Chapter 13
Fourier Braids

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Abstract By the closure operation, knots can be represented by cyclic braids, which can be unfolded as periodic complex valued functions. Their description by Fourier series allows an approximation by finite Laurent polynomials $g(z)$. We define an algebraic discriminant $\Delta_g^n(z)$, such that an n-braid is given by those $g(z)$ satisfying the condition (S) of having all roots not on the unit circle. We study property (S) from the algebraic and topological viewpoint. Using further algebraic conditions for $g(z)$ we obtain algebraic representations of cyclic braids in thickened surfaces, which represent periodic boundary conditions.

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13.1 Cyclic Braids and the Closure Operation

We note that this research is based on earlier results of the author on algebraic constructions of knots given in [2–4, 6]. The author has also given a talk on these results in the Oberwolfach Workshop 2014 on Algebraic Structures in Low-Dimensional Topology [5].

Because of Alexander's Theorem, knots and links can be obtained by closing braids. We recall the basic notions. Let

$$C^n(\mathbb{C}) := \{(z_1, z_2, \dots, z_n) \mid z_i \in \mathbb{C}, z_i \neq z_j \ \forall i, j\}$$

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be the **ordered configuration space** of n points in the complex plane. There is a free operation of the symmetric group Σ_n on $C^n(\mathbb{C})$ and the quotient gives us the (unordered) **configuration space** $C^n(\mathbb{C})/\Sigma_n$. As a base point in these spaces we can use the configuration $(1, 2, \ldots, n)$ or the configuration $(1, \epsilon, \ldots, \epsilon^{n-1})$ with $\epsilon := e^{2\pi i/n}$.

An n-braid is a closed loop in the configuration space, i.e. a continuous function

$$b : S^1 \to C^n(\mathbb{C})/\Sigma_n.$$

In the following, we often jump between the equivalent descriptions $f : I \to X$ with $f(0) = f(1)$ and $f : S^1 \to X$ for periodic functions with values in some topological space X. Here the identification $I/_{0\sim 1} \cong \mathbb{R}/\mathbb{Z} \cong S^1 \subset \mathbb{C}$ is given by the map $t \mapsto e^{2\pi i t}$.

The set of pointed homotopy classes of n-braids starting and ending in a fixed configuration is just the **braid group**

$$Br_n := \pi_1(C^n(\mathbb{C})/\Sigma_n).$$

The group structure is given by loop sum which we denote by $\perp$, i.e. concatenation of braids. If we do not specify the initial configuration as a base point and consider free homotopy classes, then the set of free homotopy classes of braids is given by the set of conjugacy classes of the group Br_n.

In order to define the **closure** operation we use the open unit disc $\dot{D}^2$ instead of the complex plane. As the strands are images of the closed interval I, they always lie in a bounded region and hence we can shrink them to the unit disc by a suitable contraction factor for any braid. (Alternatively, we could use a fixed diffeomorphism $\mathbb{C} \cong \dot{D}^2$.) Then the n strands of the braid b are embedded in the cylinder $D^2 \times I^1 \subset \mathbb{R}^3$ by $(b(t), t)$ and the closure $\hat{b}$ of b is defined by connecting the initial points and end points of the strands in the bottom and top disc of the cylinder, where the connecting paths lie outside the cylinder and have to be 'parallel'.

Here is an equivalent (isotopic) definition of the closure operation which utilizes the torus parametrization:

$$\tau : D^2 \times I \to \mathbb{R}^3$$

$$\tau(z, t) := \begin{pmatrix} \cos(2\pi t)(2 + Re(z)) \\ \sin(2\pi t)(2 + Re(z)) \\ Im(z) \end{pmatrix}.$$

Then the closure $\hat{b}$ is defined by $\tau(b(t), t)$.

For a general n-braid b the closure does not give a knot but a link in $\mathbb{R}^3$. In fact, the strands of a braid b define a permutation $\rho(b)$ of the set of initial points. Then the closure $\hat{b}$ is a knot if and only if $\rho(b)$ is an n-cycle. We call a braid with this property a **cyclic braid**.

In the general case, the link components of $\hat{b}$ correspond to the cycles in the cycle decomposition of the permutation $\rho(b)$. The other extreme is the case of a **pure braid** which is defined by the condition that $\rho(b)$ is the identity. Because of the covering map

$$C^n(\mathbb{C}) \to C^n(\mathbb{C})/\Sigma_n$$

we get an exact sequence of fundamental groups

$$1 \to PBr_n \to Br_n \to \Sigma_n \to 1$$

where $PBr_n := \pi_1(C^n(\mathbb{C}))$ denotes the **pure braid group**.

It is also well-known that the braid group has a presentation by generators and relations

$$Br_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid R1, R2 \rangle$$

$$R1 : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall i = 1, \ldots, n-2$$

$$R2 : \sigma_i \sigma_j = \sigma_j \sigma_i \ \forall i, j = 1, \ldots, n-2, |i - j| > 1$$

where the σ_i are the standard braid generators (half-twists of the ith and $(i+1)$th strands). The homomorphism ρ is just given by sending σ_i to the transposition of i and $i+1$.

We note that n, the number of strands of a cyclic n-braid, gives a bound for the bridge number of the closure $\hat{b}$. This fact can be seen from the well-known result that the bridge number of a knot $k : S^1 \to \mathbb{R}^3$ can be defined by the following minimum. (See [8] for background in differential topology.) Let $v \in S^2$ be a direction in $\mathbb{R}^3$ and consider the projection of k on the line spanned by v, i.e. $p_{k,v} : S^1 \to \mathbb{R}$, $p_{k,v} := \langle k(t), v \rangle$ (scalar product). By transversality, it is possible to find a direction v and to change k slightly up to isotopy such that $p_{k,v}$ is a Morse-function which has the property that the finite set of singular points consist of local minima and maxima only (i.e. no saddle points) and $p_{k,v}$ takes different values on them. Then the number of local minima equals the number of local maxima because the Euler characteristics of S^1 vanishes. It is well-known that the minimal possible number $deg^M(k) \in \mathbb{N}$ of local minima (where we allow to change k up to isotopy) gives just the bridge number of the knot. Obviously it holds

$$deg^M(k) = 1 \Longleftrightarrow k \text{ is the unknot.}$$

Proposition 13.1 *Let b be a cyclic n-braid, then the bridge number of its closure $\hat{b}$ satisfies $deg^M(\hat{b}) \leq n$.*

Proof Consider the closure as above and chose v in the plane spanned by x and y. After a suitable isotopy (e.g., center the braid along a small disc around 0 such that the closed braid is contained in a small tube around the unit circle), each strand contributes with one local minimum (and one local maximum) to the singular points of $p_{k,v}$. ■

13.2 Unfolding of Cyclic Braids

A general n-braid is given by n functions $I \to \mathbb{C}$ which start and end in the same configuration and whose graphs do not intersect. For a cyclic braid b, we can give a representation by only *one* complex periodic function:

Proposition 13.2 *Concatenation of strands defines a homeomorphism from the set of cyclic n-braids to the space of continuous functions*

$$UCB_n := \{f : S^1 \to \mathbb{C} \mid f(\epsilon^k z) \neq f(z) \; \forall z \in S^1, k = 1, 2, \ldots, \lfloor n/2 \rfloor\}$$

with $\epsilon := e^{2\pi i/n}$ and $\lfloor r \rfloor$ the largest integer smaller than or equal a real number r. In particular, the set of free homotopy classes of cyclic n-braids is given by $\pi_0(UCB_n)$. The closure of a cyclic n-braid b (such that the associated function f takes values in the unit disk) is given as

$$\hat{b}(t) = \begin{pmatrix} \cos(2\pi nt)(2 + Re(f(e^{2\pi it}))) \\ \sin(2\pi nt)(2 + Re(f(e^{2\pi it}))) \\ Im(f(e^{2\pi it})) \end{pmatrix}.$$

Proof For a cyclic braid b, we pick one of the points $u \in \mathbb{C}$ of the initial configuration as a start point with time parameter $t = 0$ and then we concatenate the n strands $b_1, b_2, \ldots, b_n$ which are numerated in the order of the associated n-cycle $\rho(b)$, i.e. b_k starts at $\rho(b)^{k-1}(u)$ and ends in $\rho(b)^k(u)$. This defines a function

$$b_1 \perp b_2 \perp \ldots \perp b_n : [0, n] \to \mathbb{C}$$

which starts and ends in the same point u, see Fig. 13.1 (drawing by the author). Rescaling this function to [0, 1] and using $I/_{0\sim 1} \cong S^1$ we obtain a periodic function

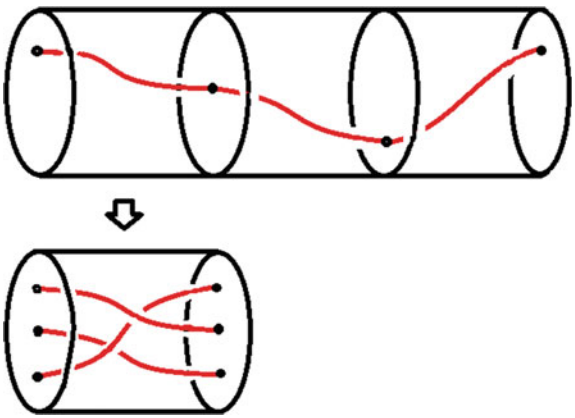

Fig. 13.1 Unfolding of a cyclic three-braid

$f : S^1 \to \mathbb{C}$. Now, multiplication of z by ϵ^k is just a time shift for the function f which changes the start point to the kth strand (according to the cyclic order given by $\rho(b)$). Hence the condition $f(\epsilon^k z) \neq f(\epsilon^l z) \forall z \in S^1$ is equivalent to the condition that the strands b_{k+i} and b_{l+i} do not intersect for all i. Because of $\epsilon^n = 1$ and symmetry, we only need the condition $f(\epsilon^k z) \neq f(z) \forall z \in S^1, k = 1, 2, \ldots, \lfloor n/2 \rfloor$ in order to guarantee that the n strands $b_1, b_2, \ldots, b_n$ do not intersect. If a periodic function $f : I \to \mathbb{C}$ satisfies this condition it can in turn be interpreted as a cyclic braid by defining the kth strand as the (rescaled) restriction of f to the segment $[k/n, (k+1)/n]$. This transformation from cyclic n-braids to functions in UCB_n and back is clearly bijective and continuous which proves the statement on π_0. As the closure can be given by $\hat{b}(t) = \tau(b(t), t)$ and as $\hat{b}$ winds n times around the z-axis, the last formula follows. ■

We call the transformation of a cyclic n-braid b to the associated function $f(z) \in UCB_n$ the **unfolding** of the braid (and 'UCB' is the abbreviation for 'unfolded cyclic braids'). Moreover, the defining property of UCB_n can be formulated with the ***n*-th discriminant** of the function $f(z)$

$$\Delta_f^{(n)}(z) := \prod_{k=1}^{\lfloor n/2 \rfloor} (f(\epsilon^k z) - f(z))$$

by the condition $\Delta_f^{(n)}(z) \neq 0$ for all $z \in S^1$.

As an example, we consider the **torus knot**

$$T_{n,m} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$$

$$T_{n,m}(z) := \begin{pmatrix} \cos(2\pi nt)(2 + \cos(2\pi mt)) \\ \sin(2\pi nt)(2 + \cos(2\pi mt)) \\ \sin(2\pi mt) \end{pmatrix}$$

with $(n, m) = 1$. By definition, $T_{n,m}$ is the closure of a cyclic braid $b_{m,n}$ with n strands which 'wind around' m/n times starting from the initial configuration $(1, \epsilon, \ldots, \epsilon^{n-1})$ (see Fig. 13.2 as an example, drawing by the author). In particular, the Morse index satisfies $deg^M(T_{n,m}) \leq n$. As $T_{n,m}$ and $T_{m,n}$ are isotopic in $\mathbb{R}^3$, it holds also $deg^M(T_{n,m}) \leq m$. Hence $deg^M(T_{2,m}) = 2$ for $m > 1$ odd because these torus knots are known to be non-trivial.

Hence the unfolded function has winding number m around the core of the torus and is given by $f(z) = z^m$. Here, the discriminant is given by $\Delta_{z^m}^{(n)}(z) = \prod_{k=1}^{\lfloor n/2 \rfloor} z^m(\epsilon^{km} - 1)$ which is non-zero on S^1.

As another example, we consider the **figure-eight knot** k_4 (see Fig. 13.3, drawing by the author) which can be obtained as the closure of the cyclic 3-braid given by the braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$. In particular $deg^M(k_4) \leq 3$, but it is well-known that actually $deg^M(k_4) = 2$.

Fig. 13.2 Torus knot $T_{3,8}$

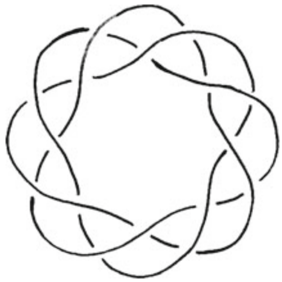

Fig. 13.3 Figure-eight knot

In order to construct a 'nice' explicit unfolding $f(z)$ for the figure-eight knot, we note that we should first construct an unfolding $g(z)$ for the cyclic 3-braid $\sigma_1\sigma_2^{-1}$ (which closes to the unknot!) and then we can set $f(z) := g(z^2)$. The reason is that the braid word for $f(z)$ is the square of the braid word for $g(z)$ and squaring of z has the effect that we run two times through the unfolding function $g(z)$. Now, inspection shows that $g(z) = (z + \bar{z})(1 + z - \bar{z})$ is an element of UCB_3 and serves as an unfolding for $\sigma_1\sigma_2^{-1}$. (It is remarkable that the image of $g(z)$ in $\mathbb{C}$ gives a lemniscate, i.e. the true figure-8 curve!) Hence

$$f(z) = (z^2 + \bar{z}^2)(1 + z^2 - \bar{z}^2) = z^4 - \bar{z}^4 + z^2 + \bar{z}^2$$

gives an unfolding for the figure-eight knot. The discriminant is given by $\Delta_f^{(3)}(z) = (z^4 + \bar{z}^2)(\epsilon - 1) + (-\bar{z}^4 + z^2)(\bar{\epsilon} - 1)$ with $\epsilon = e^{2\pi i/3}$.

The preceding examples show that certain manipulations on unfolding functions have a geometric meaning for the corresponding cyclic braids. Now we list some connections in this direction. Note that $\bar{z} = z^{-1}$ for all $z \in S^1$.

Proposition 13.3 *For cyclic n-braids b and their associated unfoldings* $f(z) \in UCB_n$*, the following correspondences hold true:*

unfolding:	*cyclic n-braid:*
$a + f(z)$	*translation by a*
$af(z)$	*rotation/dilatation by a*
$\overline{f(z)}$	*mirror braid*
$z^n f(z)$	*Dehn twist*
$f(\bar{z})$	*time reversal, inverse braid*
$f(uz)$	*time/phase shift by t*
$f(z^k)$	*k-fold concatenation*

where $a \in \mathbb{C}$, $u = e^{2\pi it} \in S^1$ *and* $(n, k) = 1$.

Proof The statements on translation, rotation/dilatation and mirror image are clear as they hold in $\mathbb{C}$. If we multiply $f(z)$ by z^n, then the new function is also an element in UCB_n and in the corresponding braid, each strand is multiplied by the function z which just yields a full Dehn twist of the whole braid. We note that a multiplication by a power z^k with $0 < k < n$ would in general not give back a cyclic braid (this can also be checked with the defining property $\Delta_f^{(n)}(z) \neq 0$ which then would not be respected). The statements on $f(\bar{z})$ and $f(uz)$ are also clear as they hold in S^1. The last statement for $f(z^k)$ follows as z^k gives a k-fold covering of S^1. ■

One can ask if the *concatenation of two braids* also corresponds to a similar operation for their unfoldings. Unfortunately, this seems not to be the case. The reason is that the concatenation of cyclic braids in general does not yield again a cyclic braid, as this does not even hold on the level of permutations. In order to obtain from a cyclic braid b again a cyclic braid, we could concatenate b with a pure braid c. Now a pure braid has no unfolding but is given by n loop functions $c_1, c_2, \dots, c_n$ which do not intersect. Then the concatenation $b \perp c$ is again a cyclic braid which has a (rescaled) unfolding $b_1 \perp c_1 \perp b_2 \perp c_2 \perp \dots \perp b_n \perp c_n$ and there seems to exist no nice way to express this on the level of unfolded functions.

Also the *Markov stabilization* seems to have no nice description using unfolded functions. Recall that the Markov stabilization of a cyclic n-braid to a cyclic $(n+1)$-braid is defined by concatenation with $\sigma_n^{\pm 1}$. After closure, it corresponds to the first Reidemeister Move.

13.3 The Winding Number of a Cyclic Braid

Now we introduce the winding number of a cyclic n-braid using its unfolding $f(z)$. As the 'kth phase difference'

$$\delta_k(z) := f(\epsilon^k z) - f(z)$$

has to be non-zero on S^1 for all $k = 1, 2, \dots, \lfloor n/2 \rfloor$, we can define its winding number (induced maps on the fundamental group $\pi_1(S^1) = \mathbb{Z}$)

$$deg(\delta_k : S^1 \to \mathbb{C}-\{0\} \simeq S^1) \in \mathbb{Z}.$$

Proposition 13.4 *The winding numbers $deg(\delta_k)$ all are equal. The winding number of the discriminant $deg(\Delta_f^{(n)} : S^1 \to \mathbb{C}-\{0\})$ is given by $deg(\Delta_f^{(n)}) = deg(\delta_f) \cdot \lfloor n/2 \rfloor$.*

Proof For $\alpha : S^1 \to \mathbb{C}-\{0\}$ note that $deg(\alpha(z)) = deg(\alpha(uz))$ where $u \in S^1$ is any phase shift. Now we have

$$f(\epsilon^k z) - f(z) = (f(\epsilon^k z) - f(\epsilon^{k-1} z)) + (f(\epsilon^{k-1} z) - f(z))$$

where the first bracket on the right side is just δ_1 with a phase shift of ϵ^{k-1} and the second bracket is δ_{k-1}, thus

$$\delta_k(z) = \delta_1(\epsilon^{k-1} z) + \delta_{k-1}(z).$$

We prove the following general result: If $\alpha, \beta : S^1 \to \mathbb{C}-\{0\}$ have the property that also their sum $\alpha + \beta$ takes values in $\mathbb{C}-\{0\}$ then it holds $deg(\alpha) = deg(\beta) = deg(\alpha + \beta)$ for their winding numbers. For the proof, consider the diagram

$$\begin{array}{ccc} \mathbb{C}-\{0\} \times \mathbb{C}-\{0\} & \stackrel{+}{\to} & \mathbb{C} \\ \cup & & \cup \\ (\mathbb{C}-\{0\} \times \mathbb{C}-\{0\}) - D & \stackrel{+}{\to} & \mathbb{C}-\{0\} \end{array}$$

where $D := \{z, -z \mid z \in \mathbb{C}-\{0\}\}$ is the anti-diagonal of $\mathbb{C}-\{0\}$. With $\mathbb{C}-\{0\} \simeq S^1$, it is straightforward to check that $(\mathbb{C}-\{0\} \times \mathbb{C}-\{0\}) - D \simeq S^1$, that the inclusion in $\mathbb{C}-\{0\} \times \mathbb{C}-\{0\}$ induces in π_1 the diagonal $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, and that addition in the lower line of the diagram induces the identity map of $\mathbb{Z}$. The general result follows as (α, β) take values in $(\mathbb{C}-\{0\} \times \mathbb{C}-\{0\}) - D$ by assumption.

Now the statement on $deg(\delta_k)$ follows by induction from the above splitting of δ_k. The statement on the winding number of the discriminant follows as complex multiplication $\mathbb{C}-\{0\} \times \mathbb{C}-\{0\} \to \mathbb{C}-\{0\}$ induces addition in π_1. ■

Definition: This number $deg(\delta_f) \in \mathbb{Z}$ is called the **winding number** of the cyclic n-braid.

13.4 Finite Fourier Approximations

In this section we will consider approximations of the functions in UCB_n by Fourier sums $\sum c_k e^{2\pi ikt}$, i.e. by Laurent polynomials $\sum c_k z^k$. We recall that a continuous periodic function $f : S^1 \to \mathbb{C}$ can be approximated by a finite Fourier sum given any **error bound** $r > 0$, i.e. there exists a Laurent polynomial with

$$|f(z) - \sum_{N^- \leq k \leq N^+} c_k z^k| < r$$

for all $z \in S^1$. We call (N^-, N^+) (with $c_{N^-} \neq 0$ and $c_{N^+} \neq 0$) the **bidegree** of the Laurent polynomial $g(z) := \sum_{N^- \leq k \leq N^+} c_k z^k$.

When approximating an unfolding $f(z) \in UCB_n$ we have to be careful that we stay in the same component of UCB_n.

Definition: For $f(z) \in UCB_n$ we define its **width** by

$$w(f) := \min\{|f(\epsilon^k z) - f(z)| \mid \forall z \in S^1, k = 1, 2, \dots, \lfloor n/2 \rfloor\},$$

which is a positive real number.

Lemma 13.1 *A continuous function $g : S^1 \to \mathbb{C}$ which approximates $f(z)$ by an error bound smaller than the width $\frac{1}{2}w(f)$ is a function in UCB_n which lies in the same path component as $f(z)$, i.e. $f(z)$ and $g(z)$ yield (freely) isotopic cyclic n-braids.*

Proof $w(f)$ is non-zero as $|f(\epsilon^k z) - f(z)| > 0$ on S^1 and the minimum is realized by some z as S^1 is compact. By the triangle inequality, $g(z)$ also satisfies $g(\epsilon^k z) - g(z) \neq 0$ for all $z \in S^1$, $k = 1, 2, \dots, \lfloor n/2 \rfloor$. More general, this is true for all functions h_t in the linear homotopy $h : S^1 \times I \to \mathbb{C}$, which is defined by $h_t(z) := f(z) + t(g(z) - f(z))$. Hence all h_t lie in UCB_n which shows that f and g lie in the same path component of UCB_n and hence are associated to isotopic cyclic n-braids. ■

In particular, this holds for a suitable approximation of $f(z)$ by a Laurent polynomial $g(z) = \sum_{N^- \leq k \leq N^+} c_k z^k$. For $N^-, N^+ \in \mathbb{Z}$, denote by

$$\mathbb{C}[z, z^{-1}]^{(N^-, N^+)}$$

the complex $(N^+ - N^- + 1)$-dimensional vector space of Laurent polynomials $g(z)$ with bidegree (n^-, n^+) such that $N^- \leq n^- \leq n^+ \leq N^+$. We also denote for $N \in \mathbb{N}$ the complex $(2N + 1)$-dimensional vector space of Laurent polynomials of order $\leq N$ by

$$\mathbb{C}[z, z^{-1}]^{\pm N} := \mathbb{C}[z, z^{-1}]^{(-N, +N)}.$$

Now we define:

$$UCB_n^{\pm\infty} := UCB_n \cap \mathbb{C}[z, z^{-1}]$$

$$UCB_n^{(N^-, N^+)} := UCB_n \cap \mathbb{C}[z, z^{-1}]^{(N^-, N^+)}$$

$$UCB_n^{\pm N} := UCB_n \cap \mathbb{C}[z, z^{-1}]^{\pm N}$$

This gives a stratification of the space UCB_n of cyclic n-braids

$$UCB_n^{\pm 1} \subset UCB_n^{\pm 2} \subset UCB_n^{\pm 3} \subset \dots \subset UCB_n^{\pm\infty} \subset UCB_n.$$

Corollary 13.1 *Finite Fourier approximation defines a sequence of isotopy types of cyclic n-braids*

$$\pi_0 UCB_n^{\pm 1} \to \pi_0 UCB_n^{\pm 2} \to \pi_0 UCB_n^{\pm 3} \to \cdots \to \pi_0 UCB_n^{\pm \infty} \approx \pi_0 UCB_n.$$

Every cyclic n-braid b can be represented up to isotopy by a Laurent polynomial.

Definition: We call the smallest possible order N of a Laurent polynomial representing a cyclic n-braid b up to isotopy its **Fourier degree** $deg^F(b)$.

As an example, the 'canonical' cyclic n-braid $b_{n,m}$ which gives the torus knot $T_{n,m}$ as a closure has $deg^F(b_{n,m}) \le m$ because z^m serves as an unfolding of b. As the torus knots $T_{n,m}$ and $T_{m,n}$ are isotopic in $\mathbb{R}^3$, we see that deg^F is primarily an invariant of cyclic n-braids (like the number of strands). For the cyclic 3-braid $b = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ with closure the figure-eight knot, we get $deg^F(b) \le 4$ by our computation of an unfolding in the preceding section.

In fact, L. Kauffman [1] and A. Trautwein [12] defined the **Fourier degree for knots** in a similar way. A knot can be considered as a periodic function $k : \mathbb{R}^1 \to \mathbb{R}^3$ and it is also possible to approximate such functions by finite Fourier sums, although the details are a little different as k takes values in real 3-dimensional space, not in $\mathbb{C}$ as in our case. Then Kauffman's Fourier degree $Deg^F(k)$ of a knot type is defined as the minimal Fourier order (i.e. largest frequency) that one needs for a Fourier approximation for k (up to isotopy). Here is a connection between our Fourier degree for cyclic braids and that of Kauffman for knots:

Proposition 13.5 *Let b be a cyclic n-braid and $\hat{b}$ its knot closure. Then it holds*

$$Deg^F(\hat{b}) \le n + deg^F(b).$$

Proof Let $d := deg^F(b)$. By definition, there is a unfolding of b given by a Laurent-Polynomial $g(z) = \sum_{-d\le k\le d} c_k z^k$. Now we use the explicit formula for the closure map

$$\hat{b}(t) = \begin{pmatrix} \cos(2\pi nt)(2 + Re(\sum_{-d\le k\le d} c_k e^{2\pi ikt})) \\ \sin(2\pi nt)(2 + Re(\sum_{-d\le k\le d} c_k e^{2\pi ikt})) \\ Im(\sum_{-d\le k\le d} c_k e^{2\pi ikt}) \end{pmatrix}.$$

Because of the trigonometric sum and product formulas which can be derived from $e^{2\pi i(r+s)t} = e^{2\pi i(r)t}e^{2\pi i(s)t}$, the largest frequency in a product of two periodic functions is the sum of largest frequencies of the factors. Hence the first two coordinates contain maximal frequencies of order $n + d$, whereas d is the largest frequency in the last coordinate. ■

This result can be interpreted in the following way: The maximal frequency in the unfolded braid is enlarged by n because of the n-fold winding of the braid around the z-axis in order to get the closure.

13.5 Property S and the Discriminant Variety

Now we consider the discriminant varieties in the spaces of Laurent polynomials

$$\Delta_n^{\pm\infty} := \{g(z) \in \mathbb{C}[z, z^{-1}] \mid \Delta_g^{(n)}(z) \text{ has a zero on } S^1\},$$

$$\Delta_n^{\pm N} := \{g(z) \in \mathbb{C}[z, z^{-1}]^{\pm N} \mid \Delta_g^{(n)}(z) \text{ has a zero on } S^1\},$$

which are the complements of the subspaces $UCB_n^{\pm\infty}$ and $UCB_n^{\pm N}$.

This leads us to consider the following open subset S of complex polynomials:

$$S := \{p(z) \in \mathbb{C}[z] \mid p(z) \neq 0 \text{ on } S^1\}.$$

By definition, S is the complement in $\mathbb{C}[z]$ of a codimension-1 discriminant variety

$$\Delta := \{p(z) \in \mathbb{C}[z] \mid p(z) \text{ has a zero on } S^1\}.$$

As every Laurent polynomial $g(z)$ can be written as $g(z) = z^{N^-} p(z)$ with a uniquely defined complex polynomial $p(x) := z^{-N^-} g(z)$ (with $p(0) \neq 0$), the condition of having no zeros on S^1 depends only on the polynomial $p(z)$ which can be considered as the essential part of $g(z)$.

Here is a result which connects the winding numbers of $p(z)$ and $g(z)$ (as functions from S^1 to $\mathbb{C}-\{0\}$) with the number of zeros of $p(z)$ in the unit disc.

Proposition 13.6 *The winding numbers of $p(z) \in S$ and $g(z) = z^{N^-} p(z)$ are given by*

$$deg(p) = \text{number of zeros of } p \text{ in the open unit disc } \dot{D}^2$$

$$deg(g) = deg(p) + N^-.$$

Proof Let $a(x), b(x) \in \mathbb{C}[z]$ be polynomials which are non-zero on S^1. As complex multiplication $\mathbb{C}-\{0\} \times \mathbb{C}-\{0\} \to \mathbb{C}-\{0\}$ induces addition in π_1, we see that $deg(ab) = deg(a) + deg(b)$. If b has no zeros on the unit disc, we get a homotopy $b : D^2 \to \mathbb{C}-\{0\}$ of $b : S^1 \to \mathbb{C}-\{0\}$ to the constant map $b(0) \in \mathbb{C}-\{0\}$, hence $deg(b) = 0$. Assume that a is normed and has all zeros on the open unit disc, i.e. $a(z) = \prod_{i=1}^d (z - z_i)$ with all $z_i \in \dot{D}^2$. Then for $t \in I$ we define

$$a_t(z) := \prod_{i=1}^{d} (z - tz_i) \in \mathbb{C}[z]$$

which gives a homotopy $a_t : S^1 \times I \to \mathbb{C}-\{0\}$ from a to $a_0(z) = z^d$. Hence $deg(a) = deg(z^d) = d$. Now every polynomial $p(z)$ can be split as $p(z) = a(z)b(z)$ with a and b as above and the statement on $deg(p)$ follows. The last statement on $deg(g)$ follows just from $g(z) = z^{N^-} p(z)$. ■

Now we consider S from the algebraic viewpoint. Given a polynomial $p(z) \in \mathbb{C}[z]$, we look for an algebraic invariant δ_p which detects the properties $p \in S$ or $p \in \Delta$. This leads to the following construction. We assume that p is normed with decomposition into linear factors $p(z) = \prod_{i=1}^{d}(z - z_i)$. Then we define

$$\delta_p(z) := \prod_{i=1}^{d}(z - z_i\bar{z}_i).$$

By definition, $\delta_p(z)$ is a polynomial of the same order as $p(z)$ with zeros the real numbers $z_i\bar{z}_i$. Hence $\delta_p(1) = 0$ is equivalent to the condition that $p(z)$ has a zero on S^1, i.e. $p \in \Delta$. Thus

$$\delta_p := \delta_p(1) = \prod_{i=1}^{d}(1 - z_i\bar{z}_i) \in \mathbb{R}$$

serves as an invariant we are looking for. Unfortunately, δ_p cannot be expressed as a polynomial invariant in the coefficients of $p(z)$ and their complex conjugates. It is not possible to apply here the fundamental theorem for symmetric polynomials (in contrast to e.g. $\prod_{i=1}^{d}(z - z_i^2)$) because $\mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]^{\Sigma_n}$ is not the polynomial ring in the elementary symmetric functions on the z_i and that on their conjugates $\bar{z}_i$.

As an example, we consider S in low degrees. Clearly, a linear complex polynomial $p(z) = z + a_0$ is in S if and only if $a_0 \notin S^1$. Already the case of a quadratic complex polynomial $p(z) = z^2 + a_1 z + a_0$ demonstrates the difficulty to describe S by conditions on the coefficients. We have $a_1 = -(z_1 + z_2)$ and $a_0 = z_1 z_2$, whereas

$$\delta_p = (1 - z_1\bar{z}_1)(1 - z_2\bar{z}_2) = 1 - (z_1\bar{z}_1 + z_2\bar{z}_2) + z_1\bar{z}_1 z_2\bar{z}_2.$$

Now $z_1\bar{z}_1 z_2\bar{z}_2 = a_0\bar{a}_0$, but $a_1\bar{a}_1 = (z_1\bar{z}_1 + z_2\bar{z}_2) + (z_1\bar{z}_2 + z_2\bar{z}_1)$ and there is no way to express the mixed sum by polynomials in the coefficients a_0, a_1 and their conjugates. Instead it is possible to give δ_p by a complicated real algebraic function of a_0 and a_1.

13.6 Cylic Braids Avoiding Fixed Links and Knots in Spaces with Periodic Boundary Conditions

We have seen that cyclic n-braids are given by unfoldings $f(z)$ which can be chosen as Laurent polynomials

$$UCB_n^{\infty} = \{f(z) \in \mathbb{Z}[z, z^{-1}] \mid \Delta_f^{(n)}(z) \neq 0 \text{ on } S^1\}.$$

Now we consider the additional condition

$$f(z) \neq 0 \text{ on } S^1$$

which means that we exclude the soul $\{0\} \times S^1$ from the solid torus $D^2 \times S^1$. Then the closure operation $b \mapsto \hat{b}$ gives us a knot in $\mathbb{R}^3 - (S^1 \times \{0\})$ which itself is diffeomorphic to an open full torus with an interior point (corresponding to infinity) removed. If we compactify $\mathbb{R}^3$ to S^3, we do not have to remove this inner point, but removing finitely many points from the space where a knot, link or braid is embedded does not change the isotopy classes of embeddings. Hence the algebraic space

$$UCB^{\infty}_{n,1} := \{f(z) \in \mathbb{Z}[z, z^{-1}] \mid f(z)\Delta_f^{(n)}(z) \neq 0 \text{ on } S^1\}.$$

serves as a model for knots in the solid torus. See the work of S. Lambropoulou [7] for more details on the theory of knots in thickened surfaces.

More generally,

$$UCB^{\infty}_{n,m} := \{f(z) \in \mathbb{Z}[z, z^{-1}] \mid \prod_{k=0}^{m-1}(f(z) - k)\Delta_f^{(n)}(z) \neq 0 \text{ on } S^1\}$$

is the algebraic space of unfolded n-braids which avoid in $S^1 \times \mathbb{C}$ the unlink with m components given by $S^1 \times \{k\}$ for $k = 0, 1, \ldots, m-1$. Their homotopy classes just form the subset of cyclic braids in the relative braid group $Br_{n,m}$, see [7]. Hence $UCB^{\infty}_{n,m}$ is a model space for certain knots in the ambient space $\mathbb{R}^3 - (S^1 \times \{0, 1, \ldots, m-1\})$ which is diffeomorphic to the complement of m solid tori in $\mathbb{R}^3$ which are unknotted and unlinked.

In order to obtain knots in the thickened torus $S^1 \times S^1 \times I$, we have to modify our construction by using the Hopf link H in $S^1 \times \mathbb{C}$ instead of the trivial 2-link $(S^1 \times \{0, 1\})$. The reason is that the complement of H is diffeomorphic to the thickened torus (minus the point at infinity). As the Hopf link can be constructed by the embedding $z \mapsto \{0, z\}$ and as each of the n strands has to avoid it, the unfolded cyclic braid has to avoid the n-fold Hopf link. Thus the algebraic space

$$UCB^{\infty}_{n,H} := \{f(z) \in \mathbb{Z}[z, z^{-1}] \mid f(z)(f(z) - z^n)\Delta_f^{(n)}(z) \neq 0 \text{ on } S^1\}$$

is a model for knots in the thickened torus $S^1 \times S^1 \times I$.

In particular, these algebraic spaces approximate knots in the spaces $S^1 \times I \times I$ and $S^1 \times S^1 \times I$ with one- and two-periodic boundary conditions. I.e. these spaces are formed from the cube I^3 by identifying one or two antipodal pairs of faces. It would be interesting to model by this method more general knotted configurations in spaces with periodic boundary conditions. This could produce new connections of Fourier series and Laurent polynomials to applications of knot theory in polymer physics, see [9–11].

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