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Gabriel J. Stylianides
Keiko Hino *Editors*

Research Advances in the Mathematical Education of Pre-service Elementary Teachers

An International Perspective

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Preface

This volume is situated in the important area of research that investigates issues related to the mathematical education of pre-service elementary teachers (e.g., Chapman and Shuhua 2017; Lin and Acosta-Tello 2017; Song 2017). This area has attracted significant attention internationally for different reasons. One reason relates to the crucial role that elementary teachers have in setting the foundations for students' future learning of mathematics (e.g., Ball 1993; Kaput et al. 2007; Schifter 2009; Stylianides 2016). Another reason relates to the difficulties that many elementary teachers face with different mathematical concepts and the counterproductive beliefs they have about mathematics (e.g., Stylianides and Stylianides 2014; Wu 2017). A third reason for which the mathematical education of pre-service elementary teachers matters greatly is that it is during this period that pre-service teachers begin to form a basis of their future and ongoing quest about their expertise in mathematical instruction (e.g., Ma 1999; Jacobs and Spangler 2017; Li and Kaiser 2011).

The volume examines new trends and developments in research related to the mathematical education of pre-service elementary teachers, and explores the implications of these research advances for theory and practice in teacher education. It is organized around the following four overarching themes. Each theme includes four main chapters and a concluding chapter acting as a commentary on the theme overall. Although several chapters address issues that span across themes, practical considerations related to the organization of the volume necessitated a best-fit approach.

- *Theme 1:* Pre-service teachers' mathematics-content and mathematics-specific pedagogical preparation (Chaps. 1–5);
- *Theme 2:* Professional growth through activities and assessment tools used in mathematics teacher preparation programs (Chaps. 6–10);
- *Theme 3:* Pre-service mathematics teachers' knowledge and beliefs (Chaps. 11–15); and
- *Theme 4:* Perspectives on noticing in the preparation of elementary mathematics teachers (Chaps. 16–20).

We will not say much about the content of the four themes as the commentary chapters connected to each theme (Chaps. 5, 10, 15, and 20, respectively) provide an in-depth discussion of the four main chapters in the theme and identify issues/ideas that emerge from the entire collection. In brief, the chapters in Theme 1 emphasized the importance of tasks to promote professional growth (Chap. 1), explored pre-service teachers' knowledge of mathematics-specific domains such as proportional reasoning (Chaps. 2 and 3), and examined the role of computer technology in geometry (Chap. 4).

The chapters in Theme 2 presented current and emerging challenges related to elementary education programs. The mathematical background and related program experiences of pre-service teachers in the United States were analyzed to illustrate some of the issues for preparing elementary school teachers (Chap. 6). Effective characteristics of learning environments were discussed through prospective teachers' activities of designing nonroutine mathematical problems (Chap. 7). The pre-service teachers' procedural and conceptual knowledge of fractions were assessed and found to be in need of more attention in teacher education programs (Chap. 8). Simulation assessment was designed to reveal pre-service teachers' eliciting and interpreting capabilities that are crucial aspects of interactional practice of teaching (Chap. 9).

The chapters in Theme 3 provided different approaches to the mathematical knowledge and beliefs for teaching. They investigated mathematical knowledge for teaching and evaluation in the particular domain of argumentation (Chap. 11), explored self-efficacy as it relates to pre-service teachers' mathematical backgrounds (Chap. 12), considered different measures of knowledge and beliefs including teachers' beliefs for topic-specific knowledge (Chap. 13), and examined prospective teachers' learning opportunities for teaching to diverse sets of students (Chap. 14).

The chapters in Theme 4 provided multiple perspectives on the ability to notice and the development of that ability amongst prospective teachers. The analysis of a roleplay activity with pre-service elementary school teachers involving the use of a calculator was used to illustrate the complexity of learning to notice and learning to act in the moment (Chap. 16). Writing narratives was used as a successful way to help pre-service teachers develop their skill of noticing students' mathematical thinking (Chap. 17). Pre-service teachers' skills to recognize, identify, and make instructional decisions were examined in a context in which prospective teachers were provided with opportunities to engage in noticing practices (Chap. 18). Finally, three practices by the teacher educators were identified as they connect pre-service teachers' learning to the practice of teaching mathematics to students (Chap. 19).

We will say a few words now about how the volume came about. The volume includes a selection of expanded and improved versions of papers presented at the 13th International Congress on Mathematical Education (ICME 13) in Hamburg (Germany, 2016), under the auspices of Topic Study Group 47 (TSG 47) titled the "Pre-service Mathematics Education of Primary Teachers". The two of us were the cochairs of TSG 47 and we shared the responsibility of the organization of the TSG activities with three other team members—Katja Eilerts (Germany), Caroline Lajoie (Canada), and David Pugalee (USA)—all of whom are involved in this volume, either as authors or commentators.

The 16 main chapters in the volume were selected from a total of 66 contributions to TSG 47 in ICME 13. Based on the review process we followed in our TSG, the 66 contributions were divided into different categories depending on how highly they were rated by other participants and members of the organizing team. The most highly rated contributions comprised 19 regular presentations, each of which included an 8-page paper. We invited the authors of those 19 papers to contribute an improved and expanded version of their paper as one of the main chapters in the volume. Sixteen of these author teams accepted our invitation and, following at least two rounds of review, each of those papers was accepted for publication in the volume. The four commentary chapters are written by internationally acclaimed scholars we invited based on their expertise on the corresponding theme.

In summary, the volume includes contributions from researchers working in 11 different countries and offers a forum for discussion of and debate on the state of the art related to the mathematical preparation of pre-service elementary teachers internationally. In presenting and discussing the findings of research conducted in different countries, the volume offers also opportunities to readers to learn about teacher education practices used around the world, such as innovative practices in advancing or assessing teachers' knowledge and beliefs, similarities and differences in the formal mathematics education of teachers, types of and routes in teacher education, and factors that can influence similarities or differences.

Last but not least, we wish to thank all the participants of TSG 47 in ICME 13 for their contributions and especially the chapter authors, the reviewers, and commentators for their constructive feedback and insights, and the Monograph Series Editor (Gabriele Kaiser) for her support throughout the preparation of this volume.

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Contents

Part I
Pre-service Teachers' Mathematics-Content and Mathematics-Specific Pedagogical Preparation

Chapter 1
Using Mathematics-Pedagogy Tasks to Facilitate the Professional Growth of Pre-service Elementary Teachers

Fou-Lai Lin and Hui-Yu Hsu

Abstract We used mathematics-pedagogy tasks (MPTs) to design content and methods (pedagogy) courses to facilitate the professional growth of pre-service elementary teachers, especially those who did not study in mathematics-related areas. MPTs, together with the use of relevant theories, enable pre-service elementary teachers to coordinate the learning of mathematics, student cognition, the sequence of mathematics content arranged in the curriculum, and teaching activities designed in textbooks. For those pre-service teachers studying in non-mathematics areas, the learning of mathematics should be the starting point, as it enables them not only to understand the mathematics but also to build personal learning theories that can subsequently be applied to realize student cognition. The integration of mathematics and student cognition becomes the foundation for pre-service elementary teachers to comprehend curriculum arrangement and textbook design. In this chapter, we discuss and exemplify the notion of MPTs using examples implemented in two teacher education courses (one content course and one methods course).

Keywords Mathematics-pedagogy task (MPT)
Pre-service elementary teachers · Professional growth · Content course
Methods course

1.1 Rationale of the Study

In the past decades, educational research has increasingly emphasized the development of teachers' competence in mathematics teaching. In teacher preparation programs, mathematics content and methods (pedagogy) courses are the core for preparing pre-service elementary teachers in mathematics and pedagogical content

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knowledge, which can support follow-up learning during their practicum experience. Although research has confirmed the effects of teacher preparation on teacher quality and student performance (e.g., Boyd et al. 2009), the nuances of such effects and the extent to which and in which ways the two courses influence the actual mathematics teaching and student learning outcomes (Wilson et al. 2001) remain unclear.

One major continuing debate over the subject matter concerns the relationship between elementary teachers' mathematics knowledge and teaching quality and how much mathematics the teachers need to learn. Li et al. (2008) showed that elementary teacher preparation programs, especially those in East Asian countries (e.g., China), attempt to increase the number of mathematics courses as much as possible, on the premise that pre-service teachers' teaching performance improves along with their increased knowledge of mathematics. Another evolving issue relates to the ways in which the two courses can be integrated to maximize pre-service teachers' learning. Elaborating ways to integrate the two courses is the focus of this chapter.

Researchers have attempted to conceptualize mathematics teacher preparation with respect to the two courses. Simon (1994) proposed a recursive framework based on the theories of constructivism and didactical situation; he argued that teacher preparation must provide pre-service teachers with opportunities to participate as students in mathematics learning. Pre-service teachers' mathematics experience in turn will enable them to recognize the nature of mathematics and develop personal theories of learning mathematics, which will support them in considering their classroom teaching. Simon (1994) particularly emphasized that a conceptual understanding of mathematics is essential to the success of both mathematics learning and teaching, which can be nurtured through exploring a broader range of problem situations. Meanwhile, Artzt (1999) articulated the view that the relationship between pre-service teachers' mathematics learning and their instructional practices is a mutually reinforcing cycle through which their pedagogical power and conceptions of mathematics can be enhanced.

In conceptualizing teacher preparation with respect to the integration of the two courses on the content and methods for pre-service elementary teachers in Taiwan, this chapter is aligned with the views of Simon (1994) and Artzt (1999). Such views emphasize the development of pre-service teachers' personal learning theories of mathematics. This chapter argues further that the learning of mathematics must also provide pre-service teachers with opportunities to reflect on how mathematics is structured and developed, in addition to developing pre-service teachers' conceptual understanding. The more important issue is the opportunity to draw an analogy between self-learning of mathematics and the learning of students, so that the connection to children's concept development can be perceived more easily. In particular, the mathematics content at the elementary school level that pre-service elementary teachers will teach poses a challenge to educators in teaching the subject. On the one hand, while elementary mathematics content is easy for pre-service elementary teachers in Taiwan, they do not think it is important to realize the structure and development of the content. Taiwanese pre-service elementary teachers usually can retrieve elementary mathematics easily to solve problems without reasoning about the underlying processes. For example, such teachers are good at calculating fraction-multiplication problems but may not be able to provide mathematical reasons for the calculations. Consequently, a mathematics content coursefocusing on the elementary mathematics level is unlikely to facilitate pre-service elementary teachers' conceptual understanding of mathematics. On the other hand, if the content course focuses on advanced mathematics, it is highly likely that pre-service elementary teachers, especially those studying in non-mathematics areas, will have difficulty in understanding the content. As a result, the learning of advanced mathematics may not contribute effectively to pre-service elementary teachers' competence in teaching. In this respect, it is necessary to determine which level of mathematics and what kinds of mathematics problems can support pre-service teachers most effectively in understanding elementary mathematics and realizing the insights of mathematical structure.

Moreover, how pre-service teachers transform their personal learning theories into teaching is another essential issue. We argue the importance of providing pre-service elementary teachers with opportunities to coordinate the learning of mathematics, student cognition, the sequence of the mathematics content in the curriculum, and the design of teaching activities in textbooks. Student cognitive behaviors must play an intermediate role in facilitating teachers to make the transition between mathematics learning and classroom teaching. The understanding of mathematics and student cognition enables pre-service elementary teachers to realize the rationale for the arrangement of the mathematics content in the curriculum and thereby criticize the strengths and weaknesses of the design of teaching activities in textbooks. Toward these goals, various mathematics-pedagogy tasks (MPTs) are used to integrate the course on the subject into that of the methods.

MPTs emphasize both mathematics and pedagogy. This is aligned with the terms of "content knowledge" and "pedagogical content knowledge" coined by Shulman (1986), which subsequently evolved into a more deliberate conception of "mathematical knowledge for teaching" (Ball 2003). The conception of mathematical knowledge for teaching consists of six substantial kinds of knowledge that are necessary for high-quality teaching. In fact, MPT is not a brand-new notion in mathematics education. Several researchers have proposed similar conceptions. For example, Stylianides and Stylianides (2010) proposed the conception of "pedagogy-related mathematics tasks" and argued that mathematics tasks should entail pedagogical space for teacher learning. Watson and Mason (2007) proposed the idea of mathematics-related tasks and identified the use of mathematical task and activity as having potential in facilitating teacher learning of subsequent pedagogy.

Baturo et al. (2007) also suggested teacher education (TE) tasks and clarified the three levels of tasks: (1) technical level involving actual classroom practices for the tasks; (2) domain level with respect to content and pedagogies for the mathematics domain of the tasks; and (3) generic level specific to mathematical structures and pedagogical approaches that operate across domains. They further argued that teacher growth occurs when teachers transcend a mathematical focus and pedagogical activity within TE tasks, enable the translation to classroom teaching and student learning, and transfer to other mathematics topics. Baturo et al. also stated that TE tasks can ensure that the mathematics topic can be covered fully and arranged in an appropriate learning sequence. Moreover, TE tasks also enable deeper insights, as the topic can be collated and affiliated with other areas of mathematics; in this way, student learning is linked to teachers' awareness of the learners, including their personal theories and differences in learning mathematics. While similar conceptions that have been proposed in the literature vary, we decide to use the notion of MPT as it highlights both mathematics and pedagogy as well as the coordination between the two.

A MPT served as a teaching unit for our study to plan and coordinate the content and methods courses in the preparation of pre-service elementary teachers. We used each MPT to enable pre-service elementary teachers to develop their knowledge of mathematics, student cognition, the arrangement of mathematics content in the curriculum, and teaching activities designed in textbooks, as well as to coordinate the four aforementioned kinds of knowledge. Generic examples of MPTs from four mathematical domains (number and measurement, space, algebra, and probability and statistics), along with relevant theoretical frameworks, were created in an attempt to facilitate pre-service elementary teachers in looking into the broader context of particular cases (Mason and Pimm 1984). In turn, generic examples including theoretical frameworks were intended as thinking models to enable pre-service elementary teachers to synthesize a broad range of teaching and learning of mathematics topics. As Fuller (1969) suggested, the concerns of new teachers initiate from themselves and then pass outside, so it is crucial for pre-service elementary teachers to be aware of self-learning practices in mathematics and draw an analogy to how students learn. Afterwards, they can realize the rationale for the arrangement of the mathematics content in the curriculum and the design of teaching activities in textbooks.

1.2 Background

In this section, we further describe the evolution of elementary teacher preparation programs in Taiwan and the background of this study.

1.2.1 Background of Teacher Preparation in Taiwan

Taiwan is an island located southeast of Mainland China, from which it is separated by the Taiwan Strait. Historically, the teacher education system in Taiwan could be described as a uniform monopolized system, as teacher colleges and universities were the institutions responsible for the preparation of future teachers. Eventually, following drastic economic, political, and social changes, several waves of education reform were initiated. In 1994, the Legislative Yuan of Taiwan issued a new policy that diversified the institutions for teacher preparation; in this way, teacher universities are no longer the only legitimate authorities in setting up preparation programs for future teachers, but other comprehensive universities in Taiwan may also be authorized to do so (Fwu and Wang 2002). On the one hand, the policy meant the end of "dictatorship" in education, reflecting a diverse society where people can express multiple perspectives. On the other hand, the policy also strongly affected the preparation programs for future teachers in Taiwan's teacher universities. The diversification of institutions for teacher preparation ended the opportunities for a guaranteed job. At present, pre-service elementary teachers must pass a national examination to obtain teacher certificates and then find teaching positions themselves. As a result, teacher universities can only recruit pre-service elementary teachers with a much lower academic performance than prior to 1994. As most pre-service elementary teachers have lower academic performance and a high percentage of them study in non-mathematics areas, this situation creates challenges in teacher preparation. It is very likely that most pre-service elementary teachers can perform mathematical procedures, but they may not understand mathematical concepts well. Many have not even had pleasant mathematics learning experiences. One pre-service elementary teacher stated:

> I do not like mathematics…When I was in elementary school, I did not understand what my teachers were saying…I tried to remember the lessons on mathematics. When I could not remember them, I would give up studying this subject. Now I have to learn it again…It is really a nightmare…I am also very afraid of becoming a mathematics teacher. [Transcript from mathematics content course]

The above pre-service elementary teacher is an example of those who are not good at mathematics. Such a situation presents a challenge to educators to help such pre-service elementary teachers consider mathematics naturally and perceive the insights of mathematics. Developing pre-service elementary teachers' profound understanding of mathematics (Ma 1999) can be much more complicated than simply requiring a mathematics-relevant degree or incorporating more courses on the subject matter (Wilson et al. 2001). Another challenge in teacher preparation is that such pre-service elementary teachers have little knowledge of student cognition and to an even smaller extent consider the understanding of student cognition as important. This can be observed in the following reaction of another pre-service elementary teacher:

> To me, I do not think analyzing students' cognitive behaviors is hard…I can easily guess students' cognitive behaviors…not all but some…You know…we have a lot of different students in classes…it is impossible to know them all…I like to design instructional activity more than to only analyze students' cognitive behaviors. [Transcript from methods course]

A good understanding of student cognition is key to successful teaching, but not all pre-service elementary teachers share such a perspective. Although pre-service elementary teachers can come up with a few answers related to student cognition, their knowledge is still limited. There are pre-service elementary teachers who do not consider students central to mathematics teaching, and the learning of students' cognition is strongly related to the arrangement of mathematics content in the curriculum and textbook design. Therefore, their limited knowledge of mathematics and student cognition and their lack of appreciation for the work of analyzing student cognitive behaviors present a challenge to educators. Thus, we designed and searched for MPTs that can develop a more profound understanding of mathematics among pre-service elementary teachers and provoke their awareness of student cognition and its importance in designing teaching activities.

1.2.2 Background of the Study

The study was developed based on the implementation of MPTs in both courses (the content courseand the methods course) in the elementary teacher preparation program of a teacher university in Taiwan. Each course lasted for 18 weeks, and each week had a two-hour lesson. With a limited time schedule that aimed to develop in pre-service elementary teachers the kinds of knowledge needed for teaching, the two courses did not intend to cover all the mathematics topics included in elementary mathematics textbooks. Rather, it aimed to develop holistic and coherent thinking models that enabled pre-service elementary teachers to reason across a broad range of pedagogical problems involving mathematics, student cognition, arrangement of mathematics content in curriculum, and teaching activities designed in textbooks. Through these models, we expected pre-service elementary teachers to individually plan a lesson that integrates the four aforementioned kinds of knowledge into the design.

At the core of developing these two courses is active learning. The learning strategy adopted is conjecturing, which is the backbone of mathematics learning (Mason et al. 1982). As mathematics can be described in its essence as a subject involving relations and patterns, conjecturing those relations and patterns enables pre-service elementary teachers to perceive a mathematical structure across different mathematics content. In addition, pre-service elementary teachers must also conjecture student cognition, the sequence of mathematics content arranged in the curriculum, and teaching activities designed in textbooks. With the help of conjecturing strategy, pre-service elementary teachers can coordinate different kinds of knowledge for teaching, thereby fostering the bridging of theories into practices. The courses on content and methods were arranged across different semesters. Pre-service elementary teachers must take the content course first, followed by the methods course. In the following section, we provide examples of MPTs and explain the ways by which they can facilitate pre-service elementary teachers' learning.

1.3 Examples of Mathematics-Pedagogy Tasks (MPT)

The tasks we present in this section are examples of MPTs meant to function as generic examples (Mason and Pimm 1984) to facilitate pre-service elementary teachers in developing thinking models and connections across mathematics, student cognition, arrangement of mathematics content in curriculum, and teaching activities designed in textbooks. Although a MPT may involve more than one kind of knowledge for teaching, each MPT also has characteristics specific to mathematics, student cognition, curriculum arrangement, and textbook designs. This chapter focuses on elaborating characteristics of MPT. The interplay among educators, MPTs, and pre-service elementary teachers is not covered in the chapter.

1.3.1 Mathematics

Swan (2007) identified five purposes of learning mathematics for elementary teachers. These are the following: (1) developing fluency while recalling facts and skills; (2) interpreting concepts and representations; (3) developing strategies for investigation and problem solving; (4) awareness of the nature and values of education system; and (5) appreciation of the power of mathematics in society. Swan also highlighted the need to foster generation and re-examination of mathematics concepts through reflection and discussion. Stylianides and Stylianides (2010) suggested pedagogy-related mathematics tasks that not only can facilitate pre-service elementary teachers in understanding the essence of mathematics but also create a pedagogical space for applying that knowledge in mathematics teaching. These scholars elaborated the features of pedagogy-related mathematics tasks, including the following: (1) a primary mathematical object; (2) a focus on important elements of mathematics knowledge for teaching; (3) a secondary but substantial pedagogical object and a corresponding pedagogical space.

Working on mathematics contributes to pre-service elementary teachers' understanding of mathematics, including those purposes described above. In addition, MPTs must also provide the pre-service teachers with opportunities to realize a mathematical structure that can help them draw an analogy to student learning of a particular mathematical concept. It is also important to develop the epistemological beliefs of pre-service elementary teachers regarding the knowledge of how mathematics is structured and developed, how it is gained, its degree of certainty, and the limits and criteria for determining this knowledge (Perry 1981). In particular, in relation to facilitating pre-service elementary teachers who do not study in mathematics-related areas, we consider MPTs that can facilitate pre-service elementary teachers in drawing an analogy between personal mathematical thinking and that of students. Mathematics is a subject involving patterns and relations in which systematic thinking is considered important. To develop systematic thinking, the three phases (entry, attack, and reflection of work on mathematics) proposed by

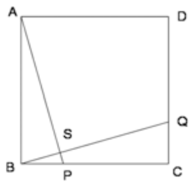
Mason et al. (1982) are adopted. A conjecturing approach enables pre-service elementary teachers to go through the three phases and realize a certain degree of generalization and specialization of mathematics. Additionally, supporting pre-service elementary teachers in knowing different kinds of strategies (e.g., analogy, generic example, visualization) to attack a problem is also a goal of the courses. Here is an example:

> David has an electronic watch. It informs the time by different voice arrangements. Low voice refers to 0, whereas high voice means 1. When the watch signals voice Low Low Low Low High, it means it is 1 o'clock now. What time is it when the watch speaks Low High High High High? Explain how you found the answer.

This MPT is for pre-service elementary teachers exploring the base-2 place value structure. Exploring and conjecturing mathematical structure is cognitively demanding for pre-service elementary teachers, as they are not familiar with generalizing from diverse examples to figure out the embedded structure. Working on this kind of MPT allows pre-service elementary teachers to understand the essence of mathematics and recognize that such an essence can serve as a model for thinking to analyze a variety of mathematics content. After pre-service elementary teachers solve the problem successfully, we draw their attention to the student learning of the base-10 system, which is important in elementary mathematics. The experiences of working in the base-2 system allow pre-service elementary teachers to notice how developing the concept of a mathematical system can be cognitively demanding for young students and to realize the underlying reasons that account for different kinds of student errors involving place values (e.g., algorithm errors with respect to addition, subtraction, multiplication, and division). Table 1.1 offers another example elaborating how a MPT can facilitate teachers in learning different problem-solving strategies.

A routine way to solve the problem in Table 1.1 is with a two-column proof. Nevertheless, pre-service elementary teachers are expected to identify different strategies to analyze the problem. First, teachers must not only rely on the given diagram to analyze the problem but also realize the dynamic nature of points P and Q and understand how that nature influences the problem. When seeing P and Q as dynamic points, pre-service elementary teachers have the opportunity to specialize the problem. For example, when points P and Q move to points C and D,

Table 1.1 An example of a MPT that is related to diverse ways of thinking geometrically

Two ants are moving along the sides of a square. One ant starts from point B and walks to point C; the other starts from point C and walks to point D. If both ants move at the same constant speed and points P and Q represent their respective locations find the measure of ∠ASQ	

respectively, the problem becomes that of finding the measures of an angle formed by intersecting diagonals. An alternative is to move P and Q back to points B and C so that the problem becomes one of recognizing the angle measures of a square. A rigid transformation by rotation is another strategy to visualize the congruence of △ABP and △BCQ and find the answer to the problem. Discussing the diverse strategies to solve the problem enables pre-service elementary teachers to generalize among different problems and identify the kinds of competencies that are core in student learning.

1.3.2 Student Cognition

One crucial goal of the two courses is to enable pre-service elementary teachers to focus on student learning and treat student cognitive behaviors as central to teaching practice. The use of MPTs facilitates pre-service teachers in understanding student cognition (e.g., misconceptions) and then connecting it to the arrangement of mathematics content in curriculum and teaching activities designed in textbooks, which significantly influence the development of mathematical concepts (Resnick et al. 1989). To aid in doing so, we used data related to student cognition collected from national surveys in Taiwan and those that have been reported in the literature. The use of data from Taiwan facilitates pre-service elementary teachers in recognizing student cognitive behaviors specific to the Taiwanese educational environment and teaching context.

The MPT example presented here was used to enhance pre-service elementary teachers' understanding of those aspects of student cognition specific to the concepts of triangles and the inclusive relationship among different kinds of triangles. The data were from a national survey of Taiwanese middle school students on identifying acute, equilateral, and isosceles triangles. When discussing the MPT with pre-service elementary teachers, this study first asked them to conjecture students' responses. For example, pre-service elementary teachers must conjecture the percentage of students who answered the diagram in Fig. 1.1 as an acute triangle and the reasons they gave.

From conjecturing students' responses and the reasons for their responses, pre-service elementary teachers were given the opportunity to reflect on their understanding of student thinking. A high percentage of pre-service elementary

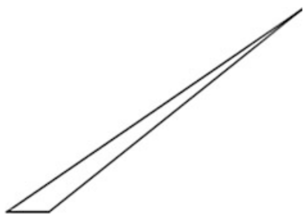


Fig. 1.1 An acute triangle

teachers believed that most middle school students should be able to recognize the triangle shape, as they thought the problem is not cognitively demanding, especially when the concept is learned in elementary school. However, the data showed that about 50% of Taiwanese middle school students could not answer the problem correctly. The situation of Taiwanese students' actual performance creates a cognitive conflict in pre-service elementary teachers. According to Bell (1993), cognitive conflict refers to the situation of persistent application of a concept to other fields, which consequently produces a disequilibrium in one's mind. Bell explained that teaching involves the arrangement of activities that can elicit students' cognitive conflicts and then help them refine their earlier primitive and cruder concepts. In the courses, after pre-service elementary teachers noticed the inconsistency between the self-conceptions of students and the actual data, they were required to answer the following questions:

1. Why did a high percentage of students give incorrect answers to this problem? Please explain your reasons.
2. Does the relevant mathematics content in the curriculum cause the misconception of students? If yes, how?
3. Do the teaching activities designed in textbooks cause the misconception of students? If yes, how?

A good knowledge of student cognition is among the keys to quality teaching (Carpenter et al. 1989, 1999), but pre-service elementary teachers often lack knowledge of this kind (Shinno et al. this volume). In most cases, pre-service elementary teachers use their prior knowledge to interpret student learning. To this end, we attempted to use MPTs that provide pre-service elementary teachers with opportunities to produce inconsistency in student cognition, so that pre-service teachers can refine their understanding of student cognition and consequently acknowledge the importance of treating student learning, especially misconceptions, as an important factor to consider in designing teaching activities.

In turn, the cognitive conflict in student cognition facilitates the reflection of pre-service elementary teachers on how mathematics topics are arranged in the curriculum and how teaching activities are designed in textbooks, as well as the reasons for students' misconceptions. For instance, the "sharp" angle at the top of the triangle in Fig. 1.1 might be one of the reasons for students' misconceptions. The literal meaning of the mathematical term "acute triangle" (銳角三角形) in Chinese may be another reason. The Chinese word 銳角 means "sharp angle," which may lead students to think that one acute angle implies an acute triangle. Thus, students do not see the need of verifying all three angle measures in a triangle even though an acute triangle is defined in mathematics textbooks as a triangle with three acute angles. The MPT also enables pre-service elementary teachers to know that students are inclined to name a triangle using only one concept (e.g., an equilateral triangle cannot be an isosceles triangle), which consequently causes difficulties in learning inclusive relationships.

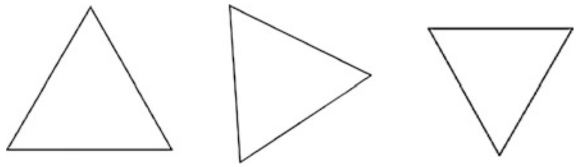


Fig. 1.2 Three equilateral triangles

More importantly, the triangles in Fig. 1.2 offered pre-service elementary teachers the opportunity to recognize how the curriculum arrangement influences students' formation of mathematical concepts (Stein et al. 2007). As can be observed in the diagram, three equilateral triangles are oriented differently. The national data show that about 50% of middle school students could not recognize the equilateral triangles as isosceles regardless of how the diagrams are oriented. However, an even higher percentage (about 75%) of students do not think an equilateral triangle is also an acute triangle. The main underlying reason for the performance differences can be attributed to the two systems of categorizing triangles in textbooks; one is by its angles (e.g., right angle), the other by its sides (e.g., isosceles triangle). In line with the national curriculum, Taiwan mathematics textbooks place the two categorization systems of triangles in different chapters, which results in teaching and learning the two systems separately. In this regard, students have fewer opportunities to reason out the connection between the two systems, which may be the main reason accounting for the misconception. Researchers have also pointed out that curriculum arrangement significantly influences the development of mathematics concepts for students, which is among the important lessons that pre-service elementary teachers must learn (Resnick et al. 1989; Stein et al. 2007).

1.3.3 Curriculum Arrangement

MPTs also aim to facilitate pre-service elementary teachers in knowing the rationale for curricula involving different domains (e.g., quality and number, geometry, and algebra) and the ways of arranging topics and units across different grade levels. We noted that pre-service elementary teachers may not pay attention to the arrangement of mathematics content in curriculum and naïvely think that the arrangement does not make much difference, especially when they believe that they know elementary mathematics well. Pre-service elementary teachers need to know that the principles of designing a curriculum are aligned with cognitive development, generally arranged from concrete to abstract, and with the degree of cognitive complexity involved in each topic. Another key issue in MPTs is that these tasks possess the potential for pre-service elementary teachers to see how MPTs can develop and connect mathematics across different topics and grade levels.

The MPT example elaborated here was revised based on Skemp's example of rectangular numbers (Skemp 1983), which provided students the opportunity to develop concept images involving addition, subtraction, and multiplication, and then learn the concepts of prime numbers, composite numbers, and number factorization. The MPT requires students to arrange some of a set of 50 coins into different rectangular shapes. Because of the task's manipulative nature and its basis in diagrammatic representations (Bruner 1966), the MPT allows students to develop abstract thinking (e.g., defining prime numbers and number factorization) by applying diverse strategies (e.g., adding or multiplying numbers). In this regard, even young students can try the activity themselves to understand the mathematics embedded. Experience with a concrete manipulative task is similar to what Dienes (1973) termed structuralism for teaching, which highlights the substantiality of a well-structured learning activity in developing students' abstract concepts and the connections among different mathematics content.

In the courses, the teachers were required to try the activity and figure out what problem-solving strategies they came up during the activity. They were then asked to conjecture the problem-solving strategies that elementary students may use. Pre-service teachers particularly needed to pay attention to different students' strategies in relation to their prior mathematics knowledge. After that, we aided the pre-service teachers in constructing models involving the cognitive development of students by analyzing the problem-solving strategies proposed. For pre-service elementary teachers, the mathematics activity in the MPT enables them to work on the mathematics and then make conjectures on what cognitive behaviors the students might have. The MPT also allows those teachers to see how a mathematics activity can be implemented to develop student competence across different mathematics topics. As a result, pre-service elementary teachers can use these experiences with the MPT to understand the rationale of curriculum design.

1.3.4 Textbook Design

MPTs were used to nurture pre-service elementary teachers' competence in understanding the design of teaching sequences in textbooks. Researchers have indicated that teachers intend to teach based on their prior learning experiences rather than the textbooks (Freeman and Porter 1989). When teachers say that they teach based on textbooks, they likely mean the mathematics problems themselves rather than the whole teaching activity (e.g., exploration, making conjectures) designed in the textbooks. Teachers need to comprehend the textbooks and realize their strengths and weaknesses; in this way, they will learn how to modify the teaching activities to meet their students' needs.

To this end, we adopted the reading comprehension strategy proposed by Yang and Lin (2012) to facilitate pre-service elementary teachers in understanding textbooks. This instructional strategy aims to facilitate students in understanding geometry proofs using *questioning for structuring schemas*, *predicting for triggering relevant knowledge, clarifying for modifying schemas, summarizing for restructuring schemas,* and *reflecting for readjusting schemas and revealing* *cognitive processes*. The instructional strategy was modified into three stages in the current work, with the goal of facilitating pre-service elementary teachers' understanding of textbooks.

The first stage, similar to that proposed by Yang and Lin (2012), asked pre-service elementary teachers to read textbooks by questioning, predicting, clarifying, summarizing, and reflecting. This process ensured pre-service elementary teachers' understanding of mathematics. The second stage required them to conjecture students' cognitive behaviors specific to questioning, predicting, clarifying, summarizing, and reflecting processes. This stage compelled pre-service elementary teachers to connect self-learning and student learning from a reading comprehension perspective. The third stage asked pre-service elementary teachers to analyze different textbooks in terms of their similarities and differences and their relation to student learning. The comparison of different textbooks afforded pre-service elementary teachers the opportunity to clarify the rationales behind designing teaching activities in textbooks. For example, after pre-service elementary teachers analyzed students' responses on the concept of triangles and their inclusion relations (see Fig. 1.1 and its explanation), one of the follow-up activities in the courses was to analyze the chapters related to triangle classifications in the textbooks using the three aforementioned stages.

1.4 Conclusion

Ball (2003) asserted that teaching is a highly complex activity. Meanwhile, content and methods courses designed in teacher preparation programs are often too fragmented to nurture pre-service elementary teachers' competence in teaching mathematics. This gap created educational challenges for teacher preparation, especially when the two courses need to take into account pre-service elementary teachers' prior knowledge in mathematics and pedagogy as well as the limited course time allotted. This chapter elaborated the notion of MPTs and how MPTs can be used as intermediate tools to coordinate the two courses in facilitating the learning of pre-service elementary teachers, especially those with non-mathematics degrees.

We presented various MPTs to reveal distinct characteristics they can have in relation to mathematics, student cognition, the arrangement of mathematics content in the curriculum, and the teaching activities designed in textbooks. The MPT examples we presented provide opportunities for pre-service elementary teachers to integrate different kinds of knowledge for mathematics teaching and transform their learning into the design of teaching activities themselves. A follow-up study we plan to conduct will analyze how pre-service elementary teachers plan a lesson after taking the two courses in order to understand the extent to which MPTs influence those teachers' competence in mathematics teaching.

A number of researchers have asserted that mathematics should be treated as central to the development of mathematics teaching competence. Stylianides and Stylianides (2010) further elaborated the importance of special kinds of mathematical knowledge and the essential pedagogical space that mathematics tasks should entail for pre-service teachers to learn. We took a further step in articulating a way of bridging the content and pedagogy of mathematics through the use of MPTs to facilitate pre-service elementary teachers in coordinating the various kinds of knowledge needed regarding mathematics, student cognition, and curriculum and teaching activities in textbooks. We expect that MPTs can enhance pre-service elementary teachers' mathematical knowledge for teaching (Hill et al. 2005; Ball et al. 2008). In particular, teaching consists of a series of in-the-moment choices (Mason and Davis 2013) in which the core is teachers' awareness of mathematics, student cognition, the curriculum, and teaching activities. MPTs may have the potential to optimize the in-the-moment pedagogical choices made by pre-service elementary teachers, which is worthy of further investigation.

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Chapter 2
Investigating the Relationship Between Prospective Elementary Teachers' Math-Specific Knowledge Domains

Roland Pilous, Timo Leuders and Christian Rüede

Abstract Notwithstanding long term efforts to differentiate between domains of mathematics-related teacher knowledge, there is no doubt that different forms and aspects of teacher knowledge are interrelated and mutually influence each other. However, the nature of this relation is still open to scholarly debate. First, we give an overview of empirical studies that investigated the relation between different domains of mathematics teachers' knowledge, notably, the domains of "content knowledge" and "pedagogical content knowledge". We demonstrate that the research on the relationship turns out to be multifaceted and we point to the need of cognitive orientated research on the integration of knowledge domains. Second, we present our own ongoing research on the integration of prospective elementary teachers' math-specific knowledge domains by describing our use and analysis of task-based interviews. Preliminary findings indicate that our approach can help to identify mental processes that illuminate the integration of math-specific knowledge domains.

Keywords Elementary school mathematics · Pedagogical content knowledge
Pre-service teachers · Integration · Cognitive processes

2.1 Introduction

One of the most influential ideas in describing professional knowledge for teaching is the distinction between mathematical *content knowledge* (CK) and mathematics-related *pedagogical content knowledge* (PCK) (Shulman 1986; Bromme 1992; Ball et al. 2008). But what is the relation between these knowledge

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domains? To answer this question it is necessary to understand what is meant by "relation". Unfortunately, the term is interpreted and used in many different ways. Before we present our own research, we thus attempt to identify and outline different approaches of research on the relation of mathematics-related CK and PCK. Hereby, we seek to provide a basis in order to classify our own ongoing research and its implications with respect to the field of research.

Among these approaches we identify two main strands: (1) Relations are expressed by theoretically discussing conceptualizations of PCK and referring either to pedagogical or to content-related aspects; and (2) Existing conceptualizations of content and pedagogic content are used to investigate teacher knowledge and the relation between its components empirically. According to the first strand, all conceptualizations of PCK and its components express a relation with CK in a specific way. For example, Shulman defined PCK as the "particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman 1986, p. 9). Thus, from his point of view PCK can be considered as a special form of CK. However, he also referred to it as that "special amalgam of content and pedagogy" (Shulman 1987, p. 8), a metaphor that expresses a more complicated quality of the relation (an amalgam is a rather tight mixture of elements, which results in a material with new qualities). Ball et al. (2008) presented a more empirically grounded conceptualization of mathematical knowledge for teaching. As above, their definitions of the components of PCK express their view about the relation between both. For example their definition of knowledge of content and teaching combines knowing about teaching and knowing about mathematics. By such conceptualizations, however, relations are often expressed implicitly or remain vague (e.g. PCK as an amalgam). Moreover, Depaepe et al. (2013) pointed out in their review article that there is no consensus about the components that PCK covers. However, in all recently discussed theoretical models PCK comprises at least knowledge on students' cognitions, instructional strategies, and representations (ibid.).

According to the second strand, relations are investigated empirically and based on predefined conceptualizations of CK and PCK. In the following section, different lines of empirical research within this strand are discussed in detail.

2.2 Overview of Empirical Studies on the Relation Between Content Knowledge and Pedagogical Content Knowledge

We differentiate between four lines of research where relations are investigated empirically: (a) Relation as correlation between CK and PCK measured by some form of test; (b) Relation as parallel development of CK and PCK, where one is described as a condition for the other; (c) Relation as a form of integration of CK and PCK; and (d) Relation as a form of association of CK and PCK. Studies that address the quality of instruction or the student progress (dependent on CK and PCK) are not included in our overview. We are aware of the fact that it is not always easy to distinguish between the lines of research mentioned above, especially since they are not fully disjoint. Nonetheless, we consider the differentiation useful to give a systematic overview. Next, we discuss each of these four lines of research separately.

2.2.1 Relation as Correlation Between CK and PCK

The studies in this line of research investigate relations quantitatively by means of written tests and score correlations. For example, Blömeke et al. (2010a, b), Hill et al. (2004) and Krauss et al. (2008) showed that it is possible to distinguish CK and PCK by statistical analysis of test behavior. In that sense, the term "relation" focuses on the question whether it is reasonable to distinguish domains of knowledge at an interindividual level (through factor analysis or by considering the discriminant validity of tests). Their findings also reveal a strong correlation between the constructs, indicating that both constructs are at least highly connected. However, the picture of correlations for prospective elementary teachers is not as clear as it is for prospective secondary teachers (Blömeke et al. 2010a). Depending on the design of research, additional forms of relations between CK and PCK on the one hand and further characteristics on the other hand were addressed. For instance, Krauss et al. (2008) investigated CK and PCK in dependence of school levels, teaching experience or personal theories about mathematics.

Although correlations between CK, PCK and further variables cannot be interpreted in a causal way, the correlative findings are often considered to support certain assumptions, e.g. about the development of PCK and CK, where one is seen as a possible condition for the other. We will refer to these interpretations in the next sub-section.

2.2.2 Relation as Co-development of CK and PCK

The abovementioned studies by Hill et al. (2004) and Krauss et al. (2008) supported the view on CK as a necessary or at least a facilitating condition for PCK. For example, Krauss et al. (2008) reported that secondary teachers in academic tracks scored better in pedagogic content items to the extent that they scored better in content items. Moreover, Even (1993) for example contends that many of the tested prospective secondary teachers did not have an appropriate mathematical concept of function (CK) which, as a consequence, may have led to inappropriate knowledge of the "vertical line test" as a rule for students to check for the univalence aspect of functions (PCK). The importance and necessity of content knowledge has also been investigated by a number of qualitative studies (cf. Ball et al. 2001): for instance, findings reveal the relevance of conceptual understanding (CK) for teaching methods as well as for the generation and analysis of representations and explanations (PCK).

However, Krauss et al. (2008) reported that some secondary teachers in non-academic tracks reached high scores in PCK measures even though they scored low in items associated with mathematics content. This indicates that CK rather facilitates the development of PCK instead of being a necessary condition in a strict logical sense. For teachers in non-academic tracks low scores in PCK always went along with low scores in CK. Additionally this result was interpreted in a way that "strong content knowledge quasi'protects' against a low level of pedagogical content knowledge" (p. 244, translation by the author).

Can CK be considered as a sufficient condition for PCK? Krauss et al. (2008) showed that secondary teachers in academic tracks scored higher in PCK measures compared to teachers in non-academic tracks although the teachers in academic tracks received less education in PCK. This may support the view that CK is even sufficient for PCK. However, referring to Capraro et al. (2005) the condition is not sufficient for pre-service elementary teachers: "having profound mathematical understanding does not ensure pre-service teachers [to] develop pedagogical content knowledge" (p. 108).

How do prospective teachers acquire knowledge about content and pedagogic content? Kleickmann et al. (2017) conducted an experimental study in order to test the assumptions that (a) teachers construct PCK from CK and PK in a process of amalgamation, (b) CK is a necessary condition and facilitates PCK development, and (c) CK is sufficient for teachers' PCK development. The study indicates "that there are different pathways to PCK development" (p. 17). Lin and Hsu (this volume) refer to the question in their discussion of the use of mathematics-pedagogy tasks to facilitate the development of mathematics-related knowledge domains. They propose to implement tasks in teacher preparation courses which offer opportunities to address knowledge about content, student cognition, curriculum and textbook design concurrently. For instance, they present tasks which may not only help to promote prospective teachers' mathematical understanding but also facilitate them to make analogies to student learning.

2.2.3 Relation as a Form of Integration of CK and PCK

With the term "integration" we refer to the ways of how (prospective) teachers' CK and PCK come together and inform teachers' behavior in teaching-specific contexts or in the course of teaching. Hence, investigating the integration requires analyzing and differentiating teachers' behavior and the knowledge that becomes evident in the process. Escudero and Sanchez (2007), for example, analyzed videotaped lessons of two experienced secondary teachers and described the teachers' behavior and the integrated knowledge. Similarly, Speer and Wagner (2009) identified "component practices of analytic scaffolding" and analyzed the knowledge "needed to enact these practices" (p. 557). For instance, they worked out that the examined undergraduate mathematics teacher (who had good mathematical knowledge) was unable to use the students' contributions as an opportunity for analytic scaffolding due to his lack of PCK.

Rowland et al. (2005) analyzed videotaped lessons of pre-service elementary teachers with the aim "to locate ways in which they drew on their knowledge of mathematics and mathematics pedagogy in their teaching" (p. 255). They identified four dimensions, called the "knowledge quartet", which can be used to observe prospective teachers' knowledge in practice. The knowledge quartet can be a useful tool for the discussion of knowledge domains between prospective teachers' and their mathematics teacher educators.

2.2.4 Relation as a Form of Association of CK and PCK

One form of association can be named "verbal association". Hereby, we denote a relationship operationalized as a connection between CK and PCK (or its components) that results from the proximity of teachers' utterances (which refer to elements of CK and PCK) in time. In this way, it is possible to count knowledge domains (or components of knowledge domains) per utterance in an interview or per episode in a lesson (enumerative approach). To our knowledge, so far no study was conducted in order to map out the relation of mathematics-related knowledge this way. However, Park and Chen (2012) used a model of PCK for teaching science (called the Pentagon model) and applied the enumerative approach to the teaching of four high school biology teachers. They worked out that the occurrence of "the components was idiosyncratic and topic-specific [and that] Knowledge of Student Understanding (KSU) and Knowledge of Instructional Strategies and Representations (KISR) were central" in the episodes (p. 930). In a similar vein, the so called Epistemic Network Analysis is applied by Weiland et al. (2015) in order to investigate "connections" between elements of mathematics content knowledge. This exploratory approach seems to enable further insights in teachers' organization of CK. However, there are no final results available yet.

A different approach to describe the association of CK and PCK is presented by Lehrer and Franke (1992). To derive "conditional relationships" among CK and PCK they applied personal construct psychology and the logic of fuzzy sets to the study of two experienced elementary teachers: after central knowledge "constructs" were identified (and later assigned to CK, PCK, or general pedagogical knowledge) the teachers were asked to rate the constructs to be more or less important (true) to each of the presented fraction problems. Fuzzy logic was applied to receive the strengths of implications (associations) between the constructs. The further analysis of the teaching revealed that teachers' individual implications correspond to their actions in the context of the classroom.

2.2.5 Summary

The overview demonstrates that the research with focus on relations between (mathematics-related) CK and PCK turns out to be multifaceted. Most of the previous work seems to focus on correlations or the co-development of knowledge domains. However, findings do not reveal a clear picture and there is still an ongoing debate of whether the findings are influenced by the respective methodology, such as by the types of tasks used in the measurement. According to Buchholtz et al. (2014) strong empirical correlations between CK and PCK reported in the literature can be ascribed to operationalizations of PCK that closely relate to CK.

As reported by Depaepe et al. (2013) only few studies address the integration or association of knowledge domains (Escudero and Sanchez 2007; Speer and Wagner 2009; Lehrer and Franke 1992). Thus, if we seek a deeper understanding of *how* aspects of content and pedagogic content play together when (prospective) teachers are faced with the demands of teaching, it is necessary to further investigate the integration of CK and PCK. The quantity of coincidences of knowledge domains (or components) per utterance or episode does not really help us to understand how CK and PCK come together. Rather, it would be necessary to concentrate both on mental processes *and* the respective knowledge domains or components that relate to those processes. However, existing studies did not fully recognize integration at a cognitive level. For this reason, we think it is of particular interest to further investigate the integration of knowledge domains and their components that way.

Our overview of empirical studies classifies existing studies aiming at the relationship between mathematics-related CK and PCK. In consequence, in the next section we are able to present our focus of research with respect to the gap of research outlined above and in contrast to the other lines of research. As pointed out already, the lines of research are not entirely independent. For instance, investigating the integration of knowledge domains may also lead to new insights according to the association of knowledge domains (because integration goes beyond association) or the development of PCK (because the identification of mental processes may give hints on how PCK can be acquired). Thus, the overview also offers an opportunity to refer back to existing studies when we interpret our preliminary and future findings with respect to the other lines of research in the discussion section.

2.3 Focus of Research

Referring to the gap of research outlined above, our aim is to investigate the integration of mathematics-related CK and PCK. As we mentioned earlier, by “integration” we understand the ways of how CK and PCK inform (prospective) teachers’ mental processes when they deal with typical demands in teaching-specific contexts or in the course of teaching. To us, it is particularly important to identify and describe mental processes that help to make clear how domains of knowledge come together. Of course, it is not possible to observe mental processes directly, such as to remember, understand, apply, analyze, evaluate, or create (Stern 2017). However, it is possible to analyze observable behavior in order to generate hypotheses about mental processes in which different mathematics-related knowledge domains are activated. Describing the integration of mathematics-related knowledge domains that way is a key aspect if we seek to improve our understanding of (professional) knowledge structures of (prospective) teachers.

It is not possible to investigate the integration without investigating the mathematics-related knowledge that (prospective) teachers use. According to Liljedahl et al. (2009), knowledge domains may become more and more integrated or even unified with time. At the same time, the integration of knowledge domains may depend on the amount of professional training and the experience of (prospective) teachers. Our aim is to shed more light on mathematics-related knowledge of prospective elementary teachers and its integration in the first phase of mathematics teacher education (i.e., during their bachelor degree courses). However, it is not our aim to investigate the development of knowledge domains or the development of its integration.

In the first phase of teacher education the knowledge domains initially emerge and often cover incomplete or incorrect knowledge. Thus, we are not only interested in knowledge that can be considered as correct from a normative point of view (professional knowledge), but we also consider incomplete, subjective, or experience-based knowledge. For this reason, it is not appropriate to use an existing conceptualization of mathematics-related knowledge. Rather it seems necessary to develop an empirically-grounded and adapted conceptualization which applies to our prospective teachers and which can be used to further investigate the integration.

We pose the following research questions:

(1) Which domains of prospective elementary teachers' mathematics-related knowledge can be distinguished in their reasoning while solving pedagogical tasks in mathematics teaching (as posed in bachelor-degree courses)?; and
(2) Which are the mental processes of prospective elementary teachers' that are based on different domains of knowledge?

2.4 Task-Based Interviews

We assume that the knowledge domains can best be investigated in situation-specific contexts. Thus, we considered the use of tests or questionnaires inappropriate. The method of lesson observation seem to fit better but does not offer the possibility to address further questions in relation to mathematics and the teaching of mathematics. Considering teaching in vivo also is not appropriate since the participants of the study are prospective elementary teachers (in their second to fourth semester) who typically have very little teaching experience and issues of classroom management may be predominant.

In previous studies the use of task-based interviews was applied successfully with respect to the investigation of teacher knowledge and beliefs (e.g. Ball 1988; Biza et al. 2007; Ma 1999). We apply this method including the task structure proposed by Biza et al. (2007) because the tasks are common to the training of prospective teachers. In the semi-structured interviews excerpts from textbooks and other curriculum materials are used as a basis for questions concerning tasks of teachers which are typical for arithmetic teaching in second to sixth grade. The list of core tasks presented by Bass and Ball (2004) served as a guiding framework. It comprises the following tasks where mathematical work is involved:

> [S]etting and clarifying goals, evaluating a textbook's approach to a topic, selecting and designing a task, re-scaling tests, choosing and using representations, analyzing and evaluating student responses, analyzing and responding to student errors, managing productive discussions, figuring out what students are learning […] (Ball and Bass 2004, p. 296).

For instance, we use the so-called Multiplication Poster by Wittmann (1998), shown in Fig. 2.1, as a representation of a holistic approach to the multiplication table and link it to the following scenario: "Suppose you plan to use the Multiplication Poster with your second graders. Reflect on your ideas and reasoning. How do you proceed?" Depending on the responses the interviewer poses questions such as: "What do you think about?" "What is the Multiplication Poster about?" "What about it could be easy or difficult for your students?" and "What are the goals you want them to reach?" Following the task structure mentioned above and in order to cover as many tasks from the list as possible, we further asked prospective teachers to examine and respond to fictional student solutions and errors.

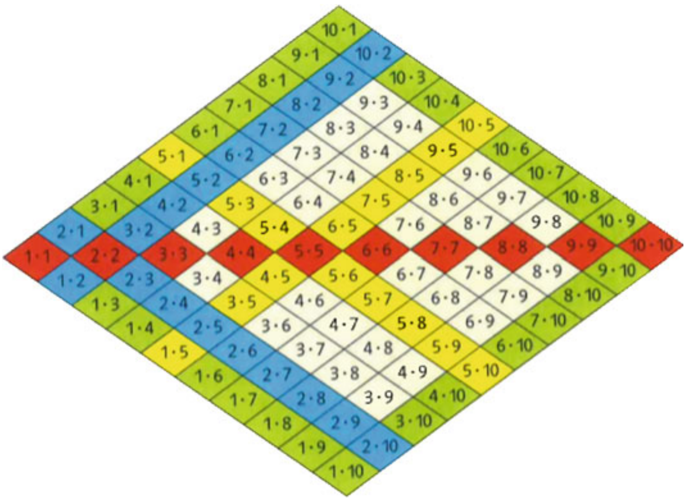

Fig. 2.1 Multiplication Poster © Klett und Balmer AG, Verlag, Zug 2007

2.5 Sampling and Analysis

Acquisition (theoretical sampling) and analysis of data (constant comparison) is based on the approach of Grounded Theory (Strauss and Corbin 1996) which is adequate for the purpose of developing theory from empirical interview data. The analysis of data is carried out using the software MAXQDA 11.

2.5.1 Participants

Our qualitative design with a small sample risk overemphasizing individual cases since prospective teachers' abilities and backgrounds are very diverse. To avoid this problem, we drew the sample following the maximum variance principle (Patton 1990). Six prospective teachers participated in our interviews which lasted between forty and sixty minutes each. Three of them were enrolled in the second semester (and attended the mathematics-related courses "Elementary Algebra and Arithmetic" and "Mathematical Thinking of Children"), one of them in the third semester (who additionally attended the course "Geometry and Applied Mathematics"), and two of them in the fourth semester (who additionally attended the course "Planning and Implementation of Mathematics Instruction"). All of them were bachelor students in an elementary education program in Switzerland.

2.5.2 Analysis

All six interviews were transcribed in full. Open and axial coding were taken out simultaneously (because they go hand in hand). Nonetheless, for reasons of readability we will outline both steps separately. Both steps of analysis are still in progress.

Open coding involves a constant comparison of data and asking questions like "On what knowledge does the interviewed person draw back in the situation?" These are core elements of this step which very closely sticks to the data. The goal is to explore the knowledge that the participants refer to in the interviews. We assign codes to segments of data if a certain concept is reflected (a piece of knowledge) according to our interpretation. The meaning of concepts represented by the codes is established by comparing data and writing memos. Memos also include discussions about the dimensions of concepts (such as frequency, depth, or duration).

The step of open coding is taken out highly inductive. This means that the names of codes are mainly developed in vivo (names are derived directly or with little variation from the data) or from the data in the sense that concepts are grounded in the data and named by the researcher during the analytical process (Strauss and Corbin 1996, p. 49). In order to remain open to the data a sentence-by-sentence coding is applied to the transcripts. Codes are not required to be disjoint.

Axial coding denotes the process of relating concepts and categories back together. Similar concepts are grouped and summarized under categories (knowledge domains) according to shared properties (Strauss and Corbin 1996, p. 47). At first, assigning these codes to categories and thus to (components of) knowledge domains is based on existing models of teachers' mathematics-related knowledge (theoretical sensitivity) (e.g. Ball et al. 2008; Krauss et al. 2008; Rowland et al. 2005; Shulman 1986). We use this generic structure as an initial point to further apply the method of constant comparison. Moreover, the assignation of codes to categories takes place irrespectively of content-related correctness. Hence, data are not evaluated as right or wrong, but as being of a certain kind of knowledge.

In the following, we give some examples illustrating codes, codings, and the assigning of codes to categories. Some of these illustrating codings are presented again as parts of interview sequences when we describe how codes (and categories) relate to each other in mental processes (see preliminary findings section).

2.5.2.1 Knowledge About Students' Cognitions

The following two codes are developed in vivo and irrespectively of being appropriate or not with respect to the represented concept (usually 7×3 is not considered to be most difficult). They are both assigned to knowledge about students' cognitions because they deal with the question what students will find hard or confusing (cf. Ball et al. 2008).

Code	Example (coding)
7×8 and 7×3 are most difficult	"I think that 7×8 and 7×3 are most difficult."
Multiplication Poster is confusing at first	"All at once [there are all multiplications up to 10 in the Multiplication Poster]. That is confusing. I mean the students cannot easily assess it."

2.5.2.2 Curricular and Teaching Related Knowledge

The notation of "core multiplications" concerns a certain approach to the multiplication table which is popular in Germany and Switzerland. It denotes the strategy of reducing an arbitrary multiplication to multiplications with factors 1, 2, 5 or 10 (core multiplications), e.g. $7 \times 8 = 5 \times 8 + 2 \times 8$, by applying the distributive property. The core multiplications are examined in detail from second grade on. This strategy is contained by many textbooks and standards for second grade.

Code	Example (coding)
Core multiplications	"For example, second graders work intensively with the 1, 2 and 5 series. In classes you normally start with them and build on them."

The utterances “second graders work intensively with” and “you normally start with them and build on them” were taken as indicators for teaching-related knowledge [cf. curricular knowledge about textbooks and standards, Shulman (1986)]. Alternatively, one may interpret this approach as a common way of making the multiplication table comprehensible. As in this case, it is not always easy to distinguish between curricular and teaching-related knowledge [cf. “knowledge of content and teaching”, Ball et al. (2008)]. One may also think about assigning the code additionally to content knowledge. However, the focus on curricular aspects and the absence of mathematical analysis is an argument against it.

2.5.2.3 Content Knowledge

When codes refer to mathematical facts, concepts, procedures or other more syntactical forms of knowledge, they were assigned to content knowledge.

Code	Example (coding)
Breaking apart	“One can use the 2 and 5 series as a basis. When you know the 5 series you can go ahead from here. For 9×8 you don’t have to count 9 so often. You can take 5×8 and then proceed.”
Associative law	“It doesn’t matter where you start from. $3 \times 3 \times 3$ is the same as 3×9 or 9×3.”

The first code example was assigned to content knowledge because the concept of core multiplications (see above) is addressed here from a mathematical point of view (the strategy of “breaking apart” is explained with an example without referring to the distributive property explicitly). The second code example again refers to a fundamental mathematical concept, the associative law.

According to the second research question we seek to investigate the integration of concepts by detecting the participants’ mental processes. We ask questions such as “How does the interviewed person proceed in this situation; how does the person make a connection between different domains of knowledge?” and go through all the interviews again. We tried to find answers by writing memos and analytic stories considering the paradigm, the dimensions and the questions that enhance theoretical sensitivity proposed by Strauss and Corbin (1996).

2.6 Preliminary Findings

As mentioned above, the analysis is still in progress. For the moment, we are able to present preliminary findings. Despite its preliminary character, these findings are meaningful in order to illustrate that our analysis is appropriate to reconstruct mental processes which appear to be informed by (math-specific) knowledge domains. First, we refer to our empirically grounded differentiation of knowledge domains. Second, we refer to the integration of knowledge domains by presenting to examples of mental processes.

2.6.1 Prospective Elementary Teachers' (Math-Specific) Knowledge Domains

So far, the analysis revealed four different domains of knowledge which we briefly describe in the following (for examples of codes relating to the respective knowledge domain see analysis section).

- *Knowledge about students' cognitions*: This domain of knowledge combines knowledge of mathematics and the learners of mathematics. It covers mathematics-related knowledge about (task) difficulties, conceptions and misconceptions, behavior patterns and "thinking paths" of elementary students. However, it does neither refer to "pure" mathematical knowledge even though it is used to identify student errors nor to knowledge about students which is not math-specific.
- *Curricular and teaching-related knowledge*: Curricular knowledge is knowledge about teaching standards, teaching programs, textbooks and other associated materials. It deals with the question which demands or representations are included and how they are sequenced in the standards, textbooks etc. Teaching-related knowledge is knowledge about the use of valuable representations of mathematics in terms of temporal sequences in the context of classroom teaching.
- *Content knowledge*: This domain includes substantial and syntactical as well as conceptual and procedural knowledge about the subject content which underlies the teaching of mathematics in elementary classes (in Switzerland first to sixth grade). Ideally, content knowledge is profound according to Ma (1999). However, we avoid making further claims about its quality in order to make it possible to capture any kind of content knowledge (since our focus of research is on the integration of knowledge domains).
- *Didactical knowledge*: This is the only domain of knowledge which is not content-related. It comprises general pedagogical and psychological knowledge such as knowledge about learning theories, forms of teaching and learning, teaching strategies, didactical principles or methods of assessment.

Knowledge about students' cognitions, curricular knowledge, and teaching-related knowledge can be considered as dimensions of our empirically grounded conceptualization of prospective teachers' PCK which closely sticks to types of knowledge known from the literature. However, differences remain. For example, so far we were not able to distinguish between curricular and teaching related knowledge (see above) or between "specialized content knowledge" and "common content knowledge" as proposed by Ball et al. (2008). In addition and in contrast to conceptualizations of professional knowledge, we also assigned incomplete, subjective, or experience-based knowledge to the respective domains. The findings presented above more serve the purpose of creating the necessary foundation to investigate the integration of prospective teachers' knowledge instead of representing a new contribution to the research of (professional) knowledge.

2.6.2 *Mental Processes in Which Different Mathematics-Related Knowledge Domains are Activated*

In the following we present two examples of prospective elementary teachers' mental processes which involve the activation of different mathematics-related knowledge domains. The mental processes are illustrated by sequences of codes and codings which appear in the original sequence as it has been found in the interview. Partially, the same codes and codings were discussed separately in the analysis section in order to illustrate the development of codes and its assigning to categories.

2.6.2.1 Evaluating Typical Task Difficulties from a Mathematical Point of View

This type of process involves mathematical analysis and thus the activation of content knowledge to evaluate the difficulty of tasks for children. In the example below, the prospective teacher first relies on his content knowledge when evaluating which tasks may be easy or hard. He identifies prime numbers. Probably he assumes multiplications with prime numbers to be more difficult because prime numbers have no factors apart from 1 and itself (and thus cannot be computed as easy as others by applying strategies such as breaking apart). In a second step, he argues that multiplications with the prime numbers 2 and 5 are easy. According to the multiplication with 5, he again relies on content knowledge by referring to the special role of 5 in the decimal system. According to the doubling, he knows that this is easy for second graders. Finally, this leads to the judgment that the 3 and the 7 series are most difficult (knowledge about students' cognitions).

Code sequence	Example (coding)
Prime numbers role of 5 in the decimal system	"Well, 7 is a prime number. And 5 of course is also a prime number. But it is the half of 10. Thus you can handle 5 more easily. 3 also is a prime number.
Doubling is easy	And 2 and 1 series are easy for the students at that level.
3 and 7 series are most difficult	So, I think the 3 and 7 series are most difficult for the pupils."

2.6.2.2 Remembering Content Knowledge in the Function of Illustrating Curricular or Teaching-Related Knowledge

In most of the interviews it appeared that content knowledge was mainly activated in a context of expressed curricular or teaching-related knowledge. In this case, the participants first relied on their curricular or teaching-related knowledge. For instance, they talked about the sequencing of tasks or topics in class, about teaching materials and so on. In the following, content knowledge was activated in the function of illustrating these statements by explicitly connecting the sequencing of tasks or the use of materials in class with mathematical concepts. In the example below, the student indicated the mathematical concept of "breaking apart" with an example in order to illustrate the prior statements. Interestingly, content knowledge was activated very often in the context of articulated curricular or teaching-related knowledge. Conversely, participants never relied on content knowledge primarily (i.e., in order to fully analyze the mathematical potential of the Multiplication Poster) or activated pedagogical content knowledge in the context of articulated content knowledge.

Code sequence	Example (coding)
Core multiplications	"For example second graders work intensively with the 1, 2 and 5 series. In classes you normally start with them and build on them.
Breaking apart	One can use the 2 and 5 series as a basis. When you know the 5 series you can go ahead from here. For 9×8 you don't have to count 9 so often. You can take 5×8 and then proceed."

2.7 Discussion

Although the analysis is still in progress, we find hints that it is possible to investigate the mathematics-related knowledge and its integration by applying the described method. We presented examples of our preliminary findings which illustrate the reconstruction of mental processes.

In their bachelor degree courses the participants of our study attended courses which have a strong emphasis on either content knowledge (such as "Elementary Algebra and Arithmetic" or "Geometry and Applied Mathematics") or pedagogical content knowledge (such as "Mathematical Thinking of Children" or "Planning and Implementation of Mathematics Instruction"). Moreover, the participants had no or limited teaching experiences. Therefore, it can be said that (to a certain extent) the initial teacher education program dealt with the dimensions of subject matter and pedagogic content discretely. Nonetheless, it becomes apparent that the reconstructed mental processes demonstrate how prospective elementary teachers integrate mathematics-related knowledge even at an early stage of preparation. Moreover, the participants never primarily and predominantly relied on content knowledge in these processes so far. We interpret these preliminary findings to oppose the widespread view that domains of knowledge develop independently at first and become more integrated with time.

As reported in the overview section, some studies focused on the conditions of the development of PCK. According to Krauss et al. (2008), for instance, CK facilitates the development of PCK. However, statements like this are formulated in a fairly general manner and do not really help us to understand how cognitive processes look like. Thus, the future findings of our qualitative analysis may help to complement previous studies. For instance, the example we discussed earlier with respect to the process of "Evaluating typical task difficulties from a mathematical point of view" can be interpreted to demonstrate qualitatively the possibility of developing knowledge of students' cognitions by relying on content knowledge.

Even though we did not analyze the (content) knowledge in terms of correctness or profoundness, we interpreted the reconstructed mental processes afterwards with respect to these qualities. For example, in many processes the full mathematical potential of the Multiplication Poster remains unrecognized which is probably due to limited content knowledge. We conclude that it may be valuable to connect mathematical concepts with pedagogy and pedagogical content at an early level of teacher preparation. The use of mathematics-pedagogy tasks proposed by Lin and Hsu (this volume) can be considered as a possible approach to meet this objective. Admittedly, it is necessary to be careful with generally formulated implications, among other things because teacher preparation programs in other countries or at other school levels may have different requirements.

After completing our analysis it may be necessary to stabilize, specify or complement findings by refining the sample. Moreover, we plan to compare our findings for prospective elementary teachers with the integration of knowledge domains of mathematics teacher educators. By interviewing teacher educators we hope to make professional knowledge structures visible in terms of their mental processes.

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Chapter 3
A Self-study of Integrating Computer Technology in a Geometry Course for Prospective Elementary Teachers

Jane-Jane Lo

Abstract This chapter documented a mathematics teacher educator's development in the domain of technological pedagogical content knowledge (TPACK) through the examination of the changes she made to the way technology was used in the course and the rationales behind those changes over four semesters. Challenges and opportunities that arose from making these changes were also identified. Such an account opens dialogues among mathematics teacher educators to critically examine our uses of technology in courses for future teachers. Implications for teacher education were also discussed.

Keywords Technology · Self-study · Geometry
Prospective elementary teachers

3.1 Background

Over the last few decades, there has been an increased presence of technology in K-12 classrooms. The National Council of Teachers of Mathematics (NCTM 2011) in the United States published a position statement on the role of technology in the teaching and learning of mathematics, which states: "It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication" (p. 1). Several meta-analysis studies have found a moderate but significantly positive effect of computer technology on mathematics achievement (e.g., Cheung and Slavin 2011; Li and Ma 2010). In particular, the uses of virtual manipulatives and dynamic geometric environments such as Cabri, GeoGebra, and Geometer's Sketchpad have been found to be effective in supporting the teaching and learning of geometry and measurement concepts at the elementary school level (e.g., Battista 2007; Moyer-Packenham and Westenskow 2013). However, the use of such computer

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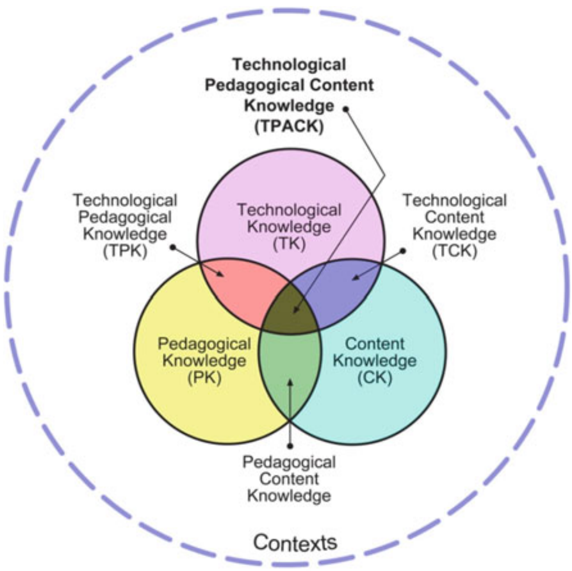


Fig. 3.1 TPACK framework and its knowledge components (Koehler and Mishra 2009, p. 63; http://tpack.org). Reproduced by permission of the publisher, © 2012 by tpack.org

technology as part of regular mathematics instruction is still relatively rare in elementary classrooms for a variety of reasons, such as the lack of regular access to technology and the teachers' insufficient *Technological Pedagogical Content Knowledge* (TPACK) (Mishra and Koehler 2006). As seen in the diagram in Fig. 3.1, TPACK is in the intersection of content knowledge (CK), pedagogical knowledge (PK), and technological knowledge (TK).

Research studies have shown that the development of TPACK takes time and requires deliberate, well-planned experiences starting with the content and method (pedagogy) courses in teacher education programs (Goos 2005). In this self-study, I reflected on the challenges and opportunities that arose from my attempts to integrate computer technology in a geometry course for prospective elementary teachers in the United States by examining my use of computer technology over a period of four semesters. The visual-based dynamic and interactive features of the computer technology were used to help prospective teachers gain a deeper understanding of the geometric and measurement properties and relationships of 2-D and 3-D shapes. It was hoped that such experiences would also increase the probability that these prospective teachers would use computer technology in their own future teaching, even though this was not part of the present study. In terms of the TPACK framework, the primary goal of the course was the development of content knowledge (CK), and the secondary goal was the development of technological content knowledge (TCK). The primary goal of the current study was to document and reflect upon my own development in the domain of TPACK as I contemplated my decisions in choosing the technologies based on my knowledge of them, incorporating each technology into the lesson plan, and facilitating prospective teachers' mathematics learning using those technologies. Such an account offers insights into my own growth of TPACK and opens dialogues among mathematics teacher educators to critically examine our uses of technology in courses for future teachers.

3.2 Conceptual Framework

Zbiek et al. (2007) distinguished two types of mathematical activities: technical and conceptual, when investigating the role of technology in mathematics education. Technical activity is about "taking actions on mathematical objects or on representations of those objects" (p. 1170). For example, the built-in transformation tools in GeoGebra can reflect, rotate, translate, or dilate a given shape with a few clicks. Conceptual activity involves reasoning, communicating, and making mathematical connections between mathematical structures and ideas. For instance, the ease of displaying multiple graphs or composing and decomposing shapes can support the development of a deeper understanding of the relationship between the areas and perimeters of various geometric shapes. Both types of activities are needed in developing deep mathematics understanding through computer technology. Sarama and Clements (2009) hypothesized seven unique interrelated affordances of computer technology in knowledge development: (1) bringing mathematical ideas and processes to the conscious awareness; (2) encouraging and facilitating complete, precise explanations; (3) supporting mental actions and objects; (4) changing the very nature of the manipulative; (5) symbolizing mathematical concepts; (6) linking the concrete and symbolic with feedback; and (7) recording and replaying students' actions. They suggested that a good use of computer technology should capitalize on one or more of these affordances. I followed this suggestion when planning the uses of technology in this geometry course. Later, I reflected upon my uses of technology by identifying the types of affordances I actually provided to my students.

Drawing upon the work by Zbiek et al. (2007), Dick (2008) and Bos (2009) discussed three types of fidelity to be considered when using technology in mathematics classrooms: pedagogical fidelity, mathematical fidelity, and cognitive fidelity. *Pedagogical fidelity* means that technology should be used in such a way that facilitates the active participation of creating and making sense of mathematics objects. In my own teaching, I tried to conduct my class in a way that was compatible with the Essential Mathematics Teaching Practices as identified by Leinwand et al. (2014), so that students had the opportunity to develop the mathematical practices as outlined by the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010). *Mathematical fidelity* means that the mathematics content presented by technology needs to be accurate and faithful to both the static and dynamic forms of underlying mathematical properties, so that it will behave as expected after mathematical actions. *Cognitive fidelity* refers to the degree to which the external representation embedded in the technology matches the mental representation of the learners using the technology. In my teaching, I considered it important to evaluate each piece of computer technology carefully for its mathematical accuracy and potential to help prospective teachers develop a deeper understanding of the mathematical properties and relationships—this understanding included both the "what" and the "why."

3.3 Method

Self-study of Teacher Education Practices refers to a special kind of action research that teacher educators undertake in order to make explicit their knowledge of practice by systematically examining their own practices (Loughran 2004; Vanassche and Kelchtermans 2015). Such self-study starts with a dilemma or question about one's own teaching practice. The goal of self-study research is to figure out a way to better understand and manage the dilemma. An important distinction between reflection and self-study is that self-study starts with reflection and strives to make the understanding public and educative so that it can be examined and transformed by other teacher educators (Bullough and Pinnegar 2001; Loughran 2004). As indicated by a review of self-study literature published between 1990–2012, there is no single methodology for self-study; rather it uses research methods and techniques such as conducting interviews and analyzing audio- or videotapes of instructional activities, autobiographical reflections, and student feedback/assignments (Vanassche and Kelchtermans 2015). Similarly to Marin (2014), in my self-study (hereafter referred to as "study" for simplicity) I adopted an "inquiry as a stance" approach and acknowledged that this is a long and complex process that involves both learning new knowledge and practices and unlearning old ones (Cochran-Smith 2003).

In this study, I sought to answer the following two research questions:

(1) How did my uses of computer technology change over four semesters and what were the reasons for making those changes?
(2) What challenges and opportunities were brought about by my attempts to use computer technology in this course?

I hope to use what I have learned from this experience to have a conversation with other mathematics teacher educators who share the same interest in integrating computer technology in their own teaching or experience the same challenge when doing so. Collectively, we will then be more conscious of the rationales and consequences of the instructional decisions we make. Such knowledge will be helpful in identifying key features of computer usages that can be studied further for their effects on supporting prospective teachers' mathematics learning.

3.3.1 Setting

The study took place at a university in the Midwest of the United States where all prospective elementary school teachers are required to take a specially designed geometry course.[1] In addition to the focus on reasoning and communication, this course is designated as technology-intensive, where prospective teachers can learn mathematics with technology regularly throughout the course. Prior to the start of this study, the technology used in the course included a TI-73 (a graphing calculator specially designed for middle school students) used throughout the course, a 2.5-day Geometer's Sketchpad (GSP) unit on triangle and quadrilateral constructions, and a 2-day Scratch unit on angles and regular polygons. Discussion with previous instructors of the course revealed two main objectives of the GSP and Scratch units: (1) introducing rich technology tools to the prospective teachers with the hope that they might use them in their future teaching; and (2) helping prospective teachers to gain a deeper understanding of the mathematics topics embedded in the activities, such as properties and relationships of special triangles and quadrilaterals, angles, and regular polygons.

3.3.2 Data and Analysis

The primary data of this study included my daily lesson plans and reflection notes of the lessons in which computer technology was used or mentioned as well as the corresponding coursepack pages from Fall 2013 (baseline data) to Spring 2015 for the same semester-long geometry course for prospective elementary teachers that I taught four times. The secondary data included student work from one project and two exam questions that had computer-related components from each semester during the same time period. Table 3.1 summarizes the primary data sources.

With the primary data, I first catalogued all the uses of computer technology for each of the semesters. Then I applied comparative analysis, as outlined by Strauss (1987), to identify the emerging themes and used them to generate hypotheses about the rationales behind those changes. I then continued to look for evidence, both confirming and disconfirming, for those hypotheses. The goal was to generate models that would be "useful to us, which have support in the data, that fit or interact productively with our larger theoretical framework, and that give us a sense of understanding by providing satisfying explanation about hidden processes underlying the phenomena in an area" (Clement 2000, p. 559). The analysis of student projects and exam questions provided additional insights into this decision-making

[1] I would like to acknowledge the other instructors' influence on many of the decisions and reflections discussed in the rest of the chapter as we shared the same course materials, including assessments, and met weekly to discuss various course-related matters.

Table 3.1 Primary data sources

Semester/number of students	Fall 2013 (n = 30)	Spring 2014 (n = 28)	Fall 2014 (n = 27)	Spring 2015 (n = 33)
Lesson plans and reflection notes	4.5 lessons and 3 notes	6 lessons and 5 notes	9 lessons and 12 notes	14 lessons and 11 notes

process. I then reflected on the challenges and opportunities that arose from my attempts to integrate computer technology into this course.

3.4 Findings

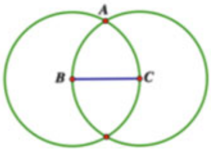
In this section, I first describe the changes I made in each semester with the reasons that have prompted those changes. I then discuss the challenges and opportunities that arose from implementing those changes.

3.4.1 Initial Change in Spring 2014: Earlier and More Integrated Experience

Historically, the goals of using the GSPs and Scratch units in this course were to develop prospective teachers' technological proficiency in using these two pieces of computer technology while gaining a deeper understanding of the properties and relationships of special polygons, including their exterior and interior angles. For example, after learning the basic construction tools through the step-by-step tutorials of constructing isosceles triangles and squares, prospective teachers were expected to construct other special shapes, like right triangles and rhombi, by using both their technological knowledge of GSP and their content knowledge of the properties of 2-D shapes such as congruent sides and right angles. However, I noted in my teaching journals in Fall 2013 that many prospective teachers struggled with both aspects, despite the follow-up class discussions on the mathematical principles behind various constructions.

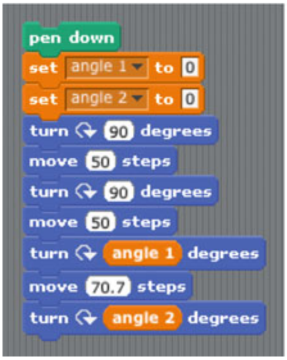
Prospective teachers' performance on a GSP-based assessment on the midterm exam was far from satisfactory. Figure 3.2 shows the assessment item and the result distributions. Similar results can be seen from a Scratch assessment item displayed in Fig. 3.3. In the Scratch assessment only 21% of the prospective teachers were able to take on the perspective of the object that carried out the motions and considered the orientation of the turn, while 57% of them answered 45° for both angles. In other words, 57% of the prospective teachers had yet to master an important mathematical idea of the Scratch programming which was to take on the spatial perspective of another object.

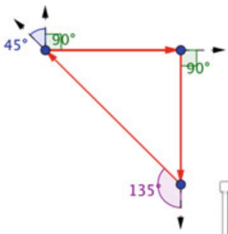
When connecting points A, B, and C with line segments, what type of triangle does the student construct? Justify your response WITHOUT using any angle or side length measures.



Correct	16 (57%)
Partially correct	4 (14%)
Totally Incorrect	8 (29%)

Fig. 3.2 Sketchpad assessment item and student performance





Correct	6 (21%)
Partially Correct	12 (57%)
Totally incorrect	6 (21%)

N=24, Fall 2013

Fig. 3.3 Scratch assessment item and student performance

After reflecting on my use of computer technology in Fall 2013, I made a decision to give earlier and more instructional time to both programs by developing additional in-class activities and out-of-class exercises. The goal was to provide more opportunities to develop both prospective teachers' technological and content knowledge and to make connections between the two (Mishra and Koehler 2006). For example, prospective teachers used the special quadrilaterals they constructed through GeoGebra[2] to explore the properties of diagonals and symmetries, and the relationship between areas and perimeters on parallelograms that have the same bases and heights. The built-in measurement tools and the ability to use "dragging" to create multiple examples quickly made it possible for prospective teachers to form and test their conjectures efficiently. Such capabilities afforded students with opportunities to bring mathematical ideas and processes to the conscious awareness and support mental actions and objects (Sarama and Clements 2009).

[2]Many factors prompted the switch from Geometer's Sketchpad to GeoGebra; a discussion of these factors is beyond the scope of this chapter.

3.4.2 The Main Changes in Fall 2014: Shifting from Using the Tools Directly to Using the Ready-Made Applets

When reflecting on the prospective teachers' overall learning at the end of the Spring 2014 semester, I recognized that a significant number of the prospective teachers in our courses had conceptual gaps in fundamental ideas of measurements grounded in physical activities. For example, while many of them understood the triangle inequality relationship (see Fig. 3.4 for an assessment item), they were not able to identify the correct side lengths because they counted the number of holes instead of the number of unit lengths between holes, a typical error made by elementary school students (Hiebert 1981).

To address this issue, major changes were made to the content, sequence, and instructional approaches of the course. A unit called *Tools for Investigation* was created for the first three weeks of the semester. During this unit, the prospective teachers spent time on developing basic concepts and vocabulary that arose naturally from classifying 3-D shapes.

To make room in the curriculum for this new unit, a difficult decision was made to remove Scratch from the course. This decision was made also because I recognized that the Scratch unit was rather isolated from the rest of the course, which made it harder to develop the technical expertise that was needed to support the conceptual development. Furthermore, instead of having the prospective teachers construct various types of triangles and quadrilaterals through step-by-step tutorials as they did in the previous semester, they were given already-constructed Shape Makers (Battista 2001; see examples in Fig. 3.5) that retained their properties under dragging motions. This was helpful in investigating questions such as "What are the common properties of all the shapes created by this particular Shape Maker?" or in determining the validity of the hierarchical relationships such as "Every square is also a rectangle and a rhombus." I also made this decision hoping to shift the prospective teachers' attention from the technical activity to the conceptual activity when using GeoGebra (Zbiek et al. 2007).

In addition, in Spring 2014, I attended the annual conference of the Society for Information Technology and Teacher Education (SITE) conference. While there, I learned about the interactive resources readily available online that had been used successfully by other mathematics teacher educators in their mathematics courses for prospective teachers. Selected online applets from websites such as

Hayes is using Polystrips to explore the question "Can any three side lengths make a triangle?" Using the two lengths provided with the given Polystrips, what are the possible lengths of the third Polystrip that is longer or equal in length to the other two side lengths? Explain.

Fig. 3.4 An assessment item on the concept of triangle inequality

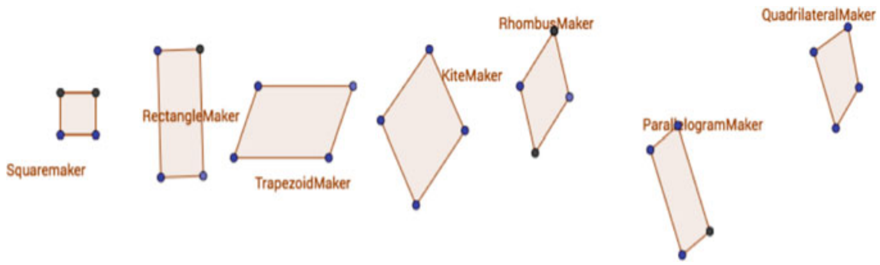


Fig. 3.5 Special quadrilateral makers

Annenberg Learner (https://www.learner.org/interactives/?=MATH) and Illuminations (https://illuminations.nctm.org/), as well as videos from Art of Problem Solving (https://www.artofproblemsolving.com/) were introduced throughout the course for in-class explorations, daily homework, and extended writing projects. I paid special attention to the affordances when choosing the online applets (Sarama and Clements 2009). For example, a math interactive from Annenberg Learner (see Fig. 3.6) allowed students to inspect the properties of prisms by rotating and by folding-unfolding them. Such visuals helped the prospective teachers to make better connections between the 2-D and 3-D representations of prisms, which were needed to develop the formula for surface areas. This applet changed the very nature of the manipulative and supported the development of mental actions and objects (Sarama and Clements 2009). Furthermore, the uses of these websites planted seeds in the prospective teachers' development of technical pedagogical content knowledge (Mishra and Koehler 2006) since all these websites were originally developed as teaching resources and professional development sites for mathematics teachers.

3.4.3 The Main Changes in Spring 2015: Focusing on GeoGebra Applets

The effects of the changes made in Fall 2014 were mixed. The instructors soon found that not all online applets were created equal. Some required more instructional time than anticipated to help the prospective teachers figure out how to use them, some were too open-ended, and some were cognitively too complex for some prospective teachers. To address these issues, we turned to the huge collections of GeoGebra explorations created by mathematics teachers around the world housed in GeoGebra Materials (https://www.geogebra.org/materials/) to identify ready-made applets that could be used to enhance prospective teachers' experiences of specific learning goals. Most of these applets used action buttons and sliders to control dynamic visual effects, which made it possible for the prospective teachers to focus more on making sense of the mathematical ideas behind the animations than worrying about the technical details. For example, GeoGebra files such as the

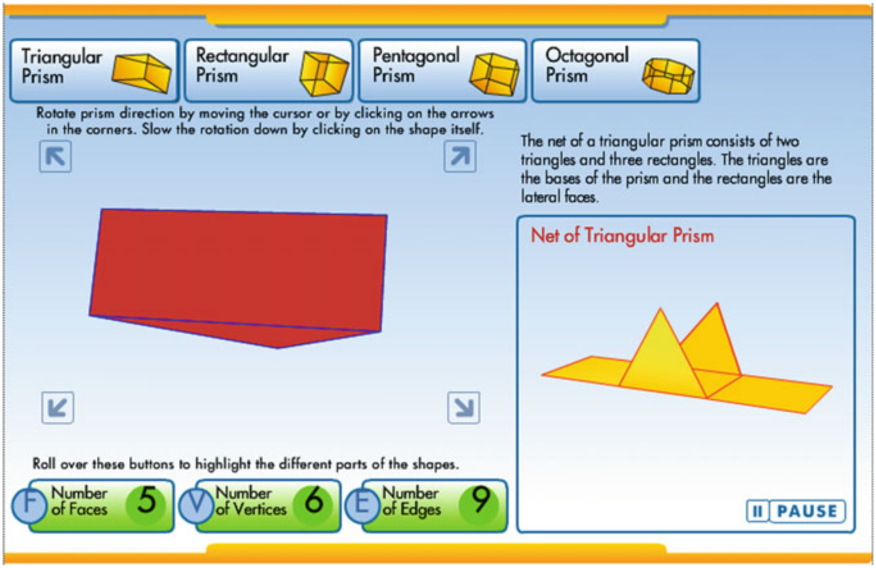


Fig. 3.6 Prism explorations from http://www.learner.org/interactives/geometry/3d_prisms.html. From Geometry: 3D Shapes, used with permission by Annenberg Learner, www.learner.org

one in Fig. 3.7 were used to deepen prospective teachers' concepts of angles. This applet contains two games. The first game, illustrated in Fig. 3.7, provides an opportunity for students to estimate the measure of a given angle. Each player takes a turn to guess the "set angle." The second game asks students to "draw" an angle for the given angle measure by stopping the spinner, which slowly opens up from angle 0° to 360°. Both games help prospective teachers develop the conceptual image of "an angle as a turn" and a better sense of the relative sizes of various angles. GeoGebra tools like this one were introduced throughout the semester and became an integral part of the course.

In addition to using the ready-made GeoGebra applets such as the Angle Spinner Game, the instructors also started to create new GeoGebra applets or to modify the existing ones to better support prospective teachers' learning. In all, 31 GeoGebra applets were used in the course in Spring 2015 to address a wide variety of topics in the course, such as special angles on parallel lines, transformations, areas and perimeters of polygons, and similarity.

3.4.4 Challenges and Opportunities

Reflecting back on my uses of computer technology over four semesters, I can see three main progressions. The first one was a shift from isolated to more integrated uses. Prior to the study, Geometer's Sketchpad and Scratch were introduced and

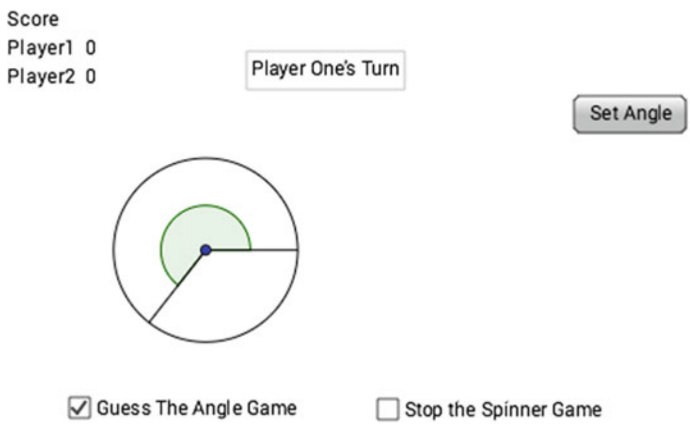


Fig. 3.7 Angle spinner game with scores at https://www.geogebra.org/m/Crj4aUwd

used only during their respective units. They were not brought up in class outside of those units. At the end of the study, GeoGebra applets were used throughout the entire semester to investigate various concepts. The second progression was a shift from using multiple software and websites to a single software: GeoGebra. The third one was a continuous effort in shifting prospective teachers' focus from technical to conceptual activities. My decisions were primarily prompted by careful analyses of the prospective teachers' performance on both the formative and summative assessments as seen in Figs. 3.2 and 3.3.

Many challenges to support prospective teachers' learning remain. These challenges give rise to the need for more systematic investigations and thus opportunities to extend the boundaries of mathematics education research. I will discuss two specific challenges with examples.

The first challenge relates to the connections among the physical, virtual, and mental activities that remain elusive for me; this makes it challenging to determine the sequence and the balance between using physical versus computer-based activities to address the same concepts. For example, circumference is a difficult concept with a challenging formula for prospective teachers. Prior to the use of computer technology, working in small groups of 3 or 4, prospective teachers were given a set of paper circles of various sizes. They were first told to measure the circumference (C) and diameter (D) of the circle and calculate the ratio of $\frac{C}{D}$. The whole-class discussion was conducted after each group posted its results on the board. Many issues came up during the discussion, such as the issues of precision and possible causes of the errors and variations, before the eventual realization that all the credible results were close to 3. The corresponding discussions took 30–40 min of class time to accomplish. Now consider the GeoGebra worksheet in Fig. 3.8.

This simple GeoGebra worksheet took away all need to discuss issues related to the physical measurements and, at the same time, focused prospective teachers'

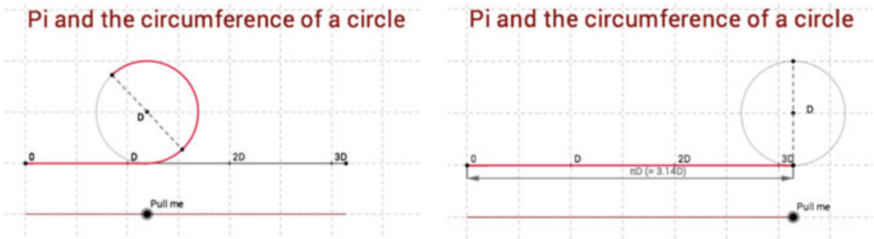


Fig. 3.8 A GeoGebra worksheet from http://tube.geogebra.org/material/simple/id/58826

attention on the main objective behind the 30-min activity, that is, to identify that the ratio between the circumference and diameter was a little over 3. It also highlighted the idea of circumference as the "distance around the circle" by visualizing the "unrolling" action, which few of the prospective teachers used as the basis to measure the circumference of a given circle. By adding another "slider," the prospective teachers could experience the same action with circles of various sizes, which led to the conclusion that this relationship existed for all circles. So the questions arose: Which experience, the physical or the visual one, will lead to a more robust conceptual understanding of the relationship between the circumference and diameter of a circle? Does the answer depend on individual learning styles? In my most recent experience teaching this topic, I used the physical experience first and then the visual one to solidify the idea. But I do not always have time to do both for all topics. So how should decisions such as this be made? Looking at the issue from a broader perspective, the root of these issues points to the insufficient information about the prospective teachers' development of basic geometry and measurement concepts in both contexts, which is needed to address the issue of cognitive fidelity.

The second challenge relates to the fact that each prospective teacher, with his or her own laptop or tablet, can engage in dynamic explorations that, although open new possibilities for them, also create new challenges. For example, I found it quite challenging to engage prospective teachers in class discussion when the online explorations were more open-ended, because typical applets did not allow prospective teachers to record or re-examine their trials as the changes occurred rapidly with each click or drag. Consider the online interactive in Fig. 3.9 and the two corresponding questions that the prospective teachers were asked to solve: (1) What is the smallest possible perimeter using 12 square tiles? (2) What is the largest possible perimeter?

While the ease of having the computer quickly calculate the perimeter facilitated the pattern finding for the first question, it also left few traces of their trial-and-error process and little time to ponder and reflect before making the next move. This often made it quite challenging to facilitate the discussion of the second question, because many figures will lead to the same largest perimeter. It soon became clear

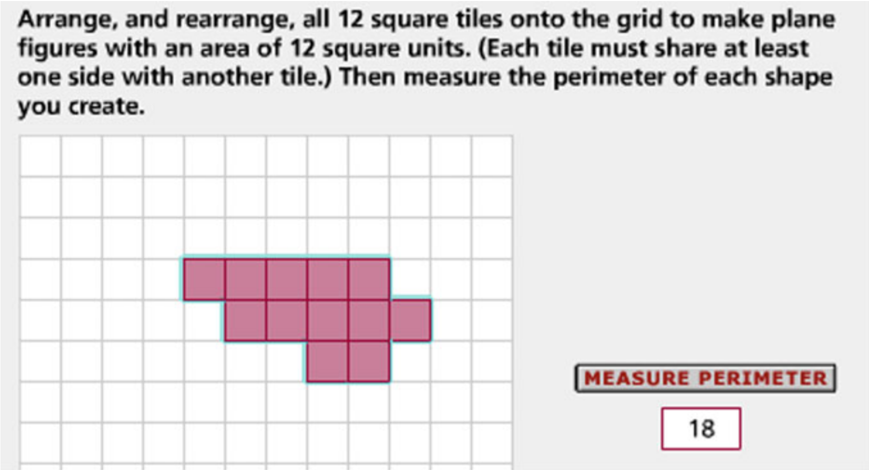


Fig. 3.9 Fixed area problem at http://www.learner.org/courses/learningmath/measurement/session9/part_a/constant.html. From Learning Math: Measurement, used with permission by Annenberg Learner, www.learner.org

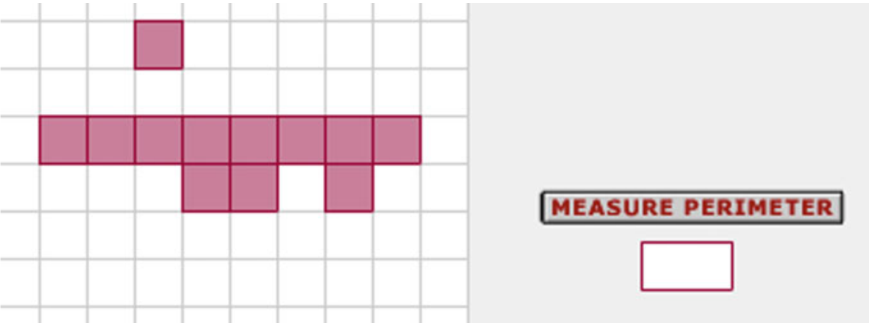


Fig. 3.10 Shapes for posting the focusing questions

to me that it would be necessary for the prospective teachers to record the results of their explorations on paper to facilitate the class discussion on the common characteristics of these different figures. My reflection after the lesson suggested that there should be more focused questions, such as asking the prospective teachers to figure out how the placement of the additional square in Fig. 3.10 might create different perimeters (20, 22, or 24 units) of the resulting shapes and why. This new insight also contributed to the justification of why a 3 by 4 rectangle would result in the minimum perimeter.

3.5 Discussion

In this paper, I reflected on my own uses of technology, over four semesters, in a geometry course that lasted one semester each time and was specifically designed for prospective elementary school teachers. I analyzed the major changes in the ways I used technology in this course, documented the reasons for those changes, and noted the opportunities and challenges that arose from the process. While concerns over prospective teachers' mathematics content knowledge was the primary impetus for all the changes, my responses had different characteristics. These differences reflect a trajectory of my growth in TPACK. As a novice in technology use, my initial response was "the more the better." As my own expertise in technology use grew, I became aware of the pedagogical issues, such as choosing technology to support the learning goals rather than learning technology as a learning goal in itself, and making curricular decisions regarding what to add, remove, and modify. My students and I were able to explore more mathematical patterns and relationships with the availability of computer applets than we were able to do without the technology. For example, properties and relationships of special quadrilaterals such as symmetry, diagonals, congruent sides and angles, and parallel and perpendicular lines really came alive for me and many of my students when using the Shape Makers. In recent semesters, my colleagues and I have started developing a GeoGebra book on "similarity" that prospective teachers can explore on their own to learn about the concepts of dilation and mathematical similarity and the relationship between them. Therefore, we are moving toward the role of technology maker, admittedly in a very basic way. This journey has been similar to the five-stage developmental process identified by Niess and her colleagues when learning to integrate a particular technology in the teaching and learning of mathematics: recognizing (knowledge), accepting (persuasion), adapting (decision), exploring (implementation), and advancing (confirmation) (Niess et al. 2009).

The growth of my TPACK knowledge occurred in the context of my own teaching that was supported by my colleagues. Prospective teachers in our classes had the opportunities to see us making on-the-spot decisions to use GeoGebra to facilitate class discourses on a particular challenging mathematical concept or to alter the lesson plans when facing technological difficulties. We believe that such experiences were crucial in helping prospective teachers develop their own TPACK. We have observed an increased number of prospective teachers using GeoGebra to construct examples or counterexamples when asked to justify their answers to questions such as "Are all equilateral triangles/rectangles/rhombi similar?" More research studies are needed in terms of studying the trajectory of the development of TPACK.

So far, I have addressed only the first four out of the seven unique interrelated affordances of the computer technology in knowledge development hypothesized by Sarama and Clements (2009) that are consistent in my course: (1) bringing mathematical ideas and processes to the conscious awareness; (2) encouraging and facilitating complete, precise explanations; (3) supporting mental actions and objects; and (4) changing the very nature of the manipulative. I still need to find better ways to help prospective teachers use computer technology to symbolize their mathematical ideas, make connections between concrete and symbolic, and record/replay their actions for individual reflection and whole-class discussion. To take advantage of more affordances of computer technology, I plan to use the new GeoGebra group feature that supports easy collaborations among students and colleagues. Specifically, the new GeoGebra group allows sharing and commenting on each other's GeoGebra creations and offers the capability to add questions and provide feedback to interactive GeoGebra tasks (http://www.geogebra.org/blog/2015/12/geogebra-groups/).

My journey from a novice technology user to a technology maker was compatible with the learning-by-design approach proposed by Koehler and his colleagues (Koehler et al. 2011). They suggested that "through engaging in pedagogical design activity with technology around specific content areas teachers not only gain knowledge of content, pedagogy and technology (and their relationships), they also engage in dialogue and collaboration to develop and scaffold their own learning" (p. 152). Other mathematics teacher educators, with deeper content and pedagogical knowledge than those of typical K-12 teachers, may also be capable of engaging in similar activity without formal professional development. In this chapter, I offered the account of my attempts to integrate computer technology in a geometry course for prospective teachers to open dialogues among mathematics educators who are also working on integrating computer technology into their own teaching. These types of reflective dialogues are critical to the continuing pursuit of excellence in mathematics teacher education in the context of technological advancement.

Looking at the challenges I faced when integrating computer technology in a mathematics course for prospective teachers from a broader perspective, the root of these challenges points to the need for more information about how to effectively integrate computer technology in supporting prospective teachers' mathematics learning, which is needed to address the issue of pedagogical fidelity and which would inform my own development of TPACK. The two main challenges I encountered—keeping a healthy balance between the physical and computer-based activities with respect to the time constraints, and promoting meaningful dialogues among students who generated multiple answers rapidly without much time for pondering and reflection—pointed to a new research direction, similar to those identified by Lin and Hsu (this volume) and Pilous et al. (this volume). More studies are needed to investigate the nature of the technology, the way to use it, and the relationship between technology and prospective teachers' learning of geometry and measurement. Furthermore, this study was not able to answer the question of how some of the changes we observed in the prospective teachers' in-class actions and writing were brought about by the changes in the computer use. This is another limitation of this study that needs to be investigated further with future research studies that include student reflection and interview data.

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Chapter 4
Pre-service Elementary Teachers' Generation of Multiple Representations to Word Problems Involving Proportions

Ryan D. Fox

Abstract How well can pre-service elementary teachers answer word problems? Furthermore, can they represent the same answer in multiple ways? To answer these questions, I conducted a study with four pre-service elementary teachers answering word problems that incorporate proportional reasoning to investigate the strengths and opportunities for growth. I found three pre-service elementary teachers generated different representations of the correct solution: writing proportions to solve by cross-multiplication, drawing pictures to solve by repeated addition, and creating tables to solve by percents. One pre-service elementary teacher did struggle to produce effective strategies to solve some of the presented word problems.

Keywords Pre-service elementary teachers · Mathematical knowledge for teaching

For the past 30 years, education researchers have used various terms to identify and explicate a specialized knowledge for teaching mathematics: starting with Shulman's (1986) Pedagogical Content Knowledge and continuing with Mathematical Knowledge for Teaching (Ball et al. 2008), Profound Understanding of Fundamental Mathematics (Ma 1999), and Mathematics for Teaching (Davis and Simmt 2006). For pre-service elementary teachers (PSETs), the search for such a specialized knowledge base feels more elusive and challenging. Mathematics teacher educators deal with the intertwined issues of PSETs' lack of confidence regarding mathematics and length of time away from mathematics classes (Goulding et al. 2002). Without proper content knowledge and confidence in the subject, teaching mathematics well to elementary students becomes a problem of worldwide significance (Vula and Kingji-Kastrati, this volume; Shaughnessy and Boerst, this volume; Lin and Hsu, this volume; Pilous et al., this volume). Yet, mathematics teacher educators persist. To support the mathematical development of PSETs,

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mathematics teacher educators engage in programs of teaching and research to address this important issue. This chapter discusses one small step toward this goal.

The research question guiding this study is: how can a pre-service elementary teacher generate multiple representations of a solution to apply her knowledge of proportional reasoning to a sequence of contextual problems? This study addresses the research question by examining the quantity and quality of representations the PSET generates. Three of the four participants provided responses of similar quality and quantity. The fourth participant struggled to produce the same quantity and quality.

4.1 Relevant Literature

As an activity for students, generating multiple solution paths to a single question is consistent with procedures with connections, a high-level cognitive demand task within Stein et al.'s (2000) framework. In their explanations to procedures with connections tasks, Stein and colleagues suggest multiple representations use "visual diagrams, manipulatives, [and] symbols…[to make] connections among multiple representations…to develop meaning" (p. 16). However, in previous works, researchers have shown PSETs struggle to demonstrate this desirable activity. Depaepe et al. (2015) showed PSETs struggled to answer questions correctly involving fractions. Because of their content struggles, PSETs could not access the appropriate pedagogical content knowledge to support students' different representations to solutions to test questions.

Researchers have shown the progress PSETs made to develop stronger mathematical knowledge for teaching. Baek et al. (2017) found PSETs could generate many representations, particularly using pictures, to word problems that involved fractions. However, not all PSETs could answer questions correctly. Although PSETs made progress in performing a valuable activity, they often struggled with challenging mathematical content: coordinating multiple units within a single question. Stohlmann et al. (2015) started with PSETs who did not possess conceptual understanding of topics from the elementary curriculum. After a course focusing on multiple representations, PSETs changed their beliefs on teaching to include an emphasis on conceptual understanding and meaning making.

Turner and Rowland's (2011) work on the Knowledge Quartet can describe the nature of a specialized knowledge of teaching elementary mathematics. As Turner and Rowland mentioned, "[while] we believe certain kinds of knowledge to be desirable for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher…*ought* to know" (p. 197, emphasis in original). Their work is an extension of the work of Rowland et al. (2005). The earlier study examined the application of university students' mathematical knowledge developed from their teacher preparation program to their clinical experiences at the end of their preparation program. The four categories they elaborated were foundations, transformations, connections, and contingencies. Turner and Rowland (2011) defined foundations as pre-service teachers' "knowledge [and] understanding… in preparation (intentionally or otherwise) for their role in the classroom" (p. 200). This component of the quartet is the only one not defined in terms of the practice of teaching. Instead, foundational knowledge is a collection of networks of information a teacher develops before his or her own teaching career begins. Foundational knowledge is generally knowledge PSETs acquired before they begin their teacher preparation program. In order to develop multiple solution paths, PSETs need access to a single solution path. Answering a question correctly from the elementary mathematics curriculum would comprise part of a PSET's foundational knowledge.

As PSETs begin their teacher preparation program, they develop the next component of the Knowledge Quartet, transformation. Rowland et al. (2005) describe transformation knowledge as a "focus on knowledge-in-action as *demonstrated* in planning to teach and in the act of teaching itself" (p. 261, emphasis in original). Transformational knowledge could be developed through the generation of multiple representations to a solution. As school students, PSETs solved many word problems that involve setting up a proportion with an unknown quantity and determining the value of the unknown. Such work would be classified as foundational knowledge. Transformational knowledge could involve explaining other connections between quantities in the proportion or providing illustrations to encourage students to visualize the quantities involved in the proportion.

In my previous works (Fox 2012, 2013), I examined how two pre-service elementary teachers solved word problems involving proportions. The two participants, Stephanie and Hope, had two contrasting approaches in the solutions to the problems. My original intention in selecting participants was to have Stephanie and Hope serve as opposing ends of performance on these word problems, with other participants fitting somewhere in between. Stephanie took a rather consistent approach to solving the problems (Fox 2012). In her desire to be as "clear" as possible, Stephanie repeated a three-step algorithm as a solution to each question: re-writing of the scale from the problem, a sub-division of the number line segment into the appropriate number of parts, and adding the wholes and parts to get to the final correct answer. Hope's approaches were less consistent than Stephanie's (Fox 2013). Hope wanted to find an approach that could answer all word problems. However, when the numbers involved in the problems changed—from whole numbers to fractions to mixed numbers—Hope's attempts did not transition well. She did develop an algorithm to getting the right answer in working with mixed numbers. She could not reproduce the algorithm when reflecting on her work in a later interview. In this chapter, I want to outline the work of two other participants and find connections across participants.

4.2 Method

This study involves the same method as outlined in previous reports (Fox 2012, 2013). All four participants in this study are PSETs. All four participants were in their second year of their undergraduate careers when they participated in the study. All had completed the university's one required mathematics content course required for their teacher preparation programs. None of the participants had taken a mathematics methods (pedagogy) course.

I interviewed each participant four times. The four-interview sequence was a modification of an interview sequence suggested by Seidman (2006). In detailing a three-interview sequence that could be applied to all social sciences, Seidman (2006) mentioned key features of each interview:

> The first interview established the context of the participants' experience. The second allows participants to reconstruct the details of their experience within the context in which it occurs. And the third encourages the participants to reflect on the meaning their experience holds for them. (p. 17)

Four participants completed my four-interview sequence. In Interview 1, I asked about the participant's background in mathematics and desire to teach elementary school. I concluded the interview with five word problems. After each word problem was a question for reflection. I asked follow-up questions to probe for additional information from the participant's reflection. I asked the participants ten word problems each in Interviews 2 and 3. Together, these interviews extend Seidman's middle interview into two separate interviews: each interview involved different details of mathematical content. In Interview 4, each participant reflected on her experiences in the Interviews 1, 2, and 3. I posed no word problems in Interview 4. Because the focus of this chapter is on the participants' mathematics, data from Interview 4 is not included in this chapter.

The four interviews in this study satisfy Wilson's (2013) definition of semi-structured interviews: "[t]he semi-structured interview…allows some standardization of questions and also the freedom to explore and add new questions as unexpected topics emerge" (p. 41). I asked each participant the same word problems. When having the participant explain her response, I would break away from the interview guide to explore why the participant wrote down what she did. The participant's responses to my reflection question led to additional questions that I could not foresee asking before the interview.

In this study I used two categories of word problems, which are described in Table 4.1. During Interview 1, I asked three Road Map and two Floor Plan word problems. For all five word problems, the number of miles and the number of feet are whole numbers. Across the twenty word problems in Interviews 2 and 3, I asked ten Road Map and ten Floor Plan word problems. Five Road Map word problems contain miles represented as a fraction between 0 and 1. Five Floor Plan word problems contained feet represented as a fraction between 0 and 1. Five Road Map word problems contained a number of miles written as a mixed number; five Floor Plan word problems contained a number of feet written as a mixed number.

Table 4.1 Word problems given to each participant throughout the first three interviews

Category	Stem of word problem	Total number of questions asked
Road Map	Let's say that I am looking at a map and the map has printed on it, "1 inch = ___ miles". How far apart are two towns if they are ___ inches apart on the map? How do you know your answer is right?	13
Floor Plan	If I had a drawing of a floor plan of a house, and the plan has printed on it, "1 inch = ___ feet". How wide is a [room] if the [room] is ___ inches wide on the plan? How do know your answer is right?	12

The word problems posed in this study would satisfy Crespo's (2003) introduction regarding mathematical tasks: "Even the most routine of mathematical activities can be constructed into a worthwhile mathematical experience when posed in such a way as to engage students in mathematical inquiry" (p. 244).

I recorded all interviews using audio and video recording devices. The video recording devices captured the written responses of the participant; the audio recording devices captured the discussion between each participant and me. During the interviews, I took field notes to capture my initial impressions of the participants' responses and to assist with later analyses.

To provide additional analysis of the participants' work, I developed ternary diagrams to map the performance of each of the four participants. I reviewed each of the participant's written responses to the 25 word problems. I coded a participant's response to each word problem using one of three codes: without a correct response, a correct response with a new representation, or a correct response with a previously used representation. I placed a participant's distribution of codes on the same ternary diagram to determine if any differences existed in the rate of codes applied across the four participants.

4.3 Results

In this section, I begin by providing a summary of results from Fox (2012, 2013) for Stephanie and Hope. I then provide more detailed results from two other participants, Brooklynne and Arielle. I selected the third and fourth participants as representatives of most PSETs' performance in the same mathematics course.

4.3.1 Stephanie

Stephanie's answers (Fox 2012), revealed the same process used for responses to word problems with proper fractions and mixed numbers in Interviews 2 and 3.

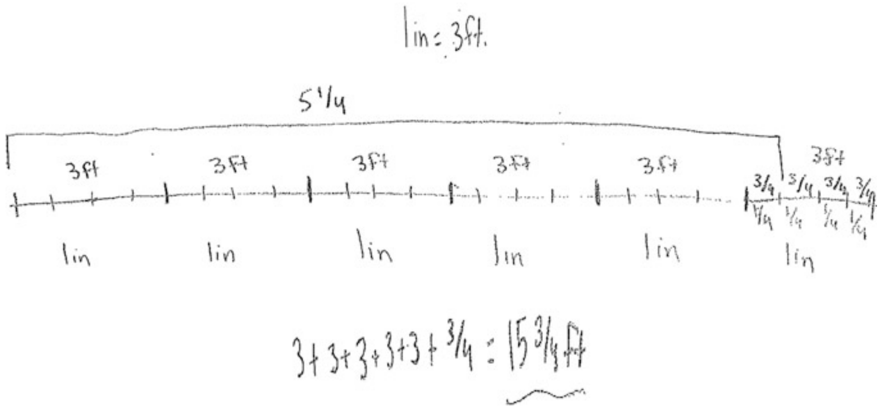

Fig. 4.1 Stephanie's representations of solutions (Fox 2012)

Each response always included a re-writing of the scale, the use of a number line to represent both the number of inches and the number of corresponding number of feet or miles, depending if the word problem was from either the Road Map or Floor Plan category. When using fractions, Stephanie represented both the numerator and denominator as number of pieces on the number line. An example of Stephanie's work on a Floor Plan word problem from Interview 3 can be found in Fig. 4.1.

4.3.2 Hope

Two themes in Hope's work are struggle and success (Fox 2013). In her written work, Hope did not provide correct final answers to six of the 25 word problems: one during Interview 1 and five during Interview 2. During Interview 3, Hope found success by repeating one algorithm that worked for word problems with mixed numbers. An example of Hope's success can be seen in Fig. 4.2, a Floor Plan question with 1 in. = 3 ft and the room being 6 2/5 in. wide.

4.3.3 Brooklynne

Brooklynne provided correct answers to her written responses for all 25 questions. In Interview 1, Brooklynne correctly answered the word problems, and explained answers as if she were thinking of her future students. Brooklynne wrote a paragraph explanation for each reflection. An example from Interview 1 can be seen in Fig. 4.3.

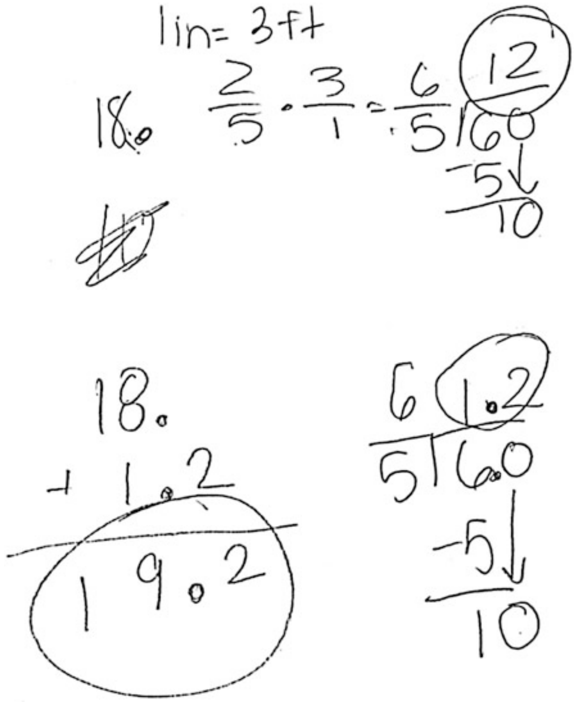

Fig. 4.2 Hope's successful repetition of an algorithm to answer problems in Interview 3

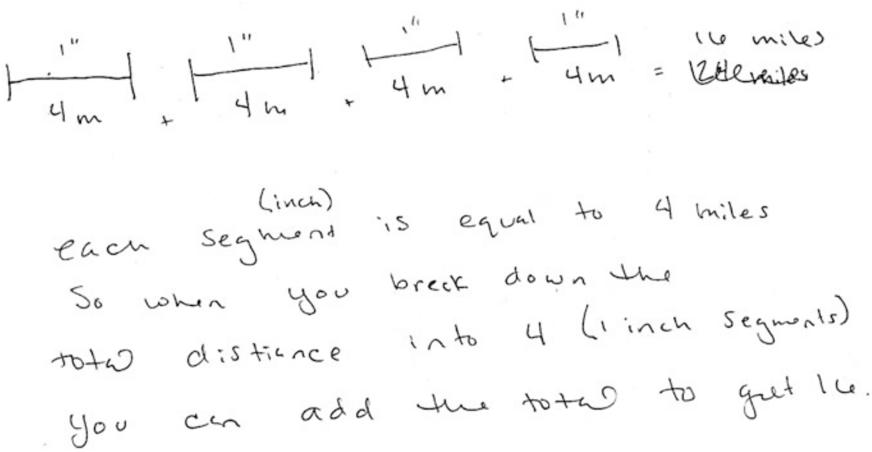

Fig. 4.3 Brooklynne's written reflection to a Road Map word problem in Interview 1

However, Brooklynne did not provide the same picture and written explanation strategy to all word problems in this interview. Her initial solution to the first word problem (a Road Map question filling in the blanks with 1 in. = 1 mile and 6 in. apart) was to provide a solution as if she was explaining the solution to a peer: set up a proportion and use cross-multiplication to determine the unknown value. Brooklynne acknowledged drawing pictures to represent the solution could be a more desirable alternate to the solution for younger students than the one she had:

> I think you could easily draw this out on a board….So you can draw one inch equals one mile…and then you can add them altogether and say six miles.

In Interview 2, Brooklynne represented whole-number multiplication as the repeated addition of whole numbers. In a Road Map word problem, Brooklynne correctly identified how using the scale, 1 in. = 4 miles, can be used to find the distance between two towns if they are only one-fifth of an inch apart: divide one unit by five to represent one fifth of an inch, and then do likewise for each picture representing the four miles. Brooklynne drew rectangles to represent the solution involving word problems from the Floor Plan category, as seen in Fig. 4.4. When I probed to ask why she drew two-dimensional pictures for floor plans, instead of straight-line distances between cities on a map, Brooklynne said:

> I think because this [question] is saying it's wide. It's like…a two-dimensional measurement, around….Like, wide, or length, or something like that.

In Interview 3, Brooklynne used different strategies to facilitate the computations in her solutions. In Fig. 4.5, Brooklynne used repeated addition to multiply four and three quarters by four. When determining the sum of three quarters four times, Brooklynne added two groups of partial sums to get three. She added that answer to four fours to get the final answer of nineteen.

In a reflection to a different Road Map word problem, Brooklynne said:

> I did the same thing for this one, but, I just used point two five instead of, um, point five. Because it was a fourth instead of a half. And then I just did the same thing where I added them all up, and they equaled one. And then I added all threes to equal twelve. And then I got thirteen.

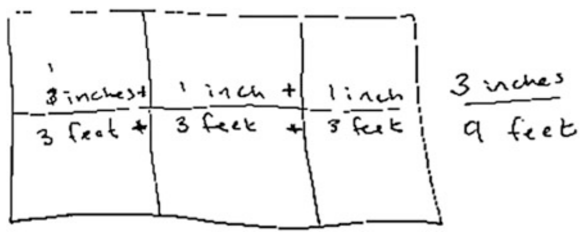

Fig. 4.4 One example of Brooklynne's rectangles to represent a Floor Map word problem

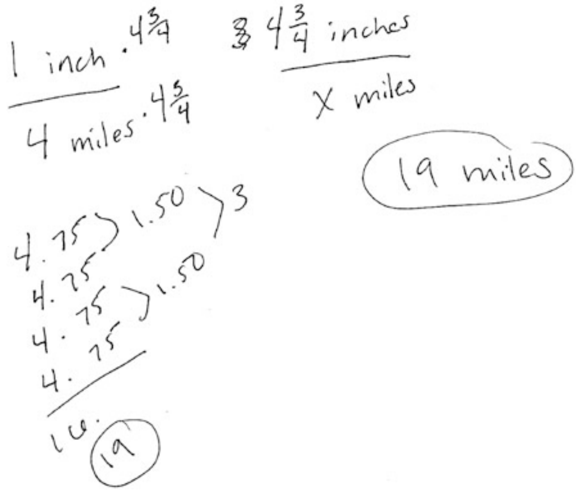

Fig. 4.5 A collection of Brooklynne's solutions and representations

4.3.4 Arielle

Across the first three interviews, Arielle provided the correct answers to all 25 word problems. Figure 4.6 includes examples of Arielle's representations for the final answer: proportions, repetitions of the scale value, and a table of values. In the figure, the proportions is for a Road Map word problem in Interview 1. The repetition of the scale value is for a Floor Plan word problem in Interview 3. The table of values is for a Road Map word problem in Interview 3. Arielle referenced her use of three different approaches for one solution to a Road Map question in Interview 2:

> First I drew out….So, half of one is a half. So half of four is two. So that's how I kind of saw it right away. But I still drew the picture. And I still wrote out what I thought down there when I just saw at first. Like, one divided by a half equals a half. Four divided by a half is two. And then I did the cross multiply. So I kind of checked it three times.

For the final word problem in Interview 3 (a Floor Plan question with 1 in. = 3 ft and the room being 6 2/5 in. wide), Arielle noticed that, for the same scale of floor plans, a larger number of inches in the floor plan corresponded to a larger number of feet on a floor plan. The previous Floor Plan word problem had the same scale, but used a drawing that was 2 2/5 in. wide. Arielle reflected on her use of number sense in her reflection to this word problem:

> Six and two fifths is at least double two and two fifths. So, I knew it had to be at least greater than sixteen.

Arielle saw patterns from previous questions that would help her in answering later questions. Additionally, Arielle noticed my convenient choice of numbers to get whole number answers. During Interview 3, Arielle commented:

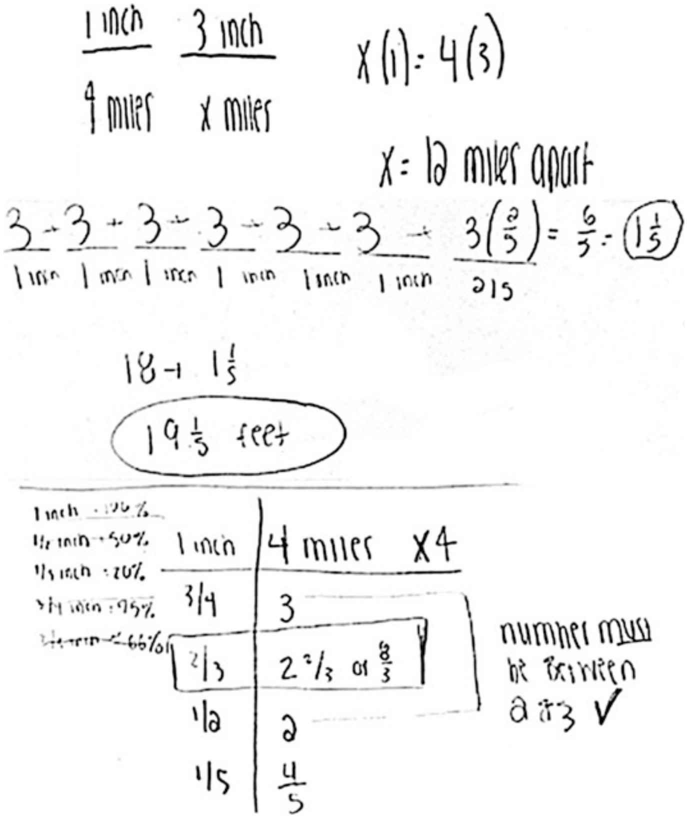

Fig. 4.6 Collection of Arielle's representations of solutions

> So, what's three fourths of four? So. I got the twelve over four. And I divided that. And It's three miles. And at first I was looking at it like why is it coming out so evenly? And then I went back and looked and I was like, wait. It's four and four. Like three fourths. One. Two. Three. Four. I should have just one, two, three miles.

4.4 Discussion

In this section, I will discuss the work of the four participants together and the potential for future work. Because these PSETs did not have much interaction with the topics of fraction multiplication and proportions recently, I believe some participants would struggle generating a single—let alone generating multiple—representations of the solution to the problems. Hope's work seemed to fit that belief. Brooklynne's work showed more correct answers and more representations of those correct answers than what Hope's work showed. Stephanie's work involved more representations than Brooklynne's work. Arielle's work involved more representations than any other participant.

4.4.1 Examining the Work of All Four Participants

For all participants, the knowledge employed to answer all questions is consistent with the codes found in the foundations component of the Knowledge Quartet (Turner and Rowland 2011): "overt subject knowledge" and "reliance on procedures" (p. 200). Three of the participants also demonstrated an activity found in the transformations component: "choice of representation" (p. 200). Arielle's reference connecting the numbers used in one word problem to numbers used in prior word problems could be consistent with Turner and Rowland's code for connections, "decisions about sequencing" (p. 201). As other researchers in this volume (e.g., Lajoie, this volume) determined, the contingency component of the Knowledge Quartet depends on the strength of the other three components of the Quartet.

Stephanie and Arielle provided three representations to a single solution. They differed in how they used the three representations. In most solutions, Arielle used the same strategy as Stephanie of representing the length as repetitions of the given scale. Stephanie's three steps followed the following sequence: re-write the scale, draw the picture to scale, then add up the corresponding values. Arielle's three steps seemed to inform each other: the picture, the chart, and the proportion all provide different contexts toward the same answer. In Interview 2, Arielle reflected on helping her younger brother with mathematics homework. She provided an interesting insight into how her preparation to become a teacher presented itself when helping her brother:

> And I explain things to him [in] so many different ways. And he sometimes gets frustrated because I'm talking to him like he is in second grade. But I am not doing it on purpose. It's just kind of like the classes I am taking.

Stephanie's approach is to create the one best explanation that would support as many students as possible in a single explanation. Arielle wants to support the mathematical development of as many students as possible by presenting different approaches and encouraging students to use the one approach that they would want to implement. Both PSETs exhibit components of a transformational knowledge: behavior "directed towards a pupil (or a group of pupils)…which follows from deliberation and judgement informed by foundation knowledge" (Turner and Rowland 2011, p. 201).

Table 4.2 provides the distribution of word problems that involved correct representations using new representations, correct responses using previously used representations, or without a correct representation. Because each participant attempted all 25 problems, the sum of each row will be 25.

Table 4.2 Distribution of responses by participant across three categories

Participant	Number of questions without a correct response	Number of questions with a correct response but a previously used representation	Number of questions with a correct response and a new representation
Brooklynne	0	16	9
Stephanie	0	17	8
Hope	6	14	5
Arielle	0	14	11

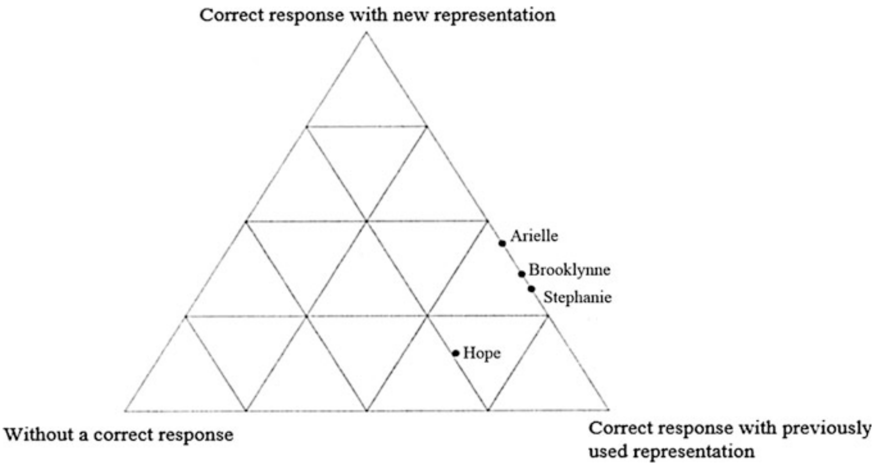

Fig. 4.7 Illustrating distribution of participants' responses in a ternary diagram

In Fig. 4.7, I placed the data from Table 4.2 on a ternary diagram. Placement of a participant's dot on the vertex of the triangle indicates all of a participant's responses received the same code. Each segment on the interior of the triangle represents one-quarter of the responses assigned that code. For example, placement of a participant's dot on the horizontal line closest to the vertex represents 75% of word problems attempted by a participant receiving the code correct response with a new representation. The next horizontal line going down represents 50% of word problems attempted by a participant receiving the category correct response with a new representation. The horizontal line farthest away from the vertex represents 25% of word problems attempted by a participant receiving the category correct response with a new representation. If a participant's collection of responses did not receive codes from a category, then the participant's placement would be the side of the triangle opposite of the vertex with that code. Because no response received the code without a correct response, three participants' locations on the diagram are on the side of the triangle opposite from the vertex representing the category without a correct response.

Based on their performances in their content course, I conjectured Brooklynne and Arielle would be in a similar location in this diagram, with Stephanie using a greater number of representations and Hope far fewer representations than these two. Figure 4.7 shows three participants' results are closely connected: Hope's location on the diagram is removed from the other three. Hope did not provide correct written responses to 24% of the word problems across Interviews 1, 2, and 3. As a result, her placement on the diagram is closest to the segment representing 25% of the word problems receiving the code without a correct response. Placement of the three participants in the diagram between the 50 and 75% segments of all questions being correct but using previous representation suggests to me that I may have made an artificial distinction between the work of Brooklynne and the pairing of Stephanie and Arielle. The dots for these three participants are on the same side of the triangle and in between the same endpoints for the correct response with representation code. Brooklynne, Stephanie, and Arielle created approximately the same number of representations in their written work.

4.4.2 Extending the Study

Although this study examined a small number of participants, the results here encourage discussions on the work of mathematics educators in preparing future elementary teachers. For example, in what ways can mathematics educators dispel common mathematical misconceptions held by PSETs? In Brooklynne's Interview 1, she connected whole number multiplication to repeated addition. She said, upon reflection, this was an effective strategy in explaining her solutions. In Interview 2, Brooklynne's comment about using division for smaller values—going, for example, from one to one-half—creates effective solution strategies for this particular word problem. However, do these reflections perpetuate mathematical myths that division makes the quotient smaller and multiplication is equivalent to repeated addition? Brooklynne's final answers are correct, but beliefs about the nature of multiplication and division give mathematics educators opportunities to encourage PSETs to explain and justify solutions. Providing multiple representations to a solution could encourage PSETs to examine their own misconceptions.

Additional discussion points for this study include ways mathematics educators support PSETs to extend and enrich their mathematical knowledge. How can mathematics educators support students like Hope? She was the only participant in this study not to answer all of the questions correctly. By expanding the number of participants, mathematics educators could identify additional PSETs in need of support in developing mathematical content. A PSET would likely struggle to develop multiple representations if the PSET cannot provide a correct solution. By recognizing some PSETs already possess a stronger foundation of mathematical knowledge, mathematics educators could incorporate enrichment opportunities to build a deeper, more connected network of knowledge (Ma 1999). In the future, how could mathematics educators support students like Stephanie and Arielle?

Having access to multiple representations permits PSETs to see the same problem from different perspectives. Without those multiple representations, PSETs could revert to a single solution approach (Gupta et al., this volume), demonstrating the same misconceptions they caution their students not to make (Ryan and McCrae 2005/2006).

Because elementary teachers around the world are trained to teach all academic subjects (Fennell, this volume; Lin and Hsu, this volume; Vula and Kingji-Kastrati, this volume), PSETs have a limited amount of time to prepare to teach mathematics. In that time, mathematics teacher educators need to find the right combination of research and practice to support the mathematical development of PSETs (Lo, this volume). With appropriate mathematical and pedagogical knowledge bases, mathematics teacher educators can present PSETs important aspects of teaching to support the mathematical development of children.

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Chapter 5
Pre-service Teachers' Mathematics-Content and Mathematics-Specific Pedagogical Preparation

Ana Donevska-Todorova, Martin Guljamow and Katja Eilerts

Abstract This chapter considers the opportunities for growth and integration of mathematical content- and mathematics-specific pedagogical content knowledge in pre-service elementary school teachers. Throughout an analysis of four contributions we present similarities and differences in the theoretical and methodological approaches which refer to prevalent knowledge concepts like *content knowledge*, mathematics-specific *pedagogical content knowledge*, *technological pedagogical content knowledge*, *mathematical knowledge for teaching*, and the *Knowledge Quartet*.

Keywords Pre-service teachers · Elementary mathematics · Mathematics content knowledge · Mathematics pedagogical content knowledge

5.1 Introduction

Mathematics education of pre-service elementary school teachers continues to be an interesting theme not only in national but also in international research discussions. The 15th ICMI Study, The Professional Education and Development of Teachers of Mathematics (Even and Ball 2009), had its focus on teachers' possession and permanent development of mathematical knowledge as a prerequisite for students' opportunities to learn mathematics and opened a wide field for new investigations. For example, the number of corresponding topic study groups (TSG) increased from one in ICME 12 to three in the recent ICME 13, thus confirming the high relevance of the topic and interest of the community members actively involved: TSG 45 Knowledge in/for teaching mathematics at primary level, TSG 47 Pre-service

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mathematics education of primary teachers, and TSG 49 In-service education and professional development of primary mathematics teachers. Although the themes of the three TSGs may seem alike on the first look, there are some characteristics, which give the TSG 47 a different ‘flavor’. Not only that the group was interested in contributions that are relevant in an international context but it also offered enough room for research ideas open to any type of institution where teacher education may take place. Further, it considered similarities and differences of formal education systems for future mathematics teachers across a variety of nations and regions and also factors which influence this diversity and stimulate innovative paths and forms of education programs. Considering theoretical perspectives in the studies presented within the group, we aim to point out some common features related to the applied *theoretical frameworks* and *methodologies* by the authors. In spite of the fact that the received studies have mainly a national character and inform about ‘local’ contexts in the German speaking countries, USA and Taiwan, for example, there are commonalities that may be relevant for the wider cross-cultural audience. One such interesting common feature may be *pre-service teachers’ opportunities to learn* during (and after) their professional education. This chapter does not consider social, economic or political circumstances as preconditions or opportunities to learn but it refers to the long-lasting simultaneous gain and integration of both teachers’ mathematics-content and mathematics-specific pedagogical knowledge which we elaborate further in the next sections.

5.2 Theoretical Background

In this section we want to give the reader a general idea of the conceptual frameworks that the contributions to this theme draw upon. Research investigations on existing connections between mathematical *content knowledge* (CK) and mathematics-specific *pedagogical content knowledge* (PCK) of teachers continue to grow ever since the initial work of Shulman (1986). For example, recent international comparative research on connections between CK and PCK has come to different results, with evidence for correlations between CK and PCK in the case of prospective teachers in secondary schools in contrast to a less clear interdependence concerning elementary school teachers (Blömeke et al. 2010). Notwithstanding that CK and PCK represent different domains on pre-service teachers’ mathematical knowledge, the existing relationships between them are multifaceted and deserve more attention in the research agenda. The large number of models alone which represent various components of PCK outline the complexity of the connections between these knowledge categories. Furthermore, they point to challenges for investigations of mental processes grounded on different components of teacher knowledge while activating one or more of them in a particular teaching situation. Moreover, the inclusion of *information and communication technology* (ICT) in the teaching of mathematics has brought new perspectives on and needs for an overarching conception of subject matter, pedagogy and technology integration (Niess 2005). We come back to this point again throughout an analysis of an empirical study by Lo (this volume).

It comes as no surprise that all four of the contributions presented to this theme build their theoretical basis on Shulman's famous content categories. The once postulated "blind spot" (Shulman 1986, p. 7) concerning *content* has long since been overcome. Instead, within the last decades the mathematics education research community has made considerable progress on Shulman's initial request "for a more coherent theoretical framework" (ibid., p. 9). One of the most prominent attempts within the field of (elementary) mathematics teacher education is that by Deborah Loewenberg Ball and colleagues at the University of Michigan which is referenced by three of the four contributions to this theme. Through extensive qualitative analyses of "the regular day-to-day, moment-to-moment demands of teaching [mathematics]" (Ball et al. 2008, p. 393) the Michigan group developed a practice-based theory of *mathematical knowledge for teaching* (*MKT*), which integrated all of Shulman's content knowledge categories and concretized six relevant sub-domains. The first category, *subject matter knowledge* (*SMK*), comprises *common content knowledge* (*CCK*), *horizontal content knowledge* (*HCK*) and *specialized content knowledge* (*SCK*) whereas the other category, *pedagogical content knowledge* (*PCK*), contains *knowledge of content and students* (*KCS*), *knowledge of content and teaching* (*KCT*) and *knowledge of content and curriculum* (*KCC*) (ibid.). Although the Michigan group admits openly that "it is not always easy to discern where one of our categories divides from the next" (ibid., p. 403), Hill et al. (2005) could demonstrate predictive validity of MKT for students' learning gains and thus its practical relevance. One of the contributions to this theme refers strongly to the *Knowledge Quartet*, an alternative conceptual framework by Rowland et al. (2005) of the University of Cambridge. Whereas the Michigan group directed their main focus on different domains of mathematical knowledge relevant for teaching the Knowledge Quartet puts emphasis on the identification of situations in which mathematical content knowledge (CK and PCK) surfaces in the teaching process in elementary school classes. A qualitative analysis allowed for the classification of the following four categories with identification codes, which have since then been used in teacher training as "a means of reflecting on teaching and teacher knowledge, with a view to developing both" (Rowland et al. 2005, p. 257): *foundation*, *transformation*, *connection* and *contingency*. For example, *foundation* refers to the knowledge, understanding and beliefs of mathematics prior to actual teaching experience, whereas the other three dimensions "focus on knowledge-in-action" (ibid., p. 262) with *transformation* representing the next step in teacher trainee development or in Shulman's words "[…] the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful" (1987, p. 15).

5.3 Thematical Discussion with Reference to Four Exemplary Studies

Drawing upon the theoretical considerations above, in this section we report about the findings of the following four studies that are grouped in this volume under Theme 1. We discuss the fourth study in a separate sub-section from an additional perspective to further explicate the alignment of the conceptual (content) framework to the demands of digitally supported mathematics education.

- Using mathematics-pedagogy tasks to facilitate professional growth of elementary pre-service teachers (Fou-Lai Lin and Hui-Yu Hsu).
- Investigating the relationship between prospective elementary teachers' math-specific knowledge domains (Roland Pilous, Timo Leuders, and Christian Rüede).
- Pre-service elementary teachers generation of multiple representations to word problems involving proportions (Ryan Fox).
- A self-study of integrating computer technology in a geometry course for prospective elementary teachers (Jane-Jane Lo).

Putting the four contributions to this theme in a coherent order promised to be a challenge, since at first glance at least thematically there did not appear to be much common ground. However, starting from the background of the authors connecting aspects became apparent: All of the authors are mathematics teacher educators and represent exemplary combinations of practice and research. Also, the contributions by and large confirm two major claims of a research review by Adler and colleagues (2005) that within researching mathematics teacher education "small-scale qualitative research predominates" (p. 368) and that most inquiries are "conducted by teacher educators studying the teachers with whom they are working" (p. 371). Although presenting the only more or less conceptual contribution, *Fou-Lai Lin* and *Hui-Yu Hsu* draw on students' expressions and task-design examples to describe their attempts to shape the mathematics courses of their elementary teacher program in Taiwan. *Roland Pilous* and *colleagues* conducted task-based interviews regarding arithmetic with pre-service elementary teachers (n = 6) to get a better understanding of the relationship between mathematics-specific knowledge domains. Equally task-based but with a very different methodological approach, *Ryan Fox* investigates the strategies of pre-service elementary teachers (n = 4) through four-interview sequences respectively to identify "strengths and opportunities for growth" and at the same time deduce consequences for the practice as a teacher educator. The reflection of her own practice as a teacher educator, especially as a novice of instruction with dynamic geometry software (DGS) tasks and tools in a technology-intensive geometry course, constitutes the focus of the concluding self-study (n = 1) by *Jane-Jane Lo*. The most striking common ground of the works of all authors presented in this theme, is the essential of (mathematics) teacher education research: A sincere ambition to contribute to "*understanding* how teachers learn, and from what opportunities, and under what conditions" and "*improving* teachers' opportunities to learn" (Adler et al. 2005, p. 363; italics in original). *Opportunity to learn* (OTL) has become an important factor within international comparative studies "as an explanation of differences in achievement and as a cross-national variable of interest in its own right" (Floden 2002, p. 237). With regards to our theme the second notion can be linked directly to questions concerning the design of teacher education programs, e.g. for scholars and policy makers who might be interested to see what aspects of mathematics-content or mathematics-specific pedagogy preparation other countries include in their curricula. However, the definition of OTL according to Floden (2002) has taken many forms since its introduction by Husén (1967), and "may refer to aspects of curricula, instructional materials, instructional experiences, or time available for instruction" (Tatto and Senk 2011, p. 124). Taking aspects such as these into account, reports of low emphasis on OTL related to mathematics content and mathematics pedagogy content in elementary teacher education programs across the world (Tatto et al. 2009) sound alarming. The descriptions of the situation in Taiwan by Lin and Hsu (this volume) seem to underline this impression. According to them pre-service elementary teachers in their teacher university only attend two mathematics-related courses within the program. Furthermore, many of their candidates seem to show little conceptual understanding of mathematics and little comprehension of the role of student cognition for learning. This delivers valuable insider views, and might enrich discussions about international comparative results on elementary school teachers in mathematics. These results, while illustrating outstanding achievements of prospective primary teachers concerning CK and PCK in Taiwan, at the same time referred to a relative weakness to diagnose student understanding (Blömeke and Delaney 2012).

There is much consensus among the scientific community that tasks are one if not the major OTL in mathematics teacher education (e.g. Clarke et al. 2009; Sierpinska and Osana 2012; Zaslavsky 2007). Furthermore, the causal relationship between task and (mathematical) activity of everyone involved serve as a sound explanation for the manifold uses of tasks in the context of mathematics (teacher) education and its research (Watson and Mason 2007). The four contributions of this theme, that will be introduced more closely in the subsequent paragraphs, may give an idea of the broad range of task use: tasks as a tool for course development and design (Lin and Hsu), as the central research tool in task-based interviews (Pilous et al., Fox) and task design and testing to investigate teaching practice (Lo).

The conceptual contribution by Lin and Hsu (this volume) offers enlightening insights into pre-service teacher education for elementary schools in a teacher education university in Taiwan with a focus on the only two mathematics-related components—subject matter and pedagogy course. Hence, the mathematics curriculum of the elementary teacher preparation program in question reflects the prevalent separation into CK and PCK. The authors' research project consists of a holistic attempt to bridge the former separation with a comprehensive implementation of *mathematics-pedagogy tasks* (MPTs) with reference to the works of Watson and Mason (2007), Baturo et al. (2007) and Stylianides and Stylianides (2010) concerning task-based course design. In this context, MPTs for pre-service teachers serve as an inspiring means to actively engage with mathematical problems in order to enable participants to make analogies from their own learning processes to key aspects of student cognition and to experience and comprehend the didactical significance of curriculum arrangement and textbook design. Therefore, MPTs might be understood as an attempt to find an adequate and often postulated “balance between CK and PCK” (Baumert et al. 2010, p. 167). Lin and Hsu deliver examples of their MPTs to exemplify the didactical conception and prospective outcomes within the course context. One can only hope for a subsequent study on the effects of this approach in the near future.

Pilous and colleagues (this volume) have provided an overview of studies differentiating relations between CK and PCK according to a possible correlation, co-development, integration or association. Aiming to examine the integration of mathematics-related CK and PCK, the authors have set three objectives, nevertheless reported findings mainly on one of them due to an ongoing analysis, and an intended extention of their study. The authors’ future objectives are to develop a model of procedures for describing types of integration of different components of CK and PCK. Yet, based on qualitative analysis of task-based interviews with six pre-service teachers in Switzerland about arithmetic in grades two to six, they suggest a method in order to generate a wide set of data for further investigations and practical implementations. The method relies on *open*, *axial* and *selective coding* of data and categorization of certain kinds of knowledge such as *knowledge about students’ cognition, curricular and teaching related knowledge, content knowledge and didactical knowledge*. As mental processes activated by different mathematics-related knowledge domains, the authors have identified two, “evaluating typical task difficulties from a mathematical point of view” and “remembering content knowledge in the function of illustrating curricular or teaching-related knowledge.” These results anticipate consideration of future integration of knowledge domains of mathematics teacher educators with an emphasis on the necessary conscious application in similar programs in other countries.

Fox (this volume) presents an explorative study following a very minute qualitative empirical approach to identify mathematical knowledge of pre-service teachers for elementary schools and their ability to apply this knowledge to answer word problems that incorporate proportional reasoning. The theoretical background is based essentially on the works of Ball et al. (2008) regarding the concept of *Mathematical Knowledge for Teaching* (MKT) and related findings within a classroom-based approach concerning *the Knowledge Quartet* by Turner and Rowland (2011). However, with regards to the latter Fox could focus mainly on foundational knowledge since this concurs with the knowledge base which can be presumed within participating pre-service teachers prior to a teaching career. The methodological paradigm of *explanatory teaching* (Steffe and Thompson 2000) was realized within four-interview sequences with each of the four participants and allowed for an in-depth participatory experience whose retrospective analysis was strongly promoted by audio and video recordings, the written responses to the mathematical tasks of the participants and field notes. Following up on prior research results on two participants, Fox was able to expand his already detailed picture on the underlying mathematical knowledge base of pre-service teachers and their task-solving performance. The results point to important matters of heterogeneity—already well known in school classrooms—concerning mathematical knowledge and performance of pre-service teachers, which question existing undifferentiating course designs. For example, significant mathematical misconceptions concerning numbers and operations could be revealed on the one hand and strong achievement potentials on the other. Respective findings could constitute an important contribution to prospective (mathematics) teacher program development.

5.4 Teaching Mathematics with Technology

The International Society for Technology in Education (ISTE 2000) announced new standards which confront teachers with an enrichment of their professional practice by considering technology-enhanced instruction and communication. While the implementation of technology in K-12 mathematics classrooms has an increasing tendency, research findings point out moderate, though positive, effects in the learning of mathematics, e.g. in elementary geometry. Questions related to teaching, teacher education and teachers' professional development, which refer to the use of digital resources in elementary schools, are ongoing and of significance in the current debates. Such questions are often tackled within the theoretical construct of technological-pedagogical and content knowledge (TPACK or formerly TPCK) (Koehler and Mishra 2005). Taking into consideration studies that point out specification of TPACK for the purposes of explorations in mathematics education in particular, M-TPACK may be more beneficial for addressing concrete problems and offer more in-depth insights into concrete mathematical theme. Mishra and Koehler (2006) have suggested the *learning by design approach* which provides teachers with unique *opportunities for learning, knowledge construction and meaning making*. In her self-study of teacher education practices, as a special type of action research, Lo (this volume) has applied this approach in her own teaching and attempted to develop a geometry course for prospective elementary school teachers, which would integrate digital resources. The transition from one dynamic geometry software (DGS) to another for designing learning materials have turned out to be challenging. Eventually, this resulted in the decision to develop a GeoGebra book for mathematical concepts like dilatation and similarity to promote engagement in GeoGebra groups for collaborative learning. By considering the rationales and implications of such changes, the author has reported a personal growth of awareness of the goals of making curricular decisions rather than setting the learning about technology as a goal of itself. It may be the case that the content knowledge about technology (CT) has led to a development of pedagogical content knowledge (PCK) by reconsidering how technologies have brought mathematical ideas and processes to an awareness or have supported mental manipulations of geometrical objects. In her self-reflection, the author has also noted an advantage of substituting a multiple- with single software usage during a three semester-long instruction and a shift from prospective teachers' foci on technical to conceptual aspects. The remark, "In my most recent experience teaching this topic, I used the physical one first and then the visual one to solidify the idea. But I do not always have time to do both for all topics" (Lo, this volume) shows the challenge of balancing between physical vs. / and computer-based activities. It validates the question on how could a decision be made in order to "maximize content learning in context and to develop the knowledge, skills, and attitudes identified in the standards" (ISTE 2000, p. 1). A second challenge is facing the students' difficulty in following, recording and re-examining quick changes in own trial and error processes, also reported in other studies. An issue that arises from this self-study is whether and how such individual progress of a teacher educator could be addressed systematically for prospective designs of professional development programs for teachers.

5.5 Conclusions

In this chapter we have looked at pre-service teachers' mathematics-content preparation as well as their mathematics-specific pedagogical preparation as discussed within the four chapters grouped under Theme 1, examining those from both theoretical and methodological perspectives. We have addressed opportunities to learn from a point of view of different forms, approaches and programs for teacher education, rather than from a political or socio-economical aspect. The four contributions exemplify such opportunities. They present explicit research interests in mathematical content- and mathematical specific pedagogical knowledge that is relevant to elementary school teachers and draw upon well-known theoretical frameworks. The first two studies in our analysis (Fou-Lai Lin and Hui-Yu Hsu; Roland Pilous and colleagues) strongly refer to the constructs of CK and PCK and the third one (Ryan Fox) frames within Mathematical Knowledge for Teaching (MKT) and the Knowledge Quartet. The fourth contribution (Jane-Jane Lo) implies an additional component related to technology usage (TPCK) and also action research in order to describe and reflect the researcher's own learning paths as a novice teacher. There are also similarities between the contributions from a methodological point of view. Pilous et al. and Fox have developed task-based interviews to serve their qualitative analyses. Over a period of three semesters Lo has created, implemented and self-reflected on course materials and lesson plans which served her comparative analysis of action research. This shows that, besides research and practice aspects in all of the studies, there also appear design components from different types which researchers and/or teachers undertake in their own teaching and learning processes.

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Part II
Professional Growth Through Activities and Assessment Tools Used in Mathematics Teacher Preparation Programs

Chapter 6
Preparing Elementary School Teachers of Mathematics: A Continuing Challenge

Francis (Skip) Fennell

Abstract This chapter analyzes the mathematical background and related program experiences of pre-service elementary education candidates (defined here as grade levels K–6) in the United States. The chapter includes an annotated review of selected international perspectives which have the potential to influence this topic [International perspective derived from recent international research and selected papers from ICME 13 TSG 47 (Hamburg Germany July 24–31, 2016)]. The analysis traces historical elements of elementary teacher preparation in the United States, considers the importance of mathematical knowledge for teaching at the elementary school level, current issues related to teacher accreditation in the U.S. and internationally, and the potential of elementary school mathematics specialists. It concludes by presenting current and emerging challenges related to elementary education programs, including supply and demand, program quality and the need for ongoing research, with particular attention to the impact of mathematical knowledge for teaching, elementary mathematics specialist models, and preparation provider impact and integrity.

Keywords Mathematical background of elementary teachers
Elementary mathematics specialists

6.1 How Are Elementary Teachers Prepared: Historical Background

Challenges related to the content background of pre-service elementary school teachers in the United States (grade levels K–6), and to some extent internationally, can be traced back to the history of the normal school movement (Labaree 2008). Normal schools originated in the 19th century in France and had a singular purpose—the preparation of teachers. In the United States the most influential

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normal schools were state-supported. The first normal school in the U.S. was the Lexington (MA) Normal School which opened in 1839, and was eventually moved and, like most normal schools, expanded beyond its original purpose. It is now Framingham (MA) State University. There were many normal schools in the U.S., some examples include: Bowling Green State Normal School, now Bowling Green State University; California State Normal School, now UCLA; Howard Normal and Theological School for the Education of Teachers and Preachers, now Howard University; San Diego Normal School, now San Diego State University.

As their name suggests normal schools were expected to set the norm for good teaching. The course of study, typically just two years for the pre-service elementary candidate, included a mixture of liberal arts courses, for content area background, and specialized training in pedagogy, which included arithmetic (forerunner of the elementary mathematics methods course), the study of educational history, philosophy, and psychology, and supervised practice teaching. It must be noted that to this day, elementary classroom teachers in the United States and many other countries (e.g. Japan, Finland) are prepared as generalists. That is, such candidates are expected to be equally knowledgeable, and perhaps equally passionate about, reading/language arts/literacy, mathematics, science and social studies—the four major content areas of the elementary school curriculum. Many, if not most, candidates also have to complete coursework and related experiences in the arts and physical education. This model, the generalist model for elementary teacher preparation, is derived from its normal school roots.

6.2 Knowledge for Teaching

In the United States, departmentalized elementary schools became popular in the early 1900s (Bunker 1916), in an effort to ensure that content focused teachers taught all of the mathematics (or science or social studies) at a particular grade level, typically Grades 4 through 6 (Becker and Gleason 1927). While cities like New York, Chicago, and St. Louis adopted departmentalization of their elementary schools, so did many rural and suburban school districts (Pierce 1935). From the 1930s to the present there have been swings of interest in departmentalization at the elementary school level. The 1930s represented a decade of debate between those advocating for self-contained classrooms and those advocating for departmentalization. The 1950s interest, particularly post Sputnik, in mathematics and science education led to increased interest in departmentalized elementary schools. During the 1970s, projects like the Developing Elementary Mathematics Enthusiasts Project were created to identify school-based mathematics "enthusiasts" who cared enough about the importance of mathematics to assist their colleagues by serving as after school and before school mentors and generally providing the mathematics support for the building (Fennell 1978). The implementation of No Child Left Behind (2001) and the current interest in the *Common Core State Standards for Mathematics* (NGA and CCSSO 2010) has heightened the interest in mathematics-specialized teachers within a departmentalized elementary school (Hood 2009). However, the self-contained classroom remains the most common structure for school and class organization at the elementary school level in the United States (Hood 2009).

In the 1980s, scholars began to investigate "knowledge for teaching," criticizing earlier research on teaching effectiveness for ignoring the subject matter and its transformation into the content of instruction (Shulman 1986; Ball and Bass 2003; Ball et al. 2008). Initially, this line of research analyzed the actions of teachers in classrooms or outcomes of interviews with teachers, rather than survey data and test scores. The focus was on identifying kinds of knowledge relevant for the teaching of mathematics, rather than mathematical knowledge in general. Mathematical knowledge for teaching is the mathematical knowledge needed for teaching the subject, and includes tasks involved in teaching (e.g. student and teacher use of representations, questioning, monitoring student learning, providing feedback to students) and the mathematical demand of these elements within any lesson (Ball et al. 2008). See related perspectives on mathematical knowledge for teaching in Shinno et al. (this volume); Jacobson et al. (this volume); and Celik et al. (this volume).

More recently, particularly given the generalist model for the preparation of elementary school teachers in the United States, the National Mathematics Advisory Panel (NMAP), charged by then President George W. Bush, with reviewing scientific evidence and making recommendations on improving mathematics education, made the recommendation that the mathematics background of prospective elementary teachers, defined here as the coursework and related experiences provided at the pre-service level, be strengthened as one means for improving teacher effectiveness (NMAP 2008). An important part of this recommendation, which was one of forty-five of the Panel's recommendations, was that prospective and inservice teachers be given ample opportunities to learn mathematics for teaching. "That is, teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (NMAP 2008, p. xxi). The Panel noted that research is needed to create a sound basis for the mathematics preparation of elementary teachers at the pre-service level, stating that outcomes of varied approaches be evaluated by using reliable and valid measures of their effects on prospective teachers' instructional techniques and, most importantly, their effects on student achievement. While research, both within the United States and internationally, is needed to validate this important NMAP recommendation, content knowledge for teaching is an accepted consideration for pre-service teacher education at the elementary school level in the United States and internationally.

The *Mathematical Education of Teachers II* (*MET II)* report (CBMS 2012) makes recommendations for the mathematics that teachers, including elementary teachers, should know and how they should come to know it. The *MET II* report reiterates and elaborates the themes from the first *Mathematical Education of Teachers* report (CBMS 2001). These include: proficiency with school mathematics is necessary but not sufficient mathematical knowledge for a teacher and that the mathematical knowledge needed for teaching differs from that of other professions.

The *MET II* report recommended that prospective elementary teachers be required to complete at least 12 semester-hours of coursework involving fundamental ideas of elementary mathematics, their early childhood precursors, and middle school successors. The mathematical knowledge for teaching at the elementary school level suggested in the *Mathematical Education of Teachers II* report particularly addressed the Standards for Mathematical Practice and appropriate content domains of the *Common Core State Standards for Mathematics* (NGA and CCSSO 2010). The content domains include: Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Number and Operations—Fractions, Measurement and Data, and Geometry. It was also noted that since elementary classroom teachers prepare their students for the middle grades (typically grades 6–8) in the United States, coursework for elementary teachers should also attend to how the mathematical concepts and understandings at the elementary level build to those at the middle grade level. So, it is expected that pre-service elementary candidates in the U.S. should also understand important elements within the middle grade content domains of Ratio and Proportional Relationships, The Number System, Expressions and Equations, and Statistics and Probability. The *MET II* report further suggests that number and operations, treated algebraically with attention to properties of operations, should occupy about 6 of the 12 credit hours required, with the remaining 6 credit hours devoted to additional concepts of algebra (e.g., expressions, equations, sequences, proportional relationships, and linear relationships), and to measurement, data, and geometry.

Recent research related to the mathematical background and preparation of elementary teachers, and presented within Topic Study Group (TSG) 47 of the International Congress on Mathematical Education—13, included the following issues which, to this writer, appeared to be particularly relevant to pre-service elementary teacher preparation programs, and the focus of this chapter:

- Günes (2016) investigated the mathematics background and perceived mathematics self-efficacy of pre-service elementary candidates in Turkey. The findings of this descriptive study indicated that the pre-service teacher's prior mathematics experiences affected their self-efficacy. One implication of the study would be that pre-service programs in elementary education may consider the assessment of self-efficacy in mathematics as a component of the admissions process for elementary education programs.
- Research by Shaughnessy and Boerst (2016) involved the use of simulation assessments to identify the extent to which elementary pre-service candidates, in the United States, knew and understood particular mathematics content and were able to both elicit and interpret student thinking within the simulation activity. The specific features of this research have implications for screening and assessing applicants for pre-service elementary education programs.

- Lin and Hsu (2016) used mathematics-pedagogy tasks (MPT) in their research to design a mathematics content and methods course for elementary pre-service teachers in Taiwan. The use of mathematics-pedagogy tasks helped to integrate mathematics content and pedagogy, with MPT tasks facilitating candidate understanding of mathematics, student cognition, curriculum, and the design of instructional activities. To this writer, the promise of mathematics-pedagogy tasks is the obvious connection between mathematics content, pedagogy, learning, and instruction. Such tasks may be a way to connect particular components of a candidate's pre-service preparation in mathematics.
- Lajoie's (2016) research analyzed observations made during lessons in which her pre-service elementary education candidates' role-played responses to particular mathematics content tasks. One task involved interpreting responses from calculators, and the other involved the digits found when dividing a whole number by 8, 6, 7, and 10. The role-playing tasks, and the responses, engaged the teacher candidates in considering their own mathematical content knowledge as well as their awareness of in-the-moment decision making with regard to, in particular, their responses. This project, undertaken in Canada, illustrates the potential of role-playing as an instructional technique helpful in determining a candidate's depth of content knowledge for the tasks selected and used within either a mathematics content or methods course or a field-based practicum associated with such courses.
- A study by Vula and Kingil-Kastrati (2016) involving pre-service elementary teacher candidates in Kosovo indicated a lack of understanding of pre-service teachers regarding conceptual knowledge of fractions. The results showed that pre-service teacher candidates displayed better fraction knowledge on procedures compared to their understanding of fraction concepts. Furthermore, candidate limitations of fraction representations had an impact on their understanding of operations with fractions, as well as, candidate capability to reason and interpret those solutions. This research not only identifies a critical mathematics content topic need for pre-service elementary teachers, but suggests particular areas of concern regarding candidate proficiency that pre-service programs in elementary education should address. Studies by Pilous, Lo, and Fox (this volume) also address the mathematics content preparation of pre-service elementary educators.
- Yang (2016) compared the curriculum structure of pre-service elementary education programs in the United States and China, with a particular emphasis on mathematics. She noted that in programs in China the curriculum structure of mathematics teacher education is knowledge-centered which puts emphasis on mathematical content knowledge and educational theory knowledge. However, the low proportion of fieldwork practice opportunities causes the lack of pedagogical content knowledge in actual classroom settings, which is the reason why the Department of Education in China requires all newly hired elementary teachers to take part in unified pre-job training before they start teaching. By contrast, the curriculum structure of pre-service elementary teacher education in the United States emphasizes the coordinated development of both academic

education and teaching practice. Such programs attach great importance to the practice and application of mathematics content and pedagogy more so than focusing on the mathematics content background of their candidates. It appears as if the professional development of Chinese elementary mathematics teachers begins after entry into the teaching profession, while, in the United States, the professional development of prospective teachers starts, to a large extent, before entry into the teaching profession.

6.3 Accreditation Challenges

Concerns related to the mathematical content and pedagogical background of pre-service elementary teachers has also become an area of emphasis within teacher accreditation in the United States. This is particularly important given the number of programs which prepare and certify elementary teachers in the U.S. First, however, some background is needed. In 2013, the National Council for Accreditation of Teacher Education (NCATE) and Teacher Education Accreditation Council (TEAC) consolidated as the Council for the Accreditation of Educator Preparation (CAEP). When a teacher certification program has been accredited, it demonstrates that it has met standards set by organizations representing the academic community, professionals, and other stakeholders. Programs complete such accreditation reviews every 7–10 years. CAEP formally accredits teacher education in the United States. However, since such accreditation is a voluntary undertaking by the program provider, CAEP is accountable for the accreditation of less than 50% of all the teacher education programs which actually provide teacher certification in the United States. Each year educator preparation programs, CAEP accredited or not, are required to submit the number of graduates, often referenced as program completers, to the United States Department of Education. An analysis of a recent United States Department of Education *Title II Education Act Report* (2014) showed that over 1700 programs prepared elementary teachers and close to 900 programs prepared early childhood (Nursery School-grade 3) teachers in the United States.[1] In contrast, internationally, Finland has 11 teacher education programs all within 8 of Finland's universities, two normal universities in Shanghai prepare elementary teachers, while Japan has 1300 teacher education providers (Jensen et al. 2016).

In the United States, CAEP standards for particular areas of teacher certification (e.g. elementary education, science education, mathematics education) are

[1]This chapter addresses issues related to mathematics education for pre-service elementary (K–6) teacher education and does not, in a very direct way, address the development, research, and challenges related to early childhood (N-3) teacher education.

developed by Specialty Area Associations. For instance, the National Council of Teachers of Mathematics (NCTM) is the Specialty Area Association responsible for developing, maintaining and updating standards and reviewing educator preparation provider reports for the initial certification of middle and high school mathematics teachers and also for the certification of elementary mathematics specialists, which is a post baccalaureate endorsement or "add on" to a candidate's existing teacher certification. The Specialty Area Association for elementary education, for the past twenty-five years, has been the Association for Childhood Education International (ACEI). Very recently, ACEI decided to no longer serve as the elementary teacher preparation Specialty Area Association for CAEP. At this writing CAEP is developing pre-service elementary teacher preparation standards, and once these are approved, CAEP intends to review elementary programs within an educator preparation provider's (EPP's) program review as part of the CAEP accreditation process for elementary education programs in the United States. These soon-to-be-approved, by CAEP, elementary standards include standards related to learning, content and curricular knowledge, assessing, planning and instruction, and developing as a professional. The standard for content and curricular knowledge includes the following key element related to mathematics:

> 2.b—Candidates demonstrate and apply understandings of major mathematics concepts, algorithms, procedures, applications and mathematical practices in varied contexts, and connections within and among mathematical domains as presented in the supporting explanation for the CAEP Mathematics Content for Elementary (K–6) Teachers (CAEP 2017).

The supporting explanation for the standard above defines elements of particular mathematics content domains and the Standards for Mathematical Practice, largely drawn from the *Mathematical Education of Teachers II* report (CBMS 2012), which was discussed earlier. Visit http://cbmsweb.org/MET2/index.htm for the full *MET II* report. The CAEP elementary standards are in the final stages of review and editing and will, hopefully, be approved by CAEP Board of Directors in December, 2017. These standards, once approved, will continue to support the preparation of elementary school teacher candidates, grades K–6, as generalists (responsible for teaching all subjects) in the United States.

It should be noted that the Association of Mathematics Teacher Educators (AMTE), an organization dedicated to the improvement of mathematics teacher education, K–12 recently released *Standards for Mathematics Teacher Preparation* (AMTE 2017). This manuscript presents standards describing a national vision for the initial preparation of all teachers, Pre-Kindergarten through grade 12, who teach mathematics. These standards, while not directly related to certification or accreditation, should be a valuable resource for those involved in preparing elementary classroom teachers and also for those in the United States and others internationally, who have the responsibility for writing accreditation reports and for providing professional development for teachers of mathematics.

6.4 Elementary Mathematics Specialists

In 1981, the NCTM recommended that state certification provide for a teaching credential/endorsement for elementary mathematics specialists. Then, in 1984, NCTM President John Dossey called for elementary mathematics specialists in an article in the *Arithmetic Teacher* (Dossey 1984). From these early beginnings, most likely based on the limited background in mathematics of generalist-prepared elementary teachers, the importance and potential of the elementary mathematics specialist began to emerge in the United States. Recommendations about the need for elementary teachers with interest and expertise in mathematics continued to appear in a range of publications. For instance, the National Research Council's *Everybody Counts* (1989) noted the following:

> The United States is one of the few countries in the world that continues to pretend – despite substantial evidence to the contrary – that elementary school teachers are able to teach all subjects equally well. It is time that we identify a cadre of teachers with special interest in mathematics and science who would be well prepared to teach young children both mathematics and science in an integrated, discovery-based environment. (p. 64)

Mathematics educators continue to advocate for elementary mathematics specialists (Fennell 2006; Gojak 2013; Lott 2003) and for research addressing the impact of the mathematics specialist. The NCTM's research brief entitled *Mathematics Specialists and Mathematics Coaches: What Does the Research Say?* was published in 2009. The more recent *Impact of Mathematics Coaching on Teachers and Students* (NCTM 2015) includes a review of 24 studies related to the impact of mathematics specialists. In the 2009 NCTM research brief, McGatha noted that Gerretson et al. (2008) found that using elementary mathematics teachersto focus only on mathematics instruction allowed them to have more time for planning and also allowed them to focus their professional development. The *National Mathematics Advisory Panel* report (2008) recommended that: "research be conducted on the use of full-time mathematics teachers in elementary schools" (p. xxii). This recommendation was based on the Panel's findings relative to the importance of teacher content knowledge and their recognition, as noted earlier, that most pre-service elementary teacher education programs do not address the teaching and learning of mathematics in sufficient depth. The Panel's recommendation has become an important call for elementary mathematics specialists in the United States, and is often used to help justify state certification proposals and continuing research.

States have taken an interest in providing certification, typically as an endorsement to existing teacher certification, for elementary mathematics specialists. At this writing approximately 20 states have approved programs, within their state, to offer elementary mathematics specialist certification. Additionally, a number of graduate programs in the U.S. offer elementary mathematics specialist programs and certification. See the Elementary Mathematics Specialists and Teacher Leaders Project website at http://www.mathspecialists.org for a review of all state certifications for elementary mathematics specialists.

The Association of Mathematics Teacher Educators (AMTE) created the *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs* (AMTE 2009). The AMTE Standards were revised in 2013 to align with recommendations of the *Mathematical Education of Teachers II* (2012) and the mathematical content standards and standards for mathematical practice of the *Common Core State Standards for Mathematics* (NGA and CCSSO 2010). These standards helped to guide and influence the writing of the *NCTM/CAEP Standards* (NCTM 2012) for elementary mathematics specialists.

Significantly impacting the continuing call and need for elementary mathematics specialists was the joint position statement (NCTM 2010) by the National Council of Teachers of Mathematics (NCTM), the National Council of Supervisors of Mathematics (NCSM), the Association of Mathematics Teacher Educators (AMTE) and the Association of State Supervisors of Mathematics (ASSM). The position statement provided below may also be accessed at http://www.nctm.org/Standards-and-Positions/Position-Statements/The-Role-of-Elementary-Mathematics-Specialists-in-the-Teaching-and-Learning-of-Mathematics/.

> The AMTE, ASSM, NCSM, and NCTM recommend the use of Elementary Mathematics Specialists (EMS professionals) in pre-K–6 environments to enhance the teaching, learning, and assessing of mathematics to improve student achievement. We further advocate that every elementary school have access to an EMS. Districts, states or provinces, and institutions of higher education should work in collaboration to create (1) advanced certification for EMS professionals and (2) rigorous programs to prepare EMS professionals. EMS professionals need a deep and broad knowledge of mathematics content, expertise in using and helping others use effective instructional practices, and the ability to support efforts that help all pre-K–6 students learn important mathematics. Programs for EMS professionals should focus on mathematics content knowledge, pedagogical knowledge, and leadership knowledge and skills.

The Elementary Mathematics Specialists & Teacher Leaders (ems&tl) Project, established in 2009, and supported by The Brookhill Institute of Mathematics was created to address issues related to and in support of elementary mathematics specialists in the U.S. and internationally. The Project, in collaboration with Maryland-based mathematics specialists, examines the challenges elementary mathematics specialists face, which informs project efforts supporting mathematics specialists nationally and to a lesser extent, internationally. Recent annual surveys of the regionally-based ems&tl elementary mathematics specialists revealed that three of the four participating school district's specialists spend close to 10% of their time working with new and first year teachers (see Table 6.1). This is beyond their day-to-day responsibilities related to professional learning, and service to teachers and students (Fennell et al. 2016). While not definitive, this finding certainly raises questions about the background and classroom readiness of new and beginning elementary teachers and their teaching of mathematics.

Not So Elementary—Primary School Teacher Quality in Top-Performing Systems (Jensen et al. 2016) was released weeks prior to ICME 13. The report analyzed, among other topics, the mathematics preparation of elementary school candidates in Shanghai, Hong Kong, Japan, Finland, and the United States.

Table 6.1 Elementary mathematics specialist responsibilities (estimated % of time spent)

School district	Professional development	Direct service—teachers	Direct service—students	Administrative duties	New and 1st year teachers	Other
Baltimore County (n = 3)	30	18	3	15	10	23
Carroll County (n = 11)	30	32	16	8	9	5
Frederick County (n = 13)	29	22	31	10	5	3
Howard County (n = 19)	40	31	6	10	9	4

Interestingly, in both Shanghai and Hong Kong pre-service candidates are prepared as specialists. In Shanghai candidates can be prepared in 1-2 subjects as their speciality areas and in Hong Kong candidates may specialize in mathematics. Both countries provide for specialization, within elementary school mathematics, at the pre-service level.

6.5 Current and Emerging Challenges

As we consider the mathematics content and pedagogical background of pre-service elementary teachers in the United States and internationally, there are two looming challenges for educator preparation providers (EPP's). The first is that many states in the U.S. produce more elementary teachers than needed. In fact, a recent Professional Education Data System (PEDS) survey, in the U.S., indicated that 31% of all degrees in education were in the field of elementary education, (AACTE 2013). The national and state documented surplus of elementary teachers in the U. S. raises questions related to the qualifications of such candidates as well as college or university proposals for restricting the number of candidates. Would this be a time to consider content area (i.e. mathematics) specialization within pre-service elementary education programs in the United States? Would such programs survive? Internationally, the ratio of applicants to acceptances in teacher education programs in Finland, Canada, Singapore, and Shanghai is between 6-1 and 10-1. These countries also have a surplus of teachers but the candidates are highly qualified (Tucker 2012).

A related challenge was indirectly referenced earlier in this chapter. As noted, an analysis of a recent United States Department of Education *Title II Education Act*

Table 6.2 Top 5 U.S. providers: elementary teachers prepared 2014

Educator Preparation Provider	Number of Candidates Prepared
University of Phoenix (AZ)	1226
A+ Texas Teachers	1059
Wilmington (DE) University	901
Grand Canyon University (AZ)	858
Texas State University	609

Report (2014) demonstrated that over 1700 educator preparation programs prepare elementary classroom teachers in the United States. Table 6.2 presents the programs which prepared the most elementary candidates in 2014.

It should be noted that neither the University of Phoenix, an online teacher preparation provider, nor A+ Texas Teachers are "brick and mortar" institutions of higher education. Additionally, even a brief analysis of the sheer number of candidates produced by the top five educator preparation providers in Table 6.2 should raise questions about program integrity, in particular, the use of full-time and adjunct faculty members, the frequency and quality of field experiences and the number of students regularly enrolled in mathematics content and pedagogy courses and related experiences. It must also be noted that four of the top ten elementary education providers in the United States, including the first and second ranked providers, are not nationally accredited.

It is imperative that a sharply focused research lens analyzes and addresses issues regarding the impact of mathematical knowledge for elementary classroom teaching, and related experiences, at the pre-service level, which may include some of the issues raised by Topic Study Group—47 during ICME 13. Institutions of higher education and online teacher education providers should also consider a specialist model for preparing elementary classroom teachers at both the pre-service level and as an endorsement to existing teacher certification, and, in particular, continue and expand on research on the role and impact of elementary mathematics specialists. Finally, a thorough and ongoing analysis of elementary teacher supply and demand as well as accreditation and the integrity of certification programs for pre-service elementary educators within the United States and internationally should be a priority.

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Chapter 7
Designing Non-routine Mathematical Problems as a Challenge for High Performing Prospective Teachers

Marjolein Kool and Ronald Keijzer

Abstract Designing non-routine mathematical problems is a challenging task, even for high performing prospective teachers in elementary teacher education, especially when these non-routine problems concern knowledge at the mathematical horizon (HCK). In an experimental setting, these prospective teachers were challenged to design non-routine HCK problems. Interaction with peers, feedback from experts, analyzing HCK problems to find criteria, building a repertoire of prototypes, a cyclic design process, experts who are themselves struggling in designing problems were the most important and effective aspects of the learning environment to rise from this explorative study.

Keywords Mathematics education · Pre-service teachers · Mathematical knowledge for teaching · Problem solving · High performing prospective teachers

7.1 Introduction

All prospective elementary school teachers in the Netherlands (Kindergarten–grade six) have to pass a mathematics test in their third year in college. For some of these prospective teachers this test is hard to pass, while others experience hardly any problem passing the test. This case study focuses on this group of high performing prospective teachers. Characteristic for these high achievers is that they passed both the mathematics entrance test and the third-year test smoothly, that they are also successful in the other subjects of their study and are in need of additional challenging activities in their study. Some of these prospective teachers wanted to learn

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how to construct test-like mathematical problems. They did this voluntarily in their leisure time, because they like mathematics and want to become better at it. We will describe their learning process. But before doing so, we will write about the test.

The third-year test is based on notions of mathematical knowledge for teaching as formulated by Ball et al. (2008), and mainly contains non-routine problems. Non-routine problems are problems that cannot be solved by an algorithm or other straightforward means of solution at the student teachers' disposal (Kantowski 1977; Schoenfeld 1985). To solve non-routine problems student teachers need to extend their mathematical knowledge in order to construct 'new' problem approaches. The third-year test also contains a second type of non-routine problems. These are problems that can be solved using a standard problem approach, however doing this would be cumbersome and time-consuming. Student teachers have to construct efficient approaches to solve these problems and to finish the test within the given time.

See for instance the following problem:

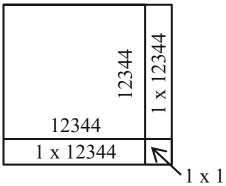
Compare 12,344 × 12,344 with 12,345 × 12,345.

The difference between these products is …

a. *2 × 12,344*
b. *2 × 12,344 + 1*
c. *2 × 12,345*
d. *2 × 12,345 + 1*

To solve this problem some student teachers will choose the standard problem approach and subtract the product of 12,344 × 12,344 from the product of 12,345 × 12,345. It takes a lot of time to do this and thereafter they have to compare their result with the several answer options, which is also time-consuming.

Student teachers who recognize 12,345 × 12,345 as $(12{,}344 + 1)^2 = 12{,}344^2 + 2 \times 12{,}344 \times 1 + 1^2$ can easily find option b as the right answer, but most of our student teachers don't have this knowledge available and need to construct a 'new' problem approach, for instance drawing a diagram.



This is an elegant and efficient problem approach, but you need to be creative to construct such a strategy by yourself. For most student teachers solving both types of non-routine mathematical problems is quite challenging. Practice is needed to prepare for the test. However, test-specific practice materials are hardly available. To fill this gap, in the college years 2013–2014 and 2014–2015 groups of high performing prospective elementary teachers designed mathematical problems for their peers who were preparing for the mathematics test.

Both groups of high performing student teachers analyzed, evaluated and designed non-routine problems concerning all three types of mathematical subject matter knowledge as distinguished by Ball et al. (2008), namely Common Content Knowledge (CCK), that is the subject-specific knowledge needed to recognize and solve mathematics problems in day-to-day-life, Specialized Content Knowledge (SCK), the—for a teacher—professional mathematical knowledge to understand, assess and evaluate the mathematical productions of students, and Horizon Content Knowledge (HCK), the knowledge that exceeds the mathematics of the school type the professional is teaching. HCK contains knowledge of how mathematical topics are related over the span of mathematics included in the curriculum. Generally these HCK problems focus on the structure of mathematical phenomena and are somewhat distant from mathematics in daily life and in daily teaching practice. The first group of high performing prospective teachers did in the end learn to design non-routine CCK and SCK problems, but at the end of the year they were unable to design non-routine problems that needed HCK (Kool and Keijzer 2015). The two expert problem designers who guided the group used their expertise in focusing the 2014–2015 group on designing CCK, SCK **and** HCK non-routine problems.

Designing problems at this high level is difficult and not a requirement for becoming an elementary teacher. It requires that one takes the perspective of the problem solver and wonders what he or she could do or think, and which different problem approaches he could choose. Doing so will enrich the designer's mathematical knowledge and it will provide a deeper mathematical understanding, which in its turn supports better mathematics teaching. The high performing student teachers started this task with much enthusiasm, because it challenged them at their own high level. Moreover they wanted to provide their peers who had not yet passed the test with suitable training materials.

7.2 Solving and Designing Non-routine Mathematical Problems

Non-routine problems are problems that cannot be solved by an algorithm or other straightforward means of solution at the problem solver's disposal. The problem solver needs to construct new problem approaches. This involves that once having solved the problem this specific type of problem can become a routine problem. Learning mathematics means that you will always encounter new non-routine problems. Students have to learn to construct new problem approaches. Knowledge, skills, metacognition and self-confidence are valuable for this process and the use of heuristics can be useful too (Verschaffel et al. 1999).

Heuristics are general advice, search rules, rules of thumb and informal approaches that might help problem solvers when solving non-routine mathematical problems. Drawing a diagram or model, looking for patterns, making suppositions, working backwards, and simplifying a problem are examples of heuristics.

Heuristics do not guarantee that a problem will be solved, but they offer a chance to find a solution that is sought for. A problem solver can use heuristics to explore non-routine problems (Verschaffel et al. 1999).

Solving non-routine problems is challenging, but designing these problems is even more demanding. Generally high performing students like to reach for demanding goals like this one, but in working on such a challenge they have specific needs. On the one hand, these high performers need sufficient subject specific input, they want to be a member of a learning community, they feel responsible for tasks and want to decide how cooperatively formulated goals could and should be achieved (Van Tassel-Baska 1993; Borasi et al. 1999; Swan et al. 2002; Heller et al. 2005; Feldhusen 2005; Subotnik and Jarvin 2005; Scager 2013). On the other hand, high performers generally need 'scaffolds' from experts who show good examples, provide feedback, evaluate and appreciate their work, and support them in performing tasks they cannot (currently) perform independently (Bain 2004; Van Geert and Steenbeek 2005; Brixler 2007; Frey and Fisher 2010). This support is needed but should fade over time, because not having it do so will have a negative impact on the high performers and influence their motivation to participate in the task (Keller 2010). These general characteristics of high performers also apply to the eight prospective teachers in this study.

7.3 Research Question

We showed above that, there was a need to develop a learning environment, including goals, roles, peers, teachers, sources, meetings and appointments, in a course setting for one year for the 2014–2015 group of high performing prospective teachers, that would support them in designing non-routine HCK problems (besides CCK and SCK problems). As a consequence the research question is:

What are characteristics of a learning environment for high performing prospective elementary school teachers that support them in designing non-routine HCK problems?

7.4 Method

This case study tries to uncover the development of a group of eight high performing prospective elementary teachers (Yin 2009). As the context of this study is designing a prospective teachers' learning environment, we will use ideas from design research to generate and analyze the study's data (Van den Akker et al. 2006). The design of the learning environment is described in Table 7.1, where

Table 7.1 Interventions and HLT for learning to design non-routine HCK problems

Interventions in the learning environment	Hypothetical learning trajectory (HLT)
Prospective teachers analyze and evaluate examples of non-routine HCK problems using a provisional criteria list based on the perspective of the problem solver	Prospective teachers become familiar with these non-routine problems; they learn about the mathematics involved and what these problems require from the problem solver
Prospective teachers are asked to look for non-routine HCK problems using the criteria list; these problems are discussed	Prospective teachers learn to choose the perspective of the problem solver and use this to recognize non-routine HCK problems
Prospective teachers analyze and evaluate non-routine HCK problems and improve and extend the criteria list in terms of problem solving characteristics (heuristics) and they collect a list of prototypes of HCK problems	Prospective teachers become familiar with the characteristics of non-routine HCK problems and heuristics used in solving these, they can recognize the quality of the problems and they build a repertoire of prototypes of HCK problems
Prospective teachers participate in masterclasses from an experienced designer of non-routine problems, focusing on the cyclic process of designing non-routine problems (production–evaluation–improvement); prospective teachers see the expert sometimes struggling designing the problems	Prospective teachers reflect on the cyclic process of designing non-routine problems, try to follow the same process in designing non-routine HCK problems and feel supported in their struggle in designing non-routine HCK problems

interventions are connected to student teachers' hypothetical learning trajectory, HLT (Simon 1995). The interventions are based on reflection on and evaluations of experiences in the 2013–2014 group (Kool and Keijzer 2015).

The group met seven times spread over the college year. Typical activities during these seven meetings included discussing examples of problems, rewriting examples of problems, explicating criteria for the intended mathematical problems, attending masterclasses delivered by an experienced test item designer (second author), providing and receiving feedback on designed problems. The authors of this chapter attended the meetings as experts. During the meetings with the group one of the experts led the session, while the other took field notes. These notes were analyzed to see whether and how the prospective teachers reacted to interventions and whether they developed as was predicted in the HLT. In addition, research data was collected through:

- analyzing problems the prospective teachers designed during the year,
- asking prospective teachers to reconstruct and describe the (cyclic) design process of some of their problems,
- a structured group interview with the prospective teachers about aspects of the learning environment they find stimulating or frustrating.

7.5 Results

During the year, eight high performing student teachers participated in the project. After some time they succeeded in designing non-routine problems about daily life experiences (CCK) and teaching experiences (SCK). The student teachers did not spontaneously design HCK problems and when they were stimulated to do so they only produced a few routine HCK problems instead of non-routine problems. They explained that their experience with this kind of problems, which play a smaller part in the teacher education curriculum, was insufficient. It turned out that they could solve this kind of problems, but most of the time appeared unable to produce more than one approach. They did have little experience with HCK problems. In the beginning their repertoire of examples and problem approaches concerning HCK was poor, and not flexible enough to design this kind of non-routine problems.

In this section we sketch how high performing prospective teachers developed over time. We first show how they developed criteria for non-routine HCK problems and how this led to a first set of prototypes of these problems. In the next stage they were offered scaffolds set by experts, which over time were removed, once most of the student teachers developed skills for redesigning existing problems in new non-routine ones.

7.5.1 Criteria for and a Collection of Prototypes of Non-routine HCK Problems

To make student teachers more familiar with non-routine HCK problems the experts provided suitable examples. They stimulated and helped student teachers to solve these problems in different ways and to analyze the character of the problems. They asked the student teachers to change their perspective to that of the problem solver and imagine which heuristics student teachers could use. In doing this the group constructed the following criteria for non-routine HCK problems. These problems:

- challenge the students to reason, backtrack, abstract, generalize, declare, explain, prove and justify, to look for the best problem approach and ask why this is the best approach,
- challenge the students to discover mathematical structures (rules, patterns),
- require that the students understand thoroughly the knowledge, skills and problem approaches they are using.

The high performing student teachers who constructed these criteria concluded that they partially overlap, and that a good HCK problem does not have to meet all the criteria. This self-constructed list helped them to recognize HCK problems, to find HCK problems in textbooks and study materials and to evaluate them.

They discussed for instance the question of whether and why the following problem was a non-routine HCK problem:

Given: John calculates the greatest common divisor of two numbers. He multiplies both numbers with the number a. Question: What do you know about the greatest common divisor of the two products?

Using their criteria, the student teachers decided that this was a good non-routine HCK problem, because knowing the algorithm to find the greatest common divisor of two numbers is not enough to solve this problem. One has to understand the concept of the greatest common divisor. Some reasoning is required in solving the problem. Of course, a conjecture can be checked filling in several numbers, but this will not provide proof for all sets of two numbers.

After four meetings, the student teachers became familiar with non-routine HCK problems. They could determine these, they had solved, analyzed and evaluated many examples, and they improved and used their list of criteria. In spite of that, designing non-routine HCK problems was still difficult for them. However, after five meetings, after another reflection on the criteria list, they generally were able to make variations of existing problems and they decided that it could be helpful to collect prototypes of HCK problems. These included problems that:

- ask for why or why not. *Why* $6{:}\,\frac{1}{2} = 6 \times 2$?
- present a proposition and ask under what conditions the proposition is correct. *When does the product of two primes have exactly 3 divisors?*
- ask if and why a certain proposition is always correct. *If you add three consecutive numbers your sum will always be odd. Is this true?*
- ask to predict the effect of a given mistake, like mixing up the digits in a number, or interchanging two operations. *I want to multiply two fractions. I only multiply the numerators and forget to multiply the denominators. What will be the effect on the result?*
- give the result of a calculation and ask what the calculation has been. *What number do you need to multiply 14 with to get 18 as a result?*
- ask to apply mathematical knowledge in an expanded situation. *You know the sum of the angles of a triangle, what will be the sum of the angles of a hexagon?*

Although the student teachers made some variations on the prototypes, they were still struggling with the task and gave up when they could not generalize the underlying principle or idea or could not design a new problem based on the same underlying principle or idea.

7.5.2 *"Scaffolds" from Experts*

During masterclasses one of the experts showed the student teachers that even experts can struggle when designing non-routine problems. Student teachers learned that it is almost impossible to design a perfect non-routine HCK problem out of the blue. Designing a problem starts with an idea, a basic problem, a mathematical concept. A cyclic process of evaluating and improving the initial idea a number of times is necessary to achieve the final product.

In the masterclasses the expert started designing by using an existing problem, sometimes just a routine problem and wondered aloud: "What can I do to change this into a non-routine problem?" He modelled intended design activities by showing how he could find his inspiration in the above list of prototypes. In doing so he spoke explicitly about what was going on in his mind, like: "Perhaps I can replace numbers by letters (variables), turn the problem upside down by giving the answer and asking for the question, or withhold some information." Each of his designed problems was evaluated by the student teachers using the list of criteria, wondering what the problem solver has to do to solve the problem: Is this a non-routine HCK problem, is it an interesting one, can we improve it? The student teachers experienced that the first attempts of the expert were not always perfect. He needed some rounds of evaluating and improving to design a better problem and sometimes he really had to struggle to finally be successful. The following impression of a masterclass illustrates this.

At the start of his masterclass, the expert always asked the student teachers to choose a mathematical topic they wanted some support on in designing problems. This time they chose repeating decimal fractions. They were familiar with fractions that equal standard decimals like 0.333… and 0.111… They tried to make non-routine variations on these examples and designed problems like finding the fractions that equals 0.555… and 0.777… but were not satisfied with their productions because they were still routine and too easy. Indeed, the problem solver can see at a glance that these numbers were multiples of 0.111… . These problems were unsuitable to prepare for the test and they wonder if the expert could design a suitable variation on these problems. The expert could not immediately satisfy their desire. To start with, he asked them to find a fraction that equals 0.123123123… The student teachers struggled with this problem and were not very successful in solving it until one of them demonstrated a problem approach he had learned once: If $a = 0.123123123\ldots$ then $1000a = 123.123123123\ldots$ and therefor $999a = 123$. So $a = \frac{123}{999} = \frac{41}{333}$. Afterwards the student teachers evaluated that this was an interesting problem but that it was not suitable to prepare for the test, because if a problem solver knows this problem approach this problem has become a routine problem but if he doesn't know this problem approach it is too hard and perhaps impossible to construct a suitable problem approach himself. The expert was invited to find a better variation. He tried the following one: "Find a fraction that equals 0.1333…" After working on this problem, the student teachers could appreciate this example because they discovered and constructed a new problem approach: $0.1333\ldots = 0.333\ldots - 0.2 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. At the same time they regretted that the standard problem approach was still usable in solving this case. One of the student teachers suggested an adjustment of the problem: "Write 0.1333 as a subtraction of two fractions." This was a good attempt, but unfortunately it was still possible for the problem solver to use the standard problem approach to find $\frac{2}{15}$ as a result and after that he had countless possibilities to construct subtractions ending on $\frac{2}{15}$. Finally, the expert designed: “Write 0.13333 as a subtraction of two fractions with numerator 1.” This final adjustment caused the problem to meet the requirements and after that it inspired the student teachers to design many variations like: “Write 0.2666… as an addition of two fractions with numerator 2” and “Write 0.291666… as a subtraction of two fractions with the property that the denominator of the first fraction equals the numerator of the second.” Each designed problem was evaluated by the student teachers and they were quite satisfied about the problem structure they finally constructed together with the expert. For instance, the last example was appreciated because it cannot be solved by the standard problem approach, the specific conditions make the problem challenging, but the average student teacher should have enough mathematical knowledge to construct his own problem approach.

For most of the student teachers it was an eye opener and an encouragement to see that even an expert in designing problems needs time, support and perseverance to design a good problem.

7.5.3 A Cyclic Designing Process

After participating in the masterclass the student teachers’ analyses focused on the mathematical activity that the designer expected from the problem solver and how this was supported in their design of non-routine HCK problems. Further, the design was viewed with the criteria for the problems formulated by the prospective teachers in mind. Moreover, in analyzing the designs the experts regarded what they intuitively considered to be good designs and related their ideas with arguments brought forward in the discussions in the group of prospective teachers. The experts discovered that the student teachers were more likely to vary on one of the prototypes to design non-routine HCK problems, that they used the criteria list to discuss and evaluate their productions and that they tried harder to improve their first attempts. In other words, the student teachers tried to follow the cycle of designing, evaluating and improving problems and produced several stages of a problem before the final stage was reached (Fig. 7.1).

For instance, student teacher Lieke developed the following HCK problem in stages. Her development is prototypical for the learning process of the prospective teachers in this group. At the fourth meeting she presented this design:

The price of a pair of shoes was increased by 20%. At the end of the season the price was decreased by 20%. What do you know of the new price compared with the original one?

A. *The new price is higher than the original price.*
B. *The new price is lower than the original price.*
C. *The new price equals the original price.*

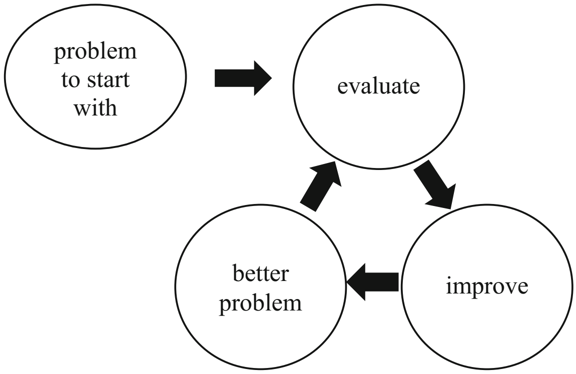


Fig. 7.1 The cyclic process of designing and improving non-routine mathematical problems

The student teachers discussed whether this was a HCK problem. First they thought this was a day-to-day-life problem (CCK) but in the end they realized that although the situation could be realistic it was in the first place a problem that asks for reasoning about a mathematical phenomenon in itself. But they were not convinced that the problem solver really has to reason to solve this problem. If you pick a price for the shoes, you can calculate and find the answer. Some student teachers remarked that this problem is part of their basic knowledge. They recognized the problem and knew the answer. So they decided that this was a routine HCK problem. Lieke used the feedback to improve her problem. At the fifth meeting she appeared with this variation:

Sale in the shoe shop: all prices are lowered by 20%! The shop assistant makes a mistake, she decreases the price of my new shoes by 20 euro, and that means that she gives me 5 euro too much discount. What was the original price of my new shoes?

The group of student teachers agreed that this was a real non-routine problem for their peers. One needs to backtrack and in constructing the problem approach the problem solver has to thoroughly understand calculation with percentages. That makes this a suitable non-routine HCK problem for student teachers. Lieke was happy with this judgment, and this experience motivated her to design a variation that was even more challenging. At the sixth meeting she presented this variation of the problem:

A shoe shop had a very special way of giving a discount. If you buy two pairs of shoes, for the cheapest pair you will get this discount: subtract 20 euros and then decrease the price by 20%. For the most expensive pair they first decrease the price by 20% and then subtract 20 euros. A customer buys two pairs of shoes and she ends up paying the same price for both pairs. What was the difference in euros between the original prices of the two pairs of shoes?

The student teachers recognized the quality of this problem and decided that this variation requires mathematical reasoning typical for non-routine HCK problems. The development of Lieke's designing work shows how interaction, feedback and evaluating problems could stimulate varying on problems and improving them. The student teachers recognized this and wanted feedback on all their designed problems from peers and experts. They experienced that it is useful to give the initial idea a chance to grow. This does not guarantee success, but the approach of throwing away each initially rejected idea and starting with a new one is far less successful. In the end half of the student teachers were able to design non-routine HCK problems. They still found it difficult, but several times they could bring their struggle to a good end. The other student teachers still need more or less support from the experts to design suitable HCK problems.

7.5.4 Evaluation of the Learning Environment by the Participants

At the end of the year the eight student teachers evaluated the trajectory they followed in a group interview to find out which aspects of the learning environment were valuable in reaching the final result. During the group interview the student teachers were asked to mention aspects of the learning environment that were frustrating and other aspects that were supportive and stimulating. The following aspects were mentioned by at least two of the eight student teachers.

During their learning-trajectory student teachers found it frustrating that the experts let them experience:

- that they have not encountered enough non-routine HCK problems during their study, that their HCK repertoire was too small,
- that although they could solve HCK problems, their HCK was not rich and deep enough to design HCK problems.

Student teachers found it stimulating and supportive to discover:

- that designing non-routine HCK problems is a cyclic process of repeated evaluating and improving a starting problem or first idea,
- that a list of criteria of non-routine HCK problems can be used to evaluate problems from the perspective of the problem solver and improve these,
- that a list of prototypes of HCK problems can give inspiration to start the cyclic process of designing problems,
- that feedback from experts can help to go on in the cyclic process of designing problems,
- that interaction and discussion with peers can be valuable too,
- that even experts struggle when designing non-routine HCK problems.

7.6 Conclusion and Discussion

Finally, all of the participating (n = 8) student teachers developed their mathematical knowledge and their designing capacities; as said before, only half of the participants (n = 4) reached the final goal and could design non-routine HCK problems independently. Student teachers with a poor and less flexible HCK and HCK problem repertoire still needed (a lot of) support to do this, but even these student teachers admitted that they found it valuable for their future profession as a teacher to have solved, analyzed, evaluated so many HCK problems. After participating in the project they seemed able to design HCK problems with expert help.

This small scale case study suggests that high performing prospective teachers who want to learn to design non-routine HCK problems will need scaffolds and feedback to achieve this challenging goal. We saw that one way of reaching this is a learning environment which provides support and input from expert designers, and targets both criteria of non-routine HCK problems and a repertoire of HCK problem types. Following a cyclic process of designing problems is helpful, as is experiencing that designing non-routine HCK problems is hard even for expert designers. These features of the learning environment were not effective for all of the eight participating high performing student teachers. This raises new questions concerning characteristics of the participants and the learning environment.

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Chapter 8
Pre-service Teacher Procedural and Conceptual Knowledge of Fractions

Eda Vula and Jeta Kingji-Kastrati

Abstract This chapter assessed pre-service teacher knowledge of fraction interpretations and their ability to demonstrate procedural and conceptual knowledge of adding and subtracting fractions. The sample included 58 pre-service teachers in Kosovo. The twenty tasks given in the study's test were related to fraction concepts, fraction addition and fraction subtraction. It was found that the pre-service teachers had limited knowledge regarding different fraction interpretations. It was also found that they had limited knowledge on showing the explanation of the procedures of adding and subtracting fractions. This chapter discusses the findings taking into considerations the context in which the study was conducted and provides suggestion for future research.

Keywords Pre-service teachers' knowledge · Fractions
Procedural knowledge · Conceptual knowledge

8.1 Introduction

During the last two decades, a huge focus has been dedicated to the preparation and development of the qualified teachers (Ball et al. 2005, 2008). What knowledge a teacher must have and what kind of professional background one must have to become a teacher, has been and is continuing to be discussed by many researchers (Depaepe et al. 2015; Ball et al. 2005; Shulman 1986). Hill et al. (2005) suggested that teachers need more than just proficiency in mathematical skills; they need mathematical understandings which will enable them to provide their students with explanations, analyse students' responses/answers, and use the right methods to present different concepts.

According to Ball et al. "Teachers must know rationales for procedures, meaning for terms, and explanations for concepts" (as cited in Van Steenbrugge et al. 2014,

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p. 143). Their knowledge should be focused on the subject (content) knowledge and the pedagogical content knowledge, as well as on their connections (Shulman 1986; Ball et al. 2008; Wilson 2010). While these connections of teachers' knowledge continue to build on Shulman's work, Ball et al. (2005) identified two categories of content knowledge, *common content knowledge*—the basic skill that a mathematically literate adult should posess and *specialized content knowledge* as a specific and detailed knowledge of mathematics required to teach it. On the other hand, Chinnappan and Forrester (2014) argued that it is a necessary need for future teachers to develop a strong body specialized content knowledge. They showed that the use of representationally rich instruction within an existing teacher education program has a significant impact on the interconnection between content knowledge and pedagogical content knowledge (Chinnappan and Forrester 2014). Thus, closer attention must be paid to the preparation of teachers in order to help them gain the deep knowledge, suggested because pre-service teacher education is a critical time for deepening teachers' knowledge (Ma 1999). The courses, and related experiences, dedicated to pre-service elementary teachers, should help them to create the link between procedures and the concepts related to these procedures that they will teach.

The aim of this study is to assess pre-service teachers' knowledge of fractions based on the perspective of procedural and conceptual knowledge.

We have selected the content domain of fractions because of its importance in elementary mathematics and because fractions are considered as an essential skill for future mathematics success (Rittle-Johnson et al. 2001; National Mathematics Advisory Panel 2008; Van Steenbrugge et al. 2014).

8.2 Procedural Knowledge, Conceptual Knowledge and Understanding Fractions

Fractions present a critical and at the same time a most complex set of concepts and skills within mathematics (Behr et al. 1993; Charalambous and Pitta-Pantazi 2007; Van Steenbrugge et al. 2010). The misconceptions that students have about fractions, both in terms of fractions as numbers and how to operate with fractions, relates particularly to the way fractions are represented and how they are taught (Barmby et al. 2009). Many studies have shown that teachers also have difficulties in understanding fractions and lack the ability to explain the rationale of a procedure or the underlining conceptual meaning of such procedures (Lin et al. 2013; Van Steenbrugge et al. 2014), which directly influences the learning of fractions by students (Van Steenbrugge et al. 2014).

Early sub-constructs theories postulated that integrating the qualities of multiple perspectives were crucial to the understanding of fractions (Moseley 2005). According to Behr et al. (1993), fractions can be interpreted as a part-whole comparison, an indicated division or quotient, a ratio, as an operator and as a measure. The part-whole construct refers to a continuous quantity or a set of discrete objects or a region partitioned into parts of equal size. It refers to how much of an object/region or set is represented by the fraction symbol (Kieren 1976). Tasks typically used to measure this kind of conceptual knowledge include the shading parts of objects. On the other hand, the quotient construct considers a fraction as a result of a division of two whole numbers (Charalambous and Pitta-Pantazi 2007). The ratio perspective is based on comparing separate quantities and usually presented as a:b or a/b. The operator construct reflects a function that transforms line segments, figures, or numbers. And lastly the measurement interpretation refers to the fact that fractions are numbers, and can be ordered on a number line (Kieren 1976). Thus, various theoretical models have been proposed for understanding fractions in the elementary school (Behr et al. 1993; Charalambous and Pitta-Pantazi 2007).

Multiple interpretations make fraction concepts more concrete and understandable, and aids in the development of a student's conceptual knowledge. Such understandings contribute to the variability in successfully carrying out fraction computation procedures. Therefore, it is important for teachers to have a deep-level of conceptual and procedural knowledge in order to deal adequately with fraction problems in their classroom.

Byrnes described conceptual knowledge as "relational representations" which "consists of two or more represented entities that are mentally linked through a relation of some sort" (as cited in Hallett et al. 2010, p. 396). By contrast, procedural knowledge involves following a sequence of certain defined actions, algorithms, rules or the computational skills needed to solve problems, that is meant to produce a right answer to a problem (Eisenhart et al. 1993; Hallett et al. 2010). Chinnappan and Forrester (2014) provide support on the argument that procedurally driven fraction knowledge has limited value and, indeed, could impede the development of the specialized content knowledge and pedagogical content knowledge necessary for mathematics teaching. Teacher's having a conceptual knowledge is crucial to explain the interrelationship of ideas and they give a logical understanding to the procedures related to fraction operations (Eisenhart et al. 1993). This includes understanding of the properties of fractions: their magnitudes, different interpretations, principles, and notations.

Several researchers (Ball 1990; Newton 2008; Van Steenbrugge et al. 2014) reviewed pre-service teacher's difficulties involving procedural and conceptual knowledge of fractions. Stoddart et al. (1993) found that pre-service teachers demonstrated from 37 to 98% accuracy among questions on procedural skills, but only 5–10% accuracy among more conceptually based questions. They showed that pre-service teachers displayed better fraction knowledge on procedures rather than concepts. Some other studies have examined the impact of instructional methods in pre-service teachers' conceptual and procedural fraction knowledge. In his extensive analysis of pre-service teachers' knowledge of fractions, Newton (2008) showed that after taking a course that explicitly linked fraction concepts and procedures, pre-service teachers performed better and showed a deepened understanding of fractions. Based on Newton's suggestion for the examination of correct methods in order to review the teachers' knowledge, Chinnappan and Forrester (2014) analyzed the impact of a representational reasoning teaching and learning approach on pre-service teachers' procedural and conceptual knowledge with fractions. Within the context of pre-service teachers' fraction knowledge, Lin et al. (2013), argued that an open approach to instruction helps pre-service teachers improve their procedural and conceptual knowledge of fractions, while Rittle-Johnson et al. (2001), proposed that procedural and conceptual knowledge should be developed in an iterative fashion in order to improve the fraction knowledge.

These findings support the argument that using appropriate methods for instruction involving fractions within teacher preparation programs could improve the development of procedural and conceptual knowledge with fractions. These knowledges would be necessary for pre-service teachers and their future teaching.

8.3 Purpose and Research Questions

The aim of the study is to assess pre-service teacher's knowledge about different interpretations of fractions and their procedural and conceptual knowledge on presentations of fraction, addition and subtraction. Specifically, this study addresses the following questions:

1. How do pre-service teachers perform in their use of different interpretations of fraction concept?
2. Do pre-service teachers master fraction procedural knowledge at a higher level than their conceptual knowledge when they represent fraction addition and subtraction?

8.4 Method

8.4.1 Participants

A total of 58 pre-service teachers (about 88% of the participants younger than age 24, and approximately 96%, female) from Prishtina University in Kosovo, voluntarily participated in this study. They were asked to complete a fractions knowledge test. In Kosovo, elementary school teachers are all-round teachers or generalists, and therefore pre-service teachers must be prepared in all school subjects, including mathematics. The Elementary Bachelor's degree program is a 4-year qualification. During the 4-years period, among subjects and pedagogical courses, pre-service teachers also complete practice teaching. Two subject matter courses are taught in the first year and they are oriented in directions toward the discipline rather than teaching. The course on teaching mathematics is taught in the last year (4th) of the elementary teachers' program.

8.4.2 Procedure

The fractions knowledge test was developed and administrated to measure pre-service teachers' performance of fraction knowledge. The 20 items of the test were used in previous studies (Charalambous and Pitta-Pantazi 2007; Lin et al. 2013). The test was divided in two subsets of tasks, and there was no limited time for administration. The first 12 tasks used in the test aimed to examine the pre-service teachers' knowledge of the different interpretations of fractions: fractions as part-whole comparisons, including area/regional and discrete models (items 1–4), fractions as indicated division or quotient (items 6, 9), fractions as a ratio (item 5), fractions as an operator (items 7, 8) and fractions as a measure (items 10–12). Fraction addition and fraction subtraction were present in 8 other tasks. Four of the tasks required only procedural knowledge of fractions and the other four tasks assessed the conceptual knowledge of fractions. A task exemplified as "Solve the problem $\frac{1}{3} + \frac{1}{2} =$" is deemed as a procedural knowledge task; another task as "Explain how you determined your answer by giving an illustration or representation for $\frac{1}{4} + \frac{2}{3} =$" is deemed as a conceptual knowledge task. Representative samples of the tasks used in the test appear in the Appendix.

A descriptive as well the qualitative analyses were conducted on our dataset, to answer our research questions. All 20 tasks were scored dichotomously: correct/incorrect and returned test forms were scored and analyzed by two researchers.

8.5 Results

To answer the first research question: *How do pre-service teachers perform in their use of different interpretations of fraction concept?* we first focused on the correctness of the pre-service teacher responses and then their qualitative analyses.

As is shown in the Table 8.1, the pre-service teachers have a misconception of fractions even for the fraction as a part of whole, which is very common in all school levels. The results showed that pre-service teachers have difficulty in solving the problems that require conceptual meaning compared to the ones which can be solved using the common models or by simple procedures. The pre-service teachers in this study demonstrated their highest level of fraction knowledge with items which represented fractions as a part of whole. The tasks which required the pre-service teachers to use the area/regional and discrete model (tasks 1, 2 and 4) were solved by most of the participants (71.8–79.4%). Even though, we have

Table 8.1 Average score for the correct answers

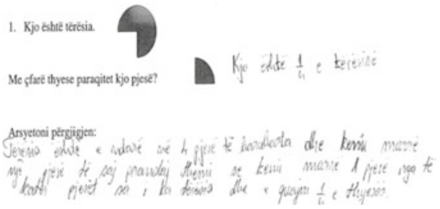
Tasks	Sub-C	N	Correct (%)	Incorrect (%)
1	P-W	58	72.2	27.8
2	P-W	58	71.8	28.2
3	P-W	58	31.4	68.6
4	P-W	58	79.4	20.6
5	Rat.	58	53.4	56.6
6	Quot.	58	84.8	15.2
7	Oper.	58	57.2	42.8
8	Oper.	58	83.1	16.9
9	Quot.	58	66.7	33.3
10	Meas.	58	57.5	42.5
11	Meas.	58	67.6	32.4
12	Meas.	58	71.8	28.2
Valid N		58		

shown that even in this case their conceptual knowledge is not sufficient when presentation of 'the whole' is changed.

Only 31.4% of participants answered correctly, after they were asked to present the part of an "irregular" shape as a fraction (task 3). In most of the rationales of the answers provided, students have considered the circular surface as a whole. This misconception could be due to the fact that the rectangular and circular "pie" piece models are mostly used as a model for explaining the fractions' concept. The fact that a single fractional 'part-whole' concept can take on different appearances seems to be incomprehensible to pre-service teachers (Fig. 8.1).

It was shown that the participants also demonstrated misconceptions when they were asked to interpret fractions as a measure (Tasks 10–12). Most of them were able to recognize and show the fraction by noting that point on the number line, or they were able to visualize fractions and locate them on a number line (Fig. 8.2a). But, the low performance was observed in the tasks which were used to assess relational understanding. Only about half of participants correctly answered the task which asked them to place the number 1 on the number line (Fig. 8.2b). The pre-service teachers were not familiar with these tasks which required them to show knowledge related to unit forming. It appears that they had a misunderstanding related to conceptual meaning of the fractional parts of units and couldn't represent the relationship between the size of the unit and the count in the number line.

The pre-service teachers did not have sufficient knowledge to represent fractions in different ways. Gaps that were found in the answers are also related to other sub-constructs of fraction interpretations. Only 53.4% of the pre-service students correctly answered the task which required a ratio interpretation of fractions and only 57.2% showed that they can use the fraction as an operator. Some of the participants used the equation, without any rationale, as is shown in the responses to task 7: *Which is the output quantity if the input quantity is equal to 12?* (Fig. 8.3).



This is ¼ of the whole.

Explain how you determined your answer.

The unit is divided into four equal pieces and we took one piece from it therefore we say that we took one piece from all four pieces that unit has and we present it as ¼ of a whole.

Fig. 8.1 A task 3 sample where the student demonstrates a misconception

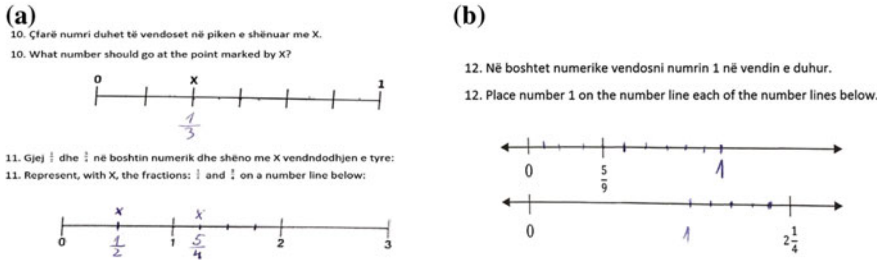
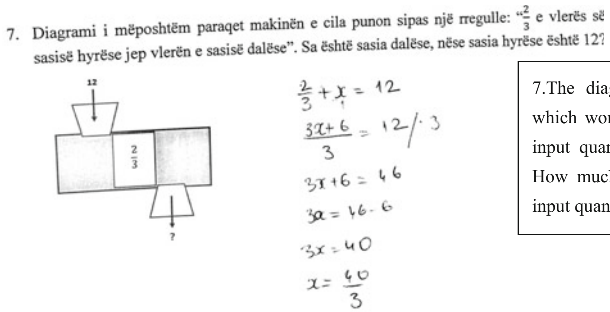


Fig. 8.2 **a** A task 10 and 11 sample answer. **b** A task 12 sample answer



7.The diagram below shows the vehicle which works under a rule: "the 2/3 of the input quantity gives the output quantity". How much will be the output quantity if input quantity is 12?

Fig. 8.3 An incorrect interpretation in the sample task 7

On the other hand, in task 8, even though the task was related with operator subconstruct, the pre-service teachers demonstrated a high level of success (83.1%). They found that it is easier to answer the question based only on the use of division. Similarly, in the task 6, most of their answers were correct (84.8%). We noted that, in a similar way, the pre-service teachers used their prior knowledge of fraction simplification rules.

Second research question: *Do pre-service teachers master fraction procedural knowledge at a higher level than conceptual knowledge when they represent and solve fraction operations problems?*

Table 8.2 Correct and incorrect answers for procedure and conceptual knowledge for adding and subtraction of fractions

Items	Proc/conc	N	Correct (%)	Incorrect (%)
13	Proc. (13–16)	58	86.2	13.8
14		58	74.1	25.9
15		58	77.6	22.4
16		58	79.3	20.7
17	Conc. (17–20)	58	12.1	87.9
18		58	19.0	81.0
19		58	5.2	94.8
20		58	10.3	89.7
Valid N		58		

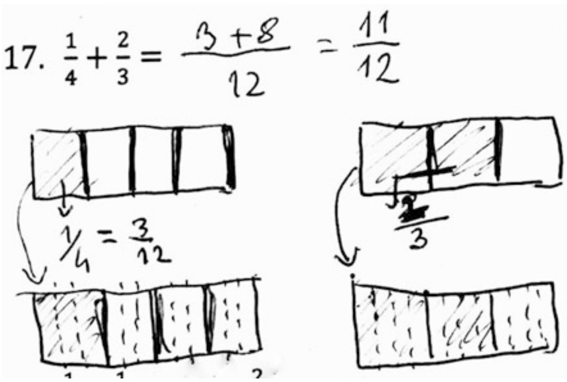


Fig. 8.4 An example of interpretation for sample task 17

Generally, the pre-service teachers had a very low performance on the tasks which required an explanation.

A considerable number of responses did not demonstrate connection between procedural and conceptual knowledge. While total average of correct answers for procedural knowledge tasks (tasks 13, 14, 15 and 16) was 79.3%, for conceptual knowledge (tasks 17, 18, 19, 20) total average of correct answers was only 11.65% (Table 8.2).

The example show how the participant explained correctly the procedure on fractions' addition (Fig. 8.4).

Results show that the pre-service teachers know how to apply the procedure on addition and subtraction of fractions without an explanation of why the procedures work. Only a few of them could provide a conceptual explanation for the procedures associated with adding and subtracting of fractions.

8.6 Discussion

The purpose of this study was assessing: (1) pre-service teacher use of different interpretations of fraction concepts and (2) their ability to demonstrate procedural and conceptual knowledge of adding and subtracting fractions. The study was based on our assumption that future teachers need to develop a strong body of specialized content knowledge (Ball et al. 2005), which should take into consideration the nature of procedural-conceptual knowledge (Newton 2008; Chinnappan and Forrester 2014; Lin et al. 2013).

Lin et al. (2013) and Van Steenbrugge et al. (2014) have argued that teachers have difficulties in understanding fractions and lack of ability to explain the rationale of procedures involving fractions or the underlining conceptual meaning. The research described in this chapter confirmed that pre-service teachers in Kosovo also have limited knowledge of fractions concepts, fraction representations, and the ability to explain procedures involving adding and subtracting fractions. In the report of the National Mathematics Advisory Panel (2008) it was considered that one key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line. Based on the results, we can see that participants in this chapter have shown limited knowledge with the representation of fraction concepts using the number line. Almost all of participants showed their answers based on their prior routine procedural knowledge involving fractions, which were mostly rule-bound (Ball 1990) without any explanation. They do not have the required experience to use the different fraction interpretations because during their practice teaching in their schools, they use elementary mathematics textbooks, which mostly present the fractions as a part of a whole and as an operator (Vula et al. 2015). Also, the fractions and their interpretation are barely presented in the program for the elementary pre-service teacher education.

The gaps that the pre-service teachers demonstrated, require the need to strengthen the training of pre-service teachers with appropriate conceptual knowledge related to fractions (Depaepe et al. 2015), because their future instruction, as a teacher, will have a direct impact on the knowledge and understanding acquired by their students (Charallambous and Pita-Pantazi 2007).

One of the limitations of this study was the constitution of the sample. They were asked to participate voluntary in the tasks without any information about the study. Therefore, the implications of this modest study are such that they indicate a need for further research, both to confirm these findings and to investigate the appropriate approaches for teaching fractions, which can be integrated into methods courses for pre-service teachers.

8.7 Conclusion

We can conclude that “learning about prospective teachers and developing strategies for working with them—can be pursued in tandem” (Ball 1988, p. 40).

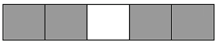
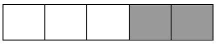
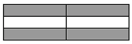
The results of this study indicate the need for more attention to fractions in mathematics courses within teacher preparation programs. These programs should support elementary pre-service teacher needs in preparing them for their professional future in teaching. Based on the findings and discussions presented in this chapter, one may assume that elementary teacher education programs should use appropriate instruction and enhance the development of connections between procedural and conceptual knowledge related to fractions. Such instruction should be based on formal knowledge of fractions, while focusing on fraction concepts, logical relationships, and on the pedagogical knowledge for teaching fractions (Chinnappan and Forrester 2014; Lin et al. 2013; Newton 2008).

Further research should also involve inservice teacher perspectives in order to improve the relationship between courses for didactics of elementary mathematics at the university-level and professional development programs regarding teaching fractions.

Appendix

Fraction Knowledge Tasks

1. Which of the following correspond of $\frac{2}{3}$?

(a)  (b)  (c) 

2. What part of each circle is coloured?

(a)  (b) 

3. This is the a whole. 

   What fraction represents this piece? 

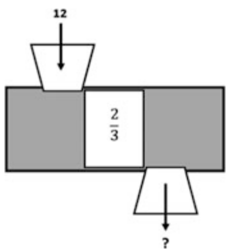
4. How can you best define the fraction that represents number of coloured circles in the set below? Give at least two fractions which represent the figure below and explain how you determined your answer.

5. Based on the figure below, who gets more pizza, the boys or girls? How can boys and girls share the pizza equally? Express in fractions how many slices of pizzas did take each of them?

6. Two boys have the same amount of money. One decides to save $\frac{1}{4}$ of his money and the other boy saves $\frac{5}{20}$ portion of his money. What do you think is the correct way to represent a comparison of the amount saved by the boys?

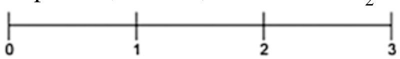
   (a) $\frac{5}{20}$ is bigger than $\frac{1}{4}$
   (b) $\frac{1}{4}$ is bigger than $\frac{5}{20}$
   (c) $\frac{5}{20}$ and $\frac{1}{4}$ are equal

7. The diagram below shows the vehicle which works under a rule: "the 2/3 of the input quantity gives the output quantity". How much will be the output quantity if input quantity is 12?

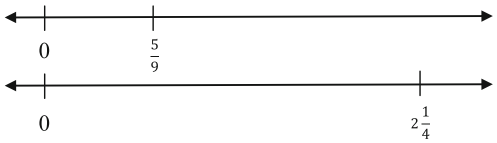


8. $\frac{1}{3}$ of which number is the number 5?
9. How many halves ($\frac{1}{2}$) are there in six wholes? Illustrate your reasoning with a figure.
10. What number should go at the point marked by X?

11. Represent, with X, the fractions: $\frac{1}{2}$ and $\frac{5}{4}$ on a number line below.



12. Place number 1 on the number line each of the number lines below.



Calculate:

13. $\frac{1}{3}+\frac{1}{2}=$ 14. $7\frac{5}{8}+4\frac{1}{2}=$ 15. $\frac{5}{6}-\frac{1}{3}=$ 16. $3\frac{2}{3}-1\frac{1}{2}=$

Perform the following operations and justify solutions using any form of fraction presentation

17. $\frac{1}{4}+\frac{2}{3}=$ 18. $2\frac{1}{3}+\frac{1}{9}=$ 19. $\frac{3}{5}-\frac{1}{2}=$ 20. $3\frac{1}{4}-\frac{2}{3}=$

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Chapter 9
Designing Simulations to Learn About Pre-service Teachers' Capabilities with Eliciting and Interpreting Student Thinking

Meghan Shaughnessy and Timothy Boerst

Abstract This chapter focuses on the design of simulation assessments to learn about pre-service teachers' capabilities with eliciting and interpreting student thinking. We present a simulation assessment and show what a performance on that assessment can reveal about a pre-service teacher's eliciting and interpreting skills, as well as their mathematical knowledge for teaching. We consider the specific design features that make it possible to appraise pre-service teachers' capabilities.

Keywords Elementary teaching · Eliciting and interpreting student thinking
Practice-based teacher education · Mathematical knowledge for teaching
Teaching simulation

9.1 Introduction

The increasing emphasis on practice-based teacher education in the United States has resulted in a focus on assessments that provide information about pre-service teachers' abilities to actually **do** the core tasks of teaching. This means combining instructional techniques and skills together with complex specialized knowledge of the content and insights into students' thinking and development. Such assessments match the new practice-focused learning goals of teacher education. Research suggests that specific feedback about practice increases pre-service teachers' ability to use feedback to improve their practice (Grossman 2010).

Many approaches to assessment have focused on appraising pre-service teachers in real contexts of practice, such as in field placements and during student teaching.

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They have included microteaching, field-based performance tasks, and systematic field observation of lessons (e.g., Hammerness et al. 2005; Elliott 2003). Observation tools have been developed (e.g., Danielson 2007) and portfolios (e.g., Darling-Hammond and Pecheone 2010) have been used as means to gather information about teachers' skills.

A more recent addition has been the use of simulations to assess pre-service teachers' developing skills. Simulations are used in many other professional fields as a means to assess skill with the practices of the profession. For example, in many medical schools, doctors in training engage in simulations of physical examinations, patient counseling, and medical history taking by interacting with "standardized patients," adults who are trained to act as patients who have specified characteristics. Evaluation of medical students' interactions with standardized patients makes possible common and sustainable appraisal of candidates' knowledge and skills (Boulet et al. 2009). Simulations have not been widely used in education in the U.S., but there is growing interest in their usage for learning (Dieker et al. 2014; Dotger 2015) and assessment (Shaughnessy and Boerst, 2017). Although the use of simulations in education may provoke skepticism, simulations address challenges inherent in field-based assessments, provide a sustainable and fair way to assess pre-service teachers' knowledge and capabilities, and offer a complement to other forms of assessment in which contextual variables impact implementation and in turn affect ability to assess pre-service teachers' skills (Shaughnessy et al. accepted). Here, we focus on simulations as a means of learning about pre-service teachers' developing capabilities.

This chapter aims to advance work on assessments of teaching practice in teacher education by focusing on the design of simulation assessments to appraise pre-service teachers' developing capabilities. We focus on the teaching practices of eliciting student thinking and interpreting student thinking. Eliciting student thinking makes the nature of students' current knowledge available to the teacher. This is essential for engaging students' preconceptions and building on their existing knowledge in instruction (Bransford et al. 2000). Interpretation is just as crucial because teachers must be able to comprehend students' ideas and their implications for subsequent teaching. Eliciting and interpreting are foundational skills for formative assessment, which has been shown to substantially impact student learning (Wiliam 2010). There has been much recent attention to developing skill in this area (e.g., Gupta et al., this volume).

Although many teaching practices could be examined, we believe that eliciting and interpreting student thinking are particularly important foci because what students think is foundational to teaching. Skilled teaching builds on and is responsive to students' understandings. Second, these practices are foundational to many other teaching practices (e.g., skillfully leading a discussion is dependent upon being able to elicit student thinking). Based on our experience designing and using simulations, we will show the potential of such an assessment to evoke, document, and appraise pre-service teachers' skills and the design decisions entailed in developing such an assessment. Throughout, we use *pre-service teacher* to refer to individuals who are enrolled in a teacher preparation program and *student* to refer to children in elementary school classrooms.

9.2 A Simulation Assessment for Assessing Teachers' Capabilities

In our assessment, pre-service teachers engage in three stages of work. First, pre-service teachers are provided with student work on a problem and given 10 min to prepare for an interaction. The task for the pre-service teacher during the interaction is to determine the process the student is using to solve the problem and the student's understanding of the core mathematical ideas involved in the process.

Second, pre-service teachers interact with a "student." The role of the "student" is carried out by a teacher educator whose words and actions are guided by a detailed profile of a particular student's thinking and rules that govern this student's interactional norms. To ensure standardization of the role, the "student" is trained to follow the highly specified rules for reasoning and responding, including responses to questions that are commonly asked by pre-service teachers. Pre-service teachers have five minutes to interact with the "student," eliciting and probing the "student's" thinking to understand the steps she took, why she performed particular steps, and her understanding of the key mathematical ideas involved.

In the third part, pre-service teachers respond verbally to a set of questions that are designed to probe their interpretations of the "student's" process and understanding and their prediction about the "student's" performance on a similar problem. The assessment takes approximately 25 min and is scored in the moment based on criteria for proficient performance, including mathematically and pedagogically key aspects.

9.3 Considerations in the Design of Simulation Assessment

Three considerations guide our design of the simulation assessment. First, we must identify and articulate the focus of an assessment. That is, to elaborate the teaching practice that we are appraising (e.g., eliciting student thinking) through a decomposition of the practice (Grossman et al. 2009). The decomposition reflects what it means to "do" this aspect of teaching. Our approach to decomposition starts with identifying requisite parts of the focal teaching practice. Importantly, the goal is to determine a set of techniques associated with the practice that can be taught to novices and appraised (Boerst et al. 2011). For example, eliciting student thinking is a teaching practice whereas formulating a question to pose to a student is one of a set of techniques that are implied in the more complex practice of eliciting student thinking. The work of decomposing a teaching practice is influenced by the work of Cohen et al. (2003) who depict the work of teaching as interactions with students and content in learning environments. In this view, teachers must integrate simultaneous and flexible attention to content, and to students as they engage with that content, in contexts that influence the nature of the work.

Second, we consider the assessment situation. Because we seek evidence of capabilities with teaching practice, assessment situations must be designed to prompt and document the teaching skills of teachers. The mathematical knowledge for teaching (Ball et al. 2008) entailed in the situation must be carefully considered as a part of this design work. In other words, we must design situations which allow pre-service teachers to demonstrate their capabilities with teaching practice in connection with content that students learn and use. Further, our design must create residue of interactive teaching practice that might otherwise be fleeting or unavailable.

Third, teacher education assessments requires assessors to make inferences based on things that pre-service teachers say, do, or make to hypothesize about what they know or can do more generally (Mislevy et al. 2004). Once we have documented pre-service teachers' performances in an assessment situation, we must make inferences about pre-service teachers' skills based on their performances. To make such inferences, we draw upon our conceptions of teaching practice (in this case, eliciting and interpreting student thinking) and how pre-service teachers develop teaching proficiency, as well as research on the mathematical knowledge needed for teaching (Ball et al. 2008). In sum, our assessment development process considers teaching practice itself and how it can be decomposed for the purposes of assessment, the assessment situation and the opportunities it creates for pre-service teachers to demonstrate their skills, and the practice-focused developmental frame that supports inferences about pre-service teachers' skills.

9.4 Constructing the Situation to Reveal Pre-service Teachers' Eliciting Capabilities

In our simulation assessment, the "student" profile (see Fig. 9.2) is crucial both for providing opportunities for pre-service teachers to demonstrate their capabilities with eliciting student thinking and for enacting the assessment. There are three main considerations in the design of the student profile: (a) the mathematics topic; (b) the characterization of the student's process and understanding; and (c) the student's way of being. We next describe each of these considerations.

9.4.1 The Mathematics Topic

The mathematics content embedded in the student work sample shapes pre-service teachers' opportunities to demonstrate their capabilities with eliciting and interpreting student thinking. When designing the assessment scenario, we select mathematics content that is high-leverage for elementary mathematics teaching (Shaughnessy et al. 2012) to provide insight into pre-service teachers' capabilities in the context of mathematics content that we expect them to understand well.

9.4.2 The Characterization of the Student's Process and Understanding

Our knowledge of teaching and the knowledge, skills, and dispositions that pre-service teachers bring to teacher education has led us to identify a second set of features to consider in the design of the assessment: the student's process for solving the problem, the student's understanding of the process and related mathematical ideas, and the accuracy of the student's answer.

A fundamental diagnostic problem of teaching is that students use an array of methods that often stretch beyond those that teachers prefer or even understand themselves. As we noted earlier, teaching requires a learner-centered orientation where teachers actively seek information about student thinking, especially in situations where the approach is unfamiliar. This is particularly demanding for pre-service teachers who are likely to know less about non-standard approaches.

It is crucial that teachers are able to determine the **processes** that students use to solve mathematics problems. In the strand of number and operation these processes include standard algorithms, alternative algorithms, and invented approaches. In our experience, pre-service teachers in the U.S. are often highly proficient with standard algorithms, but their understandings of these processes are tacit and often either not well developed or not well remembered, following over a decade procedural-focused use. Further, pre-service teachers are often unaware of alternative approaches. As a result they often have less of a sense of what is important to ask when students are using alternative algorithms or invented strategies and may revert to directing the student to more familiar territory through prompts such as, "why aren't you doing … [referencing an element of the standard algorithm]." Even when students use the standard algorithm, pre-service teachers face other challenges, such as not eliciting pertinent information from students due to assumptions that they make about what students think about parts of the process. Thus, standard algorithms, alternative algorithms, and invented approaches all provide productive arenas for assessing skill in eliciting and interpreting student thinking.

In terms of our focus on **understanding**, research indicates that it is crucial to track on students' understandings of processes that they are using (Fuson 2003; Steffe and Cobb 1988). At that start of a teacher education program, pre-service teachers track more on students' processes than on their understanding of that process (Shaughnessy and Boerst, 2017). Thus, we have found that it is important to articulate the student's understandings in the profile and to track on pre-service teachers' skill with eliciting those understandings from the student in the simulation.

With respect to **accuracy** of the answer (i.e., the correctness of the final answer), we have found that our pre-service teachers are more likely to ask questions about answers that are wrong than answers that are right. Further, pre-service teachers may be likely to discount processes and understandings when faced with an incorrect answer. This may lead them to generate interpretations that fail to capture what students do know and are able to do. Of course these categories are interrelated. For instance, pre-service teachers may be less likely to ask about the understanding behind correct answers, perhaps presuming that understanding must be there to produce the correct answer. In sum, for each assessment, we articulate the student's process, understanding, and accuracy as a critical set of assessment features.

9.4.3 The Student's Way of Being

Students differ in terms of how they think or approach mathematics problems. But just as importantly for the work of eliciting student thinking, they differ in terms of their dispositions, interactional styles, and use of language. We have termed these unique personal traits, the "student's way of being." In a recent study conducted in classrooms, we found that about one-third of students (N = 44) gave a full explanation of their process for solving a problem after being asked just one question about their written work by a pre-service teacher (Shaughnessy et al. accepted). Further, almost all of these students articulated their understanding of the process and core mathematical ideas without being prompted. In classrooms, students do of course vary in how much they share about their thinking. But for an assessment, having "students" disclose relatively little about their process and understanding unless directly asked makes it possible to learn more about pre-service teachers' eliciting skills. When students are reserved, pre-service teachers have to ask more questions, which makes their skill with the practice of eliciting student thinking more visible. We explicitly design for the student's way of being because of its impact on teacher-student interactions and the nature of eliciting and interpreting that can happen.

9.4.4 The "Student" Profile

We summarize information about the mathematics topic, the characterization of the student's process and understanding and the student's way of being in a "student"

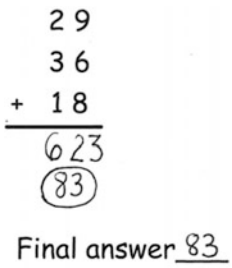

Fig. 9.1 A student's work on a multi-digit addition problem

profile. In the example assessment, we selected multi-digit addition. Specifically, the problem: 29 + 36 + 18 (see Fig. 9.1). In this example assessment, the "student" uses an algorithm, sometimes known as the column addition method, to solve the problem. The "student" adds the digits in each column (2 tens + 3 tens + 1 ten = 6 tens) and (9 ones + 6 ones + 8 ones = 23 ones). The "student" interprets the 623 in the written work as 6 "tens" and 23 "ones." The "student" knows that 23 ones can also be thought of as 2 tens and 3 ones. Then, the "student" combines the 6 tens and the 2 tens (from the 23 ones). This yields the final answer of 83. The "student" has

Mathematics topic: Multi-digit addition

Characterization of the student's process and understanding:

- **The student's process:** The student is using the column addition method for solving multi-digit addition problems, the student is working from left to right.
- **The student's understanding of the ideas involved in the problem/process:** The student has conceptual understanding of the procedure including why combining is necessary (and when and how to combine).
- **Other information about the student's thinking, language, and orientation in this scenario:** The student talks about digits in columns in terms of the place value of the column. The student uses the term "combining" to refer to trading/carrying/regrouping.

The student's way of being: The standardized student does not make errors with basic arithmetic combinations. The standardized student gives the least amount of information that is still responsive to the preservice teacher's question.

Specific responses based on the identified mathematics topic, characterization of the student's process and understanding, and the student's way of being (*a subset of them*):

Preservice teacher prompt	Response
What did you do first?"	"I added the tens: 2 + 3 + 1 and I got 6."
"**How** did you get from 623 to 83?" or, "How did you get 8?"	"I had to combine the 6 and the 2."
"Why did you need to combine those numbers?"	"Because they're both tens."

Fig. 9.2 An excerpt from the "Student" profile

conceptual understanding of the procedure and the final answer is correct. This profile also includes scripted responses to anticipated questions and these responses are based on what has been articulated with respect to the mathematics topic, the characterization of the student's process and understanding, and the student's way of being. Figure 9.2 contains an abbreviated version of a "student" profile.

9.5 Considering a Pre-service Teacher's Eliciting Performance

We next present a vignette based on a pre-service teacher's performance. This pre-service teacher begins the interaction by asking the "student" to talk about his process. In the vignette, we use T to refer to the pre-service teacher and S to refer to the "student."

T: I was wondering when looking at the problem where you started? What numbers did you start with?

S: So, I added the tens. So I added the two, and the three, and the one and I got six.

T: Okay. And how did you know that was six?

S: 'Cause I know my facts. I mean, so two and three makes five, and one more makes six.

T: Okay. And that's why you wrote down the six right there?

S: Yeah, that's right.

The pre-service teacher elicits that the "student" first added the digits in the tens column, the sequence in which the "student" added the numbers within the tens column, and the sum that resulted (6). She also elicits that the student believes that he is "adding the tens." The pre-service teacher continues to ask questions about the process.

T: And then what was your next step?

S: Then I added up the ones.

T: Okay. And how did you add up the ones?

S: So, nine, and six, and then eight to get twenty-three.

T: Okay. What if you started by adding eight, six, and nine? Would you still get twenty-three?

S: Yes. It doesn't matter which way you do it.

T: Alright. And then you have the twenty-three here [points to it]. So, what does that twenty-three mean?

S: That's twenty-three ones.

At this point, the pre-service teacher has elicited that the second step in the process was to add the digits in the ones column and the order in which they were added. She has also pressed to see whether the "student" believes that the sum will be the same if the digits are added in a different order. She continues to ask questions, focused on the combining step.

T: Twenty-three ones? Alright. And then how did you get that eighty-three?

S: Well, when you add stuff, you can't have more than one digit in a place– in an answer. Like this just looks wrong. So, you have to regroup it so that the answer will look right.

T: So, how did you regroup it?

S: So, this twenty-three, this two right here is two tens. And this, like I told you before, was also tens [pointing to the 6]. So you put the tens together and that's how you get the eight tens.

T: You got the eight tens. So, did you add six plus twenty-three ones?

S: I added the six plus the twenty ones to give me eight tens. Then I still had the three ones.

The pre-service teacher continued to press on why the "student" knew to combine the six and the two.

T: Three ones. So why didn't you add the twenty-three? How did you know that the two meant tens when you just told me before you had twenty-three ones?

S: Yeah, so in twenty-three ones, this part of it is ten and this part of it is the ones that are left after you made all your tens.

T: And how do you know that's two tens?

S: Because when you're adding it and you get past nine, then the next number is gonna be in the teens, so that you know that that digit is– actually stands for ten.

T: Okay. And then eighty-three is your final answer. What does that eight mean?

S: Eighty.

T: And then that three, what is that three referring to? What does that three mean? What is that value?

S: Three ones.

By the end of the interaction, the "student" has revealed why he combined the six and the two and his understanding of the value of the eight and the three in eight-three.

9.5.1 Scoring of the Eliciting Performance

We conceive of the work of eliciting student thinking as involving: (a) formulating questions designed to elicit and probe student thinking; (b) posing questions; (c) listening to and interpreting what students are saying; and (d) developing additional questions to pose (TeachingWorks 2016). This work is iterative. It involves teachers listening to and interpreting what students are saying, generating and posing questions to learn more about the student thinking, listening to and interpreting what students are saying and so forth. Teachers make sense of what students know and can do based on evidence from interactions and other artifacts of student work.

Importantly, students are at the center of this work. It is their thinking which is sought and intended to be understood, and the work is situated in mathematical contexts that focus dialog, shape interpretation, and influence follow-up questions.

Because the simulations make use of highly specified protocols for the student's processes, understandings, and ways of being, we are able to use observational checklists as scoring tools as a simulation unfolds. Our observational checklists for the eliciting portions of the assessment are based upon an articulation of "high-quality" eliciting of student thinking. For example, high-quality eliciting of student thinking entails launching the interaction in a way that focuses on the mathematics of the student's approach (i.e., formulating and posing an initial question designed to elicit student thinking); developing additional questions which are focused on eliciting the student's process for solving the problem and probing the student's understanding of the process and of key mathematical ideas; listening to the student which can be demonstrated through the posing of additional questions which are tied to things that the student says and does; and the posing of questions. The checklist includes specific things that the pre-service teachers might do (e.g., Elicits where the 8 comes from) and specific responses that the "student" provides based on their preparation and training (e.g., I combined the 6 and the 2) when prompted by the pre-service teacher.

Formulating an initial question designed to elicit student thinking	
✔	Asks the student what he or she did or thought about when solving the problem
Developing additional questions to elicit and probe the student's thinking	
✔	Elicits where the 6 comes from *(2 tens + 3 tens + 1 ten)*
✔	Elicits where the 23 comes from *(9 + 6 + 8)*
✔	Elicits the sequence of adding tens first and then adding ones
✔	Elicits a description of the combining/regrouping (*I combined the 6 and the 2)*
✔	Probes the student's understanding of the value of components of the 623 (*e.g., 6 is 6 tens)*
✔	Probes the student's understanding of why combining is necessary *(e.g., because the 6 and the 2 are both tens)*
Listening to the student	
✔	Asks questions tied to specific things that the student **did** (i.e., questions about the student's writing)
✔	Attends to and takes up specific ideas that the student **talks about** (includes revoicing)
Posing questions	
✔	Refrains from directing the student to a different process (in a way that competes with the student's initial process)
✔	Refrains from making evaluative statements
✔	Prompts the student fluently (e.g., does not have lengthy pauses between questions; asks clear questions and does not need to rephrase them multiple times, etc.)

Fig. 9.3 Abbreviated scoring checklist for eliciting: example performance

As summarized in Fig. 9.3, in the simulation, we are able to see evidence of this pre-service teacher's skill in formulating an initial question ("What numbers did you start with?") that is general, open-ended, and focused on an important piece of the mathematics at hand. We also have evidence of the pre-service teacher's skill in posing the question to a student, where skilled delivery is sensitive to how students might hear and respond to the question. While we are not able to directly see the pre-service teacher's skill in interpreting the student's thinking in the moment, we are able to see that follow up questions are responsive to what the student has said, which is an indicator of a pre-service teacher listening to a student. Further, the questions focus strategically on particular ideas that the student has shared/not shared such as parts of the process about which the student has said little and mathematical ideas that related to the student's process (e.g., whether it is possible to add the numbers in a column in a different order).

We use specific pre-service teacher performances and trends across the performances to improve our articulation of high-quality eliciting within a particular scenario and by implication the components of the scoring tool. For example, pre-service teachers might repeatedly probe a student's understanding of a particular mathematical idea that we had not initially identified on the observational checklist, but that seems quite reasonable to include. We also use their performances to improve the student role protocol so that the student will engage in the situation in ways that allow pre-service teachers to demonstrate their eliciting skills. Our goal is to design the situation such that we are able to appraise the eliciting and interpreting skills of our pre-service teachers. If pre-service teachers are not probing the student's understanding of particular parts of the process or incorrectly interpreting the student's understanding, we do not assume that our pre-service teachers are not skilled at eliciting and interpreting student thinking. Instead, we consider whether we need to make changes in the way that the "student" responds to specific questions. Even subtle shifts to the "student's" language can make it more likely that a pre-service teacher would ask important questions about understandings. We use the performances to identify changes that we believe will increase the likelihood that pre-service teachers are able to demonstrate their eliciting skills.

9.6 Constructing the Situation to Reveal Pre-service Teachers' Interpreting Capabilities

The follow-up interview is designed to assess pre-service teachers' capabilities with interpreting student thinking and their mathematical knowledge for teaching. Interpretation is the work that teachers do to give meaning to what they see and hear. Two crucial areas for interpretation are: (1) the student's process, and (2) the student's understanding of that process and the underlying mathematical ideas. The follow-up interview is designed to focus on both of these aspects, including the use of evidence to support the interpretations. Pre-service teachers are asked to talk about what they learned from the simulation about the student's process for solving the problem. Later, in the context of a related problem for which pre-service teachers anticipate the student's process, we ask pre-service teachers to anticipate student understanding. We ask about specific mathematical ideas and/or steps in the process because in earlier work we found that asking a targeted question can reveal more about the capabilities of pre-service teachers than a general question.

At the same time, the follow-up interview is constructed to reveal evidence of pre-service teachers' mathematical knowledge for teaching. We target four aspects. First, we elicit whether pre-service teachers can solve the problem themselves and judge the accuracy of the student's solution. Second, we ask pre-service teachers to construct a problem that they could use to confirm their understanding of the student's process. We learn whether pre-service teachers are able to identify the features of the task, including the traits of the numerical example, that must remain consistent to confirm the student's process or understanding. Third, pre-service teachers are asked to apply the student's process to a similar problem that we provide. Fourth, pre-service teachers are asked to generalize whether the process will generate a correct answer for a particular category of problems, and why.

9.6.1 Considering a Pre-service Teacher's Interpreting Performance

The questions and the pre-service teacher's responses to them are summarized in Table 9.1.

9.6.2 Scoring of the Interpreting Performance

We use an observational checklist as the interview unfolds. The observational checklist, completed for the example assessment, is shown in Fig. 9.4. It shows that this pre-service teacher is able to describe the student's process and to anticipate the student's understanding of two key mathematical ideas, using evidence from the interaction with the "student." Further, this pre-service teacher demonstrates developed mathematical knowledge for teaching through generating a follow-up problem which can be used to confirm the student's process and articulating a rationale for that problem, applying the student's process to a similar problem, and thinking critically about the mathematics of the student's process and the mathematical cases to which it will generalize.

Table 9.1 A pre-service teacher's responses to follow up questions

Question	Pre-service teacher's response
Was the student's answer correct?	Yeah. Eighty-three is correct.
Describe the process the student used to get the sum.	He started with the tens column adding the two, and the three, and one to equal six and he wrote down six, and then he moved on to adding the nine, plus the six, plus the eight and he knew that equaled twenty-three. So, he wrote the twenty-three down, but since he knew that there was more than one number in that place that he had to add the next–the six and the two together to get eight because he had twenty-three ones.
If you were to pose an additional problem for this student to complete that would help you confirm what you learned about the student's process, what problem would you pose? Why?	I would do twelve, twenty-six, and sixty-eight. Because if he starts with the tens column again, he's going to get nine. And then, moving on to the ones column, he'd get sixteen. I could see like how he deals with–if he knows still that this one goes to that nine.
Based on your interaction with the student, how do you think the student would solve this (*showing a similar problem, 27 + 48*) problem if the student used the same process as in the first problem?	The student would probably start with saying two plus four. He knows it equals six. And then the seven plus the eight, he knows it equals fifteen. And then adding the one and the six to be seven. So seventy-five.
What would the student say was the value of each of the digits here? *[point to the 6, 1, and 5]*	He would say it's sixty, and another ten, and then five ones. That's what he said when I asked about the other problem: he said 6 tens and 2 tens and 3 ones.
What would the student understand about why the answer cannot be left as this? *[point to 615]*	Because he would, once again, think that this number looks off, that it doesn't look right. He knows that you can't have more than one number in a place. That's what he said.
Will this process always produce a correct answer for addition problems with 2-digit numbers? Why or why not?	Yes. Because instead of carrying the one over to here, he acknowledges it over there and adds it in that way. So the one that that extra ten is still getting added in the end.

9.7 Simulation Assessments: The Potential and Next Steps

As illustrated in this chapter, simulation assessments hold promise for assessing pre-service teachers' pre-service teachers' developing capabilities with important interactional practices of teaching, including eliciting and interpreting student thinking. But for the use of such assessments to become more widespread, there needs to be additional conversation in the field about the design and use of simulation assessments. This chapter is designed to support such conversations.

In our current work, we are continuing to explore the design of simulation assessments. In our early work, we designed assessment simulations relying on the wisdom of practice, that is, insights generated through our own experiences

Explains the process of the student		
✔	Indicates that 83 is the correct answer	
✔	Summed digits in the tens column to get 6	
✔	Summed digits in the ones column to get 23	
✔	Combined the 6 and the 2	
✔	Summed digits in the tens column first	
Generates a follow up problem to confirm the student's process		
✔	Produces a problem that requires a combining step	
✔	Articulates why this problem would help confirm the student's process	
Anticipates student's response to a follow-up problem based on evidence		
✔	Explains that the student would add the tens (2 + 4) and record 6 in the tens column	
✔	Explains that the student would add 7 + 8 and record 15 in the ones column	
✔	Demonstrates or shares that the student would start with the tens column	
✔	Indicates that the student would combine 6 tens and 1 tens to get 7 tens	
✔	Indicates that the student would produce 75 as the final answer	
Anticipates student's understanding of key ideas on a follow-up problem based on evidence		
✔	The values of the 6, 1, and 5 (only needs to provide evidence for the values of 6 and 1)	✔ uses Evidence to support
✔	Why it is necessary to combine the 6 and the 1	✔ uses Evidence to support
Mathematical knowledge for teaching		
✔	Generalizes that the student's process (i.e., the column addition method), when properly executed, would work for all two-digit addition problems	
✔	Articulates why the student's process always works	

Fig. 9.4 Abbreviated scoring checklist for the follow-up interview: example performance

working with students, analyzing data collected from students, knowing a variety of ways students approach different mathematical situations. These insights have allowed us to articulate how students at a given grade level could reasonably be expected to talk about the problem and the ways in which they could reasonably be able to convey their understanding. We used these insights to construct the student profile after specifying the mathematical topic/practice, characterization of the student's process and understanding, and the student's way of being. Currently, we are exploring ways to draw on two additional sources of information for our design work: (1) interviews with students around the selected problem; and (2) learning progressions research which details how students at a particular point in a learning progression understand particular content. These are promising possibilities for strengthening the development of the student profile.

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Chapter 10
Professional Growth Through Activities and Assessment Tools Used in Mathematics Teacher Preparation Programs

JeongSuk Pang

Abstract Given the significance of teacher education for the professional growth of pre-service elementary school teachers, this chapter of commentary begins with the remaining challenges of mathematics teacher education programs. It then highlights what should be emphasized in teacher education programs: diagnostic activities, conceptual foundation of essential topics, mathematical connections, and core instructional skills. The chapter comments on effective learning environment and simulation assessment in terms of how to teach in mathematics teacher education. This chapter closes with an expectation of further analysis of teacher education programs across different education systems.

Keywords Professional growth of pre-service teachers · Mathematics teacher education program · Elementary mathematics specialist · Essential topics in elementary mathematics

10.1 Introduction

Teacher education programs have a crucial impact on the professional growth of pre-service teachers as mathematics teachers (Association of Mathematics Teacher Educators [AMTE] 2017; Borko et al. 2000). Teacher education programs provide pre-service teachers with the last opportunity to both learn mathematics and how to teach mathematics before entering into the teaching profession. Ma (1999) suggests that the teacher preparation period may serve as the force to break the vicious circle

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Chapter of Commentary on Theme 2.

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formed by low-quality schooling and low-quality teaching of mathematics. In short, teacher education programs for pre-service teachers matter greatly. Given this, to what extent and how does a teacher education program maximize the opportunity for professional growth necessary for teaching? Despite the fairly extensive studies on teacher education, there is no easy answer to this basic question.

The four chapters under Theme 2, Activities and Assessment Tools Used in Mathematics Teacher Preparation Programs, highlight diverse and critical features of the preceding question. Building on these chapters, as well as other references, this brief commentary consists of three sections. The first section of this chapter describes the remaining challenges within teacher education programs, in order to set a basis of what and how to teach in the programs. The second section points out what should be emphasized in teacher education programs. Finally the third section deals with what kinds of activities or tools may be employed in the programs.

10.2 Challenges of Mathematics Teacher Education

10.2.1 Elementary Teacher Preparation: Generalist Model Versus Specialist Model

What do we expect pre-service elementary school teachers to learn through teacher education programs? In a historical analysis of teacher preparation programs in the U.S. context, Fennell (this volume, Chap. 6) claims the need for elementary mathematics specialists, with the importance of mathematical knowledge for teaching, instead of the predominant generalist model by which elementary school teachers are educated to teach all or many subjects. On the one hand, the generalist model for elementary teacher education would be plausible if we consider the main objective of elementary education not as seeking profound knowledge of every subject matter but as learning fundamental ideas of each discipline, along with basic human education. On the other hand, the generalist model comes to be problematic when the knowledge of specific mathematical concepts and principles of elementary school teachers is often weak or even inadequate (e.g., Ball 1990; Ma 1999; Vula and Kingji-Kastrati, this volume, Chap. 8). This raises a critical issue of how the teachers can facilitate students' conceptual learning.

Could the specialist model for elementary teacher preparation be the solution? Finding the answer in international contexts is not easy because it is related not only to the main objective of elementary education but also to various socio-cultural factors such as the quality of schooling, the quality of teachers, the number of teachers, and social expectation of teaching profession. The call for elementary mathematics specialist models in the U.S. context seems urgent because of unsatisfactory mathematical knowledge of teachers. As Fennell (this volume, Chap. 6) describes, about 20 states in the U.S. have been changing to offer elementary mathematics specialist certification. In fact, pre-service elementary school teachers in top-performing systems such as Shanghai and Hong Kong are educated to be specialists (Jenson et al. 2016).

However, a challenging issue is determining in which subjects specialists are needed at the elementary school level. If mathematics educators call for elementary mathematics specialists, their counterparts in other content-intensive subject matters such as science or language arts may also do so. It is still common in other education systems, such as Korea and Singapore, for the main subjects including mathematics to be taught by the teacher in charge at the elementary school level, with other skill-oriented subjects such as music, fine art, and physical education taught by specialized teachers respectively (Li et al. 2008).

Another related issue is that recently much attention has been given to integrated approaches such as STEM (i.e., Science, Technology, Engineering, and Mathematics) education or core competency-based education as opposed to the current excessive segmentation into each discipline. For instance, if core-competencies are emphasized throughout schooling, pre-service teachers need to be educated to foster not only foundational literacies including numeracy and scientific literacy but also critical thinking and problem-solving, creativity and innovation, communication, collaboration, and so on (Partnership for 21st Century Skills and the American Association of Colleges of Teacher Education 2010; World Economic Forum 2015). Given this, to what extent and in which contexts do we agree on subject-matter specialists at the elementary school level?

It may not be a matter of choosing either the generalist model or the specialist model for pre-service elementary school teachers. For instance, pre-service elementary school teachers have one or two specialized subjects within the general teacher education program. For instance, pre-service elementary school teachers in Korea choosing mathematics as a concentration are required to take seven to ten courses of mathematics during their junior or senior years (Pang 2015). These courses, totaling 20–22 credit hours, are about 1/7 the credits required for a bachelor's degree in education. Usually mathematics concentration programs in the Korean context consist of both pure mathematics and mathematics education courses, in order for pre-service teachers both to deepen their disciplinary knowledge of mathematics and to enrich their pedagogical content knowledge. In this context, another related issue is on which the greater emphasis should be placed, content knowledge or pedagogical knowledge for teaching mathematics. The point is that pre-service elementary school teachers may lay the foundation of their expertise in one or two specific subjects, even though they are still educated to be generalists.

10.2.2 Variations in Teacher Certification Requirements and Testing

A challenge facing mathematics teacher education is the large variation of accreditation and its certification requirements of teacher education programs. Fennell (this volume, Chap. 6) raises accreditation challenges of teacher education programs, showing that the first and second ranked providers of teacher preparation programs in the United States (i.e., University of Phoenix and A+ Texas Teachers) are not accredited. According to Fennell (this volume, Chap. 6), the United States has over 1700 teacher education programs for elementary school teachers and Japan has 1300 providers, whereas Finland has 11 teacher education programs. The number of teacher education programs and their concomitant variations do not have to be negative. However, a lack of accreditation or certification of such programs may be problematic if we expect our pre-service elementary school teachers to be well-prepared for teaching.

According to Kool and Keijzer (this volume, Chap. 7), all pre-service elementary school teachers in the Netherlands have to pass a mathematics test during the third year of their teacher education period. This screening process seems effective, specifically when we consider many studies showing the lack of mathematical knowledge of pre-service elementary school teachers.

Another option is to take a teacher employment test after successful completion of teacher education programs, such as those taken by teachers in Japan, Taiwan, and Korea (Li et al. 2008). Such a test is viewed as a powerful lever for influencing what is emphasized in a teacher education program. For instance, as the written test items in the national teacher employment test in Korea are typically embedded in various classroom scenarios, teacher education programs tend to provide pre-service teachers with an opportunity to understand curricular emphases or instructional methods with relation to real teaching contexts (Pang 2015). Mathematical knowledge needed for teaching (Ball et al. 2008) has been emphasized in the recent tests over purely mathematical knowledge in general. More recently, the emphasis in the test of performance assessment by lesson planning and lesson implementation leads teacher education programs to underline a competent performance of teaching subject matters (Pang 2015).

The emphases in teacher education programs may be changed according to socio-cultural factors in which such programs are embedded. As Fennell (this volume, Chap. 6) emphasizes, what matters is whether or not systematic and ongoing research is conducted on the nature of such programs and their effectiveness in terms of teacher learning. There is not yet sufficient research to reach agreement regarding the specific knowledge, skills, and dispositions that will help pre-service elementary school teachers be well-prepared for their future teaching.

10.3 What to Teach in Mathematics Teacher Education

10.3.1 Diagnostic Activities of What to Teach

As mathematics is hierarchical in nature, teachers tend to assess what and how much their students understand before teaching a new mathematical topic. In a similar vein, teacher educators need to assess the prior knowledge, skills, and dispositions of pre-service teachers. A study by Vula and Kingji-Kastrati (this volume, Chap. 8) reveals where to focus in teacher education programs. The pre-service teachers in the study revealed different levels of understanding in the multiple meanings of fractions. We may simply assume that pre-service teachers understand the part-whole meaning of a fraction better than other interpretations such as the quotient or the measurement interpretation, because the part-whole meaning of a fraction is usually first addressed and emphasized in schooling (Reys et al. 2009). However, Vula and Kingji-Kastrati indicate that this is not the case, showing that the pre-service teachers' knowledge varies within the same part-whole meaning of a fraction, according to the test items. Specifically, the pre-service teachers did worst in figuring out the part of a whole displayed in an irregular shape. An understanding of the relational nature of a fraction is needed in figuring the part of a given whole. The pre-service teachers' knowledge was also different regarding other meanings of a fraction. These findings compel the design of diagnostic assessment tools for pre-service teachers to be more elaborate and specific.

Diagnostic activities for pre-service teachers do not need to be limited to mathematical knowledge but to include pedagogical content knowledge. For instance, Vula and Kingji-Kastrati (this volume, Chap. 8) assessed not only how the pre-service teachers performed addition and subtraction of fractions but also to what extent they were able to justify their solutions with multiple models. The study by Shaughnessy and Boerst (this volume, Chap. 9) exposes another effective tool for assessing the specific pedagogical content knowledge of pre-service teachers. In assessing what pre-service teachers say and do regarding a target teaching practice, teacher educators need to infer what pre-service teachers already know and what they have yet to learn in teacher education. The point is that, when pre-service teachers' knowledge, skills, and dispositions are diagnosed, it becomes more evident to decide where the focus in a teacher education program should lie. In this respect, Newton (2008) showed that pre-service teachers were able to develop a deeper, more sophisticated understanding of fractions after taking a course that put a specific emphasis on connecting fraction concepts with procedures. Similarly, Pang (2011) demonstrated that the pre-service elementary school teachers in her study initially tended to analyze a lesson through a general perspective of teaching practice, but later were able to develop mathematics-specific analysis ability through taking a course designed to provide them with opportunities to develop these specific skills.

10.3.2 Conceptual Foundation of Essential Topics in Elementary Mathematics

Despite the significance of teacher education programs, many studies report that such programs are not usually successful in providing pre-service elementary school teachers with an opportunity to develop sufficient depth of knowledge to teach mathematics. Vula and Kingji-Kastrati (this volume, Chap. 8) show that the pre-service elementary school teachers in their study had a limited understanding with regard to multiple meanings of fractions and the conceptual basis behind the addition and subtraction of fractions. Conceptual understanding of mathematical ideas is necessary for developing procedural fluency (National Research Council 2001). This implies that more important is in what ways pre-service teachers have knowledge of mathematics. Simply knowing the facts and applying that knowledge to a specific problem context is not enough. Pre-service teachers need to understand the interconnected nature of such facts and to explain them with appropriate models (AMTE 2017; Ma 1999).

In fact, the lack of knowledge of pre-service elementary school teachers found by Vula and Kingji-Kastrati (this volume, Chap. 8) is not surprising, specifically regarding conceptual knowledge over procedural knowledge. The foremost issue is how a teacher education program has to deal with such a lack of conceptual understanding of mathematical ideas held by pre-service elementary school teachers. It is impossible in a teacher education program to teach all the topics that have been reported as weak content areas for pre-service elementary school teachers. As described above, most pre-service elementary school teachers are educated to be generalists so that a teacher education program does not have much room to intensely focus on mathematics among multiple subject matters. This, in turn, makes it difficult for pre-service teachers to develop a profound understanding of mathematical knowledge. What do pre-service elementary school teachers need to know? How much do they need to know? Recently, National Council of Teachers of Mathematics [NCTM] presents the *essential understanding series* for teachers to enrich their own knowledge of particularly important topics for specific grade bands such as rational numbers in Grades 3–5 (Barnett-Clarke et al. 2010). As such, the series targets mathematical knowledge that teachers need to know.

10.3.3 Mathematical Connections Across Grade Bands or School Levels

Content knowledge for teaching is the main component regarding what to teach in mathematics teacher education for pre-service teachers. In fact, AMTE (2017, p. 2) assumes that "learning to teach mathematics requires a central focus on mathematics." A related aspect which may be easy to miss is a mathematical connection of knowledge across grade bands or school levels. As Fennell (this volume, Chap. 6, p. 4) states, The *Mathematical Education of Teachers II* report recommends that "prospective elementary teachers be required to complete at least 12 semester-hours of coursework involving fundamental ideas of elementary mathematics, *their early childhood precursors, and middle school successors*." The conceptual foundation of essential topics in elementary mathematics would be strengthened if pre-service elementary school teachers could connect such foundation with mathematics that students encounter earlier or later in school. Longitudinal connections in students' learning will make the mathematical background of pre-service elementary school teachers solid and effective. For instance, pre-service elementary school teachers are expected to understand the importance of algebraic thinking over the simple dichotomy between arithmetic in elementary school and algebra in secondary school (NGA and CCSSO 2010). Such teachers will be ready to deal with basic operations algebraically with specific focus on the common properties of operations and the relations between operations (Blanton et al. 2011).

Kool and Keijzer (this volume, Chap. 7) contribute to the monograph in two unique ways: (a) focusing on *horizon content knowledge* (HCK) rather than on common content knowledge (CCK) or specialized content knowledge (SCK) among subject matter knowledge; and (b) involving high-performing pre-service teachers. Here HCK is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al. 2008, p. 403). A teacher education program needs to provide pre-service teachers with an opportunity to develop their HCK, as long as they are expected to teach mathematics as interconnected set of concepts across grade bands or school levels. Kool and Keijzer (this volume, Chap. 7) report that designing a non-routine HCK problem is more challenging than designing a CCK or SCK problem even for high-performing pre-service teachers. Although designing a high-level problem is not necessary for pre-service elementary school teachers, as the researchers point out, doing so helps them deepen their own mathematical knowledge and enrich their understanding of the problem solver. Furthermore, such a challenging task may improve the disposition of high-performing pre-service teachers towards mathematics, particularly their satisfaction from mathematics.

10.3.4 Core Instructional Skills Combined with Knowledge of Content and Knowledge of Students

Regarding what to teach in mathematics teacher education, the contribution by Shaughnessy and Boerst (this volume, Chap. 9) turns our attention from the mathematical content knowledge to mathematics teaching practices. Given the recently increasing emphasis on practice-based teacher education, what to teach in a teacher education program should be directly connected to the core tasks of teaching (Ball and Forzani 2009). Shaughnessy and Boerst (this volume, Chap. 9) focused on the capabilities of pre-service teachers in eliciting and interpreting student thinking. In fact, eliciting and interpreting student thinking is one of eight mathematics teaching practices that have been identified as "a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM 2014, p. 9). The researchers further decomposed, for instance, the work of eliciting student thinking to make it accessible for pre-service elementary school teachers: "(a) formulating questions designed to elicit and probe student thinking; (b) posing questions; (c) listening to and interpreting what students are saying; and (d) developing additional questions to pose" (p. 7).

Articulating what eliciting and interpreting student thinking means helps pre-service teachers pinpoint where to focus while interacting with students and content. The simulation assessment that Shaughnessy and Boerst (this volume, Chap. 9) designed implies that pre-service teachers need to learn how to combine detailed instructional skills together with mathematical knowledge of a specific topic and knowledge of student thinking of that topic.

10.4 How to Teach in Mathematics Teacher Education

10.4.1 Effective Learning Environment for Pre-service Teachers

How to teach is as important as what to teach in mathematics teacher education. As mentioned, pre-service teachers have an opportunity to learn how to teach mathematics by their own engagement into diverse activities in a teacher education program. Kool and Keijzer (this volume, Chap. 7) provide us with a fascinating learning environment of how to support pre-service teachers who need help in designing non-routine HCK problems. Such an environment included diverse features such as: (a) providing many examples of non-routine HCK problems, (b) asking pre-service teachers to solve the problems in different ways and to analyze their characteristics, (c) encouraging them to take the perspective of problem solver and to anticipate heuristics from that perspective, (d) emphasizing interaction with peers, (e) urging them to build a repertoire of prototypes of HCK problems, (f) providing feedback from experts and showing an expert's own struggling process in designing HCK problems, and (g) facilitating them to be engaged in a cyclic design process of evaluating and improving the original data.

These characteristics of the learning environment in Kool and Keijzer (this volume, Chap. 7) have at least three implications regarding how to teach in mathematics teacher education. Firstly, the pre-service teachers in the study had an opportunity to solve HCK problems for themselves and to take the perspective of problem-solver. This activity balances emphasizing both content knowledge and pedagogical knowledge for pre-service teachers. Even though the problem-solver in the study was another pre-service teacher, anticipating heuristics from the problem-solver perspective will help them pay attention to the various approaches of students. In fact, anticipating multiple responses from students to a challenging task is a foundational practice for effective mathematics teaching (Smith and Stein 2011). Secondly, the pre-service teachers in the study interacted with peers and experts over a substantial period of time. The experience of being an active member of a learning community in a teacher education program is significant, if we regard teaching not as an isolated personal aspect but as a collaborative communal activity to ensure the quality long-term practice of continuous development (Stigler and Hiebert 1999). Finally, the pre-service teachers in the study were engaged in the cyclic process of designing and improving mathematical problems, while watching an expert's own struggling process. This engagement may help them regard themselves as career-long learners and appreciate an iterative process of lesson planning, implementation, and reflection on it to improve classroom expertise. By providing a stimulating and supportive learning environment in a teacher education program, pre-service elementary school teachers will be better prepared to establish similar learning environments in the classroom they enter as professionals.

10.4.2 Simulation Assessment as a Way to Foster Classroom Expertise of Pre-service Teachers

A central issue in teacher education is how pre-service teachers apply what they have learned in a teacher education program to their actual classroom teaching. Many studies report a widening gap between what is expected through teacher education and what is implemented in the classroom (Borko et al. 1992; Pang 2003). A way to overcome such a gap is to provide pre-service teachers with an opportunity to learn how to teach mathematics while engaging in authentic teaching activities which combine both mathematically and pedagogically key aspects. In this respect, Shaughnessy and Boerst (this volume, Chap. 9) exemplify in what ways pre-service teachers are able to be involved in the work of eliciting and interpreting student thinking with the mathematical topic of multi-digit addition. In fact, the researchers merit specific simulations that can reveal pre-service teachers' skills with eliciting and interpreting student thinking, along with their mathematical knowledge for teaching. As the researchers suggest, simulations can be employed in teacher education not only to assess pre-service teachers' capabilities but also to evaluate their developing skills.

A mathematics method course in a teacher education program deals with various instructional approaches to teach mathematics. However, assessing the teaching practices of pre-service teachers is more difficult for teacher educators than measuring their mathematical knowledge. On the one hand, Shaughnessy and Boerst (this volume, Chap. 9) make a great contribution in that they provide us with a prototype of how to design a simulation assessment in order to evoke, document, and assess pre-service teachers' capabilities, together with a vignette of a pre-service teacher's performance on the assessment. On the other hand, designing such a simulation assessment remains difficult for most teacher educators because it requires considerable efforts in articulating the target of assessment, gaining specific information about learners related to the target (i.e., characterization of the student's process and understanding as well as the student's way of being), and inferring a chain of reasoning connecting the two.

10.5 Concluding Thoughts

The beginning is always important. Teacher education programs for pre-service teachers serve a basis for classroom expertise which will be further developed throughout the teaching career. This chapter of commentary focuses both on what to teach and on how to teach in such programs. The main key aspects in the programs include (a) conceptual foundation of essential topics in elementary mathematics, (b) mathematical connections of such topics in the school curriculum, and (c) basic instructional skills such as eliciting and interpreting student thinking for effective teaching with relation to essential topics students need to know. Another related issue is evaluating what to teach and how to teach in teacher education programs by diagnosing and monitoring pre-service teachers' knowledge, skills, and dispositions in a sustainable way.

A teacher education program provides pre-service teachers with various learning experiences through mathematics content courses, methods course, clinical settings, etc. What pre-service teachers experience in the programs may serve as a foundation to strengthen and to sustain the teaching profession. It is impossible to expect that pre-service teachers will teach well unless they learn how to teach mathematics competently through their teacher education programs. The four chapters under Theme 2 provide teacher educators with an opportunity to re-think the direction, content, and methods of teacher education. Building on this line of study, further diverse, ongoing, and systematic analysis of teacher education programs across different education systems is expected.

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Part III
Pre-service Mathematics Teachers' Knowledge and Beliefs

Chapter 11
An Investigation of Prospective Elementary Teachers' Argumentation from the Perspective of Mathematical Knowledge for Teaching and Evaluating

Yusuke Shinno, Tomoko Yanagimoto and Katsuhiro Uno

Abstract The purpose of this study is to investigate prospective elementary school teachers' mathematical process knowledge related to argumentation. To achieve this, we focus on prospective teachers' mathematical argumentation as a key aspect of the mathematical knowledge teachers need for teaching. By referring to the framework of mathematical knowledge for teaching, we pay special attention to "process knowledge" instead of "content knowledge." The study involves 136 prospective teachers at a national university in Japan who performed a task requiring the evaluation of several incorrect solutions to a realistic problem. The results show that most prospective teachers have difficulties in evaluating or assessing children's incorrect solutions. This study contributes to the field on a conceptual and a methodological level. Regarding the conceptual framework, we suggest the importance of teachers' process knowledge in teaching and evaluating, particularly in relation to mathematical argumentation and, regarding methodology, we create a way to help participants notice children's incomplete thinking.

Keywords Prospective elementary teachers · Argumentation · Mathematical process · Mathematical knowledge for teaching · Realistic problem
Teachers' evaluating

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11.1 Introduction

Mathematical knowledge for teaching at the elementary level is a relatively new research topic in the Japanese mathematics education community. From a traditional perspective, the importance of mathematical knowledge for teaching at the elementary level differs from the knowledge required of specialists in mathematics (Wittmann 2001). To approach the issue of mathematical knowledge in a professional context, we consider mathematical processes, rather than mathematical content, as indicative of elementary teachers' mathematical knowledge for teaching at elementary schools. Specifically, we pay particular attention to their argumentation, which can be seen as an essential for mathematical activity, because this allows us to see that mathematical reasoning and proving involve some pattern or structure associated with the argument. In other words, argumentation as a mathematical process, even if it is represented informally, can be seen as a vehicle of mathematical structure. Devlin (1994) clearly stated this point as follows:

> The choice of language, whether symbolic, verbal, or even pictorial, might affect the length of the proof, or the ease with which you can understand it, but it does not affect whether the argument does or does not constitute a proof. (Devlin 1994, pp. 37–38)

Argumentation, reasoning, and proving are crucial mathematical processes in mathematics at all school grades and have been studied as such (e.g., Hanna and de Villiers 2012; Harel and Sowder 2007; Mariotti 2006; Reid and Knipping 2010; Stylianou et al. 2009). The meaning of argumentation and related notions (e.g., reasoning, proving, proof) have often been discussed in the field of mathematics education research (e.g., Mariotti 2006; Reid and Knipping 2010; Stylianides et al. 2016). In the present study, we use the term "argumentation" based primarily on the considerations expressed in Stylianides et al. (2016) as follows:

> There seems to be a fairly shared understanding among researchers about the meaning of *argumentation*, a term which is generally used to describe the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false. (Stylianides et al. 2016, p. 316)

In addition, the argumentation we examine in this study is closely related to teachers' pedagogical activities; therefore, we focus on teachers' mathematical argumentation rather than that of the students. Although some previous literature also mentions the relationship between argumentation and proof, we do not pay much attention to this distinction here because our emphasis is on the argumentation of teachers that can be situated in their regular classroom practice in elementary schools. However, it is necessary to briefly discuss other related notions because "argumentation is often situated in the context of a broader mathematical activity which has been described using different terms (e.g., proving or reasoning-and-proving)" (Stylianides et al. 2016, p. 316). In the literature review by Reid and Knipping (2010), while different definitions of "proof" in mathematics education occur, "proof" as a convincing argument can be related to "proving" as follows: "If 'proof' is used to mean a convincing argument, then 'proving' usually refers to convincing someone of something" (Reid and Knipping 2010, p. 30). For some researchers, this usage of "proof" also refers to a reasoning process (Reid and Knipping 2010).

In our study, proving, reasoning, and other mathematical processes such as interpreting or representing are used to refer to students' mathematical processes rather than those of teachers. Although a number of studies have focused on students' perspectives towards argumentation, reasoning, or proving, some studies have reported on (elementary) teachers' knowledge of reasoning and proving (e.g., Herbert et al. 2015; Martin and Harel 1989; Simon and Blume 1996; Stylianides and Ball 2008; Stylianides et al. 2007, 2013). According to these earlier studies, many elementary school teachers have limited mathematical knowledge of reasoning and proving. We think that teachers' argumentation skills can be essential not only in teaching mathematical processes, such as reasoning and proving, but also in evaluating and assessing students' knowledge at all levels and of all content throughout school mathematics.

In our previous studies (Shinno et al. 2012; Yanagimoto et al. 2013), presented at ICME-12 in Seoul, we focused on a proving problem in relation to the concept of parity to investigate prospective teachers' argumentation. In order to make mathematical processes related to reasoning and proving more central in mathematics classroom lessons, we focus in the present study on a realistic problem that involves fruitful mathematical processes such as mathematizing, reasoning, proving, and interpreting. Because many Japanese teachers believe that realistic problems can be effective in fostering students' competencies, such problems have been widely used in mathematics classrooms in Japanese elementary schools. However, teachers may often face difficulties in evaluating or assessing students' representations of the mathematical processes they applied to the problem. Thus, although earlier studies reported that many elementary school teachers have a fragile knowledge of reasoning and proving, we think that further research is needed to examine another aspect of the knowledge of mathematical processes. Namely, we argue that more attention should be paid to teachers' argumentation skills in teaching and evaluating.

Our research position may contribute to addressing the following challenges, mentioned by Stylianides and Ball (2008) as a direction of further research in this domain: "to investigate the mathematical knowledge involved in teachers' decisions about whether and when to transform a regular problem into a proving task" (p. 329). To help students learn mathematical processes, teachers have to evaluate or assess their students' explanations, which can be presented in the form of (oral or written) words, objects, and figures. In order for teachers to understand students' various (correct or incorrect) explanations and respond to them, it is important to consider teachers' argumentation skills, which come into play when transforming a regular problem into a proving problem in classroom teaching situations. Therefore, we discuss both the mathematical knowledge needed for teaching and the mathematical argumentation skills needed for evaluation by focusing on prospective elementary school teachers in Japan (Shinno et al. 2012; Yanagimoto et al. 2013).

In the present study, we adopt the well-known conceptual framework of *Mathematical Knowledge for Teaching*, formulated by Ball and her colleagues (Ball et al. 2008; Hill et al. 2008), and apply this framework to discuss teachers' process and content knowledge (Foster et al. 2014). The purpose of this study is to investigate prospective elementary school teachers' mathematical process knowledge as related to argumentation by using a realistic problem.

11.2 Mathematical Knowledge for Teaching

Research on knowledge for teaching began with Shulman's Presidential Address at the 1985 American Education Research Association (AERA) annual meeting (Shulman 1986), which categorized teacher knowledge into the three principal domains of content knowledge, curricular knowledge, and pedagogical content knowledge. Based on Shulman's proposal, Ball et al. (2008) elaborated the so-called *Mathematical Knowledge for Teaching* (MKT) model in terms of the diagram summarized in Fig. 11.1.

According to Ball et al. (2008), MKT consists of *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). SMK, the left side of the diagram in Fig. 11.1, comprises *specialized content knowledge* (SCK), which includes mathematical knowledge and skills that are uniquely used in teaching contexts, and *common content knowledge* (CCK), which is used in settings other than teaching. PCK, the right side of the oval, comprises *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). Although Fig. 11.1 conceptualizes mathematical knowledge of different kinds needed for teaching, as Ball et al. (2008) describe, it may be difficult to make explicit distinctions between domains in practice:

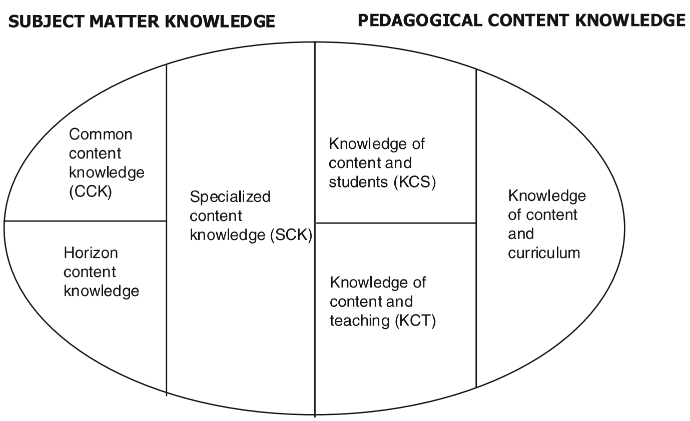


Fig. 11.1 Domain of mathematical knowledge for teaching (Ball et al. 2008, p. 403)

> Consider the example of analyzing a student error. A teacher might figure out what went wrong by analyzing the error mathematically. What steps were taken? What assumptions made? But another teacher might figure it out because she has seen students do this before with this particular type of problem. The first teacher is using specialized content knowledge, whereas the second is using knowledge of content and students. (Ball et al. 2008, p. 403)

In the present study, we pay special attention to SCK and how it relates to KCS and KCT. Ball et al. (2008) pointed out several mathematical tasks for teaching that draw on SCK, such as "presenting mathematical ideas" and "responding to students' 'why' questions." In teaching tasks that relate to SCK, we focus on "evaluating the plausibility of students' claims" and "giving or evaluating mathematical explanations," as we think that these aspects of SCK are closely related to the argumentative knowledge and skill of teachers. On the one hand, we think that it is important to recognize the deep connection between SMK (mainly SCK) and PCK (mainly KCS and KCT) in an actual classroom situation. On the other hand, when we utilize Fig. 11.1 as a conceptual framework for researchers, it is also important to clarify different domains of knowledge for teaching with regard to a specific mathematical task. In our study, we claim that classifying prospective teachers' knowledge or difficulties in terms of this framework can be undertaken within researchers' practice rather than prospective teachers' practice. Therefore, we would like to illustrate SCK in relation to KCS and KCT through the realistic mathematical problems discussed in the next section.

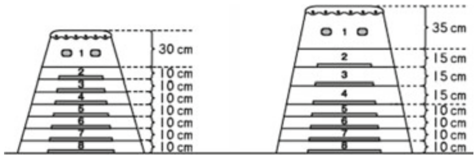
Foster et al. (2014) recently conceptualized the domain of mathematical knowledge for teaching in terms of mathematical processes such as representing, analyzing, interpreting, and communicating, replacing every occurrence of the word "content" in Fig. 11.1 with the words "concept and process." We agree with Foster et al.'s (2014) emphasis on "process" because the notion of argumentation is deeply related to mathematical processes. They also state: "We do not suggest that process and content are dichotomous; on the contrary, we take the view that concepts and processes together constitute the content. We believe, however, that mathematical processes have been relatively neglected." (Foster et al. 2014, p. 98) Based on this perspective, even though the focus of our study is on SCK, KCS, and KCT, we pay special attention to "process knowledge" instead of "content knowledge." In the next section, we identify specific mathematical processes that are involved in a problem used in our study.

11.3 Methodology

In this section, we introduce the instruments we used in this study and explain them in terms of the categories of mathematical knowledge for teaching and the relevant mathematical processes involved in the problem. The participants and the data collection process are also described.

At Yuriko's school, there are two types of vaulting horse, small and medium. The top layer of the small vaulting horse is 30 cm high, and each layer from the second through the eighth is 10 cm in height. For the medium vaulting horse, the top layer is 35 cm high and each layer from the second through the fourth layer is 15 cm in height. For each layer from the fifth through the eighth, the height is 10 cm.

Small Vaulting Horse (8 layers) **Medium Vaulting Horse (8 layers)**



Yuriko and her friends want to set up the medium vaulting horse so that its height will be 70 cm. Is it possible to set up the medium vaulting horse so that it has a height of 70 cm? Explain why you chose that answer using words and numbers.

Fig. 11.2 The vaulting horse problem (NIER 2012) (slightly adapted excerpt from the translation by IMPULS 2014)

11.3.1 The Problem Used in the Study

To investigate prospective teachers' mathematical knowledge for teaching and evaluating, we use a realistic problem originally developed for the national assessment of academic ability in Japan (National Institute for Education Policy Research 2012). The problem, which is shown in Fig. 11.2, was included in the national assessment for sixth-grade students in April 2012.

This problem, which we call "the vaulting horse problem," is closely connected with Japanese sixth-grade students' actual experiences in physical education classes. In fact, "physical fitness with the vaulting horse" is one part of the official teaching content of physical education in the Japanese national curriculum.[1] This means that the context of the problem, which is related to activities with the vaulting horse, is very familiar to Japanese elementary school students.

According to the national assessment report (NIER 2012), 27% of sixth-graders gave the correct answer to this problem. Compared with other problems in the same assessment test, this score is not very good. The essential part of the problem was to explain "the reason why it is *not* possible to set up the medium vaulting horse so that it has a height of 70 cm." Although the original item in the national assessment consisted of three questions, we focus only on the second question listed in Fig. 11.2. In order to give a correct explanation, children must interpret and represent the information included in a realistic situation that is needed to make sense of the problem.

[1]An English translation of the Japanese national curriculum can be retrieved from the following link: http://www.mext.go.jp/en/policy/education/elsec/title02/detail02/1373859.htm.

11.3.2 Linking with SCK, KCS, and KCT

This problem involves two main processes, interpreting a real-life issue from the data given and representing (explaining) it using one's knowledge. In particular, it includes "the order in which layers are set up," based on real-world logic, and "the height of each layer" as a given condition. Although these aspects are not always clear to children, it is important that they be able to interpret and represent them and use them as assumptions in order to answer the problem. While we consider the processes used by children to arrive at the correct solution to this problem to be the primary concern of teaching and evaluating, we also feel that evaluating different solutions given by children and assessing (modifying) their inadequate or incorrect solutions are important parts of SCK that relate to KCS and KCT. For example, regarding KCS, Ball et al. (2008) state that "teachers must also be able to hear and interpret children's emerging and incomplete thinking as expressed in the ways that pupils use language" (p. 401). In this sense, teachers' argumentation for teaching mathematics can be a crucial factor in assessing or evaluating students' process knowledge.

The results of the national assessment reported that 30% of sixth-graders answered incorrectly by stating, "it is possible to set up 70 cm." They failed to consider the order and height of each layer. Although some children answered correctly by saying that "it is not possible to set up a medium vaulting horse of 70 cm," most of them had difficulty explaining why. This may be due to the children's simplistic understanding of the problem. Therefore, when it comes to KCT, teachers should be able to assess and explain children's incorrect solutions so that the children become aware of their own incomplete thinking. This kind of teachers' response can be considered an argumentative skill needed for teaching mathematics. In fact, the national assessment report introduces an example of a lesson plan using the above problem (Fig. 11.3). The teacher's way of handling the following situation provides a basis for the children to engage with their difficulties or errors.

Child 1: By stacking the first, second, fifth, and sixth layers, it is possible to set up a medium vaulting horse with a height of 70 cm.

Child 2: If we add the fifth layer after the second layer, it will be like this figure. Therefore, it is not possible to set it up.

Teacher: It is important to consider your experience in preparing a vaulting horse. Is it possible to set up a vaulting horse with a height of 70 cm by stacking these in order, such as the first, the second, the third, the fourth…?

Child 3: It is not possible to set it up by stacking them in order.

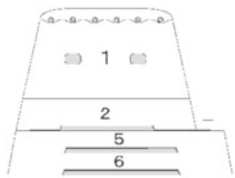


Fig. 11.3 An example of a lesson idea (NIER 2012) (translated by the authors)

11.3.3 Data Collection

In this investigative study, conducted in 2015, the participants were 136 prospective elementary teachers. Most of the participants were second- or third-year undergraduate students at a national university in Japan. Prospective elementary school teachers enrolled in the university's Faculty of Education, where four-year training is provided for teacher preparation, have the opportunity to study both the contents and the methods of each school subject such as mathematics, social studies, and the arts.[2] In the period during which the survey was undertaken, 136 prospective teachers who had already taken the content courses of mathematics for teaching, including both second- and third-year undergraduates, had just begun to learn the methods of mathematics for teaching. As a result, we expect there to be no significant differences between the second- and third-year undergraduates in terms of their learning experiences in teaching mathematics, as they had similar mathematical experiences in their training program. Therefore, in our analysis we do not distinguish the data collected from participants in different years of the program.

Although the study involved no actual classroom lesson, we created a task that allows participants to focus on children's particular difficulties in the problem. In this study, we considered the four answers to the problem listed in Table 11.1 as typical incorrect solutions. These four answers were chosen by referring to the national assessment report (NIER 2012), its criteria for correct solutions, and its examples of children's incorrect solutions. In order to investigate the participants' explanations of the reason why it is *impossible* to set up the medium vaulting horse so that it has a height of 70 cm, which is the main characteristic of the problem as mentioned above, we did not consider statements saying it was possible to set up the vaulting horse in the particular way.

Using the typical incorrect responses listed in Table 11.1, we wanted to discuss the participants' SCK and KCS when evaluating written solutions to the problem. Therefore, we requested that the participants evaluate each incorrect solution to the problem, and then we analyzed their argumentation in doing so. We first asked them to mark the above answers as "correct" or "incorrect" and then to write a reason justifying their decision. In doing so, the participants were expected to provide detailed responses, which can be considered feedback to the (imaginary) child who made each (incorrect) solution.

To examine the participants' KCT, we chose one class with 30 participants and asked the participants to complete the extra task of responding to the following questions related to answer E below. This extra task was conducted one week after the initial task, which included answers A–D in Table 11.1.

[2]More general official descriptions of teacher training in Japan are available at the following link: https://www.nier.go.jp/English/educationjapan/index.html.

Table 11.1 Typical incorrect answers subjected to evaluation by prospective teachers

A	Because the medium vaulting horse consists of one 35 cm, three 15 cm, and four 10 cm, it is not possible to set up one with a height of 70 cm
B	Even adding these up, it is not possible to set up a 70 cm high medium vaulting horse
C	If the medium vaulting horse consists of three layers, 35 + 15 + 15 = 65 cm. Another 5 cm are missing; hence, it is not possible to construct one
D	If the medium vaulting horse consists of four layers, it becomes 80 cm high. Hence, it is not possible to set up one 70 cm high

[E] If the medium vaulting horse consists of the first three layers, the total height becomes 65 cm. Hence, it is not possible to set it up with a height of 70 cm.

Q1. What are the points in the above answer that make it incomplete?

Q2. What kind of counterexample would you show the child for him/her to become aware of the incorrectness? Please fill in the following blank: If________________________________, then it is possible to set it up with a height of 70 cm.

Answer E is incorrect because it does not mention the height of the fourth layer (15 cm) or the total height of all four layers (80 cm). We consider Q1 to be related to SCK and KCS because it is similar to the earlier question with four incorrect answers (Table 11.1). On the other hand, we think that Q2 is related to KCT; that is, by showing a possible counterexample to the claim that "it is *possible* to set up a medium vaulting horse 70 cm high by adding a fourth layer 5 cm high to the three layers," the teacher may assess the child's incomplete thinking. In mathematics or logic, a counterexample is a specific exception to a universal proposition that shows it to be false. The notion of counterexamples in this chapter can be pedagogically important. This kind of counterexample can be pedagogically useful in making children aware of the implicit assumptions in their thinking. Therefore, the participants' responses to Q2 may enable us to analyze their KCT.

11.4 Results

Table 11.2 shows the results of the study (all percentages included in Table 11.2 are rounded to the nearest integer). The numbers in the table that correspond to "correct" and "incorrect" indicate the results of the participants' markings of each

Table 11.2 Results of the study (N = 136)

<table>
<tr><td colspan="2" rowspan="2">Correct</td><td colspan="2">A</td><td colspan="2">B</td><td colspan="2">C</td><td colspan="2">D</td></tr>
<tr><td colspan="2">4 (3%)</td><td colspan="2">8 (6%)</td><td colspan="2">96 (71%)</td><td colspan="2">30 (22%)</td></tr>
<tr><td rowspan="2">Incorrect</td><td>Adequate reason</td><td rowspan="2">132 (97%)</td><td>106 (78%)</td><td rowspan="2">128 (94%)</td><td>44 (32%)</td><td rowspan="2">40 (29%)</td><td>27 (20%)</td><td rowspan="2">106 (78%)</td><td>88 (65%)</td></tr>
<tr><td>Inadequate reason</td><td>26 (19%)</td><td>84 (62%)</td><td>13 (10%)</td><td>18 (13%)</td></tr>
</table>

student response. For example, we can see in the table that four participants considered answer A "correct" and 132 participants considered it "incorrect." The numbers that correspond to "adequate reason" and "inadequate reason" indicate the results of our analysis of the participants' written responses.

It is important to note that 71% of the participants decided that answer C was correct. In the justification of answer C, they were expected to refer to the height of the fourth layer (15 cm). However, only 20% of the participants addressed this point adequately. It is also important to note that 62% of the participants could not adequately justify their reasons for stating that answer B is incorrect; in this case, they were expected to mention the height of each layer and the way the vaulting horse is set up.

The following responses were considered to provide an adequate reason for the incorrectness of the four answers:

[A] 35 + 15 + 10 + 10 = 70. In this way, it is possible to achieve a height of 70 cm. In this case, it is necessary to address the order and height of each layer.

[B] For example, if we add the layers from the bottom, we can reach a height of 70 cm by adding the sixth layer (10 + 10 + 10 + 10 + 15 + 15 = 70). So, it is possible.

[C] You need to explain the next layer. The point of the missing 5 cm does not make sense because it is not mentioned anywhere that the next layer should be 15 cm.

[D] It is necessary to add a sentence like, "even taking one layer away, it ends up at 65 cm."

11.4.1 Difficulties in SCK and KCS

The presented problem involves mathematical processes such as interpreting a situation and representing an idea. The results suggest that the prospective teachers faced difficulties in considering these processes. The following responses were considered inadequate in explaining the incorrectness of the four answers, because they neglected to mention certain assumptions, such as the order of the layers and the height of the fourth layer (15 cm). We chose the following responses as illustrative of inadequate reasons because each response was thought to represent typical difficulties that many participants faced.

[A] You need to calculate and find an exact value of the height, and then explain the reason in detail.

[B] This answer is not sufficient because it is hard to understand the way of adding these up concretely.

[C] There is no rule stating that the medium vaulting horse should consist of three layers.

[D] This does not mention the calculation needed to reach 80 cm.

We do not suggest that these participants did not understand the correct solution to the problem; it seems that they had enough procedural knowledge to solve the original problem in Fig. 11.2. We expected, though, that they as problem solvers could produce a correct answer to the problem. We think that in evaluating the given answers, they may not have recognized the necessity of pointing out the assumptions explicitly. In other words, we consider the four responses by participants above to involve difficulties in giving feedback to children about points they may have missed. This in turn implies that the SCK and KCS of participants who gave these answers are not solid and that their argumentative skills for evaluating students' mathematical processes relevant to the problem are rather weak.

11.4.2 *Difficulties in KCT*

The 30 participants in the follow-up task had already studied adequate reasons for evaluating A, B, C, and D. Nevertheless, only seven of these 30 participants gave a valid evaluation of answer E when responding to Q1 and when giving a counterexample in response to Q2. Below is an example of an adequate response to Q1; this response is adequate since it mentions the fact that a child who answered E may not understand the importance of the order of the layers. In responding to Q1, the only thing the participants were expected to do was to mention that the description of the height of the fourth layer (15 cm) or the height of four layers (80 cm) should have been given in the answer.

> [E_Q1] There is no description that a vaulting horse can be set up by stacking the layers in order.

On the other hand, in responding Q2, the following incomplete responses demonstrate the participants' difficulties with providing counterexamples. These responses cannot be considered as having adequately addressed answer E because neither response addresses the actual content of answer E.

> [E_Q2] If you used the first, second, fifth, and sixth layers for setting up, then…
>
> [E_Q2] If the third and fourth layers are each 10 cm high, then…

In order to improve children's awareness of their own incomplete thinking, it is important for a teacher to provide an effective counterexample, which is where KCT comes into play. Here we are not able to conclude that these results constitute a general model for the participants' difficulties in KCT, for the 30 participants were assigned the extra task after providing adequate responses to the first four answers regarding SCK and KCS. Nevertheless, as far as this particular case is concerned, the results may imply that difficulties in KCT are highly relevant to difficulties in SCK and KCS because the seven participants who responded adequately to Q1 were also able to provide an effective counterexample in Q2.

11.5 Concluding Remarks

This study contributes to the field on a conceptual and a methodological level. First, regarding the conceptual framework, we suggested that the importance of teachers' argumentation in evaluating is closely related to SCK, KCS, and KCT. Taking mathematical processes such as reasoning, proving, and interpreting into consideration, we paid special attention to "process knowledge" instead of "content knowledge." Based on this perspective, we focused on a realistic problem, the vaulting horse problem, in order to investigate the prospective teachers' argumentation. Second, regarding methodology, we created a way to make participants notice children's incomplete thinking. This method can provide an opportunity for the participating prospective teachers to consider aspects of the argumentation in the given problem. This can also clarify prospective teachers' difficulties in evaluating solutions to problems and assessing children's incomplete thinking. Based on this method, we identified some challenges in understanding and responding to children's incomplete thinking. In this concluding section, we make some further remarks about the findings of the study.

In the previous section, we showed some difficulties related to SCK, KCS, and KCT. How do we interpret these results from the perspective of mathematical argumentation for teaching and evaluating? We believe that evaluating children's explanations, such as A through D as shown in Table 11.1, and adequately pointing out their (in)correctness are important aspects of SMK that are unique to teaching. Although most of the participants were able to respond correctly to a problem like the vaulting horse problem, they had difficulty as teachers (evaluators) in adequately evaluating, assessing, and noticing children's incorrect solutions and incomplete thinking. Table 11.3 summarizes our observations and shows implications for teaching practice, based on Ball et al.'s (2008) conceptual framework for MKT and the findings of our study. Specifically, in terms of the three categories of knowledge, Table 11.3 describes which argumentative skills can be required for elementary school teachers in evaluating children's mathematical processes and what difficulties or challenges can be faced by the teachers in their classroom instruction.

By looking back to what we mentioned at the beginning of the chapter, our attempts may contribute to investigating the mathematical knowledge involved in teachers' decisions when transforming a regular problem into a proving task (Stylianides and Ball 2008). At first glance, the vaulting horse problem shown in Fig. 11.2 might not appear to be a proving task. However, an important aspect of the problem is that it requires explaining "the reason why it is *not* possible." Answering this type of question may entail expressing the assumptions or conditions included in the problem. Although in this study we provided an evaluating setting that allows prospective teachers to focus on children's particular difficulties related to proving or explaining processes in a realistic problem, this opportunity may also occur in an actual classroom interaction. This kind of opportunity implies that, in order for teachers to attend to children's difficulties, they may need to recognize proving aspects which could be included in the original problem.

Table 11.3 Summary of the study based on Ball et al.'s (2008) framework for MKT

	Prospective argumentation for …	Difficulties in …
SCK: *Specialized content knowledge*	Figuring out children's incorrect solutions and explaining why they are incorrect	Identifying the incorrectness by focusing on the assumptions of the problem
KCS: *Knowledge of content and students*	Evaluating and assessing children's incomplete thinking as expressed in their own words	Assessing mathematical processes involved in children's answers
KCT: *Knowledge of content and teaching*	Giving a counterexample to make children aware of their own incomplete thinking	Responding on the basis of children's incomplete thinking by using a counterexample

As a next step, we need to refine the study based on the outcomes shown in Table 11.3 as follows. Although we focused on mathematical processes, such as interpreting and representing, by using a real-life situation, we also need to use other problem situations to examine other processes, such as analyzing and communicating. It is also important to consider the relationship between our findings and the importance of teachers attending to children's strategies, interpreting children's understanding, and deciding how to respond based on that understanding (e.g., Jacobs et al. 2010). In fact, the role of teacher noticing is also discussed in several chapters in this monograph (e.g., Gupta et al., this volume; Ivars and Fernández, this volume). In particular, the pedagogical notion of counterexamples that we considered a task in this study can be important for deciding how and when to respond to children in a classroom situation. We believe it is important to elaborate this notion in the context of teacher noticing. Finally, although the method of the present study did not pay much attention to any actual classroom lessons (e.g., lesson observations, videos, scenarios, etc.), it seems that most of the findings in the study can be potentially applied to describing or improving teaching practice in elementary schools. In order to do so, there is a need for further research on prospective or in-service teachers' argumentation in actual classroom interactions.

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Chapter 12
The Mathematics Backgrounds and Mathematics Self-efficacy Perceptions of Pre-service Elementary School Teachers

Gönül Güneş

Abstract This study aims to investigate the mathematics backgrounds and mathematics self-efficacy perceptions of pre-service elementary school teachers. It explores the existence of a link between the pre-service elementary school teachers' years of university education and their mathematics self-efficacy. Finally, the study seeks to reveal the links between the pre-service elementary school teachers' mathematics self-efficacy and other variables concerning their mathematics backgrounds. The study employed a survey method. The sample consisted of 209 pre-service teachers (66 male and 143 female). The study found that the mathematics self-efficacy perceptions of pre-service teachers do not change as they progress towards their university degree. However, variables concerning the mathematics backgrounds of the pre-service elementary school teachers affect their perceptions of mathematics self-efficacy. Therefore, individuals who intend to become teachers can be expected to perform better in their teaching practices if they have a high level of mathematics self-efficacy prior to commencing elementary school teacher education.

Keywords Self-efficacy · Mathematics
Pre-service elementary school teachers · Background story

12.1 Introduction

Numerous factors play a role in determining achievement levels in education, including the students themselves, teachers, curricula, administrators, education experts, education technology, learning environments and finance. However, teachers are definitely the most decisive factor in students' achievement levels. Teachers' qualifications in terms of content knowledge and teaching approaches

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have a directly affect student learning. Some (e.g., Rimm-Kaufman and Sawyer 2004) argue that the attitudes and priorities of teachers are closely related with their behaviors and practices in their classes.

The educational system's ability to teach individuals to meet its stated requirements depends on the presence of well-trained and qualified teachers (Özden 2011). For teachers to be considered qualified in their profession, they should have a robust command of general culture as well as content knowledge (Shulman 1986), not to mention adequate teaching skills. These capabilities should be coupled with affective skills such as attitudes, beliefs, self-efficacy, perceptiveness and motivation.

12.2 Conceptual Framework

A review of affective characteristics, hitherto a factor investigated by studies on teacher training, provided new insights into efficacy in teaching. In particular, self-efficacy began to draw attention as one of the affective variables concerning the training of teachers (Akay and Boz 2011). The concept of self-efficacy stems from Bandura's social cognitive theory. Bandura (1997) defines self-efficacy as: "people's judgments of their capabilities to organize and execute the courses of action required to attain designated types of performances" (p. 391). According to Bandura (1997), confidence in one's self-efficacy plays a major role in shaping individual behavior and is built on four pillars. These are mastery teaching experiences, vicarious experiences, social persuasion, physiological and emotional states. Mastery teaching experiences refer to the experiences individuals accumulate. The results of previous experiences with a task affect individuals' confidence in their ability to perform similar tasks (Bandura 1997). Vicarious experience, on the other hand, refers to indirect experience one gathers by watching others (parents, teachers, siblings and peers). The performance results of others observed by individuals can make a positive or negative impression on self-efficacy perceptions. Social persuasion refers to positive or negative feedback individuals receive from people close to them (family, teachers, and friends) with respect to their performance. Physiological and emotional states refer to well-being and moods. Individuals suffering high levels of anxiety and stress may, at times, feel inadequate to perform a given task. In short, the factors that determine self-efficacy perception levels are past experiences, the ability to make use of role-models, praise and persuasion and psychological states. In this context, the personal experience of individuals has positive and negative effects on their self-efficacy perceptions.

Self-efficacy can also be defined as individuals' perceptions of and belief in their ability to cope with various circumstances and to perform successfully in a given activity (Senemoğlu 2005). Teachers who does not deem themselves effective in their field cannot perform their duties as required, may avoid relevant activities and hence achieve a limited level of effectiveness in teaching activities. According to Gibson and Dembo (1984), teachers with higher levels of self-efficacy perception allocate more time to academic activities, try harder to teach and develop more effective learning environments. A large number of studies have found that self-efficacy reflects significant achievement levels in mathematics (Street et al. 2017; Pietsch et al. 2003; Pajares and Graham 1999; Wolters and Pintrich 1998; Pajares and Kranzler 1995; Pajares and Miller 1994; Randhawa et al. 1993; Schunk 1981).

Considering the fact that elementary school teachers in particular should have a basic level of efficacy in a number of fields (language, mathematics, science, social sciences, etc.), teachers lacking self-efficacy in their field cannot positively influence the achievement levels of their students. Teachers' perceptions of self-efficacy affect on their work in the classroom. Every attempt teachers make to contribute to student learning should involve the feeling of self-efficacy. Thus, well-trained elementary school teachers should have higher levels of mathematics self-efficacy perception. Smith (1996) states that a strong sense of efficacy supports teachers' efforts to face difficult challenges and to persist in the face of adversity. High levels of self-efficacy enable the use of cognitive strategies for the students, foster a conviction of successful completion of activities and make it possible to develop alternatives to solve problems (Stevens et al. 2004). Teachers' self-efficacy perception can also shape their belief in their ability to achieve the desired results with the students through their skills and competences (Tschannen-Moran and Hoy 2001).

The dimensions of self-efficacy required for elementary school teachers are related to mathematics as well. Mathematical concepts and the relationships between them are interrelated and gradual to a certain extent (Durmuş 2007). During the learning and teaching of mathematics, each concept is built on top of others. The failure to take into account the gradual nature of the relationships between the concepts, due to a lack of preliminary information or qualifications on part of individuals, may negatively affect learning. Teachers' backgrounds and self-efficacy perceptions in these dimensions significantly influence students' mathematics learning (Smith 1996; Işıksal and Çakıroğlu 2006). In-depth command of every mathematics topic the teacher is supposed to teach is necessary (NCTM 2000). Various studies have found that the teacher's competence is the single most important factor determining mathematics learning outcomes on part of the student (NCTM 2000; Romberg and Carpenter 1986). Teachers' ability to convey their content knowledge, provided that it is adequate, to learners, is yet another must for effective mathematics education (Gürbüz et al. 2013; Ball 1988, 1990; Davis and Simmt 2006; Hill et al. 2005; Shulman 1986, 1987; Tchoshanov 2011). Therefore, a review of the mathematics self-efficacy perceptions of pre-service elementary school teachers and an analysis of the relation they have with their mathematics backgrounds is crucial in terms of understanding specific parts of the educational process and its improvement.

This study was carried out to reveal the relationship between the mathematics backgrounds and mathematics self-efficacy perceptions of pre-service elementary school teachers. It investigated these research questions:

(1) What kinds of mathematics learning experiences are in the educational backgrounds of pre-service elementary school teachers?
(2) What are the mathematics self-efficacy perceptions of pre-service elementary school teachers?
(3) Is there a statistically significant relationship between pre-service elementary school teachers' year of study and their mathematics self-efficacy perceptions?
(4) Is there a statistically significant relationship between pre-service elementary school teachers' mathematics self-efficacy perceptions and their mathematics backgrounds?

12.3 Method

This study used the survey method to achieve its objective of determining the relationship between the mathematics backgrounds and mathematics self-efficacy perceptions of pre-service elementary school teachers. As applied by this study, the survey method is useful because it presents a clear picture of an existing state of affairs.

12.3.1 Participants

The participants were pre-service elementary school teachers being trained at a university in northeastern Turkey. The participants were selected using layered sampling, in order to come up with a number of students sufficient to represent each year (first, second, third and fourth) of study in the faculty. The sample included 209 pre-service teachers, 66 males and 143 females. Of the pre-service teachers, there were 44 freshmen, 47 sophomores, 59 juniors and 59 seniors.

12.3.2 Instruments

The data collection tools employed were the Mathematics Background Form and the Mathematics Self-Efficacy Scale. Both are included in Appendix. The following section will describe them.

12.3.2.1 The Mathematics Background Form

The Mathematics Background Form was used to determine the pre-service elementary school teachers' predispositions towards mathematics, the assistance they get when studying mathematics and any remarkable memories they have about mathematics. The form was also used to gain insight into their self-awareness regarding their qualifications in the field and their views about teaching mathematics. The form asked if the pre-service teachers received private tutoring for mathematics courses, or if they received assistance from another individual (e.g., teacher or another person competent in mathematics) when studying mathematics. The form was used to investigate the level of self-efficacy in mathematics courses as perceived by the pre-service teachers, and asked about their own perception of competence in terms of their mathematics content knowledge with questions about the stages of pre-service teachers' mathematics skills development (NCTM 2000) and their mathematics backgrounds. Many studies of students' affective characteristics regarding mathematics (Koca and Lee 1998; Mandacı-Şahin 2007; Hoffman and Spatariu 2008; Stevens et al. 2013) have used similar scales. For example, a study of students' opinions of, development in and attitudes towards mathematics (Mandacı-Şahin 2007) investigated the specific factors shaping the current state of affairs and explored students' mathematics backgrounds. The form was prepared, and the questions were finalized in consultation with two experts with doctorates in mathematics education.

12.3.2.2 The Mathematics Self-efficacy Scale

The Mathematics Self-Efficacy Scale was used to gain insight into the prospective teachers' self-efficacy perceptions and opinions about mathematics teaching. The scale was developed by Umay (2001) and contains 14 items. It is based on three factors: mathematics personality perception, behavioral awareness about mathematics and ability to translate mathematics into life skills. The Cronbach's alpha reliability factor of the scale is 0.82. The items on the scale are 8 positive and 6 negative statements and are scored on a scale of 1–5. The participants were asked to state the degree to which they agreed with the items by marking one of these options: always, most of the time, sometimes, rarely or never.

12.3.3 Analysis

The data gathered using the Mathematics Self-efficacy Scale were analyzed with SPSS-16 software. The changes in the grade point averages and mathematics self-efficacy perceptions of the pre-service teachers by year of study were analyzed using one way analysis of variance (ANOVA). The changes in their opinions about mathematics teaching, along with the changes in their perceptions of self-efficacy in mathematics teaching, their opinions about mathematics proficiency and the assistance they received when studying mathematics were also analyzed using one way ANOVA. The independent-samples t-test was used to determine the relationship between the courses they liked and did not like and their self-efficacy perceptions in mathematics. The threshold for statistical significance was $p < 0.05$.

The data gathered using the Mathematics Background Form were also studied using content analysis. Then, the opinions were categorized and presented by percentage and frequency. The responses to each question on the Mathematics Background Form were grouped according to compared and contrasted develop a coding scheme. For instance, responses to: "Have you had any unforgettable experience in mathematics courses to date? Could you please describe it?" were first coded as affirmative or not for the first question and then as positive or negative for the second. Thus, real life cases served as examples of the pre-service teachers' positive and negative experiences.

12.4 Results

This section presents the findings of the analysis of research data for the mathematics backgrounds of pre-service teachers and their self-efficacy levels in mathematics.

12.4.1 The Mathematics Backgrounds of the Pre-service Teachers

The first step to understanding the mathematics backgrounds of pre-service elementary school teachers was to review the findings from the analysis of the data gathered using the Mathematics Background Form. The following section presents the results of this analysis of data to determine the mathematics backgrounds of the pre-service teachers. First, their cumulative grade point averages were investigated. The percentages and frequency data for the cumulative grade point averages of the pre-service teachers by year of study are shown in Table 12.1.

Table 12.1 The cumulative grade point averages (cGPAs) of the pre-service teachers

cGPA range	Year				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
0.00–2.00	4 (9)	1 (3)	1 (2)	0 (0)	6 (3)
2.01–2.50	5 (11)	6 (13)	6 (10)	3 (5)	20 (10)
2.51–3.00	12 (27)	19 (40)	33 (56)	30 (51)	94 (45)
3.01–3.50	18 (41)	21 (45)	15 (22)	25 (42)	79 (38)
3.51–4.00	5 (11)	0 (0)	4 (7)	1 (2)	10 (5)
Total	44	47	59	59	209

The cGPAs of freshmen and sophomores clustered above 3.00/4.00. On the other hand, the averages of junior and senior year students were found to be, more often than not, in the 2.50–3.00 range. One can conclude that the students get higher grade point averages in the earlier stages of their undergraduate education.

The Mathematics Background Form was used to investigate the pre-service teachers' interest in mathematics courses. The analysis of the responses to: "Please list your favorite courses," are shown in Tables 12.2 and 12.3.

Table 12.2 shows that 67% of pre-service teachers chose mathematics among their favorite courses. The remaining 33% did not consider it a favorite course. Therefore, more than half of the pre-service teachers seemed to have favorable outlooks towards mathematics courses.

Table 12.3 shows that the vast majority (89%) of pre-service teachers did not choose mathematics as their least favorite course. Only a small minority (11%) did so. One can conclude that the overwhelming majority of pre-service teachers did not consider mathematics a dislikeable course. Indeed, as noted above, more than half of them considerer mathematics their favorite course.

Data for the question if pre-service teachers received help when studying mathematics is shown in Table 12.4. It shows that 51% responded that they did not, 22% rarely got help, and 28% needed frequent assistance.

The responses to the question about whether the prospective teachers had any memorable incidents—positive or negative—in their mathematics backgrounds are presented in Table 12.5. Of the participants, 62% had no memorable experience with mathematics, while 35% mentioned unforgettable memories they had in mathematics courses in the past, and 3% passed on this question. Only 20% of the

Table 12.2 The percentage of the pre-service teachers who chose mathematics as their favorite course

Favorite courses	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Includes math	25 (57)	32 (68)	42 (71)	41 (70)	140 (67)
Does not include math	19 (43)	15 (32)	17 (29)	18 (31)	69 (33)
Total	44	47	59	59	209

Table 12.3 The percentage of the pre-service teachers who chose mathematics as their least favorite course

Least favorite courses	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Math	8 (18)	4 (9)	5 (9)	6 (10)	23 (11)
Not math	36 (82)	43 (92)	54 (92)	53 (90)	186 (89)
Total	44	47	59	59	209

Table 12.4 Help needed by the pre-service teachers when studying mathematics

Help	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Frequently	12 (27)	11 (23)	15 (25)	20 (34)	58 (28)
Never	29 (66)	27 (57)	32 (54)	18 (31)	106 (51)
Rarely	3 (7)	9 (19)	12 (20)	21 (36)	45 (22)
Total	44	47	59	59	209

Table 12.5 Memorable incidents in the mathematics backgrounds of the pre-service teachers

Memorable incidents	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Yes	11 (25)	19 (40)	18 (31)	26 (44)	74 (35)
No	31 (71)	27 (57)	41 (70)	30 (51)	129 (62)
No comment	2 (5)	1 (2)	0 (0)	3 (5)	6 (3)
Total	44	47	59	59	209

Table 12.6 The pre-service teachers' assessment of their mathematics knowledge efficacy

Mathematics knowledge efficacy	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Sufficient	16 (36)	19 (40)	36 (61)	28 (48)	99 (47)
Insufficient	22 (50)	17 (36)	13 (22)	21 (36)	73 (35)
Moderately sufficient	6 (14)	11 (23)	10 (17)	10 (17)	37 (18)
Total	44	47	59	59	209

students who mentioned a memory of a mathematics course gave a detailed description. A review of these experiences found that half were positive experiences and half were negative. Here are a few of the positive responses: "I got perfect 100 in four consecutive exams in the first year of secondary school." "I solved problems on the blackboard for a full hour." "I was able to solve a problem better than the teacher." Here are some negative memories: "A substitute teacher for our elementary school teacher in third grade shouted at a friend of mine who failed to solve a problem." "I had a female teacher who told me that I was rotten and beyond any hope."

The pre-service teachers were asked if they had considered their mathematics knowledge efficacy sufficient. The findings are shown in Table 12.6. Of the pre-service teachers, 47% considered themselves sufficient, 18% moderately sufficient and 35% insufficient. Only a minority of pre-service teachers considered their mathematics knowledge efficacy insufficient.

Table 12.7 The pre-service teachers' opinions about teaching mathematics

The teaching of mathematics	Year of study				
	1st	2nd	3rd	4th	Total
	f (%)	f (%)	f (%)	f (%)	f (%)
Able to teach well	36 (82)	37 (79)	49 (83)	46 (78)	168 (80)
Try to teach well	7 (16)	6 (13)	6 (10)	10 (17)	29 (14)
Would not be able to teach well	1 (2)	4 (9)	4 (7)	3 (5)	12 (6)
Total	44	47	59	59	209

The pre-service teachers' opinions about teaching mathematics are shown in Table 12.7. Of them, 80% responded that they would be able to teach well, 14% said that they would try to teach well, and only 6% said that they would not be able to teach well. This indicates that the vast majority of the pre-service elementary school teachers believed that they will be able to teach mathematics well.

12.4.2 Data on the Pre-service Teachers' Perceptions of Self-efficacy in Mathematics

The responses provided by pre-service elementary school teachers for the Mathematics Self-efficacy Scale found that the vast majority (76%) deemed their self-efficacy to be high. Table 12.8 shows that the average score of the pre-service elementary school teachers' responses to the items on the Mathematics Self-efficacy Scale was 3.73. Their lowest mean score was 2.72 for the item: I can find my way with mathematical constructs and theorems and discover little details here and there, and their highest average score was 4.39 for the item: I find my confidence wavers when studying mathematics. These findings indicate that the pre-service elementary school teachers have a good self-efficacy perceptions in mathematics.

The results of the ANOVA test for the relationship between certain variables concerning the mathematics backgrounds of the pre-service teachers and their mathematics self-efficacy levels are shown in Table 12.9, which shows that there is no statistically significant difference between the self-efficacy perceptions of pre-service teachers in various years of the program ($F = 1.389$, $p = 0.247 > 0.05$). Likewise, no statistically significant difference was observed between the pre-service teachers' attitudes towards teaching mathematics with reference to year of study ($F = 0.276$, $p = 0.843 > 0.05$). As Table 12.9 shows, no statistically significant variance was observed in the pre-service teachers' perceptions of self-efficacy in mathematics and the existence of memorable incidents in mathematics classes ($F = 1.290$, $p = 0.278 > 0.05$).

The existence of any statistically significant variance between the pre-service teachers' perceived self-efficacy in mathematics and certain variables regarding their history with mathematics (their opinions about teaching mathematics,

Table 12.8 The pre-service teachers' perceptions of self-efficacy

Perceptions of self-efficacy	Year of study				
	1st	2nd	3rd	4th	Total
Frequency	44	47	59	59	209
Arithmetic mean	3.63	3.68	3.81	3.76	3.73

Table 12.9 ANOVA test results

		Sum of squares	df	Mean square	*F*	*p*
The pre-service teachers' mathematics self-efficacy perceptions with reference to their year of study	Between groups	1.029	3	0.343	1.389	0.247
	Within groups	50.625	205	0.247		
The pre-service teachers' views about teaching with reference to their year of study	Between groups	0.421	3	0.140	0.276	0.843
	Within groups	104.134	205	0.508		
The pre-service teachers' mathematics self-efficacy perceptions and the existence of a memorable experience they had in their own mathematics classes	Between groups	0.639	2	0.319	1.290	0.278
	Within groups	51.015	206	0.248		
The pre-service teachers' mathematics self-efficacy perceptions with reference to their opinions regarding the teaching of mathematics	Between groups	3.628	2	1.814	7.782	0.001
	Within groups	48.026	206	0.233		
The pre-service teachers' mathematics self-efficacy perceptions with reference to their opinions regarding their proficiency in mathematics	Between groups	14.040	2	7.020	38.445	0.000
	Within groups	37.614	206	0.183		
The pre-service teachers' mathematics self-efficacy perceptions with reference to their need for outside help when studying mathematics	Between groups	3.951	2	1.975	8.530	0.000
	Within groups	47.704	206	0.232		

mathematics proficiency, help received when studying mathematics) was investigated using the one way ANOVA test. As Table 12.9 shows, the analysis found a statistically significant ($F = 7.782$, $p = 0.001 < 0.05$) variance between self-efficacy in mathematics and pre-service teachers' opinions about teaching mathematics. The pre-service teachers who believed that they will be very good at teaching mathematics were found to express higher levels of self-efficacy in mathematics. There was another statistically significant variance between the pre-service teachers' perceived self-efficacy in mathematics and their opinions of their proficiency in mathematics ($F = 38.445$, $p = 0.000 < 0.05$). The opinions of the pre-service teachers who said that their proficiency in mathematics is sufficient were markedly different from the opinions of pre-service teachers who reported otherwise. The difference in favor of the pre-service teachers who consider their proficiency level sufficient was found using Tukey's HSD range test. A statistically significant variance was observed with reference to the pre-service teachers' self-efficacy perceptions and their need for help when studying mathematics ($F = 8.530$, $p = 0.000 < 0.05$). The opinions of the pre-service teachers who reported no need for help when studying mathematics significantly differed from those of the pre-service teachers who needed help with certain issues or most of the time. The difference favored the pre-service teachers who did not need help when studying mathematics.

Independent-samples t-testing was used to see if any statistically significant relationship exists between the self-efficacy perceptions of pre-service teachers and certain variables regarding their mathematics backgrounds (thinking of mathematics as their favorite course, thinking of mathematics as their least favorite course). A statistically significant ($t = 6.164$, $p = 0.000 < 0.05$) relationship was observed between the pre-service teachers' perceived self-efficacy in mathematics and their choice of mathematics as their favorite course. The relationship favors the pre-service teachers whose favorite course was mathematics. Another statistically significant relationship was observed between perceived self-efficacy in mathematics and choosing mathematics as their least favorite course ($t = -6.341$, $p = 0.000 < 0.05$). The difference was in favor of the pre-service teachers who reported mathematics as their least favorite course.

12.5 Conclusion

The study found that the mathematics self-efficacy perceptions of pre-service elementary school teachers improve somewhat as they progress in the program, but not at a statistically significant level. It is only natural that a significant change is not observed with reference to year in the program, given the fact that the pre-service elementary school teachers enrolled in the program exhibit high levels of mathematics self-efficacy perception in all years (first, second, third and fourth) of the program.

A high level of self-efficacy perception leads to individuals setting high goals for themselves, as well as to consistency in their decisions, ultimately shaping their cognitive processes and bringing about a higher level of motivation. The results of this study can be interpreted to signify a high level of motivation on part of the pre-service elementary school teachers, who exhibit a determined outlook in terms of the achievement of their objectives.

Pişkin and Durmuş (2010) also observed a high level of mathematics self-efficacy perception in their study with pre-service elementary school teachers. A robust level of mathematics self-efficacy perception on part of the pre-service teachers is a most welcome finding since it will positively affect their performance, their ability to resolve the problems they face as teachers and getting students to like mathematics. In a study on the self-efficacy perceptions of pre-service mathematics teachers, Işıksal and Çakıroğlu (2006) found that their mathematics self-efficacy perceptions varied from year to year. This finding is not consistent with those of this study. Apparently, the mathematics self-efficacy perceptions of pre-service mathematics teachers and pre-service elementary school teachers move on different trajectories with reference to year of study.

This study did not find a statistically significant relationship between pre-service elementary school teachers' perspectives on teaching mathematics and year of study. Even though no statistically significant difference was observed with reference to their year in the program, the pre-service elementary school teachers' views regarding the teaching of mathematics were found to be highest in the third year, indicating a perception of ability to teach well. This finding may relate to the mathematics teaching course offered in the third year of the program. Işıksal and Çakıroğlu's (2006) conclusions are also similar to these findings. They also found that the pre-service teachers' perceptions regarding the teaching of mathematics did not vary by the year of study in the program.

The study concluded that a statistically significant relationship existed between pre-service teachers' mathematics self-efficacy perceptions and their views regarding teaching mathematics as well as certain variables regarding their experiences with mathematics. A statistically significant variance was observed in the pre-service teachers' mathematics self-efficacy perceptions with reference to their favorite course, least favorite course, help needed when studying mathematics and proficiency in mathematics. Bandura (1986) noted that efficacy perception is shaped by individuals' past experiences. Therefore, certain variables concerning pre-service teachers' experience with mathematics naturally affect their self-efficacy perceptions. Ural (2015) mentions individual performance in previous mathematics assignments as the most significant factor in mathematics self-efficacy perception.

Yet another interesting result of the study lies in the finding that most of the pre-service elementary school teachers deem themselves proficient in mathematics. Only a small number of pre-service teachers reported an insufficient level of mathematics proficiency. Naturally, pre-service teachers reporting a sufficient level of proficiency believe in their competence to teach mathematics.

This study found that the pre-service teachers' past experiences with mathematics affected their mathematics self-efficacy perceptions and their competence to teach mathematics. Therefore, high levels of mathematics self efficacy perception can improve the effectiveness of training for pre-service elementary school teachers. The recommendations to be made in this context include testing efficacy in mathematics and a preliminary assessment of candidates for pre-service teacher training programs. Teacher training activities and programs are certainly crucial and require due attention to ensure the professionalism and competence of future generations (Lin and Hsu this volume).

Appendix

(This scale was applied originally in Turkish. Here, a translation is provided for information purposes only.)

Dear Prospective Teachers
This scale is developed within the framework of a scientific study. Your answers will be held in absolute confidence, and no personal assessment will be applied. The responses provided will be analyzed on a collective basis. Truthful and factual answers you will provide are crucial for the achievement of research objectives, as well as the quality of the study. In this sense, it is essential for the validity of the study, that answers provided for specified cases reflect the actual state of affairs.

Thank you for your time and participation.

Mathematics Background Form

Gender			
Year 1–4		Grade point average (GPA)	
Please state your favorite courses			
Please state the courses you dislike most			
Do you need help studying mathematics? What is the frequency of asking for help?			
Do you deem yourself qualified enough for mathematics courses?			
Do you deem your mathematics knowledge level adequate? If not, who do you think is responsible for this?			
Have you had any unforgettable experience in mathematics courses to date? Could you please describe it?			
Do you believe that you can teach mathematics well when doing your job?			

Mathematics Self-Efficacy Scale

Please mark the most applicable option for the following statements.

	Never	Rarely	Sometimes	Often	Always
1. I believe I am able to make effective use of mathematics in my daily life					
2. I think in mathematical terms when planning my day/time					
3. I believe mathematics is not the right occupation for me					
4. I deem myself competent in terms of problem solving in mathematics					
5. I can solve any mathematical problem if I work on it long enough					

(continued)

(continued)

	Never	Rarely	Sometimes	Often	Always
6. I feel like I am taking incorrect steps when solving problems					
7. I panic when I face unexpected circumstances while solving a problem					
8. I wander around mathematical constructs and theorems, and can come up with small discoveries					
9. I know how to proceed when I come across some new issue in mathematics					
10. I believe mathematics competence to match that of my colleagues is simply beyond me					
11. I deem the time spent with problem solving mostly as a waste					
12. I realize my self-confidence levels fall as I study mathematics					
13. I can easily help people around me with their questions on mathematics					
14. I can propose mathematical solutions to any problem in life					

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Chapter 13
Mathematics Teachers' Knowledge and Productive Disposition for Teaching: A Framework and Measure

Erik Jacobson, Fetiye Aydeniz, Mark Creager, Michael Daiga and Erol Uzan

Abstract Mathematics teacher education aims both to increase knowledge (cognitive constructs) and to instill productive disposition (affect-related constructs) for teaching mathematics. Prospective teachers' knowledge and productive disposition are theoretically intertwined and together make up Mathematical Proficiency for Teaching (MPT). Although both aspects of MPT represent simultaneous goals in university classes, most research has focused on one kind of outcome or the other, and those studies that address both often use separate, disconnected measures. In this chapter we discuss the MPT framework and describe a novel MPT survey that simultaneously measures pedagogical content knowledge (i.e., knowledge) and teaching self-efficacy and motivation beliefs (i.e., productive disposition) for teaching the topic of multidigit addition and subtraction. We describe our use of the survey measure to investigate how one methods (pedagogy) class contributed to elementary teachers' MPT for this topic. Our results from a cross-sectional study and a longitudinal follow-up show the survey measure is psychometrically well-behaved, measures substantially different constructs in spite of a narrow content focus, and characterizes strengths and limitations of the specific methods class in question. Implications for research and theory are discussed.

Keywords Pedagogical content knowledge · Teaching self-efficacy
Teacher education · Motivation · Productive disposition · Mathematical proficiency for teaching

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13.1 Introduction

Knowledge and productive disposition for teaching mathematics are theoretically intertwined to compose *Mathematical Proficiency for Teaching* (MPT; Jacobson 2013), a worthwhile and multifaceted goal for teacher education. *Productive disposition* in the MPT framework is defined as the set of affect-related constructs that have empirical relationships with students' mathematics learning (Jacobson and Kilpatrick 2015), and it includes emotions, attitudes, and beliefs (cf., Philipp 2007). Important findings have accrued from past teacher education research focused on single outcomes (e.g., pedagogical content knowledge for mathematics, Baumert et al. 2010; motivation beliefs for fractions, Newton 2009; mathematics self-efficacy, Güneş this volume). By contrast, the MPT framework defines effective teacher education holistically in relation to multiple outcomes, and looks toward synthetic results.

An important open question is how changes in one component of MPT (e.g., pedagogical content knowledge) are related to changes in other components (e.g., motivation beliefs). A related question concerns the role of instructional experience in MPT development (Jacobson 2017a). Similar questions have been raised from different perspectives. For example, Cobb et al. (1990) challenged the causal linearity implied by questions about whether teachers' beliefs or their practice changed first. They argued from a social-constructivist perspective that teachers' beliefs and teaching practices may be dialectically related. Change in beliefs can support change in practice, but change in practice is necessary for lasting change in beliefs. In our work, we are alert to non-linear relationships between the components of MPT.

A second set of important open questions involve the scope of the mathematical content involved in measures of teacher knowledge and productive disposition. These constructs can be defined for mathematics in general (e.g. mathematics anxiety; Wood 1988), for broad domains of mathematics (e.g., knowledge of algebra for teaching; McCrory et al. 2012), or more narrowly in terms of specific topics (e.g., motivation for fraction multiplication and division; Jacobson and Izsák 2015). The field has yet to systematically explore how relationships between knowledge and productive disposition constructs differ by domain (e.g., algebra vs. geometry), by topic (e.g., whole number multiplication vs. fraction multiplication), and when constructs differ in mathematical scope (e.g. elementary content knowledge for teaching and fraction anxiety). It is possible that the relationships between constructs may depend in important ways on the mathematical scope at which constructs are measured.

Questions about how teacher knowledge and disposition are related over time and questions about how these relationships differ by mathematical scope are significant because the answers would inform the design of mathematics teacher education that aims to increase MPT. If teachers experience different knowledge or productive disposition relationships for different topics or domains of mathematics, then teacher educators cannot assume that learning from activities focused on one topic or within one domain will transfer to other topics or domains. Existing measures of knowledge and productive disposition constructs that involve large content areas (e.g., all of elementary mathematics) offer limited opportunities to investigate how the interactions between changing knowledge and disposition differ by mathematical topics. Thus, new topic-specific measures are needed to address the questions about mathematical scope.

In this chapter, we discuss the MPT framework (Jacobson 2013; Jacobson and Kilpatrick 2015; Jacobson 2017a) in greater detail than it has been described in previous publications. We also report on two studies that used a novel MPT survey measure to explore how elementary Prospective Teachers (PTs) in a specific teacher preparation program developed components of MPT.

In this work, we pioneered a coordinated measurement strategy that focused on a narrow slice of mathematical content to more closely examine relationships between MPT components. Focusing on a narrow content area also made our research problem more tractable. As mathematics education researchers, we wanted to understand the role of a specific mathematics methods (pedagogy) class in MPT development, but we were concerned that if MPT developed differently for different mathematical topics, then the variety of interactions between knowledge and productive disposition across different topics would obscure the interactions we wanted to understand. By examining MPT more narrowly, we aimed to understand how the components of MPT developed together for the specific case of multidigit addition and subtraction in the context of a specific teacher preparation program. We return to this important methodological issue in the concluding discussion.

Our focus on multidigit arithmetic in the studies we report was strategic. A major goal for the elementary teacher preparation program we studied is that PTs become familiar with both student thinking and viable instructional routines for multidigit addition and subtraction. Existing research suggested that PTs struggle to understand and explain these mathematical ideas (e.g., Borko et al. 1992; Thanheiser 2009). In the teacher preparation program, this goal is largely addressed during the methods class which emphasizes the practice of mathematics teaching and that PTs take after completing the required mathematics classes. Thus, we examined what PTs know and believe about place value, standard algorithms, instructional representations, and student thinking in the context of the methods class.

13.2 Theoretical Framework

The Mathematical Proficiency for Teaching (MPT) framework is a conceptual framework that aims to synthesize all mathematics-related teacher knowledge and teacher affect constructs that are consequential for student mathematics learning (Jacobson 2013, 2017a; Jacobson and Kilpatrick 2015). This framework extends the work of Kilpatrick et al. (2001) who described mathematical proficiency for students as "a composite, comprehensive view of successful mathematics learning" (p. 116) with five intertwined strands: four strands related to knowledge and a fifth strand called productive disposition that included beliefs about mathematics. Kilpatrick et al. (2001) also argued that analogous components of teachers' knowledge and productive disposition were similarly intertwined; "just as mathematical proficiency itself involves interwoven strands, teaching for mathematical proficiency requires similarly interrelated components" (p. 380). Significantly, the MPT framework builds on the assumption from Kilpatrick et al. (2001) that these strands are interdependent; they interact and develop together.

The five strands of teaching for proficiency from Kilpatrick et al. (2001, p. 380) are listed below:

(1) conceptual understanding of the core knowledge required in the practice of teaching;
(2) fluency in carrying out basic instructional routines;
(3) strategic competence in planning effective instruction and solving problems that arise during instruction;
(4) adaptive reasoning in justifying and explaining one's instructional practices and in reflecting on those practices so as to improve them; and
(5) productive disposition toward mathematics, teaching, learning, and the improvement of practice.

In the MPT framework (Fig. 13.1), the five Kilpatrick et al. (2001) strands are categorized into the cognitive domain of *knowledge* (strand 1), the affect-related domain of *productive disposition* (strand 5), and the situated, socially-enacted domain of *instruction* (strands 2, 3, and 4). This categorization is useful for research in part because it emphasizes similarities and differences between the strands, and explains why strands in different domains require different methods of investigation in MPT research. The domains of knowledge and productive disposition comprise psychological constructs (i.e., presumed to be "in the mind" of the teacher), but the domain of instruction is necessarily enacted and therefore inescapably situated and social. Instruction involves not just the teacher but also the students and the mathematical ideas at stake (Cohen et al. 2003). Because the mathematics being taught and learned can be conceptualized at different ways (i.e., as mathematics in general, as a broad domain like algebra, or as a specific topic like fraction multiplication), the MPT framework attends explicitly to the mathematical scope of constructs.

The organization provided by the MPT framework is also useful because it suggests how MPT domains might be interrelated. We conjecture that knowledge and productive disposition support or constrain how a teacher enacts instruction, and that instruction shapes a teachers' knowledge and productive disposition. These relationships between domains are theoretical claims, and, as such, a major goal of research using the MPT framework is to reject or refine these relationships through empirical investigation.

The categories in the MPT framework, shown in Fig. 13.1, articulate the goals of mathematics teacher education and the theorized relationships suggest novel lines of research on how these goals might be achieved. Note that knowledge and

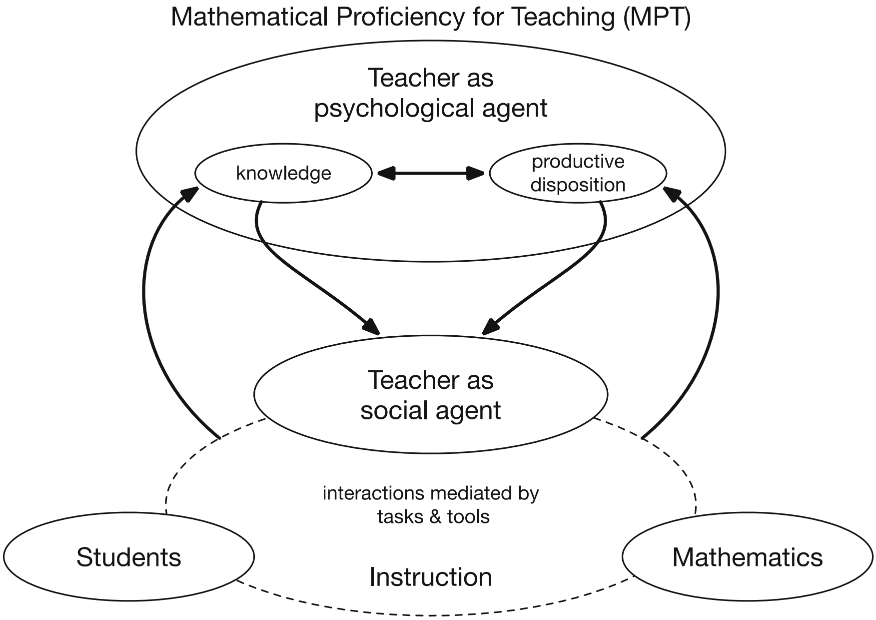


Fig. 13.1 The mathematical proficiency for teaching framework. Figure by Jacobson (2017b), available online at https://doi.org/10.6084/m9.figshare.4793422.v1 under a CC-BY4.0 license

productive disposition together are termed mathematical proficiency for teaching (MPT), and a major goal of research using this framework is to specify the relationship between MPT and instruction. The primary theoretical claim is that knowledge and disposition are interdependent and develop together in reciprocal relation with instruction. On the one hand, teachers' skill with particular teaching practices is shaped by the teacher's knowledge and productive disposition. But this relationship is also reciprocal: an individual's history of instructional activity supports and constrains what a teacher knows, believes, and feels both about her or himself as a teacher and about the nature of mathematics teaching and learning.

The MPT framework differs from perspectives that group teacher beliefs together with teacher knowledge. One such framework is the Knowledge Quartet (e.g., Rowland et al. 2005), which bundles "espoused beliefs about mathematics, including beliefs about why and how it is learnt" (p. 260) with propositional knowledge of mathematics within the category of *foundation knowledge*. By contrast, the MPT framework holds beliefs as conceptually distinct yet intertwined with knowledge. In this regard, MPT is similar to Fennema and Franke's (1992) view of teachers' knowledge as inherently interactive with teachers' beliefs.

Another aspect of the MPT framework emphasizes change. As Fig. 13.2 illustrates, teachers' knowledge, productive disposition, and instruction are not considered as fixed or immutable but rather as inherently dynamic characteristics best understood in explicit relation to the passage of time. Furthermore, we conjecture

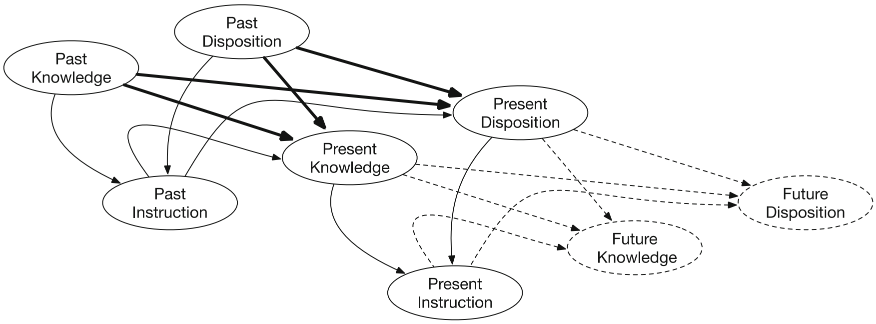


Fig. 13.2 The Mathematical Proficiency for Teaching framework emphasizes interactions between knowledge, disposition, and instruction that unfold over time. Figure by Jacobson (2017c), available online at https://doi.org/10.6084/m9.figshare.4793416.v2 under a CC-BY4.0 license

that the psychological domains of MPT mediate the relationship between past instruction and current instruction so there is no direct link. Our theoretical focus on change has methodological implications. Reciprocity and interaction describe processes, thus measures at multiple time points provide useful data to understand how MPT processes unfold over time.

The MPT framework helps reveal how little is currently known about the trajectories of change for the components of MPT both across different MPT domains and in regard to mathematical scope. A priori, it seems implausible to us that each component would follow the same trajectory even as PTs follow the same course of study. The descriptive question central to the present study is how components change over time, and the key comparative question is whether there are similar changes across components in relation to the same learning opportunities. In particular, we compared PTs before and after the methods class along several different components of MPT.

13.2.1 Selected MPT Components

We created a theory-based survey to measure MPT for multidigit addition and subtraction by designing an integrated survey to measure simultaneously the constructs of pedagogical content knowledge, teaching self-efficacy, and motivation. We chose these constructs because we wanted to judge the methods class relative to a diverse set of consequential outcomes. All three constructs have been linked empirically with mathematics learning in prior research yet differ from each other substantially in meaning. The survey itself is described in the methods section. In this section, we describe each construct we selected to include in the MPT survey.

First, we focused on teachers' Pedagogical Content Knowledge (PCK, Shulman 1986), which has become an important area of interest internationally (e.g., Cai 2005; Schmidt et al. 2011). PCK as Shulman described it has two components: (1) selecting and using instructional representations and (2) appraising students' conceptions and reasoning. Both of the components of PCK that Shulman identified were addressed by our survey measures for a narrow slice of mathematical content, multidigit addition and subtraction. Past survey measures of teachers' mathematics PCK were designed to measure large domains of mathematics (e.g., number and operation in the elementary grades; *Learning Mathematics for Teaching Project*, not dated). Many of these measures have been correlated with students' mathematics achievement. For example, Hill et al. (2005) found that teachers' mathematical knowledge for teaching—a related construct—was correlated with student achievement and had effects on the order of those associated with socio-economic status. Baumert et al. (2010)—using a different measure of PCK—also found a substantial positive effect for teacher PCK on student learning gains.

Second, we focused on teaching self-efficacy beliefs ("If I try hard, I can figure out how to teach it."; e.g., Tschannen-Moran and Hoy 2001). The more general construct of self-efficacy plays a central role in the social-cognitive theory of psychology (Bandura 1977, 1986). In this theory, self-efficacy is the key factor of human agency. Self-efficacy is predicted to influence "how much effort will be expended and how long it will be sustained in the face of obstacles and aversive experiences" (Bandura 1977, p. 191). For the purposes of our study, we were interested in teaching self-efficacy because there are strong links between teaching self-efficacy and students' achievement and motivation (see Tschannen-Moran et al. 1998 for a comprehensive review). Teaching self-efficacy is likely to form early in teachers' careers and remain relatively difficult to change later on, thus the development of teaching self-efficacy is a crucial goal for programs preparing new teachers.

Third, we focused on three belief constructs composing motivation (Newton 2009): anxiety, value, and self-concept of ability. These constructs, based on the expectancy-value theory of achievement motivation, explain persistence and performance using individuals' "beliefs about how well they will do on the activity and the extent to which they value the activity" (Wigfield and Eccles 2000, p. 68). Newton (2009) found that PTs developed motivation for fractions in a content class, providing evidence that teacher education can influence PTs' disposition. Jacobson and Izsák (2015) found that the relationship between middle grades teachers' knowledge and their instructional use of drawn representations depended entirely on motivation for using drawn representations of fraction multiplication and division. This finding suggests that students are unlikely to benefit if teacher motivation for effective instructional practices does not increase alongside the teacher knowledge required to support these practices. In short, the components of MPT used in this study encompassed several different achievable and consequential outcomes for elementary mathematics teacher education.

13.3 Methods

13.3.1 Context

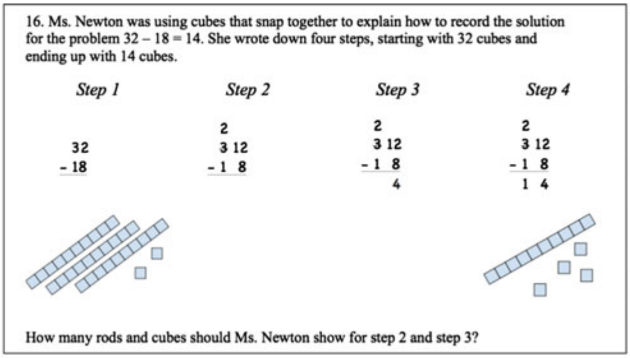
The studies we report in this chapter focus on a specific methods class within a single teacher preparation program. This choice is aligned with our immediate purpose of developing measures that can be used by teacher educators to improve their own practice rather than our long-term purpose of finding general patterns to support broad claims about how teachers learn. Thus, we begin with a short description of some of the features of the teacher preparation program and the methods class that made them well suited to our investigation of MPT for the mathematical topic of multidigit addition and subtraction. Based on prior experience we expected PTs' understanding of this topic to be relatively stronger than their understanding of other mathematical topics covered in the methods class. We wanted to pick a mathematical topic for which a strong mathematical knowledge supported their ability to learn PCK.

PTs enrolled in the methods class had already completed two mathematics classes for elementary teachers: a class focused on number and operation and a class focused on geometry. The methods class used mathematical content areas from the elementary school curriculum as sites to study mathematics learning and teaching. The focal content areas include counting, place value, operations with multidigit numbers, and operations with fractions and decimals. As an example of studying learning, PTs in the methods class studied a learning progression for counting (Clements and Sarama 2009) and completed assignments to identify different kinds of counters based on video clips or written student work. As an example of studying teaching, PTs practiced specific questioning strategies to elicit student thinking and support classroom mathematics discourse (e.g., Jacobs and Ambrose 2008). The goals of the class include greater PCK to support effective teaching practices (e.g., using representations) and increased teaching self-efficacy and motivation to enact these practices.

About 4 weeks of the 16-week class focused on multidigit arithmetic, with two weeks focused on addition and subtraction. The key foci of this section of the class for studying learning included additive problems structures, place value and regrouping, non-standard student strategies, and standard algorithms. The key foci for studying teaching included the instructional use of representations such as number lines and base ten blocks, both to record non-standard student strategies and to explain standard algorithms. Figure 13.3 shows two survey items that were similar to tasks PTs worked on in the methods class during the unit on multidigit addition and subtraction.

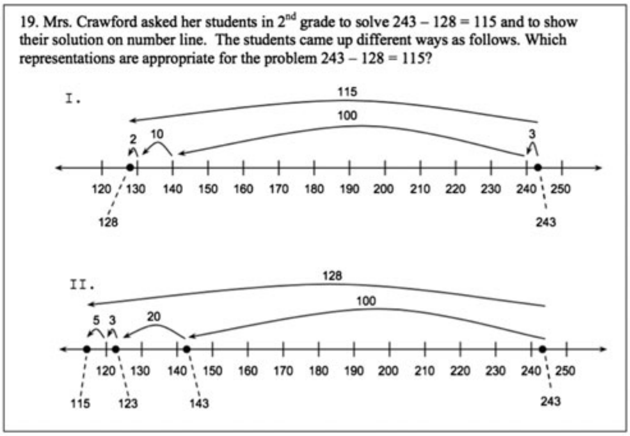
Some readers may wonder how the university methods class involves *instruction*, the conjectured context for change in knowledge and productive disposition in the MPT framework. PTs enrolled in the methods class were also enrolled in a field

(a)


16. Ms. Newton was using cubes that snap together to explain how to record the solution for the problem 32 – 18 = 14. She wrote down four steps, starting with 32 cubes and ending up with 14 cubes.

How many rods and cubes should Ms. Newton show for step 2 and step 3?

(b)


19. Mrs. Crawford asked her students in 2nd grade to solve 243 – 128 = 115 and to show their solution on number line. The students came up different ways as follows. Which representations are appropriate for the problem 243 – 128 = 115?

(c)

Please CIRCLE ONE OPTION to rate your agreement with the following statements:

	Strongly Disagree		Disagree		Agree		Strongly Agree
I am good at answering questions like this one.	1	2	3	4	5	6	7
I often feel nervous when I try to answer questions like this one.	1	2	3	4	5	6	7
Elementary teachers should know how to answer this question.	1	2	3	4	5	6	7
If I try hard, I can usually figure out questions like this one.	1	2	3	4	5	6	7

Fig. 13.3 Sample PCK items (**a**, **b**) from the MPT survey measure that illustrate the content area of multidigit addition and subtraction. The motivation and teaching self-efficacy survey questions (**c**) assessed self-concept of ability, anxiety, value, and teaching self-efficacy, and they used PCK questions to describe the mathematical scope of the construct. Figure by Jacobson (2017d), available online at https://doi.org/10.6084/m9.figshare.4793437.v1 under a CC-BY4.0 license

experience that focused on mathematics for 6 weeks of the semester. During this period, groups of 2 or 3 PTs taught or observed weekly lessons and interviewed students about the content they were teaching. Although PTs were working with different grade levels, most lessons were focused on number and operation. Thus, many but not all PTs were able to practice what they had learned about multidigit addition and subtraction in the university methods class during their concurrent field experience. For these reasons, the methods class provided opportunities for PTs to engage in approximations of practice (Grossman et al. 2009; Grossman 2011). We conjecture that these experiences along with rehearsals in the methods class led to the change in MPT that we observed.

13.3.2 Measures

We developed the PCK measure in a three-step process of drafting and revising items, conducting item-response interviews, and expert review. After an initial review of literature, each researcher developed several PCK items to examine the two components of PCK specified by Shulman: (1) selecting and using instructional representations (Fig. 13.3a) and (2) appraising students' conceptions and reasoning (Fig. 13.3b). Each item was discussed and revised several times. Next, 60-min, videotaped item-response interviews were conducted with 15 PTs, who were asked to answer each item and explain why they selected their answer. The research group reviewed the interview data and made revisions to problematic items. Then, six faculty members and two graduate students who are elementary mathematics teacher educators at institutions in four different US states reviewed the survey for content validity; they found all items were appropriate.

Most productive disposition constructs are measured by survey questions that describe the mathematical scope in a short phrase. For example, a question measuring mathematics self-concept of ability is "How good at *math* are you?" (emphasis added; Wigfield and Meece 1988). To adapt domain-specific measures to be topic-specific, Newton (2009) changed the wording to "How good at *fractions* are you?" (p. 93). Our innovation in measure design was to use PCK items to describe the mathematical scope.

The survey paired each multiple choice PCK item with four 7-point rating questions (1: Strongly disagree to 7: Strongly agree; see Fig. 13.3c). Each PCK item was followed by the same 4 questions: three measuring motivation beliefs (anxiety, value, and self-concept of ability), and the fourth measuring teaching self-efficacy. Thus, the measures of motivation and teaching self-efficacy were aligned item-by-item with the measure of knowledge. We took low scores on anxiety and high scores on the other outcomes to indicate greater MPT.

13.3.3 Study Design

In Study 1, we invited all PTs who had taken a content or methods class at the university in the last two years to participate and 119 completed the survey (21% response rate); of these, 54 had taken the methods class and 65 had not. We used item response theory to assess the 28-item PCK instrument, and found that the instrument displayed no evidence of item or person misfit and had acceptable reliability ($\alpha_{knowledge} = 0.77$). The teaching self-efficacy scale and the three motivation scales were highly reliable ($\alpha_{self\text{-}efficacy} = 0.95$, $\alpha_{anxiety} = 0.98$, $\alpha_{self\text{-}concept} = 0.94$, and $\alpha_{value} = 0.96$). We used bivariate correlations to compare the constructs, and MANCOVA to compare PTs who had taken the methods class with the rest of the PTs in the sample.

In Study 2, we selected 40 PTs with a range of mathematical ability (as assessed by their instructor) from those enrolled in the methods class, and 33 of them completed surveys at the beginning and end of the semester (83% response rate). We used paired *t*-tests so that each participant functioned as her own control, and we analyzed the differences in pre- and post-scores for PCK, teaching self-efficacy, and the three motivation constructs. Achieved power for this study design with 33 participants was 0.8 for detecting a difference of 0.5 standard deviations.

13.4 Results

All of the measures we used were focused narrowly on the topic of multidigit addition and subtraction. A first step was investigating whether the measures plausibly reflected different constructs in spite of their common focus. The low-to-moderate bivariate correlations in Table 13.1 reveal that the constructs measured by the MPT survey were substantially different from each other and reflect a range of PT outcomes. This structure of relationships was an important validation of our theoretical perspective that the constructs of MPT are inter-related yet distinct.

Table 13.1 Correlations between measures (*n* ranges from 104 to 119 because of missing values)

	Motivation			Teaching
	Anxiety[a]	Value	Self-concept	Self-efficacy
Knowledge	−0.20*	0.29**	0.14	0.16
Anxiety[a]		−0.12	−0.43***	−0.29**
Value			0.33***	0.43***
Self-concept				0.64***

[a]Anxiety (a negative attribute) was reverse scored relative to other MPT constructs
*$p < 0.05$, **$p < 0.01$, ***$p < 0.001$

Teaching self-concept and teaching self-efficacy beliefs were the most closely related, but even these were only correlated at 0.64, implying that these variables shared about 41% of their variance. As expected, all correlations with anxiety (a negative attribute) were negative. Three correlations were not statistically significant: anxiety and value, knowledge and self-concept, and knowledge and teaching self-efficacy. It was reasonable to us that anxiety and value were not strongly related because value—the belief that teachers in general should be able to do something—is rather distinct from one's feelings of anxiety about doing something oneself. We were more surprised that knowledge was not strongly related to self-concept of ability or teaching self-efficacy. Perhaps, as others have argued (e.g., Hill 2007), teachers are not very good judges of their own knowledge for teaching.

13.4.1 Study 1: Those Who Had Taken the Methods Class Differed in MPT from Those Who Had Not

We compared PTs who had taken methods with those who had not by examining means for each group. We noticed that PTs who had taken the methods class had more knowledge and less anxiety for teaching multidigit addition and subtraction, but seemed to have similar means for the other outcomes. We used MANCOVA to test whether the MPT differences between these groups were statistically significant. Without covariates, the two groups did not have a statistically significant difference (Wilks = 0.91, approx. $F = 1.77$, $df = 5$, $p = 0.12$). However, after adding two covariates (self-reported achievement in high school and self-reported achievement in university), there was a statistically significant difference between groups (Wilks = 0.88, approx. $F = 2.48$, $df = 5$, $p = 0.03$). Covariates account for some of the variability in response, and thus allow for more accurate comparison of groups that differ on covariates.

The statistically significant result of differences between the two groups (i.e., those who had taken the methods class and those who had not) revealed that taking the methods class was related to more knowledge and less anxiety. However, the difference between the two groups was only evident once self-reported achievement was taken into account. The findings from Study 1 emphasize the possibility that PTs may differ from year to year in prior academic preparation. A different study design was needed to investigate MPT change in light of this possibility.

13.4.2 Study 2: Methods Class Improved Some but not All Facets of MPT

Because background characteristics are an important part of the story from Study 1, we conducted a follow-up study (Study 2) with another group of students.

Table 13.2 Comparison of measures before and after the methods class

	Before methods class		After methods class		Difference
	M	*SD*	*M*	*SD*	t (df = 32)
Knowledge	−0.32	0.76	0.07	0.65	2.70**
Anxiety	3.70	1.32	3.58	1.50	−0.43
Value	6.17	0.69	6.18	0.77	0.08
Self-concept of ability	4.97	0.94	5.27	1.08	1.90*
Self-efficacy	4.93	0.83	5.33	0.76	2.91***

*$p = 0.066$, **$p < 0.01$, ***$p < 0.001$

By surveying the same PTs before and after the methods class, each PT served as her or his own comparison. This design removed the possibility that differences in background could account for the differences in observed outcomes.

In Table 13.2, we report the pre- and post-class survey scores on each MPT construct. Paired *t*-tests revealed statistically significant increases in knowledge and in teaching self-efficacy beliefs. Comparing motivation beliefs did not reveal a consistent or strong pattern of change. Anxiety decreased and self-concept of ability increased as we had hoped, but these differences were not statistically significant at the 0.05 level. The notion of 'value' showed little change, perhaps because pre-class survey ratings were already very high on the 7-point scale. Overall, our results suggest two things: during the methods class PTs developed knowledge and teaching self-efficacy beliefs, but they did not develop motivation beliefs.

13.5 Discussion and Conclusion

In Study 2 there was a statistically significant increase in PCK among the same students compared at the beginning and end of the methods class. Thus, we confirmed our finding in Study 1 that PCK for multidigit addition and subtraction was higher for PTs who had taken the methods class. Similarly, in Study 1 we observed less anxiety for teaching the topic among PTs who had taken methods and in Study 2 we noted a decrease in anxiety from the beginning to the end of the methods class but it was not statistically significant. Although there was no evidence that the methods class influenced teaching self-efficacy beliefs in Study 1, we did find evidence for this relationship in Study 2: teaching self-efficacy increased significantly from the beginning to the end of the methods class. The methods class was intended to increase all of the MPT outcomes we measured, but these findings suggest that the methods class improved some of the MPT outcomes (notably PCK) but had less influence on others, such as motivation beliefs. The findings provided useful feedback for improving the course. The challenge for the ongoing redesign of the class is to add or modify activities in order to improve motivation without abandoning the aspects that were important for increasing knowledge. In this way, our survey results provided a useful characterization of a specific methods class at one university relative to its goals.

The empirical findings of these studies coupled with prior theoretical work on PCK and teaching self-efficacy provide justification for the pathway from instruction to knowledge and disposition emphasized by the MPT framework. We attribute the increase in PCK to the focus in the methods class on representations and student thinking for the target content area of multidigit addition and subtraction. Similarly, the large increase in teaching self-efficacy beliefs was encouraging because teachers with higher teaching self-efficacy are often more effective (Tschannen-Moran et al. 1998; Tschannen-Moran and Hoy 2001). The rise in teaching self-efficacy was plausible to us because PTs designed and taught lessons in peer groups during the field experience that accompanied the methods class. Experiences in which individuals enact a particular activity and receive feedback (either from others or by reflecting on the experience themselves) are classified as mastery experiences; such experiences can be a source of change in self-efficacy beliefs about the activity (Bandura 1986; Usher and Pajares 2008).

Based on linked measures that focus on a narrow range of mathematical content, the empirical results of the two studies we report have implications for refining the MPT theoretical framework. Chief among these is the conclusion that some components of MPT may develop without simultaneous change in others. In particular, we found that increased PCK for multidigit addition and subtraction does not guarantee increased motivation for teaching the same topic. This relationship is likely topic-specific; existing broad measures may not have been sensitive to it. Developing MPT measures for other topics will help us understand how common it is that knowledge develops without motivation.

We argue that all components of MPT may be tied directly to the mathematical topic being taught. For example, a teacher's PCK, teaching self-efficacy, and motivation may be different for multidigit subtraction than for quadrilateral geometry. The theoretical support for this empirical possibility is clearest for teaching self-efficacy. Bandura (1986) argued that self-efficacy beliefs are most relevant in the specific situation or activity for which the beliefs are held. If teachers think of different topics in mathematics as separate rather than connected, their knowledge and beliefs for one topic such as multidigit subtraction may not match their knowledge and beliefs for other topics such as geometry. Although the pattern of change we observed for multidigit addition and subtraction may not hold for other mathematical topics in the curriculum, our work pioneers means of distinguishing between knowledge and beliefs with respect to specific mathematical topics. We hope this work will lead to improved mathematics education.

Gains in teacher knowledge are an important outcome of teacher education but there is evidence that increased knowledge is not sufficient without increased motivation. Jacobson and Izsák (2015) analyzed teachers' knowledge and motivation for using drawn models of fractions and compared outcomes of these measures with the same teachers' self-reported use of drawn models for fraction instruction. They found that the strong, direct relationship between knowledge and instructional use disappeared when motivation was taken into account. In other words, knowledge may shape instruction only to the extent that teachers are also motivated. Their work raises a critical question about how teachers learn: Is it possible for teacher education to be successful at increasing teachers' knowledge yet even so have limited impact on teachers' motivation? Our finding of change in knowledge without simultaneous change in motivation suggests this is true for the particular topic and in the specific methods class we studied. Although the same relationship may not hold for other topics or in other methods classes, the implication for both mathematics teacher education research and practice is clear: attention to multiple outcome measures is important.

Future research that will focus on other topics and on classes in other teacher education programs may be able to shed light on the generality of the patterns we found, and qualitative research underway with the 33 students in Study 2 may help reveal why the learning opportunities in the particular methods class we studied supported increased knowledge and self-efficacy beliefs but not motivation. Attending to a diverse set of outcomes provided a more comprehensive account of the strengths and limitations of the methods class and raised several questions that merit further investigation. Most importantly, the findings we report suggest it is misguided to assume that changes in knowledge guarantee changes in disposition, even when these constructs are measured relative to the same mathematical topic.

We hope that future research on prospective teacher education will attend to multiple simultaneous outcomes, including knowledge and disposition. Several potentially important constructs are presently being explored, such as PTs' perceptions of preparation for teaching diverse learners (Çelik et al. this volume). As the field expands its attention to novel constructs, care should be taken to coordinate these new foci with teacher education outcomes that have already been studied. The MPT framework suggests several questions to be explored. First, to what extent are the changes in knowledge and disposition topic-specific or more general? We used a narrow measure in one content area because we were concerned that more general measures would not be sufficiently sensitive. Future work might compare trends in PT knowledge and disposition for two separate mathematics topics to determine how change in teacher knowledge and dispositions are connected to the underlying mathematical topics. Second, how do experiences teaching change PTs' knowledge and beliefs? Jacobson (2017a) used cross-sectional survey data from the Teacher Education Development Survey in Mathematics to show similar relationships between the kinds of field experience PTs experienced during teacher preparation and both their mathematics content knowledge and beliefs about mathematics teaching and learning at the end to teacher preparation. Future work might examine such change using observation and interview methods to describe the underlying processes of change.

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Chapter 14
Prospective Mathematics Teachers' Opinions About Their Opportunities for Learning How to Teach to a Diverse Group of Students

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Abstract This study aimed to analyze prospective elementary mathematics teachers' opinions about how often teacher training programs provide opportunities for learning how to teach to diverse groups of students. The study employed survey method. The sample used consists of 1386 prospective elementary mathematics teachers in their last year of training at 21 state universities in twelve regions of Turkey. One-Way ANOVA test was applied to reveal how prospective teachers' responses about the opportunities they had for learning how to teach to diverse students vary among regions with different levels of development. The study found that the prospective teachers in Turkey 'sometimes' get the opportunity to learn about teaching to diverse groups of students. ANOVA analysis revealed that the prospective teachers' opinions about learning to teach to diverse groups of students vary significantly between different regions of Turkey. In other words, the universities in Turkey cannot be characterized as homogenous in terms of learning opportunities for teaching to a

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diverse group of students. Hence, more diligence and detailed analysis is called for when interpreting the results of comparative international studies on education, where countries are often construed as homogeneous wholes.

Keywords Prospective mathematics teachers · Learning opportunities Diverse students · Knowledge for teaching

14.1 Introduction

Increasingly multicultural societies of the 21st Century naturally result to some major changes on the education front. Today, the level of diversity and the needs of the individuals who receive formal education, including higher education, are more emphasized (Cabello and Burstein 1995; Moore and Hansen 2011). Today's classrooms require teachers to educate students who are different in terms of their abilities, language, culture and more (Richards et al. 2007). The results of international comparative studies to assess achievement levels of the students (e.g. PISA 2003 and PISA 2006) revealed that the factors such as student background, socio-economic context and migration status exhibit high levels of correlation with student performance (OECD 2010). In this context, successful education programs should address diversity as a source of potential growth rather than an obstacle to students' performance (OECD 2010). The capabilities and flexibility levels of teachers, which are functions of their knowledge of and beliefs about teaching, play major parts in doing so. Jacobson et al. (this volume) claim that knowledge for and beliefs about teaching are theoretically intertwined to compose proficiency for teaching.

The teacher's knowledge for teaching, which is one of the basic components of the proficiency for teaching, in essence, includes content knowledge as well as pedagogical content knowledge (Ball et al. 2008; Fennema and Franke 1992; Shulman 1986). It is related to how the content is taught and entails an awareness of the students' existing level of knowledge on the subject, insight into the concepts they have difficulty in understanding, and the identification and utilization of the strategies, methods and techniques required to overcome such difficulties (Shulman 1986). A glance at this definition reveals an implicit reference to the design and application of the learning-teaching process through insight into the students (Shulman 1986). Thus, the ability to recognize the strengths, weaknesses, limitations, and special needs of the learners, whose backgrounds differ in various aspects, and who have special education needs, is also an important element of the knowledge of teachers. In other words, teachers must create a classroom culture that supports their students from diverse backgrounds as well as for all other students, and provide them with the best opportunities to facilitate learning. At this junction, a definition of diversity, the key concept of this chapter, is called for.

In the widest sense, *diversity* refers to differences among groups of people and individuals, in terms of ethnicity, socioeconomic status, gender, uncommon traits, abilities, language, religion etc. (Moore and Hansen 2011). In an ever globalizing world, the availability of getting education and living in another country as an option, increasing numbers of immigrants in the developed world in particular, running away from wars and poverty, as well as the increasing incidence of diagnosed learning or physical disabilities, are among the fundamental reasons making today's classrooms increasingly diverse (Moore and Hansen 2011). Increasing diversity in the classroom, as a trend, is taking place against the background of changing expectations from teachers. In such a classroom, the teacher would be expected to be aware of her own cultural identity and prejudices, be inclined to learn about the worldviews of different cultures, develop culture-sensitive teaching methods and provide equality of opportunity in education for all individuals who may have different characteristics and needs, and finally be able to engage in planning of the contents and process with reference to the diversity of students (Banks 2004; Başbay and Bektaş 2009; Demir 2012; Gay 2002; Richards et al. 2007; Zeichner 1992). In other words, teachers have a responsibility to all their students, in terms of providing equal opportunities to achieve to the best of the students' abilities (Richards et al. 2007). In this context, some scholars (Cole 2008; Saravia-Shore 2008; Villegas and Lucas 2002) describe general teaching strategies which could be effective in educating diverse groups of students. These strategies include, among others, the following: (i) demonstrating high expectations for diverse groups of students, (ii) making optimal use of students' backgrounds to enhance learning, (iii) creating culturally compatible learning environments, (iv) using a multicultural teaching approach, cooperative learning strategies, and alternative assessments, (v) encouraging interdisciplinary teaching, and (vi) adopting constructivist views on learning. Designing a learning-teaching environment which does not leave out any students (regardless of diversity) will be a product of the teacher's teaching knowledge and skills (Başbay 2014). Therefore, pre-service and in-service teacher training programs should prepare teachers for increasing diversity in their classrooms.

Current studies on teaching to diverse groups of students reveal that the majority of teachers are not particularly successful in building learning environments for diversity (Gay 2002). In conjunction with this finding, Teaching And Learning International Survey (TALIS), the first OECD-backed international survey of teachers from 23 countries, shows that over 80% of teachers reported some level of need for professional development to improve their teaching in a multicultural setting (OECD 2010). Also, one in seven teachers reported having a high level of need for professional development of this kind (OECD 2010). It is clear that there is a need to improve teachers' level of professional development for teaching to a diverse group of students. Training teachers in that perspective requires multi-faceted efforts due to the necessity of change in both their teaching practices and their behavior as a product of their beliefs.

One would also expect the curriculum to support the teachers for successful teaching to diverse groups of students. But, often, conventional curriculum is incapable of registering the diversity in the audience, and responding accordingly. For example, Cirik (2014) investigated the relations between the objectives of

Turkish primary school (grades 1–8) curricula and multiculturalism, and found connections between the objectives of curricula and multiculturalism to be quite low. This point is crucial since, where the curriculum falls short in addressing the needs of all students, teachers must instead provide a bridge and support their students from different backgrounds (Richards et al. 2007). Thus, teachers must be prepared to identify the strengths, weaknesses, and special needs of such a diverse group of students, and be aware of potential shortcomings of the curriculum in addressing such distinct requirements. Furthermore, teachers must learn to use effective teaching strategies in meeting the needs of diverse groups of students (Cole 2008; Moore and Hansen 2011).

In this context, teacher training programs play an important role in the accumulation of teaching knowledge for the group of diverse students. According to many professors engaged in such programs, the opportunity to learn about teaching to diverse students is a crucial component of any teacher training program (Tatto et al. 2012). Hermans (2002) notes preparing prospective teachers to multicultural classroom environments among the leading requirements of teacher-training programs. It is crucial for the teachers to be able to develop appropriate learning environments to reach out to potentially diverse groups of students in the class, and to have a perspective of personal differences as sources to enrich learning environments, rather than as sources of problems. In the US, the National Council for Accreditation of Teacher Education (NCATE) notes experience in teaching to diverse student populations as one of the six fundamental standards applicable to teacher training (NCATE 2008). In this light, Rao (2005) emphasizes the need for teacher training programs offering courses on developing multicultural classroom environments and providing prospective teachers the opportunity to work with diversity students. Therefore, the prospective teachers' opinions about the existence of learning opportunities for effective planning of teaching to diverse groups of students are among the crucial indicators of the success of teacher training programs, hence a natural object of research interest.

A glance at the literature reveals an emphasis on the increasing diversity in societies all over the globe, and a focus on teacher competences in multi-cultural educational environments. Yet, the actual number of studies investigating this subject is small (Demir 2012; Grant et al. 2004; Parker-Jenkins et al. 2004). The systematic review by Parker-Jenkins et al. (2004) shows the absence of a large body of rigorous empirical research on teacher training with reference to diversity. Indeed, the studies that do exist are general commentaries rather than analyses of specific areas of content (e.g. mathematics, sciences, history). Such a state of affairs, in turn, reveals the need to come up with the existing picture of each area of content, as well as a list of the concrete needs in a given area. Against this background, the present study tries to shed light on the field of mathematics, in relation to teaching to diverse groups of students, in the context of Turkey which is often considered a multi-cultural country given also the large number of immigrants it received in recent years. From this perspective, Turkey serves as an example involving diverse groups of students.

One should also note the importance of taking precautions to identify and solve existing problems regarding teaching in multicultural environments at a fundamental level. Therefore, starting on this quest with the teacher training programs in which the knowledge of the teacher is formally structured is crucial. Also, Fennell (this volume) suggests that research on the quality of teacher training programs with reference to specific aspects of the processes and objectives of the programs, in national and international scale, should be accorded priority.

For all these reasons, this study aims to analyze prospective elementary mathematics teachers' opinions in Turkey about how often the teacher training program provides them with opportunities for learning how to teach to diverse groups of students.

14.1.1 A Review of the Contents of Elementary Mathematics Teacher Training Programs in Turkey, from the Perspective of Teaching to a Diverse Group of Students

In Turkey, teacher training is provided by faculties of education at the university level. In this context, elementary mathematics teacher training program is a four-year one, leading to an award of a bachelor's degree. The teacher training programs in Turkey were revised substantially in 1997, as part of an initiative introduced by the Higher Education Council of Turkey (HECT), to reflect the contemporary perspectives on teacher training. In the process, a framework program was published to impose certain standards in terms of the courses offered, contents of the courses, credits etc., between various faculties of education. The framework program was revised once again in the period 2007–2010, and devotes 50% of the time spent towards the degree to courses on content and content knowledge, 30% to courses on the teaching profession itself, and 20% to liberal education (HECT 2007). Furthermore, the faculties were given liberty to offer elective courses up to approximately 25% of the overall credit counts, providing a level of flexibility among the programs (HECT 2007). The framework program stipulates a 'Special Education' course under the teaching profession courses category, to be offered in the fourth year of the elementary mathematics teacher training program. An in-depth analysis of the contents of that course reveals a focus on the characteristics of children with mental, hearing, visual, or physical disabilities, who suffer from linguistic or communication disorders, special learning disabilities, attention deficit and hyperactivity, or who are autistic or gifted. It then proceeds to a discussion of the principles concerning their education (HECT 2015). Yet, the 'Special Education' course does not, in and of itself, address the characteristics or education of a group of children diverse in terms of their gender, cultural or socio-economic characteristics. Furthermore, the course focuses on general education, rather than the mathematics education.

14.2 Methodology

14.2.1 Participants

The sample of the study consisted of 1386 prospective elementary mathematics teachers in their last year of training, from 21 state universities in twelve different regions in Turkey. 975 (70.35%) of the participants are female and 404 (29.15%) are male. 7 participants (0.50%) did not state their gender. The universities included in the analysis were selected with reference to the Nomenclature of Territorial Units for Statistics (NUTS) level 1. NUTS is a geocode standard for referencing the subdivisions of countries for statistical purposes. Population, the cultural structure, and the development status of the regions are among the criteria taken into account in determining NUTS levels (Taş 2006; TurkStat 2015). In this sense, one can argue that, as a classification system, NUTS reflects the socio-economic structure of Turkey. NUTS level 1 stipulates twelve regions (TR1, TR2, …, TR9, TRA, TRB, TRC) with different socio-economic backgrounds. The scheme is expected to reflect a more or less accurate picture of the country. Table 14.1 shows each region (with codes) in NUTS level 1, as well as the number of universities and students (prospective teachers) included in the sampling from these regions.

Table 14.1 Distribution of prospective mathematics teachers by region

NUTS-level 1 (12 regions)	Codes	Number of universities	Number of students	Gender (females/males)
Istanbul Region	TR1	1	38	28/9
West Marmara Region	TR2	1	99	70/29
Aegean Region	TR3	3	189	136/50
East Marmara Region	TR4	1	38	34/4
West Anatolia Region	TR5	2	62	56/6
Mediterranean Region	TR6	2	105	72/33
Central Anatolia Region	TR7	3	219	161/58
West Black Sea Region	TR8	1	63	51/12
East Black Sea Region	TR9	2	260	194/65
Northeast Anatolia Region	TRA	2	213	137/75
Central East Anatolia Region	TRB	1	44	16/28
Southeast Anatolia Region	TRC	2	101	57/43
Total		21	1431[a]	1012[a]/412[a]

[a]Total number of participants before data reduction

14.2.2 Instruments and Data Collection

This study was conducted using the survey method, and study focuses on prospective teachers' opportunities for learning how to teach mathematics to a diverse group of students. Specifically, the *Teaching for Diversity* instrument, which was developed by the Teacher Education and Development Study in Mathematics (TEDS-M) Project (Tatto et al. 2008), was used to collect the data. The instrument was adapted into Turkish by the researchers. The instrument contains 6 items shown in Table 14.2. All the items were rated on a 4-point scale. Each item asked prospective teachers to indicate how frequently (1 = never, 2 = sometimes, 3 = occasionally, 4 = often) they had the opportunity to learn how to teach to a diverse group of students (e.g. students with behavioral and emotional problems, students with learning disabilities, gifted students; students from diverse cultural backgrounds, students with physical disabilities, students from poor or disadvantaged backgrounds). The validity and reliability of the instrument were analyzed with reference to a group of 370 prospective elementary mathematics teachers selected from 3 state universities. In this study, the Cronbach alpha for internal reliability of the instrument was found to be 0.89.

14.2.3 Data Analysis

In this study, each item on the instrument were scored 1 for 'never', 2 for 'sometimes', 3 for 'occasionally', and 4 for 'often'. Then the individual scores for each item were added up to produce the overall score for the teacher in question. These scores were then used to calculate average scores for individual regions and the whole country (Turkey). Data were analyzed using SPSS 17.0 software, and were screened with reference to the assumptions of parametric statistics. Normality and homogeneity of variances were tested at a multivariate level. Furthermore, one-way ANOVA test and Tukey HSD post hoc analysis were applied to reveal how prospective teachers' responses about the learning opportunities to teach to a diverse group of students vary among regions defined in NUTS.

14.3 Results

14.3.1 Turkish Prospective Mathematics Teachers' Opinions About the Opportunity to Learn to Teach to Diverse Students

Table 14.2 presents the distribution of prospective mathematics teachers' opinions on how often the teacher training program provides opportunities for them to learn

Table 14.2 Prospective mathematics teachers' opinions about each item in the *Teaching for Diversity* instrument

In your teacher training program, how often did you have the opportunity to learn to do the following?	M	SD
A. Develop specific strategies for teaching to students with behavioral and emotional problems	2.21	0.93
B. Develop specific strategies and curriculum for teaching to students with learning disabilities	2.20	0.93
C. Develop specific strategies and curriculum for teaching to gifted students	2.02	0.91
D. Develop specific strategies and curriculum for teaching to students from diverse cultural backgrounds	2.03	0.89
E. Accommodate the needs of students with physical disabilities in your classroom	2.07	1.00
F. Work with children from poor or disadvantaged backgrounds	2.19	0.97

how to teach to a diverse group of students. Turkish prospective mathematics teachers reported in general that the teacher training programs provide them with fewer opportunities to learn how to "*Develop specific strategies and curriculum for teaching to gifted students*" (M = 2.02), "*Develop specific strategies and curriculum for teaching to students from diverse cultural backgrounds*" (M = 2.03), "*Accommodate the needs of students with physical disabilities in your classroom*" (M = 2.07) compared to the items to "*Work with children from poor or disadvantaged backgrounds*" (M = 2.19), "*Develop specific strategies and curriculum for teaching students with learning disabilities*" (M = 2.20), and "*Develop specific strategies for teaching students with behavioral and emotional problems*" (M = 2.21).

14.3.2 Do Prospective Teachers' Opinions About the Learning Opportunities Provided for Teaching to Diverse Students Differ Significantly Between Regions in Turkey?

The distribution of the responses provided by prospective teachers, across regions, for each item in the instrument is provided in Table 14.3. Turkish prospective teachers' opinions on how often teacher-training programs provide opportunities for learning how to teach to a diverse group of students produced a mean score of 2.12. The prospective mathematics teachers' total scores in 6 regions [TR1 (Istanbul), TR2 (West Marmara), TR3 (Aegean), TR5 (West Anatolia), TR7 (Central Anatolia), TRB (Central East Anatolia)] were above country average, while scores in the remaining 5 regions [TR4 (East Marmara), TR6 (Mediterranean), TR8 (West Black Sea), TR9 (East Black Sea), TRC (Southeast Anatolia)] were below country average. The scores for prospective teachers from the TRA (Northeast Anatolia)

Table 14.3 Average scores for prospective teachers' opinions on items in terms of regions

NUTS level 1	n	Item A	Item B	Item C	Item D	Item E	Item F	Total score
		$\bar{x}$	$\bar{x}$	$\bar{x}$	$\bar{x}$	$\bar{x}$	$\bar{x}$	$\bar{x}$
TR1	37	2.38	2.54	2.22	2.19	2.46	2.38	2.36
TR2	97	2.24	2.26	2.11	2.13	2.18	2.33	2.21
TR3	185	2.24	2.28	2.11	2.03	2.09	2.29	2.17
TR4	30	2.13	1.80	1.70	1.77	1.90	1.87	1.86
TR5	59	2.53	2.63	2.32	2.37	2.46	2.57	2.48
TR6	101	2.09	2.23	2.09	1.91	2.03	2.17	2.09
TR7	216	2.45	2.43	2.24	2.28	2.35	2.43	2.36
TR8	62	1.92	1.76	1.60	1.95	1.79	1.81	1.81
TR9	255	2.00	1.96	1.75	1.82	1.86	1.91	1.88
TRA	209	2.19	2.22	2.06	2.02	2.08	2.14	2.12
TRB	44	2.20	2.32	2.02	2.16	2.14	2.34	2.20
TRC	91	2.22	1.94	1.87	1.83	1.66	2.08	1.93
Total	1386	2.21	2.20	2.02	2.03	2.07	2.19	2.12

Table 14.4 The results of the ANOVA test applied on learning how to teach to diverse groups of students

Source	Sum of squares	*df*	Mean square	*F*	Significant difference
Between groups	1786.820	11	162.438	8.431*	**TR1** > TR8, **TR1** > TR9, **TR1** > TRC, **TR2** > TR8, **TR2** > TR9, **TR3** > TR8 **TR3** > TR9, **TR5** > TR4, **TR5** > TR8, **TR5** > TR9, **TR5** > TRC, **TR7** > TR4, **TR7** > TR8, **TR7** > TR9, **TR7** > TRC, **TR9** > TRA
Within groups	26473.487	1374	19.267		

$^*p < 0.01$

region is equal to the country average. Table 14.3 shows how those prospective teachers' opinions about the learning opportunities for teaching to a diverse group of students vary from region to region.

The results of the ANOVA test applied in order to establish if the differences were statistically significant or not are presented in Table 14.4. ANOVA analysis revealed that the prospective teachers' opinions about learning how to teach to diverse groups of students exhibit significant differences with reference to the development levels of regions in Turkey [F (11-1374) = 8.431; $p < 0.01$]. Post hoc analysis was also applied to identify the groups which differ in this context. As Levene test results did not lead to a homogenous pattern of variance [F (11-1374) = 3.236; $p < 0.05$], multiple comparisons were based on the results of Tamphane' T2 test.

According to Table 14.4, post hoc analysis revealed significant differences in favor of TR5 (Western Anatolia) and TR7 (Central Anatolia), in comparison to TR4 (East

Marmara), TR8 (West Black Sea), TR9 (East Black Sea) and TRC (Southeast Anatolia). Significant differences in favor of TR1 (Istanbul) compared to TR8, TR9, TRC were also observed. Further significant differences were found presenting a favorable picture in TR2 (West Marmara) and TR3 (Aegean) compared to TR8 and TR9.

14.4 Discussion and Conclusion

Many educators agree on the necessity of developing the teacher training programs with the infrastructure to clarify how mathematics could be taught to increasingly diverse groups of students. This study investigated Turkish prospective elementary mathematics teachers' opinions about how often the teacher training programs in Turkey provide opportunities for learning how to teach to a diverse group of students. The results indicate that mathematics teacher training programs in Turkey do not provide sufficient learning opportunities to prospective teachers in terms of developing specific strategies and curriculum for teaching to students from diverse cultural backgrounds, as well as to gifted students, not to mention accommodating the needs of students with physical disabilities. The study found that prospective mathematics teachers in Turkey were sometimes given the opportunity to learn about working with children from poor families or disadvantaged backgrounds, and developing specific strategies and curriculum for teaching to students with learning disabilities, or specific strategies for teaching to students with behavioral and emotional problems. Results show that prospective mathematics teachers in Turkey sometimes have the opportunity to learn about teaching to a diverse group of students. The findings run parallel those of previous studies carried out in Turkey (Polat and Kılıç 2013; Ünlü and Örten 2013; Kaya 2014). Kaya (2014), in a study with 64 prospective teachers receiving training at a state university, found that prospective teachers did not consider themselves sufficiently equipped with the skills to contribute to their teaching at environments characterized by diversity.

Many international studies also note that teacher training programs in many countries suffer from similar shortcomings. According to TEDS-M, prospective primary school mathematics teachers in most European countries and some Asian countries had means closer to or lower than the midpoint on the scale used in this study (Tatto et al. 2012). In the same vein, prospective elementary school teachers in Germany, Norway, and Poland were reported having never or only occasionally been given opportunity to learn about how to teach to a diverse group of students (Tatto et al. 2012). Moreover, the results for prospective teacher training programs for secondary schools in Chinese Taipei, Georgia, Germany, Norway, Oman, Poland, the Russian Federation, Singapore, and Switzerland were even more striking: the prospective mathematics teachers reported that they rarely or never had the opportunity to learn about how to teach to a diverse group of students (Tatto et al. 2012). Only prospective mathematics teachers in Botswana, the Philippines, and the United States were found to enjoy better learning opportunities compared to prospective teachers from the rest of the countries, in terms of teaching to a diverse group of students. One can, thus, deem the problem to be a global one.

Diversity is a concept enormous in scope, referring to groups with students from different cultures, as well as different levels of talent, different needs in terms of special education, or students with physical disabilities or behavioral/emotional problems. When developing teacher training programs with a view to coming up with solutions to this global problem about diversity in education, one should certainly take the extraordinary scale of the issue into consideration. In this context, Rao's (2005) proposal that all teacher training programs should entail courses on the education of diverse groups of students makes sense.

Currently the elementary mathematics teacher training program in place in Turkey includes just one course on teaching to diverse groups of students (although there may be some elective courses offered at individual universities). As noted in the introduction section, the said single course is a theoretical one, discussing merely the general principles regarding the education of diverse groups of students. The conclusions reached, that the prospective teachers have only limited experience in terms of learning to teach to diverse students, are probably brought about by the very limited opportunities the particular course provides to prospective teachers. OECD (2010) concurs with this finding, arguing that teacher training programs include some form of diversity training, but it is often in the form of a single module or an elective course, which does not have a major, lasting impact on the teachers' careers.

Rao (2005) proposed a three-stage model for a course to be offered to remedy this deficiency. The course designed for the final year of the teacher training program is expected to (i) offer theoretical knowledge to prospective teachers, (ii) require prospective teachers to develop practices, and finally (iii) provide opportunities for field work in multicultural environments. These basic stages could also be supported by various strategies [encourage exploration and investigation, use students' prior knowledge, use culturally relevant materials as a springboard for mathematics instruction, encourage collaborative problem solving, offer an enriched curriculum and challenging activities, etc.; for more information see D'Ambrosio and Kastberg (2008)] to promote equality and achievement in mathematics for diverse groups of students. Yet one can forcefully argue that the training prospective teachers involved in the present study are receiving corresponds only to the first stage of this model. As a result, the teachers do not have the opportunities to develop and implement appropriate teaching practices for diverse students. Indeed under appropriate instructional conditions, diverse groups of students can learn just as well as any other children.

Preparing prospective teachers for multicultural teaching environments is among the fundamental responsibilities of teacher training programs (Hermans 2002). Yet another means to achieve this goal is to ensure that the trainers of teachers serve as models for prospective teachers by coming up with a learning-teaching process to reflect the diversity in the classroom environment. Teresa and Pivera (2004) state that such an approach would be much more effective compared to teaching awareness of diversity to prospective teachers. Moreover, Smolen et al.'s (2006) study on the trainers of teachers revealed that professors who provide teacher training often meet students from diverse cultural backgrounds, that they believe in the necessity for curricula sensitive to culture, but that they have difficulties in terms of implementing these. This last finding suggests that, above anything else, the relevant competences of trainers of teachers need improvement.

This study also found that the prospective teachers' opinions about learning how to teach to diverse groups of students presented significant differences with reference to the development levels of (NUTS) regions in Turkey. The findings suggest that prospective mathematics teachers in regions with a higher level of development in Turkey were found to have better learning opportunities in this context, compared to prospective teachers in other regions.

Finally, the results of this study lead to certain local and global conclusions and suggestions. First of all, one can conclude that Turkish universities (or the regions of these universities) are not homogenous in terms of designing opportunities for prospective teachers to learn about teaching to diverse groups of students. Future studies may investigate the nature of the learning opportunities concerning diversity and the factors causing differentiation at the local level. Such analyses would provide valuable insights to evidence-based proposals to improve teacher training programs. Secondly, the results obtained from this research reveal that there are differences between individual regions of the country, even in terms of a single variable (opportunities for learning how to teach to diverse students). Hence, one cannot be careful enough in reading the results obtained from a comparative education study performed at the local level, and interpreting them in the light of other international comparative studies. In international comparative studies in education, the countries are generally considered as homogenous reference units, without any regard for any differences which may exist within individual countries. The question of "whether it is possible to compare countries' educational systems as homogenous units" is one of the important theoretical debates in the field of comparative education studies. The results of this study suggest that this question cannot be answered very easily, especially when we consider countries which could be characterized as multicultural (such as Turkey).

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Chapter 15
Pre-service Mathematics Teachers' Knowledge and Beliefs

Andreas J. Stylianides and Seán Delaney

Abstract The notions of mathematics teachers' knowledge and beliefs have been conceptualized in manifold ways in the literature. Notwithstanding these different conceptualizations, however, the point stands that mathematics teachers' knowledge and beliefs are important factors to consider both in the study of classroom instruction in mathematics and in thinking about the goals, curriculum, or organization of the education of pre-service mathematics teachers. In this commentary we discuss how the four preceding chapters in this section of the book contribute to this body of research. Specifically, the four chapters contribute, collectively, to the broad issue of describing, elaborating, or conceptualizing kinds of mathematical knowledge and beliefs that are important for the education of pre-service elementary teachers. In doing so, they raise interesting challenges for the curriculum of teacher education and research in this area.

Keywords Knowledge · Beliefs · Teacher education · Elementary

A large body of research in mathematics education and mathematics teacher education has examined different issues related to mathematics teachers' knowledge and beliefs, how they are acquired, how they are changed, and how they affect student learning. The research focuses in some cases on one of these constructs—either knowledge (e.g., Ball et al. 2008) or beliefs (e.g., Philipp 2007)—and in other cases on the interplay between the two (e.g., Drageset 2010). As we will illustrate shortly, there is no canonical definition of mathematics teachers' knowledge or beliefs in the relevant literature. However, a common thread that permeates this literature is that mathematics teachers' knowledge and beliefs (however construed) are associated with teachers' instructional decisions and thus influence students' opportunities to learn mathematics. According to Wilson and Cooney (2002),

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> regardless of whether one calls teacher thinking beliefs, knowledge, conceptions, cognitions, views, or orientations, with all the subtlety these terms imply, or how they are assessed, e.g., by questionnaires (or other written means), interviews, or observations, the evidence is clear that teacher thinking influences what happens in classrooms, what teachers communicate to students, and what students ultimately learn (p. 144).

The various conceptualizations of teachers' mathematical knowledge and beliefs that are available in the literature reflect different perspectives researchers have brought to the study of these constructs. Ponte and Chapman (2008) conceptualized *knowledge* broadly "to refer to a wide network of concepts, images, and intelligent abilities possessed by human beings, including beliefs and conceptions" (p. 233), and they distinguished between two main and partly overlapping kinds of teacher knowledge: *knowledge of mathematics*, which has a referent in the field of mathematics, and *knowledge of mathematics teaching*, which has a referent in professional practice. The perspective of Ponte and Chapman (2008) to consider *beliefs* as an element of the broader construct of *knowledge* is reflected also in the writings of other researchers. For example, Philipp (2007) viewed knowledge as comprising the special class of beliefs that are "held with certainty" (p. 259). Similarly, Furinghetti and her colleagues associated beliefs with individuals' subjective (personal) knowledge in contrast with the kind of objective (official) knowledge accepted within a community (Furinghetti and Pehkonen 2002), or as a main component of a teacher's "practical knowledge" (Furinghetti and Morselli 2011).

Other researchers considered separately the constructs of teachers' knowledge and teachers' beliefs and offered categorizations of each thus illuminating different (often complementary) aspects of their complex and multifaceted natures. Regarding teachers' knowledge, following Shulman's (1986) influential work, several frameworks have been developed to describe important components of the content knowledge that teachers of mathematics draw on, or need to have, as they manage the demands of their professional practice. Two examples of such frameworks are the Mathematical Knowledge for Teaching framework (Ball et al. 2008) and the Knowledge Quartet framework (Rowland et al. 2009). These frameworks have supported several strands of research in the area of mathematics teachers' content knowledge, such as research on the relationship between teachers' content knowledge and students' achievement (Hill et al. 2005), on deepening teachers' content knowledge for teaching (Turner and Rowland 2011), or on organizing the mathematical preparation of pre-service teachers in teacher education (Stylianides and Stylianides 2014b).

Regarding teachers' beliefs, several frameworks have been developed to categorize those beliefs primarily according to their objects. Furinghetti and Morselli (2011) noted that "[t]he objects of mathematics teachers' beliefs may be internal (themselves as persons, as learners, as teachers) or external (the nature of mathematics, the nature of mathematics teaching and learning)" (p. 589). An example of a framework addressing internal-objects beliefs is Bandura's (1977, 1997) framework of self-efficacy beliefs as used in research on teachers' self-efficacy beliefs in specific mathematical domains such as problem solving (e.g., Stylianides and Stylianides 2014a) or on teachers' self-efficacy beliefs about mathematics teaching more generally (e.g., Philippou and Christou 1998). An example of a framework addressing external-object beliefs is Ernest's (1989) framework identifying different teacher roles in the classroom, such as facilitator or instructor. A major strand of research on mathematics teachers' beliefs has explored connections between teachers' beliefs and their teaching practice, though inconsistencies were often documented between the two thus motivating the development of further conceptualizations about the nature of teachers' beliefs, such as viewing teachers' beliefs as sensible systems (Leatham 2006).

The bottom line of this brief overview of research on mathematics teachers' knowledge and beliefs is that, despite their manifold conceptualizations in the literature, the constructs of knowledge and beliefs are important factors to consider both in the study of classroom instruction in mathematics and in thinking about the goals, curriculum, or organization of the education of pre-service mathematics teachers. The four chapters in this section of the book add considerably to this body of research; they illustrate some of the avenues currently being pursued within it and identify some that merit further investigation, with a particular focus on pre-service elementary teachers.

Specifically, the four chapters contribute, collectively, to the broad issue of describing, elaborating, or conceptualizing kinds of mathematical knowledge and beliefs that are important for the education of pre-service elementary teachers. In doing so, they raise some interesting challenges for the curriculum of teacher education and research in this area. We organize our commentary around three sections according to whether the focus of the reviewed chapters is on teacher knowledge only (Çelik; Shinno et al.) or teacher beliefs only (Güneş), or on the interplay between the two (Jacobson et al.). We acknowledge that the focus of our commentary on teacher knowledge and beliefs inevitably downplays some important contributions made in the chapters that did not fit directly within the scope of our commentary. We will allude to some of these contributions as we discuss each chapter in the following sections or in the final section where we will consider implications of the four chapters for teacher education research and practice.

15.1 Teachers' Knowledge

Shinno et al. use teachers' classroom use of argumentation as the lens through which to analyze teachers' mathematical knowledge. For them teachers' argumentation skills include how teachers evaluate students' solutions, how they assess and modify incomplete or incorrect solutions, and how they explain incorrect solutions. In short, Shinno et al. believe that teachers' argumentation skills underpin the teaching of mathematical practices and the assessment of students' knowledge of mathematics across grade levels and content areas.

In their study prospective teachers were asked to evaluate and respond in writing to four typical incorrect responses to the vaulting horse mathematics problem.

The researchers mapped the prospective teachers' anticipated responses to the framework of mathematical knowledge for teaching developed by Ball et al. (2008). They conjectured that responding to students' incorrect responses would draw on prospective teachers' specialized content knowledge (SCK) and their knowledge of content and students (KCS). The researchers added a fifth incorrect student response to draw some more on SCK and KCS and to tap into a third kind, knowledge of content and teaching (KCT).

In the prospective teachers' analysis of children's responses, some responses the researchers deemed incorrect were judged by many prospective teachers to be correct. Furthermore, where prospective teachers correctly classified responses as incorrect, many provided inadequate justification for deeming the responses to be incorrect. Notwithstanding the shortcomings in evaluating students' responses, Shinno et al. believe the prospective teachers understood the correct solution and that their difficulties rested with poor argumentative skills, which the researchers see as symptoms of shortcomings in prospective teachers' SCK and KCS. Some prospective teachers failed to provide a suitable example to counter an incorrect student response, which is seen as illustrating difficulties with their KCT.

The authors present a compelling case for expanding the Ball et al. (2008) framework of mathematical knowledge for teaching. Their conjecture about the relationship between teachers' mathematical argumentation and their mathematical knowledge is worthy of further investigation.

In addition, the study raises some interesting questions. For example, it is stated that "most of the participants were able to respond correctly to a problem like the vaulting horse problem" but no data are presented to support this. Confirming that prospective teachers hold common content knowledge (CCK) of this problem would help interrogate further the findings of the study. Such information could be valuable because some implied conditions of the problem are not explicitly stated in the written text. For example, the wording does not specify that the top 35 cm layer must be part of any solution. The absence of that condition makes a response of layers 3 to 8 potentially credible. Nor does the wording of the problem explicitly state that only consecutive layers are permitted in the final solution. The absence of that condition makes, for example, the solution of layer 1, 3, 5 and 7 possible.

Although the realistic nature of the problem should rule out those solutions, such solutions may seem plausible to students or prospective teachers who are unfamiliar with gymnastics. We are told that Japanese children use the vaulting horse in physical education classes and know how it works, but they may not automatically transfer that knowledge to the mathematics problem or they may think that different conditions could apply in a mathematics context to ensure the numbers work out, for example. More generally, in school settings where children have diverse social or cultural experiences, explicit contextual information to accompany realistic problems may help children solve problems (Boaler 1993; Wijaya et al. 2014) and consequently help teachers evaluate their errors.

Throughout the chapter Shinno et al. refer to "mathematical processes" such as argumentation, reasoning, proving, interpreting, representing, communicating, mathematizing and explaining. With regard to students' learning, these overlap with what the Common Core State Standards in the United States (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010) refer to as "mathematical practices." Further discussion would be welcome about whether the phrase "mathematical processes" is more helpful in describing teachers' knowledge or if the term "mathematical practices" adequately captures their knowledge too.

Like Shinno et al., Çelik et al.'s conception of teacher knowledge includes knowledge of content and students (KCS) and in many, if not all, contemporary settings, KCS must include extensive knowledge of student diversity. Students differ in terms of ability, language, culture, special needs, socio-economic status and so on, and teachers must be able to recognize such factors and respond to them through various teaching approaches, including several listed by Çelik et al. Although educators are increasingly aware of interrogating teachers' personal cultural identity and prejudices, and their implications for teaching, many teachers have difficulty accommodating diversity in their classrooms. Conventional resources, including curricula, frequently provide little support for teachers in accommodating diversity and may make further demands on teachers to adapt the resources for their students (Dowling 1998).

One potential source of knowledge of mathematics and students for teachers is their initial teacher education experience. With this in mind, Çelik et al. study the perceptions of almost 1400 prospective elementary teachers in the final year of their teacher education programme in 21 universities from all over Turkey about their opportunities to learn how to teach diverse students. An instrument developed for the Teacher Education and Development Study in Mathematics (TEDS-M) was adapted by the authors for the study.

The framework for initial teacher education in Turkey requires that students take a special education course which focuses on the general (as opposed to mathematical) learning needs of students with a range of disabilities, linguistic or communication disorders, who have autism and who are exceptional achievers. Given the existence of this required course, when prospective teachers were surveyed about their perceptions of opportunities given to them to develop strategies or curriculum, or to accommodate or work with various categories of students, on average prospective teachers believed that they had such opportunities "sometimes". Differences in responses across regions in Turkey were identified.

Based on their findings, the authors conclude that insufficient learning opportunities are provided to prospective teachers to develop specific strategies and curriculum for teaching children of particular diverse backgrounds. In light of findings from the TEDS-M study, they conclude that a similar situation exists in many countries other than Turkey. In order to support teachers in teaching for diversity, the authors endorse the work of Rao (2005) and others to suggest that prospective teachers should take standardized college courses on multicultural teacher education, a practicum component should be linked to multicultural coursework and that prospective teachers should experience a lengthy, mentored internship in a school with a university partnership. Furthermore, teacher educators in their own classes should act as role models for prospective teachers. However, the authors caution against overgeneralizing about the specific needs of educators in given countries because needs can vary from region to region within one country.

No doubt the authors have echoed an important finding about prospective teachers' perceptions of their preparedness for teaching diverse students in diverse settings. The need for such preparation becomes more important as teaching becomes a more mobile profession (e.g., Appleton et al. 2006) and given the role teachers play in evaluating materials for their sensitivity to cultural and other differences (Dowling 1998). Indeed because teaching is a cultural activity, approaches to differentiation of teaching have been interpreted differently across cultures (Stigler and Hiebert 1999). An important question that remains to be answered is what kind of experiences would help prepare prospective teachers to better teach curriculum subjects such as mathematics to students who are diverse in many ways. Another way of expressing this is to seek further elaboration of what constitutes KCS with specific reference to mathematics. Answering this question will further refine our conception of mathematical knowledge for teaching.

15.2 Teachers' Beliefs

The chapter by Güneş focuses on pre-service elementary teachers' beliefs. Specifically Güneş studied the relationship between pre-service elementary teachers' mathematical backgrounds and their perceptions of self-efficacy in mathematics, including the variation (or lack thereof) of these perceptions during a 4-year teacher education program. Thus the overarching aim of the research has been to describe this particular kind of teacher beliefs in the context of teacher training and in relation to factors that may be associated with it. According to the classification of teacher beliefs by Furinghetti and Morselli (2011) that we described earlier, Güneş' focus was on internal objects of teachers' beliefs, i.e., themselves as teachers.

The conceptual framework of the research drew primarily on Bandura's (1977, 1997) seminal work on self-efficacy beliefs, which, broadly speaking, describe an individual's own perceptions of his or her capacity to cope with certain situations or to perform certain tasks. A well-established body of research, some of which is discussed in the chapter, has shown that an individual's self-efficacy beliefs play a role in how well he or she actually copes with relevant situations or performs relevant tasks. Indeed, according to a hypothesis set forth by Bandura (1977) and substantiated by more recent studies, "expectations of personal efficacy determine whether coping behavior will be initiated, how much effort will be expended, and how long it will be sustained in the face of obstacles and aversive experiences" (p. 191). Güneş points out that teachers' self-efficacy beliefs have particular significance, for these beliefs can influence not only teachers' performance of their professional duties but also their students' opportunities to learn, even the students' own self-efficacy beliefs. An obvious factor associated with teachers' *mathematics* self-efficacy beliefs is their *mathematical* backgrounds, as the actual strength of these backgrounds, or teachers' perceptions of their strength, can shape the extent to which teachers deem themselves competent in mathematics. Thus Güneş' choice to investigate the relationship between these two factors—mathematics self-efficacy beliefs and mathematical backgrounds—comes as no surprise.

To carry out the investigation, Güneş used a cross-sectional survey research design with a sample of 209 pre-service elementary teachers in a Turkish university with all four years of a teacher education program represented in the sample. The finding about an association between participants' mathematics self-efficacy beliefs and mathematical backgrounds was to be expected. However, we consider worthy of careful consideration the finding that there was no statistically significant difference in participants' (generally positive) self-efficacy beliefs across the different years of the teacher education program and that participants attributed their (positive) perceptions of their self-efficacy in mathematics to their past experiences with mathematics. Based on this finding Güneş recommends the screening of pre-service teacher candidates for high-level mathematics self-efficacy beliefs before they may be admitted to teacher education programs. This is a sensible recommendation but caution needs to be exercised before its possible adoption.

Indeed, the recommendation may be misconstrued as reflecting an implicit acceptance of a failure of teacher education to support notable improvements in pre-service teachers' beliefs over the course of their training. Güneş' research does not allow the secure investigation of this issue in the context of the particular teacher education program, primarily due to the study's cross-sectional design: although pre-service teachers' beliefs appeared to be relatively stable across the four years of the program, the participants in the study were surveyed at a specific point in time rather than longitudinally. It is also possible that the measure of self-efficacy beliefs that was used in the research was not sensitive enough to changes at the upper level of self-efficacy perceptions. What we can say, though, is that Güneş' research raises the important issue about the impact that specific teacher education programs have, or can have, on pre-service teachers' mathematics self-efficacy beliefs, and how this impact may be enhanced. Encouragingly, research has shown that there *are* individual interventions, or teacher education programs more generally, that achieved a notable impact on pre-service elementary teachers' beliefs, including their self-efficacy beliefs (e.g., Philippou and Christou 1998; Stohlmann et al. 2014; Stylianides and Stylianides 2014a). Thus the question seems to be more *how*, rather than *whether*, impact can be achieved or enhanced.

15.3 Interplay Between Teachers' Knowledge and Beliefs

The chapters that we have discussed thus far focused on specific components of either knowledge or beliefs, and offered useful insights into the respective bodies of research in mathematics teacher education. The chapter by Jacobson et al. presents a more integrative approach to the study of teachers' knowledge and beliefs, viewing the two as distinct but also as inherently interactive. Using the rather encompassing framework of *Mathematical Proficiency for Teaching* (*MPT*), which Jacobson and colleagues introduced in previous publications and revisit and further elaborate in the chapter, Jacobson et al. report an innovative way to conceptualize and describe (measure) major interrelated components of teachers' knowledge and beliefs and their change during a teacher education course for pre-service elementary teachers in the United States.

The MPT framework conceives of mathematical proficiency for teaching as including mathematics-related teacher knowledge and teacher belief constructs that are consequential for students' mathematics learning. These include the constructs of pedagogical content knowledge(Shulman 1986), teaching self-efficacy, and motivation beliefs pertaining to anxiety, self-concept, and value. The belief-related constructs in the framework are denoted by the term *productive disposition* for teaching, which together with teacher *knowledge* define MPT. The framework has promise not only because of the theoretically robust way it depicts the dynamic organization of teacher knowledge and belief constructs, but also because it allows the generation of hypotheses, explorable empirically, about the interrelationship and change of these constructs and about their role in how a teacher enacts instruction. Of course, a prerequisite for such empirical explorations is a rigorous operationalization of the constructs comprising MPT so as to allow valid and reliable measurement of the constructs. This is a methodologically challenging task that Jacobson et al. have undertaken commendably for the constructs of MPT in the specific area of multidigit addition and subtraction.

In particular, Jacobson et al. describe an MPT survey they developed for measuring pedagogical content knowledge, teaching self-efficacy, and motivation beliefs for teaching the topic of multidigit addition and subtraction, and discuss their use of the survey to investigate how pre-service elementary teachers in a teacher education course in the United States developed components of MPT. Two studies—a cross-sectional and a longitudinal follow-up—allowed the authors to demonstrate that the survey is psychometrically well-behaved, it measures related but substantially different constructs (as they had posited theoretically), and it offers a useful tool to evaluate the extent to which a teacher education course promotes the various components of MPT. With regard to the particular course, they found that pre-service teachers' knowledge and self-efficacy beliefs developed during the course but their motivation beliefs did not. On the basis of these findings they conjectured that some components of proficiency may develop independently from other co-requisite or seemingly-related components. If this is indeed the case, an important implication for teacher education is that explicit attention needs to be paid to all components of proficiency for teaching a specific topic.

Another important question emerging from this research is whether the patterns of change in teachers' knowledge and beliefs are topic-specific. If they are, another important implication for teacher education follows: the need for a customized approach to the needs of teachers' proficiency for teaching different topics. Whether and how such a customized approach could be put into practice is a serious point for consideration, especially in light of the fact that the design of an approach for a specific topic would require in advance administration of an MPT survey for the specific topic of the prospective beneficiaries (pre-service teachers) of the approach.

The situation is further complicated by the fact that currently only one MPT survey for a specific topic (multidigit addition and subtraction) appears to be available.

15.4 Concluding Remarks

The four chapters we have commented on here emphasize what we already know about inadequacies in prospective teachers' mathematical proficiency for teaching and they highlight new areas that are worthy of the field's attention in this regard. Collectively and in conjunction with wider research in the field, they emphasize the urgency of addressing complex questions in relation to the pre-service education of teachers. What should be prioritized in designing a curriculum for the mathematical preparation of teachers? In what ways should this curriculum be topic-, grade-, country- or even region-specific? Can pre-service teacher education programs bring about changes in teachers' knowledge, beliefs, or their mathematical proficiency for teaching more generally? If so, how can pre-service teacher education programs bring about or amplify such changes? What minimal belief- or knowledge-related foundation is necessary or desirable in recruiting prospective teachers? What part of the teacher education curriculum should be prioritized for pre-service teacher education and what part can be postponed because it is of lower immediate priority or because it is more successfully learned on the job? What impact do various aspects of teachers' knowledge or beliefs have on children's learning? And finally, is it reasonable for an elementary teacher to be knowledgeable in mathematics and a range of other curriculum subjects or will children, even at a young age, learn more if they are taught by specialist teachers of mathematics?

The evidence base is growing on which such questions might be addressed and the four chapters contribute to such evidence. In addition to gathering more evidence, an orientation to addressing some of the questions raised above that have deep implications for teachers' knowledge and beliefs would be challenging but would ultimately be of immense benefit to the field.

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Part IV
Perspectives on Noticing in the Preparation of Elementary Mathematics Teachers

Chapter 16
Learning to Act in-the-Moment: Prospective Elementary Teachers' Role-Playing on Numbers

Caroline Lajoie

Abstract In this chapter, I report observations made during a three hour lesson in which forty pre-service elementary school teachers enrolled in mathematics method course on primary arithmetic at UQAM (Université du Québec à Montréal, Québec, Canada) prepared, performed and discussed a role-play involving the use of a calculator. I use those observations (including extracts) to illustrate the complexity of learning to notice and to act in-the-moment. I also use those observations to illustrate the potential of role-play for sharpening the awareness and the ability to notice of all students enrolled in the course, whatever role they play during the lesson (elementary school teacher, elementary school pupil or observer).

Keywords Role-play · Knowing to act in-the-moment
Noticing in-the-moment · Contingent situations · Decimal notation
Pre-service elementary school teachers

16.1 Introduction

Role-play involves staging a problematic situation with characters taking roles. It may be used to fulfill various objectives such as therapeutic objectives, personal and professional training objectives, or may be used as a pedagogical method (Mucchielli 1983). The premise of role-play is to have persons become active characters in a given situation. The objective of role-play when used in teaching contexts is to lead student-actors and other students to learn something about the characters and/or the situation (van Ments 1989, p. 16). In one of our mathematics method courses in UQAM (Université du Québec à Montréal), for example, students take the part of a teacher while others act as pupils, and they improvise, in an informed way, around a pupil's question or production, the use of teaching material,

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the use of calculators, and so on (Lajoie and Pallascio 2001; Lajoie 2010; Maheux and Lajoie 2011; Lajoie and Maheux 2013).

Since role-play requires from pre-service teachers to become active actors in teaching situations, instead of simply imagining or analysing such situations, it provides a relevant, original approach to teacher education and to research on teacher education. As reported by Zazkis and Jamshid Nejad (2014, p. 68), however, despite its known advantages, role-playing in teacher education is underdeveloped and most authors who report on its implementation most often do it in the form of self-reports or anecdotal evidence of experiences.

Role-play as a pedagogical approach in teacher education can be categorized as an "approximation of practice" (Grossman et al. 2009a, b). Other similar approaches in the same category, such as "rehearsals" (Lampert and Graziani 2009), have received more attention from research (for research on rehearsals, see for example Lampert et al. 2013; Kazemi et al. 2016). However, to my knowledge, research on those approaches has not examined their potential for the development of the ability to notice, which is precisely what I intend to do in this chapter, in the case of role-play.

16.2 Elements of a Conceptual Framework

Teaching involves various kinds of knowing (e.g. Shulman 1987), among which "pedagogical" and "content knowledge" attracted much attention in the community. However, beside knowledge "about" teaching and mathematical concepts, a growing attention is given to what some call "know-how" or "knowing-to act in the moment" (Mason and Spence 1999; Mason and Davis 2013): the ability to draw on various knowledge in response to actual situations. Mason and Spence (1999) suggested that such ability "depends on the structure of attention in the moment, depends of what one is aware of" (p. 135). "Structure of attention" is concerned with what is attended to and how it is attended to (Mason 2003).

> Educating this awareness is most effectively done by labelling experiences in which powers have been exhibited, and developing a rich network of connections and triggers so that actions 'come to mind'. No-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform (Mason and Spence 1999, p. 135).

Experiencing situations in which a rich network of triggers and connections come about and can be rendered explicit is considered fundamental to this development. Also, it is important for the students to be in the presence of someone who is aware of the *awarenesses* (Mason and Spence 1999).

> Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment. That is why it is so vital for students to have the opportunity to be in the presence of someone who is aware of the awareness that constitute their mathematical 'seeing' (Mason and Spence 1999, p. 151).

Mason and Spence (1999) also suggest that intentional preparation benefits the bringing to mind 'in the moment' of possibility for action.

Following the approach developed at UQAM, students are given "roles" to prepare in teams, then they improvise with classmates in front of the whole class, without any script, while others observe, and finally a whole class discussion including the teachers' educator follows. Hence, students *experience* various teaching situations, from the point of view of a teacher, a pupil, or an observer, and become actors in lessons instead of simply imagining or analysing these (as is often the case in other teacher education approaches). Also, students *examine* what comes out of improvisation so to prepare oneself for future action, to *anticipate,* for example by identifying alternative course of action.

Such importance to improvisation is given in this particular mathematics teaching course because we, as teachers' educators, beleive that learning comes with doing. Hence, we are interested in developing our students' *knowing-to* act in the moment and not only in developing their *knowledge-about* mathematics teaching and learning. As Mason and Spence (1999) would put it:

> (…) knowing-to act when the moment comes requires more than having accumulated knowledge-about. It requires relevant knowledge to come to the fore so it can be acted upon. That is what *knowing-to* captures for us (p. 139).

In this chapter, I use extracts involving pre-service teachers role-playing to illustrate the potential of role-play for contributing to sharpen pre-service teachers *in-the-moment noticing* (Mason and Davis 2013), thus educate their *awareness*, which in turn can contribute to a more responsive *in-the-moment pedagogy* (Mason and Davis 2013).

16.3 Method (Research Context and Design)

This study is a part of a research program on teacher education. Forty pre-service elementary school teachers enrolled in a 45 h undergraduate course (3 h weekly) on primary arithmetic mathematics method were involved in this study. They were in their second year of a four-year concurrent program. In their first year, they had completed a 45 h mathematics course dedicated to the preparation of elementary school teachers and oriented towards problem solving in arithmetic, probabilities and geometry. The course used for this study, for which I was one of the teachers (there were several groups of forty and one was under my responsibility), was designed around ten different *role-plays* (Lajoie 2010) on various arithmetical topics including numeration, operations, fractions and decimal numbers.

In the course, each role-play is organized in four moments:

1. The 'theme' on which students will need to role-play is introduced (introduction time).

2. Teams comprising four students are informed of the main objectives of role-play, a problematic situation involving pupils and an elementary school teacher is presented, and all the teams prepare for role-play not knowing beforehand who will play a pupil role or a teacher role (preparation time). Didactic instructions are given to teams orally or through written form in order to help them resolve the problematic situation and students may be required, beforehand (at home), to read research papers on mathematical concepts, on pupils' conceptions about mathematics, on pupils' errors in mathematics, or else. At the end of preparation time, based on what could be observed during that time, and in order to avoid pre-arranged role-play, the teacher-trainer chooses the teams who are required to delegate actors for teacher and pupil roles, making sure that the actors come from different teams.
3. Then comes the play itself (play time), where students chosen by the instructor come in front of the classroom and improvise a teacher-pupil(s) interaction. Depending on the role-play, the play might involve one pupil, a few pupils or, like in the case reported here, the whole class.
4. Finally, there is a whole classroom discussion on the play (discussion time). During this discussion, the actors' performances (teacher's and pupils'), the problematic situation itself, the apprenticeships realized through role-play, etc., can be examined.

Importantly, students have a preparation time to *anticipate*, to consider what might happen between a teacher and his/her pupil(s), but the role-play is essentially improvisational since there is no script and students learn only minutes before the play if they will be performing that day, and what role they have to take.

During one semester, the three hours role-plays in my class were videotaped using two cameras (one was fixed, capturing the whole classroom setting and interactions; the other was in the hands of a colleague who walked in the class during preparation time, capturing part of the work of the different teams). For this chapter, I focus on the play itself (play time), and on the whole group discussions (discussion time) for both tasks of the role-play presented in Fig. 16.1. That particular role-play, involving the calculator, took place during the second half the term.

16.4 Results

In this section, I describe two events that occurred while role-playing, as well as the discussion related to each, and offer my analysis in terms of the elements of my conceptual framework. I deliberately chose not to separate in two sections the events' descriptions and excerpts from my analysis. It appeared to me that my analysis would be easier to follow if it were embedded in the description and excerpts. The two events I selected involved two different teacher-actors, Marie and Max.

Role-play theme

Your pupils [10-12 years old] are working on a few mathematical problems using their personal calculator [they are not using the same model of calculator]. In so doing, there are many computations that they need to do. The different models of calculators do not always produce the same results, which surprises them since they are all very confident with the way they used their machine! Also, some results they obtain, surprise them. What's happening? You want your pupils to address this question, and others they might have, by themselves! Some of the computations they need to do are as follows:

1) 123 456 x 456 789 =

2) What are all the different possibilities for digits after the coma when you divide a whole number by 8 ? by 6 ? by 7 ? by 10 ? (Found in Caron 1999)

Note for the play :

In teams, prepare yourselves to either play the role of a *pupil*, or that of a *teacher*. For each play, there will be one teacher and ten observers (one per team). The other students will act as *pupils*. Every *teacher* will have a few minutes to explore a task with the whole class. Every *pupil* in the class will have a calculator.

Fig. 16.1 Role-play theme

In the context of role-play, "contingent" situations (in reference to "contingency"—one of the four components of the Knowledge Quartet (KQ) described by Rowland et al. 2005) are expected to happen in such a context, and "unexpected opportunities" for the teacher-actor are actually expected to occur.

> (…) teaching also involves attending to students' questions, anticipating some difficulties and dealing with unexpected ones, taking advantage of opportunities, making connections, and extending students' horizons beyond the immediate tasks. In short, teaching involves dealing with unpredictable, *contingent* events in the classroom (Rowland and Zazkis 2013, p. 138).

A first criteria for selecting the two events that will be discussed here was their "contingent" nature (Rowland and Zazkis 2013), which placed the teachers (Marie and Max) in the position of having to improvise, even if they had had some preparation time prior to the play, and the fact that both events offered a learning opportunity of a mathematical kind to the pupils involved in the play (and in fact to all the pre-service teachers in the class!). A second criteria was the fact that in the first case, related to task 1) in Fig. 16.1, the opportunity was taken by the teacher (Marie) whereas in the second case, related to task 2) of Fig. 16.1, the opportunity was missed or at least put aside by the teacher (Max). Finally, as a third criteria, in both cases those opportunities were discussed during the whole class discussion.

"*At the tenth power* or *times 10 at the tenth power?*": An opportunity taken by Marie.

Figure 16.2 shows the different answers that were given to Marie by the class when she asked the pupils to use their calculator in order to compute 123,456 × 456,789.

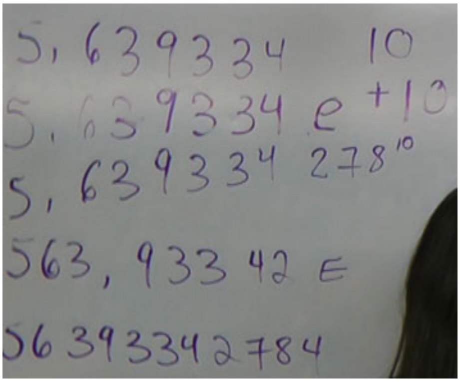

Fig. 16.2 123,456 × 456,789 on different calculators

Marie, while writing the different results on the board, questions the students in order to write those results exactly as they appear on their calculator. When Max gives his result (third one shown on Fig. 16.2), Marie's reaction suggests that she might not have anticipated that particular representation of the product while preparing for the play with her teammates.

Max. 5.639334278 to the tenth power.
Marie. [Writing 5.639334278^{10} on the board and pointing to a possible space between the last digit and the power (10)] and here, was there anything else or it was just like that?

After writing the different results on the board, Marie questions students. First, she questions them specifically on the meaning of the second answer, in which the expression e^{+10} is used. Students answer that e stands for "exponent", and e^{+10} for "exponent ten" [exposant dix]. After being questioned also on the first and third results, students conclude that in each case the notation used at the end of the number means the same, that is "exponent ten".

Marie then asks: "For you, what does 'exponent ten' mean?" Max, who is the one who gave the third answer on the board (see Fig. 16.2) answers "times ten". Clearly not expecting that answer, Marie repeats "times ten" but writes "10^{10}" on the board and then says "ten at the tenth power". This might lead one to think that she puts aside Max's answer. However, she immediately comes up with questions for him:

Marie. Here [pointing the first result (see Fig. 16.2)], what would you do … with this here, exponent ten?
Max. Times 10.
Marie. Times 10? So what would it change in the number?
Max. It would move the coma [dot] one position further [on the right].
Marie. So for you, it means times 10 only? If it is exponent 10, it is times 10. Do you all agree?
The class. No!

Marie. No? How could we correct his answer?
Pupil. It is at the 10th power but since we are in base 10, it is like 10 times … if it were at the 8th power I would move the coma 8 times but since it is at the 10th power I will move it 10 times, not just once.
(…)
Frank. I have a question concerning the 10th power. For us [his preparation team], 2 to the 10th power (…) would be 2 times 2 times 2 times 2 … 10 times. So here why isn't it 5.639334… times 5.639334 … like this 10 times?
Marie. Ok. To be frank, you lost me.
Frank. Well, for example, 4 at the 3rd power. (…) 4 times 4 times 4. (…) In this case, we do 10 times 10 times 10 … whereas the number is 5.639334. It should not be times 10. Should be times the number itself.
Marie. Actually, the trick here, when we have a number with a coma [dot] and many decimals after the coma [dot], is very easy. Instead of making long computations, we know that when we have exponent a number [here exponent 10] we can simply move the coma [dot] the number of times that is indicated on your calculator [here, ten]. Does this answer your question?
Frank. Actually, it would then be times 10. The n to the nth power, we multiply the number by itself. This is what is bugging me.
Marie. But here [pointing at the board], it is 10 at the 10th power, not 10.
A pupil, somewhere in the back of the class. It is times 10 at the 10th power!
Marie. That's it! It is multiplied by 10 at the 10th power. This is what it is. Here, it is implicit [on the calculator]. It is not indicated.

Marie's question to the class ("For you, what does 'exponent ten' mean?"), which was answered at first by Max, will have an important impact on the rest of the play since the class, with Marie's help, will struggle to make sense of this expression, in that particular context, up until the end of the play.

What exactly was Marie *aware of* when she decided, *in the moment*, to ask this particular question to the class? During the discussion following the play, Marie admits that she did not have herself a clear answer to that particular question. Hence, she deliberately took a risk, which suggests that she might have been *aware of* an opportunity for the class (and probably for her as well) to learn something from that particular question.

During the whole class discussion, the actual meaning of the expression "to the tenth power", its difference with the expression "times 10 to the 10th power", and the sense behind "moving the coma [dot] ten times" is discussed. Also, I (the instructor) feel the need to say a few words about the *scientific notation* in order for my students to "recall" what it is and what it means. However, even after my intervention, students, including Marie, do not sound as if they fully understand the expression given by the calculator.

Tom. [Addressing his question to Marie] How did you know it was ten at the tenth power? Me … I wouldn't have known what to answer had I been in front of the class.

Marie. Well, I was also embarrassed with that. But I think it might be because we are in base ten. Since they [the calculators] are not explicit on that, we have to take for granted that it is in base ten. I did not see it either so this is what I decided to do.

Hence, even at that point, the conclusion reached at the end of the play involving Marie ("It is multiplied by 10 at the 10^{th} power") is not clear for everyone, including for Marie, which indicates that there would have been other acts to perform by Marie during the play or that there would have been a need to pursue even more the discussion after the play in order for the class to end up with a better understanding of the whole situation.

"1 is actually a thousand": An opportunity put aside by Max.

In order to bring the class to answer the question 2 shown in Fig. 16.1 (What are all the different possibilities for digits after the coma [dot] when you divide a whole number by 8? by 6? by 7? by 10?), Max brings the class to make the computations shown in Fig. 16.3 with their calculator and writes everything (the divisions and the quotients) on the board.

In doing so, as he will admit during the whole class discussion that will follow the play, he wants the class to realize that there is a "cycle", that if the list were to be continued, the decimal developments would repeat themselves, in the same order. Of course, Max's "cycle" could have been more obvious had he asked for a few more computations but it did not seem to cause any problem to the pupils.

Max (teacher). What can you observe?
Justine. After the coma [dot], either we have 1 digit, 2 or 3.

Clearly not expecting this answer, Max ignores it and repeats the question. As for the pupil, she does not insist, even if her answer simply reflects the fact that she interprets the question differently, as if she had to find the number of possibilities for the number of digits (instead of the possibilities for the digits).

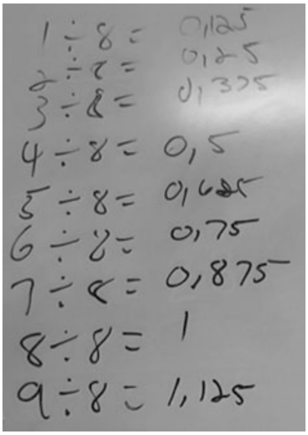

Fig. 16.3 Max's board

Max. Yes. But what can you observe?
Frank. What I noticed, spontaneously, is that we have multiples of 125. First one, 125. Second one looks like 0.25 but is in fact 250.
Max. Very good. Can everyone see that? [He transforms 0.25 to 0.250 on the board]. (…) Do we, at 8 [for 8 ÷ 8], if we have 1 [as quotient]… Is that a possibility? Where is 125 in that case?
Frank. Actually, it is not 1. If we put the coma [dot], it is 1000.
Max. Ah … Bravo! [He writes 1.000 on the board]. But is it really 1000 or 1?
Frank. Well it is not a thousand …
Max. There are zeros … Yes I understand what you mean but it is NOT a thousand.
Max. So … [Class is laughing]. The others do not understand however … All this to say that you destabilized me a little bit there.

For a moment, Max seems to be following Frank's reasoning. He even congratulates him twice! However, his remark at the end of the excerpt suggests that it is not entirely the case. Also, instead of questioning Frank, he rejects his answer.

During the collective discussion that follows the play, one of Frank's teammates, Tom, quickly raises his hand and reveals that he does not agree with the decision made by Max regarding Frank's answer. Tom explains that, while preparing for the play, his team established a relation with fractions, noticing, for example, that 1 divided by 8 was "0 and 1/8", and that 9 divided by 8 was "1 and 1/8", which brought them to the conclusion that there was 8 possible different decimal developments corresponding to "1/8, 2/8, 3/8, …" and adds that this relation helped the team make "more sense" of the situation.

Aware of the discomfort created in the class during the play by Max's reaction to Frank's assertion ("1 is a thousand"), I (the instructor) jump on Tom's remark to make explicit the relation between the different ways of representing a reminder when dividing by 8:

Instructor. When we divide by 8, the possible reminders are 0, 1, 2, 3, 4, 5, 6 and 7, which can be expressed 0/8, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8 and 7/8, and that is exactly what you see here [while pointing on 0.125 on the board] … 125 *thousandths* is 1/8, 250 *thousandths* is 2/8 …
Instructor. And what you said earlier [Frank], that 1 is a thousand, is actually true! It is a thousand *thousandths*! And what is a thousand *thousandths* if we compare it to 125 *thousandths*?
The class. Eight times.
Instructor. That's it. Eight times. Eight *eighths*. So what you said, Frank, was not crazy! You took Max by surprise though …
Max. Well yes. I was looking for *thousandths* but I never found it in my head. So I decided to put this aside and continue.

This last remark made by Max is very interesting. What he says is that, during the play, in the moment, he tried to make sense of that *thousand* coming from Frank but he just could not. The fact that one is equal to a thousand tousandths did not come to his mind when needed. Hence, he deliberately chose to reject Frank's answer, he stopped asking for pupils' answers and he switched to telling and explaining. Fortunately, at least one student *noticed*, during the play, and brought the situation to the fore during discussion.

16.5 Discussion/Conclusion

As reported by Gupta, Soto, Dick, Broderick and Applegate (this volume), professional noticing of children's mathematical thinking has received much attention in the mathematics education community (there was even a discussion on that particular topic at ICME13) but most of the research on that topic has been focused on practicing teachers. Recently, though, as noted by the same authors, "the value of incorporating professional noticing in teacher education programs to engage and prepare PST's before entering their own classroom is gaining momentum". In this chapter, I chose to examine the potential of role-play for developing pre-service elementary school teachers' ability to notice.

The previous analysis illustrates that role-playing with pre-service student teachers can contribute to educate awareness of all the students, whatever role they play (teacher, pupil, observer). Preparation time gives every student the opportunity to anticipate and interpret students's thinking and imagine himself/herself acting appropriately as a teacher (making instructional decisions based on students' thinking, such as posing questions in order to probe or to extend students' thinking) or as a pupil, thus sharpening his/her noticing in the moment. Then comes the play, during which the teacher needs to act in the moment, making instructional decisions based on students' thinking, while others either interact with the teacher (as pupils) or observe. Hence, during the play, every one in the class has multiple occasions to attend to pupils' answers/questions/strategies/difficulties (which involves, as stated by Dick (2013), noticing "mathematically significant details"), interpret their thinking and imagine himself/herself acting appropriately if he/she were the teacher, thus again multiple occasions to educate his/her awareness. Finally, during whole class discussion, students can share what they noticed during the play, realize (with some help from the instructor or from their teammates) what they missed to notice and discuss acts that were performed and those that could/should have been performed. Hence, discussion time offers occasions to become collectively and explicitly aware, which, again, might increase students' sensitivity to notice more richly in the future, a phenomenon termed by Dick (2013), in a different but quite similar context, as "collective influence" (p. 60).

Educating awareness in a role-play context, however, necessitates students' involvement at every moment. Not only students must be aware, during the play, of possibilities for the teacher to act (or act differently than the way he/she did in the moment), but they must also be willing, afterwards, to share what they noticed with the others and discuss openly about acts that were performed or acts that could have been performed.

In both situations that were analysed previously, for example, Frank and Tom's involvement played a very important role in bringing the teachers, Marie and Max, but also the whole class, become aware of certain acts to perform. In the first case, Frank is very active during the play, as a pupil. By his questions, he forces Marie, and the whole class, to realize that there is a problem with the use of the expression "to the tenth power" because it does not mean what many pupils seem to think it means. In the second case, Frank is willing to share his answer to question 2, as a pupil, with Max, in front of the whole class. However, when Max rejects Frank's answer, Frank does not insist. Later, during the discussion, Tom, followed by the instructor, leads the class to see what Frank meant when he suggested that 1 was in fact a thousand, which brings the class to conclude that Max should have acted differently.

Frank's involvement in those two cases does not seem however to be of the same nature. In the first case, Frank is clearly aware of the fact that there is a problem with the expression "to the tenth power" … He admits that he does not fully understand what is happening mathematically and, by his questions, pushes Marie to act. In the second case, since Max does seem to understand Frank's reasoning during the play, Frank does not insist, probably not wanting to put Max in an awkward position.

Many mathematics educators and researchers in mathematics education will agree on the fact that teaching a mathematical subject goes way beyond the complexity of the mathematical subject itself (see for example Lajoie and Maheux (2013) on the complexity of teaching division and Grossman, Hammerness and McDonald (2009a, b) or Lampert et al. (2013) on teaching as a complex practice)! As shown in the previous extracts, Marie and Max had many decisions to make during the play that involved much more than understanding, for themselves, the mathematics involved in the situation. Marie, for example, had to decide how to write/translate on the board the results that were given to her verbally, what result to examine first and how, what results to compare or contrast, whether or not she should reject any of the results, how to respond to questions that were asked to her regarding the many different results given by the calculators, how to investigate, interpret and extend Max's and Frank's thinking, and so on. However, the complexity of the mathematical subject itself did play an important role on the way Marie and Max acted in the moment. When the contingent situations happened, both Marie and Max were aware of the fact that there was an act to perform but, probably (at least partly) because they did not fully understand the mathematics involved, they missed some mathematically significant details during their interaction with pupils and ended up acting in a way that was not fully satisfactory, either for them, for the pupils involved, or for the observers.

In this chapter, only a few excerpts involving a few students were used to illustrate the potential of role-play for developing pre-service elementary school teachers noticing. Deeper and more sophisticated analysis are now needed in order to determine in what ways, to what extent, and under which conditions, pre-service teachers develop their ability to notice in-the-moment, as well as their knowing-to act in-the-moment, through role-play.

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Chapter 17
The Role of Writing Narratives in Developing Pre-service Elementary Teachers' Noticing

Pedro Ivars and Ceneida Fernández

Abstract Previous research has shown evidence of pre-service teachers' development of the skill of noticing students' mathematical understanding when they participate in online debates, throughout meetings with colleagues or watching video clips. The aim of our study is to analyze if writing narratives and receiving feedback from a university tutor could help pre-service teachers develop this skill during their teaching practices at schools. In their narratives, pre-service elementary teachers had to identify critical events related to students' mathematical understanding, describe and interpret them and make new instructional decisions to support the conceptual development of students. Results suggest that writing narratives with this focus and the tutor's feedback helped pre-service teachers structure their attention on students' mathematical understanding. In this sense, our study provides information about how the skill of noticing students' mathematical understanding can be developed in teacher education programs.

Keywords Noticing · Narratives · Pre-service teachers
Students' mathematical understanding

17.1 Introduction and Theoretical Background

Classroom situations and interactions occur, simultaneously, overlapping one another and hindering teachers in their attempt to attend to them all with the same intensity. In this context, teachers should be able to identify and focus their attention on classroom situations or interactions that could be potentially enriching to develop students' learning (Mason 2002; Sherin and van Es 2005; van Es and

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Sherin 2002). The NCTM (2014) has claimed for a change in the way of dealing with interactions in the classroom, by stating "finding the mathematics in students' comments and actions, considering what students appear to know in light of intended learning goals and progressions and determining how to give the best response and support to students on the basis of their current understanding" (p. 56). Therefore, teaching involves observing students, listening attentively to their ideas and explanations, planning objectives and using the information to make instructional decisions. This perspective calls for a greater flexibility of teachers in recognizing students' understanding while they are teaching (van Es and Sherin 2002) and suggests the development of the skill related to being aware of what happens in their classrooms and how to manage it (Mason 2002, 2011).

Over the past decade, a line of research has been carried in an attempt to respond to this new perspective (for a review, see Stahnke et al. 2016). These studies identify the skill of noticing as a way of providing effective responses in the classroom, making instructional decisions and being able to adapt them to a particular situation that arises in the middle of the instruction and cannot be pre-planned.

17.1.1 The Noticing Skill and Its Development

Noticing has been shown as an important skill for teachers (Mason 2002). Although the skill of noticing has been conceptualized from different perspectives (Mason 2002, 2011; Sherin et al. 2011), all of them emphasize the importance of identifying the relevant aspects in teaching and learning situations and interpreting them to make teaching decisions. Pre-service teachers' development of this skill during teacher education programs has become an important issue, over the past decade, but this is not an easy task. The noticing skill could be developed by moving from a focus on teachers' actions to students' conceptualizations and by moving from evaluative comments to interpretative comments based on evidence (Bartell et al. 2013; van Es 2011).

Previous researches have shown some characteristics of prospective and pre-service teachers' development of this skill, principally through the use of video clips (Coles 2013; Santagata et al. 2007; van Es and Sherin 2002). For example, van Es and Sherin (2002, 2008), through the use of video clips of classrooms interactions, showed changes in what teachers were able to identify and how they discussed what they had found, moving their discussions from assessment to interpretation based on evidence. Similarly, Coles (2013) showed that the use of video clips enabled teachers to rebuild classroom interactions in chronological order (accounts of) to interpret them later, providing evidence (accounts for) without judgments.

In this study, we are going to focus on the skill of noticing children's mathematical understanding.

17.1.2 The Skill of Noticing Children's Mathematical Understanding

In this study, we focused on the conceptualization of this skill given by Jacobs et al. (2010) as a set of three interrelated skills:

- attending to students' strategies: if pre-service teachers attend to mathematical details in students' strategies,
- interpreting students' mathematical understanding: if pre-service teachers' reasoning, about students' understanding, is consistent with both the details of the students' strategies and the research on students' understanding and,
- deciding how to respond on the basis of students' mathematical understanding: if pre-service teachers use what they have learned about the students' understanding, of a specific situation, and if their reasoning is consistent with the research on students' understanding.

In this context, previous research has emphasized that this skill can begin to develop in teacher education programs using different contexts and specific mathematical domains (Callejo and Zapatera 2016; Choy 2016; Coles et al. 2013; Fernández et al. 2012; Lajoie, this volume; Sánchez-Matamoros et al. 2015; Schack et al. 2013; Walkoe 2015). For instance, Fernández et al. (2012), showed that virtual debates are an appropriate instrument to promote the development of the skill of noticing students' mathematical understanding since pre-service teachers moved, from describing general strategies to give evidence of how students were developing the proportional reasoning. Coles et al. (2013) postulated that meetings between in-service elementary teachers, sharing the work they were doing at school, developed their skill of noticing children's mathematical understanding. In the domain of early numeracy, Schack et al. (2013), using a module with video, containing students' interviews, showed that pre-service teachers were able to develop the skill of noticing children's mathematical understanding. Walkoe (2015) concluded that participating in a video club helped teachers more consistently attend to student algebraic thinking and to reason about students' thinking in deeper ways. Recently, Choy (2016) showed the development of teachers' processes of noticing as they go through a lesson cycle, examining what teachers notice when they design tasks. The theoretical model generated in this last research provides lenses to recognize the productive noticing. With regard to the skill of proposing next-step instructional decisions, Gupta et al. (this volume), showed that pre-service teachers struggled with providing specific next steps tending to gravitate towards traditional teaching ideas.

These studies have shown that the development of this skill is not an easy task in different contexts. Following this line of research, our study examines how the act of writing narratives can help pre-service teachers to develop the skill of noticing.

17.1.3 Writing Narratives

Writing has been identified as a mediator of learning since "writing as process-and-product possesses a cluster of attributes that correspond uniquely to certain powerful learning strategies" (Emig 1977, p. 122). From this perspective, writing is understood as a powerful tool for knowledge construction whose primary function is to mediate recall and reflection (Wells 1999) by its abstractness, compared to speaking, which forces the writer to act more intellectually. For Wells, this characteristic implies that writing develops "the abstract, rational mode of thinking that is considered to be the endpoint of mental development" (p. 278). When somebody has to write for others, he/she needs to know and understand a topic better (in order to communicate accurately). Thus, writing is seen as a tool for collaborative reflection and for problem solving at the same time (Llinares and Valls 2009; Schrire 2006).

Narratives could be considered as the "primary form by which human experience is made meaningful" (Polkinghorne 1988, p. 1), thus, narratives could be an efficient tool in helping us recognize and understand how teachers act in professional contexts as well as how they organize their work. In the field of mathematics education, Chapman (2008) postulated that narratives allow teachers to express their practical understanding of mathematics teaching. A narrative is a story that tells a sequence of events that are significant for the author and has an internal logic that makes sense to him/her (Chapman 2008; Ponte et al. 2003). From this perspective, pre-service teachers could be seen as storytellers of their own stories and of the others during their periods of practices at schools, since narratives are seen as "a key form through which individuals come to know themselves, construct their lives, and make sense of their experiences" (Chapman 2008, p. 17). Moreover, narratives are seen as a tool that allows pre-service teachers to create explicative schemes through the analysis of the teaching-learning interactions and the reflection about these interactions (Schultz and Ravitch 2013). In this sense, writing narratives about classroom interactions and mathematical topics might allow pre-service teachers to become aware of important mathematical details of teaching and learning situations.

Our hypothesis was that if pre-service teachers wrote narratives focused on describing in detail a teaching-learning situation, in which they think that students' mathematical understanding has been developed, and then received feedback from a university tutor on how to improve and structure their way of noticing, pre-service teachers would focus their interpretations on students' mathematical understanding rather than on general aspects of the classroom. Our research question is:

- Do writing narratives and receiving feedback from a university tutor help pre-service teachers develop the skill of noticing students' mathematical understanding?

17.2 Method

17.2.1 Participants and Context

The participants were 22 pre-service elementary school teachers enrolled in the last year of their four-year-degree to become elementary teachers. They were in their last period of teaching practices at elementary schools. The first part of their practice, two out of eight weeks, was a period of observation and in the second part, the following six weeks, they had to implement a didactic unit developed by them. In both periods, they had to identify, describe and interpret noteworthy interactions in the classroom focusing their attention on students' mathematical understanding. Before their teaching practice at school, pre-service teachers had completed two mathematics education courses related to numerical and geometrical sense and a mathematics method course.

17.2.2 Instrument: The Narratives

During each period of their teaching practice, pre-service teachers wrote a narrative about some critical events that they had identified as being related to how students were developing the mathematical competence. We provided pre-service teachers with specific prompts:

- *Describe "in detail" the mathematics teaching-learning situation*. The task (curricula contents, materials, resources…). What did the elementary school students do? For example, you can indicate some students' answers to the task, difficulties…What did the teacher do? For instance, you can describe the methodology and some aspects of the interactions.
- *Interpret the situation*. Indicate the mathematical objectives of the task and how the task was implemented. Indicate some evidence of students' answers that show that the elementary school students had achieved the objectives (students' understanding of the mathematical content) and/or the difficulties they had.
- *Complete the situation*. Complete the situation indicating how you will continue in order to help students develop other aspects of the mathematical competence identified (that is, in order to support students in their conceptual development).

Once pre-service teachers had written the first narrative, during their observation period, they shared it with their university tutor by email who returned a written feedback a week later. In this feedback, the tutor took into account that the noticing skill could be developed by moving from general descriptions of the classroom to interpretative comments based on evidence of students' mathematical understanding. Afterwards, pre-service teachers wrote a second narrative of their own teaching practice.

The written feedback can be seen as a dialogic process of knowledge construction (Andriessen et al. 2003; Mitchell 2003) since "good feedback can significantly improve learning processes and outcomes, if delivered correctly" (Shute 2008, p. 154). Accordingly, construction of knowledge could happen through the process of dialogic argumentation, that occurs when different perspectives are being examined and the purpose is to reach agreement on acceptable claims of course actions such as in on-line debates (Fernández et al. 2012; Llinares and Valls 2010) or doing peer reviews of the written text (Swain et al. 2002). Therefore, the written feedback can help pre-service teachers to focus their attention on the important mathematical details of students' strategies, providing evidence of students' understanding and teaching decisions based on students' understanding.

17.2.3 Analysis

The two narratives written by pre-service teachers, before and after the feedback, were analyzed individually by three researchers looking for evidence of how pre-service teachers noticed students' mathematical understanding (what pre-service teachers had identified, how they had interpreted the events identified and which teaching decisions had provided). We discussed agreements and disagreements as we shared what we had found as evidence. We briefly explain, below, what we consider to be evidence of how pre-service teachers noticed students' mathematical understanding:

- If in the descriptions of students' responses, pre-service teachers included mathematically important details. For example, if pre-service teachers thought about whole-number operations and they commented on how children counted and used the decomposition of numbers and the arithmetical properties to represent and manipulate quantities.
- If in the interpretations of students' mathematical understanding, pre-service teachers addressed students' understanding and linked students' understanding with specific details (mathematical elements) of the situation. For example, when pre-service teachers made sense of the details of a student strategy and note how these details reflected what the student understood in specific situations.
- If in the decisions about the next instructional steps, pre-service teachers anticipated in their reasoning students' hypothetical strategies and provided specific tasks to define the new instructional situation.

Afterwards, we identified changes between both narratives relating to what they had identified, how they interpreted it and which teaching decision was provided and how feedback from the university tutor supported these changes.

17.3 Results

From the analysis of the data, two main results emerged. Firstly, the act of writing narratives helped pre-service teachers focus their attention on the mathematically important details of the situation and on students' mathematical understanding. Secondly, receiving feedback from the university tutor helped some pre-service teachers, who had difficulties in interpreting students' understanding or making teaching decisions in their first narrative, change the way in which they interpreted students' understanding in their second narrative.

17.3.1 Writing Narratives About Critical Events in the Lesson

Narratives written by pre-service teachers showed evidence of how they began to notice students' mathematical understanding. In the first narrative, pre-service teachers described interactions between the elementary teacher, students and mathematical knowledge, showing evidence of how they made sense of the mathematically important details of students' answers. Ten out of 22 pre-service teachers had difficulties in providing evidence of students' understanding. In these cases, pre-service teachers provided general descriptions and did not link their interpretations with specific mathematical elements of the situation. The other 12 pre-service teachers provided evidence of students' understanding, but 5 of them did not provide teaching decisions based on students' understanding (only 7 provided teaching decisions). Nonetheless, the fact that more than half of the pre-service teachers were able to interpret students' mathematical understanding seems to indicate that writing narratives helped them focus their attention on specific mathematical elements and on students' mathematical understanding. Below, we present an excerpt of a narrative written by a pre-service teacher, in the period of observation, showing how she noticed students' mathematical understanding.

Pre-service teacher 08, in her first narrative, described the context, mathematical contents, methodology and the activity which was problem solving in a fourth grade elementary school classroom (Fig. 17.1).

After presenting the activity, she described the situation and students' difficulties as follows:

> The first question didn't raise any particular difficulty, […] all students knew that the total amount was the sum of all expenses over the past four months. Regarding the second question, most of the students had problems because they believed they had to multiply the sum of the values by the number of months, which was four.
>
> They also had difficulties with the third question. Some students answered that they had to divide the sum of expenses of each month by 60 and others that they had to divide 60 by the number of days in each month.

The graph represents the expenses made by the council to maintain the natural pool (during the last year)

1395€	1530€	1643€	1519€
May	June	July	August

1. How much money have they spent in total in the four months?

2. What was the average daily cost in May? What about June? In which month was the average daily cost the lowest?

3. This year, the council has taken a measure: charging people for the use of the pool if the average daily cost exceeds 60€. In which months had people been charged for the use of the pool last year?

Fig. 17.1 Activity included in the narrative of pre-service teacher 08

The pre-service teacher focused her attention on the specific mathematical elements of the situation highlighting the students' difficulties in understanding the word problem, in relation to data that was not explicit, or that was not necessary to solve the problem. Therefore, she interpreted students' mathematical understanding generating a possible explanation about the students' difficulties, as reflected in the following excerpt:

> In the second question, students did not take into account the days of the month (30 or 31) because this data didn't appear explicitly in the word problem.
>
> In question three, students thought that if a number appeared in the word problem, they would necessarily have to include it in their calculations.

Then she provided the following teaching decision, based on her interpretation of students' mathematical understanding:

> Insisting on the meaning of the division as: distribute into equal groups. They do not identify when to use a division in a problem. We could ask them for example: If expenses varied from day to day, could we solve the problem using the same operation?

This latter teaching decision focused on identifying the use of a division is related to her interpretation of the students' mathematical understanding, and it reinforces the specific learning objective of understanding the word problem. Extracts such as these seem to indicate that writing narratives helped pre-service teachers notice students' mathematical understanding.

17.3.2 Receiving Feedback from the University Tutor

After having received feedback from the university tutor, 14 out of 22 pre-service teachers were able to provide evidence of students' understanding and made teaching decisions based on them. Consequently, 7 pre-service teachers who had difficulties in interpreting students' understanding or making teaching decisions in the first narrative changed the way in which they interpreted students' understanding (taking into account that only 7 pre-service teachers were able to provide evidence of students' understanding and made teaching decisions in the first narrative). Furthermore, they provided more detailed tasks in their teaching decisions after the feedback (although making teaching decisions according to students' understanding was the most difficult task for them). Below, we present some excerpts from the narratives of a pre-service teacher which showed how writing narratives and receiving feedback from the university tutor helped him improve the skill of noticing students' mathematical understanding.

17.3.2.1 The First Narrative and Feedback from the Tutor

The pre-service teacher 11 described the context (25 fourth grade elementary school students) and the task that involved a problem in which students had to interpret a football league table and answered some questions that required making comparisons and some operations (Fig. 17.2).

Then he wrote the interaction between the teacher and some students who gave incorrect answers to question 4:

Student 1: Dolphins won 6 matches
Student 2: No, I don't think so… They won 4 matches and tied 1
Teacher: You should read the problem again and try to figure out how to solve it…

And subsequently, he interpreted students' understanding providing general comments:

Table shows the results of a Football League. As you know a match could be won (W = 3 points), tied (T = 1 point) or lost (L= 0 points)

	W	T	L	Total
Wolves	2	2	4	8
Eagles	4	2	2	
Bats	2	4	2	
Parrots	3			12
Dolphins				13

1. How many matches did each team play?
   a) 4 b) 8 c) 12 d) 13
2. How many points did the Eagles win?
   a) 4 b) 8 c) 14 d) 25
3. How many matches did the Parrots tie?
   a) 0 b) 1 c) 2 d) 3
4. If Dolphins only lost one match, how many matches did they win?
   a) 1 b) 2 c) 3 d) 4
5. Which team won the championship?
   a) Dolphins b) Eagles c) Bats d) Parrots

Fig. 17.2 The problem used

> The difficulty here was the fact that some students were lost or they didn't know what kind of operation they had to do. Thereafter, they didn't participate.

The pre-service teacher identified the procedure engagement in the task but he had difficulties in interpreting students' understanding providing only general teaching and learning descriptions. For instance, this pre-service teacher did not notice that student 1 and 2 had difficulties in finding a decomposition of the number 13, relating the matches tied or won and the points obtained (13 are the points obtained in the 8 matches played). Student 1 had not considered that winning 6 matches is impossible since 6 matches $\times$ 3 points = 18 points, and Dolphins has obtained a total of 13 points. Student 2 had not taken into account the number of matches played. Although this student had obtained a decomposition of the number 13 ($13 = 4 \times 3 + 1$), he had not considered that they have played 8 matches. Therefore, both students did not consider the matches played and the relation between the matches tied or won and the points obtained.

The paragraph provided by the pre-service teacher was highlighted and commented by the tutor:

> You should analyze more in depth the wrong answers given by the students. Why did students answer in that way? Which difficulties did they encounter? What did the teacher do to avoid these difficulties?

This pre-service teacher provided a teaching decision based on a general comment in the narrative: "*it would be better to answer this kind of problem individually than with all the class*".

Again, the tutor gave the following feedback:

> In this part of the narrative it would be better to suggest a concrete action to complete the activity proposed in the classroom. The type of activity that you think would help students avoid their difficulties with the mathematic content and would help them develop their mathematical understanding.

As can be observed in the excerpts above, firstly, the university tutor was trying to get the pre-service teacher to focus its attention on students' conceptualizations instead of on teacher's actions. Thus, she wrote, "*Why did they* (children) *answer in that way?*" and later, the tutor tried to avoid evaluative comments by writing "[…] *they didn't know what kind of operation they had to do*" and focus the pre-service teacher's attention on interpretative comments based on evidence. In this way, the tutor wrote, "*Which difficulties did they encounter? What did the teacher do to avoid these difficulties?*" Finally, as the tutor realized that the pre-service teacher had completed the situation with an instructional decision based on general comments, she suggested that "*a concrete action to complete the activity proposed in the classroom*" would help students avoid "*their difficulties with the mathematical content and would help them develop their mathematical understanding*".

17.3.2.2 The Second Narrative

In the second narrative (after feedback from the tutor), pre-service teacher 11 focused on the meaning of supplementary and complementary angles. Firstly, he described his interaction with two of his students:

> Although students said that everything was understood, I could see in their faces that something was wrong. Then I asked two students to go to the blackboard and to draw supplementary and complementary angles of an obtuse and acute angle respectively that I had drawn… They doubted and tried writing the measure of the angles I'd drawn.

Next this pre-service teacher interpreted the situation highlighting important mathematical elements (two angles that form together an angle of 180° are supplementary and two angles that form together an angle of 90° are complementary) and the relevance of the process of visualization:

> … students were not able to visualize that two angles together could form another one […] Then I knew that they hadn't understood anything and asked them to draw a straight angle and a right angle. When they had drawn this, I asked them to overlap my angles into theirs and painted each of them in different colors for them to visualize that when two angles put together form an angle of 180º they are supplementary and if they form an angle of 90º they are complementary.

This pre-service teacher noticed that the two students have difficulties visualizing that two angles together could form another one (and that they could be supplementary or complementary) in his writing "*students were unable to visualize that two angles together could form another one…*". This is his interpretation of why students hesitated to draw supplementary and complimentary angles of obtuse and acute angles that he had drawn and they tried to write the measure of the angles that he had drawn.

This pre-service teacher, after identifying the difficulties of the students and interpreting them, by linking his interpretation with the initial representation of the concept that he had used, he suggested an activity with manipulatives (Meccano or simply toothpicks) as his next teaching action. This activity is focused on students' development of the process of visualization, so the activity proposed is based on students' mathematical reasoning:

> I will plan activities with manipulatives such as Meccano® or simply toothpicks in which students will have the opportunity to engage in the construction of different angles and visualize supplementary and complementary angles.

Excerpts of the narratives of the pre-service teacher 11, both during the observation period and during their teacher practice period (after feedback from the tutor), show us how this pre-service teacher was identifying, interpreting and making instructional decisions about critical events in mathematics teaching situations. In his first narrative, he described the task and some interactions in the teaching and learning process between the teacher and the students, but he did not interpret the mathematical elements that students had shown in their answers and the relationship between those and the students' mathematical understanding.

This pre-service teacher's descriptions and instructional decisions were based on general comments or actions. However, in the second narrative, not only did this pre-service teacher described the situation and interpreted students' mathematical understanding by identifying their difficulties but also gave an instructional decision related to the students' difficulties.

17.4 Discussion

The aim of this research is to analyze how writing narratives and receiving feedback from a university tutor help pre-service teachers develop the skill of noticing students' mathematical understanding during their teaching practices at schools. Based on our results, we underline that narratives seem to be a useful instrument in helping pre-service teachers focus their attention on specific mathematical elements and on students' mathematical understanding since, in the first narrative, 12 out of 22 pre-service teachers interpreted students' mathematical understanding. In this sense, we can see the act of writing as a mediator in pre-service teachers' learning (Ivars and Fernández 2016; Wells 1999). However, only 7 out of these pre-service teachers who interpreted students' mathematical understanding were able to make a teaching decision considering how they had interpreted the students' understanding. Therefore, the skill of deciding how to respond on the basis of students' mathematical understanding is the most difficult one to acquire in teacher education programs (Choy 2016; Tyminski et al. 2014).

Nevertheless, in their second narrative, after receiving feedback from the university tutor, 7 out of 15 pre-service teachers who had difficulties in interpreting students' understanding or making teaching decisions in the first narrative, changed the way in which they interpreted students' understanding. In this sense, our results suggest, that the feedback provided by the tutor helped pre-service teachers focus their attention on students' mathematical understanding in a more detailed way and to justify their teaching decisions. So, our study provides information about a context that can help pre-service teachers develop the skill of noticing students' mathematical understanding in teacher education programs.

The shifts observed between both narratives suggest that the process of dialogic argumentation fostered the construction of knowledge of pre-service teachers, improving their learning processes and outcomes (Andriessen et al. 2003; Hattie and Timperley 2007; Mitchell 2003; Shute 2008). This process of dialogic argumentation was established between the tutor when doing the review of the first narrative (Swain et al. 2002) to give a written feedback, and the pre-service teachers when they wrote their second narrative taking into account the tutor's feedback. Therefore, this dialogic process facilitated that pre-service teachers focus their attention on the important mathematical details of students' strategies that helped them provide evidence of the students' understanding and teaching decisions based on their interpretation of students' understanding. Consequently, our results suggest that writing narratives in a context in which pre-service teachers receive feedback from their tutors allows them to begin to frame the practical situations (Smith 2003) through the cognitive processes of attending and interpreting the students' mathematical understanding.

Although other factors such as instruction received by pre-service teachers during the degree are assumed to influence the nature of their narratives (Chapman 2008), changes in pre-service teacher descriptions and interpretations of students' mathematical understanding suggest that writing narratives and receiving university tutors' feedback helped pre-service teachers structure their attention on students' mathematical understanding.

These claims generate additional research questions. For example, if an online debate, where pre-service teachers would be able to share their narratives with other pre-service teachers and the university tutor, could also help them to focus their attention on interpreting students' mathematical understanding. These online discussions will also represent a source of learning on how students understand mathematics. This is similar to the "Math-Talk Learning Community" (Hufferd-Ackles et al. 2004) that is, a "classroom community in which the teacher and students use discourse to support the mathematical learning of all participants" (p. 82). This community would be created online to support pre-service teachers' learning of mathematical knowledge for teaching and how to use this knowledge to notice students' mathematical thinking.

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Chapter 18
Noticing and Deciding the Next Steps for Teaching: A Cross-University Study with Elementary Pre-service Teachers

Dittika Gupta, Melissa Soto, Lara Dick, Shawn D. Broderick and Mollie Appelgate

Abstract Teaching is a complex endeavor which requires teachers to make decisions based on children's thinking. Pre-service teachers (PSTs) training to be future teachers need experiences to increase their ability to notice, understand, and analyze children's responses/work to make sense of children's mathematical thinking. This chapter examines elementary PSTs' skills to recognize, identify, and make instructional decisions in their teacher preparation programs when provided with opportunities to engage in noticing practices in their mathematics methods courses. Qualitative analysis using open coding was used to analyze data sets collected from three different universities across the U.S. to find commonalities and themes. Results of the study provide insight into the effectiveness of the instructional activities and PSTs' next-step instructional decisions.

Keywords Professional noticing · Children's mathematical thinking
Elementary pre-service teachers · Student work analysis

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18.1 Introduction

The U.S. National Council of Teachers of Mathematics (2000) has indicated that to improve instruction teachers must be aware of their own and their children's actions and consider how these actions affect children's mathematical learning. van Es and Sherin (2008) state that teachers can improve their noticing by changing what they notice, reasoning about student work, and making decisions. In the case of pre-service teachers (PSTs), noticing children's mathematical thinking can be particularly difficult as they are in the midst of learning content and pedagogy. PSTs need to be exposed to ideas from frameworks such as professional noticing of children's mathematical thinking, which consists of attending to children's strategies, interpreting children's understanding, and deciding how to respond based on children's understanding (Jacobs et al. 2010) as a foundation for understanding and analyzing student work. Providing PSTs opportunities to notice, understand, and analyze student responses, specifically children's written work, allows them to make sense of the complex teaching environment and prepare for making decisions in the moment.

This research study investigated PSTs' skills of attending and interpreting children's mathematical understanding and deciding the next steps in instructional decisions in their teacher preparation mathematics methods course. The researchers in the study taught the same lesson and activities to PSTs at three different institutions across the U.S. in order to foster the act of professional noticing and deciding the instructional next steps. Using PSTs' learnings from multiple sources and different types of locations provides a greater number of cases and a wider range of settings which lends strength to the researchers' findings. The purpose of the research study is to focus on the instructional decision component of professional noticing for individual children and then extend to a whole class setting.

18.2 Literature Review

Professional noticing of children's mathematical thinking has received much attention in the mathematics education community. Understanding how and what teachers notice and how it translates into practice can provide the means to better prepare teachers (Jacobs et al. 2010; Star and Strickland 2008; van Es 2011). Past research has shown that learning to "notice" children's mathematical thinking is a practice that needs to be purposefully developed (Santagata 2011; Star and Strickland 2008; van Es 2011).

Much of the noticing research has focused on practicing teachers rather than PSTs, however, the value of incorporating professional noticing in teacher education programs to engage and prepare PSTs before entering their own classroom is gaining momentum. Philipp (2008) discussed the need to construct settings for PSTs to grapple with children's mathematical thinking while in the context of working with real children. Such settings can include video and/or student work analysis. For PSTs, the majority of research on professional noticing has been conducted through watching and analyzing videos (Barnhart and van Es 2015; Schack et al. 2013). Only a few have looked at PSTs' noticing using student artifacts (e.g. Dick 2017; Fernandez et al. 2013), despite the fact that PSTs need assistance learning how to analyze student work (Bartell et al. 2013; Crespo 2000; Spitzer et al. 2011).

Simply looking at student work does not ensure that PST learning will occur (Ball and Cohen 1999; Bartell et al. 2013). When PSTs begin to analyze student work, they tend to (a) describe the work as right or wrong without specifically attending to the mathematics present (Crespo 2000; Goldsmith and Seago 2011; Shaughnessy and Boerst this volume; Spitzer et al. 2011); (b) draw on personal experience (often their own traditional instruction) and/or knowledge to fill in gaps in children's work (Bartell et al. 2013; Goldsmith and Seago 2011; Spitzer et al. 2011); and (c) not know what to do with non-standard or surprising solutions (Bartell et al. 2013; Crespo 2000; Goldsmith and Seago 2011). Since PSTs must be taught how to analyze student work, the researchers contend that the professional noticing framework provides a lens for the PSTs to focus their analysis. By having PSTs consider multiple student work samples of the same problem, they are engaged with a real teaching practice. Asking PSTs to determine how to make whole class instructional decisions when children approach the same problem differently is yet another important skill that PSTs need to develop.

18.3 Purpose

This research study has thus been undertaken to examine PSTs' noticing using student work with a focus on making sound next-step instructional decisions first for individual children and then for the whole class. Using the professional noticing framework proposed by Jacobs et al. (2010), this current research study focuses on the third skill of deciding the next steps after attending to and interpreting individual children's work. The researchers' rationale of choosing to focus on this skill is that though PSTs can be taught how to make next-step decisions for individual children (Dick 2013), being able to make an instructional decision for a whole group/class is still underdeveloped and needs more attention. Thus, this research study extends the professional noticing framework to noticing and deciding for both individual children and then to a whole class. To fill this gap, this research study was conducted at three different institutions across the U.S.

18.4 Research Questions

The research questions that provide focus to the study are:

1. What instructional decisions do elementary PSTs make for individual children as they engage in analysis of student work samples via professional noticing analysis?
2. What instructional decisions do elementary PSTs make for a whole class by analyzing a set of individual student work samples?

18.5 Methodology

18.5.1 Participants

The participants comprised of elementary PSTs enrolled in their first mathematics methods course at three different institutions of higher education across the U.S. The institutions varied in enrollment sizes from 3,600 to 36,000 students and the class sizes ranged from 12 to 30 PSTs per class. The PSTs were enrolled in a four year elementary education program at two institutions and in a post-baccalaureate multiple subject (elementary) credential program at the third university. The demographics at the different institutions were comparable to the nation's current enrollment in elementary teacher education programs with a majority of the participating PSTs being white and female. In total, 95 PSTs participated in the study: Institution-1 had 28 PST participants, Institution-2 had 17 PST participants, and Institution-3 had two classes of PST participants with 50 total PSTs.

18.5.2 Research Design

The research design chosen for this study was a collective case study method (Creswell 2007). Merriam (1998) recommends the use of case study to "gain an in-depth understanding of the situation and meaning of those involved" (p. 19). Since the focus of this research study was to examine PSTs' thinking and learning as a result of exposure to the noticing framework along with carefully designed instruction to foster understandings, the collective case study methodology assisted in answering the questions related to the depth and relevance of next-step decision making via the use of professional noticing.

18.5.3 The Lesson

To engage PSTs in the professional noticing framework, especially making sound next-step instructional decisions, the researchers planned and taught a common lesson at their respective institutions. After analysis of the curriculum at the different institutions, the need for PSTs to engage more deeply with ideas and understandings surrounding multiplication emerged, and a common lesson around this topic was developed. This lesson was introduced at each of the institutions by eliciting responses from PSTs about their conceptions of multiplication and then watching video clips of children solving single digit multiplication problems (Carpenter et al. 2015). Four video clips were selected that represented young children's multiplication solution strategies along a trajectory (Carpenter et al. 2015, p. 51). Exposing the PSTs to a trajectory of children's responses was a means to develop PSTs' Mathematical Knowledge for Teaching (Hill et al. 2008) regarding multiplication. After each video clip, PSTs discussed and shared what the children did to solve the problem (attend), what children understood and their level of sophistication (interpret), and what could be a next step for that child (decide). While the researchers carefully designed the lesson to elicit the concepts from the professional noticing framework (Jacobs et al. 2010), they did not explicitly teach or use the professional noticing language (attend, interpret, decide) with the PSTs.

After discussions of various children's solution strategies for single digit multiplication, the PSTs were presented with *The Case of Mr. Harris and The Band Concert*, a classroom case study (NCTM 2014), in which the children solved the problem:

> The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, the class needs to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area? (NCTM 2014, p. 27)

The PSTs were first asked to solve the problem as children would and then analyze student work (see Fig. 18.1 for examples of the student work) from the lesson with the specified goal of making next-step instructional decisions for each student and finally, the whole class. This particular case study was selected from the NCTM Principles to Action Toolkit (NCTM, n.d.) because it was focused on the concept of multiplication, and included several student work samples for the same problem that the researchers could use to prompt PSTs to think about individual and whole class next-step instructional decisions. All PSTs discussed student work in small groups, answered questions based upon the noticing framework, noted their ideas on poster-sized paper, did a gallery walk to compare their ideas with each other, and finally shared out as a whole class.

Because the initially designed lesson included only one opportunity for the PSTs to engage with the practice of making next-step instructional decisions, the researchers developed an additional homework assignment and piloted the assignment at one of the institutions. It should be noted that the PSTs who completed this assignment did not receive any additional instruction or feedback from their instructors regarding professional noticing; the assignment provided more time to practice the noticing framework. For this assignment, PSTs, working in pairs, analyzed a different set of student work for a different multiplication problem

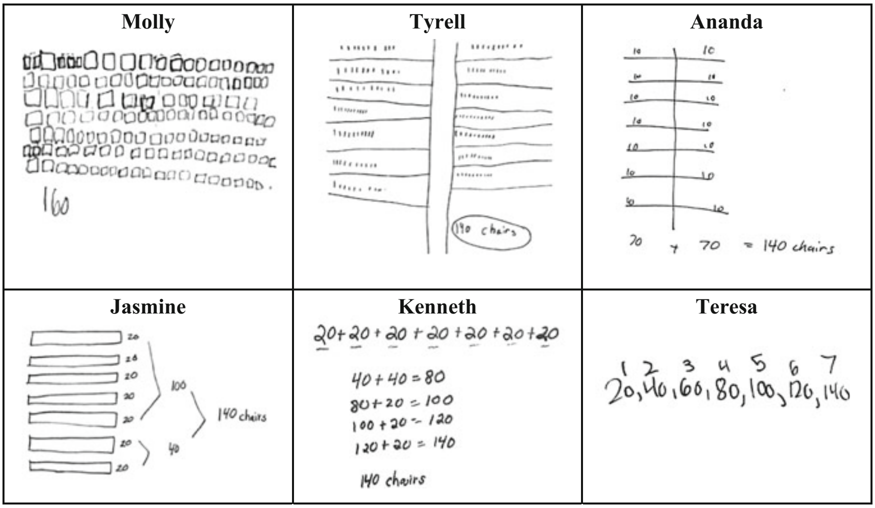

Fig. 18.1 Student work samples from the *Case of Mr. Harris* (NCTM, n.d.) Reprinted with permission from The Principles to Actions Toolkit, copyright 2017, by the National Council of Teachers of Mathematics. All rights reserved.

(Drake et al., in preparation). As before, the researchers chose this set of student work because of its focus on concepts of multiplication and that fact that it included different student responses to a problem. PSTs were first required to apply the noticing framework to five individual student work samples, and then to consider the student work as indicative of a whole class in order to decide on an instructional next step for the whole class. The problem and the student work samples are located in Fig. 18.2.

18.5.4 Data Collection and Analysis

Data were analyzed using a collective case study methodology (Creswell 2007), wherein each work sample collected from PSTs at each institution was treated as a separate case. Different cases from the three institutions were analyzed together for holistic understanding of PSTs' thinking about instructional decisions for individual children and for a whole class. Data sources included notes from class discussions of the *Case of Mr. Harris and The Band Concert* (NCTM, n.d.), PSTs' written products when analyzing student work, transcriptions of PSTs' discussions, and PST's work in pairs analyzing a different sample of children's work on multiplication.

Using qualitative methodology for this collective case study allowed the researchers to explore the relationship between the implementation of the developed lesson and the PSTs' understanding of next-step decisions in a real-world perspective (Yin 2014). Analyzing data from PSTs' work at different institutions gave

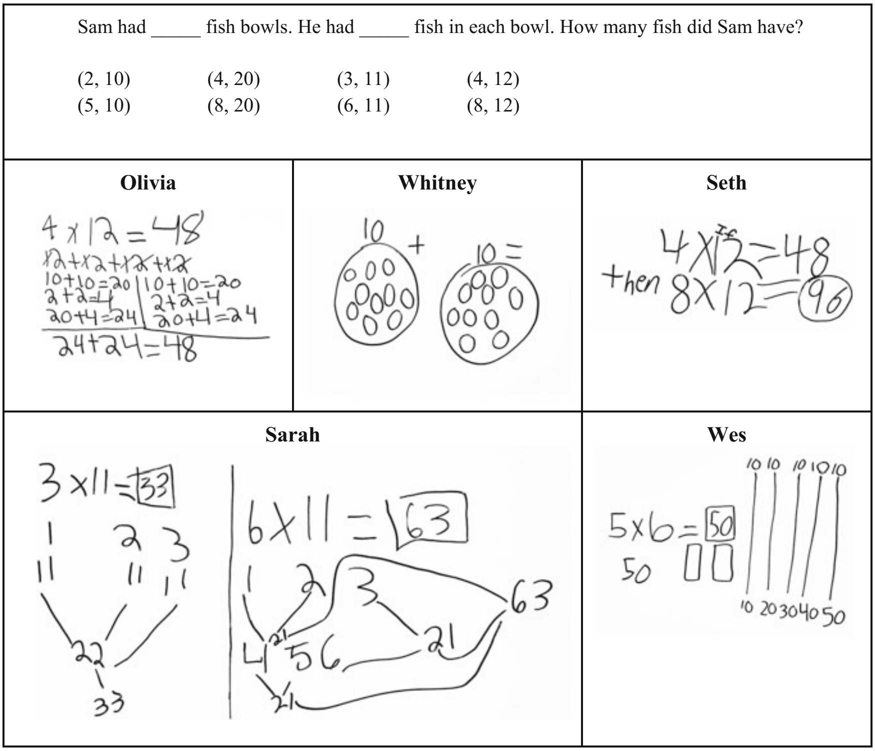

Fig. 18.2 Additional assignment and selected student work samples (Drake et al., in preparation)

a more in-depth and critical understanding of PSTs' responses about making next-step instructional decisions. Open coding was used by researchers to first analyze each work sample collected from PSTs at their respective institutions resulting in initial themes. Afterwards, the data were analyzed together by all researchers. Initial codes guided the analysis and emerging, divergent themes were added as new codes to answer the research questions.

18.6 Results

Collective case study analysis revealed four distinct themes relating to PSTs' suggestions on next-step decisions for individual children and the whole class: (a) gravitation towards traditional teaching ideas, (b) vague next-step suggestions, (c) desire for written number sentences, and (d) focus on strategy progression. Themes for individual and whole class next-step instructional decisions were similar.

18.6.1 Gravitation Towards Traditional Teaching Ideas

Out of all the emerging themes, PSTs' gravitation towards traditional teaching was the most common and significant in the data analysis. Specifically, these ideas included a focus on memorization, using mental calculations or number facts, vertical organizational structure of written multiplication solutions, and using the "times" sign as the next-step decision for individuals and the whole class. In the following example, a more traditional emphasis on computational speed arose. A group of PSTs at Institution-1, suggested that Ananda (see Fig. 18.1) "could use skip counting by 20s which would decrease problem solving time." This shows not only the PSTs desire for speed, but also the fact that the PSTs wanted to see Ananda use groups of 20s, which does not acknowledge the fact that Ananda's strategy directly modeled the problem situation in that there was an auditorium with a center aisle, a key component to her solution process.

Other PSTs indicated a desire to "see" children show their work in a more traditional format. For example PST pair #14 at Institution-2 suggested that for Olivia (see Fig. 18.2), "The next step I would give her would be to work on multiplication problems vertically. This will be a better format for her to see how the problem works." This quote indicates the PSTs' focus on traditional procedures and the algorithm, and may also show that the PSTs did not understand Olivia's strategy.

Analysis also revealed that PSTs were focused on "showing" children how to solve the problem. Seven of the seventeen PSTs at Institution-2 who completed the additional noticing assignment, indicated they would in some manner *show* children how to solve the problems. For example, PST pair #1 commented that, "I would help *show* Oliva how to solve the larger number equations without having to break up the problem." These results indicated that *showing*, a more traditional view, was *teaching* for many of the PSTs.

The same results were seen when looking at the PSTs' whole class decision making. For example, pair #8 at Institution-2 suggested that the whole class should work on, "Memorization of multiplication facts for preparation of multiplying two digit numbers [and] set up and solve multiplication problems without addition." Also, pair #15 indicated that they should "introduce them [the whole class] all to the ones facts table in multiplication. This will help them focus solely on actually multiplying the numbers and teach them the concept they are almost grasping." Overall, the PSTs desired to see the whole class "do" what they would have done when they were in school. While concerning, these ideas are not surprising considering that PSTs often base their analysis of student work on their personal experiences which tend to be more traditional (Bartell et al. 2013; Goldsmith and Seago 2011; Spitzer et al. 2011).

18.6.2 Vague Next-Step Decisions

The analysis further revealed that although PSTs indicated greater ability to determine instructional next-step decisions as they analyzed children's work, the decisions were mostly vague. For example, when discussing the next steps for an individual student in *Mr. Harris*' class, a group of PSTs at Institution-3 suggested that Molly "work on organization" while another group at Institution-3 suggested that Ananda "try math in her head" but did not provide specific reasons or thoughts (see Fig. 18.1 for Molly's and Ananda's work).

Similar results were seen for the PSTs instructional decisions for a whole class. PSTs at all three institutions made content and pedagogical whole class next-step decisions without specificity. At Institution-1, during the whole class discussion for *Mr. Harris*, PSTs suggested pushing children to use the next strategies in the strategy progression. PSTs did not provide clear instructional strategies as to how they would accomplish this goal, such as a specific story problem with number choices they would pose. The instructor at Institution-1 indicated the difficulty of getting the PSTs to focus more specifically on what they would do. The same results were seen at Institution-2 with whole class decisions on the second assignment. Pair #16 stated, "Give a variety of numbers. Teach [children] different ways to break down harder numbers. Have student focus on anchor numbers first to help them." Pair #9 suggested the whole class "work out a multiplication problem and adding [sic] bigger numbers." These decisions, as well as the majority of the responses received, were vague.

There were a few PSTs at each institution that were transitioning towards providing more specific pedagogical decisions for individual children. For example, when considering Kenneth's work, a group of PSTs from Institution-1 provided specific information on values to use and how to have Kenneth move forward in his knowledge of multiplication (see Fig. 18.3). While there was more specificity as to how Kenneth could use the distributive property to solve the problem, the PSTs were still vague as to how to get him to engage with this property of operation.

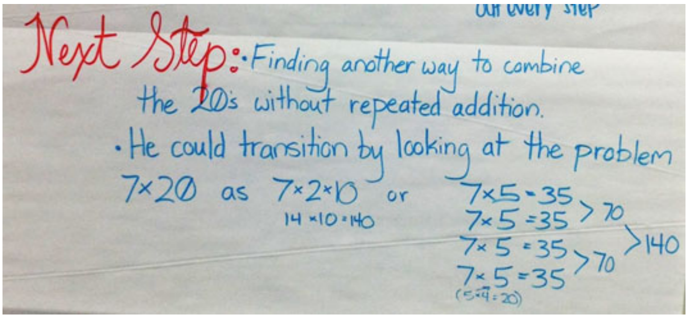

Fig. 18.3 The next step for Kenneth created by a group of PSTs from Institution-1

As another example, PST pair #8 from Institution-2 said, "The next step for Whitney is to pair her with a student that uses repeated addition and have that student help Whitney" (see Fig. 18.2 for Whitney's work). Again, while the PSTs provided a specific pedagogical decision to group children together to exchange ideas about their solution strategies, they did not specify questions or scaffolds to ensure the children worked together. Overall, PSTs at all institutions tended to provide vague next-step instructional decisions for individual children but not so much for the whole class.

18.6.3 Desire for Written Number Sentences

Another significant theme that emerged from the analysis was PSTs' desire for written number sentences. Though this could be related to traditional view of teaching, this emerged as a theme on its own because PSTs' desire for number sentences did not only emerge in next-step decisions but also in helping children clarify their thinking and reflect on their errors. For example, when discussing Kenneth's work (see Fig. 18.1), PSTs at Institution-3 indicated their desire to see him connect his work to multiplication by writing a number sentence. Figure 18.4 shows part of the PSTs' written poster. Similarly, PST pair #2 at Institution-2 commented that "I would show Whitney how skip counting relates to multiplication that way she will not feel the need to add everything. She will be able to write a number sentence using multiplication."

Overall, the PSTs equated written number sentences with the multiplication symbol as more sophisticated than the solution strategies present in the children's work. This belief is inferred from the PSTs' poster at Institution-1 seen in Fig. 18.5. Jacobs and Ambrose (2008) discussed the importance of helping children generate a number sentence "linked to their interpretation of the problem" (p. 265), which is what this PST group appears to have done since the PSTs' number sentences represent Jasmine's work. However, many PSTs indicated a desire for stand-alone number sentences not connected to the children's strategies. For example, two of the three groups at Institution-2 indicated that the next steps for each child was to write number sentences but without any connection to the context of the problem or how each child approached the solution. Consistently, across all institutions, many PSTs held the conception that the children were not "doing multiplication" until

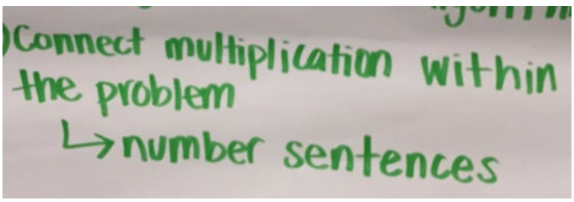

Fig. 18.4 PST next-step decision for Kenneth from Institution-3

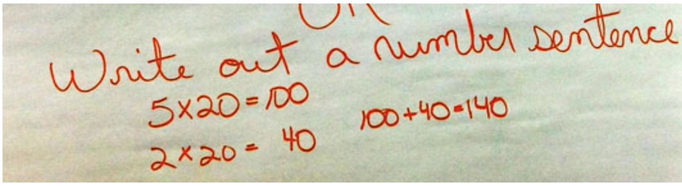

Fig. 18.5 PST next-step decision for Jasmine from Institution-1

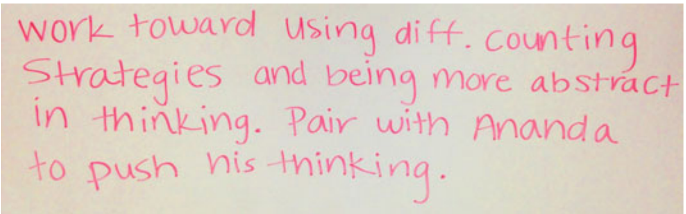

Fig. 18.6 Next steps for Tyrell from Institution-3

they knew how to write out a number sentence that included the multiplication symbol.

While this desire for number sentences was prevalent amongst individual instructional decisions, it was only provided once as a whole class instructional decision by PST pair #2 at Institution-2. Their whole class decision was "writing number sentences using multiplication. Break down sentences to relate skip counting to multiplication." Like most of the examples above, the desire for number sentences was devoid of connections to the children's strategies.

18.6.4 Focus on Strategy Progression

The final theme that emerged from the analysis was PSTs' focus on strategy progression which was a focus of the common lesson. As intended, PSTs at the three institutions often related what they learned in the method's course lesson to their next-step decisions. A group of PSTs from Institution-3 indicated that the next step for Tyrell would be to pair him with another student and "be more abstract" (see Fig. 18.6). In their suggestion, the PSTs identified the next level of sophistication in the solution strategy trajectory as being a counting strategy, however, they did not take into consideration what Tyrell had already done. For instance, he correctly grouped the chairs by ones in sections of ten but rather than writing the number ten, as Ananda did, this student represented each chair individually (see Fig. 18.1). Hence, perhaps before moving to a counting strategy, Tyrell would benefit from working with tens.

Another group from Institution-1, suggested that Tyrell "represent the tallies as numbers, if he counted by ones, then count by 10s or 20s." In this comment, PSTs made explicit connections between what the student did (solved using tally marks) and specifically indicated how he could advance in the solution strategy progression. Rather than Tyrell representing each chair by ones, using the number 10 to represent the group would be more efficient.

Similar results were seen for the PSTs' next-step decisions for the whole group related to strategy progressions. PSTs at Institution-1 discussed what they would do with *Mr. Harris*' class and decided that they would group children together based on the strategy they used. They indicated this would allow them to push the children to use more sophisticated strategies. But as previously mentioned, PSTs were not specific as to how they would accomplish this goal. These general comments about strategy progression persisted with PSTs at Institution-2. PST pair #11 wrote a general suggestion that described the next step for five children she analyzed,

> If these five [children] were my class, the next step I would address for this class would be how to go from skip counting to partitioning strategies. I would do this because only a few [children] attempted partitioning strategies and those that did didn't necessarily solve the problem correctly. I would show them partitioning strategies by relating it to the skip counting that the majority of the [children] already understand (PST pair #11).

This particular pair of PSTs provided a rationale for suggesting partitioning strategies for the children because this was not a common strategy, and one could infer that the PSTs believed it is a more sophisticated strategy that children should learn. However, it was not clear in what ways this partitioning strategy would be used or implemented. In contrast, PST pair #4 was more specific and provided a whole-class decisions based on strategy progression. They would, "introduce strategies each kid used to the whole class. Teach [children] to solve using Oliva's approach and have Oliva help."

Overall, it appeared that the majority of PSTs who focused on strategy progressions did so without much consideration of the children's current understanding, but rather believed that since it was the next level in the trajectory, they should push the children towards that goal without providing specific information as to how to do so.

18.7 Conclusions and Future Work

Exposing PSTs to the ideas behind the professional noticing framework provided them with an understanding and a means of support to analyze children's mathematical thinking which they need as future teachers. Though growth in understanding is still needed, engaging PSTs in the *Mr. Harris* case study (NCTM, n.d.) and instructional activities that focused on professional noticing assisted PSTs in gaining an understanding of the noticing framework, specifically making instructional decisions for individuals and a whole class. As revealed through the data, PSTs tended to gravitate towards traditional teaching ideas and on the mathematical learning progressions presented during the lesson without taking into consideration children's solution strategies when proposing next steps. The PSTs also struggled with providing specific next steps, such as a detailed problem to pose next or the type of support needed to move the whole class forward. This resonates with the research by Ivars and Fernández (this volume) who found that within the narratives written by PSTs, it is difficult for PSTs to provide evidence for children's understanding and they often provide general descriptions with limited connection between their interpretations and the mathematical situation.

Making sound, evidenced-based instructional decisions is tough. Therefore, PSTs need their instructors to expose them to the intricacies of instructional decision making and provide them support as they practice this important skill. Professional noticing of children's mathematical thinking is a skill that can help PSTs analyze student work of any mathematical topic and provides a foundational base for analyzing children's thinking. In subsequent iterations of this lesson that the researchers have taught, the researchers included more explicit introduction and exposure to the professional noticing framework by having PSTs read an article about the framework (Thomas et al. 2015) as well as explicitly using the terms in the lesson (attend, interpret, and decide) as the researchers analyzed student work samples. Moreover, instructors of methods and/or content courses do not have sufficient time to cover all elementary mathematics topics and hence, the need to engage PSTs in professional noticing becomes even greater. The work the researchers have undertaken has PSTs engaged in noticing student work samples removed from a real-time classroom setting so they have the opportunity to reflect and focus on children's mathematical thinking. This could be thought of as a precursor to having PSTs engage in in-the-moment noticing as was studied by Lajoie (this volume).

In conducting this study, the three methods instructors at the different institutions along with two other content instructors from additional institutions spent time reevaluating the effectiveness of the lesson. The collaboration of methods and content instructors brought different perspectives to the lesson in pedagogy and content. Li and Castro Superfine (this volume) recommend mathematics teacher educators (MTEs) work to develop their own knowledge through analysis of their own teaching which grounds the researchers' future work. Thus, the researchers are currently analyzing the lesson using lesson study methodology (Lewis et al. 2009).

The researchers realized that to make the lesson and the learning more effective for the PSTs, several changes needed to be made in the lesson. One of the changes would be to introduce and explicitly teach the noticing framework to the PSTs. It was also realized that there was a need for PSTs to identify and articulate instructional goals of lessons prior to deciding on the next instructional decisions. This lesson showed us that MTEs must also change the way they instruct and question the PSTs as they work with them by focusing the PSTs' attention on posing a specific task for a whole class and making next-step instructional decisions (Appelgate et al. 2016; Broderick et al. 2016). The hope is that by focusing on refining the lesson, the researchers will be able to better engage PSTs with both the mathematical content and more of the critical thinking related to professional noticing that they need in order to make thoughtful next-step decisions.

Furthermore, it is the hope that with the implementation of a carefully studied and redesigned lesson, PSTs will begin making important shifts in certain growth indicators at an earlier time in their careers as opposed to realizing them later on (Jacobs et al. 2010). These growth indicators include: a shift from describing general strategies children use in tasks to being able to decipher key characteristics that describe important mathematical understandings, a shift from general statements about learning to statements specific to children's mathematical knowledge, a shift from a traditional following of the curriculum to deciding the next instructional steps based on children's thinking, current knowledge and anticipated strategies, and a shift from general suggestions for improvement to articulating specific strategies for growth as a result of noticing and analyzing children's work.

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Chapter 19
Understanding the Work of Mathematics Teacher Educators from a Knowledge of Practice Perspective

Wenjuan Li and Alison Castro Superfine

Abstract The work of mathematics teacher educators (MTE) is far from understood. In this study, we explore the nature of knowledge drawn upon MTEs as they connect pre-service teachers' content learning to the practice of teaching mathematics to children. Using data from a two-year project focused on the professional development of university-based teacher educators, we illustrate MTEs' work of teaching pre-service elementary teachers mathematics. In doing so, we identified three MTE practices of connecting to teaching practice, including MTEs supporting pre-service teachers in making sense of and remedying children's errors, MTEs modeling for pre-service teachers how to modify a mathematical task appropriate to children's current level of understanding, and MTEs engaging pre-service teachers to consider children's common conceptions. The nature of knowledge MTEs draw upon in their teaching practices is discussed.

Keywords Mathematics teacher educators · Mathematics content course
Connect to teaching practices · Pre-service elementary teachers

19.1 Introduction

Mathematics teacher educators (MTEs) are professionals who work with practicing and/or pre-service teachers (PSTs) to develop and improve mathematics teaching (Jaworski 2008). In pre-service teacher education in particular, we use the phrase "mathematics teacher educators" to refer to individuals who are primarily responsible for the mathematical preparation of PSTs. Such individuals include mathematicians, education researchers, graduate students, mathematics educators and

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classroom teachers, all of whom have different professional backgrounds and bring to bear a variety of expertise in their work with PSTs (Bergstrom and Grevholm 2008). Despite the critical role MTEs play in preparing PSTs for teaching, the field of teacher education lacks an evidentiary base for understanding the knowledge demands of teacher educators' work. As Jawsorski (2008) argues, an evidentiary base for understanding MTEs' knowledge can provide powerful learning opportunities that can enhance MTEs' knowledge and their own professional development.

While there is a small, yet growing body of research focused on MTEs, much of the extant research in mathematics teacher education has focused on the education of PSTs. For example, what PSTs need to know, what knowledge PSTs bring into teacher education programs, and how course curricula can be designed to improve PSTs' learning have been extensively documented (e.g., Ball et al. 2008; Roland, this volume; Swars et al. 2007; Li and Kulm 2008; Hiebert et al. 2003). However, there has been a sparse synthesis of what MTEs need to know and do in order to support PSTs in teacher education coursework. In this study, we shift the focus of research on mathematics teacher education from what PSTs learn to what the work of MTEs is as they supported PSTs' learning. In our prior study (Castro Superfine and Li 2014a), we explored the knowledge entailed by teaching mathematics in ways specific to teaching PSTs, and illustrated different forms of knowledge observed across different MTEs' practice. In particular, we observed MTEs drawing on different forms of knowledge when connecting PSTs' content learning to teaching practice. Building on our prior research, in the current study we take a different perspective on MTE knowledge and further investigate a form of MTE knowledge we identified in our prior work, and attempt to unpack what makes this form of MTE knowledge unique. Specifically, we focus on the following research question: What is the nature of the knowledge that MTEs draw on as they connect PST's learning of mathematics to teaching children mathematics?

19.2 Theoretical Framework

To explore the nature of the knowledge that MTEs draw upon in their work with PSTs, we first consider what constitutes knowledge. Acknowledging that there are various perspectives, we begin with Cochran-Smith and Lytle (1999) in positing that knowledge supporting the practice of teaching is highly situated and intimately related to individual practice. Cochran-Smith and Lytle (1999) describe three prominent conceptions of teachers' knowledge: *knowledge for practice*, *knowledge in practice* and *knowledge of practice*. They defined knowledge for practice as formal knowledge and theory, such as, subject matter knowledge, pedagogical knowledge, and theories of learning. Additionally, taking courses in a university or participating in professional development workshops, for example, are vehicles for developing knowledge for practice. Knowledge in practice is considered as knowledge that is embedded in practice or in teachers' own reflections on their practice. Teachers' knowledge in practice is often documented in the form of teacher narratives or reflective accounts about their teaching practice. Finally, knowledge of practice is a public form of knowledge generated as teachers collaborate within a broader community (such as, teacher researchers) to address problems related to teaching practice. As Goodwin et al. (2014) argue, knowledge of practice "…bridges the externally consumed 'formal knowledge' related to teaching…(knowledge-for-practice) with the internally generated knowledge embedded in practice (knowledge-in-practice)." (p. 286). In other words, a knowledge of practice perspective can illuminate ways in which MTEs leverage their formal knowledge (i.e., mathematics, pedagogy, classroom teaching) in their daily work of teaching mathematics in ways needed for teaching PSTs. It is this perspective of knowledge that we take in this study.

The elaboration of MTEs' knowledge of practice offers insight into how MTE can further improve their work and better support PSTs' learning. In the analysis of his own professional development, Tzur (2001), for example, proposes a series of four levels of focus that teacher educators progress through on their way to becoming mentors of teacher educators. The four-foci model of teacher educator development includes learning mathematics, learning to teach mathematics, learning to teach teachers, and learning to mentor teacher educators. There is a hierarchy among these four levels. Initially an MTE may attend to questions about mathematics, as they progress through the levels, the questions come to emphasize the meaning of learning mathematics and then the meaning of mathematics teaching. Moreover, consistent with the emphasis on the importance of knowledge of practice, Bergsten and Grevholm (2008) examine MTEs' practices of linking between theoretical course work and teaching practices, and MTEs' knowledge associated with these practices. Specifically, drawing on their reflections, the researchers discuss ways in which MTEs can prepare PSTs to continue to learn from their own practice, and describe how MTEs can leverage PSTs' academic studies to establish a context for PSTs to make connections between the theories they are studying and their teaching experiences. In addition, Lo (this volume) documents the changes in the design of a geometry course for PSTs over a period of three semesters, and reflects on the challenges and opportunities regarding the improved design of the geometry course. The researcher highlights the development of technological pedagogical content knowledge as evidenced in the changes in course design. Together, these studies take a self-study approach to make public MTEs' own reflections about their work and transform their knowledge in practice into knowledge of practice that can guide MTEs' professional development. As a complement to this line of inquiry, in this study, we distance the role of MTE researchers from the role of MTEs and take an outsider perspective to make public one type of MTE instructional practices, namely, connecting mathematics learning to teaching children mathematics.

The practice of connecting mathematics learning to teaching children mathematics is a unique part of the work of teaching teachers. Mason (1998) suggests the work of MTEs involves helping PSTs recognize how to relate what they are learning to teaching children. Specifically, he suggests that the work of MTEs involves developing and enhancing different levels of awareness in PSTs as opposed to simply helping them learn the content that needs to be learned. PSTs need to be able to engineer instructional situations in which children experience a shift in their attention where they (i.e., children) become aware of ideas and concepts of which they were previously unaware. Consequently, the work for MTEs is to develop PSTs' understanding of certain mathematical ideas and concepts, and develop PSTs' awareness of how to connect what they are learning to teaching. For example, MTEs must not only develop PSTs' ability to evaluate the transparency of mathematical ideas in representations for themselves as learners, but also support PSTs in recognizing why evaluating the transparency of representations is important for planning lessons and selecting representations that will support the development of student's understandings. Indeed, we observed the use of what we defined as the practice of connecting to teaching practice in different MTEs' teaching practice (Castro Superfine and Li 2014a; Li and Castro Superfine 2016). Thus, the work of MTEs involves helping PSTs become aware of how what they are learning about mathematics is connected to children's mathematical thinking. Moreover, Mason's (1998) research suggests different knowledge and skills that are necessary for teaching teachers in particular, what Zopf (2010) refers to as mathematical knowledge for teaching teachers. According to Zopf (2010), mathematical knowledge needed for teaching teachers is different from teaching children in the following ways. First, children have informal understandings of mathematics whereas PSTs have more formal understandings of mathematics. Second, the mathematics content that is being taught is different: teachers teach mathematics while MTEs teach mathematical knowledge for teaching (Ball et al. 2008). Finally, the purposes of teaching are different. While children often learn mathematics in order to participate in school and society writ large, PSTs learn mathematical knowledge for teaching in order to teach children.

As the aforementioned research suggests, MTEs need knowledge that is specific to teaching teachers in order to carry out their work. In this study, we posit that scrutinizing MTEs' knowledge in practice can illuminate the nature of their knowledge of practice they draw upon in their work with PSTs. Thus, we aimed to examine MTEs' in-the-moment teaching when they connect PSTs' mathematics learning to the work of teaching mathematics to children.

19.3 Methods

The present study is a part of a larger research project focused on understanding and enhancing MTEs' teaching practice in teaching university-level mathematics content courses for elementary PSTs. The six MTEs, four female and two male, were recruited from two- and four-year colleges and universities in the Midwestern United States. MTEs attended to a series of professional development workshops to discuss aspects of PSTs' learning, to examine teacher educator practices that are supportive of PSTs' development of mathematical knowledge needed for teaching, and to reflect on their own practice and collaborate with other teacher educators (Castro Superfine and Li 2014b). Each MTE was interviewed and asked how their content courses for elementary PSTs are typically designed and implemented throughout the 2-year project. Three MTEs and a majority of their PSTs agreed to be videotaped during the content courses. MTEs and their PSTs who consented were videotaped during 4 lessons after they had participated in the professional development workshops. Each lesson lasted 1.5 h. In the videos, MTEs were teaching one of the compulsory content courses required by elementary teacher preparation programs at their institutions. The content of the course includes a focus on area, perimeter and volume of plane and solid figures, extended solutions of general polygons, as well as statistics and probability.

The data source for our study is videotaped classroom observations from three of the participating MTEs. To unpack the practice of connecting PST's learning of mathematics to teaching children mathematics, we selected two MTEs' videotaped classroom observations. Both MTEs have experience in teaching primary and secondary school mathematics, and their classroom videos exhibit several instances of the practice of connecting mathematics learning to teaching children mathematics. The total length of the videos are 12 h.

Similar to other analyses of mathematics teaching (e.g., Ball et al. 2008), we employed a practice-based theory of mathematical knowledge for teaching. Thames (2008) provides further specification of the methods and design of such a practice-based analytic approach to examine the mathematical demands of the work of teaching, an approach that is grounded in the discipline of mathematics. The type of analysis we employed is "top-down" in that it operates with a set of hypotheses about the particular practices of teaching mathematics to PSTs and also "bottom-up" in that it closely examines what is happening in the university classroom. As such, our analysis is empirically grounded, constantly basing the hypotheses and claims on evidence from the data.

As the goal of the present study is to understand the nature of MTEs' knowledge as used in their teaching practice, we adopted the inductive method (Derry et al. 2010) to conduct the video analysis. First, we viewed the videos several times and highlight critical episodes wherein MTEs made connections between mathematics and teaching children. Next, we conducted an in-depth analysis by describing the nature of the knowledge used by MTEs in these episodes. Specifically, each researcher viewed each video multiple times and highlighted critical episodes individually with the research question in mind. Then, each critical episode was reviewed and discussed by the authors collaboratively, focusing on the following questions: What is the MTE doing? What may be potential pedagogical and mathematical purposes of the instructional moves observed in those episodes? What is the formal knowledge or mathematical knowledge for teaching (Ball et al. 2008), such as, knowledge of children and mathematics, associated with these instructional moves? These questions were developed as the researchers were viewing the videos and discussing the critical episodes. This set of questions guided the researchers to clarify the context in each critical episode and researchers' assumptions about MTEs' instructional intent. It also grounds inference-making about MTEs' knowledge on close examination of MTEs' practices. In addition, the researchers used the refined categories of subject matter knowledge (i.e., common content knowledge, specialized content knowledge, and horizon content knowledge) and pedagogical content knowledge (i.e., knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum) from the mathematical knowledge for teaching framework (Ball et al. 2008) as a tool to capture the subtle nature in MTEs' knowledge. Six critical episodes were validated in order to address our research question. We then applied the constant comparative method (Strauss and Corbin 1998) to the episodes with the purpose of discerning the nature of the knowledge observed in MTEs' practice.

19.4 Findings and Discussion

Three critical episodes with high clarity and representativeness are presented in this section. The rest of three episodes are excluded due to the similar nature of MTE knowledge identified in those episodes. The three episodes illustrate how MTEs support PSTs in making sense of and remedying children's errors, how MTEs model for PSTs how to modify a mathematical task appropriate to children's current level of understanding, and how MTEs engage PSTs to consider children's common conceptions. We first describe the context in which the three episodes happened. We then highlight what we observed the MTEs doing to connect mathematics learning to teaching practice. In doing so, we discuss the nature of the knowledge MTEs seemed to be drawing on in those episodes.

19.4.1 Supporting PSTs in Making Sense of, Remedying Children's Errors

The first episode involves making sense of and remedying children's mathematical errors wherein the MTE provided an opportunity for PSTs to discuss the nature of children's errors and ways in which, as a teacher, PSTs might address those errors. In this episode, three different PSTs present their work on a problem that involved errors in children's thinking. The problem is from the course textbook (Beckmann 2011), which includes two different children's solutions to a mathematics problem that required finding the mean of the number of small candies based on a given dot-plot graph. The problem says: "The dot-plot represents the number of small candies found in several packets. John found the mean number of the candies in the packet by calculating this way:

$$\frac{21+22+23}{3}=22$$

Anne found the mean number of candies in a packet by calculating this way:

$$\frac{4+7+5}{3}=5\frac{1}{3}$$

Is either of these methods correct? If not, explain what is wrong, and explain how to calculate the mean number of candies in the packets correctly" (Beckmann 2008, p. 809). The accompanying dot-plot graph has the title of "the number of small candies found in several packets". The x-axis of the dot-plot shows numbers 20, 21, 22, 23, 24 and 25. There are zero dot above 20, four dots above 21, seven dots above 22, five dots above 23, and zero dot above 24 and 25.

During the PSTs' presentation, the MTE pointed out that such mistakes in the problem were common types of mistakes that children might make.

MTE: You wanna pay attention to Problem number 8 because this is a common mistake that children can make when they're working with averages.

PST 1: The first one, John is saying that there was one bag of candies for each amount. So that's what she's calculating is one bag instead of saying that there's four bags of 21, he only said there's only one bag of 21, one bag of 22, and one bag of 23.

PST 2: For the second one, Anne thought that 4 people found 21 and 7 people found 22, and then 5 people found 23. So she averaged 4, 7 and 5. But the correct answer is…There's 21, 21, 21, 21 [points to the four dots above 21 in the dot-plot], so if you add 21 four times. And then you would add 22 seven times, and then you'd add 23 five times. And you'd add all those. And you divide by 16 because there's a total of 16 of the dots. And then your average would be 22.

PST 3: So the problem was to find the average number of candies, and in the first one they didn't count all of the candies that are given. And in the second one, they didn't count the candies. They were counting the people. So that's why these [solutions] are both incorrect.

After the PSTs explained children's incorrect solutions, the MTE further engaged PSTs in making sense of the errors by pressing PSTs to analyze potential confusions that John might have. John got the correct answer 22, but his reasoning is invalid. By bringing up John's potential confusion, the MTE illustrated that children who have correct answers might not have valid reasoning and provided an opportunity for PSTs to examine why John's reasoning is invalid.

MTE: Now, what if John argues with you and says, "I'm right, I got the same number". And he is gonna say, "I got 22, you got 22. Why am I wrong?" What are you gonna say to him? Somebody from the audience here, explain what they [PST1, PST2, PST3] said. John has an answer of twenty-two. He is gonna argue with you. What are you gonna say?

PST 4: Well, it's not what the problem is asking for. They're asking for the mean of the number of candies, I guess. So, like the average…. I don't know how to explain it.

MTE: So, what do you do to help her?

PST 1: …You could tell the student. And you could show them the graph, and tell them, "Well, you're only seeing that there's that many [points to the top dot above 21] when you put down one 21. You can literally show them, "For twenty-two, you're only seeing that there's this many [points to the top dot above 22]…so, when you add…that's not gonna add up to the number of people because you've literally just divided by three people and that's where you got the three from. They're literally taking all of these people away [points to the extra dots above 21, 22, and 23] and you're just saying that there are three people.

PST 3: You're not using all the data that's given.

MTE: Okay. So you're using the word "people" and I don't see that word, "people", anywhere in the problem.

PST 5: It's kind of confusing, because this graph doesn't show "people."

PST 1: It's not about people. It's about packets. So this packet has 21 candies. This packet has 22. This one has 23….This person found 21 in this packet, but it's really packets. So, sorry if that was confusing.

PST 6: Maybe the student doesn't know how to read the… dot-plot figure?

MTE: Well, that's always a possibility, but does it look like he knows how to compute the averages? From what he did?

PSTs: Yeah.

As PST 1 and 6 pointed out, John did the problem incorrectly because he did not pay attention to what the dots in the graph represented. The 4 dots above the packets with 21 candies indicates that 4 packets have 21 candies. John's calculation of the mean number did not show the total number of candies divided by the total number of packets. The ambiguity of what the dots represent is also evidenced in the PSTs' presentation when some of the PSTs mistakenly refer the number of dots to the number of people. As a summarization, the MTE then confirmed that John did not have a complete understanding of the dot-plot, which might be the reason that he did not exhibit valid reasoning. Furthermore, the MTE highlighted the importance of precise language in PSTs' explanations.

MTE: It looks like he might not know how to read the dot-plot. You're right. I mean, that could be the problem, right? But do you understand that when children get the same answer as the correct answer, but do it the wrong way, you have to be able to convince them of why they're wrong. And they will not bend very easily. You understand? And so I've pushed on the word "people" because the word "people" is not in the problem. You have to be very careful and use the language of the problem. So each dot represents a packet that has candy in it, and the first column says every packet has 21 pieces. You follow? So you have to be very clear with your explanations.

In this episode, the MTE pressed PSTs to make sense of the potential reasoning underlying the errors, asked PSTs to consider how they, as a teacher, would remedy the error, and model the use of precise language to avoid potential confusion. In doing so, the MTE seemed to draw on not only her subject matter knowledge, on finding averages and reading graphs, but also her knowledge of children's thinking, namely that children's correct answers might not be a result of valid mathematical reasoning.

This episode illustrates the commonality between school teachers' and MTEs' knowledge of finding averages and reading graphs, as well as knowledge about the characteristics of children's reasoning. In addition, however, the MTE exhibits a different type of knowledge as she pressed PSTs to make sense of children's mathematical reasoning. That is, the MTE seems to know that PSTs' analysis of children's reasoning does not reveal the nature of the reasoning but only evaluate the validity of the reasoning. Beside the knowledge of content and PSTs, the MTE exhibited knowledge of content and teaching PSTs, which allow her to spot PSTs' confusion about the unit as they interpret the dot-plot and to lead PSTs away from using the incorrect unit "people".

19.4.2 Modeling for PSTs How to Modify a Mathematical Task Appropriate to Children's Current Level of Understanding

The second episode involves modifying a task so as to provide a learning opportunity appropriate for children's current level of understanding. The MTE pressed PSTs to consider children's prior knowledge in relation to the mathematics that lies just beyond children's current understanding. In this episode, PSTs were asked to find the areas and perimeters of a set of regular and irregular geometric shapes. To develop a deep understanding of area and perimeter, the MTE employed a hands-on approach in this lesson. PSTs were asked to cover the shapes using unit squares to find area. The perimeter of shapes was phrased as a bug crawling along the sides of the shapes. The following problem presents a 2×4 rectangle that is divided into two congruent triangles along a diagonal. The problem asks PSTs to find the area and perimeter of the shaded triangle (see Fig. 19.1). After most of the PSTs successfully found the perimeter of the shaded triangle using the Pythagorean Theorem, the MTE asked PSTs to find the perimeter without using the Pythagorean Theorem. By doing so, not only did he provide PSTs opportunities to learn the mathematics that PSTs need for themselves (i.e., find perimeter of a right triangle using the Pythagorean Theorem), but he also made connection between mathematics learning and future teaching by having PSTs consider how to find the perimeter of a right triangle if the Pythagorean Theorem were not taught yet.

A rectangle is divided into two triangles along a diagonal. Find the area and perimeter of the shaded triangle.

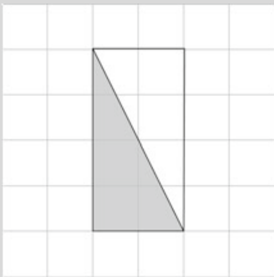

Fig. 19.1 Shaded triangle in a rectangle

When leading the discussion on how to find the perimeter of the shaded triangle, the MTE pressed PSTs to consider how a teacher might engage children in a problem that requires knowledge beyond their current level of understanding. He then modeled two different ways to engage children solving the problem without using the Pythagorean Theorem.

MTE: Could you be the bug, start here [points to the left upper point on the rectangular]? How much did I walk, if I go straight down here [traces down the long leg of the shaded triangle]?

PSTs: Four centimeters.

MTE: Four centimeters, plus…[traces across the short leg]

PSTs: Two.

MTE: And plus…[traces the hypotenuse]. We are running into a problem? It's diagonal, but most kids want to say the length will be…?

PSTs: Four.

MTE: (to PST 7),[1] what do you think?

PST 7: I know you said not to use formulas, but… Since it's a right triangle, you need to use the Pythagorean theorem, although a younger kid would not know how to do that.

MTE: Yeah, exactly. So with younger kids, would you have them find the perimeter for a shape like this one?

PSTs: No.

[1]All participants' names are pseudonyms.

MTE: Certainly not, I don't want for my young kids to stumble at it because the only way to find the exact perimeter is to use the Pythagorean theorem to get that length. I gotta use a formula to figure out what it is. But you can have kids to think about questions like, "Is the length four?" or "Is it longer than four, or is it shorter than four?"

PSTs: Longer than four.

MTE: Longer than four. Great. So if that's four units down along the long leg, would it take me a longer distance to walk over there? Yeah, a little longer, so it's gonna be four point something. So you either need to give kids that length, because they don't know yet how to figure it out using a formula. We have to just be a little bit careful with what kids are able to do. Or we can have kids physically measure it, use measurement to solve the problem.

In this episode, the MTE is connecting PSTs' work on perimeter to ways to engage children with the key mathematical ideas, but in ways appropriate for their current level of understanding. The MTE offered two ways in which a teacher might modify the task such that children can explore the concept of perimeter of a right triangle without confronting the need for using the Pythagorean theorem to find the length of the hypotenuse. One way is to modify the question asked in the problem. As an alternative to finding the perimeter, the MTE suggested that the task could be modified to ask children to determine whether or not the perimeter is larger or smaller rather than to determine a certain numerical value. The other way is to modify the expectations in the task, either to get the length ready for them or allow children to use less rigorous methods to figure out the length. In doing so, the MTE is exhibiting subject matter knowledge and drawing on his knowledge of children's thinking as it relates to teaching.

What makes salient the uniqueness of MTEs' knowledge is that the MTE models the practice of adapting mathematics tasks based on children's prior knowledge. In doing so, the MTE articulates two strategies for adapting the task (i.e., modify the question asked in the problem; provide additional information in the problem), knowledge that arguably most school teachers possess. In addition, however, the MTE also exhibits his knowledge of teaching PSTs, that modeling can engage PSTs in discussing situations with which PSTs have little experience with or familiarity.

19.4.3 Engaging PSTs to Consider Children's Common Conceptions

The third episode involves considering children's common mathematical conceptions or ways of thinking wherein the MTE provided an opportunity for PSTs to think about "good" wrong answers or vague answers that children may offer for a given problem. In this episode, PSTs were exploring the area and perimeter of various 2-dimensional shapes. After working on problems about finding areas of irregular shapes, the MTE invited PSTs to present their work to the class as if they were teaching the problems to children.

MTE: Let's get a couple of people to talk us through when we are thinking about area or perimeter here. As you're going up there, I really want you to think about how you are going to explain it to children. Most children say area is…?

PSTs: Length times width.

MTE: Length times width. There's a good wrong answer for what area means. That works for what type of shapes?

PSTs: Rectangles and squares

MTE: It works for rectangles or squares, but does it work for every shape? Certainly not, okay? What's a vague answer? Area is…

PST 8: How much stuff is inside.

MTE: Yeah, how much stuff is inside. Okay. Great. And we want to try to get more specific than that, ok? And so, Dan, what could you, what would you say?

PST 9: Um, how many squares are inside the shape?

MTE: Yeah, great! How many squares fill up the shape. And does it matter? What if it's a triangle?

PST 9: It doesn't matter.

MTE: It's still how many squares fill up the shape. But what if it's a circle?

PSTs: Still how many squares fill up the circle?

MTE: Still how many squares fill up the shape. Great. (To PST 9) So can you start us off and talk us through that first one, what you got for area and perimeter. Show us how you would teach it to kids.

In this episode, the MTE is connecting PSTs' work on area and perimeter problems to children's conceptions about the topic. The MTE had PSTs consider a good wrong answer and a vague answer that children might offer in response to the question about what is area. In discussing what a good wrong answer may be, the MTE made explicit that children often think of area as length times width because they often limit their thinking to rectangles and squares without considering other shapes. When the MTE asked about what a vague answer might be, he pointed out that the ambiguous response, "How much stuff?" should be made more precise with mathematical words such as "squares." In addition, when discussing a good wrong answer, the MTE pressed PSTs to verify that the definition worked for other shapes, not only for rectangles and squares. Taken together, in this example, the MTE is exhibiting subject matter knowledge, specifically that length times width is a way to calculate area of rectangles and squares. It is not, however, a sufficient definition of area since this formula is not applicable to the range of other shapes children might encounter in the school curriculum. In addition, the MTE is exhibiting knowledge of children's mathematical thinking (i.e., common misconceptions and informal conceptions about area) in facilitating PSTs' consideration of children's common conceptions about the topic.

This episode illustrates a distinctive instance often seen in MTEs' work but not in school teachers' work. Even though school teachers have knowledge of children's misconceptions of area, they are likely to address the misconceptions through a hands-on activity, such as, measuring the area of shapes with grid paper. In contrast, in the case of PSTs, because PSTs generally have more complex mathematics knowledge structures than children do, the MTE draws on PSTs' knowledge about the area formula varying for different geometric shapes they know and guides them to mathematically valid definitions of area.

19.5 Conclusion

While researchers generally agree that the work of MTEs involves working with PSTs to improve and develop the teaching of mathematics, the knowledge MTEs use in their work is far from understood. In this study, we employ a knowledge of practice perspective to bridge formal knowledge about teaching with MTEs' knowledge embedded in practice. We describe three episodes in which they connected PSTs' mathematics learning to teaching practice: how MTE's help PSTs to make sense of and remedy children's errors; how MTEs model for PSTs how to modify a mathematical task appropriate to children's current level of understanding; and how MTEs engage PSTs to consider children's common conceptions. In doing so, we unpack the knowledge that MTEs' draw on while interacting with PSTs in university-based courses. In this section, we first highlight the unique aspect of MTEs' knowledge in this analysis as compared to the work of K–12 teachers, and discuss what makes MTEs' knowledge of teaching mathematics to PSTs unique. Then, we discuss how this analysis of MTEs' knowledge in practice, particularly, their in-the-moment teaching, contributes to an evidentiary base for understanding MTEs' work more broadly.

This study provides empirical evidence of the uniqueness of MTEs' knowledge as they connect PSTs' content learning to children's mathematics learning. The three episodes analyzed suggest that MTEs not only know the subject matter knowledge, children's mathematical thinking, teaching mathematics to children as school teachers do, but also know how to connect PSTs' mathematics learning to the work of teaching children mathematics. Specifically, the first episode illustrates MTEs' unique knowledge of building on children's reasoning to develop PSTs' content knowledge (i.e., finding averages, reading graphs) and knowledge of content and children (i.e., children's correct answer might not from valid reasoning); The second episode shows MTEs' unique knowledge of modeling to develop PSTs' content knowledge (i.e., finding perimeter, Pythagorean Theorem) and knowledge of content and teaching (i.e., modifying mathematical task). The third episode illustrates a distinctive instance of MTEs' knowledge of capitalizing on PSTs' subject matter knowledge about area formula varying for different geometric shapes and guiding them to develop mathematically valid definitions of area. According to the unique MTE knowledge observed, we note that the uniqueness of MTEs' knowledge stems from how knowledge is used and for whom. It is perhaps not surprising that there is so much commonality between MTEs' and teachers' mathematics knowledge, for example, the common content knowledge that children, teachers, and MTEs should know about, or the knowledge of children's mathematical thinking that teachers and MTEs draw on in their teaching. The three episodes in this study, however, illustrated that the ways in which MTEs use their knowledge of children's mathematics thinking and their knowledge of teaching children mathematics are distinct from how teachers would use this knowledge. Therefore, we consider MTEs' knowledge not merely parallel to teachers' knowledge, but rather a variation of and expansion on teachers' mathematics knowledge for teaching.

In addition, this study illustrated that a fine-grained analysis of MTEs' in-the-moment teaching can be a useful tool to explore the unique knowledge drawn upon by MTEs in their work with PSTs. On one hand, MTEs' in-the-moment teaching represents the immediate and daily work MTEs engage in, or MTEs' knowledge in practice, and the moment-to-moment actions of teaching were influenced largely by MTEs' knowledge, or MTEs' knowledge for practice. Thus, examining in-the-moment teaching allows researchers to identify the unique practice in MTEs' work that is likely to lead to the formulation of distinctive aspects of MTE knowledge. On the other hand, fine-grained analysis of MTEs' moment-to-moment interactions with PSTs provides details on how MTEs' work differs from school teachers. This type of analysis is likely to lead to the identification of subtle knowledge forms that MTEs use even when they seem to be doing work similar to school teachers.

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Chapter 20
Perspectives on Noticing in the Preparation of Elementary Mathematics Teachers

David K. Pugalee

Abstract The development of practice is at the heart of educational programs for pre-service teachers and professional development opportunities for educational professionals. A key focus of this work is to 'make sense' of the classroom including students' thinking, discussions, and work so that this sense making provides useful contexts and information for the teacher to interpret as a basis for instructional decisions. This idea of 'making sense' is replete with different perspectives on exactly what it means to make sense. Noticing is a process that provides an approach to this sense making. In this chapter, van Es and Sherin's (J Technol Teacher Educ 10:571–596, 2002) definition explicated through three critical components of noticing will be used to better understand how noticing supports pre-service mathematics teachers make sense of the complexity of classroom events. These three components are (a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions (p. 573).

Keywords Noticing · Instructional Practice · Teaching
Elementary pre-service teachers · Pedagogy · Mathematics Methods

20.1 Introduction

Classrooms are complex environments where multifaceted decisions shape the teaching and learning landscape. Refining pedagogical practice requires 'noticing' salient features of instruction and how those features affect instruction. Understanding how noticing develops and how shifts in teachers' noticing

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behaviors occur is critical to understanding this process as a vehicle for positive instructional change. One definition of noticing views it as a process whereby particular features are mentally isolated resulting in the creation and re-presenting of metal records for those features, and then from those features identify particular regularities, properties, features, and/or conceptual objects (Hohensee 2016; Lobato et al. 2013). Philipp et al. (2014) posit that noticing is different from knowledge and beliefs since noticing involves an interactive and practice-based process rather than naming a cognitive resource. Noticing in mathematics education, as such, focuses on how teachers interact with instructional situations which are practice-based thereby making the nature of noticing complex and demanding to develop. The complex process of 'noticing' is characterized by three central components: (a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom events (van Es and Sherin 2002, p. 573). These three central components serve as a useful framework to describe current perspectives on noticing related to elementary mathematics teacher preparation. These three components are not always exclusive and the following perspectives sometimes overlap; however, the discussions explicate primary ideas relative to each of these central components to the act of noticing.

20.2 Identifying What Is Important

What is identified as important about classroom situations through noticing reflect Shulman's perspective (1987) that successful teachers must develop good subject matter knowledge; content and general pedagogy; understanding of the curriculum, learners, and learner characteristics, knowledge of educational contexts; and awareness of educational ends, purposes, values as well as their philosophical and historical foundations. Osmanoglu et al. (2015) conducted a study that included investigating what prospective elementary mathematics teaching noticed. The pre-service mathematics teachers' reflections and interviews showed that what was attended to reflected specific teacher knowledge domains and that noticing increased over time. Through data analysis three main issues reflecting specific domains of teacher knowledge were identified: Pedagogical Content Knowledge (PCK), General Pedagogical Knowledge (GPK), and Curriculum Knowledge (CK). These domains reflect the critical elements that are the focus of noticing. Pedagogical content knowledge (Shulman 1987; Osmanoglu et al. 2015) focuses on how to teach and includes subject matter content and how to best represent ideas and concepts to facilitate student understanding. General pedagogical knowledge relates to more generic skills such as classroom management and organization strategies. Curriculum knowledge deals with how topics in the curriculum are arranged to maximize learning and how resources are used as part of organizing instruction. The researchers found that video-case discussions provided the teacher candidates with opportunities to raise awareness of teacher knowledge and its relationship to effective teaching. The researchers posit that tools to support pre-service teachers in understanding the teaching process are critical in improving noticing skills.

Lajoie (this volume) explicates this complexity of learning to act in-the-moment as a process through a study focused on pre-service teachers' role-playing the use of a calculator in developing number concepts. In a methods course students took the role of teacher while others simulated the role of students. The role playing supported an educational awareness of the student, learner, and observer roles. Through considerations of preparation for the role play, the pre-service teachers could imagine the appropriate actions of both the teacher and the pupil thus sharpening their noticing in the moment. A critical component of this process was the discussion or 'debrief' where the discussions promoted collective and explicit awareness as participants focused on what was performed in the moment and what could have been performed to strengthen student learning. Role-playing appeared to build a culture of openness allowing the pre-service teachers to identify what was noteworthy in the classroom simulation.

Ivars and Fernandez (this volume) explore the use of writing narratives in the development of pre-service teachers noticing. The narratives involved describing, interpreting, and completing a situation related to practices at schools. Through the writing pre-service teachers described and interpreted critical events related to students' mathematical understanding. At first the teacher candidates demonstrated personal struggles in providing evidence of student understanding. After feedback on the narratives, pre-service teachers were then able to give specific details and examples of student understanding. The narratives provided a useful tool that promoted how the pre-service teachers focused on mathematical thinking and justifications for their next teaching actions. The researchers propose future studies focusing on online debate as a support for the teacher candidates to focus attention on interpreting students' understanding. The focus on writing narratives introduces tools which are less frequently employed in developing noticing behaviors. The importance of writing to noticing is promoted by Mason (2002) who identified three levels of noticing. The first level is 'ordinary' noticing in which one's memory is jogged when someone else provides a cue. The second level is 'marking' where one makes a sufficient notice to remark upon something to someone else. The third level is 'recording' where one makes a note of something, typically in written form which promotes describing or defining. According to Mason, each of these levels requires added focus in order to enact it. Thus the written narratives used by Ivars and Fernandez provide a tool to reach the highest levels of noticing.

Research on pre-service mathematics teachers' noticing also identifies some challenges in terms of the nature of what is attended to or noticed by the candidates. A study of 169 student teachers' video recorded lessons (Vondrová and Žalská 2015) identified the mathematics specific phenomena in their lessons. The study found that pre-service teachers give limited attention to the mathematics content in their lessons, they are likely to notice specific mathematics phenomena which are deemed as not important by experts, and their ability to notice specific mathematics phenomena does not differ significantly across stages of a preparation program. Tenable explanations for some of these findings include the possibility that important events are inherently more difficult to notice and the most attention grabbing attributes may not be those that are most important in effective instruction as identified by experts. Another possibility is that students may not have developed the skills allowing them to distinguish between important and less important features of a lesson. These possibilities underscore the importance of carefully selected experiences in programs of study that address these challenges and improve pre-service mathematics teachers' ability to attend to important mathematics phenomena.

20.3 Making Connections

Once components of classrooms are identified as noteworthy, educators engage in reflective thinking to make connections between those noteworthy components and the teaching and learning principles that they represent. This is a complex cognitive process involving teachers' situation specific skills. A review conducted of empirical mathematics education research (Stahnke et al. 2016) related to these situation specific skills provides some key insights. One of the most frequent frameworks employed in these studies was teachers' noticing or teachers' professional vision. The analysis found that the expertise of teachers as well as their experiences positively influence noticing; mathematical knowledge performs an important role particularly in the interpretation of student work; teaching in the moment decisions are influenced by knowledge, beliefs, and goals; aspects of knowledge and beliefs predict situation specific skills which are related to instructional practice; and teachers have difficulty interpreting tasks and describing their learning potential. It should come as no surprise that pre-service teachers' expertise and experience are significant in their noticing. Pre-service mathematics teachers also experience roadblocks in their perception and interpretation of student work with these skills tied to their level of mathematical knowledge. The importance of connections between teacher's mathematical knowledge, beliefs, and goals and the act of noticing points not only to the supports selected to foster noticing but also how those supports and their content are organized. These areas of concern contribute to inhibited decision making. The Stahnke et al. study (2016) found deficits in terms of teachers' abilities to propose relevant and effective instructional strategies that would foster students' conceptual understanding in mathematics. Both a teacher's mathematics content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK) are predictors of situation-specific skills including planning of actions which are correlated with instructional quality (also see Dunekacke et al. 2015; Blömeke et al. 2015).

Gupta et al. (this volume) link noticing and deciding as next steps for instructional decisions in research with pre-service teachers in mathematics methods and content courses. The researchers sought to fill a gap in noticing of pre-service teachers using student work and linking next steps to the whole class. Findings suggest that pre-service teachers' next-step decisions for individuals and the whole class tend toward traditional teaching, are vague suggestions, focus on a desire for written number sentences, and emphasize strategy progression. The data demonstrates the difficulty pre-service teachers face in making meaningful connections between classroom interactions and relevant principles of teaching and learning represented in the instructional decisions (van Es and Sherin 2002). The researchers also found that the pre-service teachers focused on strategies progression but did not demonstrate a consideration for the level of understanding in the students' responses. The pre-service teachers appeared to believe that students should move toward the next level in the associated trajectory. The researchers continue their work, currently focusing on engaging pre-service teachers on a specific task as a whole class. The goal is to support growth indicators in noticing that include a shift from describing general strategies employed by students to understanding and interpreting key characteristics important to mathematical understanding, a shift from general statements related to learning to specific statements specific to student's mathematical knowledge, a shift from traditional teaching perspectives characterized by traditional following the curriculum to instructional decisions based on children's thinking, current knowledge and anticipated strategies, and a shift to formulating specific strategies that result from noticing and analysis of student work.

These connections are explicated in Ball's (Ball et al. 2008) concept of 'knowledge of content and students' (KCS), which is a subcomponent of pedagogical content knowledge (PCK). In this subcomponent, content knowledge and knowledge of students are connected as the primary force undergirding teaching actions. Teachers anticipate the likely mathematical thinking of students including potential misconceptions and confusion. According to Stephan et al. (2016), anticipating how students will reason about a mathematics problem including a process of imaging both conceptions and misconceptions and how those might become public through instructional actions designed to promote learning for all students. In this imaging of the mathematics lesson, the teacher connects their content and pedagogical knowledge to anticipate possible student solutions and to plan a sequence of instructional actions based on these anticipated student outcomes so that student thinking is used as a vehicle to support effective inquiry teaching. These connections are illustrated in a study by Barnes and Solomon (2013) involving a teacher development program focused on developing a deeper understanding of pedagogical subject knowledge in mathematics through researching their practice and development of a critical reflexivity. Noticing was used as a tool for support in which the teacher records microincidents in the classroom and engages in subsequent reflection designed to facilitate drawing back from immediate practice so the teacher notices things previously overlooked or by habituation become insignificant. This perspective from the teacher as a participant enables sensitivity to context and promotes a response to learner. Small teaching episodes with subsequent reflection allow the teacher to take on the role of 'researcher from the inside'. This role moves the participant beyond examining micro incidents within their teaching practice to noticing at a macro strategic level. This case study demonstrated that skilled ‘researcher from the inside’ is able to switch between these two types of noticing so that they are able to establish awareness at both the pupil level and the wider strategic level. Teachers who connect practice with principles of teaching and learning are able to investigate the teaching and learning that occurs in their own classrooms, question and challenge the existing discourses and practices, and experiment with new ideas and reflect on the outcomes at both the micro and macro levels.

20.4 Reason About Classroom Interactions

As teachers develop their ability to notice they become agents to transform their own mathematical teaching practice. According to McDuffie et al. (2014), it is important for prospective teachers to move from attending primarily to their own instructional moves and describing what they noticed to becoming aware of significant interactions and reason about the effects of these interactions on students’ mathematical learning. The four lenses described by the researchers offer an in-depth analysis of what it means to notice in ways that promote reasoning about classroom interactions. The four lenses, each with a series of prompts, are teaching, learning, tasks, and power and participation. The teaching lens focuses on how the teacher elicits student thinking and responds. The learning lens emphasizes the specific mathematical understandings and/or confusions and how they are indicated in students work, talk, and—or behavior. The tasks lens considers the richness of the task and how it might be improved to maximize learning. The power and participation lens centers on who is participating and how the classroom culture values and encourages full student participation. Through the activities which served as a decomposition of practice, pre-service teachers engaged in substantive analysis and interpretation of observed events that included relevant evidence for any claims made. The data showed ability to interpret and reason about how and why teaching and learning unfolded in certain ways. The pre-service teachers increased in their noticing including an awareness of student resources and noticing ways in which those resources can support mathematics learning.

Li and Superfine (this volume) extend this conception of pre-service teachers’ noticing to a practice perspective of mathematics teacher educators. Through a knowledge of practice perspective, they unpack the knowledge that MTEs draw on to work with PSTs across their courses. Their research identifies three episodes that demonstrate how teacher educators support pre-service teachers in making sense of, remedying children’s mathematical errors; demonstrate, through modeling, the modification of a task aligned to students’ current mathematical understanding; and engage pre-service teachers to consider student’s common mathematical misconceptions. These episodes express how mathematics teacher educators unpack their knowledge while working with pre-service teachers in university-based courses. An important contribution of the research was how a fine-grained analysis of MTE’s in-the-moment teaching can be useful in identifying the unique practices in teacher educators work that are likely to lead to building distinctive aspects of mathematics teacher educator knowledge. These practices uniquely demonstrate how noticing is important to mathematics teacher educators as they explore how what they know affects their reasoning about classroom events.

A longitudinal study investigating the relationships between mathematics teacher preparation and the graduates' analysis of classroom teaching found that the participants performed significantly better on three tasks that were a focus of the program compared to topics not included in their studies (Hiebert et al. 2017). This supports a premise that teachers must first see mathematics teaching differently before they can make relevant changes to their teaching practices. This study measured teachers' tendency to notice particular aspects of mathematics teaching. One of those categories, proposing alternative methods, was the most highly correlated with the quality of teaching and the students' mathematics learning. Proposing alternative methods require both mathematics knowledge for teaching and advanced noticing skills. This in depth reasoning about classroom interactions and student learning occurs when teachers are more able to attend to the mathematics in the classroom interactions. The study highlights the critical importance of an emphasis on moves that support students' conceptual understanding where key mathematical ideas are made explicit and students have opportunities to grapple with those ideas. The researchers conclude that "at the most advanced level, noticing the mathematics involves cause—effect reasoning that suggests changes, where appropriate, to the mathematics discussed in the classroom to improve the conceptual learning opportunities for students" (p. 9).

20.5 Conclusion

Noticing is an essential element in effective teaching and the improvement of practice. Mason (2002) argues that noticing is a collection of practices that enables us to develop sensitivity to recognize opportunities which arise in our practice. Noticing results in our ability to formulate alternative options as we reason about classroom interactions and students' learning. Current perspectives on noticing described in this chapter provide multiple ideas about engaging pre-service mathematics teachers as they attend to mathematics and instructional phenomena, connect the specifics of classroom interactions with principles of teaching and learning, and use what they know as they reason about classroom interactions (van Es et al. 2017; van Es and Sherin 2002). Vondrová and Žalská (2015) identified a major challenge for pre-service teachers in attending to the mathematics content in their lessons. Several tools do demonstrate promise in developing noticing including writing narratives (Ivars and Fernandez, this volume), role-playing (Lajoie, this volume), and video cases (Osmanoglu et al. 2015). In terms of making connections between classroom interactions and teaching and learning principles, a review of empirical studies (Stahnke et al. 2016) identified expertise, experience, mathematical knowledge, knowledge, beliefs, and goals as major factors influencing noticing. The review also identified a challenge that teachers face in interpreting tasks and describing their learning potential. Analysis of student work is a promising approach in addressing the gap for pre-service teachers in using student work to link next steps to a whole class (Gupta et al., this volume). Developing a critical reflexivity was identified by Barnes and Solomon (2013) as an effective way to develop the 'researcher from the inside' perspective that fosters the investigation of teaching and learning in one's own classrooms. McDuffie et al. (2014) identify four lenses (teaching, learning, tasks, power and participation) that provide pre-service teachers with a framework to support noticing significant classroom interactions and reasoning about them. The importance of such experiences on future teaching effectiveness was highlighted in a longitudinal study focusing on the relationship between mathematics teacher preparation and graduates' classroom teaching (Hiebert et al. 2017). Li and Superfine (this volume) extend the importance of noticing to mathematics teacher educators with research demonstrating the role of noticing in how teacher educators reason about classroom events with pre-service teachers.

Practices and instructional approaches for noticing should be at the core of mathematics education programs with a focus on the three critical components used as a framework for this chapter. Noticing supports pre-service teachers as they develop the capacity to examine the relation between student thinking, teaching practice, and mathematical content (van Es et al. 2017). The studies highlighted in this chapter underscore the challenges in developing noticing and its positive impacts on teacher practice and student learning through various tools and resources. Given the strong evidence in the literature, elementary pre-service mathematics teacher education programs should strive to develop core practices that engage teachers in developing noticing skills. Through these efforts there is the promise of significant and sustained improvements in practice.

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