

# Chapter 12

## On the Height of Knotoids

Neslihan Ggmc and Louis H. Kauffman

**Abstract** Knotoid diagrams are defined in analogy to open ended knot diagrams with two distinct endpoints that can be located in any region of the diagram. The height of a knotoid is the minimal crossing distance between the endpoints taken over all equivalent knotoid diagrams. We define two knotoid invariants; the affine index polynomial and the arrow polynomial that were originally defined as virtual knot invariants given in (Kauffman, *J Knot Theory Ramif* 21(3), 37, 2012) [6], (Kauffman, *J Knot Theory Ramif* 22(4), 30, 2013) [8], respectively, but here are described entirely in terms of knotoids in  $S^2$ . We reprise here our results given in (Ggmc, Kauffman, *Eur J Combin* 65C, 186–229, 2017) [3] that show that both polynomials give a lower bound for the height of knotoids.

### 12.1 Introduction

The theory of knotoids was introduced by V. Turaev [17] in 2012. A knotoid diagram [17] is an open ended knot diagram with two endpoints that can be located in any region of the diagram. The theory of knotoids forms a new diagrammatic theory that is an extension of the classical knot theory. In this paper, we give an exposition of two new polynomial knotoid invariants that were constructed in [3].

It is natural to examine knotoids in the context of virtual knot theory [5, 6]. Virtual knots are knots in thickened surfaces (or knot diagrams in surfaces) taken up to handle stabilization. There is a diagrammatic theory for virtual knots, as we explain briefly in this paper. The endpoints of a knotoid diagram can be connected to form what we call the virtual closure of the diagram. This way of connecting the endpoints of

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a knotoid diagram forms a well-defined map from the set of knotoids to the set of virtual knots. Virtual knot invariants can be then applied to extract knotoid invariants by using the virtual closure map.

Section 12.2 recollects the fundamental concepts of knotoids. Knotoid diagrams can be defined both in  $S^2$  and  $\mathbb{R}^2$ . There is an inclusion map between two sets of knotoids; knotoids in  $\mathbb{R}^2$  and knotoids in  $S^2$ , induced by the inclusion  $\mathbb{R}^2 \hookrightarrow S^2$ . Knotoids in  $\mathbb{R}^2$  are a part of geometric 3-dimensional knot theory, and, as such, are related to open-ended embeddings of intervals in three-dimensional space. We discuss this point of view in Sect. 2.4.

Given a knotoid  $K$  in  $S^2$ , we can ask how far apart the endpoints need to be in all instances of diagrams for the equivalence class of  $K$ . The smallest distance between two endpoints of  $K$  (in terms of number of classical crossings created while connecting the endpoints with an embedded arc that goes under) is called the *height* of the knotoid. The height is an invariant of knotoids in  $S^2$  [17]. In Sect. 12.3 we define the height of a knotoid as in [17]. We then make a remark on virtual knot theory and discuss the virtual closure map. In Sect. 3.2, we mention two conjectures from [3]. One of the conjectures asserts that there are virtual knots of genus 1 which are not in the image of the virtual closure map. We have proved this conjecture by a use of surface bracket polynomial of virtual knots/links and we will present the proof in a subsequent paper. The other conjecture asserts that the normalized bracket polynomial (the Jones polynomial) detects the trivial knotoid. This conjecture extends the well-known conjecture on the Jones polynomial of knots.

Section 12.4 is devoted to two polynomial invariants of knotoids; the affine index polynomial and the arrow polynomial. Both of these polynomials are examined by the authors in full detail in [3]. The affine index polynomial and the arrow polynomial were originally defined as virtual knot invariants [2, 8, 14]. We observe in [3] that they can be defined as invariants of knotoids by considering only knotoid diagrams. In this paper we present our main results given in [3], that consist in lower bound estimations for the height of knotoids via these two polynomials. We give sketches of the proofs for our results. We end the section with examples for the use of these estimations.

## 12.2 About Knotoids

A *knotoid diagram*  $K$  in  $S^2$  or in  $\mathbb{R}^2$  is defined as a generic immersion,

$$K : [0, 1] \rightarrow S^2 \text{ or } \mathbb{R}^2 \text{ such that}$$

- $K$  has finitely many transversal double points. These points are endowed with over/under-crossing data and called the *crossings* of  $K$ .
- The images of 0 and 1 are two points distinct from each other, and from any of crossings of  $K$ . These two points are the *endpoints* of a knotoid diagram and called the *tail* and the *head* of  $K$ , respectively. A knotoid diagram is always oriented from its tail to its head.

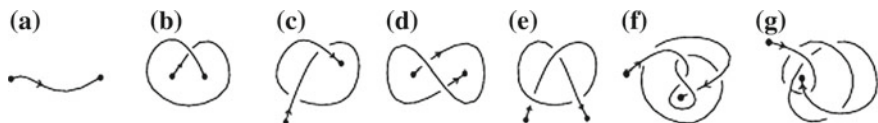


Fig. 12.1 Some examples of knotoid diagrams

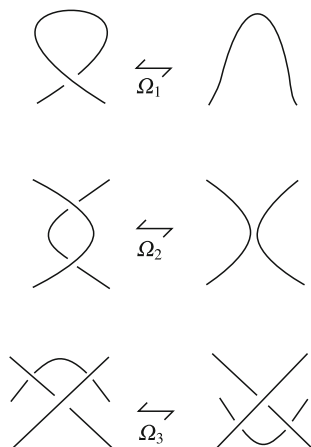


Fig. 12.2  $\Omega$ -moves

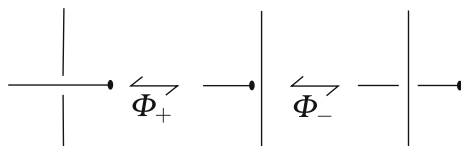


Fig. 12.3 Forbidden knotoid moves

The trivial knotoid diagram is an embedding of the unit interval into  $S^2$  (or in  $\mathbb{R}^2$ ). It is depicted by an arc without any crossings as shown in Fig. 12.1a.

The three Reidemeister moves shown in Fig. 12.2, are defined on knotoid diagrams, and denoted by  $\Omega_1, \Omega_2, \Omega_3$ , respectively. These moves modify a knotoid diagram within a small disk as shown in the figure without utilizing the endpoints. The moves in Fig. 12.3 that consist of pulling an endpoint over or under a strand, are the *forbidden over and under moves*, and denoted by  $\Phi_+$  and  $\Phi_-$ , respectively. It is clear that if both  $\Phi_+$  and  $\Phi_-$  moves were allowed, any knotoid diagram could be turned into the trivial knotoid diagram.

The  $\Omega_{i=1,2,3}$ -moves plus isotopy of  $S^2$  generate an equivalence relation on knotoid diagrams in  $S^2$  (for knotoid diagrams in the plane we consider the isotopy of the plane). A *knotoid* is defined to be an equivalence class of all equivalent knotoid diagrams up to this equivalence relation. The set of all knotoid classes in  $S^2$  and in  $\mathbb{R}^2$  are denoted by  $\mathcal{K}(S^2)$  and  $\mathcal{K}(\mathbb{R}^2)$ , respectively.

There is a well-defined map between the two knotoid sets,

$$\iota : \mathcal{K}(\mathbb{R}^2) \rightarrow \mathcal{K}(S^2),$$

that is induced by the inclusion  $\mathbb{R}^2 \hookrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$ . Any knotoid in  $S^2$  can be represented by a knotoid diagram in  $\mathbb{R}^2$  by pushing a representative diagram in  $S^2$  away from the  $\infty \in S^2$ . Considering the equivalence class of this planar representation in  $\mathcal{K}(\mathbb{R}^2)$ , there is a well-defined map  $\rho : \mathcal{K}(S^2) \rightarrow \mathcal{K}(\mathbb{R}^2)$ . It is clear that  $\iota \circ \rho = id$  so that  $\iota$  is surjective. However, there are examples of nontrivial knotoids in  $\mathbb{R}^2$  which are trivial in  $\mathcal{K}(S^2)$  showing that the map  $\iota$  is not injective. An example for this is given in Fig. 12.1b. The knotoid diagram in the figure which represents a nontrivial planar knotoid [17] although it represents the trivial knotoid in  $S^2$ . In this paper, we study the knotoids in  $S^2$  and we mean knotoids in  $S^2$  unless otherwise stated.

**Definition 12.1** Let  $\mathcal{M}$  be a category of mathematical structures (e.g. polynomials, Laurent polynomials, the integers modulo five, commutative rings, groups,  $\dots$ ). An *invariant* of knotoids is a mapping  $I : \mathcal{K}(S^2) \rightarrow \mathcal{M}$  such that equivalent knotoid diagrams map to equivalent structures in  $\mathcal{M}$ .

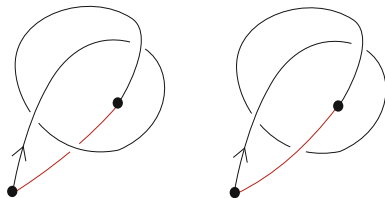
### 12.2.1 Knots via Knotoid Diagrams

In [17] the study of knotoid diagrams is suggested as a new diagrammatic approach to the study of knots in three dimensional space  $\mathbb{R}^3$  in the following way. The endpoints of a knotoid diagram can be connected with an embedded arc in  $S^2$  that is declared to go under each strand it crosses. In this way we obtain an oriented classical knot diagram in  $S^2$  representing a knot in  $\mathbb{R}^3$ . The resulting knot diagram is called the *underpass closure* of the knotoid diagram. Note that the arc connecting the endpoints is unique up to isotopy. We say that a knotoid diagram  $K$  represents a classical knot  $\kappa$  if  $\kappa$  is represented by the underpass closure of  $K$ .

Alternatively, the endpoints of a knotoid diagram in  $S^2$  can be connected with an embedded arc in  $S^2$  which is declared to go over every strand it crosses. The resulting oriented knot diagram is called the *overpass closure* of the knotoid diagram. Note that this type of connection arc is also unique up to isotopy. The underpass closure and the overpass closure of a knotoid diagram may represent inequivalent knots. For instance, the knotoid diagram given in Fig. 12.4 represents a trefoil via the underpass closure and represents the trivial knot via the overpass closure. In order to have a well-defined representation of knots via knotoid diagrams, we should fix the closure type. The closure type is chosen to be the underpass closure and the following proposition follows.

**Proposition 12.1** ([17]) *Two knotoid diagrams  $K_1$  and  $K_2$  represent the same classical knot if and only if they are related to each other by finitely many  $\Omega_{i=1,2,3}$ -moves,  $\Phi_-$ -moves (forbidden under moves) and isotopy of  $S^2$ .*

**Fig. 12.4** Two types of closures



Given a knot in  $\mathbb{R}^3$ , we take an oriented diagram of the knot in  $S^2$ . Cutting out an underpassing arc which may contain no crossing or one or more crossings from this diagram, results in a knotoid diagram that represents the given knot. In fact, any knot in  $\mathbb{R}^3$  can be represented by a knotoid diagram in  $S^2$ .

Representing knots in three dimensional space via knotoid diagrams may ease the computation or give finer estimations for knot invariants. See [17] for the knot group computation via knotoid diagrams and for the lower bound estimation for the Seifert genus of knots.

### 12.2.2 Knotoids as an Extension of Knot Theory

There is a well-defined injective map,

$$\alpha : \{\text{Knot Diagrams in } S^2 / \langle \Omega_1, \Omega_2, \Omega_3 \rangle\} \rightarrow \mathcal{K}(S^2),$$

where  $\langle \Omega_1, \Omega_2, \Omega_3 \rangle$  denotes the equivalence on knot diagrams given by the three Reidemeister moves defined on knot diagrams. Let  $D$  be an oriented knot diagram in  $S^2$ . Cutting out an open arc of  $D$  which is apart from the crossings of  $D$  results in a knotoid diagram with two endpoints in the same local region of the diagram. The map  $\alpha$  is induced by assigning  $D$  to the resulting knotoid diagram. It is verified in [17] that the knotoid obtained via the map  $\alpha$  does not depend on the knot diagram chosen or the arc that is cut out from the knot diagram. Therefore  $\alpha$  is a well-defined map. It is also verified in [17] that the map  $\alpha$  is injective.

**Definition 12.2** Knotoids that are in the image of the  $\alpha$  map, are called *knot-type knotoids*.

A knot-type knotoid has at least one diagram in its equivalence class whose endpoints lie in the same local region of the diagram. Such a diagram is called a *knot-type knotoid diagram*. The Fig. 12.1a, b and e set some examples of knot-type knotoid diagrams.

**Definition 12.3** The knotoids that are not in the image of  $\alpha$ , are called *proper or pure knotoids*.

The endpoints of a proper knotoid can lie in any but different local regions of the diagram for any its representative diagram. The set of knotoids in  $S^2$  can be regarded as a union of these two types of knotoids, that is,

$$\mathcal{K}(S^2) = \{\text{Knot-type knotoids}\} \cup \{\text{Proper knotoids}\}.$$

The map  $\alpha$ , being an injective map, gives a one-to-one correspondence between the set of equivalence classes of knots and the set of knot-type knotoids. Thus the theory of knotoids extends the theory of classical knots (knots in  $\mathbb{R}^3$ ). Notice that by the allowance of the  $\Phi_-$ -move on knotoid diagrams, the theory of knotoids becomes an equivalent theory to the theory of classical knots.

### 12.2.3 The Algebraic Structure on the Set of Knotoids

A multiplication operation is defined on the set of knotoids,  $\mathcal{K}(S^2)$  in [17] as follows. Let  $k_1, k_2$  be two knotoids represented by two knotoid diagrams  $K_1$  and  $K_2$ . Take 2-disk neighborhoods  $B_1, B_2$  of the head of  $K_1$ , and the tail of  $K_2$ , respectively, such that both discs intersect the diagrams along a radius. We glue  $S^2 - \text{Int}(B_1)$  to  $S^2 - \text{Int}(B_2)$  through a homeomorphism taking  $\partial B_1$  to  $\partial B_2$  and carrying the single intersection point of  $\partial B_1$  and  $K_1$  to the single intersection point of  $\partial B_2$  and  $K_2$ . Then  $K_1 - \text{Int}(B_1)$  meets with  $K_2 - \text{Int}(B_2)$  at one point and form a knotoid diagram in  $S^2$  that is denoted by  $K_1 K_2$ . The knotoid diagram  $K_1 K_2$  represents the product knotoid  $k_1 k_2$  in  $S^2$ .

It can be verified by the reader that the multiplication operation on knotoids is associative and the trivial knotoid is the identity element. The set of knotoids,  $\mathcal{K}(S^2)$  endowed with the multiplication forms a semigroup with identity element [17].

### 12.2.4 A Geometric Interpretation of Knotoids

It is natural to see a knotoid diagram in  $\mathbb{R}^2$  as a generic projection of an open-ended, oriented space curve. Given an open-ended, smooth, oriented curve that is embedded in  $\mathbb{R}^3$  with a generic projection to a plane. The endpoints of the curve determine two lines that pass through the endpoints and are perpendicular to the plane. The generic projection of the curve to the plane along the lines is a knotoid diagram in that plane when self-crossings of the projection curve are endowed with over and under-crossing data accordingly with the weaving of the space curve. We call an open-ended curve embedded in  $\mathbb{R}^3$  that has a generic projection to a plane, a *generic curve with respect to the plane*.

Any knotoid diagram in  $\mathbb{R}^2$  determines an open-ended oriented curve embedded in  $\mathbb{R}^3$ . Let  $K$  be a knotoid diagram in  $\mathbb{R}^2$ . The plane of the diagram is identified with  $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ . The overpasses of the diagram are pushed into the upper half-space

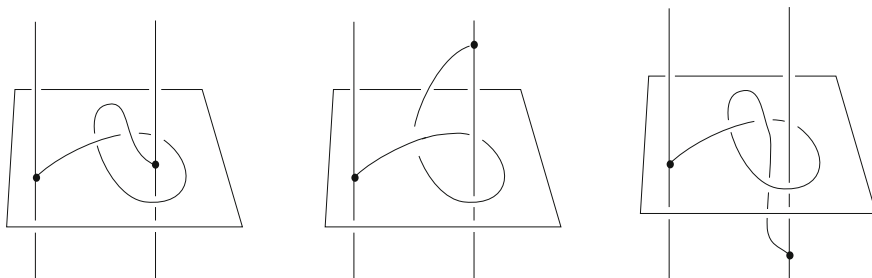


Fig. 12.5 Space curves obtained by the knotoid diagram in Fig. 12.1c

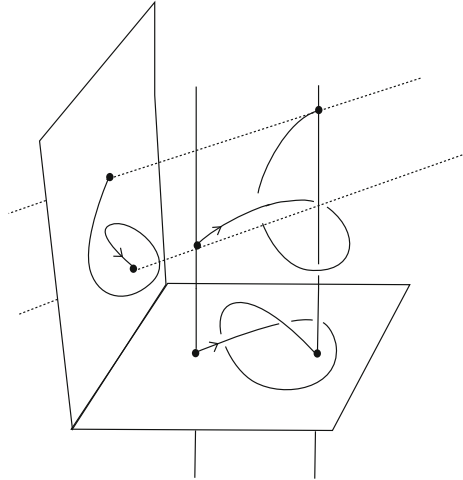
and the underpasses are pushed into the lower half-space in the vertical direction. The tail and the head of the diagram are attached to the two lines,  $t \times \mathbb{R}$  and  $h \times \mathbb{R}$  that pass through the tail and the head, respectively, and are perpendicular to the plane of the diagram. Moving the endpoints of  $K$  along these special lines gives rise to several open-ended oriented curves in  $\mathbb{R}^3$  with endpoints lying on these lines. Figure 12.5 illustrates three space curves obtained from the same knotoid diagram in the  $xy$ -plane.

**Definition 12.4** Let  $c_1, c_2$  be two open-ended, smooth, oriented curves that are embedded in  $\mathbb{R}^3$  with a generic projection to the same plane, and  $t$  and  $h$  denote their endpoints. The curves  $c_1$  and  $c_2$  are said to be *line isotopic* if there is a smooth ambient isotopy of the pair  $(\mathbb{R}^3 \setminus \{t \times \mathbb{R}, h \times \mathbb{R}\}, t \times \mathbb{R} \cup h \times \mathbb{R})$ , taking one curve to the other curve in the complement of the lines, taking endpoints to endpoints, and taking the lines to the lines;  $t \times \mathbb{R}$  to  $t \times \mathbb{R}$  and  $h \times \mathbb{R}$  to  $h \times \mathbb{R}$ .

**Theorem 12.1** ([3, Theorem 2.2]) *Two oriented curves in  $\mathbb{R}^3$  that are both generic with respect to a given plane, are line isotopic (with respect to the lines determined by the endpoints of the curves and the plane) if and only if the projections of the curves to the plane when endowed with over/under data at each self-crossing points accordingly to the weaving of the curves, are equivalent knotoid diagrams.*

Note that the equivalence classes of knotoid diagrams that are the projections of the same open-ended curve embedded in three-dimensional space, may vary with respect to the projection plane. Figure 12.6 depicts a space curve and the projections of the curve to the  $xy$ - and the  $xz$ -plane. It is clear that the  $xz$ -projection gives the trivial knotoid. The projection to the  $xy$ - plane, however, gives the knotoid diagram given in Fig. 12.1c and one can show that this diagram represents a nontrivial knotoid in  $S^2$  and in  $\mathbb{R}^2$  (see Sect. 1.3.1 and [3] for more details). In [3] we suggest that one can study the set of all knotoids assigned to one space curve for understanding the physical properties of the space curve.

**Fig. 12.6** Knotoids in different projection planes



### 12.3 The Height of a Knotoid

The *height* (or the *complexity* as used in [17]) of a knotoid diagram in  $S^2$  is the minimum number of crossings that a connection arc creates during the underpass closure. The *height of a knotoid*  $K$ ,  $h(K)$  is defined as the minimum of the heights, taken over all knotoid diagrams equivalent to  $K$ . The height is a knotoid invariant [17]. We note that we use the term height instead of complexity for knotoids to focus on this concept and allow us to use the word complexity more freely. One often refers to the complexity of a knot or a knotoid in terms of its crossing number or the virtual crossing number of the closure and other measures of how complicated is the topological type of the object. We hope that the reader will agree that this choice of terminology is useful in this case.

The height of a knotoid is preserved under the basic involutions of knotoid diagrams that are the *reversion*, *mirror image*, and the *symmetry* [17]. That is, for a knotoid  $K$ ,

$$h(K) = h(\text{mir}(K)) = h(\text{sym}(K)) = h(\text{rev}(K)).$$

The height is also invariant under the isotopy of  $S^2$  so that we can consider only planar knotoid diagrams and connection arcs in  $\mathbb{R}^2$  for the computation of the height.

**Theorem 12.2** ([17, Theorem 4.3]) *The height of a product knotoid  $k_1k_2$ ,  $h(k_1k_2) = h(k_1) + h(k_2)$  for any  $k_1, k_2 \in \mathcal{K}(S^2)$ .*

As pointed out in [17], a knotoid is of knot-type if and only if it has zero height or equivalently, a knotoid is a proper knotoid if and only if it has a nonzero height.



Thus the height is an efficient tool to measure how far a knotoid is from being a knot and for the classification of knotoids. The following conjecture has been made by V. Turaev.

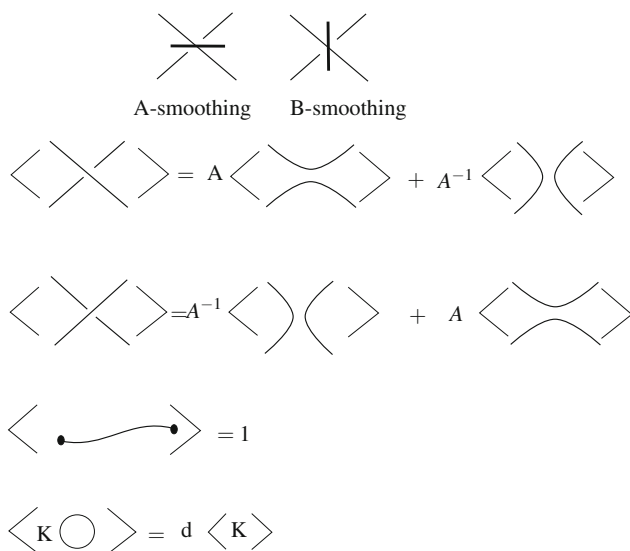
*Conjecture 12.1* [17] Minimal diagrams (with respect to the crossing number) of knot-type knotoids have zero height.

We have found a proof of this conjecture and we will give the proof in [4].

### 12.3.1 The Bracket Polynomial of a Knotoid

The bracket polynomial of a knotoid [17] is defined by extending the state expansion of the bracket polynomial of knots [9, 10]. Each crossing of a knotoid diagram  $K$  is smoothed either by *A- or B-type smoothing*, as shown in Fig. 12.7. A smoothing site is labeled by 1 if *A-smoothing* is applied and labeled by  $-1$  if *B-smoothing* is applied at a particular crossing. A *state* of the knotoid diagram  $K$  is a choice of smoothing each crossing of  $K$  with the labels at smoothing sites. Each state of  $K$  consists of disjoint embedded circular components and a single long segment component with two endpoints. The initial conditions given in Fig. 12.7 are sufficient for the skein computation of the bracket polynomial of a knotoid.

**Definition 12.5** The bracket polynomial of a knotoid diagram  $K$  is defined as



**Fig. 12.7** Skein relations of the bracket polynomial

$$\langle K \rangle = \sum_S A^{\sigma(S)} d^{\|S\|-1},$$

where the sum is taken over all states,  $\sigma(S)$  is the sum of the labels of the state  $S$ ,  $\|S\|$  is the number of components of  $S$ , and  $d = -A^2 - A^{-2}$ .

The *writhe* of a knotoid diagram  $K$ ,  $wr(K)$  is the number of positive crossings (crossings with sign  $+1$ , see Fig. 12.8) minus the number of negative crossings (crossings with sign  $-1$ ) of  $K$ . The writhe is invariant under the  $\Omega_2$  and  $\Omega_3$  moves but  $\Omega_1$ -move changes the writhe by  $\pm 1$ . The bracket polynomial turns into an invariant for knotoids with a normalization by the writhe. The *normalized bracket polynomial* of a knotoid  $K$ ,  $f_K$  is defined as the multiplication,  $f_K = (-A^3)^{-wr(K)} \langle K \rangle$  [17].

The normalized bracket polynomial of knotoids in  $S^2$  generalizes the Jones polynomial of knots in  $\mathbb{R}^3$  with the substitution  $A = t^{-1/4}$ . Note that the Jones polynomial of the trivial knotoid is trivial.

*Example 12.1* Let  $K_1$  be the knotoid diagram illustrated in Fig. 12.9 with  $wr(K_1) = +2$ . As we show in the figure, the bracket polynomial of  $K_1$ ,  $\langle K_1 \rangle = A^2 + 1 - A^{-4}$  and so  $f_{K_1} = A^{-4} + A^{-6} + A^{-10}$ . This implies that  $K_1$  is a non-trivial knotoid.

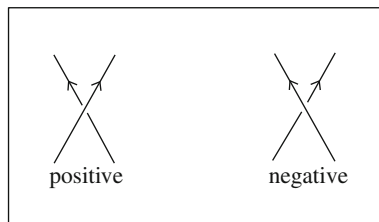


Fig. 12.8 Crossing types

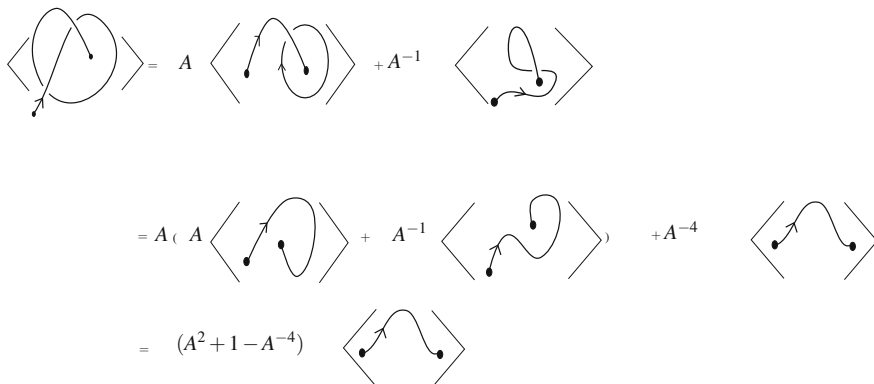


Fig. 12.9 Computation of the bracket polynomial of  $K_1$

The Jones polynomial conjecture asserts that the Jones polynomial detects the trivial knot. This well-known conjecture can be extended to the following conjecture for knotoids in  $S^2$ .

*Conjecture 12.2* The normalized bracket polynomial of knotoids in  $S^2$  (or the Jones polynomial) detects the trivial knotoid.

### 12.3.2 Briefly on Virtual Knots

The theory of virtual knots that was introduced by the econd listed author [5, 6] in 1996, studies the embeddings of circles in thickened surfaces modulo isotopies and diffeomorphisms and one-handle stabilization of the surfaces.

Virtual knot theory has a diagrammatic formulation. In the diagrammatic theory, virtual knots and links are represented by diagrams with finitely many *classical crossings* (transversal self-crossings of the underlying curve that are endowed with over/under- data) and *virtual crossings* which are neither over-crossings or under-crossings. A virtual crossing is a combinatorial structure of the diagram that is indicated by two crossing segments with a small circle placed around the crossing point. A knot/link diagram with virtual crossings is called a *virtual knot/link diagram*.

The moves on virtual knot/link diagrams are generated by the classical Reidemeister moves plus the detour move. The detour move allows a segment with a consecutive sequence of virtual crossings to be excised and replaced by any other such a segment with a consecutive virtual crossings, as shown in Fig. 12.10. Local expressions that generate the detour move are illustrated in Figs. 12.11 and 12.12.

Virtual knot and link diagrams that can be connected by a finite sequence of these moves are said to be *equivalent* or *virtually isotopic*. Corresponding equivalence classes of virtual knot/link diagrams with respect to the equivalence relation generated by these moves, are called *virtual knots/links*.

There is a one-to-one correspondence between topological and diagrammatic approach. Further information on virtual knots and their association with thickened surfaces can be found in [5–7, 12, 13, 15, 16]. Here we state the following theorem.

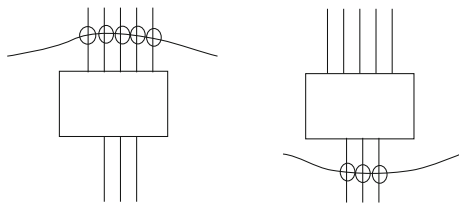
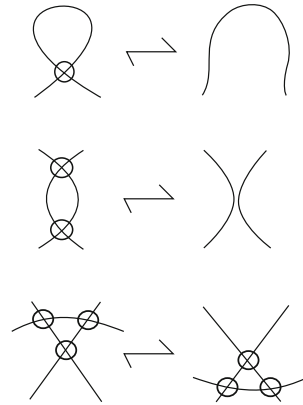
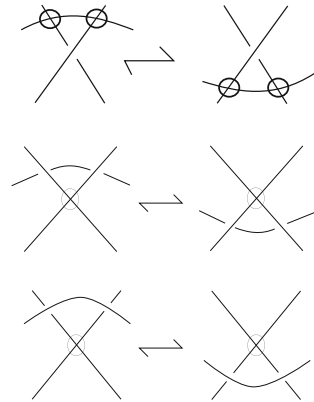


Fig. 12.10 The detour move

**Fig. 12.11** Local detour moves



**Fig. 12.12** The virtual forbidden moves

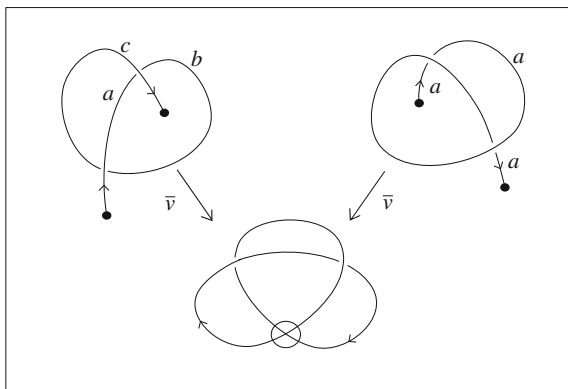


**Theorem 12.3** ([6, 11, 16, Theorems 1, 3.3, 4.1 ]) *Two virtual link diagrams are virtually isotopic if and only if their surface embeddings are equivalent up to isotopy and diffeomorphism of the surface, and addition/removal of empty handles.*

### 12.3.3 A Transition to Virtual Knot Theory

A knotoid diagram in  $S^2$  represents a virtual knot as pointed out in [17]. The endpoints of a knotoid diagram can be connected with an embedded arc in  $S^2$  in the virtual fashion, that is, a virtual crossing is created every time the connection arc crosses through a strand of the diagram. The resulting diagram is a virtual knot diagram that can also be represented in a torus without any virtual crossings by attaching a 1-handle to the sphere of the diagram which holds the connection arc.

**Fig. 12.13** Nonequivalent knotoid diagrams with the same virtual closure



Connecting the endpoints of a knotoid diagram in  $S^2$  in the virtual fashion induces a well-defined map that is called the *virtual closure map* and is denoted by  $\bar{v}$ ,

$$\bar{v} : \{\text{Knotoids in } S^2\} \rightarrow \{\text{Virtual knots of genus } \leq 1\}.$$

Figure 12.13 illustrates a pair of knotoid diagrams whose virtual closures are the same virtual knot. It can be shown that the knotoid group [17] of the left hand side knotoid diagram has the presentation  $\langle x, y | x^2 = y^3 \rangle$  (it is isomorphic to the knot group of the trefoil knot). On the other hand, the knotoid group of the right hand side knotoid diagram is isomorphic to  $\mathbb{Z}$ . Since the knotoid group is a knotoid invariant [17], we conclude that these two diagrams are nonequivalent knotoid diagrams. One other way to distinguish these two knotoid diagrams is the tricolorability of knotoids. The tricolorability of classical knots can be extended to an invariant of knotoids by applying the rules of tricolorability directly to knotoid diagrams. A knotoid diagram  $K$  in  $S^2$  is *tricolorable* if each overpassing strand of  $K$  (a strand of  $K$  that initiates at an undercrossing and terminates at the next undercrossing or terminates at the head of  $K$  or initiates at the tail of  $K$  and terminates at the next undercrossing) can be colored with one of three colors with respect to the following rules.

- At least two colors must be used.
- At each crossing, the three incident strand should be colored either with the same color or with three different colors.

In Fig. 12.13 shows that the left hand side knotoid diagram is tricolorable with the colors  $a, b, c$  and the right hand side knotoid diagram is not tricolorable; the diagram can be colored with only one color. Since tricolorability is a knotoid invariant it follows that they are not equivalent knotoid diagrams. Therefore, the virtual closure map is not an injective map [3]. We will discuss on tricolorability and in general coloring of knotoids in detail in a subsequent paper.

In our paper [3] we conjectured that there are genus 1 virtual knots that do not lie in the image of the virtual closure map. We have proved this conjecture by examining

of the surface states of the surface bracket polynomial of virtual knots which lie in the image of the virtual closure map. Our proof will appear in [4]. In this paper we state the following.

**Proposition 12.2** ([4]) *The virtual closure map is not surjective.*

The virtual closure map, being a well-defined map forms a machinery to define knotoid invariants by virtual knot invariants. In fact for any invariant of a virtual knot, denoted by  $Inv$ , we can define a knotoid invariant, denoted by  $I$  through the following formula.

$$I(K) = Inv(\bar{v}(K)), \text{ for a knotoid } K \text{ in } S^2.$$

In fact, many invariants obtained from the virtual closure can be defined directly for the knotoids in their own category, without the consideration of the virtual closure map. This is the case for the affine index polynomial and the arrow polynomial as described in the next section.

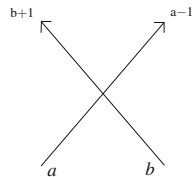
*Remark 12.1* There is a rich subject of *virtual knotoids* where we allow knotoid diagrams with virtual crossings. This subject was firstly proposed in [17] and discussed briefly in [3] and will be the subject of further papers.

## 12.4 Estimation of Height

### 12.4.1 Affine Index Polynomial of a Knotoid

The affine index polynomial of a knotoid is constructed in terms of weights assigned to crossings. The *underlying flat diagram* of a knotoid diagram is obtained by omitting the over/under-data at each crossing of the knotoid diagram and turning them into *flat crossings*. An *arc* of a flat knotoid diagram either connects an endpoint of the diagram to a flat crossing or connects a flat crossing to the next flat crossing. There is an integer labeling rule assigned to the arcs of the underlying flat diagram of a knotoid diagram which is set as follows. If an arc crosses a (flat) crossing towards right then its label is decreased by one and if it crosses a node towards left then its label is increased by one. See Fig. 12.14 for the labeling at a flat crossing.

**Fig. 12.14** Integer labeling at a flat crossing



Let  $K$  be a knotoid diagram. The labeling procedure begins by the arc that is adjacent to the tail of  $K$  and it is labeled by 0 conventionally. Note that the adjacent arcs to the tail and the head take the same label.

There are two integer outcomes by the labeling rule at each crossing of  $K$ . The *positive* and *negative weights* of a crossing  $c$ , denoted by  $w_+(c)$  and  $w_-(c)$ , respectively, are defined as the differences of the labels around  $c$ . That is,

$$\begin{aligned} w_+(c) &= a - (b + 1) \\ w_-(c) &= b - (a - 1) \end{aligned}$$

where  $a$  and  $b$  are the labels for the left and the right incoming arcs at the corresponding node to  $c$ , respectively.

The *weight* of  $c$  is defined as

$$w_K(c) = \begin{cases} w_+(c), & \text{if the sign of } c \text{ is positive} \\ w_-(c), & \text{if the sign of } c \text{ is negative.} \end{cases}$$

**Definition 12.6** The *affine index polynomial* of a knotoid diagram  $K$  is defined by the equation,

$$P_K(t) = \sum_c \text{sign}(c)(t^{w_K(c)} - 1) = \sum_c \text{sign}(c)t^{w_K(c)} - \text{wr}(K)$$

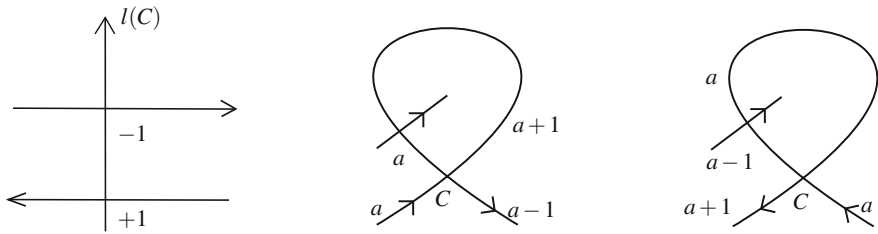
where  $\text{wr}(K)$  is the writhe of  $K$ .

**Theorem 12.4** ([3, Theorem 5.1]) *The affine index polynomial is an invariant of knotoids in  $S^2$ .*

*Proof* It is sufficient to check the change in the weights under oriented  $\Omega_{i=1,2,3}$ -moves. The labeling is uniquely inherited by these moves. It is left to the reader the check the crossing added/removed by an  $\Omega_1$ -move has zero weight. Two crossings added/removed by an  $\Omega_2$ -move have opposite weights so their contributions cancel each other. The weights of the crossings in an  $\Omega_3$ -move do not change. Therefore the polynomial remains the same under these moves.

**Theorem 12.5** ([3, Theorem 4.12]) *Let  $K$  be a knotoid in  $S^2$ . The height of  $K$  is greater than or equal to the maximum degree of the affine index polynomial of  $K$ .*

*Proof* (Sketch of the proof) The proof of the theorem relies mostly on the following observation. Each crossing of a knotoid diagram determines a unique loop (a continuous path obtained by traversing the diagram starting and ending at a crossing accordingly to the orientation of the diagram) throughout the diagram. A loop that is determined by a crossing is called the *loop at the crossing*. The algebraic intersection number of a loop at a crossing, with other strands of the diagram is equal to either the positive weight or the negative weight of that crossing, depending to the orientation of the loop. On the left hand side of Fig. 12.15, a small portion of the loop  $l(C)$  at



**Fig. 12.15** Two possible loops at the crossing  $C$  with different orientations

**Fig. 12.16** Seifert smoothing of a crossing



a crossing  $C$  is illustrated and we see the convention for the algebraic intersection number of the loop with other strands. On the right-hand side of the figure, two possible types of loops at  $C$  that may be observed accordingly to the orientation of the diagram, are shown. The algebraic intersection number of the strands shown in the figure with the loops are  $+1$  and  $-1$ , respectively. By the same figure, it can also be verified that  $w_-(C) = +1$  for the first loop that is oriented counterclockwise and  $w_+(C) = -1$  for the second loop that is oriented clockwise (See [3] for more details).

Let  $\tilde{K}$  be a representative knotoid diagram of  $K$ . A crossing of  $\tilde{K}$  is a *maximal weight crossing* if  $w_{\tilde{K}}(c)$  is maximal among the weights of crossings of  $\tilde{K}$ . Let  $c$  be a maximal weight crossing of  $\tilde{K}$ . All crossings which are seen twice along the loop at the crossing  $c$  are smoothed in the oriented way (*Seifert smoothing*), as shown in Fig. 12.16.

This smoothing results in disjoint oriented embedded circles in  $S^2$  (*Seifert circles*) and a long segment containing the endpoints with an orientation on it from the tail to head. Let  $I_{\tilde{K}}$  be the algebraic intersection number of the long segment with the Seifert circles. It is observed that,

- $|I_{\tilde{K}}| \leq$  the number of the resulting Seifert circles enclosing the endpoints,
  - $I_{\tilde{K}}$  is equal to the algebraic intersection number of the loop at  $c$  with the rest of the diagram since the crossings on the loop which contribute to the intersection number non-trivially are not smoothed. Then,
 
$$|I_{\tilde{K}}| = w_{\tilde{K}}(c),$$
  - The number of the Seifert circles enclosing the endpoints  $\leq h(\tilde{K})$ , by the Jordan curve theorem.
- Therefore we have the following inequality,
- $w_{\tilde{K}}(c) \leq h(\tilde{K})$ .



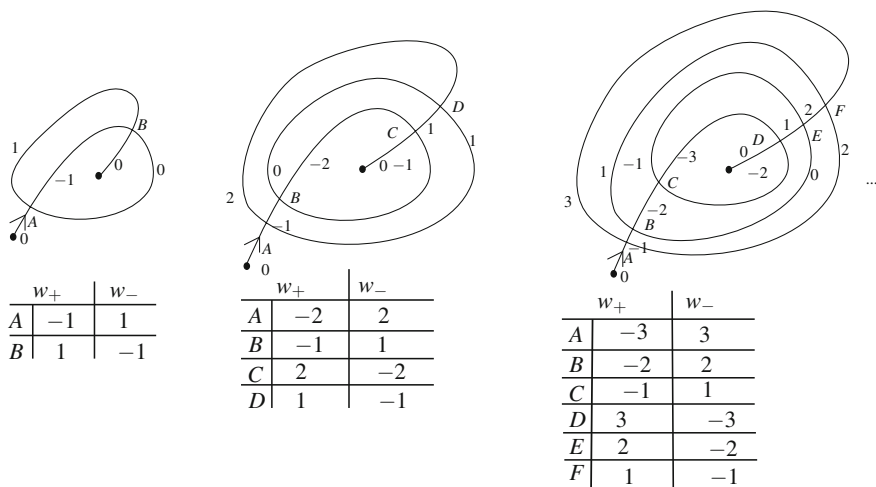
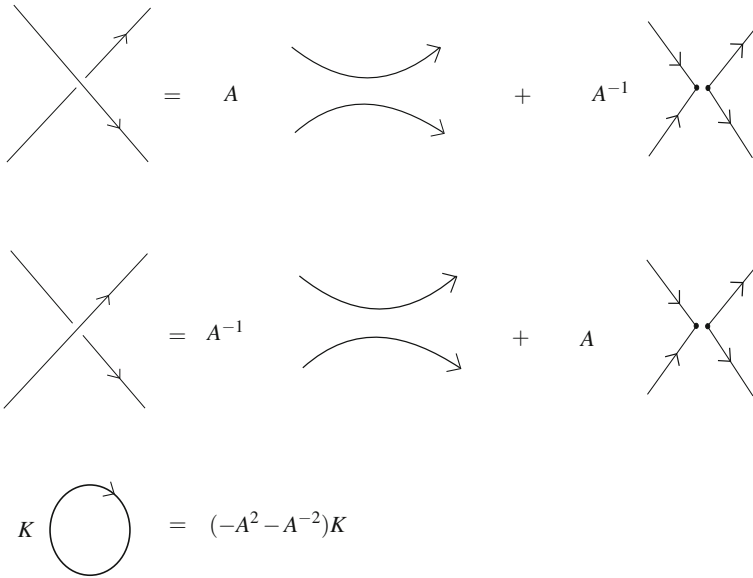


Fig. 12.17 Flat spiral knotoid diagrams

The maximum degree of the affine index polynomial is an invariant since the polynomial is an invariant. This implies that there exists a crossing at each representative diagram whose weight is equal to  $w_{\tilde{K}}(c)$ . By applying the same procedure given above to each representative diagram, we can conclude that the maximum degree of the affine index polynomial is a lower bound for the height of  $K$ .

*Example 12.2* The height of a knotoid represented with a standard  $n$ -fold spiral diagram can be read by Theorem 12.5. In particular, the affine index polynomials of the knotoid diagrams  $K_1$ ,  $K_2$  and  $K_3$  that are overlying the flat diagrams given in Fig. 12.17 with all positive crossings, are  $P_{K_1}(t) = t + t^{-1} - 2$ ,  $P_{K_2}(t) = t^2 + t + t^{-1} + t^{-2} - 4$  and  $P_{K_3}(t) = t^3 + t^2 + t + t^{-1} + t^{-2} + t^{-3} - 6$ , respectively. It can be verified by the figure that the heights of the diagrams  $K_1$ ,  $K_2$ ,  $K_3$  are 1, 2, 3, respectively. Then by Theorem 12.5 we conclude that the heights of the knotoids represented by  $K_1$ ,  $K_2$ ,  $K_3$  are 1, 2 and 3, respectively. This argument is generalized as follows. The affine index polynomial of a knotoid represented by an  $n$ -fold spiral knotoid diagram is of the form  $t^n + t^{n-1} + \dots + t + t^{-1} + \dots + t^{-(n-1)} + t^{-n} - 2n$  if all crossings of the diagram are positive. The maximal degree of the affine index polynomial is  $n$ . Then by Theorem 12.5, the height of the knotoid is at least  $n$ . The height of the  $n$ -fold spiral diagram is  $n$ . Therefore, the height of a knotoid represented by a  $n$ -fold spiral diagram is equal to  $n$ . This implies that we have an infinite set of knotoids whose heights are given by the maximal degrees of their affine index polynomials.

**Proposition 12.3** *The height of a product knotoid  $k_1k_2$ ,  $h(k_1k_2) \geq \deg P(k_1) + \deg P(k_2) \geq \deg P(k_1k_2)$ , where  $\deg P(k_i)$  is the maximal degree of the affine index polynomial of  $k_i$ ,  $i = 1, 2$ .*



**Fig. 12.18** Oriented state expansion

*Proof* The inequality  $deg P(k_1 k_2) \leq deg P(k_1) + deg P(k_2)$  can be verified by the reader. By Theorem 12.2 given in Sect. 1.3 we know that the height is an additive invariant. Then it follows that  $h(k_1 k_2) \geq deg P(k_1) + deg P(k_2)$ .

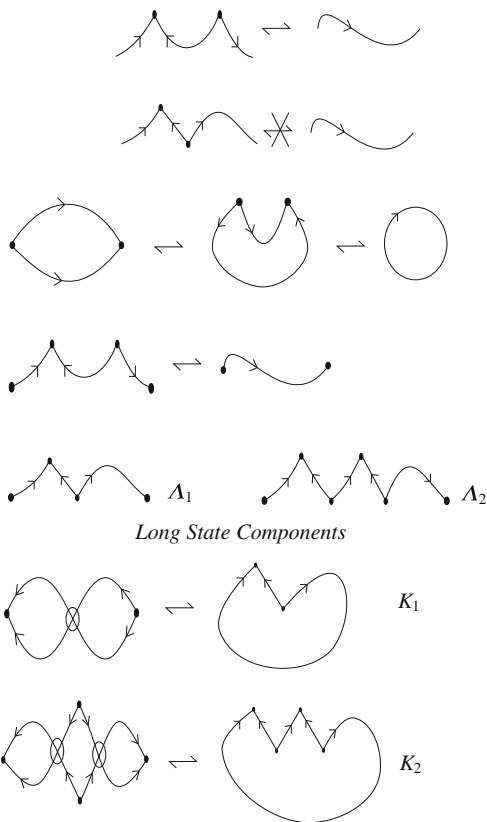
### 12.4.2 The Arrow Polynomial of a Knotoid

The arrow polynomial of a knotoid is constructed by the *oriented state expansion* of the bracket polynomial for knotoids; consisting of oriented and disoriented smoothings of crossings, shown in Fig. 12.18.

Smoothing out all the crossings of a knotoid diagram results in *oriented states* containing circular components and a long segment component. There is an extra combinatorial structure on the state components coming out by the disoriented smoothings. This new structure is seen in the form of a pair of cusps, each of cusps has two arcs either going into the cusp or going out from the cusp. Each cusp generates two angles; one is an acute, the other one is an obtuse angle. The local region which is spanned by the acute angle is called the *inside* of the cusp.

There is a list of rules given in Fig. 12.19 to reduce the number of cusps in state components. The rules essentially eliminate two consecutive cusps with insides located on the same side of the segment connecting them. As a consequence of the Jordan curve theorem, all cusps on circular state components of a knotoid diagram are eliminated so that circular components are free of cusps. Each circular and

**Fig. 12.19** Reduction rules for the arrow polynomial



the long state component without cusps contributes as a  $-A^2 - A^{-2}$  factor to the polynomial. The long state components with consecutive cusps whose insides are located at different sides cannot be saved from the cusps and contribute to the arrow polynomial as variables  $\Lambda_i$ .

**Definition 12.7** The arrow polynomial of a knotoid diagram  $K$  in  $S^2$  is defined as follows.

$$\mathcal{A}[K] = \sum_S \langle K|S \rangle (-A^2 - A^{-2})^{\|S\|-1} \Lambda_i,$$

where the sum runs over the oriented bracket states,  $\langle K|S \rangle$  is the usual vertex weights of the bracket polynomial,  $\|S\|$  is the number of components of the state  $S$  and  $\Lambda_i$  is the variable associated to the long segment component of  $S$  with irreducible cusps.

**Theorem 12.6** ([3, Theorem 5.1]) *The normalization of the arrow polynomial by  $(-A^3)^{-\text{wr}(K)}$ , where  $K$  is a knotoid diagram and  $\text{wr}(K)$  is its writhe, is a knotoid invariant.*

*Proof* The proof follows similarly with the proof for the invariance of the normalized arrow polynomial for virtual knots/links [2].

**Definition 12.8** The  $\Lambda$ -degree of a summand of the arrow polynomial of a knotoid in  $S^2$  which is of the form,  $A^m \Lambda_i$  is equal to  $i$ . The  $\Lambda$ -degree of the arrow polynomial of a knotoid is defined to be the maximum  $\Lambda$ -degree among the  $\Lambda$ -degrees of the summands.

The arrow polynomial was firstly defined as a virtual knot invariant by H. Dye and the second listed author [2] and independently by Y. Miyazawa [14]. It also utilizes the oriented state expansion and is defined in a similar way as it is defined for knotoids. Oriented state components of a virtual knot diagram consist of circular components. Circular components may have irreducible cusps as illustrated in Fig. 12.19. A circular component with  $2i$  irreducible cusps contributes as a  $K_i$ -variable to the arrow polynomial. Since oriented states of a classical knot are cusp-free, no  $K_i$ -variables occur in the arrow polynomial of classical knots [6]. Similarly, circular components of an oriented state of a knotoid are cusp-free and so no  $K_i$ -variables occur in the arrow polynomial of a knotoid in  $S^2$ . However, closing the endpoints of a knotoid diagram via the virtual closure map turns  $\Lambda_i$ -variables into  $K_i$ -variables that are assigned to the circular components obtained via this closure.

**Definition 12.9** The  $K$ -degree of a summand of the arrow polynomial of a virtual knot which is of the form,  $A^m (K_{i_1}^{j_1} K_{i_2}^{j_2} \dots K_{i_n}^{j_n})$  is equal to

$$i_1 \times j_1 + \dots + i_n \times j_n.$$

The  $K$ -degree of the arrow polynomial of a virtual knot is the maximum monomial degree in  $K$ .

**Definition 12.10** The least number of virtual crossings that a virtual knot can have among its virtual equivalence class is the *virtual crossing number*.

**Theorem 12.7** ([2, Theorem 2.3]) *The virtual crossing number of a virtual knot/link is greater than or equal to the maximal  $K$ -degree of the arrow polynomial of that virtual knot/link.*

**Theorem 12.8** ([3, Theorem 5.4]) *The height of a knotoid  $K$  in  $S^2$ ,  $h(K)$  is greater than or equal to the  $\Lambda$ -degree of its arrow polynomial.*

*Proof* Closing a knotoid diagram virtually corresponds, in the states, to closing the endpoints of the long state components in the virtual fashion. Therefore the  $\Lambda_i$ -variables assigned to long state components with surviving cusps transform to  $K_i$ -variables assigned to the circular components with surviving cusps in the arrow polynomial of the virtual knot obtained by the virtual closure. By this observation and Theorem 12.7, we have the following inequality for the knotoid  $k$ .

$$\text{The } \Lambda\text{-degree of } \mathcal{A}[K] \leq \# \text{ of virtual crossings of the knot } \bar{v}(K).$$

The least number of virtual crossings obtained by closing a knotoid diagram virtually, is equal to the height of that diagram. Thus,

$$\# \text{ of virtual crossings of the knot } \bar{v}(K) \leq h(\tilde{K}),$$

for any knotoid diagram  $\tilde{K}$  representing  $K$ . This gives the following inequality,

$$\text{The } \Lambda - \text{degree of } \mathcal{A}[K] \leq h(\tilde{K}).$$

The inequality above holds for any knotoid diagram in the equivalence class of  $K$  since the  $\Lambda$ -degree of the polynomial is invariant under  $\Omega_{i=1,2,3}$ -moves. Therefore we have,

$$\text{The } \Lambda - \text{degree of } \mathcal{A}[K] \leq h(K).$$

**Proposition 12.4** *The height of a product knotoid  $k_1k_2$ ,  $h(k_1k_2) \geq \text{deg}_{\Lambda} \mathcal{A}[k_1] + \text{deg}_{\Lambda} \mathcal{A}[k_2] \geq \text{deg}_{\Lambda} \mathcal{A}[k_1k_2]$ , where  $\text{deg}_{\Lambda} \mathcal{A}[k_i]$  is the  $\Lambda$ -degree of  $\mathcal{A}[k_i]$ ,  $i = 1, 2$ .*

*Proof* It can be verified by the reader that  $\text{deg}_{\Lambda} \mathcal{A}[k_1k_2] \leq \text{deg}_{\Lambda} \mathcal{A}[k_1] + \text{deg}_{\Lambda} \mathcal{A}[k_2]$ . By Theorem 12.2 given in Sect. 1.3 we have  $h(k_1k_2) = h(k_1) + h(k_2)$ . Then by Theorem 12.8 it follows that  $h(k_1k_2) \geq \text{deg}_{\Lambda} \mathcal{A}[k_1] + \text{deg}_{\Lambda} \mathcal{A}[k_2]$ .

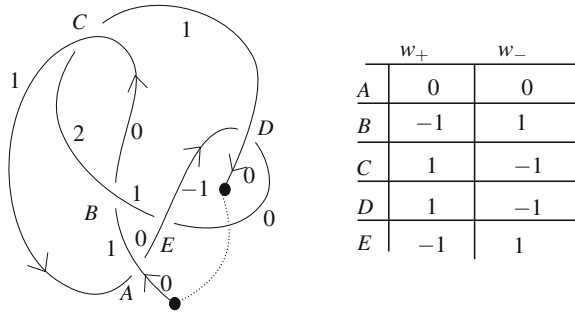
### 12.4.3 A Discussion on Two Polynomials

The arrow polynomial coincides with the bracket polynomial of knotoids [3, 17] if the  $\Lambda_i$ -variables assigned to long segment components of oriented states are set to be equal to 1. Thus the arrow polynomial is a generalization of the bracket polynomial of knotoids. The affine index polynomial uses the flat biquandle structure [8] of a knotoid diagram. It is quite a different concept than both the arrow and the bracket polynomial.

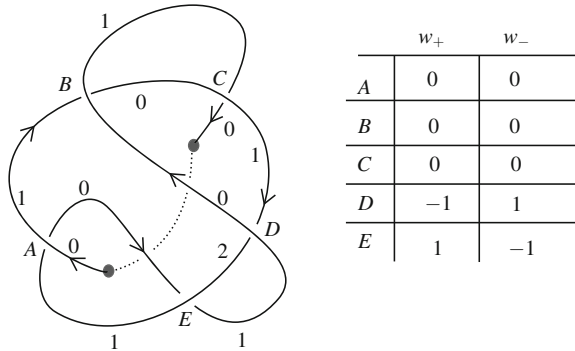
Two estimations of the height given by the affine index and the arrow polynomials are used to determine the height of many knotoids. Here we present one example where the arrow polynomial gives a more accurate estimation of the height than the affine index polynomial and another example where both polynomials are not sufficient to make an exact estimation of the height.

*Example 12.3* The knotoid diagram  $K$  in Fig. 12.20 represents the knotoid listed as knotoid 5.7 [1]. The affine index polynomial of the knotoid 5.7 is trivial (the reader can verify this by the weight chart of  $K$  given in the figure). However, the arrow polynomial of the knotoid is nontrivial. In fact we have,  $\mathcal{A}[K] = (-A^{-3} + A - 2A^5 + A^9) + (A^{-9} - 2A^{-5} + 2A^{-1} - 2A^3 + A^7)\Lambda_1$ . Thus the  $\Lambda$ -degree of the arrow polynomial of the knotoid 5.7 is 1. Also, it is clear that the height of the diagram  $K$  is one. By Theorem 12.8, it follows that the height of the knotoid 5.7 is 1 and this knotoid is in fact a proper knotoid.

**Fig. 12.20** The knotoid diagram  $K$  and its weight chart



**Fig. 12.21** The knotoid diagram  $K$  and its weight chart



*Example 12.4* The knotoid diagram  $K$  given in Fig. 12.21 represents the knotoid listed as knotoid 5.24 [1]. It can be verified that the affine index polynomial of the knotoid is  $P_K(t) = t + t^{-1} - 2$ . The arrow polynomial of the knotoid is  $\mathcal{A}[K] = (-A^{-7} + A^{-3} - A - A^5 + A^9) + (2A^{-1} - 3A^3 + A^7)A_1$ . Both the maximal degree of the affine index polynomial and the  $A$ -degree of the arrow polynomial are equal to 1. Thus the height of the knotoid is at least 1, by Theorems 12.5 and 12.8. It is clear that the diagram  $K$  has height 2. Thus we have  $1 \leq h(K) \leq 2$ . We conclude that the knotoid 5.24 is a proper knotoid but both the affine index polynomial and the arrow polynomial cannot give an exact height estimation for this knotoid.

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