# Bifurcations and Stability Regions of Nonlinear Dynamical Systems

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## 1 Stability Regions of Nonlinear Dynamical Systems

Attractors of nonlinear dynamical systems are rarely globally stable. Actually, there exists a subset of the state space, called *stability region*, composed of all the initial conditions that have trajectories approaching the attractor as time tends to infinity. Region of attraction, area of attraction and basin of attraction are other names commonly employed in the literature for stability region.

In this chapter, we will study stability regions of the following class of nonlinear dynamical systems:

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$-function. We will assume that solutions of (1) are defined for all $t \in \mathbb{R}$ and the solution of (1) passing through $x_o$ at time $t = 0$ is denoted by $\varphi(t, x_o)$.

**Definition 1** (*Invarinat Set*) An invariant set $\gamma$ is an attracting set of system (1) if there exists a neighborhood (open set) $N$ of $\gamma$ such that $\varphi(t, x) \to \gamma$ as $t \to \infty$ for all $x \in N$.

**Definition 2** (*Stability Region*) The stability region $A(\gamma)$ of an attracting set $\gamma$ of system (1) is the set:

$$A(\gamma) = \{x \in \mathbb{R}^n : \varphi(t, x) \to \gamma \text{ as } t \to \infty\}$$

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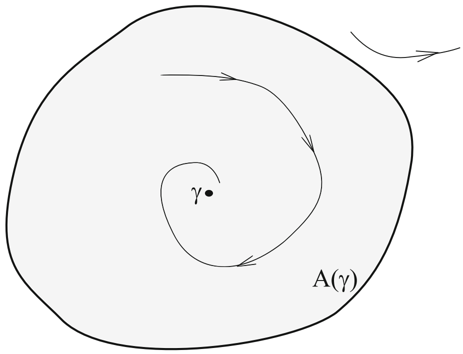



**Fig. 1** Stability region of an attracting set $\gamma$

Asymptotically stable equilibrium points and asymptotically stable limit cycles are examples of attracting sets. Figure 1 illustrates the concept of stability region. The stability region $A(\gamma)$ is an open and invariant set. Its topological boundary will be called stability boundary and will be denoted $\partial A(\gamma)$. The stability boundary is a closed and invariant set [8].

Determining stability regions is relevant in many areas of sciences. For instance, stability regions plays an important role in the assessment of transient stability in electrical power systems [8] and in the problem of immunization in biological systems [13]. In the process of determining or estimating stability regions, the determination or estimation of the stability boundary is relevant. In the next subsections, the existing theory of characterization of stability regions and stability boundaries is reviewed. Invariant sets on the stability boundary play an important role in the theory of stability boundary characterization. In Sect. 1.1, the characterization of hyperbolic equilibrium points on the stability boundary is studied while Sect. 1.2 studies closed orbits on the stability boundary.

## *1.1 Hyperbolic Equilibrium Points on the Stability Boundary*

In this section, a characterization of the stability boundary of a fairly large class of dynamical systems is developed. This class is composed of the dynamical systems that admit hyperbolic equilibrium points as the only type of critical element (minimal invariant set) on the stability boundary.

A key point to derive a characterization of the stability boundary is to understand the relationship between the critical elements and the stability region and its boundary. Theorem 1, proven in [6], establishes this relationship offering necessary and sufficient conditions for a hyperbolic equilibrium point lying on the stability boundary.

**Theorem 1** (Equilibrium Points on the Stability Boundary) [6] *Let* $x^s$ *be a hyperbolic asymptotically stable equilibrium point of (1) and* $A(x^s)$ *be its stability region. If assumptions:*

*(**A1**) All the equilibrium points on* $\partial A(x^s)$ *are hyperbolic,*
*(**A2**) The stable and unstable manifolds of equilibrium points on* $\partial A(x^s)$ *satisfy the transversality condition,*
*(**A3**) Every trajectory on* $\partial A(x^s)$ *approaches one of the equilibrium points as* $t \to +\infty$*,*

*are satisfied and* $x^*$ $(x^* \neq x^s)$ *is a hyperbolic equilibrium point of (1). Then the following statements are equivalent:*

*(i)* $x^* \in \partial A(x^s)$
*(ii)* $W^u(x^*) \cap A(x^s) \neq \emptyset$
*(iii)* $W^s(x^*) \subseteq \partial A(x^s)$.

Figure 2 illustrates the conclusions of Theorem 1. Next theorem, proven in [6], extends the characterization given in Theorem 1 by asserting the stability boundary $\partial A(x^s)$ is the union of the stable manifolds of the equilibrium points on $\partial A(x^s)$.

**Theorem 2** (Stability Boundary Characterization) *Let* $x^s$ *be a hyperbolic asymptotically stable equilibrium point of (1) and* $A(x^s)$ *be its stability region. If assumptions* $(A1)$ *and* $(A3)$ *are satisfied, then:*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x^i),$$

*where* $x^i$, $i = 1, 2, \ldots$ *are the equilibrium points on* $\partial A(x^s)$*. If, additionally,* $(A2)$ *is satisfied, then:*

$$\partial A(x^s) = \bigcup_i W^s(x^i).$$

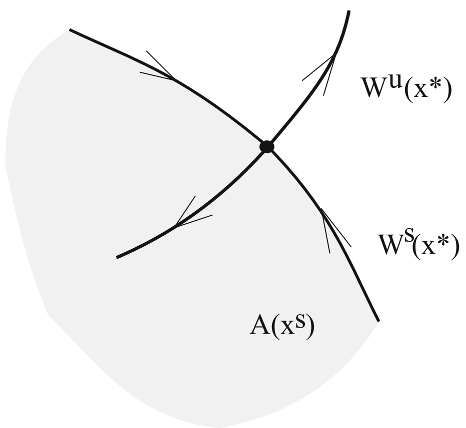



**Fig. 2** The equilibrium $x^*$ is on the stability boundary. Its stable manifold $W^s(x^*)$ lies on the stability boundary and the unstable manifold $W^u(x^*)$ has a non empty intersection with the stability region $A(x^s)$

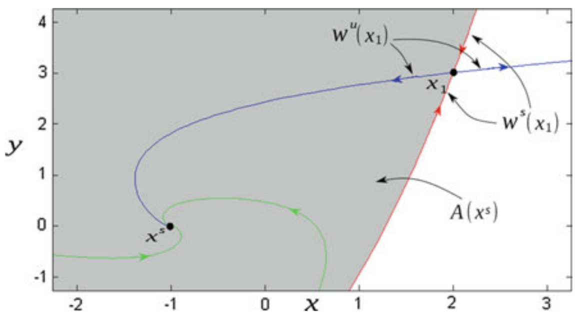



**Fig. 3** Phase portrait of system (2). The stability boundary of $x^s = (-1,\ 0)$ is formed by the stable manifold $W^s(x_1)$ of the type-1 hyperbolic equilibrium point $x_1 = (2, 3)$

The following example illustrate the characterization of the stability boundary of Theorem 2.

*Example* Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= x^2 - y - 1 \\ \dot{y} &= x - y + 1 \end{aligned} \tag{2}$$

where $(x,\ y) \in \mathbb{R}^2$. System (2) possesses two equilibrium points, an asymptotically stable equilibrium point, $x^s = (-1, 0)$, and a type-1 hyperbolic equilibrium point $x_1 = (2, 3)$ on the stability boundary $\partial A(x^s)$. Since all equilibrium points are hyperbolic, then assumption (A1) is satisfied. In agreement with Theorem 1, the unstable manifold $W^u(x_1)$ of the hyperbolic equilibrium point $x_1 = (2, 3)$ intersects the stability region $A(x^s)$ and its stable manifold $W^s(x_1)$ is contained on the stability boundary $\partial A(x^s)$. The stability boundary $\partial A(x^s)$ is the stable manifold $W^s(x_1)$ of the type-1 hyperbolic equilibrium point $x_1 = (2, 3)$, in agreement with the results of Theorem 2. See Fig. 3.

## *1.2 Closed Orbits on the Stability Boundary*

In this section, the stability boundary characterization of Sect. 1.1 is extended to accommodate periodic orbits on the boundary of stability regions.

**Definition 3** (*Critical Element*) A critical element $\phi$ of the autonomous dynamic system (1) is either a closed orbit or an equilibrium point.

The next theorem, proven in [6], establishes necessary and sufficient conditions for a critical element point lying on the stability boundary.

**Theorem 3** (Critical Element on the Stability Boundary) [6] *Let $x^s$ be an asymptotically stable equilibrium point of (1) and $A(x^s)$ be its corresponding stability region. Let $\phi$ be a critical element. If assumptions:*

***(B1)*** *All the critical elements of (1) on* $\partial A(x^s)$ *are hyperbolic,*
***(B2)*** *The stable and unstable manifolds of critical elements of (1) on* $\partial A(x^s)$ *satisfy the transversality condition,*
***(B3)*** *Trajectories on* $\partial A(x^s)$ *approach one of the critical elements of system (1) as* $t \to +\infty$*,*

*are held, then the follwoing statements are equivalent:*

*(i)* $\phi \subset \partial A(x^s)$
*(ii)* $W^u(\phi) \cap A(x^s) \neq \emptyset$
*(iii)* $W^s(\phi) \subset \partial A(x^s)$.

Theorem 3 is an extension of Theorem 1 and offers a local characterization of the boundary of the stability region in the neighborhood of critical elements. The following theorem, proven in [6], develops a global characterization of the stability boundary. Under assumptions (**B1**)–(**B3**), it asserts the stability boundary is the union of the stable manifolds of the hyperbolic critical elements on the boundary of the stability region.

**Theorem 4** (Stability Boundary Characterization) [6] *Let* $x^s$ *be an asymptotically stable equilibrium point of (1) and* $A(x^s)$ *its stability region. If assumptions (**B1**) and (**B3**) are held, then:*

$$\partial A(x^s) \subset \bigcup_i W^s(x_i) \bigcup_j W^s(\phi_j)$$

*where* $x_i,\ i = 1,\ 2,\ \ldots$ *are the equilibrium points and* $\phi_j,\ j = 1,\ 2,\ \ldots$ *are the closed orbits in* $\partial A(x^s)$*. If, additionally, assumption (**B2**) is satisfied, then*

$$\partial A(x^s) = \bigcup_i W^s(x_i) \bigcup_j W^s(\phi_j).$$

## *1.3 Energy Functions and Stability Boundary Characterization*

The characterizations of stability boundary given in Sects. 1.1 and 1.2 are given in terms of stable manifolds of critical sets. These manifolds are difficult to compute, specially in high dimensional systems. Despite that, level sets of energy functions provide concrete estimates of stability regions and stability boundaries, moreover, energy functions have important implications on the stability boundary characterization.

Consider the nonlinear dynamical system (1) and let $E := \{x \in \mathbb{R}^n\ :\ f(x) = 0\}$ be the set of all equilibrium points of (1). The following definition of energy function was firstly proposed in [7].

**Definition 4** (*Energy Function*) A $C^1$-function $V : \mathbb{R}^n \to \mathbb{R}$ is an energy function of system (1) if the following conditions are satisfied:

(i) $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$.
(ii) if $x_0 \notin E$, then the set $\{t \in \mathbb{R}_+ \ : \ \dot{V}(\varphi(t, x)) = 0\}$ has zero measure in $\mathbb{R}$.
(iii) if $V(\varphi(t, x_0))$ is bounded for $t \in \mathbb{R}_+$, then the trajectory $\varphi(t, x_0)$ is bounded for $t \in \mathbb{R}_+$.

The existence of an energy function guarantees that every bounded trajectory must approach an equilibrium point as $t \to +\infty$. As a consequence, complex behavior such as closed orbits and chaos cannot exist for systems that admit energy functions. Moreover, the existence of an energy function ensures that every trajectory on the stability boundary is bounded, although the stability boundary can be unbounded, and converges to an equilibrium point on the stability boundary as $t \to +\infty$ [6]. In other words, the existence of an energy function is a sufficient condition to guarantee assumption $(A3)$.

Next theorem, proven in [7], provides a complete characterization of the stability boundary for systems that admit energy functions.

**Theorem 5** (Stability Boundary Characterization) [7] *Let $x^s$ be a hyperbolic asymptotically stable equilibrium point of (1) and $A(x^s)$ be its stability region. If assumption $(A1)$ is satisfied and system (1) admits an energy function, then:*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x^i)$$

*where $x^i$, $i = 1, 2, ...$ are the hyperbolic equilibrium points on the stability boundary $\partial A(x^s)$.*

## 2 Persistence of Stability Regions to Parameter Variation

Complete characterizations of stability regions and stability boundaries were proven in the literature [6, 8] and the main results of this theory were presented in Sect. 1. These characterizations are given in terms of the union of the stable manifolds of the critical elements on the stability boundary. However, systems are subjected to uncertainties and parameter changes and a natural question that pops up is how these characterizations are robust with respect to parameter variation. The answer to this question is crucial to ensure that estimates of the stability region obtained by means of these characterizations are robust to parameter changes.

In this chapter, we will study stability regions of the following class of nonlinear dynamical systems:

$$\dot{x} = f(x, \lambda) = f_\lambda(x), \tag{3}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a $\mathscr{C}^1$-function and $\lambda$ is a real parameter. The trajectory of system $\dot{x} = f_\lambda(x)$ passing through $x_o$ at time $t = 0$ will be denoted $\varphi_\lambda(t, x_o)$.

Suppose $x^s_{\lambda_o}$ is a hyperbolic asymptotically stable equilibrium point of (3) for $\lambda = \lambda_o$ and let $A_{\lambda_o}(x^s_{\lambda_o})$ be its stability region. Hyperbolic equilibrium points persist to parameter changes. Consequently, it does make sense to study the stability region $A_\lambda(x^s_\lambda)$ of the perturbed equilibrium $x^s_\lambda$. Suppose assumptions (A1)–(A3) hold for $\lambda = \lambda_o$ and $x^i_{\lambda_o}$, $i = 1, \ldots, m$ are the unstable equilibrium points on the stability boundary. Then, according to Theorem 2, the stability boundary is given by:

$$\partial A_{\lambda_o}(x^s_{\lambda_o}) = \bigcup_i W^s_{\lambda_o}(x^i_{\lambda_o})$$

If assumptions (A1)–(A3) hold for all $\lambda$ in a neighborhood of $\lambda_0$ and the number of equilibrium points on the boundary is finite, it can be proven, under reasonable conditions, that the stability region and the stability boundary do not suffer drastic changes. More precisely, if an unstable equilibrium point $x^u_{\lambda_o}$ belongs to the stability boundary $\partial A_{\lambda_o}(x^s_{\lambda_o})$ of the unperturbed system, then the perturbed unstable equilibrium point $x^u_\lambda$ will persist on the stability boundary, i.e., $x^u_\lambda \in \partial A_\lambda(x^s_\lambda)$ for every $\lambda$ sufficiently close to $\lambda_o$. Consequently,

$$\partial A_\lambda(x^s_\lambda) \subset \bigcup_i W^s_\lambda(x^i_\lambda),$$

indicating that the stability boundary does not suffer drastic changes for $\lambda$ sufficient close to $\lambda_o$. However, with changes in the parameter $\lambda$, assumptions (A1) and (A2) may be violated. In these cases, drastic changes in the stability regions and stability boundaries may occur. In this chapter, we will study these changes when assumption (A1) is violated due to the appearance of two types of nonhyperbolic equilibrium points on the stability boundary: the saddle-node equilibrium point and the Hopf equilibrium point. We first develop characterizations of the stability boundary in the presence of these nonhyperbolic equilibrium points in Sect. 3 and then the stability region and stability boundary behavior due to changes in parameters in the neighborhood of saddle-node and Hopf bifurcations are studied in Sect. 4.

## 3 Non-hyperbolic Equilibrium Points on the Stability Boundary

Hyperbolicity of equilibrium points on the stability boundary is a fundamental property for the characterizations of the stability boundary developed in Sect. 1. Although the hyperbolicity of equilibrium points of a dynamical system is a generic property, i.e., it is satisfied for almost all dynamical systems, the violation of hyperbolicity condition of the equilibrium points on the stability boundary is very common when the system is subject to variation of parameters. In the analysis of voltage stability in electric power systems, for instance, the occurrence of saddle-node bifurcations on the stability boundary were reported, violating the hyperbolicity condition of the equilibrium points on the stability boundary [12].

In this section, we discuss the stability boundary characterization in the presence of two types of non-hyperbolic equilibrium points: the saddle-node and the Hopf equilibrium points. Exploring these characterizations, we also discuss on how to obtain estimates of the stability region for dynamical systems that admit energy functions.

## 3.1 Saddle-Node Equilibrium Points on the Stability Boundary

In Sect. 1.1, the properties of hyperbolic equilibriums on the stability boundary were studied. In this section, we develop necessary and sufficient conditions for a saddle-node, which is the simplest of the non-hyperbolic equilibrium points, belonging to the boundary of the stability region. In addition, we also develop a characterization of the stability boundary in the presence of saddle-node equilibrium points. The results developed in this section are a generalization of the ones proven in [3, 4]. They are also a generalization of the results presented in Sect. 1.1.

**Definition 5** (*Saddle-Node Equilibrium Point*) [17] A non-hyperbolic equilibrium point $p \in \mathbb{R}^n$ of (1) is a *saddle-node equilibrium point* if the following conditions are satisfied:

(i) $D_x f(p)$ has a unique simple null eigenvalue and none of the other eigenvalues have real part equal to zero,
(ii) $w(D_x^2 f(p)(v, v)) \neq 0$,

with $v$ as the right eigenvector and $w$ as the left eigenvector associated with the null eigenvalue.

Saddle-node equilibrium points can be classified in types according to the number of eigenvalues of $D_x f(p)$ with positive real part.

**Definition 6** (*Saddle-Node Equilibrium Type*) A saddle-node equilibrium point $p$ of (1) is called a type-$k$ saddle-node equilibrium point if $D_x f(p)$ has $k$ eigenvalues with positive real part and $n - k - 1$ with negative real part.

If $p$ is a saddle-node equilibrium point of (1), then there exist invariant local manifolds $W_{loc}^s(p)$, $W_{loc}^{cs}(p)$, $W_{loc}^c(p)$, $W_{loc}^u(p)$ and $W_{loc}^{cu}(p)$ of class $C^r$, tangent to the eigenspaces $E^s$, $E^c \oplus E^s$, $E^c$, $E^u$ and $E^c \oplus E^u$ at $p$, respectively [14, 18]. These manifolds are respectively called stable, stable center, center, unstable and unstable center manifolds. The stable and unstable manifolds are unique, but the stable center, center and unstable center manifolds may not be. Dynamic properties of these manifolds can be found in [17, 18].

### 3.1.1 Stability Boundary Characterization

In the presence of non-hyperbolic equilibrium points on the stability boundary, assumption $(A1)$ is violated and the stability boundary characterization given in Theorem 2 is not valid. In this section, a characterization of the stability boundary in the presence of hyperbolic and saddle-node equilibrium points is developed. This characterization is developed in two steps. In the first step, a local characterization of equilibrium points on the stability boundary is developed and then a global characterization of the stability boundary in terms of manifolds of equilibrium points is developed.

Let $x^s$ be an asymptotically stable equilibrium point of (1) and let $A(x^s)$ be its stability region. Consider the following assumptions:
($\mathbf{A1}'$) All the equilibrium points on $\partial A(x^s)$ are hyperbolic or saddle-node equilibrium points.
($\mathbf{A2}'$) The following transversality conditions are satisfied:

(i) The stable and unstable manifolds of equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.
(ii) The unstable manifolds of equilibrium points and the stable component of the stable center manifolds of the type-$k$ saddle-node equilibrium points, with $1 \leq k \leq n-2$, on $\partial A(x^s)$ satisfy the transversality condition.
(iii) The unstable manifolds of equilibrium points and the stable component of the center manifolds of the type-$(n-1)$ saddle-node equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.
(iv) The stable manifolds of equilibrium points and the unstable component of the center manifolds of the type-0 saddle-node equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.
(v) The stable component of the stable center manifolds of the type-$k$ saddle-node equilibrium points, with $1 \leq k \leq n-2$, and the unstable component of the center manifolds of the type-0 saddle-node equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.
(vi) The stable component of the center manifolds of the type-$(n-1)$ saddle-node equilibrium points and the unstable component of the center manifolds of the type-0 saddle-node equilibrium points on $\partial A(x^s)$ satisfy the transversality condition.

Assumptions $(A1')$ and $(A2')$ are generic properties of dynamical systems [16]. Under assumptions $(A1')$, $(A2') - (iv)$, $(v)$, $(vi)$ and $(A3)$, next theorem, proven in [1], offers necessary and sufficient conditions to guarantee that a type-0 saddle-node equilibrium point lies on the stability boundary of a nonlinear autonomous dynamical system.

**Theorem 6** (Type-0 Saddle-Node Equilibrium Points on the Stability Boundary) [1] *Let $x^s$ be an asymptotically stable equilibrium point of (1) and let $A(x^s)$ be its stability region. Suppose that assumptions $(A1^{'})$, $(A2^{'})$-$(iv)$, $(v)$, $(vi)$ and $(A3)$ are satisfied. If p is a type-0 saddle-node equilibrium point, then the following statements are equivalent:*

*(a)* $p \in \partial A(x^s)$
*(b)* $W^{c^+}(p) \cap A(x^s) \neq \emptyset$
*(c)* $W^s(p) \subseteq \partial A(x^s)$

Theorem 6 offers necessary and sufficient conditions to guarantee that a type-0 saddle-node equilibrium point $p$ belongs to the stability boundary. More precisely, it shows that the non empty intersection of the unstable component $W^{c^+}(p)$ of the center manifold with the stability region guarantees that the saddle-node equilibrium point $p$ lies on the stability boundary.

In this sense, we observe that $W^{c^+}(p)$ plays in Theorem 6 the same role of $W^u(p)$ in Theorem 1. Consequently, one can check if an equilibrium point $p$ lies on the stability boundary by checking if the unstable manifold intersects the stability region in the case of a hyperbolic equilibrium point and if the unstable component of the center manifold intersects the stability region in the case of a type-0 saddle-node equilibrium point. Figure 4 illustrates the results of Theorem 6.

Next theorem, proven in [1], offers necessary and sufficient conditions to guarantee that a hyperbolic or a type-$r$ saddle-node equilibrium point, with $r \geq 1$, lies on the stability boundary of a nonlinear autonomous dynamical system.

**Theorem 7** (Hyperbolic and Type-$r$ Saddle-Node Equilibrium Points, with $r \geq 1$ on the Stability Boundary) [1] *Let $x^s$ be an asymptotically stable equilibrium point of (1) and let $A(x^s)$ be its stability region. Suppose that assumptions $(A1^{'})$, $(A2^{'})$ and $(A3)$ are satisfied. Then:*

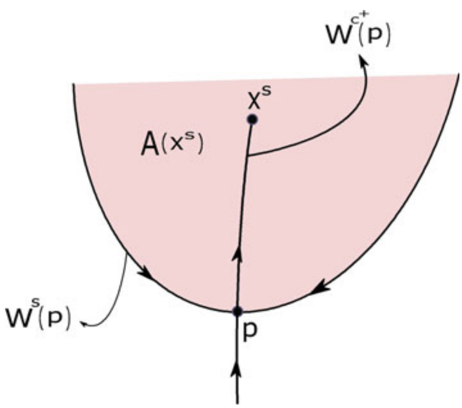



**Fig. 4** The type-0 saddle-node equilibrium point $p$ is on the stability boundary. Its stable manifold $W^s(p)$ lies on the stability boundary $\partial A(x^s)$ and the unstable component of the center manifold $W^{c^+}(p)$ has a non empty intersection with the stability region $A(x^s)$

(i) *If $p^*$ is a hyperbolic equilibrium point or a type-r saddle-node equilibrium point, with $r \geq 1$, of (1) and $(W^u(p^*)\text{-}\{p^*\}) \cap \overline{A(x^s)} \neq \emptyset$, then the following statements are equivalent:*

(a) $p^* \in \partial A(x^s)$
(b) $W^u(p^*) \cap A(x^s) \neq \emptyset$
(c) $\begin{cases} W^s(p^*) \subseteq \partial A(x^s) & \text{if } p^* \text{ is a hyperbolic equilibrium point} \\ W^{cs^-}(p^*) \subseteq \partial A(x^s) & \text{if } p^* \text{ is a type-}r \text{ saddle-node equilibrium point, } r \leq n-2 \\ W^{c^-}(p^*) \subseteq \partial A(x^s) & \text{if } p^* \text{ is a type-}(n\text{-}1) \text{ saddle-node equilibrium point.} \end{cases}$

(ii) *If p is a type-r saddle-node equilibrium point, with $r \geq 1$, of (1) and $(W^u(p) - \{p\}) \cap \overline{A(x^s)} = \emptyset$ then the following statements are equivalent:*

(a) $p \in \partial A(x^s)$
(b) $W^{c^+}(p) \cap A(x^s) \neq \emptyset$ *for some unstable component $W^{c^+}(p)$ of the center manifold.*
(c) $W^s(p) \subseteq \partial A(x^s)$.

Admitting the existence of non hyperbolic saddle-node equilibrium points on the stability boundary, generalizing the transversality condition and exploring assumption $(A3)$, Theorem 7 extends the results of Theorem 1. Observe that the same equivalences proven in Theorem 1 are still valid for hyperbolic equilibrium points even in the presence of saddle-node equilibrium points on the stability boundary.

Theorem 7 offers necessary and sufficient conditions for a type-$r$ saddle-node equilibrium point, with $r \geq 1$, lying on the stability boundary $\partial A(x^s)$. For saddle-node equilibrium points, two different situations can occur. Therefore, two cases are separately treated in Theorem 7, the case (i), in which $(W^u(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$, and the case (ii), in which $(W^u(p) - \{p\}) \cap \overline{A(x^s)} = \emptyset$.

Next theorem, proven in [1], combines the results of Theorems 6 and 7 to offer a complete characterization of the stability boundary of a nonlinear autonomous dynamical system in the presence of saddle-node equilibrium points on the stability boundary $\partial A(x^s)$.

**Theorem 8** (Stability Boundary Characterization) [1] *Let $x^s$ be an asymptotically stable equilibrium point of (1) and $A(x^s)$ be its stability region. Suppose that assumptions $(A1^{'})$, $(A2^{'})$ and $(A3)$ are satisfied. Then:*

$$\partial A(x^s) = \bigcup_i W^s(x_i) \bigcup_j W^s(p_j) \bigcup_l W^{cs^-}(z_l) \bigcup_t W^s(z_t) \bigcup_m W^{c^-}(q_m)$$

*where $x_i$ are the hyperbolic equilibrium points on $\partial A(x^s)$, $p_j$ the type-0 saddle-node equilibrium points on $\partial A(x^s)$, $z_l$ the type-k saddle-node equilibrium points on $\partial A(x^s)$, with $1 \leq k \leq n-2$, and $(W^u(z_l) - \{z_l\}) \cap \overline{A(x^s)} \neq \emptyset$, $z_t$ the type-d saddle-node equilibrium points on $\partial A(x^s)$, with $d \geq 1$ and $(W^u(z_t) - \{z_t\}) \cap \overline{A(x^s)} = \emptyset$ and $q_m$ the type-$(n-1)$ saddle-node equilibrium points on $\partial A(x^s)$, with $(W^u(q_m) - \{q_m\}) \cap \overline{A(x^s)} \neq \emptyset$, $i, j, l, t, m = 1, 2, \ldots$.*

Figure 5 which was presented in [1], shows an example of a dynamical system in $\mathbb{R}^3$, where a type-1 saddle-node equilibrium point $p$ lies on the stability boundary $\partial A(x^s)$ of an asymptotically stable equilibrium point $x^s$, but $(W^u(p) - \{p\}) \cap \overline{A(x^s)} = \emptyset$. The stability boundary $\partial A(x^s)$ is formed, according to Theorem 8, as the union of the stable manifold $W^s(p)$ and the stable manifolds $W^s(x_1)$, $W^s(x_2)$ of the unstable hyperbolic equilibrium points $x_1$ and $x_2$ that belong to the stability boundary $\partial A(x^s)$.

Figure 6 which was presented in [1], shows an example of a dynamic system in $\mathbb{R}^3$, where $(W^u(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$. The stability boundary $\partial A(x^s)$ is formed, according to Theorem 8, as the union of the stable component $W^{cs^-}(p)$ of the stable center manifold and the stable manifold $W^s(x_1)$ of the unstable hyperbolic equilibrium point $x_1$ that belongs to the stability boundary $\partial A(x^s)$.

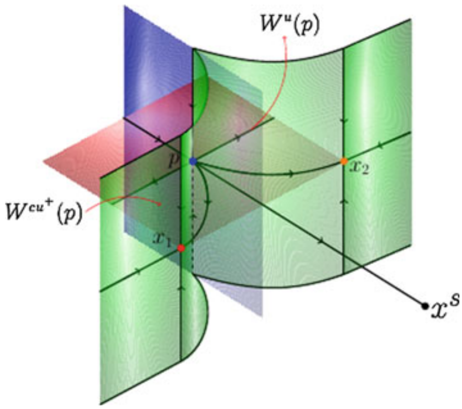



**Fig. 5** Example of a dynamical system on $\mathbb{R}^3$ where the unstable component of the unstable center manifold $W^{cu^+}(p)$ of the type-1 saddle-node equilibrium point $p$ lying on the stability boundary $\partial A(x^s)$ intersects the closure of the stability region $\overline{A(x^s)}$ and $(W^u(p) - \{p\}) \cap \overline{A(x^s)} = \emptyset$. Reprinted from [1]

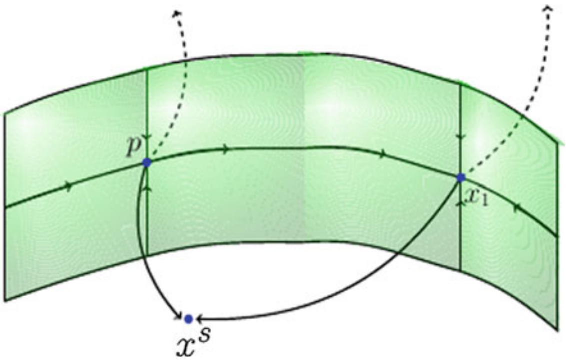



**Fig. 6** Example of a dynamical system on $\mathbb{R}^3$ where the unstable component of the unstable center manifold $W^{cu^+}(p)$ of the type-1 saddle-node equilibrium point $p$ lying on the stability boundary $\partial A(x^s)$ intersects the closure of the stability region $\overline{A(x^s)}$ and $(W^u(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$. Reprinted from [1]

### 3.1.2 Stability Region Estimation

In this section, we derive a scheme to obtain optimal estimates of stability regions via level sets of a given energy function even in the presence of saddle-node equilibrium points on the stability boundary.

The first theorem of this section was proven in [7]. It guarantees that every local minimum of the energy function on the stability boundary is attained at an equilibrium point.

**Theorem 9** (Energy Functions and Equilibrium Points I) [7] *Let $x_s$ be an asymptotically stable equilibrium point of the nonlinear dynamical system (1) and $A(x_s)$ be its stability region. If system (1) admits an energy function, then, the point on the stability boundary $\partial A(x_s)$ at which the energy function attains the minimum value must be an equilibrium point.*

The point of minimum energy on the stability boundary may not be unique. However, since the property that all equilibrium points of system (1) have distinct energy function values is generic, we can affirm that the point of minimum energy on the stability boundary is generically unique. In other words, the uniqueness of the point with minimum energy is almost always guaranteed.

Theorem 10, proven in [5], gives a characterization for the equilibrium points at which the global minimum of energy is attained over the stability boundary.

**Theorem 10** (Energy Functions and Equilibrium Points II) [5] *Let $x_s$ be an asymptotically stable equilibrium point of the nonlinear dynamical system (1) and $A(x_s)$ be its stability region. Suppose that system (1) admits an energy function. If $x^*$ is the equilibrium point with the minimum value of the energy function over the stability boundary $\partial A(x_s)$, then*

*(i) if $x^*$ is a hyperbolic equilibrium point, then $x^*$ is of the type-one;*
*(ii) if $x^*$ is a saddle-node equilibrium point, then $x^*$ is of the type-zero.*

Theorem 11, proven in [5], gives a dynamical characterization of this equilibrium in terms of its invariant manifolds. Note that this theorem holds without the transversality condition.

**Theorem 11** (Dynamical Characterization) [5] *Let $x_s$ be an asymptotically stable equilibrium point of the nonlinear dynamical system (1) and $A(x_s)$ be its stability region. Suppose that system (1) admits an energy function. If $x^*$ is the equilibrium point with the minimum value of the energy function over the stability boundary $\partial A(x_s)$, then*

*(i) if $x^*$ is hyperbolic, then $W^u(x^*) \cap A(x_s) \neq \emptyset$;*
*(ii) if $x^*$ is a type-zero saddle-node equilibrium point, then $W^{c^+}(x^*) \cap A(x^s) \neq \emptyset$*

Next theorem provides a scheme to obtain the best estimate of the stability region, via level sets of a particular given energy function, even in the presence of a saddle-node equilibrium point on the stability boundary. Its proof can be found in [5].

**Theorem 12** (Stability Region Estimation) [5] *Let $A(x^s)$ be the stability region of the asymptotically stable equilibrium point $x^s$ of (1). Suppose also that system (1) admits an energy function $V$. If $L = \min_{x \in E \cap \partial A(x^s)} V(x)$, then:*

*(i) the connected component $D(L)$ of the level set $\{x \in \mathbb{R}^n : V(x) < L\}$ containing the equilibrium $x^s$ is inside the stability region $A(x^s)$.*
*(ii) the connected component $D(B)$ of the level set $\{x \in \mathbb{R}^n : V(x) < B\}$ containing the equilibrium $x^s$ has a nonempty intersection with the complement of the stability region $A^c(x^s)$ for any number $B > L$.*

Theorem 12 ensures that, calculating all type-one hyperbolic and type-zero saddle-node equilibrium points on the stability boundary, we can obtain the best estimate of the stability region, in the form of a level set of the energy function $V$, by picking the level set with a level value that equals the value of the energy of the equilibrium point on the stability boundary which has the lowest value of energy. The choice $L$ is optimal in the sense that any level set with an energy level greater than $L$ is not contained in $A(x^s)$.

Theorem 12 generalizes the results in [7] by allowing the existence of saddle-node equilibrium points on the stability boundary. It suggests the following conceptual algorithm, which is also a generalization of the one proposed in [7], to obtain the optimal estimate of the stability region in the form of level sets of a given energy function $V$:

**Conceptual Algorithm for Stability Region Estimation**

**Step 1** Compute all the equilibrium points on $\partial A(x^s)$.
**Step 2** Identify the equilibrium point $x^{\min}$ that possesses the lowest energy among them. Let $L = V(x^{\min})$.
**Step 3** The connected component $D(L)$ of the level set $\{x \in \mathbb{R}^n : V(x) < L\}$ containing $x^s$ is the largest estimate of the stability region $A(x^s)$ in the form of a level set of $V$.

*Example* This example, proposed in [5], illustrates the application of the conceptual algorithm for stability region estimation. Consider the nonlinear autonomous dynamical system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= -x^4 + x^2 - y \end{aligned} \tag{4}$$

where $(x, y) \in \mathbb{R}^2$. Function $V(x, y, \lambda) = -\frac{x^5}{5} + \frac{x^3}{3} + \frac{y^2}{2}$ is an energy function for system (4). System (4) possesses three equilibrium points; they are $x^s = (-1, 0)$, a

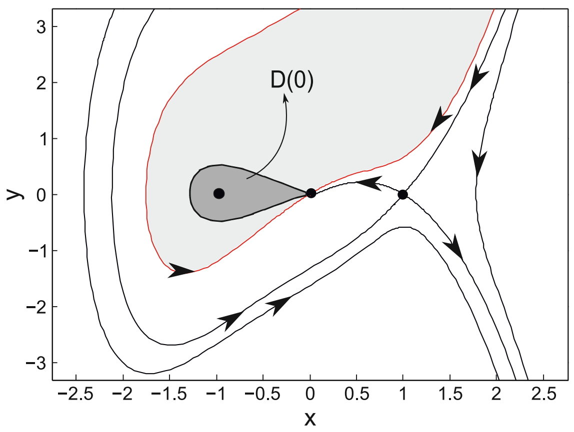



**Fig. 7** The phase portrait of system (4). The type-zero saddle-node equilibrium point (0, 0) is on the stability boundary $\partial A(-1, 0)$. The region in light gray is the stability region of $(-1, 0)$ and the dark gray region is the largest estimate obtained via level set of the energy function $V(x, y, \lambda) = -\frac{x^5}{5} + \frac{x^3}{3} + \frac{y^2}{2}$. Reprinted from [5]

hyperbolic asymptotically stable equilibrium point, $p = (0, 0)$, a type-zero saddle-node equilibrium point and $x^* = (1, 0)$, a type-one hyperbolic equilibrium point. The stability boundary $\partial A(-1, 0)$ is depicted in Fig. 7. It is formed of the stable manifold of the type-zero saddle-node equilibrium point (0, 0). The minimum of the energy function $V$ on the stability boundary $\partial A(-1, 0)$ is attained at the type-zero saddle-node equilibrium point $p$, the unique equilibrium on $\partial A(-1, 0)$. The energy function value at $p$ is $L = V(0, 0) = 0$. The connected component $D(0)$ of the level set $\{x \in \mathbb{R}^2 : V(x, y) < 0\}$ containing the asymptotically stable equilibrium point $x^s = (-1, 0)$ is completely contained in $A(-1, 0)$, see Fig. 7, and it is the largest estimate that can be obtained in the form of a level set of $V$.

## 3.2 *Hopf Equilibrium Points on the Stability Boundary*

In this section, we study the properties of another type of non-hyperbolic equilibrium point on the stability boundary, the so called Hopf equilibrium point. In particular, we develop necessary and sufficient conditions for a Hopf equilibrium point belonging to the stability boundary. Moreover, we develop a complete characterization of the stability boundary in the presence of Hopf equilibrium points. The results of this section generalize the characterization of stability boundary given in Sect. 1 and are a compilation of the results presented in [9–11].

Consider the nonlinear dynamical system (1). We can always perform a change of coordinates in system (1), shifting the equilibrium point to origin. Thus, without loosing generality, system (1) can be rewritten as

$$\dot{x} = Ax + F(x), x \in \mathbb{R}^n, \tag{5}$$

where $F$, $F(x) = \mathscr{O}(\|x\|^2)$, is a smooth function that has Taylor expansion in $x$ starting with at least quadratic terms. We can also write function $F(x)$, following the notation of [15], as

$$F(x) = \frac{1}{2}B(x,\ x) + \frac{1}{6}C(x,\ x,\ x) + \mathscr{O}(\|x\|^4)$$

where $B(x,\ y)$ and $C(x, y, z)$ are symmetric multilinear vector functions of $x,\ y,\ z \in \mathbb{R}^n$ such that

$$B_i(x,\ y) = \sum_{j,\ k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\right|_{\xi=0} x_j y_k,\ i = 1,\ \ldots,\ n$$

and

$$C_i(x,\ y,\ z) = \sum_{j,\ k,\ l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\right|_{\xi=0} x_j y_k z_l,\ i = 1,\ \ldots,\ n.$$

**Definition 7** (*Hopf Equilibrium Point*) A non-hyperbolic equilibrium point $p \in \mathbb{R}^n$ of (1) is called a Hopf equilibrium point if the following conditions are satisfied:

(i) $A = D_x f(p)$ has a simple pair of purely imaginary eigenvalues, $\pm i\omega$, and no other eigenvalue with null real part;

(ii) $l_1 \neq 0$ where $l_1$ is the first Lyapunov coefficient, which can be computed by the formula:

$$l_1 = \frac{1}{2\omega}\Re\left[\langle u, C(v, v, \overline{v})\rangle - 2\langle u, B(v, A^{-1}B(v,\ \overline{v}))\rangle\ + \langle u, B(\overline{v}, (2i\omega I - A)^{-1}B(v, v))\rangle\right]$$

where $v$ is the complex eigenvector associated with the imaginary eigenvalue $i\omega$, $u$ is the complex adjoint eigenvector of the transposed matrix $A$ associated with its eigenvalue $-i\omega$ and satisfying the normalization condition:

$$\langle u,\ v\rangle = 1,$$

where $\langle x,\ y\rangle = \sum_{i=1}^{n} \overline{x}_i y_i$ represents the inner product in $\mathbb{C}^n$, and $\Re$ is the real part of a complex number.

Hopf equilibrium points can be classified according to the sign of the first Lyapunov coefficient.

**Definition 8** (*Supercritical and Subcritical Hopf Equilibrium Point*) A Hopf equilibrium point $p \in \mathbb{R}^n$ of (1) is called a supercritical Hopf equilibrium point if the first Lyapunov coefficient $l_1 < 0$. A Hopf equilibrium point $p \in \mathbb{R}^n$ of (1) is called a subcritical Hopf equilibrium point if the first Lyapunov coefficient $l_1 > 0$.

Lyapunov coefficients are related to the asymptotic behavior of the system on the central manifold. Supercritical Hopf equilibrium points attract orbits on the central manifold while subcritical Hopf equilibrium points repel them. Furthermore, Hopf equilibrium points can be also classified in types according to the number of eigenvalues of $D_x f(p)$ with positive real part.

**Definition 9** (*Type-k Hopf Equilibrium Point*) A Hopf equilibrium point $p$ of (1) is called a type-$k$ Hopf equilibrium point if $D_x f(p)$ has $k$ ($k \leq n-2$) eigenvalues with positive real part and $n-k-2$ with negative real part.

Figure 8 illustrates the invariant manifolds for a type-0 supercritical Hopf equilibrium point in $\mathbb{R}^3$ and Fig. 9 illustrates these invariant manifolds for a type-1 supercritical Hopf equilibrium point in $\mathbb{R}^3$.

Figure 10 illustrates the invariant manifolds for a type-1 subcritical Hopf equilibrium point in $\mathbb{R}^3$ and Fig. 11 illustrates these invariant manifolds for a type-0 subcritical Hopf equilibrium point in $\mathbb{R}^3$.

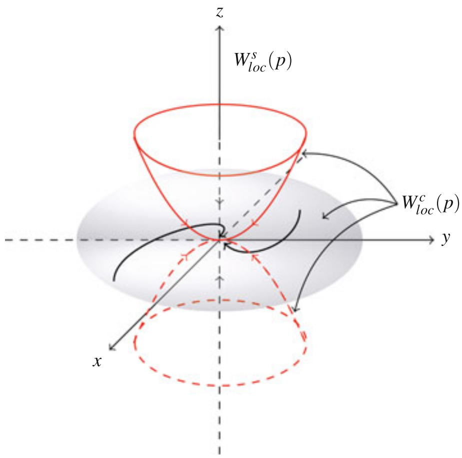



**Fig. 8** Manifolds $W^c_{loc}(p)$ and $W^s_{loc}(p)$ for a type-0 supercritical Hopf equilibrium point $p$ of system (1) in $\mathbb{R}^3$. $W^c_{loc}(p)$ is not unique. Three choices of $W^c_{loc}(p)$ are displayed in this figure. Reprinted from [9]

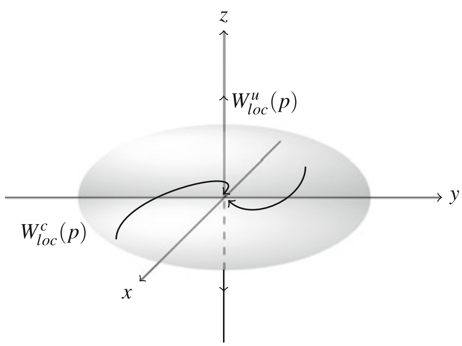



**Fig. 9** Manifolds $W^c_{loc}(p)$ and $W^u_{loc}(p)$ for a type-1 supercritical Hopf equilibrium point $p$ of system (1) in $\mathbb{R}^3$. In this case, $W^c_{loc}(p)$ is unique. Reprinted from [9]

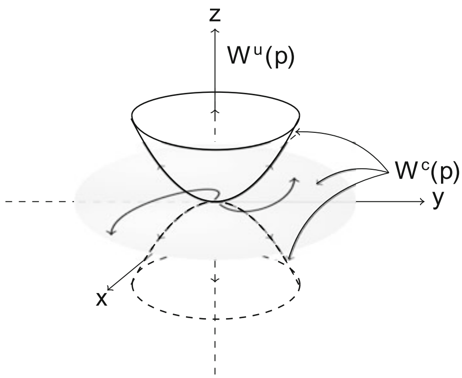



**Fig. 10** Manifolds $W^c(p)$ and $W^s(p)$ for a type-1 subcritical Hopf equilibrium point $p$ of system (1) in $\mathbb{R}^3$. $W^c(p)$ is not unique. Three choices of $W^c(p)$ are displayed in this figure. Reprinted from [10]

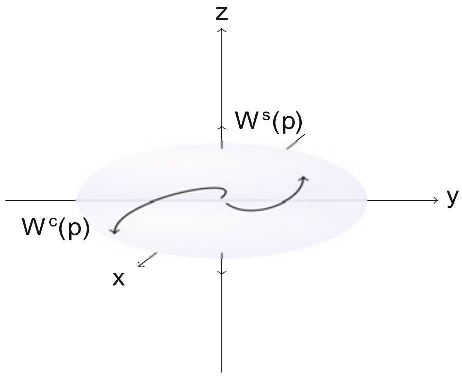



**Fig. 11** Manifolds $W^c(p)$ and $W^u(p)$ for a type-0 subcritical Hopf equilibrium point $p$ of system (1) in $\mathbb{R}^3$. In this case, $W^c(p)$ is unique. Reprinted from [10]

### 3.2.1 Stability Boundary Characterization

In this section, a complete characterization of the stability boundary in the presence of a Hopf equilibrium point on the stability boundary is developed. This characterization is developed in two steps. First we study a local characterization of the stability boundary by studying and characterizing the equilibrium points and closed orbits that belong to the stability boundary, then a global characterization is developed.

Initially, a characterization of the stability boundary considering supercritical Hopf equilibrium points is developed and then the result is extended to consider subcritical Hopf equilibrium points.

The next theorem, proven in [9, 11], provides necessary and sufficient conditions to guarantee that a supercritical Hopf equilibrium point or a hyperbolic critical element lies on the boundary of the stability region. It extends the results of Theorems 1 and 3 to accommodate closed orbits and supercritical Hopf equilibrium points on the stability boundary. These conditions are expressed in terms of the properties of their stable, unstable and center-stable manifolds.

**Theorem 13** (Critical Elements on the Stability Boundary) [9, 11] *Let $A(x^s)$ be the stability region of an asymptotically stable equilibrium point $x^s$ of (1). Let $p$ be a type-k supercritical Hopf equilibrium point, with $1 \leq k \leq n-2$, and let $\phi$ be a type-$k'$ hyperbolic critical element, with $k' \leq n$, of (1). If the assumptions:*

***(B1')*** *All the critical elements on $\partial A(x^s)$ are hyperbolic critical elements or supercritical Hopf equilibrium points;*
***(B2')*** *The stable, center-stable and/or center and unstable manifolds of the critical elements on $\partial A(x^s)$ satisfy the transversality condition;*
***(B3)*** *Trajectories on $\partial A(x^s)$ approach one of the critical elements as $t \to \infty$ are held, then:*

*(i) If $\phi$ is a type-$k'$ critical element, with $1 \leq k' \leq n$, then*

$$\phi \subset \partial A(x^s) \Longleftrightarrow W^u(\phi) \cap A(x^s) \neq \emptyset$$
$$\phi \subset \partial A(x^s) \Longleftrightarrow W^s(\phi) \subset \partial A(x^s)$$

*(ii) If $p$ is a type-k supercritical Hopf equilibrium point, with $1 \leq k \leq n-3$, then*

$$p \in \partial A(x^s) \Longleftrightarrow W^u(p) \cap A(x^s) \neq \emptyset$$
$$p \in \partial A(x^s) \Longleftrightarrow W^{cs}(p) \subset \partial A(x^s)$$

*(iii) If $p$ is a type-$(n-2)$ supercritical Hopf equilibrium point, then*

$$p \in \partial A(x^s) \Longleftrightarrow W^u(p) \cap A(x^s) \neq \emptyset$$
$$p \in \partial A(x^s) \Longleftrightarrow W^c(p) \subset \partial A(x^s)$$

The next theorem offers a complete characterization of the stability boundary when a supercritical Hopf equilibrium points lies on $\partial A(x^s)$. It is a generalization of Theorems 2 and 4 that allows the existence of closed orbits and supercritical Hopf equilibrium points on the stability boundary.

**Theorem 14** (Characterization of the Stability Boundary for Critical Elements) [9, 11] *Let $x^s$ be an asymptotically stable equilibrium point of (1) and let $A(x^s)$ be its stability region. If assumptions (**B1'**) and (**B3**) are held, then:*

$$\partial A(x^s) \subset \bigcup_i W^s(\phi_i) \bigcup_j W^{cs}(p_j) \bigcup_l W^c(q_l)$$

*where $\phi_i$ are the hyperbolic critical elements, $p_j$ the type-k supercritical Hopf equilibrium points, with $1 \leq k \leq n-3$, and $q_l$ the type-$(n-2)$ supercritical Hopf equilibrium points on $\partial A(x^s)$, $i,\ j,\ l = 1,\ 2,\ \ldots$. If assumption (**B2'**) is additionally satisfied, then*

$$\partial A(x^s) = \bigcup_i W^s(\phi_i) \bigcup_j W^{cs}(p_j) \bigcup_l W^c(q_l).$$

Theorem 14 states that the stability boundary is composed of the stable manifolds of the critical elements on the stability boundary union with the center-stable manifold of the supercritical Hopf equilibrium points on the stability boundary.

Now, we will construct the same characterization of the stability boundary when the non-hyperbolic equilibrium point on the stability boundary is a subcritical Hopf equilibrium point.

The next theorem provide necessary and sufficient conditions to guarantee that a subcritical Hopf equilibrium point or a hyperbolic critical element lies on the boundary of the stability region. These conditions are expressed in terms of the properties of its stable, center-unstable and center manifolds.

**Theorem 15** (Critical Elements on the Stability Boundary) [11] *Let $A(x^s)$ be the stability region of an asymptotically stable equilibrium point $x^s$ of (1). Let $p$ be a type-k subcritical Hopf equilibrium point, with $1 \leq k \leq n-3$, and let $\phi$ be a type-$k'$ hyperbolic critical element, with $k' \leq n$, of (1). If the assumptions:*

- ***(B1")*** *All the critical elements on $\partial A(x^s)$ are hyperbolics critical elements or subcritical Hopf equilibrium points;*
- ***(B2")*** *The stable and unstable, center-unstable and/or center manifolds of the critical elements on $\partial A(x^s)$ satisfy the transversality condition;*
- ***(B3)*** *Trajectories on $\partial A(x^s)$ approach one of the critical elements as $t \to \infty$ are held, then:*

*(i)* *If $\phi$ is a type-$k'$ critical element, with $1 \leq k' \leq n$, then*
$\phi \in \partial A(x^s) \Longleftrightarrow W^u(\phi) \cap A(x^s) \neq \emptyset$
$\phi \in \partial A(x^s) \Longleftrightarrow W^s(\phi) \subset \partial A(x^s)$

*(ii)* *If $p$ is a type-0 subcritical Hopf equilibrium point, then*
$p \in \partial A(x^s) \Longleftrightarrow W^c(p) \cap A(x^s) \neq \emptyset$
$p \in \partial A(x^s) \Longleftrightarrow W^s(p) \subset \partial A(x^s)$

*(iii)* *If $p$ is a type-k subcritical Hopf equilibrium point, with $1 \leq k \leq n-3$, then*
$p \in \partial A(x^s) \Longleftrightarrow W^{cu}(p) \cap A(x^s) \neq \emptyset$
$p \in \partial A(x^s) \Longleftrightarrow W^s(p) \subset \partial A(x^s)$

*(iv)* *If $p$ is a type-$(n-2)$ subcritical Hopf equilibrium point, then*
$p \in \partial A(x^s) \Longleftrightarrow W^{cu}(p) \cap A(x^s) \neq \emptyset$

The next theorem provides a complete characterization of the boundary of the stability region when there are subcritical Hopf equilibrium points in $\partial A(x^s)$. It is a generalization of Theorems 2 and 4 that allows the presence of closed orbits and subcritical Hopf equilibrium points on the stability boundary.

**Theorem 16** (Characterization of the Stability Boundary for Critical Elements) *Let $x^s$ be an asymptotically stable equilibrium point of (1) and let $A(x^s)$ be its stability region. If assumptions **(B1")** and **(B3)** are held, then:*

$$\partial A(x^s) \subset \bigcup_i W^s(\phi_i) \bigcup_j W^{cs}(p_j)$$

*where $\phi_i$ are the hyperbolic critical elements and $p_j$ the subcritical Hopf equilibrium points, with $1 \leq k \leq n-2$, on $\partial A(x^s)$, $i = 1, 2, \ldots$. If, additionally, assumption (**B2'**) is satisfied, then*

$$\partial A(x^s) = \bigcup_i W^s(\phi_i) \bigcup_j W^{cs}(p_j).$$

Next example illustrates the results and characterizations developed in this section.

*Example* Consider the autonomous nonlinear dynamical system proposed in [9]:

$$\begin{cases} \dot{x} = -xz^2 - y - x(x^2 + y^2); \\ \dot{y} = -yz^2 + x - y(x^2 + y^2); \\ \dot{z} = -z(z-3)(8-z); \end{cases} \tag{6}$$

where $(x, y, z) \in \mathbb{R}^3$.

System (6) has three equilibrium points, they are an asymptotically stable equilibrium point $x^s = (0, 0, 3)$, a type-1 hyperbolic equilibrium point $x = (0, 0, 8)$ and a type-1 supercritical Hopf equilibrium point $p = (0, 0, 0)$. Consequently, assumptions (**B1**$'$) and (**B2**$'$) are satisfied.

Function $V(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^4}{4} + \frac{11}{3}z^3 - -12z^2$ is an energy function for system (6). For instance, $\dot{V} = -(x^2 + y^2)z^2 - (x^2 + y^2)^2 - z^2(z-3)^2(8-z)^2 \leq 0$ and assumption (**E1**) holds. The derivative of $V$ equals zero only at equilibrium points, consequently assumption (**E2**) holds. And finally, if a solution $\varphi(t, x_0)$ is unbounded for $t \geq 0$, then $V(\varphi(t, x_0))$ is also unbounded for $t \geq 0$. As a consequence, assumption (**E3**) is satisfied and $V$ is an energy function for system (6).

The existence of an energy function implies that assumption (**B3**) is held. Consequently the assumptions of Theorems 13 and 14 are satisfied and the complete characterization of stability boundary developed in Theorem 14 also holds. The unstable manifold of the type-1 supercritical Hopf equilibrium point $p = (0, 0, 0)$ intersects the stability region of $x^s = (0, 0, 3)$, consequently, according to Theorem 13, $p$ lies on the stability boundary $\partial A(x^s)$ and the center manifold is contained in the boundary of the stability region of $x^s = (0, 0, 3)$, see Fig. 12. The unstable manifold of the type-1 hyperbolic equilibrium point $x = (0, 0, 8)$ also intersects the stability region of $x^s = (0, 0, 3)$ and therefore $x$ lies on the stability boundary of $x^s$ and the stable manifold is contained in the boundary of the stability region of $x^s = (0, 0, 3)$, according to the Theorem 13, see Fig. 12.

Figure 12 illustrates the boundary of the stability region of the asymptotically stable equilibrium point $x^s = (0, 0, 3)$. The boundary is formed, according to Theorem 14, of the union of the stable manifold of the type-1 hyperbolic equilibrium point $x = (0, 0, 8)$, the highest shaded surface passing by $x$ at Fig. 12, with the center manifold of the type-1 supercritical Hopf equilibrium point $p = (0, 0, 0)$, the lowest shaded surface passing by $p$ at Fig. 12.

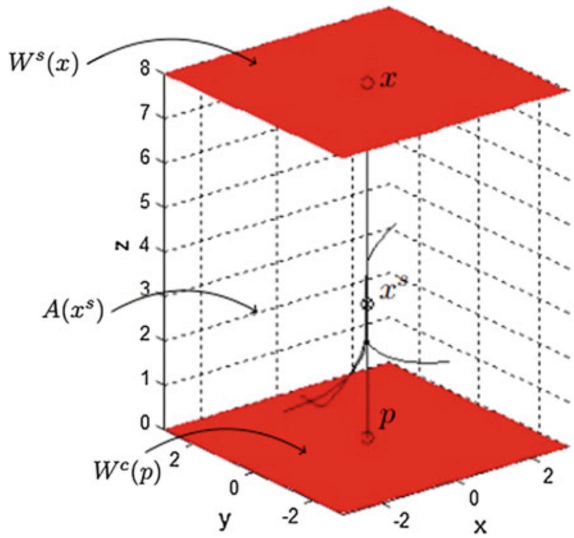



**Fig. 12** The stability boundary of the asymptotically stable equilibrium point $x^s = (0,\ 0,\ 3)$ of system (6) is composed of two surfaces, the stable manifold of the type-1 hyperbolic equilibrium point $x = (0,\ 0,\ 8)$ and the center manifold of the type-1 supercritical Hopf equilibrium point $p = (0,\ 0,\ 8)$. Reprinted from [9]

# 4 Stability Region Bifurcations

The characterization of stability boundaries derived in Sect. 1 were developed under assumptions $(A1) - (A3)$. Under parameter variation, bifurcations may occur on the stability boundary and assumptions $(A1)$ or $(A2)$ may be violated at bifurcation points. Studying the characterization of the stability boundary at these bifurcation points is of fundamental importance to understanding how the stability region behaves under parameter variation. In this section, we study bifurcations of the stability boundary that are induced by local bifurcations of critical elements on the stability boundary and, in particular, by local bifurcation of equilibrium points. It will be shown that drastic changes in the size of the stability region might occur.

## *4.1 Sadde-Node Bifurcation*

Consider the nonlinear dynamical system (3) and let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector field of class $C^r$, with $r \geq 2$.

**Definition 10** (*Saddle-Node Bifurcation Point*) The point $(p_{\lambda_0}, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ is called a saddle-node bifurcation point of system (3) if $p_{\lambda_0} \in \mathbb{R}^n$ is a non-hyperbolic equilibrium point of (3) for the fixed parameter $\lambda = \lambda_0$ and the following conditions are satisfied:

(SN1) $D_x f_{\lambda_0}(p_{\lambda_0})$ has a unique simple eigenvalue equal to 0 with $v$ as an eigenvector to the right and $w$ to the left.

(SN2) $w(D_x^2 f_{\lambda_0}(p_{\lambda_0})(v, v)) \neq 0$.
(SN3) $w((\partial f_\lambda/\partial\lambda)(p_{\lambda_0}, \lambda_0)) \neq 0$.

In other words, $(p_{\lambda_0}, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ is a saddle-node bifurcation point of system (3) if $p_{\lambda_0} \in \mathbb{R}^n$ is a saddle-node equilibrium point of (3) for a fixed parameter $\lambda = \lambda_0$ and the transversality condition $(SN2)$ is satisfied. A saddle-node bifurcation point $(p_{\lambda_0}, \lambda_0)$ will be of type $k$ if the non-hyperbolic equilibrium point $p_{\lambda_0}$ is a type-$k$ saddle-node equilibrium point. The parameter $\lambda_0$ will also be called a type-$k$ saddle-node bifurcation value.

Next theorem, proven in [17], studies the dynamical behavior of system (3) in the neighborhood of a saddle-node bifurcation point.

**Theorem 17** (Saddle-Node Bifurcation) [17] *Let $(p_{\lambda_0}, \lambda_0)$ be a saddle-node bifurcation point of (3). Then there exist a neighborhood $N$ of $p_{\lambda_0}$ and $\delta > 0$ such that, depending on the signs of the expressions in (SN2) and (SN3), there is no equilibrium point on $N$ when $\lambda \in (\lambda_0 - \delta, \lambda_0)[\lambda \in (\lambda_0, \lambda_0 + \delta)]$ and two equilibrium points $p_\lambda^k$ and $p_\lambda^{k+1}$ in $N$ for each $\lambda \in (\lambda_0, \lambda_0 + \delta)[\lambda \in (\lambda_0 - \delta, \lambda_0)]$. The two equilibrium points on $N$ are hyperbolic, more specifically $p_\lambda^k$ is of type-k and $p_\lambda^{k+1}$ is of type-$k + 1$, $k \in \mathbb{N}$. Moreover, the stable manifold of the type-k equilibrium point and the unstable manifold of the type-$k + 1$ equilibrium point intersect along an one-dimensional manifold.*

## 4.2 Saddle-Node Bifurcation on the Stability Boundary

In this section, we develop results that describe the behavior of the stability region and stability boundary in the neighborhood of a saddle-node bifurcation value. These results generalize the results of [4], which explore the behavior of the stability region and stability boundary in the neighborhood of only a type-zero saddle-node bifurcation value.

Next theorem, proven in [2], describes the local behavior of the stability boundary in the neighborhood of a type-$k$ saddle-node equilibrium point.

**Theorem 18** (Stability Boundary Behavior Near a Saddle-Node) [2] *Let $p_{\lambda_0}$ be a type-k saddle-node equilibrium point lying on the stability boundary $\partial A_{\lambda_0}(x_{\lambda_0}^s)$ of the hyperbolic asymptotically stable equilibrium point $x_{\lambda_0}^s$ of (3) for $\lambda = \lambda_0$. If assumptions $(A1) - (A3)$ are satisfied in an open interval containing $\lambda_0$, except at the type-k saddle-node bifurcation value, with $k \geq 0$, where assumptions $(A1^{'})$, $(A2^{'})$ and $(A3)$ are satisfied, and the number of equilibrium points on $\partial A_{\lambda_0}(x_{\lambda_0}^s)$ is finite, then:*

*(i) If $(p_{\lambda_0}, \lambda_0)$ is a type-zero saddle-node bifurcation point, with $p_{\lambda_0}$ lying on the stability boundary $\partial A_{\lambda_0}(x_{\lambda_0}^s)$, then there is $\beta > 0$ such that, for all $\lambda \in (\lambda_0 - \beta, \lambda_0)$, we have that*

$$p_{\lambda^0} \notin \partial A_\lambda(x_\lambda^s) \text{ and } p_{\lambda^1} \in \partial A_\lambda(x_\lambda^s)$$

*where* $p_{\lambda^0}$ *and* $p_{\lambda^1}$ *are the hyperbolic equilibrium points originated from the type-zero saddle-node bifurcation.*

*(ii) If* $(x_{\lambda_0}, \lambda_0)$ *is a type-r saddle-node bifurcation point, with* $r \geq 1$*, with* $x_{\lambda_0}$ *lying on the stability boundary* $\partial A_{\lambda_0}(x^s_{\lambda_0})$*, then there is* $\beta > 0$ *such that, for all* $\lambda \in (\lambda_0 - \beta, \lambda_0)$*, we have that*

$$y_{\lambda^r} \in \partial A_\lambda(x^s_\lambda) \ \ \textit{and} \ \ y_{\lambda^{r+1}} \in \partial A_\lambda(x^s_\lambda)$$

*where* $p_{\lambda^r}$ *and* $p_{\lambda^{r+1}}$ *are the unstable hyperbolic equilibrium points originated from the type-r saddle-node bifurcation, with* $r \geq 1$.

Theorem 18 shows that, in the occurrence of a type-$r$ saddle-node bifurcation, with $r \geq 1$, on the stability boundary, necessarily the two hyperbolic equilibrium points that coalesce and disappear at the bifurcation saddle-node belong to the stability boundary. Otherwise, the generic assumption of transversality would be violated.

The following corollary offers a complete characterization of the stability boundary in the neighborhood of a type-$k$ saddle-node bifurcation value, with $k \geq 0$.

**Corollary 1** (Characterization of the Stability Boundary in the Neighborhood of a Type-$k$ Saddle-Node Bifurcation Value, with $k \geq 0$) *Let* $p_{\lambda_0}$ *be a type-k saddle-node equilibrium point lying on the stability boundary* $\partial A_{\lambda_0}(x^s_{\lambda_0})$ *of the hyperbolic asymptotically stable equilibrium point* $x^s_{\lambda_0}$ *of (3) for* $\lambda = \lambda_0$*. If assumptions* $(A1) - (A3)$ *are satisfied in an open interval containing* $\lambda_0$*, except at the type-k saddle-node bifrucation value, with* $k \geq 0$*, where assumptions* $(A1^{'})$*,* $(A2^{'})$ *and* $(A3)$ *are satisfied, and the number of equilibrium points on* $\partial A_{\lambda_0}(p_{\lambda_0})$ *is finite, then:*

$(i)$ *For* $\lambda = \lambda_0$ *we have that*

$$\partial A_{\lambda_0}(x^s_{\lambda_0}) = \bigcup_i W^s_{\lambda_0}(w^i_{\lambda_0}) \bigcup_j W^s_{\lambda_0}(p^j_{\lambda_0}) \bigcup_l W^{cs^-}_{\lambda_0}(z^l_{\lambda_0}) \bigcup_m W^{c^-}_{\lambda_0}(q^m_{\lambda_0})$$

*where* $w^i_{\lambda_0}$ *are the hyperbolic equilibrium points in* $\partial A_{\lambda_0}(x^s_{\lambda_0})$*,* $p^j_{\lambda_0}$ *are the type-zero saddle-node equilibrium points,* $z^l_{\lambda_0}$ *the type-k saddle-node equilibrium points, with* $1 \leq k \leq n-2$ *and* $q^m_{\lambda_0}$ *the type-*$(n-1)$ *saddle-node equilibrium points in* $\partial A_{\lambda_0}(x^s_{\lambda_0})$*,* $i, j, l, m = 1, 2, \ldots$*.*

$(ii)$ *There is* $\varepsilon > 0$ *such that, for all* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$*,*

$$\partial A_\lambda(x^s_\lambda) = \bigcup_i W^s_\lambda(w^i_\lambda) \bigcup_j W^s_\lambda(y^j_{\lambda^k}) \bigcup_j W^s_\lambda(y^j_{\lambda^{k+1}})$$

*where* $w^i_\lambda$ *are the perturbed hyperbolic equilibrium points in* $\partial A_\lambda(x^s_\lambda)$*,* $y^j_{\lambda^k}$ *and* $y^j_{\lambda^{k+1}}$ *are the unstable hyperbolic equilibrium points originated from the type-k saddle-node bifurcation, with* $k \geq 0$*, that also belong to* $\partial A_\lambda(x^s_\lambda)$*,* $i, j, = 1, 2, \ldots$*.*

$(iii)$ *There is* $\varepsilon > 0$ *such that, for all* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$*,*

$$\partial A_\lambda(x_\lambda^s) = \bigcup_i W_\lambda^s(w_\lambda^i)$$

*where* $w_\lambda^i$ *are the perturbed hyperbolic equilibrium points in* $\partial A_\lambda(x_\lambda^s)$*,* $i = 1, 2, \ldots$.

*Example* Consider the following system of differential equations from [2]:

$$\begin{aligned}\dot{x} &= 1 - \lambda sen(x) - 2sen(x-y)\\ \dot{y} &= 1 - 3sen(y) - 2sen(y-x)\\ \dot{z} &= \quad\quad -z\end{aligned} \tag{7}$$

where $(x; y; z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

System (7) possesses for $\lambda_0 = 2,84$, a hyperbolic asymptotically stable equilibrium point $x_{\lambda_0}^s = (0,35; 0,34; 0)$ and a type-1 saddle-node equilibrium point $x_{\lambda_0} = (1,42; 3,39; 0)$. The type-1 saddle-node equilibrium point belongs to the stability boundary $\partial_{\lambda_0}(0,35; 0,34; 0)$. For $\lambda = 2,87$, system (7) possesses a hyperbolic asymptotically stable equilibrium point $x_\lambda^s = (0,33; 32; 0)$, a type-1 hyperbolic equilibrium point $p_{\lambda^1} = (1,14; 3,34; 0)$ and a type-2 hyperbolic equilibrium point $p_{\lambda^2} = (1,48; 3,43; 0)$. The equilibrium points $p_{\lambda^1}$ and $p_{\lambda^2}$ are originated from the type-1 saddle-node equilibrium point in a type-1 saddle-node bifurcation. Moreover, $p_{\lambda^1} \in \partial A_\lambda(0,33; 32; 0)$ and $p_{\lambda^2} \in \partial A_\lambda(0,33; 32; 0)$, according to Theorem 18, see Fig. 13.

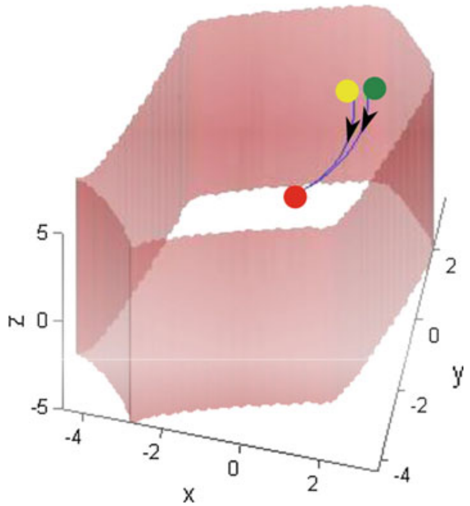



**Fig. 13** The surface in this figure is the stability boundary of the stability region of the asymptotically stable equilibrium point $x_\lambda^s = (0,33; 32; 0)$ of system (7) for $\lambda = 2,87$. The unstable equilibrium points $p_{\lambda^1} = (1,14; 3,34; 0)$ and $p_{\lambda^2} = (1,48; 3,43; 0)$, originated from the type-1 saddle-node bifurcation, belong to the stability boundary $\partial_\lambda(0,33; 32; 0)$. Reprinted from [2]

### 4.3 Hopf Bifurcation on the Stability Boundary

In this section, a characterization of the stability boundary in a small neighborhood of the parameter $\mu_0$ of a Hopf bifurcation of type-$k$, with $k \geq 1$, is developed. We begin the section establishing some concepts of the Hopf bifurcation theory.

Consider the autonomous dynamic system dependent on a parameter

$$\dot{x} = f(x, \mu), \; x \in \mathbb{R}^n, \; \mu \in \mathbb{R} \tag{8}$$

where $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a $\mathscr{C}^1$-vector field .

**Definition 11** (*Hopf Bifurcation Point*) A non-hyperbolic equilibrium point $x_{\mu_0} \in \mathbb{R}^n$ of (8), for a fixed parameter $\mu = \mu_0$, is called a Hopf equilibrium point and $(x_{\mu_0}, \mu_0)$ a Hopf bifurcation point if the following conditions are satisfied:

(i) $D_x f(x_{\mu_0})$ has a simple pair of purely imaginary eigenvalues, $\pm i\omega$, and no other eigenvalue with null real part; and
(ii) $l_1 \neq 0$, where $l_1$ is the first coefficient of Lyapunov, see [9].

Hopf bifurcation points can also be classified according to the sign of the first Lyapunov coefficient. A Hopf bifurcation point $(x_{\mu_0}, \mu_0)$ of (8) is called a supercritical Hopf bifurcation point if the first Lyapunov coefficient $l_1 < 0$ and is called a subcritical Hopf bifurcation point if the first Lyapunov coefficient $l_1 > 0$.

Hopf bifurcation points can also be classified in types according to the number of eigenvalues of $D_x f(x_{\mu_0})$ with positive real part. The Hopf bifurcation point $x_{\mu_0}$ of (8) is called a type-$k$ Hopf bifurcation point if $D_x f(x_{\mu_0})$ has $k$ ($k \leq n-2$) eigenvalues with positive real part and $n-k-2$ with negative real part.

Let $x^s_{\mu_0}$ be an asymptotically stable equilibrium point of (8) and let $A_{\mu_0}(x^s_{\mu_0})$ be its stability region for the fixed parameter $\mu = \mu_0$. Consider the following assumptions:
**(B1')** All the critical elements on $\partial A_{\mu_0}(x^s_{\mu_0})$ are hyperbolic critical elements or supercritical Hopf equilibrium points.
**(B2')** The stable, the center-stable and/or the center manifolds and the unstable manifolds of the critical elements on $\partial A_{\mu_0}(x^s_{\mu_0})$ satisfy the transversality condition.

In the following theorems, we will explore the behavior of the boundary of the stability region of the asymptotically stable equilibrium in a small neighborhood of the parameter $\mu_0$ of a type-$k$ supercritical Hopf bifurcation, with $k \geq 1$. We will assume, for the value of the supercritical Hopf bifurcation parameter $\mu_0$, the existence of only hyperbolic critical elements of system (8) at $\mu = \mu_0$, with the exception of the type-$k$ supercritical Hopf nonhyperbolic equilibrium point, with $k \geq 1$, $x_{\mu_0}$. Furthermore, in a small neighborhood of the parameter $\mu_0$, we will assume the existence of only critical elements that are the perturbed critical elements of the original system (8) at $\mu = \mu_0$. Initially, we will establish the behavior of the boundary of the stability region in the neighborhood of a type-$k$ supercritical Hopf equilibrium point with $k \geq 1$ and, then we will present a global characterization of the boundary in that neighborhood.

**Theorem 19** (Boundary of the Stability Region in the Neighborhood of a Type-$k$ Supercritical Hopf Bifurcation Point with $k \geq 1$) [11] *Let $(\mu_0, x_{\mu_0})$ be a type-k supercritical Hopf bifurcation point, with $k \geq 1$, of (8) for $\mu = \mu_0$. Suppose that the type-k supercritical Hopf bifurcation point $x_{\mu_0}$ belongs to the boundary of the stability region $\partial A_{\mu_0}(x^s_{\mu_0})$ of an asymptotically stable hyperbolic equilibrium point $x^s_{\mu_0}$ of (8) for $\mu = \mu_0$. Admit that assumptions (**B1**), (**B2**) and (**B3**) are satisfied for all $\mu$ belonging to an open interval $I$ containing $\mu = \mu_0$, except in $\mu_0$ where assumptions (**B1'**) and (**B2'**) are satisfied. Furthermore, assume that $x_{\mu_0}$ is the only nonhyperbolic equilibrium point in $\mu = \mu_0$. Suppose also that for all $\mu \in I$, all the critical elements of the perturbed system $\dot{x} = f(x, \mu)$ are perturbed critical elements originated from the system $\dot{x} = f(x, \mu_0)$. Then there is a neighborhood $U$ of $x_{\mu_0}$ and $\varepsilon_1 \geq \varepsilon > 0$ such that:*

*(i) There is a hyperbolic equilibrium point $x^H_\mu$ of type-k, with $1 \leq k \leq n-2$, in $U$ for all $\mu \in (\mu_0 - \varepsilon_1, \mu_0)$ and there are a hyperbolic closed orbit $\Omega^H_\mu$ of type-k, with $1 \leq k \leq n-2$, and a hyperbolic equilibrium point $x^H_\mu$ of type-$k+2$, with $1 \leq k \leq n-2$, in $U$ for all $\mu \in (\mu_0, \mu_0 + \varepsilon_1)$.*
*(ii) For $\mu \in (\mu_0, \mu_0 + \varepsilon)$ we have that $\Omega^H_\mu \in \partial A_\mu(x^s_\mu)$ and $x^H_\mu \in \partial A_\mu(x^s_\mu)$.*
*(iii) For $\mu \in (\mu_0 - \varepsilon, \mu_0)$ we have that $x^H_\mu \in \partial A_\mu(x^s_\mu)$.*

Theorem 19 ensures, for $\mu \in (\mu_0 - \varepsilon, \mu_0)$, the hyperbolic equilibrium point $x^H_\mu$ of type-$k$, with $1 \leq k \leq n-2$, in the neighborhood $U$, belongs to the stability boundary of $x^s_\mu$. At $\mu = \mu_0$, the equilibrium point loses hyperbolicity, leading to the emergence of a type-$k$ supercritical Hopf equilibrium point, with $k \geq 1$. The supercritical Hopf equilibrium point is on the stability boundary of $x^s_\mu$. For values of $\mu > \mu_0$, the hyperbolic equilibrium point $x^H_\mu$ of type-$k+2$, with $1 \leq k \leq n-2$ in $U$ loses stability and a hyperbolic closed orbit $\Omega^H_\mu$ of type-$(k+1)$ arises, with $1 \leq k \leq n-2$, on the stability boundary of $x^s_\mu$. Theorem 19 states that both the stability region as the stability boundary undergo changes when the parameter changes in the interval $(\mu_0 - \varepsilon, \mu_0 + \varepsilon)$. The next result establishes the characterization of the boundary of the stability region in a small neighborhood of the type-$k$ supercritical Hopf bifurcation parameter value, with $k \geq 1$.

**Theorem 20** (Characterization of the Stability Boundary in the Neighborhood of a Type-$k$ Supercritical Hopf Equilibrium Point with $k \geq 1$) [11] *Let $(\mu_0, x_{\mu_0})$ be a type-k supercritical Hopf bifurcation point, with $k \geq 1$, of (8) for $\mu = \mu_0$. Suppose that the type-k supercritical Hopf bifurcation point $x_{\mu_0}$ belongs to the boundary of the stability region $\partial A_{\mu_0}(x^s_{\mu_0})$ of an asymptotically stable hyperbolic equilibrium point $x^s_{\mu_0}$ of (8) for $\mu = \mu_0$. Admit that assumptions (**B1**), (**B2**) and (**B3**) are satisfied for all $\mu$ in an open interval $I$ containing $\mu = \mu_0$, except at $\mu_0$, where assumptions (**B1'**) and (**B2'**) are satisfied. Furthermore, assume that $x_{\mu_0}$ is the only nonhyperbolic equilibrium point in $\mu = \mu_0$. Suppose also that for all $\mu \in I$, all the critical elements of the perturbed system $\dot{x} = f(x, \mu)$ are perturbed critical elements originated from the system $\dot{x} = f(x, \mu_0)$. If $r^i_{\mu_0}$ are the critical elements in $\partial A_{\mu_0}(x^s_{\mu_0})$, $i = 1, \ldots, k$, then:*

*(i)* *For $\mu = \mu_0$ we have $\partial A_{\mu_0}(x^s_{\mu_0}) = \bigcup_i W^s_{\mu_0}(r^i_{\mu_0}) \bigcup W^c_{\mu_0}(x_{\mu_0})$.*

*(ii)* *There is $\varepsilon > 0$ such that, for all $\mu \in (\mu_0 - \varepsilon,\ \mu_0)$, $\partial A_\mu(x^s_\mu) = \bigcup_i W^s_\mu(r^i_\mu) \bigcup W^s_\mu(x^H_\mu)$ where $r^i_\mu$, $i = 1, 2, \ldots, k$ are the perturbed hyperbolic critical elements in $\partial A_\mu(x^s_\mu)$ and $x^H_\mu$ is the type-k hyperbolic equilibrium point, with $1 \le k \le n-2$, originated from the type-k supercritical Hopf bifurcation, $k \ge 1$.*

*(iii)* *There is $\varepsilon > 0$ such that, for all $\mu \in (\mu_0,\ \mu_0 + \varepsilon)$, $\partial A_\mu(x^s_\mu) = \bigcup_i W^s_\mu(r^i_\mu) \bigcup W^s_\mu(x^H_\mu) \bigcup W^s_\mu(\Omega^H_\mu)$ where $r^i_\mu$, $i = 1, 2, \ldots, k$ are the perturbed hyperbolic critical elements in $\partial A_\mu(x^s_\mu)$ and $x^H_\mu$ and $\Omega^H_\mu$ are the type-$(k+2)$ hyperbolic equilibrium point, with $1 \le k \le n-2$, and the type-k periodic orbit, with $1 \le k \le n-2$, respectively, originated from the type-k supercritical Hopf bifurcation, $k \ge 1$.*

In the next two theorems, we will present the behavior of the stability boundary of an asymptotically stable equilibrium point in a small neighborhood of the parameter $\mu_0$ of a type-*k* subcritical Hopf bifurcation, with $k \ge 1$. We will assume, for the value of the subcritical Hopf bifurcation parameter $\mu_0$, the existence of only hyperbolic critical elements of the system (8), with the exception of the type-*k* subcritical Hopf non-hyperbolic equilibrium point, with $k \ge 1$, $x_{\mu_0}$. Furthermore, in a small neighborhood of the parameter $\mu_0$, we will assume the existence of only critical elements that are the disturbed critical elements of the original system (8) in $\mu = \mu_0$. Proceeding in the same way we did in the occurrence of a supercritical Hopf bifurcation, we will establish the behavior of the boundary of the stability region in the neighborhood of a type-*k* subcritical Hopf equilibrium point with $k \ge 1$ and, then we will present a global characterization of the boundary in that neighborhood.

**Theorem 21** (Stability Boundary in the Neighborhood of a Type-*k* Subcritical Hopf Bifurcation Point with $k \ge 1$) [11] *Let $(\mu_0, x_{\mu_0})$ be a type-k subcritical Hopf bifurcation point, with $k \ge 1$, of (8) for $\mu = \mu_0$. Suppose that the type-k subcritical Hopf bifurcation point $x_{\mu_0}$ belongs to the boundary of the stability region $\partial A_{\mu_0}(x^s_{\mu_0})$ of an asymptotically stable hyperbolic equilibrium point $x^s_{\mu_0}$ of (8) for $\mu = \mu_0$. Admit that assumptions **(B1)**, **(B2)** and **(B3)** are satisfied for all $\mu$ belonging to an open interval I containing $\mu = \mu_0$, except in $\mu_0$ where assumptions **(B1")** and **(B2")** are satisfied. Furthermore, assume that $x_{\mu_0}$ is the only nonhyperbolic equilibrium point in $\mu = \mu_0$. Suppose also that for all $\mu \in I$, all the critical elements of the perturbed system $\dot{x} = f(x,\ \mu)$ are perturbed critical elements originated from the system $\dot{x} = f(x,\ \mu_0)$. Then there is a neighborhood of $x_{\mu_0}$ and $\varepsilon_1 \ge \varepsilon > 0$ such that:*

*(i)* *There is a hyperbolic closed orbit $\Omega^H_\mu$ of type-$(k+1)$, with $1 \le k \le n-2$, and a hyperbolic equilibrium point $x^H_\mu$ of type-k, with $1 \le k \le n-2$, in U for all $\mu \in (\mu_0 - \varepsilon_1,\ \mu_0)$ and a hyperbolic equilibrium point $x^H_\mu$ of type-$(k+2)$, with $1 \le k \le n-2$, in U for all $\mu \in (\mu_0,\ \mu_0 + \varepsilon_1)$.*

*(ii)* *For $\mu \in (\mu_0 - \varepsilon,\ \mu_0)$ we have that*

$$\Omega^H_\mu \in \partial A_\mu(x^s_\mu) \quad \text{and} \quad x^H_\mu \in \partial A_\mu(x^s_\mu).$$

*(iii) For* $\mu \in (\mu_0,\ \mu_0+\varepsilon)$ *we have that*

$$x_\mu^H \in \partial A_\mu(x_\mu^s).$$

Theorem 21 ensures, for $\mu \in (\mu_0-\varepsilon,\ \mu_0)$, the hyperbolic periodic orbit $\Omega_\mu^H$ of type-$(k+1)$, with $1 \le k \le n-2$, and the hyperbolic equilibrium point $x_\mu^H$ of type-$k$, with $1 \le k \le n-2$, in the neighborhood $U$, belongs to the stability boundary of $x_\mu^s$. As the parameter $\mu$ grows, the amplitude of the closed orbit decreases and approaches the type-$k$ hyperbolic equilibrium point, with $1 \le k \le n-2$ in $U$. At $\mu=\mu_0$, the periodic orbit coalesces with the hyperbolic equilibrium point in $U$, resulting in the emergence of a type-$k$ subcritical Hopf equilibrium point, with $k \ge 1$. The subcritical Hopf equilibrium point is on the stability boundary of $x_\mu^s$. For values of $\mu > \mu_0$, we have a hyperbolic equilibrium point $x_\mu^H$ in $U$, which belongs to the stability boundary. Theorem 21 states that both the stability region as the stability boundary undergo changes when the parameter changes in the interval $(\mu_0-\varepsilon,\ \mu_0+\varepsilon)$.

The next result establishes the characterization of the stability boundary in a small neighborhood of the parameter value of a type-$k$ subcritical Hopf bifurcation, with $k \ge 1$.

**Theorem 22** (Characterization of the Stability Boundary in the Neighborhood of a Type-$k$ Subcritical Hopf Equilibrium Point with $k \ge 1$) [11] *Let* $(\mu_0, x_{\mu_0})$ *be a type-k subcritical Hopf bifurcation point, with* $k \ge 1$*, of (8) for* $\mu=\mu_0$*. Suppose that the type-k subcritical Hopf bifurcation point* $x_{\mu_0}$ *belongs to the stability boundary* $\partial A_{\mu_0}(x_{\mu_0}^s)$ *of an asymptotically stable hyperbolic equilibrium point* $x_{\mu_0}^s$ *of (8) for* $\mu=\mu_0$*. Admit that assumptions **(B1)**, **(B2)** and **(B3)** are satisfied for all* $\mu$ *belonging to an open interval I containing* $\mu=\mu_0$*, except in* $\mu_0$ *where assumptions **(B1")** and **(B2")** are satisfied. Furthermore, assume that* $x_{\mu_0}$ *is the only nonhyperbolic equilibrium point in* $\mu=\mu_0$*. Suppose also that for all* $\mu \in I$*, all the critical elements of the perturbed system* $\dot{x}=f(x,\ \mu)$ *are perturbed critical elements originated from the system* $\dot{x}=f(x,\ \mu_0)$*. If* $r_{\mu_0}^i$ *are the critical elements in* $\partial A_{\mu_0}(x_{\mu_0}^s)$*,* $i=1,\ \ldots,\ k$*, then:*

*(i) For* $\mu=\mu_0$ *we have*

$$\partial A_{\mu_0}(x_{\mu_0}^s) = \bigcup_i W_{\mu_0}^s(r_{\mu_0}^i) \bigcup W_{\mu_0}^s(x_{\mu_0})$$

*(ii) There is* $\varepsilon > 0$ *such that, for all* $\mu \in (\mu_0-\varepsilon,\ \mu_0)$*,*

$$\partial A_\mu(x_\mu^s) = \bigcup_i W_\mu^s(r_\mu^i) \bigcup W_\mu^s(x_\mu^H) \bigcup W_\mu^s(\Omega_\mu^H)$$

*where* $r_\mu^i$*,* $i=1,\ 2,\ \ldots,\ k$ *are the perturbed hyperbolic critical elements in* $\partial A_\mu(x_\mu^s)$ *and* $x_\mu^H$ *and* $\Omega_\mu^H$ *are the type-k hyperbolic equilibrium point, with* $1 \le k \le n-2$*, and a type-*$(k+1)$ *periodic orbit, with* $1 \le k \le n-2$*, respec-*

*tively, originated from the type-k subcritical Hopf bifurcation, with* $k \geq 1$.

*(iii)* *There is* $\varepsilon > 0$ *such that, for all* $\mu \in (\mu_0,\ \mu_0 + \varepsilon)$,

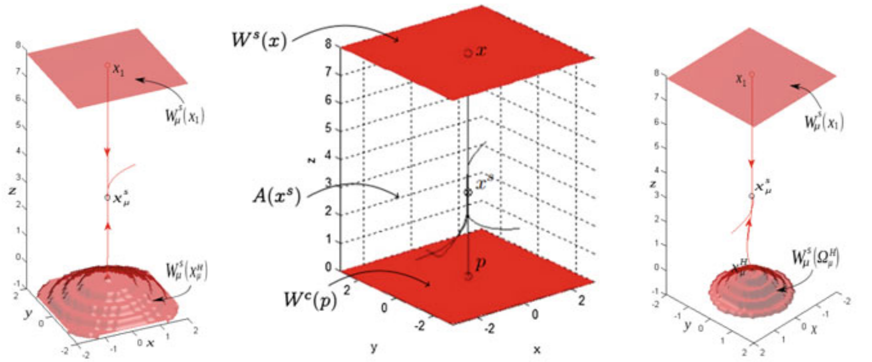
$$\partial A_\mu(x_\mu^s) = \bigcup_i W_\mu^s(r_\mu^i) \bigcup W_\mu^s(x_\mu^H)$$

*where* $r_\mu^i$, $i = 1,\ 2,\ \ldots,\ k$ *are the perturbed hyperbolic critical elements in* $\partial A_\mu(x_\mu^s)$ *and* $x_\mu^H$ *is the type-*$(k+2)$ *hyperbolic equilibrium point, with* $1 \leq k \leq n-2$, *originated from the type-k subcritical Hopf bifurcation,* $k \geq 1$.

*Example* Consider the nonlinear dynamical system from [9]:

$$\begin{cases} \dot{x} = (-z+\mu)x - y - x(x^2+y^2); \\ \dot{y} = (-z+\mu)y + x - y(x^2+y^2); \\ \dot{z} = -0.1(z+0.5(x^2+y^2))(z-3)(8-z); \end{cases} \tag{9}$$

where $(x,\ y,\ z) \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$. For $\mu_0 = 0$, system (9) has three equilibrium points, they are: a type-1 hyperbolic equilibrium point, $x_1 = (0,\ 0,\ 8)$, a type-1 supercritical Hopf equilibrium point, $x_{\mu_0}^H = (0,\ 0,\ 0)$, and an asymptotically stable equilibrium point, $x_\mu^s = (0,\ 0,\ 3)$. The boundary of the stability region of $x_{\mu_0}^s = (0,\ 0,\ 3)$ is formed by the union of the stable manifold of the type-1 hyperbolic equilibrium point $x_1 = (0,\ 0,\ 8)$ with the center manifold of the type-1 supercritical




(a) Phase picture of the system for $\mu = -0.5$.

(b) Phase picture of the system for $\mu_0 = 0$.

(c) Phase picture of the system for $\mu = 0.5$.

**Fig. 14** **a** The boundary of the stability region of $x_\mu^s = (0,\ 0,\ 3)$ is formed by the union of the stable manifold $W_\mu^s(x_1)$ with the stable manifold $W_\mu^s(x_\mu^H)$. **b** The boundary of the stability region of $x_{\mu_0}^s = (0,\ 0,\ 3)$ is formed by the union of the stable manifold $W_{\mu_0}^s(x_1)$ with the center manifold $W_{\mu_0}^c(x_{\mu_0})$. **c** The boundary of the stability region of $x_\mu^s = (0,\ 0,\ 3)$ is formed by the union of the stable manifold $W_\mu^s(x_1)$ with the stable manifold $W_\mu^s(\phi_\mu^H)$. Reprinted from [9]

Hopf equilibrium point, $x^H_{\mu_0} = (0,\ 0,\ 0)$, see Fig. 14b. For $\mu = -0.5$, the system has three equilibrium points, they are: two type-1 hyperbolic equilibrium points, $x^H_\mu = (0,\ 0,\ 0)$ and $x_1 = (0,\ 0,\ 8)$, and an asymptotically stable equilibrium point, $x^s_\mu = (0,\ 0,\ 3)$. The equilibrium point $x^H_\mu$ is originated from the type-1 supercritical Hopf equilibrium point in a type-1 supercritical Hopf bifurcation. The hyperbolic equilibrium points $x^H_\mu = (0,\ 0,\ 0)$ and $x_1 = (0,\ 0,\ 8)$ belong to the boundary of the stability region $\partial A_\mu(x^s_\mu)$, according to Theorem 19, see Fig. 14a. For $\mu = 0.5$, the system has four critical elements, they are: a type-3 hyperbolic equilibrium point, $x^H_\mu = (0,\ 0,\ 0)$, a type-1 hyperbolic equilibrium point, $x_1 = (0,\ 0,\ 8)$, an asymptotically stable equilibrium poin, $x^s_\mu = (0,\ 0,\ 3)$, and a type-1 hyperbolic periodic orbit $\phi^H_\mu$. The critical elements $x^H_\mu$ and $\phi^H_\mu$ were originated from the type-zero supercritical Hopf equilibrium point in a type-zero supercritical Hopf bifurcation. The hyperbolic equilibrium points $x_1$ and $\phi^H_\mu$ belong to the boundary of the stability region $\partial A_\mu(x^s_\mu)$, according to Theorem 19, see Fig. 14c.

## 5 Concluding Remarks

In this chapter, the body of the existing theory regarding the study of changes in the stability region due to parameter variation has been presented. These changes might be very complex and we have studied in this chapter only the ones triggered by two types of local bifurcation on the stability boundary: the saddle-node bifurcation and the Hopf bifurcation. It has been shown that these bifurcations may induce drastic changes in the “size” of the stability region, impacting on the stability of practical systems. There are many open issues to investigate to understand how stability region of general nonlinear systems behave as a consequence of parameter variation. Other types of local bifurcations on the stability boundary and global bifurcations are examples of potential themes for future research.

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