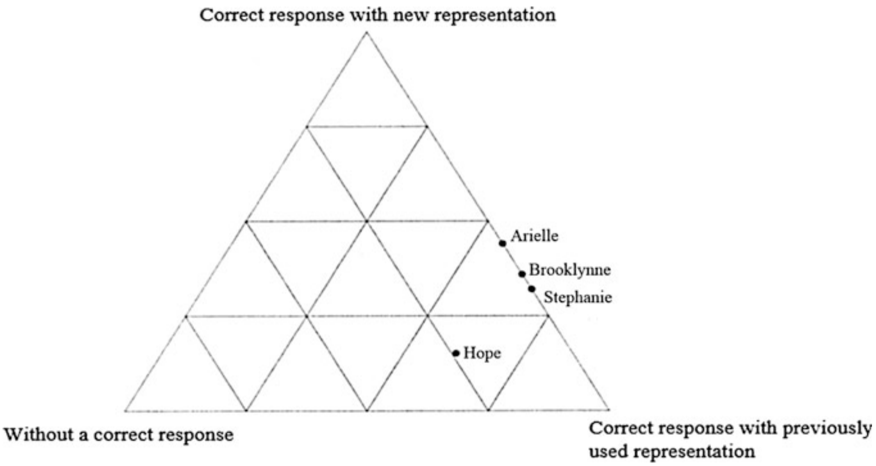

Fig. 4.7 Illustrating distribution of participants' responses in a ternary diagram

In Fig. 4.7, I placed the data from Table 4.2 on a ternary diagram. Placement of a participant's dot on the vertex of the triangle indicates all of a participant's responses received the same code. Each segment on the interior of the triangle represents one-quarter of the responses assigned that code. For example, placement of a participant's dot on the horizontal line closest to the vertex represents 75% of word problems attempted by a participant receiving the code correct response with a new representation. The next horizontal line going down represents 50% of word problems attempted by a participant receiving the category correct response with a new representation. The horizontal line farthest away from the vertex represents 25% of word problems attempted by a participant receiving the category correct response with a new representation. If a participant's collection of responses did not receive codes from a category, then the participant's placement would be the side of the triangle opposite of the vertex with that code. Because no response received the code without a correct response, three participants' locations on the diagram are on the side of the triangle opposite from the vertex representing the category without a correct response.

Based on their performances in their content course, I conjectured Brooklynne and Arielle would be in a similar location in this diagram, with Stephanie using a greater number of representations and Hope far fewer representations than these two. Figure 4.7 shows three participants' results are closely connected: Hope's location on the diagram is removed from the other three. Hope did not provide correct written responses to 24% of the word problems across Interviews 1, 2, and 3. As a result, her placement on the diagram is closest to the segment representing 25% of the word problems receiving the code without a correct response. Placement of the three participants in the diagram between the 50 and 75% segments of all questions being correct but using previous representation suggests to me that I may have made an artificial distinction between the work of Brooklynne and the pairing of Stephanie and Arielle. The dots for these three participants are on the same side of the triangle and in between the same endpoints for the correct response with representation code. Brooklynne, Stephanie, and Arielle created approximately the same number of representations in their written work.

4.4.2 Extending the Study

Although this study examined a small number of participants, the results here encourage discussions on the work of mathematics educators in preparing future elementary teachers. For example, in what ways can mathematics educators dispel common mathematical misconceptions held by PSETs? In Brooklynne's Interview 1, she connected whole number multiplication to repeated addition. She said, upon reflection, this was an effective strategy in explaining her solutions. In Interview 2, Brooklynne's comment about using division for smaller values—going, for example, from one to one-half—creates effective solution strategies for this particular word problem. However, do these reflections perpetuate mathematical myths that division makes the quotient smaller and multiplication is equivalent to repeated addition? Brooklynne's final answers are correct, but beliefs about the nature of multiplication and division give mathematics educators opportunities to encourage PSETs to explain and justify solutions. Providing multiple representations to a solution could encourage PSETs to examine their own misconceptions.

Additional discussion points for this study include ways mathematics educators support PSETs to extend and enrich their mathematical knowledge. How can mathematics educators support students like Hope? She was the only participant in this study not to answer all of the questions correctly. By expanding the number of participants, mathematics educators could identify additional PSETs in need of support in developing mathematical content. A PSET would likely struggle to develop multiple representations if the PSET cannot provide a correct solution. By recognizing some PSETs already possess a stronger foundation of mathematical knowledge, mathematics educators could incorporate enrichment opportunities to build a deeper, more connected network of knowledge (Ma 1999). In the future, how could mathematics educators support students like Stephanie and Arielle?

Having access to multiple representations permits PSETs to see the same problem from different perspectives. Without those multiple representations, PSETs could revert to a single solution approach (Gupta et al., this volume), demonstrating the same misconceptions they caution their students not to make (Ryan and McCrae 2005/2006).

Because elementary teachers around the world are trained to teach all academic subjects (Fennell, this volume; Lin and Hsu, this volume; Vula and Kingji-Kastrati, this volume), PSETs have a limited amount of time to prepare to teach mathematics. In that time, mathematics teacher educators need to find the right combination of research and practice to support the mathematical development of PSETs (Lo, this volume). With appropriate mathematical and pedagogical knowledge bases, mathematics teacher educators can present PSETs important aspects of teaching to support the mathematical development of children.

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