Chapter 5
On the Framization of the Hecke Algebra of Type $\mathtt{B}$

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Abstract We give a cross look to two framizations of the Hecke algebra of type $\mathtt{B}$. One of these is a particular case of the cyclotomic Yokonuma–Hecke algebra. The other one was recently introduced by the author, J. Juyumaya and S. Lambropoulou. The purpose of this paper is to show the main concepts and results of both framizations, giving emphasis to the second one, and to provide a preliminary comparison of the invariants constructed from both framizations.

Introduction

The idea of framization of a knot algebra was introduced by J. Juyumaya and S. Lambropoulou in [14], and it consists in adding certain new generators, called framing generators, to the original presentation of a knot algebra together with certain relations among the original generators and these new generators. One does this procedure, with the aim of constructing new invariants for framed links, and consequently for classical links, see [11, 13]. It is important to mention that the framization procedure doesn't have a structured recipe, whence it is possible to find more than one framization for the same algebra. However, since the motivation behind the procedure of framization is to obtain new polynomial invariants for (framed) knots and links, focus is always given to those framizations that produce such new invariants. Then, when we handle multiple possible framizations of the same knot algebra, we will choose the one that is more natural from a topological viewpoint, cf. [7].

The Yokonuma–Hecke algebra is the first example of framization, since it is considered as a framization of the Hecke algebra of type $\mathtt{A}$. The last 10 years the Yokonuma–Hecke algebra has earned importance in knot theory, since in [10] it was proved that such an algebra supports a Markov trace, therefore, by using the Jones's recipe, invariants for: framed links [13], classical links [11] and singular links [12] were constructed. It is worth to say, that recently it was proved that the invariants for

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S. Lambropoulou et al. (eds.), *Algebraic Modeling of Topological and Computational Structures and Applications*, Springer Proceedings in Mathematics & Statistics 219, https://doi.org/10.1007/978-3-319-68103-0_5

classical links constructed in [11] are not topologically equivalent either to the Homflypt polynomial or to the Kauffman polynomial, see [3].

On the other hand, Jones suggested that his recipe for the construction of the Homflypt polynomial might be used for Hecke algebras of other types than $\mathtt{A}$, cf. [9, p. 336]. Then, S. Lambropoulou used the Jones's recipe for the $\mathtt{B}$–Hecke algebra $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$; in fact in [15, 16] she constructed all the possible analogues of the Homflypt polynomial for oriented knots and links inside the solid torus, see also [6].

In [2] taking as model the Yokonuma–Hecke algebra, M. Chlouveraki and L. Poulain d'Andecy introduced the cyclotomic Yokonuma–Hecke algebra, denoted by $\mathrm{Y}(d, m, n)$. These algebras generalize to the Ariki-Koike algebra, and the Yokonuma–Hecke algebra. In particular, the cyclotomic Yokonuma–Hecke algebra provides a framization of the Hecke algebra of type $\mathtt{B}$, since the Ariki-Koike algebra generalizes $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$. Moreover, in [2] it was also proved that $\mathrm{Y}(d, m, n)$ supports a Markov trace, and therefore, using Jones's recipe, an invariant for framed links in the solid torus was constructed.

Recently in [5] we introduced a new framization of the Hecke algebra of type $\mathtt{B}$, denoted by $\mathrm{Y}^{\mathtt{B}}_{d,n} := \mathrm{Y}^{\mathtt{B}}_{d,n}(\mathsf{u}, \mathsf{v})$, with the principal objective to explore their usefulness in knot theory. More precisely, in this article we constructed two linear bases, a faithful tensorial representation of Jimbo type for $\mathrm{Y}^{\mathtt{B}}_{d,n}(\mathsf{u}, \mathsf{v})$, and we proved that $\mathrm{Y}^{\mathtt{B}}_{d,n}$ supports a Markov trace. Finally we defined, by using Jones's recipe, a new invariant for framed and classical links in the solid torus.

The article is organized as follows. In Sect. 5.1 we introduce the notations and background used in the paper. In Sect. 5.2 we review the main results about the cyclotomic Yokonuma–Hecke algebra given in [2], aiming to provide a preliminary comparison of the invariants constructed from both framizations. The following sections keep the order given in [5], and have as objective to show the main results obtained in that work, and also remark some differences between the framizations $\mathrm{Y}^{\mathtt{B}}_{d,n}$ and $\mathrm{Y}(d, 2, n)$.

5.1 Preliminaries

In this section we review known results, necessary for the sequel, and we also fix the terminology and notations that will be used along the article:

- The letters $\mathsf{u}, \mathsf{v}, \mathsf{v}_1, \ldots, \mathsf{v}_m$ denote indeterminates. Consider $\mathbb{K} := \mathbb{C}(\mathsf{u}, \mathsf{v})$, $\mathscr{R}_m := \mathbb{C}[\mathsf{u}^{\pm 1}, \mathsf{v}_1^{\pm 1}, \ldots, \mathsf{v}_m^{\pm 1}]$, and $\mathscr{F}_m$ the field of fractions of $\mathscr{R}_m$.
- The term *algebra* means unital associative algebra over $\mathbb{K}$
- For a finite group G, $\mathbb{K}[G]$ denotes the group algebra of G
- The letters n and d denote two fixed positive integers
- We denote by ω a fixed primitive d–th root of unity
- We denote by $\mathbb{Z}/d\mathbb{Z}$ the group of integers modulo d, $\{0, 1, \ldots, d-1\}$.
- As usual, we denote by ℓ the length function associated to the Coxeter groups.

5.1.1 *Braids Groups of Type* $\mathtt{B}$

Set $n \geq 1$. Let us denote by W_n the Coxeter group of type $\mathtt{B}_n$. This is the finite Coxeter group associated to the following Dynkin diagram

Define $\mathtt{r}_k = \mathtt{s}_{k-1} \dots \mathtt{s}_1 \mathtt{r}_1 \mathtt{s}_1 \dots \mathtt{s}_{k-1}$ for $2 \leq k \leq n$. It is known, see [6], that every element $w \in W_n$ can be written uniquely as $w = w_1 \dots w_n$ with $w_k \in \mathtt{N}_k$, $1 \leq k \leq n$, where

$$\mathtt{N}_1 = \{1, \mathtt{r}\}, \quad \mathtt{N}_k = \{1, \mathtt{r}_k, \mathtt{s}_{k-1} \dots \mathtt{s}_i, \mathtt{s}_{k-1} \dots \mathtt{s}_i \mathtt{r}_i;\ 1 \leq i \leq k-1\}. \tag{5.1}$$

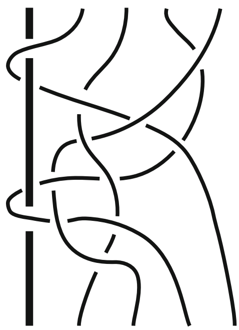
Furthermore, this expression for w is reduced. Hence, we have $\ell(w) = \ell(w_1) + \cdots + \ell(w_n)$.

The corresponding *braid group of type* $\mathtt{B}_n$ associated to W_n, is defined as the group $\widetilde{W}_n$ generated by $\rho_1, \sigma_1, \dots, \sigma_{n-1}$ subject to the following relations

$$\begin{array}{rcll} \sigma_i\sigma_j &=& \sigma_j\sigma_i & \text{for} \quad |i-j|>1, \\ \sigma_i\sigma_j\sigma_i &=& \sigma_j\sigma_i\sigma_j & \text{for} \quad |i-j|=1, \\ \rho_1\sigma_i &=& \sigma_i\rho_1 & \text{for} \quad i>1, \\ \rho_1\sigma_1\rho_1\sigma_1 &=& \sigma_1\rho_1\sigma_1\rho_1. & \end{array} \tag{5.2}$$

Geometrically, braids of type $\mathtt{B}_n$ can be viewed as classical braids of type $\mathtt{A}_n$ with $n+1$ strands, such that the first strand is identically fixed. This is called 'the fixed strand'. The 2nd,…, $(n+1)$st strands are renamed from 1 to n and they are called 'the moving strands'. The 'loop' generator ρ_1 stands for the looping of the first moving strand around the fixed strand in the right-handed sense, see [15, 16]. In Fig. 5.1 we illustrate a braid of type $\mathtt{B}_4$. Another way of visualizing $\mathtt{B}$–type braids geometrically is via symmetric braids, see [18].

Fig. 5.1 A braid of type $\mathtt{B}_4$



Remark 5.1 The group W_n can be realized as a subgroup of the permutation group of the set $X_n := \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$. More precisely, the elements of W_n are the permutations w such that $w(-m) = -w(m)$, for all $m \in X_n$. Further the elements of W_n can be parameterized by the elements of $X_n^n := \{(m_1, \ldots, m_n) \mid m_i \in X_n \text{ for all } i\}$ (see [8, Lemma 1.2.1]). More precisely, the element $w \in W_n$ corresponds to the element $(m_1, \ldots, m_n) \in X_n^n$ such that $m_i = w(i)$.

For example, $\mathtt{s}_i$ is parameterized by $(1, 2, \ldots, i+1, i, \ldots, n)$ and $\mathtt{r}_1$ is parameterized by $(-1, 2, \ldots, n)$. More generally, if $w \in W_n$ is parameterized by $(m_1, \ldots, m_n) \in X_n^n$, then

$$\begin{array}{lll} w\mathtt{r}_1 & \text{is parameterized by} & (-m_1, m_2, \ldots, m_n) \\ w\mathtt{s}_i & \text{is parameterized by} & (m_1, \ldots, m_{i+1}, m_i, \ldots, m_n). \end{array} \tag{5.3}$$

Finally we recall [8, Lemma 1.2.2], which is crucial to prove Proposition 5.1

Lemma 5.1 [8, Lemma 1.2.2] *Let* $w \in W_n$ *parameterized by* $(m_1, \ldots, m_n) \in X_n^n$. *Then* $\ell(w\mathtt{s}_i) = \ell(w) + 1$ *if and only if* $m_i < m_{i+1}$ *and* $\ell(w\mathtt{r}_1) = \ell(w) + 1$ *if and only if* $m_1 > 0$.

5.1.2 Framed Braid Groups of Type B

We start with the definition of a d–framed version of W_n.

Definition 5.1 The *d–modular framed Coxeter group of type* $\mathtt{B}_n$, $W_{d,n}$, is defined as the group generated by $\mathtt{r}_1, \mathtt{s}_1, \ldots, \mathtt{s}_{n-1}$ and $t_1, \ldots, t_n$ satisfying the Coxeter relations of type $\mathtt{B}$ among $\mathtt{r}_1$ and the $\mathtt{s}_i$'s, together with the following relations:

$$\begin{array}{ll} t_i t_j = t_j t_i & \text{for all } i, j, \\ t_i^d = 1 & \text{for all } i, \\ t_j \mathtt{r}_1 = \mathtt{r}_1 t_j & \text{for all } j, \\ t_j \mathtt{s}_i = \mathtt{s}_i t_{s_i(j)} & \text{where } s_i \text{ is the transposition } (i, i+1). \end{array} \tag{5.4}$$

The analogous group defined by the same presentation, where only relations $t_j^d = 1$ are omitted, shall be called *framed Coxeter group of type* $\mathtt{B}_n$ and will be denoted as $W_{\infty,n}$.

Definition 5.2 The *framed braid group of type* $\mathtt{B}_n$, denoted $\mathscr{F}_n^{\mathtt{B}}$, is the group presented by generators $\rho_1, \sigma_1, \ldots, \sigma_{n-1}, t_1, \ldots, t_n$ subject to the relations (5.2), together with the following relations:

$$\begin{array}{ll} t_i t_j = t_j t_i & \text{for all } i, j, \\ t_j \rho_1 = \rho_1 t_j & \text{for all } j, \\ t_j \sigma_i = \sigma_i t_{s_i(j)}. & \end{array} \tag{5.5}$$

The *d–modular framed braid group*, denoted $\mathscr{F}_{d,n}^{\mathtt{B}}$, is defined as the group obtained by adding the relations $t_i^d = 1$, for all i, to the above defining presentation of $\mathscr{F}_n^{\mathtt{B}}$.

The mapping that acts as the identity on the generators $\mathtt{r}_1$ and the $\mathtt{s}_i$'s and maps the t_j's to 1 defines a morphism from $W_{d,n}$ onto W_n. Also, we have the natural epimorphism from $\mathscr{F}_{d,n}^{\mathtt{B}}$ onto $W_{d,n}$ defined as the identity on the t_j's and mapping ρ_1 to $\mathtt{r}_1$ and σ_i to $\mathtt{s}_i$, for all i. Thus, we have the following sequence of epimorphisms.

$$\mathscr{F}_n^{\mathtt{B}} \longrightarrow \mathscr{F}_{d,n}^{\mathtt{B}} \longrightarrow W_{d,n} \longrightarrow W_n$$

where the first arrow is the natural projection of $\mathscr{F}_n^{\mathtt{B}}$ to $\mathscr{F}_{d,n}^{\mathtt{B}}$.

5.2 The Cyclotomic Yokonuma–Hecke Algebras

In this section we will review the main results obtained in [2]. First we recall the definition of cyclotomic Yokonuma–Hecke algebra

Definition 5.3 Let d, m and n positive integers. We denote by $\mathrm{Y}(d, m, n)$ to the associative algebra over $\mathscr{R}_m$ generated by framing generators $t_1, \ldots, t_n$, braiding generators $g_1, \ldots, g_{n-1}$ and the cyclotomic generator b_1 subject to the following relations

$$g_i g_j = g_j g_i \quad \text{for} \quad |i-j| > 1, \tag{5.6}$$
$$g_i g_j g_i = g_j g_i g_j \quad \text{for} \quad |i-j| = 1, \tag{5.7}$$
$$b_1 g_i = g_i b_1 \quad \text{for all} \quad i \neq 1, \tag{5.8}$$
$$b_1 g_1 b_1 g_1 = g_1 b_1 g_1 b_1, \tag{5.9}$$
$$t_i t_j = t_j t_i \quad \text{for all} \quad i, j, \tag{5.10}$$
$$t_j g_i = g_i t_{s_i(j)} \quad \text{for all} \quad i, j, \tag{5.11}$$
$$t_i b_1 = b_1 t_i \quad \text{for all}\, i, \tag{5.12}$$
$$t_i^d = 1 \quad \text{for all} \quad i, \tag{5.13}$$
$$g_i^2 = 1 + (\mathsf{u} - \mathsf{u}^{-1}) e_i g_i \quad \text{for all} \quad 1 \leq i \leq n-1, \tag{5.14}$$
$$(b_1 - \mathsf{v}_1) \ldots (b_1 - \mathsf{v}_m) = 0 \tag{5.15}$$

where the e_i's are the elements introduced in [10], that is

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}, \quad 1 \leq i \leq n-1 \tag{5.16}$$

Four linear bases are given for this algebra in [2]. We recall only one of them, which is used by Chlouveraki and Poulain D'Andecy to define a Markov trace in $\mathrm{Y}(d, m, n)$.

For $k = 1, \dots, n$, we set

$$W^{(k)}_{j,a,b} := g_j^{-1} \dots g_1^{-1} b_1^a t_1^b g_1 \dots g_{k-1},$$

where $j = 0, \dots, k-1$, and $a \in \mathbb{Z}/m\mathbb{Z}$, $b \in \mathbb{Z}/d\mathbb{Z}$. Then, we have the following result

$$B_{d,m,n} := \{W^{(n)}_{j_n,a_n,b_n} \dots W^{(2)}_{j_2,a_2,b_2} W^{(1)}_{j_1,a_1,b_1} \mid j_k = 0, \dots, k-1,\ a_k \in \mathbb{Z}/m\mathbb{Z},\ \text{and } b_k \in \mathbb{Z}/d\mathbb{Z}\}$$

is a basis for $\mathrm{Y}(d, m, n)$, see [2, Sect. 4].

Remark 5.2 To prove the above result, the authors proved first that $B_{d,m,n}$ spans the algebra $\mathrm{Y}(d, m, n)$, and then that its dimension is $(dm)^n n!$. The last result is obtained using tools of representation theory, specifically they constructed a set $\{V_\lambda\}_{\lambda \in \mathscr{P}(d,m,n)}$ of pairwise irreducible non-isomorphic representations of $\mathscr{F}_m \otimes_{\mathscr{R}_m} \mathrm{Y}(d, m, n)$ satisfying the following equation

$$\sum_{\lambda \in \mathscr{P}(d,m,n)} (\dim(V_\lambda))^2 = (dm)^n n!,$$

hence they concluded that $B_{d,m,n}$ is indeed a basis of $\mathrm{Y}_{(d,m,n)}$. Then, in particular, $\mathscr{F}_m \otimes_{\mathscr{R}_m} \mathrm{Y}_{(d,m,n)}$ is a semisimple algebra, for details see [2, Proposition 3.4] and [2, Proposition 4.6]

Using the method of relative traces (see e.g. [2]) it is also proved that the algebra $\mathrm{Y}(d, m, n)$ supports a Markov trace, which we denote by $\mathtt{Tr}_n$. More precisely, this trace is constructed from certain relative traces as follows.

Let z and $x_{a,b}$ with $a \in \{0, \dots, m-1\}$ and $b \in \{0, \dots, d-1\}$, be parameters in $\mathscr{R}_m$ such that $x_{0,0} = 1$. The relative traces $\mathtt{tr}_k : \mathrm{Y}(d, m, k) \to \mathrm{Y}(d, m, k-1)$ are given, for any $k \geq 1$, by.

$$\mathtt{tr}_k(W^{(k)}_{j,a,b} w) = z W^{(k-1)}_{j,a,b} \quad \text{if } 0 \leq j \leq k-1 \tag{5.17}$$

$$\mathtt{tr}_k(W^{(k)}_{j,a,b} w) = x_{a,b} w \quad \text{if } j = k-1 \tag{5.18}$$

where $w \in \mathrm{Y}(d, m, k-1)$.

We define

$$\mathtt{Tr}_n := \mathtt{tr}_1 \circ \dots \circ \mathtt{tr}_n.$$

Then the family $\{\mathtt{Tr}_n\}_{n \geq 1}$ is a Markov trace, for details see [2, Sect. 5].

Finally, as it is usual, using the Jones's recipe, new invariants for framed links in the solid torus are constructed, which are denoted by Γ_m, for details see [2, Sect. 6.3].

Remark 5.3 As we see in [13], to be able define the invariant Γ_m, it is necessary that the trace parameters satisfy a non-linear system of equations. In this case the system is the following

$$\frac{1}{d}\sum_{s=0}^{d-1} x_{0,-s}x_{a,b+s} = x_{a,b}E \qquad \text{for all } a \in \{0,\ldots,m-1\} \text{ and } b \in \{0,\ldots,d-1\},$$

where $E = \mathtt{Tr}_{i+1}(e_i)$, and it is called the affine E–system, also any solution of this system is referred to by saying that it satisfies the affine E–condition. This system is solved in [2, Sect. 6.5] using only standard tools of linear algebra.

5.3 The Algebra $\mathrm{Y}_{d,n}^{\mathtt{B}}$

We begin this section giving the definition of the framization of the Hecke algebra of type $\mathtt{B}$ introduced in [5], denoted by $\mathrm{Y}_{d,n}^{\mathtt{B}}$, which will be the main object of study from now on.

Definition 5.4 Let $n \geq 2$. The algebra $\mathrm{Y}_{d,n}^{\mathtt{B}} := \mathrm{Y}_{d,n}^{\mathtt{B}}(\mathsf{u},\mathsf{v})$, is defined as the algebra over $\mathbb{K} := \mathbb{C}(\mathsf{u},\mathsf{v})$ generated by framing generators $t_1,\ldots,t_n$, braiding generators $g_1,\ldots,g_{n-1}$ and the loop generator b_1, subject to the following relations

$$\begin{aligned}
g_ig_j &= g_jg_i \quad \text{for} \quad |i-j|>1, && (5.19)\\
g_ig_jg_i &= g_jg_ig_j \quad \text{for} \quad |i-j|=1, && (5.20)\\
b_1g_i &= g_ib_1 \quad \text{for all} \quad i \neq 1, && (5.21)\\
b_1g_1b_1g_1 &= g_1b_1g_1b_1, && (5.22)\\
t_it_j &= t_jt_i \quad \text{for all} \ i,j, && (5.23)\\
t_jg_i &= g_it_{s_i(j)} \quad \text{for all} \ i,j, && (5.24)\\
t_ib_1 &= b_1t_i \quad \text{for all i}, && (5.25)\\
t_i^d &= 1 \quad \text{for all} \quad i, && (5.26)\\
g_i^2 &= 1 + (\mathsf{u}-\mathsf{u}^{-1})e_ig_i \quad \text{for all } i, && (5.27)\\
b_1^2 &= 1 + (\mathsf{v}-\mathsf{v}^{-1})f_1b_1., && (5.28)
\end{aligned}$$

where the e_i's are the elements defined in (5.16) and

$$f_1 := \frac{1}{d}\sum_{s=0}^{d-1} t_1^s.$$

For $n = 1$, we define $\mathrm{Y}_{d,1}^{\mathtt{B}}$ as the algebra generated by 1, b_1 and t_1 satisfying the relations (5.25), (5.26) and (5.28).

Notice that the elements f_1 and e_i's are idempotents. In Fig. 5.2 we illustrate the generators of the algebra $\mathrm{Y}_{d,n}^{\mathtt{B}}$.

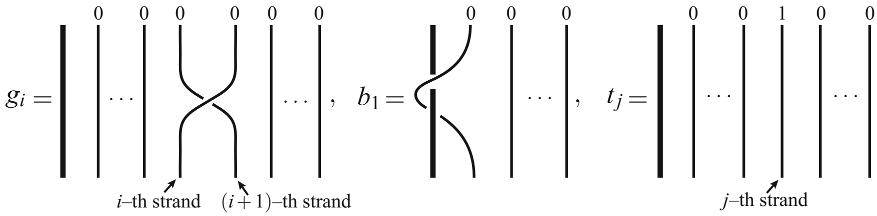


Fig. 5.2 The generators of $\mathrm{Y}^{\mathtt{B}}_{d,n}$ geometrically

Note. By taking $d = 1$, the algebra $\mathrm{Y}^{\mathtt{B}}_{1,n}$ becomes $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$. Further, by mapping $g_i \mapsto h_i$ and $t_i \mapsto 1$, we obtain an epimorphism from $\mathrm{Y}^{\mathtt{B}}_{d,n}$ to $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$. Moreover, if we map the t_i's to a fixed non–trivial d–th root of the unity, we have an epimorphism from $\mathrm{Y}^{\mathtt{B}}_{d,n}$ to $\mathrm{H}_n(\mathsf{u}, 1)$.

Remark 5.4 As we noted previously, the cyclotomic Yokonuma–Hecke algebra also provides a framization of the Hecke algebra of type $\mathtt{B}$. Specifically, if we put $m = 2$, $\mathsf{v}_1 = \mathsf{v}$ and $\mathsf{v}_2 = -\mathsf{v}^{-1}$, then $\mathrm{Y}(d, 2, n)$ is a framization of $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$. But, also we can note that the relation (5.15) doesn't involve framing elements like the defining relation (5.28) of $\mathrm{Y}^{\mathtt{B}}_{d,n}$. In fact $\mathrm{Y}(d, m, n)$ is essentially the Yokonuma–Hecke algebra of type $\mathtt{A}$ with the cyclotomic generator and relation attached. This fact makes us think that $\mathrm{Y}^{\mathtt{B}}_{d,n}$, at least algebraically, is a more natural framization for $\mathrm{H}_n(\mathsf{u}, \mathsf{v})$ than $\mathrm{Y}(d, 2, n)$, since the quadratic relation for the loop generator involves the idempotent element f_1, which plays the analogous role as e_i in the quadratic relation of the braiding generators. Then, in some way the braiding generators and the loop generator interact with the framing generators from a more homogeneous way in $\mathrm{Y}^{\mathtt{B}}_{d,n}$.

5.4 A Tensorial Representation of $\mathrm{Y}^{\mathtt{B}}_{d,n}$

In this section we define a tensorial representation of $\mathrm{Y}^{\mathtt{B}}_{d,n}$, the definition of this representation is based on the tensorial representation constructed by Green in [8] for the Hecke algebra of type $\mathtt{B}$ and following the idea of an extension of the Jimbo representation of the Hecke algebra of type $\mathtt{A}$ to the Yokonuma–Hecke algebra proposed by Espinoza and Ryom–Hansen in [4].

Let V be a $\mathbb{K}$–vector space with basis $\mathscr{B} = \{v_i^r\,;\, i \in X_n,\ 0 \le r \le d-1\}$. As usual we denote by $\mathscr{B}^{\otimes k}$ the natural basis of $V^{\otimes k}$ associated to $\mathscr{B}$.

We define the endomorphisms $T, B : V \to V$ by: $(v_i^r)T = \omega^r v_i^r$, and

$$(v_i^r)B = \begin{cases} v_{-i}^r & \text{for} \quad i > 0 \text{ and } r = 0,\\ v_{-i}^r + (\mathsf{v} - \mathsf{v}^{-1})v_i^r & \text{for} \quad i < 0 \text{ and } r = 0,\\ v_{-i}^r & \text{for} \quad r \neq 0. \end{cases}$$

On the other hand we define $G : V \otimes V \to V \otimes V$ by

$$(v_i^r \otimes v_j^s)G = \begin{cases} \mathsf{u} v_j^s \otimes v_i^r & \text{for} \quad i = j \text{ and } r = s, \\ v_j^s \otimes v_i^r & \text{for} \quad i < j \text{ and } r = s, \\ v_j^s \otimes v_i^r + (\mathsf{u} - \mathsf{u}^{-1}) v_i^r \otimes v_j^s & \text{for} \quad i > j \text{ and } r = s, \\ v_j^s \otimes v_i^r & \text{for} \quad r \neq s. \end{cases}$$

For all $1 \leq i \leq n-1$ and $1 \leq j \leq n$, we extend these endomorphisms to the endomorphisms T_i, G_i and B_1 of the n–th tensor power $V^{\otimes n}$ of V, as follows:

$$T_j := 1_V^{\otimes(j-1)} \otimes T \otimes 1_V^{\otimes(n-j)} \quad \text{and} \quad G_i := 1_V^{\otimes(i-1)} \otimes G \otimes 1_V^{\otimes(n-i-1)} \quad \text{and} \quad B_1 := B \otimes 1_V^{\otimes(n-1)},$$

where $1_V^{\otimes k}$ denotes the endomorphism identity of $V^{\otimes k}$.

Theorem 5.1 (See [5, Theorem 1]) *The mapping $b_1 \mapsto B_1$, $g_i \mapsto G_i$ and $t_i \mapsto T_i$ defines a representation Φ of $\mathrm{Y}_{d,n}^{\mathrm{B}}$ in* $\mathrm{End}(V^{\otimes n})$.

We shall finish the section enunciating Proposition 5.1, which is an analogue of [8, Lemma 3.1.4]. This proposition is used in the proof of Theorem 5.2 and describes, through Φ, the action of W_n on the basis $\mathscr{B}^{\otimes n}$.

The defining generators b_1 and g_i of the algebra $\mathrm{Y}_{d,n}^{\mathrm{B}}$ satisfy the same braid relations as the Coxeter generators $\mathtt{r}$ and $\mathtt{s}_i$ of the group W_n. Thus, the well–known Matsumoto's Lemma implies that if $w_1 \ldots w_m$ is a reduced expression of $w \in W_n$, with $w_i \in \{\mathtt{r}, \mathtt{s}_1, \ldots, \mathtt{s}_{n-1}\}$, then the following element g_w is well–defined:

$$g_w := g_{w_1} \cdots g_{w_m}, \tag{5.29}$$

where $g_{w_i} = b_1$, if $w_i = \mathtt{r}$ and $g_{w_i} = g_j$, if $w_i = \mathtt{s}_j$.

The notation Φ_w stands for the image by Φ of $g_w \in \mathrm{Y}_{d,n}^{\mathrm{B}}$.

Proposition 5.1 (See [5, Proposition 3]) Let $w \in W_n$ parameterized by $(m_1, \ldots, m_n) \in X_n^n$. Then

$$(v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_w = v_{m_1}^{r_{|m_1|}} \otimes \cdots \otimes v_{m_n}^{r_{|m_n|}}.$$

5.5 Linear Bases for $\mathrm{Y}_{d,n}^{\mathrm{B}}$

In this section we construct two linear bases for the algebra $\mathrm{Y}_{d,n}^{\mathrm{B}}$, which will be denote by C_n and D_n. The first one is used for defining a Markov trace on $\mathrm{Y}_{d,n}^{\mathrm{B}}$ (as we will see in the next section), and the second one plays a technical role for proving that C_n is a linearly independent set.

Set $\overline{b}_1 := b_1$, $\overline{b}_k := g_{k-1} \ldots g_1 b_1 g_1 \ldots g_{k-1}$, and $b_k := g_{k-1} \ldots g_1 b_1 g_1^{-1} \ldots g_{k-1}^{-1}$ for all $2 \leq k \leq n$. For all $1 \leq k \leq n$, let us define inductively the sets $N_{d,k}$ by

$$N_{d,1} := \{t_1^m, \overline{b}_1 t_1^m \,;\, 0 \leq m \leq d-1\} \quad \text{and}$$
$$N_{d,k} := \{t_k^m, \overline{b}_k t_k^m, g_{k-1}x \,;\, x \in N_{d,k-1},\ 0 \leq m \leq d-1\} \quad \text{for all } 2 \leq k \leq n.$$

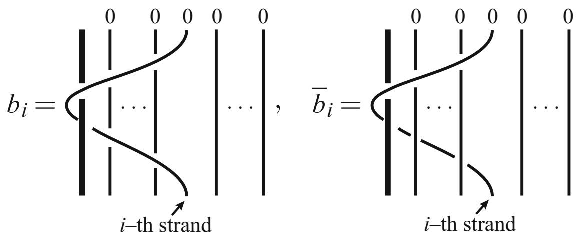


Fig. 5.3 Elements b_i and $\overline{b}_i$ geometrically

Analogously, for all $1 \leq k \leq n$ we define inductively the sets $M_{d,k}$ exactly like $N_{d,k}$'s but exchanging $\overline{b}_k$ by b_k in each case (Fig. 5.3)

Now notice that every element of $M_{d,k}$ has the form $\mathfrak{m}^{+}_{k,j,m}$ or $\mathfrak{m}^{-}_{k,j,m}$ with $j \leq k$ and $0 \leq m \leq d-1$, where

$$\mathfrak{m}^{+}_{k,k,m} := t_k^m, \qquad \mathfrak{m}^{+}_{k,j,m} := g_{k-1} \cdots g_j t_j^m \quad \text{for } j < k,$$

and

$$\mathfrak{m}^{-}_{k,k,m} := t_k^m b_k, \qquad \mathfrak{m}^{-}_{k,j,m} := g_{k-1} \cdots g_j b_j t_j^m \quad \text{for } j < k.$$

Similar expressions exist for elements in $N_{d,k}$ exchanging b_k by $\overline{b}_k$ as well, see [5, Sect. 4.1].

Remark 5.5 Observe that the above elements are the natural analogues of the elements $W^{(k)}_{j,a,b}$ given in Sect. 5.2. Indeed, $\mathfrak{m}^{+}_{k,j,m}$ (respectively $\mathfrak{m}^{-}_{k,j,m}$) correspond to $W^{(k)}_{j-1,0,m}$ (respectively $W^{(k)}_{j-1,1,m}$).

Further, we consider the set $\mathsf{D}_n = \{\mathfrak{n}_1\mathfrak{n}_2 \cdots \mathfrak{n}_n \mid \mathfrak{n}_i \in N_{d,i}\}$, which is a spanning set of $\mathrm{Y}^{\mathrm{B}}_{d,n}$ [5, Proposition 4], moreover using the representation given in the previous section, we can prove that D_n is also a linearly independent set, then we have

Theorem 5.2 D_n *is a linear basis for* $\mathrm{Y}^{\mathrm{B}}_{d,n}$. *Hence the dimension of* $\mathrm{Y}^{\mathrm{B}}_{d,n}$ *is* $2^n d^n n!$.

Sketch of the Proof of Theorem 5.2

Firstly, taking into account the structure properties of W_n given in Sect. 5.1.1, it is easy to see that we can write the basis D_n as follows

$$\mathsf{D}_n = \{g_w t_1^{m_1} \cdots t_n^{m_n} \,;\, w \in W_n,\ (m_1, \ldots, m_n) \in (\mathbb{Z}/d\mathbb{Z})^n\}.$$

for details see [5, Proposition 1].

Secondly, we shall use a certain basis $\mathscr{D}$ of V introduced by Espinoza and Ryom–Hansen in [4]. More precisely, $\mathscr{D}$ consist of the following elements:

$$u_k^r = \sum_{i=0}^{d-1} \omega^{ir} v_k^i \tag{5.30}$$

where k is running X_n and $0 \leq r \leq d-1$.

Moreover, it is not difficult to prove that:

(i) $(u_k^r)T = u_k^{r+1}$.
(ii) For a $w \in W_n$ parameterized by $(i_1, \ldots, i_n)$, we have

$$(u_1^0 \otimes \cdots \otimes u_n^0)\Phi_w = u_{i_1}^0 \otimes \cdots \otimes u_{i_n}^0.$$

Note that (ii) follows by Proposition 5.1 and (5.30).

Now, suppose that

$$\sum_{c \in \mathsf{D}_n} \lambda_c c = \sum_{w \in W_n;\ m \in (\mathbb{Z}/d\mathbb{Z})^n} \lambda_{w,m} g_w t_1^{m_1} \ldots t_n^{m_n} = 0,$$

where $m = (m_1, \ldots, m_n)$. Then applying Φ and evaluating in the element $u_1^0 \otimes \cdots \otimes u_n^0$, we have

$$\sum \lambda_{w,m}(u_1^0 \otimes \cdots \otimes u_n^0)\Phi(g_w t_1^{m_1} \ldots t_n^{m_n}) \ = \ 0$$
$$\sum \lambda_{w,m}(u_1^0 \otimes \cdots \otimes u_n^0)\Phi_w T_1^{m_1} \cdots T_n^{m_n} = 0$$

and by (ii) we obtain

$$\sum \lambda_{i,m} u_{i_1}^0 \otimes \cdots \otimes u_{i_n}^0 T_1^{m_1} \cdots T_n^{m_n} = 0,$$

where $i := (i_1, \ldots, i_n)$ runs in X_n^n and $m := (m_1, \ldots, m_n)$ runs in $(\mathbb{Z}/d\mathbb{Z})^n$. Finally, by using i) the result follows. Further we have the following corollary.

Corollary 5.1 *The representation Φ is faithful.*

Finally, we consider $\mathsf{C}_n = \{\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \mid \mathfrak{m}_i \in M_{d,i}\}$, which also is a spanning set for $\mathrm{Y}_{d,n}^{\mathtt{B}}$, this fact is proved using the computations listed in [5, Lemmas 5, 6 and 7], then as D_n and C_n have the same cardinality we deduce the following result.

Proposition 5.2 *The set C_n is a basis for $\mathrm{Y}_{d,n}^{\mathtt{B}}$.*

5.6 A Markov Trace on $\mathrm{Y}_{d,n}^{\mathtt{B}}$

In this section we show that the algebra $\mathrm{Y}_{d,n}^{\mathtt{B}}$ supports a Markov trace. This fact was proved by using the method of relative traces, cf. [1, 2]. In few words, the method consists in constructing a certain family of linear maps $\mathrm{tr}_n : \mathrm{Y}_{d,n}^{\mathtt{B}} \longrightarrow \mathrm{Y}_{d,n-1}^{\mathtt{B}}$, called *relative traces*, which builds step by step the desired Markov properties. Finally, the Markov trace on $\mathrm{Y}_{d,n}^{\mathtt{B}}$ is defined by

$$\mathrm{Tr}_n := \mathrm{tr}_1 \circ \cdots \circ \mathrm{tr}_n.$$

Let z be an indeterminate and denote by $\mathbb{L}$ the field of rational functions $\mathbb{K}(z) = \mathbb{C}(\mathsf{u}, \mathsf{v}, z)$. We set $x_0 := 1$ and from now on we fix non–zero parameters $x_1, \ldots, x_{d-1}$ and $y_0, \ldots, y_{d-1}$ in $\mathbb{L}$.

Definition 5.5 For $n \geq 1$, we define the linear functions $\mathrm{tr}_n : \mathrm{Y}^{\mathtt{B}}_{d,n} \longrightarrow \mathrm{Y}^{\mathtt{B}}_{d,n-1}$ as follows. For $n = 1$, $\mathrm{tr}_1(t_1^{a_1}) = x_{a_1}$ and $\mathrm{tr}_1(b_1 t_1^{a_1}) = y_{a_1}$. For $n \geq 2$, we define tr_n on the basis C_n of $\mathrm{Y}^{\mathtt{B}}_{d,n}$ by:

$$\mathrm{tr}_n(w\mathfrak{m}_n) = \begin{cases} x_m w & \text{for} \quad \mathfrak{m}_n = t_n^m \\ y_m w & \text{for} \quad \mathfrak{m}_n = b_n t_n^m \\ zw\mathfrak{m}^{\pm}_{n-1,k,m} & \text{for} \quad \mathfrak{m}_n = \mathfrak{m}^{\pm}_{n,k,m} \end{cases} \tag{5.31}$$

where $w := \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \in \mathsf{C}_{n-1}$. Note that (5.31) also holds for $w \in \mathrm{Y}^{\mathtt{B}}_{d,n-1}$.

Using the previous definition and the relations on $\mathrm{Y}^{\mathtt{B}}_{d,n}$, it is not difficult to prove that tr_n has the following properties:

- $\mathrm{tr}_n(XYZ) = X\mathrm{tr}_n(Y)Z, \quad$ for all $X, Z \in \mathrm{Y}^{\mathtt{B}}_{d,n-1}$ and $Y \in \mathrm{Y}^{\mathtt{B}}_{d,n}$. (5.32)
- $\mathrm{tr}_n(Xt_n) = \mathrm{tr}_n(t_n X), \quad$ for all $X \in \mathrm{Y}^{\mathtt{B}}_{d,n}$. (5.33)
- $\mathrm{tr}_{n-1}(\mathrm{tr}_n(Xg_{n-1})) = \mathrm{tr}_{n-1}(\mathrm{tr}_n(g_{n-1}X)), \quad$ for all $X \in \mathrm{Y}^{\mathtt{B}}_{d,n}$. (5.34)

for details see [5, Lemmas 9, 10 and 13] respectively.

We define $\mathrm{Tr}_n : \mathrm{Y}^{\mathtt{B}}_{d,n} \to \mathbb{L}$ inductively by:

$$\mathrm{Tr}_1 := \mathrm{tr}_1 \quad \text{and} \quad \mathrm{Tr}_n := \mathrm{Tr}_{n-1} \circ \mathrm{tr}_n.$$

Thus, we obtain directly

- $\mathrm{Tr}_n(1) = 1$
- $\mathrm{Tr}_n(x) = \mathrm{Tr}_k(x)$, for $x \in \mathrm{Y}^{\mathtt{B}}_{d,k}$ and $n \geq k$.

Let us denote Tr the family $\{\mathrm{Tr}_n\}_{n \geq 1}$. The following theorem is one of the main results of [5].

Theorem 5.3 Tr *is a Markov trace on* $\{\mathrm{Y}^{\mathtt{B}}_{d,n}\}_{n \geq 1}$. *That is, for every* $n \geq 1$, *the linear map* $\mathrm{Tr}_n : \mathrm{Y}^{\mathtt{B}}_{d,n} \longrightarrow \mathbb{L}$ *satisfies the following rules:*

(i) $\mathrm{Tr}_n(1) = 1$,
(ii) $\mathrm{Tr}_{n+1}(Xg_n) = z\mathrm{Tr}_n(X)$,
(iii) $\mathrm{Tr}_{n+1}(Xb_{n+1}t_{n+1}^m) = y_m \mathrm{Tr}_n(X)$,
(iv) $\mathrm{Tr}_{n+1}(Xt_{n+1}^m) = x_m \mathrm{Tr}_n(X)$,
(v) $\mathrm{Tr}_n(XY) = \mathrm{Tr}_n(YX)$,

where $X, Y \in \mathrm{Y}^{\mathtt{B}}_{d,n}$.

Proof Rules (ii)–(iv) are a direct consequences of (5.32). We prove rule (v) by induction on n. For $n = 1$, the rule holds since $\mathrm{Y}^{\mathtt{B}}_{d,1}$ is commutative. Suppose now that (v) is true for all k less than n. We prove it first for $Y \in \mathrm{Y}^{B}_{d,n-1}$ and $X \in \mathrm{Y}^{\mathtt{B}}_{d,n}$. We have

$$\begin{aligned}\mathrm{Tr}_n(XY) &= \mathrm{Tr}_{n-1}(\mathrm{tr}_n(XY)) \overset{(32)}{=} \mathrm{Tr}_{n-1}(\mathrm{tr}_n(X)Y)\\ &\overset{\text{(induction)}}{=} \mathrm{Tr}_{n-1}(Y\mathrm{tr}_n(X)) \overset{(32)}{=} \mathrm{Tr}_{n-1}(\mathrm{tr}_n(YX)).\end{aligned}$$

Hence, $\mathrm{Tr}_n(XY) = \mathrm{Tr}_n(YX)$ for all $X \in \mathrm{Y}^{\mathtt{B}}_{d,n}$ and $Y \in \mathrm{Y}^{\mathtt{B}}_{d,n-1}$. Now, we prove the rule for $Y \in \{g_{n-1}, t_n\}$. By using (5.33) and (5.34), we get

$$\mathrm{Tr}_n(XY) = \mathrm{Tr}_{n-2}(\mathrm{tr}_{n-1}(\mathrm{tr}_n(XY))) = \mathrm{Tr}_{n-2}(\mathrm{tr}_{n-1}(\mathrm{tr}_n(YX))).$$

In summary, we have

$$\mathrm{Tr}_n(XY) = \mathrm{Tr}_n(YX)$$

for all $X \in \mathrm{Y}^{\mathtt{B}}_{d,n}$ and $Y \in \mathrm{Y}^{\mathtt{B}}_{d,n-1} \cup \{g_{n-1}, t_n\}$. Clearly, having in mind the linearity of Tr_n, this last equality implies that rule (v) holds.

Note 5.1 As we could see in the proof of Theorem 5.3, (5.32), (5.33) and (5.34) give step by step the desired Markov properties for Tr. This fact is the principal advantage of use of the relative traces technique.

Remark 5.6 Let $\mathtt{Tr}$ the Markov trace of $\mathrm{Y}(d, 2, n)$ recalled in Sect. 5.2. Then, considering Remark 5.5, and ignoring the fact that $\mathtt{Tr}$ has a different domain, we can say that the Markov trace defined in this section “coincides” with $\mathtt{Tr}$, by taking $x_{0,m} = x_m$ and $x_{1,m} = y_m$, for all $m \in \{0, \ldots, d-1\}$.

5.7 The E–condition and the F–condition

We want to construct a new invariant for applying Jones’s recipe to the pair ($\mathrm{Y}^{\mathtt{B}}_{d,n}$, Tr). For that, as it was seen in [13], we need that the following equation holds

$$\mathrm{Tr}_{n+1}(we_n) = \mathrm{Tr}_n(w)\mathrm{Tr}_{n+1}(e_n) \qquad \text{for all } w \in \mathrm{Y}^{\mathtt{B}}_{d,n}. \tag{5.35}$$

Then we must establish sufficient conditions over the parameters $x_1, \ldots, x_{d-1}$, $y_0, \ldots, y_{d-1} \in \mathbb{L}$, such that (5.35) be satisfied.

With this goal in mind, we define the elements $E^{(k)}$ and $F^{(k)}$ as follows

$$E^{(k)} := \frac{1}{d}\sum_m x_{k+m}x_{d-m} \qquad \text{for} \qquad 0 \le k \le d-1 \tag{5.36}$$

$$F^{(k)} := \frac{1}{d}\sum_{m} x_{d-m}y_{k+m} \qquad \text{for} \qquad 0 \leq k \leq d-1. \tag{5.37}$$

where the summations over m's are regarded modulo d. Note that $E^{(0)} = \mathrm{Tr}_n(e_n)$.

By using the above definitions and the trace rules, we obtain the following results.

Lemma 5.2 *Let* $w = w' t_n^k$, *where* $w' \in \mathrm{Y}^{\mathrm{B}}_{d,n-1}$. *Then*

$$\mathrm{Tr}_{n+1}(we_n^{(m)}) = \frac{E^{(k+m)}}{x_k}\mathrm{Tr}_n(w).$$

Hence, $\mathrm{Tr}_{n+1}(we_n) = \frac{E^{(k)}}{x_k}\mathrm{Tr}_n(w)$.

Lemma 5.3 *Let* $w = w' b_n t_n^k$, *where* $w' \in \mathrm{Y}^{\mathrm{B}}_{d,n-1}$. *Then*

$$\mathrm{Tr}_{n+1}(we_n^{(m)}) = \frac{F^{(k+m)}}{y_k}\mathrm{Tr}_n(w).$$

In particular, we have $\mathrm{Tr}_{n+1}(we_n) = \frac{F^{(k)}}{y_k}\mathrm{Tr}_n(w)$.

Lemma 5.4 *Let* $w = w'\mathfrak{m}^{\pm}_{n,k,\alpha}$, *with* $w' \in \mathrm{Y}^{\mathrm{B}}_{d,n-1}$. *Then* $\mathrm{Tr}_{n+1}(we_n) = z\mathrm{Tr}_n(xe_{n-1})$, *where* $x = \mathfrak{m}^{\pm}_{n-1,k,\alpha}w'$.

Considering the previous lemmas, the following definition becomes natural.

Definition 5.6 The E–system is the non–linear system formed by the following $d-1$ equations:

$$E^{(m)} = x_m E^{(0)} \qquad (0 < m \leq d-1)$$

Any solution $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ of the E–system is referred to by saying that it satisfies the E–condition.

The elements $E^{(k)}$ and the E–system were originally introduced in [13] in order to define new invariants. Specifically, whenever the trace parameters of the Markov trace on the Yokonuma–Hecke algebra satisfy the E–system we have an invariant for framed and classical knots and links in the 3-sphere. Further, in [13, Appendix] P. Gérardin showed that the solutions of the E–system are parameterized by the non-empty subsets of $\mathbb{Z}/d\mathbb{Z}$. Now, we introduce the F–system

Definition 5.7 Assume now that $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ a solution of the E–system parameterized by the set $S \subseteq \mathbb{Z}/d\mathbb{Z}$. The F–system is the following homogeneous linear system of d equations in $y_0, \ldots, y_{d-1}$:

$$\mathrm{F}^{(m)} = y_m \mathrm{E}^{(0)} \qquad (0 \leq m \leq d-1)$$

where $\mathrm{E}^{(0)}$ and $\mathrm{F}^{(m)}$ are the elements that result from replacing x_i by x_i in (5.36) and (5.37) respectively, that is:

$$\mathrm{E}^{(0)} := \frac{1}{d}\sum_m \mathrm{x}_m \mathrm{x}_{d-m} \quad \text{and} \quad \mathrm{F}^{(m)} := \frac{1}{d}\sum_m \mathrm{x}_{d-m} y_{k+m}.$$

Also we have that $\mathrm{E}^{(0)} = \frac{1}{|S|}$, see [13, Sect. 4.3]. Thus the F–system is formed by the following equations:

$$\sum_m \mathrm{x}_{d-m} y_{k+m} - \frac{d}{|S|} y_m = 0 \qquad (0 \leq m \leq d-1). \tag{5.38}$$

For any solution $(\mathrm{y}_0, \ldots, \mathrm{y}_n)$ of the F–system we say that it satisfies the F–condition.

Remark 5.7 Note that the F–system is a particular case of the affine E–system recalled in Remark 5.3. More precisely, when the trace parameters are specialized to complex numbers $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ and $(\mathrm{y}_0, \ldots, \mathrm{y}_n)$ that satisfy the E–condition and the F–condition respectively, then these also satisfy the affine E–system for $m = 2$.

Finally using Lemmas 5.2, 5.3 and 5.4 the following theorem is obtained easily.

Theorem 5.4 *We assume that the trace parameters are specialized to complex numbers* $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ *and* $(\mathrm{y}_0, \ldots, \mathrm{y}_n)$ *that satisfy the* E*–condition and the* F*–condition respectively. Then*

$$\mathrm{Tr}_{n+1}(we_n) = \mathrm{Tr}_n(w)\mathrm{Tr}_{n+1}(e_n) \qquad \textit{for all} \qquad w \in \mathrm{Y}^{\mathtt{B}}_{d,n}. \tag{5.39}$$

5.7.1 Solving the F*–system*

The affine E–system is solved in [2] using only standard tools of linear algebra, then by Remark 5.7, in particular, it provides a solution for the F–system. Now, we give an alternative approach to solve the F–system, following the method of resolution of the E–system done by P. Gérardin, that is, by using some tools from the complex harmonic analysis on finite groups, see [13, Appendix]. We shall introduce first some notations and definitions, necessary for solving the F–system by the method of Gérardin.

We shall regard the group algebra $\Lambda := \mathbb{L}[\mathbb{Z}/d\mathbb{Z}]$, as the algebra formed by all complex functions on $\mathbb{Z}/d\mathbb{Z}$, where the product is the convolution product, that is:

$$(f * g)(x) = \sum_{y \in \mathbb{Z}/d\mathbb{Z}} f(y)g(x-y) \qquad \text{where} \quad f, g \in \Lambda.$$

As usual, we denote by $\delta_a \in \Lambda$ the function with support $\{a\}$.

Also we denote by $\mathbf{e}_a$'s the characters of $\mathbb{Z}/d\mathbb{Z}$, that is:

$$\mathbf{e}_a : b \mapsto \cos\left(\frac{2\pi ab}{d}\right) + i \sin\left(\frac{2\pi ab}{d}\right).$$

The Fourier transform $\mathscr{F}$ on Λ is the automorphism defined by $f \mapsto \widehat{f}$, where

$$\widehat{f}(x) := (f * \mathbf{e}_x)(0) = \sum_{y \in \mathbb{Z}/d\mathbb{Z}} f(y)\mathbf{e}_x(-y).$$

Recall that $(\mathscr{F}^{-1} f)(x) = d^{-1}\widehat{f}(-u)$, where $\widehat{f}(v) = \sum_{u \in G} f(u)\mathbf{e}_v(-u)$. For more properties of Fourier transform over finite groups see [17].

To solve the E–system, Gérardin considered the elements $x \in \Lambda$, defined by $x(k) = x_k$. Then, he interpreted the E–system as the functional equation $x * x = (x * x)(0)x$ with the initial condition $x(0) = 1$. Now, by applying the Fourier transform on this functional equation we obtain $\widehat{x}^2 = (x * x)(0)\widehat{x}$. These last equations imply that $\widehat{x}$ is constant on its support S, where it takes the values $(x * x)(0)$. Thus, we have

$$\widehat{x} = (x * x)(0) \sum_{s \in S} \delta_s.$$

By applying $\mathscr{F}^{-1}$ and the properties listed in the proposition above, Gérardin showed that the solutions of the E–system are parameterized by the non–empty subsets of $\mathbb{Z}/d\mathbb{Z}$. More precisely, for such a subset S, the solution x_S is given as follows.

$$x_S = \frac{1}{|S|} \sum_{s \in S} \mathbf{e}_s.$$

Now, in order to solve the F–system with respect to x_S, we define $y \in \Lambda$ by $y(k) = y_k$. Then we have $F^{(k)} = d^{-1}(x * y)(k)$. So, to solve the F–system is equivalent to solving the following functional equation:

$$x * y = (x * x)(0)y.$$

which, applying the Fourier transform, is equivalent to:

$$\widehat{x}\widehat{y} = (x * x)(0)\widehat{y}.$$

This equation implies that the support of $\widehat{y}$ is contained in the support of $\widehat{x}$. Now, set S the support of $\widehat{x}$. Then we can write $\widehat{y} = \sum_{s \in S} \lambda_s \delta_s$. Finally applying $\mathscr{F}^{-1}$ to the last equation, we get:

$$y = \frac{1}{d} \sum_{s \in S} \lambda_s \mathbf{e}_s.$$

Thus, we have proved the following proposition.

Proposition 5.3 *The solution of the* F*-system with respect to the solution x_S of the* E*–system is in the form:*

$$y_S = \sum_{s \in S} \alpha_s \mathbf{e}_s,$$

where the α_s's are complex numbers.

5.8 Knot and Link Invariants from $\mathrm{Y}^{\mathtt{B}}_{d,n}$

In this section we define invariants for knots and links in the solid torus, by using the Jones's recipe applied to the pairs $(\mathrm{Y}^{\mathtt{B}}_{d,n}, \mathrm{Tr}_n)$ where $n \geq 1$. To do that, we fix a subset S of $\mathbb{Z}/d\mathbb{Z}$ from now, and we will consider the trace parameters x_k's and y_k's as the solutions given in the previous section associated to set S. The invariants constructed here will take values in $\mathbb{L}$.

As in the classical case, the closure of a framed braid α of type B (recall Sect. 5.1.2) is defined by joining with simple (unknotted and unlinked) arcs its corresponding endpoints and is denoted by $\widehat{\alpha}$. The result of closure, $\widehat{\alpha}$, is a framed link in the solid torus, denoted ST. This can be understood by viewing the closure of the fixed strand as the complementary solid torus. For an example of a framed link in the solid torus see Fig. 5.4.

By the analogue of the Markov theorem for ST (cf. for example [15, 16]), isotopy classes of oriented links in ST are in bijection with equivalence classes of braids of type B and this bijection carries through to the class of framed links of type B.

We set

$$\lambda_S := \frac{z - (\mathrm{u} - \mathrm{u}^{-1})E_S}{z} \quad \text{and} \quad \Lambda_S := \frac{1}{z\sqrt{\lambda_S}}, \tag{5.40}$$

where $E_S = \mathrm{Tr}(e_i) = 1/|S|$. We are now in the position to define link invariants in the solid torus.

Definition 5.8 For α in $\mathscr{F}^{\mathtt{B}}_n$, the Markov trace Tr with the trace parameters specialized to solutions of the E–system and the F–system, and π the natural epimorphism of $\mathscr{F}^{\mathtt{B}}_n$ onto $\mathrm{Y}^{\mathtt{B}}_{d,n}$ we define

$$\mathscr{X}^{\mathtt{B}}_S(\widehat{\alpha}) := \Lambda_S^{n-1}(\sqrt{\lambda_S})^e \,\mathrm{Tr}(\pi(\alpha)),$$

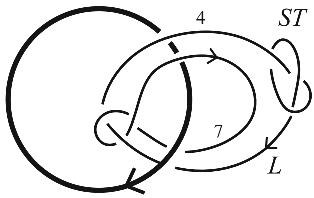


Fig. 5.4 A framed link in the solid torus

where e is the exponent sum of the σ_i's that appear in α. Then $\mathcal{X}_S^{\mathrm{B}}$ is a Laurent polynomial in u, v and z and it depends only on the isotopy class of the framed link $\widehat{\alpha}$, which represents an oriented framed link in ST.

Remark 5.8 The invariants $\mathcal{X}_S^{\mathrm{B}}$, when restricted to framed links with all framings equal to 0, give rise to invariants of oriented classical links in ST. By the results in [3] and since classical knot theory embeds in the knot theory of the solid torus, these invariants are distinguished from the Lambropoulou invariants [15]. More precisely, they are not topologically equivalent to these invariants on *links*.

Remark 5.9 As we said in the Remark 5.7, when we focus in the algebra $Y(d, 2, n)$, the affine E–condition coincide with our conditions (E– and F– condition). Then, we consider the polynomial Γ_m defined in [2, Sect. 6.3] for $m = 2$, which is given by

$$\Gamma_2(\widehat{\alpha}) := \Lambda_S^{n-1}(\sqrt{\lambda_S})^e \, \mathtt{Tr}(\overline{\pi}(\alpha)),$$

where $\overline{\pi} : \mathcal{F}_n^{\mathrm{B}} \to \mathrm{Y}(d, 2, n)$ is the natural algebra epimorphism given by

$$\rho_1 \mapsto b_1, \quad \sigma_i \mapsto g_i, \quad i = 1, \ldots, n-1, \quad \text{and} \quad t_j \mapsto t_j$$

At first sight the invariants look similar, but the structural differences between $\mathrm{Y}_{d,n}^{\mathrm{B}}$ and $\mathrm{Y}(d, 2, n)$ commented in Remark 5.4 make them differ. For example, for the loop generator twice, we have the following

In $\mathrm{Y}_{d,n}^{\mathrm{B}}$	In $\mathrm{Y}(d, 2, n)$
$\mathrm{Tr}(\pi(b_1^2)) = \mathrm{Tr}(1 + (\mathsf{v} - \mathsf{v}^{-1})b_1 f_1)$	$\mathtt{Tr}(\overline{\pi}(b_1^2)) = \mathtt{Tr}(1 + (\mathsf{v} - \mathsf{v}^{-1})b_1)$
$= 1 + \frac{(\mathsf{v}-\mathsf{v}^{-1})}{d} \sum_s \mathrm{Tr}(b_1 t_1^s)$	$= 1 + (\mathsf{v} - \mathsf{v}^{-1})y_0$
$= 1 + \frac{(\mathsf{v}-\mathsf{v}^{-1})}{d} \sum_s y_s$	

Therefore

$$\mathcal{X}_S^{\mathrm{B}}(\widehat{b_1^2}) = 1 + \frac{(\mathsf{v} - \mathsf{v}^{-1})}{d} \sum_s y_s \quad \text{and} \quad \Gamma_2(\widehat{b_1^2}) = 1 + (\mathsf{v} - \mathsf{v}^{-1})y_0.$$

Then clearly for the framed link $\widehat{b_1^2}$, the two invariants have different values, nevertheless in order to do a proper comparison of these invariants is necessary a deeper study.

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