Chapter 1
Link Invariants from the Yokonuma–Hecke Algebras

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Abstract The Yokonuma–Hecke algebras are naturally related to the framed braid group and they support a Markov trace. Consequently, invariants for various types of links (framed, classical, singular and transverse) are derived from these algebras. In this paper, we present results about these invariants and their properties. We focus, in particular, on the family of 2-variable classical link invariants that are not topologically equivalent to the HOMFLYPT polynomial and on the 3-variable classical link invariant that generalizes this family and the HOMFLYPT polynomial.

Keywords Classical braids · Framed braids · Yokonuma–Hecke algebras
Markov trace · Framed knots and links · E–condition · Classical knots and links
Transverse knots and links · Singular braid monoid · Singular knots and links
HOMFLYPT polynomial

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Introduction

The first example of construction of link invariants via braid groups is the Jones polynomial [27]. It can be defined by means of a *knot algebra*, that is a triple (A, π, τ) where A is an algebra, π is a representation of the braid group in A and τ is a Markov trace on A. The Jones polynomial is obtained via the Jones' trace on the Temperley–Lieb algebra. This construction generalizes to the HOMFLYPT (or 2-variable Jones)

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polynomial [19, 39, 43] using as a knot algebra the Iwahori–Hecke algebra of type A and the Ocneanu trace [27].

Some years ago, the Yokonuma–Hecke algebras of type A [47] received attention. These algebras have "framing" generators and they are naturally related to the framed braid group. We denote them as $\mathrm{Y}_{d,n}(q)$, where n corresponds to the number of strands of the framed braid group $\mathscr{F}_n$, $d \in \mathbb{N}$ imposes a modular condition on the framing generators and q is a non-zero complex number. For $d = 1$ the algebra $\mathrm{Y}_{1,n}$ coincides with the Iwahori–Hecke algebra. The representation theory of the Yokonuma–Hecke algebras has been extensively studied in [13, 45]. J. Juyumaya in [29] defined a unique Markov trace on the Yokonuma–Hecke algebras, denoted by tr_d, making $\mathrm{Y}_{d,n}(q)$ into a knot algebra and a natural candidate for defining framed and classical link invariants. Surprisingly, the trace tr_d could not be directly re-scaled for the negative stabilization move of the framed braid equivalence. For this, a condition was needed to be imposed on the framing parameters of tr_d that, in turn, meant that they should satisfy a certain non-linear system of equations, called the E-*system*. As it was shown by P. Gérardin [31, Appendix], the solutions of the E-system are parametrized by the non-empty subsets D of $\mathbb{Z}/d\mathbb{Z}$.

Consequently, in [31] and in [32] an infinitum of framed and classical link invariants respectively were defined, parametrized by d and the subsets D. Both of these families of invariants contain the HOMFLYPT polynomial for $d = 1$. Furthermore, the Yokonuma–Hecke algebras were proved to be suitable for defining invariants for other classes of links, such as singular links [33] and transverse links [10].

With the classical link invariants in hand, the next natural question, which remained as a long-standing open problem, was whether these invariants are *topologically equivalent* to the HOMFLYPT polynomial P in the sense that they distinguish or not the same pairs of links. The first step was taken in [12], where it was proven that these invariants do not coincide with P except for the trivial cases when $|D| = 1$ and $q = \pm 1$. The Yokonuma–Hecke algebras have a quite complex quadratic relation for the braiding generators that involves some idempotent elements, denoted by e_i, that are sums of products of the framing generators. Computations were needed, and for this purpose a computer program was developed (see [35] and http://www.math.ntua.gr/~sofia/yokonuma). Note that the invariants have been originally defined using a different presentation of the Yokonuma–Hecke algebras, denoted by $\mathrm{Y}_{d,n}(u)$, that used a more complicated quadratic relation than that of the presentation $\mathrm{Y}_{d,n}(q)$. By comparing the classical invariants on various pairs of knots and links, a conjecture for the case of knots was formulated in [10] and later proved in [11]. In both papers the new presentation $\mathrm{Y}_{d,n}(q)$ is used. More precisely, the classical link invariants from the Yokonuma–Hecke algebras coincide with P on *knots*. Note that, by this result it follows that these invariants are not topologically equivalent to the Kauffman polynomial.

In [12] the *specialized Juyumaya trace* $\mathrm{tr}_{d,D}$ that is the trace tr_d with the framing parameters specialized to a solution of the E-system was defined. Now, in [11] it has been proved that the trace $\mathrm{tr}_{d,D}$ can be computed for classical braids by five rules that involve the braiding generators of $\mathrm{Y}_{d,n}$ and the idempotents e_i, instead of the framing generators. This result makes the calculations for the trace $\mathrm{tr}_{d,D}$ easier, since the elements e_i can be considered as formal elements in the image of the classical braid group in $\mathrm{Y}_{d,n}(q)$. By the same result it also followed that the invariants $\Theta_{d,D}$ of classical links are actually parametrized by the natural numbers and can be simply denoted as Θ_d. In order to compare the invariants Θ_d to P on classical links, a new program was developed [35] that uses the five rules for the trace $\mathrm{tr}_{d,D}$ when applied to images of classical braids. This program facilitated the comparison of the invariants Θ_d and P on several P-equivalent pairs of links (that is pairs of links having the same HOMFLYPT polynomial) and, as it turned out, the invariants Θ_d are not topologically equivalent to P on *links* [11]. This fact was also proved theoretically in [11], since the invariants Θ_d satisfy the HOMFLYPT skein relation, but only on crossings involving different components, and this enabled a diagrammatic approach to the definition of the invariants Θ_d.

Remarkably, our examples of P-equivalent pairs of links distinguished by Θ_d were distinguished for all $d \geq 2$ [11]. Furthermore, in the five rules of $\mathrm{tr}_{d,D}$ when restricted to classical braids only the value $E_D := 1/|D|$ appears that depends only on the cardinality of the set D. This led to the hypothesis that the value E_D can be seen as a parameter, resulting in the construction of a new 3-variable classical skein link invariant $\Theta(q, \lambda, E)$ that is stronger than the HOMFLYPT polynomial [11]. Moreover, W.B.R. Lickorish provided in [11, Appendix B] a closed formula for the invariant Θ that shows that it is a complicated mixture of linking numbers and the values of P on sublinks of a given link, providing thus a topological interpretation for the invariant Θ (see also [36, 42]). Finally, the construction of the invariant Θ led to an analogous generalization of the Kauffman polynomial and to new state sum models, using the skein theoretical methods for Θ [36].

The interest in the Yokonuma–Hecke algebras led to the notion of framization of knot algebras [34], the Yokonuma–Hecke algebra being the basic example, as the framization of the classical Iwahori–Hecke algebra. Consequently, appropriate Temperley–Lieb-type quotients of the Yokonuma–Hecke algebras were constructed and studied in [22–25], see also [15, 16]. Furthermore, Yokonuma–Hecke algebras related to type B have been constructed [14, 18, 34], equipped with Markov traces, and related link invariants for the solid torus have been derived. Finally, a framization of the BMW algebra has also been defined and studied [4, 34].

In this paper we present results mainly from [10, 11] on the Yokonuma–Hecke algebras and link invariants derived from them. The paper is organized as follows. In Sect. 1.1 we define the Yokonuma–Hecke algebras and provide some facts about them. Then, in Sect. 1.2, we recall the definition of the Markov traces tr_d and $\mathrm{tr}_{d,D}$ on $\mathrm{Y}_{d,n}(q)$ and we discuss some properties that they satisfy. In Sect. 1.3 invariants for framed, classical, singular and transverse links are presented, while in Sect. 1.4 we study further the classical link invariants. In Sect. 1.5 the 3-variable classical link invariant Θ is presented and we discuss various ways to prove its well-definedness. Finally, in Sect. 1.6 we recall framed and classical link invariants derived from another presentation of the Yokonuma–Hecke algebras and we discuss their relation to the ones derived from the new presentation $\mathrm{Y}_{d,n}(q)$.

1.1 The Yokonuma–Hecke Algebra

In this section we recall the definition of the Yokonuma–Hecke algebra as a quotient of the framed braid group.

1.1.1 The Framed Braid Group and the Modular Framed Braid Group

The *framed braid group*, $\mathcal{F}_n \cong \mathbb{Z}^n \rtimes B_n$, is the group defined by the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the classical braid group B_n together with the framing generators $t_1, \dots, t_n$ (t_j indicates framing 1 on the j-th strand), subject to the relations:

$$\begin{array}{llll} (\mathrm{b}_1) & \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \\ (\mathrm{b}_2) & \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ (\mathrm{f}_1) & t_i t_j = t_j t_i & \text{for all } i, j \\ (\mathrm{f}_2) & t_j\sigma_i = \sigma_i t_{s_i(j)} & \text{for all } i, j \end{array} \tag{1.1}$$

where $s_i(j)$ is the effect of the transposition $s_i := (i, i+1)$ on j. Relations (b_1) and (b_2) are the usual braid relations, while relations (f_1) and (f_2) involve the framing generators. Further, for a natural number d the *modular framed braid group*, denoted $\mathcal{F}_{d,n}$, can be defined as the group with the presentation of the framed braid group, but including also the modular relations:

$$(\mathrm{m}) \quad t_j^d = 1 \quad \text{for all } j \tag{1.2}$$

Hence, $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$. Geometrically, the elements of $\mathcal{F}_n$ (respectively $\mathcal{F}_{d,n}$) are classical braids on n strands with an integer (respectively an integer modulo d), the framing, attached to each strand. Further, due to relations (f_1) and (f_2), every framed braid α in $\mathcal{F}_{d,n}$ can be written in its *split form* as $\alpha = t_1^{k_1} \dots t_n^{k_n}\sigma$, where $k_1, \dots, k_{n-1} \in \mathbb{Z}$ and σ involves only the standard generators of B_n. The same holds also for the modular framed braid group.

For a fixed d we define the following elements e_i in the group algebra $\mathbb{C}\mathcal{F}_{d,n}$:

$$e_i := \frac{1}{d} \sum_{1 \le s \le d} t_i^s t_{i+1}^{-s} \qquad (1 \le i \le n-1)$$

where $-s$ is considered modulo d. One can easily check that e_i is an idempotent: $e_i^2 = e_i$ and that $e_i\sigma_i = \sigma_i e_i$ for all i.

1.1.2 The Yokonuma–Hecke Algebras

Let $d \in \mathbb{N}$ and let $q \in \mathbb{C}\backslash\{0\}$ fixed. The *Yokonuma–Hecke algebra* q, denoted $\mathrm{Y}_{d,n}(q)$, is defined as the quotient of $\mathbb{C}\mathscr{F}_{d,n}$ by factoring through the ideal generated by the expressions: $\sigma_i^2 - 1 - (q - q^{-1})e_i\sigma_i$ for $1 \leq i \leq n-1$. We shall denote g_i the element in the algebra $\mathrm{Y}_{d,n}(q)$ corresponding to σ_i while we keep the same notation for t_j in the algebra $\mathrm{Y}_{d,n}(q)$. So, in $\mathrm{Y}_{d,n}(q)$ we have the following quadratic relations:

$$g_i^2 = 1 + (q - q^{-1})\, e_i\, g_i \qquad (1 \leq i \leq n-1). \tag{1.3}$$

The elements $g_i \in \mathrm{Y}_{d,n}(q)$ are invertible:

$$g_i^{-1} = g_i - (q - q^{-1})\, e_i \qquad (1 \leq i \leq n-1). \tag{1.4}$$

Further the elements $g_i \in \mathrm{Y}_{d,n}(q)$ satisfy the following relations:

Lemma 1.1 ([10, Lemma 1.1]) *Let* $i \in \{1, \ldots, n-1\}$. *Then:*

$$g_i^r = (1 - e_i)\, g_i + \left(\frac{q^r + q^{-r}}{q + q^{-1}}\right) e_i g_i + \left(\frac{q^{r-1} - q^{-r+1}}{q + q^{-1}}\right) e_i \qquad \text{for } r \text{ odd},$$
$$g_i^r = 1 - e_i + \left(\frac{q^r - q^{-r}}{q + q^{-1}}\right) e_i g_i + \left(\frac{q^{r-1} + q^{-r+1}}{q + q^{-1}}\right) e_i \qquad \text{for } r \text{ even}.$$

The Yokonuma–Hecke algebras were originally introduced by T. Yokonuma [47] in the representation theory of finite Chevalley groups and they are natural generalizations of the Iwahori–Hecke algebras $\mathrm{H}_n(q)$. Indeed, for $d = 1$ all framings are zero, so the corresponding elements of $\mathscr{F}_n$ are identified with elements in B_n; also we have $e_i = 1$, so the quadratic relation (1.3) becomes the well–known quadratic relation of the algebra $\mathrm{H}_n(q)$:

$$g_i^2 = 1 + (q - q^{-1})\, g_i \qquad (1 \leq i \leq n-1).$$

Thus, the algebra $\mathrm{Y}_{1,n}(q)$ coincides with the algebra $\mathrm{H}_n(q)$. The Yokonuma–Hecke algebras can be also regarded as unipotent algebras in the sense of [45]. The representation theory of these algebras has been studied in [13, 45]. In [13] a completely combinatorial approach is taken to the subject. Further, in [26] a decomposition of the Yokonuma–Hecke algebra is constructed, as a direct sum of matrix algebras with coefficients in tensor products of Iwahori–Hecke algebras of type A, which is a special case of a result of G. Lusztig [40].

Following [29, Sect. 3], the algebra $\mathrm{Y}_{d,n}(q)$ has linear dimension $d^n n!$ and the set

$$\mathscr{B}_n^{\mathrm{can}} = \left\{ t_1^{k_1} \ldots t_n^{k_n} (g_{i_1} \ldots g_{i_1 - r_1}) \cdots (g_{i_p} \ldots g_{i_p - r_p}) \;\middle|\; \begin{array}{l} k_1, \ldots, k_n \in \mathbb{Z}/d\mathbb{Z} \\ 1 \leq i_1 < \cdots < i_p \leq n-1 \end{array} \right\}$$

is a $\mathbb{C}$-linear basis for $\mathrm{Y}_{d,n}(q)$. This basis is called the *canonical* basis of $\mathrm{Y}_{d,n}(u)$. Note that, in each element of the standard basis, the highest index generator g_{n-1} appears at most once.

Now, the natural inclusions $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ give rise to the algebra inclusions $\mathbb{C}\mathscr{F}_n \subset \mathbb{C}\mathscr{F}_{n+1}$, which in turn induce the algebra inclusions $\mathrm{Y}_{d,n}(q) \subset \mathrm{Y}_{d,n+1}(q)$ for $n \in \mathbb{N}$ (setting $\mathbb{C}\mathrm{Y}_{d,0}(q) := \mathbb{C}$). We can construct an inductive basis $\mathscr{B}_n^{\mathrm{ind}}$ for $\mathrm{Y}_{d,n}(q)$ in the following way: we set $B_0^{\mathrm{ind}} := \{1\}$ and

$$\mathscr{B}_{n+1}^{\mathrm{ind}} := \{w_n g_n g_{n-1} \dots g_i t_i^k,\ w_n t_{n+1}^k \mid 1 \leq i \leq n,\ k \in \mathbb{Z}/d\mathbb{Z},\ w_n \in \mathscr{B}_n^{\mathrm{ind}}\},$$

for all $n \in \mathbb{N}$.

Remark 1.1 In the papers [12, 29–34] another presentation was employed for the Yokonuma–Hecke algebra that was using a more complex quadratic relation giving rise to more computational difficulties. We refer to this extensively in Sect. 1.6.

Remark 1.2 By the fact that the classical braid group B_n embeds in $\mathscr{F}_{d,n}$ (and in $\mathscr{F}_n$) and by relations (1.1) ($\mathrm{b}_1, \mathrm{b}_2$), there is a natural homomorphism from B_n to $\mathrm{Y}_{d,n}$, treating the framing generators t_j's as formal elements in the algebra. So, the algebra $\mathrm{Y}_{d,n}$ can be also used in the study of classical knots and links.

Note 1.1 In this paper we will sometimes identify algebra monomials with their corresponding braid words.

1.2 Markov Traces on the Yokonuma–Hecke Algebras

In this section we recall the definition of a unique Markov trace defined on the algebras $\mathrm{Y}_{d,n}(q)$, as well as a necessary condition on the trace parameters, needed for obtaining framed link invariants.

1.2.1 The Juyumaya Trace

By the natural inclusions $\mathscr{F}_n \subset \mathscr{F}_{n+1}$, which induce the inclusions $\mathrm{Y}_{d,n}(q) \subset \mathrm{Y}_{d,n+1}(q)$, and using the inductive bases of the algebras $\mathrm{Y}_{d,n}(q)$ we have:

Theorem 1.1 ([29, Theorem 12]) *For $z, x_1, \dots, x_{d-1}$ indeterminates over $\mathbb{C}$ there exists a unique linear map*

$$\mathrm{tr}_d : \bigcup_{n \geq 0} \mathrm{Y}_{d,n}(q) \longrightarrow \mathbb{C}[z, x_1, \dots, x_{d-1}]$$

satisfying the rules:

$$\begin{array}{lllll} (1) & \mathrm{tr}_d(\alpha\beta) & = & \mathrm{tr}_d(\beta\alpha) & \alpha, \beta \in \mathrm{Y}_{d,n}(q) \\ (2) & \mathrm{tr}_d(1) & = & 1 & 1 \in \mathrm{Y}_{d,n}(q) \\ (3) & \mathrm{tr}_d(\alpha g_n) & = & z\,\mathrm{tr}_d(\alpha) & \alpha \in \mathrm{Y}_{d,n}(q) \quad \text{(Markov property)} \\ (4) & \mathrm{tr}_d(\alpha t_{n+1}^s) & = & x_s\,\mathrm{tr}_d(\alpha) & \alpha \in \mathrm{Y}_{d,n}(q) \quad (1 \le s \le d-1). \end{array}$$

Note that for $d = 1$ the trace restricts to the first three rules and it coincides with the Ocneanu trace τ on the Iwahori–Hecke algebras.

Moreover, the trace tr_d satisfies the following equality:

$$\mathrm{tr}_{d,D}(a e_n g_n) = z\,\mathrm{tr}_{d,D}(a) \qquad a \in \mathrm{Y}_{d,n}(q). \tag{1.5}$$

Note 1.2 In this paper we will sometimes write $\mathrm{tr}_d(\alpha)$ for a framed braid $\alpha \in \mathscr{F}_n$, by using the natural epimorphism of $\mathscr{F}_n$ onto $\mathrm{Y}_{d,n}$. Similarly, the same holds for a classical braid $\alpha \in B_n$ by using the natural homomorphism of B_n into $\mathrm{Y}_{d,n}$.

1.2.2 The E*–System*

Using the natural epimorphism of the framed braid group $\mathscr{F}_n$ onto $\mathrm{Y}_{d,n}(q)$, the trace tr_d and the Markov framed braid equivalence, comprising conjugation in the groups $\mathscr{F}_n$ and positive and negative stabilization and destabilization (see for example [37]), in [31] the authors tried to obtain a topological invariant for framed links after the method of V.F.R. Jones [27] (using for the algebra the presentation discussed in Sect. 1.6). This meant that tr_d would have to be normalized, so that the closed braids $\widehat{\alpha}$ and $\widehat{\alpha\sigma_n}$ ($\alpha \in \mathscr{F}_n$) be assigned the same value of the invariant, and re-scaled, so that the closed braids $\widehat{\alpha\sigma_n^{-1}}$ and $\widehat{\alpha\sigma_n}$ ($\alpha \in \mathscr{F}_n$) be also assigned the same value of the invariant. However, as it turned out, $\mathrm{tr}_d(\alpha g_n^{-1})$ does not factor through $\mathrm{tr}_d(\alpha)$, that is:

$$\mathrm{tr}_d(\alpha g_n^{-1}) \stackrel{(4)}{=} \mathrm{tr}_d(\alpha g_n) - (q - q^{-1})\,\mathrm{tr}_d(\alpha e_n) \neq \mathrm{tr}_d(g_n^{-1})\mathrm{tr}_d(\alpha). \tag{1.6}$$

The reason is that, although $\mathrm{tr}_d(\alpha g_n) = z\,\mathrm{tr}_d(\alpha)$, $\mathrm{tr}_d(\alpha e_n)$ does not factor through $\mathrm{tr}_d(\alpha)$, that is:

$$\mathrm{tr}_d(\alpha e_n) \neq \mathrm{tr}_d(e_n)\mathrm{tr}_d(\alpha). \tag{1.7}$$

This is due to the fact that:

$$\mathrm{tr}_d(\alpha t_n^k) \neq \mathrm{tr}_d(t_n^k)\mathrm{tr}_d(\alpha) \quad k = 1, \ldots, d-1. \tag{1.8}$$

Forcing

$$\mathrm{tr}_d(\alpha e_n) = \mathrm{tr}_d(e_n)\mathrm{tr}_d(\alpha) \tag{1.9}$$

yields that the trace parameters $x_1, \ldots, x_{d-1}$ have to satisfy the E–*system*, the non-linear system of equations in $\mathbb{C}$:

$$E^{(m)} = x_m E \qquad (1 \leq m \leq d-1)$$

where

$$E := E^{(0)} = \frac{1}{d}\sum_{s=0}^{d-1} x_s x_{d-s} = \mathrm{tr}_d(e_i) \qquad \text{and} \qquad E^{(m)} := \frac{1}{d}\sum_{s=0}^{d-1} x_{m+s} x_{d-s}\,,$$

where the sub-indices on the x_j's are regarded modulo d and $x_0 := 1$ (see [31]). As it was shown by P. Gérardin (in the Appendix of [31]), the solutions of the E–system are parametrized by the non-empty subsets of $\mathbb{Z}/d\mathbb{Z}$. For example, for every singleton subset $\{m\}$ of $\mathbb{Z}/d\mathbb{Z}$, we have a solution of the E–system given by:

$$\mathrm{x}_1 = \exp(2\pi m\sqrt{-1}/d) \qquad \text{and} \qquad \mathrm{x}_k = \mathrm{x}_1^k \quad \text{for } k = 2, \ldots, d-1. \tag{1.10}$$

1.2.3 The Specialized Trace

Let $X_D := (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1})$ be a solution of the E–system parametrized by the non-empty subset D of $\mathbb{Z}/d\mathbb{Z}$. We shall call *specialized trace* the trace tr_d with the parameters $x_1, \ldots, x_{d-1}$ specialized to $\mathrm{x}_1, \ldots, \mathrm{x}_{d-1} \in \mathbb{C}$, and it shall be denoted $\mathrm{tr}_{d,D}$ (cf. [12]). More precisely,

$$\mathrm{tr}_{d,D} : \bigcup_n \mathrm{Y}_{d,n}(q) \longrightarrow \mathbb{C}[z]$$

is a Markov trace on the Yokonuma–Hecke algebra, satisfying the following rules:

$$\begin{array}{llcll}
(1) & \mathrm{tr}_{d,D}(\alpha\beta) & = & \mathrm{tr}_{d,D}(\beta\alpha) & \alpha, \beta \in \mathrm{Y}_{d,n}(q) \\
(2) & \mathrm{tr}_{d,D}(1) & = & 1 & 1 \in \mathrm{Y}_{d,n}(q) \\
(3) & \mathrm{tr}_{d,D}(\alpha g_n) & = & z\,\mathrm{tr}_{d,D}(\alpha) & \alpha \in \mathrm{Y}_{d,n}(q) \quad \text{(Markov property)} \\
(4') & \mathrm{tr}_{d,D}(\alpha t_{n+1}^s) & = & \mathrm{x}_s\,\mathrm{tr}_{d,D}(\alpha) & \alpha \in \mathrm{Y}_{d,n}(q) \quad (1 \leq s \leq d-1).
\end{array}$$

The rules (1)–(3) are the same as in Theorem 1.1, while rule (4) is replaced by the rule (4′). As it turns out [32]:

$$E_D := \mathrm{tr}_{d,D}(e_i) = \frac{1}{|D|}, \tag{1.11}$$

where $|D|$ is the cardinality of the subset D. Note that $\mathrm{tr}_{1,\{0\}}$ coincides with tr_1 that in turn coincides with the Ocneanu trace τ.

1.2.4 Properties of the Markov Traces

We shall now give some properties of the traces tr_d and $\mathrm{tr}_{d,D}$ [10], analogous to known properties of the Ocneanu trace τ, by considering their behaviour under the operations below. Clearly, a property satisfied by tr_d is also satisfied by $\mathrm{tr}_{d,D}$ (and by τ), but the converse may not hold.

• *Inversion of braid words.* Inversion means that a braid word is written from right to left. For $\alpha = t_1^{k_1} \dots t_n^{k_n} \sigma_{i_1}^{l_1} \dots \sigma_{i_r}^{l_r} \in \mathcal{F}_n$, where $k_1, \dots, k_n, l_1, \dots, l_r \in \mathbb{Z}$, we denote by $\overleftarrow{\alpha}$ the inverted word, that is, $\overleftarrow{\alpha} = \sigma_{i_r}^{l_r} \dots \sigma_{i_1}^{l_1} t_n^{k_n} \dots t_1^{k_1}$. On the level of closed braids this operation corresponds to the change of orientation on all components of the resulting link. The operation can be extended linearly to elements of $\mathrm{Y}_{d,n}$. The trace tr_d (and consequently also the trace $\mathrm{tr}_{d,D}$) satisfies the following property:

$$\mathrm{tr}_d(\alpha) = \mathrm{tr}_d(\overleftarrow{\alpha}).$$

• *Split links.* Let $L = L_1 \sqcup \dots \sqcup L_m$ be a split framed link, where $L_1, \dots, L_m$ are framed links. Then there exists a braid word $\alpha = \alpha_1 \dots \alpha_m \in \mathcal{F}_n$, where $\alpha_i \in \mathcal{F}_{i_j} \setminus \mathcal{F}_{i_{j-1}+1}$ for some $1 \leq i_1 < \dots < i_k \leq n$ with $i_{j+1} - i_j > 1$ such that $\widehat{\alpha} = L$ and $\widehat{\alpha_i} = \sqcup_{k=1}^{i_{j-1}} U \sqcup L_i$ for each $i = 1, \dots, m$. The trace tr_d (and consequently also the trace $\mathrm{tr}_{d,D}$) satisfies the following property:

$$\mathrm{tr}_d(\alpha) = \mathrm{tr}_d(\alpha_1) \cdots \mathrm{tr}_d(\alpha_m) \tag{1.12}$$

• *Connected sums.* Let $\alpha \in \mathcal{F}_n$ and $\beta \in \mathcal{F}_m$ for some $n, m \in \mathbb{N}$. The *connected sum* of α and β is the word $\alpha\#\beta := \alpha^{[0]}\beta^{[n-1]}$ in the framed braid group $\mathcal{F}_{n+m-1}$, where $\alpha^{[0]}$ is the natural embedding of α in $\mathcal{F}_{n+m-1}$, while $\beta^{[n-1]}$ is the embedding of β in $\mathcal{F}_{n+m-1}$ induced by the following shifting of the indices: $\sigma_i \mapsto \sigma_{n+i-1}$ for $i \in \{1, \dots, m-1\}$ and $t_j \mapsto t_{n+j-1}$ for $j \in \{1, \dots, m\}$. Upon closing the braids, this operation corresponds to taking the connected sum of the resulting framed links. It is known that the Ocneanu trace is multiplicative under the connected sum operation, that is, $\tau(\alpha\#\beta) = \tau(\alpha)\,\tau(\beta)$ if $\alpha \in B_n$ and $\beta \in B_m$. On the other hand, the trace tr_d is *not multiplicative* under the connected sum operation, due to (1.8) and (1.7) (we have $\alpha\#t_1^k = \alpha t_n^k$ and, by linear extension, $\alpha\#e_1 = \alpha e_n$). Yet, the specialized trace $\mathrm{tr}_{d,D}$ is multiplicative on connected sums, due to the E–condition (1.9), but this is only true on the level of classical braids [10]. Namely:

$$\mathrm{tr}_{d,D}(\alpha\#\beta) = \mathrm{tr}_{d,D}(\alpha)\,\mathrm{tr}_{d,D}(\beta) \text{ for } \alpha \in B_n \text{ and } \beta \in B_m.$$

For framed braids this is true only when $E_D = 1$, that is, when the set D is singleton and hence the corresponding solution X_D of the E–system is described by (1.10). Namely, for $\alpha \in \mathcal{F}_n$ and $\beta \in \mathcal{F}_m$:

$$\mathrm{tr}_{d,D}(\alpha\#\beta) = \mathrm{tr}_{d,D}(\alpha)\mathrm{tr}_{d,D}(\beta) \Leftrightarrow \mathrm{x}_1^d = 1 \text{ and } \mathrm{x}_k = \mathrm{x}_1^k \text{ for } k = 1, \dots, d-1.$$

• *Mirror images.* Let us consider the group automorphism of B_n given by $\sigma_i \mapsto \sigma_i^{-1}$. For $\alpha \in B_n$, we denote by α^* the image of α via this automorphism. We call α^* the *mirror image* of α. On the level of closed braids this operation corresponds to switching all crossings. Note that the operation mirror image applies on classical braids or links. It is known that the Ocneanu trace satisfies a "mirroring property". Namely, $\tau(q,z)(\alpha^*) = \tau\left(q^{-1}, z-(q-q^{-1})\right)(\alpha)$. However, due to (1.6), the trace tr_d does not satisfy the mirroring property, but the specialized trace $\mathrm{tr}_{d,D}$ does. Namely, observe that $\mathrm{tr}_{d,D}(g_i^{-1}) = z-(q-q^{-1})E_D$ and set a new variable λ_D in place of z. By re-scaling σ_i to $\sqrt{\lambda_D}g_i$, so that $\mathrm{tr}_{d,D}(g_n^{-1}) = \lambda_D z$, we find

$$\lambda_D := \frac{z-(q-q^{-1})E_D}{z}. \tag{1.13}$$

If we solve (1.13) with respect to the variable z, we obtain

$$z = \frac{(q-q^{-1})\,E_D}{1-\lambda_D}.$$

Hence, the trace $\mathrm{tr}_{d,D}$ can be considered as a polynomial in the variables (q,z) or in the variables (q,λ_D) by the above change of variables. Using this notation, the trace $\mathrm{tr}_{d,D}$ satisfies the following property:

$$\mathrm{tr}_{d,D}(q,z)(\alpha^*) = \mathrm{tr}_{d,D}\left(q^{-1}, z-(q-q^{-1})E_D\right)(\alpha), \quad \text{for any } \alpha \in B_n,$$

or equivalently,

$$\mathrm{tr}_{d,D}(q,\lambda_D)(\alpha^*) = \mathrm{tr}_{d,D}\left(q^{-1}, \lambda_D^{-1}\right)(\alpha), \quad \text{for any } \alpha \in B_n.$$

1.2.5 The Specialized Trace $\mathrm{tr}_{d,D}$ *on Classical Braids*

Let $\alpha \in B_n$ be a classical braid. When calculating $\mathrm{tr}_{d,D}(\alpha)$, the framing generators t_j appear only in the form of the idempotents e_i due to the application of the quadratic relation (1.3). In this case, the fourth rule of the trace $\mathrm{tr}_{d,D}$ is not applied directly, but rather indirectly using the E–condition (1.9). It has been long conjectured by J. Juyumaya that the fourth rule of the trace $\mathrm{tr}_{d,D}$, when computed on classical braids, can be substituted by rules involving only the idempotents e_i (cf. [2, 28]). Indeed, this fact has been proved in [11].

Before we proceed to the statement (Theorem 1.2), we define the subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ of $\mathrm{Y}_{d,n}(q)$ generated only by the braiding generators $g_1,\ldots,g_{n-1}$. The subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ is also the image of the natural homomorphism $\delta : \mathbb{C}B_n \to \mathrm{Y}_{d,n}(q)$ defined by $\sigma_i \mapsto g_i$, since $g_i^{-1} \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ for $i=1,\ldots,n-1$ [11]. Remarkably, for $q \neq 1$ it is also true that $e_i \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ for $i=1,\ldots,n-1$ [11].

When computing the trace $\mathrm{tr}_{d,D}$ for classical braids, we restrict ourselves on the subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$. Then the fourth rule of the trace can be substituted by two new rules as follows:

Theorem 1.2 ([11, Theorem 4.3]) *The following rules are sufficient for computing the trace* $\mathrm{tr}_{d,D}$ *on* $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$*:*

$$\begin{array}{llcll}
\text{(i)} & \mathrm{tr}_{d,D}(ab) & = & \mathrm{tr}_{d,D}(ba) & a, b \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})} \\
\text{(ii)} & \mathrm{tr}_{d,D}(1) & = & 1 & 1 \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})} \\
\text{(iii)} & \mathrm{tr}_{d,D}(ag_n) & = & z\,\mathrm{tr}_{d,D}(a) & a \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})} \quad (\textit{Markov property}) \\
\text{(iv)} & \mathrm{tr}_{d,D}(ae_n) & = & E_D\,\mathrm{tr}_{d,D}(a) & a \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})} \\
\text{(v)} & \mathrm{tr}_{d,D}(ae_ng_n) & = & z\,\mathrm{tr}_{d,D}(a) & a \in \mathrm{Y}_{d,n}(q)^{(\mathrm{br})}.
\end{array}$$

As we have seen, rule (v) holds for the trace tr_d (1.5) and rule (iv) is the E–condition (1.9). There is no analogue of this theorem for the trace tr_d since it does not satisfy the E–condition (rule iv). Notice that the value of the trace $\mathrm{tr}_{d,D}$ does not depend on the specific solution of the E–system, but only on the cardinality of the subset $D \subseteq \mathbb{Z}/d\mathbb{Z}$ due to (1.11). This fact will have important consequences on the classical link invariants defined via the trace $\mathrm{tr}_{d,D}$.

1.3 Link Invariants from the Yokonuma–Hecke Algebras

Given a solution $X_D := (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1})$ of the E–system invariants for various types of knots and links, such as framed, classical and singular, have been constructed from $\mathrm{tr}_{d,D}$ in [31–33]. The definitions of these invariants have been adapted in [10, 11] in view of the new presentation of $\mathrm{Y}_{d,n}(q)$. Moreover, we recall the construction of the transverse link invariants defined in [10].

1.3.1 Framed Links

Let $\mathscr{L}_f$ denote the set of oriented framed links. We set:

$$\Lambda_D := \frac{1}{z\sqrt{\lambda_D}}. \tag{1.14}$$

From the above and re-scaling σ_i to $\sqrt{\lambda_D}g_i$, so that $\mathrm{tr}_{d,D}(g_n^{-1}) = \lambda_D z$, we have the following Theorem, which is analogous to [31, Theorem 8]:

Theorem 1.3 ([10, Theorem 4.1]) *Given a solution* X_D *of the E–system, for any framed braid* $\alpha \in \mathscr{F}_n$ *we define for the framed link* $\widehat{\alpha} \in \mathscr{L}_f$*:*

$$\Phi_{d,D}(\widehat{\alpha}) = \Lambda_D^{n-1}(\sqrt{\lambda_D})^{\epsilon(\alpha)}\left(\mathrm{tr}_{d,D} \circ \gamma\right)(\alpha)$$

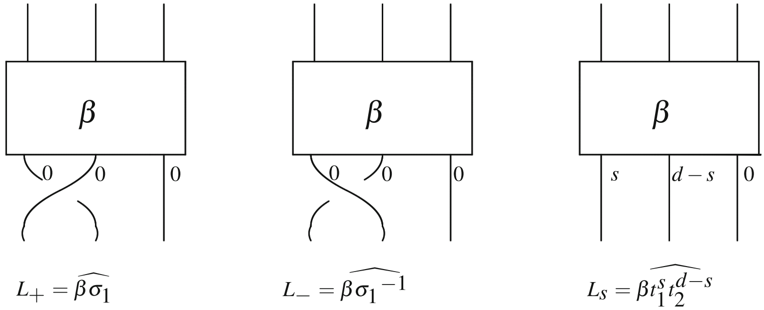

Fig. 1.1 The framed links of the skein relation in open braid form

where $\gamma : \mathbb{C}\mathscr{F}_n \longrightarrow \mathrm{Y}_{d,n}(q)$ is the natural algebra homomorphism defined via: $\sigma_i \mapsto g_i$ and $t_j^s \mapsto t_j^{s(\mathrm{mod}\, d)}$, and $\epsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Then the map $\Phi_{d,D}(q, z)$ is a 2-variable isotopy invariant of oriented framed links.

Proposition 1.1 ([10, Proposition 4.2]) *The invariant $\Phi_{d,D}$ satisfies the following skein relation:*

$$\frac{1}{\sqrt{\lambda_D}}\Phi_{d,D}(L_+) - \sqrt{\lambda_D}\Phi_{d,D}(L_-) = \frac{q-q^{-1}}{d}\sum_{s=0}^{d-1}\Phi_{d,D}(L_s), \tag{1.15}$$

where the links L_+, L_- and L_s are closures of the framed braids illustrated in Fig. 1.1.

Remark 1.3 Note that, for every $d \in \mathbb{N}$, we have $2^d - 1$ distinct solutions of the E–system, so the above construction yields $2^d - 1$ isotopy invariants for framed links.

Remark 1.4 Due to the complicated computations for the trace $\mathrm{tr}_{d,D}$ (and hence for the invariants $\Phi_{d,D}$, two computer programs have been developed for this purpose. One has been developed by M. Chmutov in Maple [10] and the other by the first author in C#. Let $w \in \mathrm{Y}_{d,n}(q)$ be a word. Both of the programs apply iteratively the quadratic relation, breaking the word w into many simpler words. Then using the relations (b_1) and (f_2) of the braid group (which also hold in the algebra $\mathrm{Y}_{d,n}(q)$) the programs reduce w into words written in split form. Finally, the four rules of the trace $\mathrm{tr}_{d,D}$ are applied and the computation ends. Note that both programs have exponential complexity with respect to $r(w)$, where $r(w)$ is the number of indices of the braiding generators in w with powers different than 0 or 1.

1.3.2 Classical Links

Let $\mathscr{L}$ denote the set of oriented classical links. The classical braid group B_n injects into the framed braid group $\mathscr{F}_n$, whereby elements in B_n are viewed as framed braids with all framings zero. So, by the classical Markov braid equivalence, comprising conjugation in the groups B_n and positive and negative stabilizations and destabilizations, and by the construction and notations above, we obtain isotopy invariants for oriented classical knots and links, where the t_j's are treated as formal generators. These invariants of classical links, which are analogous to those defined in [32] where the old presentation for the Yokonuma–Hecke algebra is used, are denoted as $\Theta_{d,D}$ and the restriction of $\gamma : \mathbb{C}\mathscr{F}_n \longrightarrow \mathrm{Y}_{d,n}(q)$ on $\mathbb{C}B_n$ is denoted as δ. Namely,

$$\Theta_{d,D}(\widehat{\alpha}) := \Lambda_D^{n-1}(\sqrt{\lambda_D})^{\epsilon(\alpha)}\,(\mathrm{tr}_{d,D} \circ \delta)(\alpha).$$

An important corollary of Theorem 1.2 is that the invariants $\Theta_{d,D}$ do not depend on d and D but only on the cardinality of the set D. Namely:

Proposition 1.2 ([11, Proposition 4.6]) *The values of the isotopy invariants $\Theta_{d,D}$ for classical links depend only on the cardinality $|D|$ of D. Hence, for a fixed d, we only obtain d invariants. Further, for d, d' positive integers with $d \leq d'$, we have $\Theta_{d,D} = \Theta_{d',D'}$ as long as $|D| = |D'|$. We deduce that, if $|D'| = d$, then $\Theta_{d',D'} = \Theta_{d,\mathbb{Z}/d\mathbb{Z}}$. Therefore, the invariants $\Theta_{d,D}$ can be parametrized by the natural numbers, setting $\Theta_d := \Theta_{d,\mathbb{Z}/d\mathbb{Z}}$ for all $d \in \mathbb{Z}_{>0}$.*

The invariants $\Theta_d(q,z)$ need to be compared with known invariants of classical links, especially with the HOMFLYPT polynomial. The HOMFLYPT polynomial $P(q,z)$ is a 2-variable isotopy invariant of oriented classical links that can be constructed from the Iwahori–Hecke algebras $\mathrm{H}_n(q)$ and the Ocneanu trace τ after re–scaling and normalizing τ [27]. In this paper we define P via the invariants Θ_d, since for $d = 1$ the algebras $\mathrm{H}_n(q)$ and $\mathrm{Y}_{1,n}(q)$ coincide and the traces τ, tr_1 and $\mathrm{tr}_{1,\{0\}}$ also coincide. Namely, we define:

$$P(\widehat{\alpha}) = \Theta_1(\widehat{\alpha}) = \left(\frac{1}{z\sqrt{\lambda_{\mathrm{H}}}}\right)^{n-1} (\sqrt{\lambda_{\mathrm{H}}})^{\epsilon(\alpha)} \left(\mathrm{tr}_{1,\{0\}} \circ \delta\right)(\alpha)$$

where $\lambda_{\mathrm{H}} := \frac{z-(q-q^{-1})}{z} = \lambda_{\{0\}}$. Further, recall that the HOMFLYPT polynomial satisfies the following skein relation [27]:

$$\frac{1}{\sqrt{\lambda_{\mathrm{H}}}}\,P(L_+) - \sqrt{\lambda_{\mathrm{H}}}\,P(L_-) = (q - q^{-1})\,P(L_0) \tag{1.16}$$

where L_+, L_-, L_0 is a Conway triple.

Contrary to the case of framed links, the skein relation of the invariants $\Phi_{d,D}(q,z)$ has no topological interpretation in the case of classical links since it introduces framings. This makes it very difficult to compare the invariants $\Theta_d(q,z)$ with the HOMFLYPT polynomial using diagrammatic methods. Further, on the algebraic level, there is no algebra homomorphism connecting the algebras and the traces [12]. Consequently, in [12] it is shown that for generic values of the parameters q, z the invariants $\Theta_d(q,z)$ do not coincide with the HOMFLYPT polynomial. In fact, they only coincide in the trivial cases where $q = \pm 1$ or $\mathrm{tr}_d(e_i) = 1$. The last case implies that the solution of the E–system comprises the d-th roots of unity. Yet, the invariants $\Theta_d(q, z)$ could be topologically equivalent to the HOMFLYPT polynomial, in the sense that they distinguish or not distinguish the same pairs of knots and links. The topological comparison of Θ_d with P has been a long-standing open problem that was eventually answered in [11] and these results are presented in Sect. 1.4 below.

Remark 1.5 A computer program has been developed by the first author in Mathematica for computing the classical link invariants Θ_d [35]. This program uses Theorem 1.2 for computing the trace $\mathrm{tr}_{d,D}$ on classical braids, that is, the elements e_i are considered as formal elements instead of a sum of products of the framing generators. Further, it uses the new presentation $\mathrm{Y}_{d,n}(q)$ of the Yokonuma-Hecke algebra, whose quadratic relation is more economical for computations. Both facts result in lower computational complexity than the programs of Remark 1.4, however the computational complexity remains exponential (see [35] for more details).

1.3.3 Singular Links

Let $\mathcal{L}_{\mathcal{S}}$ denote the set of oriented singular links. Oriented singular links are represented by singular braids that form the singular braid monoids $\mathcal{S}B_n$ [5, 7, 44]. The singular braid monoid $\mathcal{S}B_n$ is generated by the classical braiding generators σ_i with their inverses, together with the elementary singular braids τ_i that are not invertible. In [33] a monoid homomorphism was constructed that we adapt here to the new presentation $\mathrm{Y}_{d,n}(q)$ of the Yokonuma–Hecke algebra, namely:

$$\begin{array}{rcl} \eta : \mathcal{S}B_n & \longrightarrow & \mathrm{Y}_{d,n}(q) \\ \sigma_i & \mapsto & g_i \\ \tau_i & \mapsto & e_i \end{array} \tag{1.17}$$

Using the singular braid equivalence [21] (see also [38]), the map η and the specialized trace $\mathrm{tr}_{d,D}$ we obtain isotopy invariants for oriented singular links, analogous to the ones constructed in [33, Theorem 3.6], as follows:

Theorem 1.4 ([10, Theorem 4.8]) *For any singular braid $\alpha \in \mathcal{S}B_n$, we define*

$$\Psi_{d,D}(\widehat{\alpha}) := \Lambda_D^{n-1} (\sqrt{\lambda_D})^{\epsilon(\alpha)} \left(\mathrm{tr}_{d,D} \circ \eta\right)(\alpha) \,,$$

where Λ_D, λ_D are as defined in (1.13) and (1.14), η is as defined in (1.17) and $\epsilon(\alpha)$ is the sum of the exponents of the generators σ_i and τ_i in the word α. Then the map $\Psi_{d,D}(q, z)$ is a 2-variable isotopy invariant of oriented singular links.

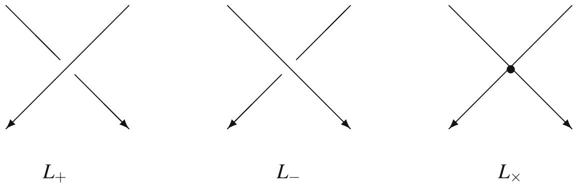

Fig. 1.2 The singular links L_+, L_- and $L_\times$

Moreover, in the image $\eta(\mathscr{S}B_n)$, we have

$$g_i - g_i^{-1} = (q - q^{-1})e_i \quad \text{for all } i = 1, \ldots, n-1,$$

which gives rise to the following skein relation (compare with [33]):

$$\frac{1}{\sqrt{\lambda_D}}\,\Psi_{d,D}(L_+) - \sqrt{\lambda_D}\,\Psi_{d,D}(L_-) = \frac{q - q^{-1}}{\sqrt{\lambda_D}}\,\Psi_{d,D}(L_\times)$$

where L_+, L_- and $L_\times$ are diagrams of three oriented singular links that are identical except for one crossing, where they are as in Fig. 1.2. Furthermore, the properties of the traces $\mathrm{tr}_{d,D}$ under inversion, split links, connected sums and mirror imaging carry through to the invariants $\Psi_{d,D}$.

1.3.4 Transverse Links

Another class of links that is naturally related to the Yokonuma–Hecke algebras is the class of *transverse links*, denoted by $\mathscr{L}_{\mathscr{T}}$. A transverse knot is represented by a smooth closed spacial curve that is nowhere tangent to planes of a special field of planes in $\mathbb{R}^3$ called *standard contact structure* (for the precise definition see for example [20]). Transverse links are naturally framed and oriented. Two links that are classically isotopic may be transversely non–equivalent. So, a topological type of framed links may consist of several different types in $\mathscr{L}_{\mathscr{T}}$. The problem is to find transverse invariants for such links.

In 1983, D. Bennequin [6] noted that the closed braid presentation of knots is convenient for describing transverse knots with the blackboard framing. For a knot K represented as a closed braid $\widehat{\alpha}$ with n strands, one can check that the self-linking number is equal to $sl(K) = \epsilon(\alpha) - n$, where $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators σ_i in the word $\alpha \in B_n$ [6]. So, the transverse knot K defines naturally an element of the framed braid group $\alpha' := t_1^{sl(K)}\alpha \in \mathscr{F}_n$. This generalizes to transverse links in the obvious manner (using the self-linking of each component).

Further, S. Orevkov and V. Shevchishin [41] and independently N. Wrinkle [46] gave a transverse analogue of the Markov Theorem, comprising conjugation in the braid groups and *only positive* stabilizations and destabilizations: $\alpha \sim \alpha\,\sigma_n$, where $\alpha \in B_n$. Now, rule (3) of the definition of the trace tr_d (Theorem 1.1) tells us that tr_d

respects positive stabilizations. Moreover, the absence of the negative stabilization in the transverse braid equivalence resembles the problem of re-scaling of the trace tr_d with respect to the negative stabilization, recall (1.6), making the Yokonuma–Hecke algebras a natural algebraic object related to the class $\mathscr{L}_{\mathscr{T}}$ of transverse links. Let L be a transverse link represented by the closure $\widehat{\alpha}$ of a braid $\alpha \in B_n$, giving rise to the framed braid $\alpha' = t_1^{r_1} \dots t_n^{r_n} \alpha \in \mathscr{F}_n$, where $r_1 + \dots + r_n = sl(L)$. We define

$$M_d(\widehat{\alpha}) := \frac{1}{z^{n-1}} \mathrm{tr}_d(\alpha').$$

Theorem 1.5 ([10, Theorem 4.11]) *The map $M_d(q, z, x_1, \dots, x_{d-1})$ is a $(d+1)$-variable isotopy invariant of oriented transverse links.*

The properties of the trace tr_d under inversion and split links carry through to the invariants M_d.

Remark 1.6 Due to the transverse braid equivalence, M_d need not take the same value on the links $\widehat{\alpha\sigma_n}$ and $\widehat{\alpha\sigma_n^{-1}}$. Hence, the re-scaling map $\sigma_i \mapsto \sqrt{\lambda_D} g_i$ is not needed any more and by consequence a quantity analogous to λ_D is not introduced. However, if we make such a re-scaling and specialize $(x_1, \dots, x_{d-1})$ to the solution $X_D = (\mathrm{x}_1, \dots, \mathrm{x}_{d-1})$ of the E–system, then the corresponding invariant of transverse links would coincide with the invariants $\Phi_{d,D}(q, z)$ of oriented framed links from Theorem 1.3.

Our original hope was that the invariants M_d would distinguish the transverse knots of the same topological type and with the same Bennequin numbers [6]. However, this turned out not to be the case, due to the following reason: any quantum knot invariant can be expressed in terms of Vassiliev invariants in a standard way (see, for example, [7] or [8]); we show that the invariants M_d can be similarly expressed in terms of Vassiliev invariants.

Proposition 1.3 ([10, Proposition 6.1]) *Let us make a substitution $q = e^h$ into the transverse knot invariant $M_d(q, z, x_1, \dots, x_{d-1})$ and consider the Taylor expansion in the power series in h. For every $n \in \mathbb{N}$, the coefficient of h^n is a Vassiliev invariant of order $\leq n$.*

The above proposition implies that the invariant $M_d(e^h, z, x_1, \dots, x_{d-1})$ of transverse knots is covered by an (infinite) sequence of Vassiliev invariants. However, the Fuchs–Tabachnikov theorem [20, Theorem 5.6] claims that any transverse Vassiliev invariant turns out to be a topological Vassiliev invariant of framed knots. For further discussion, we refer the reader to [10].

1.4 The Classical Link Invariants

1.4.1 Behaviour of the Invariants Θ_d on Knots

Let L_1 and L_2 be two links. We will say that L_1 and L_2 are Θ_d*-equivalent* (respectively P*-equivalent*) if $\Theta_d(L_1) = \Theta_d(L_2)$ (respectively $P(L_1) = P(L_2)$). A long-standing question had been how the classical link invariants Θ_d compare to the HOMFLYPT polynomial and among themselves for various values of $d \geq 2$. The aim has been to find a pair of P-equivalent knots or links that are not Θ_d-equivalent for some values of $d \geq 2$.

The first computations (using the presentation $\mathrm{Y}_{d,n}(u)$) on several pairs of P-equivalent pairs of knots and links were disappointing. However, it became evident from the computations that the invariants Θ_d are related on *knots* to the HOMFLYPT polynomial via a change of variables. This conjecture (formulated in [10]) has been proved in [11] by comparing the traces $\mathrm{tr}_{d,D}$ and τ on braids whose closures are knots. In detail:

Proposition 1.4 ([11, Proposition 5.6]) *Let $\alpha \in B_n$ be a knot. Then*

$$\mathrm{tr}_{d,D}(q, z)(\alpha) = E_D^{n-1}\, \tau(q, z/E_D)(\alpha).$$

Now, using Proposition 1.4 the conjecture relating the invariants Θ_d and P could be proved:

Theorem 1.6 ([11, Theorem 5.8]) *Given a solution X_D of the E–system, for any braid $\alpha \in B_n$ such that $\widehat{\alpha}$ is a knot, we have:*

$$\Theta_d(q, z)(\widehat{\alpha}) = \Theta_1(q, z/E_D)(\widehat{\alpha}) = P(q, z/E_D)(\widehat{\alpha}),$$

or equivalently:

$$\Theta_d(q, \lambda_D)(\widehat{\alpha}) = \Theta_1(q, \lambda_D)(\widehat{\alpha}) = P(q, \lambda_D)(\widehat{\alpha}).$$

Note that the polynomials $\Theta_d(q, \lambda_D)$ and $P(q, \lambda_{\mathrm{H}})$ coincide on knots by considering substituting the variable λ_{H} with λ_D in P. Hence, the value E_D does not appear when computing the invariants Θ_d on knots.

1.4.2 Behaviour of the Invariants Θ_d on Split Links and Disjoint Union of Knots

Let L and L' be two links. The invariants Θ_d satisfy the following property for split links:

$$\Theta_d(L \sqcup L') = \frac{1-\lambda_D}{(q-q^{-1})\sqrt{\lambda_D}E_D}\,\Theta_d(L)\Theta_d(L').$$

This equality can easily be proven using the property of the trace on split links (1.12). Now, Theorem 1.6 can be generalized to disjoint unions of knots using the multiplicative property of the invariants Θ_d on split links and Proposition 1.4. In detail:

Theorem 1.7 ([11, Theorem 6.2]) *Given a solution X_D of the E–system, for any braid $\alpha \in B_n$ such that $\widehat{\alpha}$ is a disjoint union of k knots, we have*

$$\Theta_d(q,z)(\widehat{\alpha}) = E_D^{1-k}\Theta_1(q,z/E_D)(\widehat{\alpha}) = E_D^{1-k}P(q,z/E_D)(\widehat{\alpha}),$$

or equivalently:

$$\Theta_d(q,\lambda_D)(\widehat{\alpha}) = E_D^{1-k}\Theta_1(q,\lambda_D)(\widehat{\alpha}) = E_D^{1-k}P(q,\lambda_D)(\widehat{\alpha}).$$

Now the remaining question was the case of links that are not disjoint unions of knots. In this case there was not an apparent conjecture relating the invariants Θ_d and P or relating the traces $\mathrm{tr}_{d,D}$ and τ. On the contrary, computations of the traces $\mathrm{tr}_{d,D}$ and τ on simple examples of links indicated that the invariants Θ_d and P may not be related by a change of variables [11, Sect. 6.2]. Specifically, for a 2-component link, the invariants Θ_d seemed to depend not only on the value of P of the same link, but also on the value of P of the link with the two components unlinked [11, Sect. 6.2]. Also, the behaviour of the elements e_i when computing the trace $\mathrm{tr}_{d,D}$ on simple examples was complicating the comparison of Θ_d with P on links [11, Sect. 6.3].

1.4.3 Behaviour of the Invariants Θ_d on Links – A Special Skein Relation

In order to investigate the question of Θ_d equivalence on P-equivalent pairs of links, a diagrammatic computation was not possible, since the skein relation of the framed invariants $\Phi_{d,D}$ (1.15) involves framed links and hence, there is no topological interpretation when computing the invariants Θ_d. However, if the skein relation (1.15) is applied to a *crossing involving different components*, then all the framed links of the skein relation reduce to classical links. Also, the skein relation obtained is identical to the one of the HOMFLYPT polynomial. In detail:

Proposition 1.5 ([11, Proposition 6.8]) *Let $\beta \in \mathscr{F}_n$ and $i \in \{1,\ldots,n-1\}$. Let*

$$L_+ = \widehat{\beta\sigma_i}, \quad L_- = \widehat{\beta\sigma_i^{-1}} \quad \text{and} \quad L_0 = \widehat{\beta}.$$

Suppose we apply the skein relation (1.15) of $\Phi_{d,D}$ on L_+ on the crossing σ_i and that the i-th and $(i+1)$-st strands (at the region of the crossing) belong to different

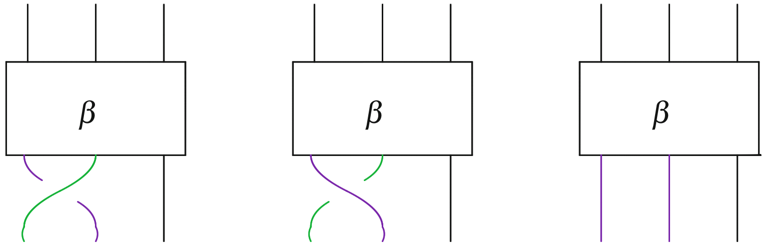

Fig. 1.3 The links in the special skein relation in open braid form

components. Then the skein relation reduces to the skein relation of the HOMFLYPT polynomial $P = P(q, \lambda_D)$ (1.16)

$$\frac{1}{\sqrt{\lambda_D}}\Phi_{d,D}(L_+) - \sqrt{\lambda_D}\Phi_{d,D}(L_-) = (q - q^{-1})\Phi_{d,D}(L_0), \tag{1.18}$$

see Fig. 1.3. Furthermore, if we take $\beta \in B_n$ *as a framed braid with all framings zero, then the above skein relation of* $\Phi_{d,D}$ *also holds for the invariants* Θ_d*, since it involves only classical links:*

$$\frac{1}{\sqrt{\lambda_D}}\Theta_d(L_+) - \sqrt{\lambda_D}\Theta_d(L_-) = (q - q^{-1})\Theta_d(L_0), \tag{1.19}$$

1.4.4 Behaviour of the Invariants Θ_d on Links – Θ_d as a Sum of HOMFLYPT Polynomials

This new special skein relation allows us to attack the problem diagrammatically. One can apply the special skein relation on mixed crossings resulting to a skein tree whose leaves consist of disjoint union of knots. Then using Theorem 1.7 one can compute the value of Θ_d on the initial link. Specifically, it has been proved inductively in [11], that this procedure can be applied:

Theorem 1.8 ([11, Theorem 6.16]) *For any* ℓ*-component link* L*, the value* $\Theta_d(L)$ *is a* $\mathbb{Q}[q^{\pm 1}, \sqrt{\lambda_D}^{\pm 1}]$*-linear combination of* $P(L)$ *and the values of* P *on disjoint unions of knots obtained by the skein relation:*

$$\Theta_d(L) = \sum_{k=1}^{\ell} E_D^{1-k} \sum_{\widehat{\alpha}\in\mathcal{N}(L)_k} c(\widehat{\alpha})\, P(\widehat{\alpha}) = P(L) + \sum_{k=2}^{\ell} (E_D^{1-k} - 1) \sum_{\widehat{\alpha}\in\mathcal{N}(L)_k} c(\widehat{\alpha})\, P(\widehat{\alpha}),$$

where $\mathcal{N}(L)_k$ denotes the set of all disjoint unions of k knots for $k = 1, \ldots, \ell$. Conversely, the value $P(L)$ is a $\mathbb{Q}[q^{\pm 1}, \sqrt{\lambda_D}^{\pm 1}]$-linear combination of $\Theta_d(L)$ and the values of Θ_d on disjoint unions of knots obtained by the skein relation:

$$P(L) = \sum_{k=2}^{\ell} E_D^{k-1} \sum_{\widehat{\alpha} \in \mathcal{N}(L)_k} c(\widehat{\alpha})\, \Theta_d(\widehat{\alpha}) = \Theta_d(L) + \sum_{k=2}^{\ell} (E_D^{k-1} - 1) \sum_{\widehat{\alpha} \in \mathcal{N}(L)_k} c(\widehat{\alpha})\, \Theta_d(\widehat{\alpha}).$$

Theorem 1.8 constitutes the first confirmation that Θ_d-equivalence does not necessarily imply P-equivalence and vice versa. Let now $d, d' \geq 2$ with $d \neq d'$. Using Theorem 1.8 one can write the invariants Θ_d and $\Theta_{d'}$ as a sum of HOMFLYPT polynomials and attempt to derive a relation connecting the two invariants. Indeed, this is possible and in fact is a generalization of Theorem 1.8:

Theorem 1.9 ([11, Theorem 6.18]) *Let $d, d' \in \mathbb{N}$. For any ℓ-component link L, the value $\Theta_{d'}(L)$ is an $\mathbb{Q}[q^{\pm 1}, \sqrt{\lambda_D}^{\pm 1}]$-linear combination of $\Theta_d(L)$ and the values of Θ_d on disjoint unions of knots obtained by the skein relation:*

$$\Theta_{d'}(L) = \Theta_d(L) + \sum_{k=2}^{\ell} \left(\left(\frac{E_D}{E_{D'}} \right)^{k-1} - 1 \right) \sum_{\widehat{\alpha} \in \mathcal{N}(L)_k} c(\widehat{\alpha})\, \Theta_d(\widehat{\alpha}).$$

It is immediate by Theorem 1.9 that Θ_d-equivalence does not necessarily imply $\Theta_{d'}$-equivalence for $d \neq d'$. However, for 2-component P-equivalent links the following result holds:

Theorem 1.10 ([11, Theorem 7.1]) *Let $d, d' \geq 2$ and let L_1 and L_2 be a pair of 2-component P-equivalent links. Then L_1 and L_2 are Θ_d-equivalent if and only if they are $\Theta_{d'}$-equivalent.*

1.4.5 Behaviour of the Invariants Θ_d on Links – A Skein-Theoretic Approach

Theorem 1.8 provides an algorithmic procedure to compute Θ_d diagrammatically as follows:

Step 1. Apply the skein relation of Proposition 1.5 on crossings linking different components until the link L is decomposed into disjoint unions of knots. An algorithmic process for achieving this is the following: we order the components of L and we select a starting point on each component. Starting from the chosen point of the first component and following its orientation we apply the skein relation on all mixed crossings we encounter, so that the arcs of this component are always overarcs. We proceed similarly with the second component changing all mixed crossing except for crossings involving the first component, and so on. In the end we obtain the split version of the original link.

Step 2. Following Theorem 1.8 and its notation, we obtain

$$\Theta_d(L) = \sum_{k=1}^{\ell} E_D^{1-k} \sum_{\widehat{\alpha} \in \mathcal{N}(L)_k} c(\widehat{\alpha})\, P(\widehat{\alpha}).$$

Step 3. Apply the skein relation (1.16) of the HOMFLYPT polynomial to obtain the value of P on $\widehat{\alpha}$ at variables (q, λ_D), for all disjoint unions of knots $\widehat{\alpha} \in \mathcal{N}(L)_k$, $k = 1, \ldots, \ell$.

Since the invariants Θ_d are well-defined via braid methods, we obtain by Theorem 1.8 the following:

Theorem 1.11 ([11, Theorem 6.19]) *The invariants Θ_d can be completely defined via the HOMFLYPT skein relation.*

1.4.6 Behaviour of the Invariants Θ_d on Links – Comparison with the HOMFLYPT Polynomial on Links

Theorem 1.8 and the special skein relation (1.19) allow us to compare diagrammatically the invariants Θ_d to the HOMFLYPT polynomial on various links.

It is known that the HOMFLYPT polynomial, being a skein link invariant, does not distinguish *mutant* knots or links. The operation of mutation on a link diagram is defined by choosing a disk that intersects the diagram at exactly four points and then rotating 180° the 2-tangle encircled by the disk. Investigating with the use of the trace $\mathrm{tr}_{d,D}$ whether the invariants Θ_d distinguish mutant knots or links would be impossible. However, using the special skein relation, it is possible to prove the following result:

Proposition 1.6 ([11, Proposition 6.5]) *Let L and L' be two mutant links. Then $\Theta_d(L) = \Theta_d(L')$.*

In order to compare the invariants Θ_d to the HOMFLYPT polynomial on examples of P-equivalent pairs of links, not isotopic to each other as unoriented link, computations were needed in order to calculate the values of the invariants Θ_d on them. Using the data from [9], 89 pairs of such links up to 11 crossings were found and we computed, using the program of Remark 1.5 [35], the values of the invariants Θ_d on them using the program of Remark 1.5. Out of these 89 pairs, there are 6 pairs of P-equivalent links that are not Θ_d-equivalent for every $d \geq 2$ [11] (Table 1.1):

Table 1.1 Six P-equivalent pairs of 3-component links that are not Θ_d-equivalent

$L11n358\{0,1\}$	$L11n418\{0,0\}$
$L11a467\{0,1\}$	$L11a527\{0,0\}$
$L11n325\{1,1\}$	$L11n424\{0,0\}$
$L10n79\{1,1\}$	$L10n95\{1,0\}$
$L11a404\{1,1\}$	$L11a428\{0,1\}$
$L10n76\{1,1\}$	$L11n425\{1,0\}$

Specifically, for these pairs the differences of the polynomials have been computed [11]:

$$\Theta_d(L11n358\{0,1\}) - \Theta_d(L11n418\{0,0\}) = \frac{(E_D-1)(\lambda_D-1)(q-1)^2(q+1)^2\left(q^2-\lambda_D\right)\left(\lambda_D q^2-1\right)}{E_D\lambda_D^4 q^4},$$

$$\Theta_d(L11a467\{0,1\}) - \Theta_d(L11a527\{0,0\}) = \frac{(E_D-1)(\lambda_D-1)(q-1)^2(q+1)^2\left(q^2-\lambda_D\right)\left(\lambda_D q^2-1\right)}{E_D\lambda_D^4 q^4},$$

$$\Theta_d(L11n325\{1,1\}) - \Theta_d(L11n424\{0,0\}) = -\frac{(E_D-1)(\lambda_D-1)(q-1)^2(q+1)^2\left(q^2-\lambda_D\right)\left(\lambda_D q^2-1\right)}{E_D\lambda_D^3 q^4},$$

$$\Theta_d(L10n79\{1,1\}) - \Theta_d(L10n95\{1,0\}) = \frac{(E_D-1)(\lambda_D-1)(q-1)^2(q+1)^2\left(\lambda_D+\lambda_D q^4+\lambda_D q^2-q^2\right)}{E_D\lambda_D^4 q^4},$$

$$\Theta_d(L11a404\{1,1\}) - \Theta_d(L11a428\{0,1\}) = \frac{(E_D-1)(\lambda_D-1)(\lambda_D+1)(q-1)^2(q+1)^2\left(q^4-\lambda_D q^2+1\right)}{E_D q^4},$$

$$\Theta_d(L10n76\{1,1\}) - \Theta_d(L11n425\{1,0\}) = \frac{(E_D-1)(\lambda_D-1)(\lambda_D+1)(q-1)^2(q+1)^2}{E_D\lambda_D^3 q^2}.$$

Note that the factor $(E_D - 1)$ is common to all six pairs. This confirms that the pairs have the same HOMFLYPT polynomial, since for $E_D = 1$ the difference collapses to zero. Further, all the computations can be found on http://www.math.ntua.gr/~sofia/yokonuma. Except for the computational results, there is a diagrammatic proof for the pair of links $L11n358\{0,1\}$ and $L11n418\{0,0\}$ in [11]. Now, we can formulate the following immediate statement:

Theorem 1.12 ([11, Theorem 7.3]) *The invariants Θ_d are not topologically equivalent to the HOMFLYPT polynomial for any $d \geq 2$.*

The proof uses recursive applications of the special skein relation (1.19), in order to construct skein trees, where only disjoint union of knots appear as leaves, for both links. Then using the split link property of Θ_d (1.20), the value of Θ_d is written as a sum of HOMFLYPT polynomials of knots. Finally, by recognizing the topological type of the knot diagrams and by finding the HOMFLYPT polynomial values for these knots, the calculation is completed. Note that the intrinsic difference in computing the invariants Θ_d and P is in the different values of these invariants on disjoint unions of knots. In particular, if K is a knot and U is the unknot, for the invariants Θ_d:

$$\Theta_d(K \sqcup U) = \frac{1-\lambda_D}{(q-q^{-1})\sqrt{\lambda_D}E_D}\Theta_d(K),$$

while for P we have that:

$$P(K \sqcup U) = \frac{1-\lambda_D}{(q-q^{-1})\sqrt{\lambda_D}}P(K).$$

Theorem 1.12 confirms that P-equivalence does not imply Θ_d-equivalence, but it cannot provide us with any indication as to whether or not the invariants Θ_d are strictly stronger than the HOMFLYPT polynomial.

1.5 A 3-Variable Generalization of the HOMFLYPT Polynomial

1.5.1 The Invariant $\Theta(q, \lambda, E)$

The program of Remark 1.5 considers the value E_D as a parameter. Moreover, the six pairs of links of Table 1.1 are distinguished by Θ_d for every $d \geq 2$, as seen of the difference of the values of Θ_d on each pair. The natural question arising is whether the value E_D can be considered as an indeterminate, allowing us to construct a link invariant generalizing both the HOMFLYPT polynomial and the invariants Θ_d. Indeed, in [11] such an invariant has been constructed:

Theorem 1.13 ([11, Theorem 8.1]) *Let q, λ, E be indeterminates. There exists a unique isotopy invariant of classical oriented links $\Theta : \mathscr{L} \to \mathbb{C}[q^{\pm 1}, \lambda^{\pm 1}, E^{\pm 1}]$ defined by the following rules:*

1. *For a disjoint union L of k knots, with $k \geq 1$, it holds that:*

$$\Theta(L) = E^{1-k}\, P(L).$$

2. *On crossings involving different components the following skein relation holds:*

$$\frac{1}{\sqrt{\lambda}}\,\Theta(L_+) - \sqrt{\lambda}\,\Theta(L_-) = (q-q^{-1})\,\Theta(L_0),$$

where L_+, L_-, L_0 is a Conway triple.

The invariant Θ distinguishes the 6 pairs of Table 1.1. Moreover, since Θ generalizes both the invariants Θ_d and the HOMFLYPT polynomial, we have the following:

Theorem 1.14 ([11, Theorem 8.2]) *The invariant $\Theta(q, \lambda, E)$ is stronger than the HOMFLYPT polynomial.*

Remark 1.7 By the above, the computer program of Remark 1.5 computes the invariant Θ.

As we shall see, the well-definedness of Θ can be proved using a variety of techniques, from diagrammatic ones to algebraic or combinatorial ones.

1.5.2 Properties of the Framed and Classical Invariants

The invariants $\Phi_{d,D}$ and Θ (and hence also the invariants Θ_d) satisfy properties analogous to the known ones of the HOMFLYPT polynomial, due to the behaviour of trace $\mathrm{tr}_{d,D}$ under inversion, split links, connected sums and mirror imaging. In detail:

- *Reversing orientation:*

$$\Phi_{d,D}(L) = \Phi_{d,D}(\overleftarrow{L}) \quad \text{and} \quad \Theta(L) = \Theta(\overleftarrow{L}),$$

  where $\overleftarrow{L}$ is the link L with reversed orientation on all components.
- *Split links:*

$$\begin{aligned} \Phi_{d,D}(L \sqcup L') &= \Lambda_D \, \Phi_{d,D}(L)\Phi_{d,D}(L') \\ \text{and} \quad \Theta(L \sqcup L') &= \frac{1-\lambda_D}{(q-q^{-1})\sqrt{\lambda_D}E}\, \Theta(L)\Theta(L'). \end{aligned} \tag{1.20}$$

- *Connected sums:*

$$\Phi_{d,D}(L\#L') = \Phi_{d,D}(L)\Phi_{d,D}(L') \quad \text{and} \quad \Theta(L\#L') = \Theta(L)\Theta(L'),$$

  where D is a subset of $\mathbb{Z}/d\mathbb{Z}$ such that $\mathrm{x}_1^d = 1$ and $\mathrm{x}_k = \mathrm{x}_1^k$ for all $k = 1, \ldots, d-1$.
- *Mirror images:*

$$\Phi_{d,D}(q, \lambda_D)(L^*) = \Phi_{d,D}(q^{-1}, \lambda_D^{-1})(L) \quad \text{and} \quad \Theta(q, \lambda_D)(L^*) = \Theta(q^{-1}, \lambda_D^{-1})(L),$$

  where L^* is the mirror image of L.

Notice that P satisfies the exact same properties as Θ except for split links, where the parameter E (or the value E_D) does not appear:

$$P(L \sqcup L') = \frac{1 - \lambda_{\mathrm{H}}}{(q - q^{-1})\sqrt{\lambda_{\mathrm{H}}}} P(L)P(L').$$

1.5.3 A Closed Formula for Θ

Before we proceed to explain how the well-definedness of the invariant Θ can be proved, we provide a closed formula for Θ, proved by W.B.R. Lickorish. More precisely, the invariant Θ is a complicated mixture of linking numbers and HOMFLYPT polynomials of sublinks. In detail:

Theorem 1.15 ([11, Appendix B by W.B.R. Lickorish]) *Let L be an oriented link with n components. Then*

$$\Theta(L) = \sum_{k=1}^{n} \mu^{k-1} E_k \sum_{\pi} \lambda^{\nu(\pi)} P(\pi L) \tag{1.21}$$

where the second summation is over all partitions π of the components of L into k (unordered) subsets and $P(\pi L)$ denotes the product of the HOMFLYPT polynomials of the k sublinks of L defined by π. Furthermore, $\nu(\pi)$ is the sum of all linking numbers of pairs of components of L that are in distinct sets of π, $E_k = (E^{-1} - 1)(E^{-1} - 2)\cdots(E^{-1} - k + 1)$, with $E_1 = 1$, and $\mu = \frac{\lambda^{-1/2} - \lambda^{1/2}}{q - q^{-1}}$.

The above formula provides us with a topological interpretation of the invariant Θ. Specifically, the invariant Θ is *completely determined by the linking matrix of a link L and the values of P on each sublink of L*. For example, on the pair of links of Theorem 1.12 the invariant Θ detects a pair of 2-component sublinks that are not P-equivalent. Specifically, the link $L11n358\{0, 1\}$ contains a disjoint union of two unknots as a sublink, whereas $L11n418\{0, 0\}$ does not; hence, there is a pair of sublinks with different HOMFLYPT polynomials.

Theorem 1.15 is proved by W.B.R. Lickorish using the special skein relation and combinatorial tools. For more details, the reader can refer to [11, Appendix B]. The above result has also been proved in [42] using representation theory techniques.

Moreover, Theorem 1.15 allows us to investigate further the question of whether Θ_d-equivalence implies $\Theta_{d'}$-equivalence or vice versa. In detail:

Proposition 1.7 ([11, Proposition 8.9]) *Let L and L′ be two n-component links that are not Θ-equivalent. Then they are not Θ_d-equivalent for $d \geq n$.*

The proof uses exclusively Theorem 1.15. In detail, for $d \geq n$ we do not lose any topological information, since all the coefficients E_k are not zero for all $k \in \{1, \ldots, n\}$. However, for $d < n$ the coefficients E_k for $k \geq d$ are zero and hence a pair of HOMFLYPT non-equivalent sublinks may not be detected. Hence, for a pair of

P-equivalent links L and L', either all the invariants Θ_d coincide and do not distinguish the links or there exists a $d \in \mathbb{N}$ with $2 \leq d < n$ such that $\Theta_d(L) \neq \Theta_d(L')$.

In [11] it is shown that Θ can indeed be defined by (1.21):

Theorem 1.16 ([11, Theorem 8.11]) *Suppose that the invariant Θ is defined by Eq. 1.21. Then, this definition is equivalent to the definition of Theorem 1.13.*

Note that the definition via (1.21) provides a quick way to compute the invariant Θ for a link L. One only needs to identify the sublinks of a link and compute Θ using the known values of P on the sublinks and the linking matrix of L.

1.5.4 The Well-Definedness of Θ

Chronologically, our first proof of the well-definedness of Θ was algebraic using the class of *tied links* [2]. However, there exist other methods for proving that Θ is well-defined. Summarizing, there are the following four equivalent methods:

1. *combinatorially*, via the closed formula of W.B.R. Lickorish (Theorem 1.15);
2. *skein-theoretically* based on Theorem 1.13 (a direct proof can be found in [36]);
3. *algebraically*, via the *algebra of braids and ties* $\mathcal{E}_n(q)$ [2] generated by $g_1, \ldots, g_{n-1}$ with the algebra of braids and ties.
4. *algebraically*, via the isomorphism of the subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ of $\mathrm{Y}_{d,n}(q)$ generated by $g_1, \ldots, g_{n-1}$ with the algebra of braids and ties $\mathcal{E}_n(q)$ for $d \geq n$ [17].

All the above methods do not involve complicated constructions such as the E–system, even though Θ contains the invariants Θ_d where the E–system is needed. Moreover, the restriction $d \geq n$ of Theorem 1.17 does not obstruct the well-definedness of Θ: for a link L written as a braid in n strands one can always choose a suitable $d \geq n$.

Now we will provide some more insight on the last two algebraic methods for proving the well-definedness of Θ using *tied links*. Tied links were introduced and studied by F. Aicardi and J. Juyumaya in [2, 3]. A *tied link* is defined as a classical link L endowed with a set of ties, containing unordered pairs of points belonging to the components of L [3, Definition 1]. Diagrammatically, one can visualize a tie as a spring connecting two (not necessarily different) components of L. The endpoints of a tie are allowed to slide along the components that they are attached to. If two ties join the same two components, one of them can be removed, and any tie on a single component can be also removed. A tie that cannot be removed is called *essential*.

Tied link invariants can be constructed using either diagrammatic or algebraic methods. In [3] such an invariant is defined using both methods. Specifically, a tied link invariant is constructed with the use of a Markov trace on the *algebra of braids and ties* $\mathcal{E}_n(q)$. The algebra of braids and ties $\mathcal{E}_n(q)$ is defined as the algebra generated by $g_1, \ldots, g_{n-1}, e_1, \ldots, e_{n-1}$ satisfying the following relations (cf. [2, Definition 1]):

$$\begin{array}{rcll} g_i g_j g_i &=& g_j g_i g_j & \text{for } |i-j|=1 \\ g_i g_j &=& g_j g_i & \text{for } |i-j|>1 \\ e_i e_j &=& e_j e_i & \\ e_i^2 &=& e_i & \\ e_i g_i &=& g_i e_i & \\ e_i g_j &=& g_j e_i & \text{for } |i-j|>1 \\ e_i e_j g_i &=& g_i e_i e_j & \text{for } |i-j|=1 \\ e_i g_j g_i &=& g_j g_i e_j & \text{for } |i-j|=1 \\ g_i^2 &=& 1+(q-q^{-1})\, e_i g_i\,. & \end{array}$$

Diagrammatically, the generators g_i correspond to the classical braiding generators and the elements e_i correspond to ties connecting the i-th and the $(i+1)$-th strands. Note that the cancellation properties of ties mentioned above are reflected in the fact that the elements e_i are idempotents. In [2, 3], a different presentation $\mathcal{E}_n(u)$ is used, where the quadratic relation is changed.

The similarity between the algebra $\mathcal{E}_n(q)$ and $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ is obvious: both can be generated by the same generators and the generators of $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ satisfy the exact same relations. However, it is not evident that these relations are enough for the subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ and whether the two algebras are isomorphic. In [17], the authors have provided a representation-theoretic proof for the isomorphism between the two algebras:

Theorem 1.17 ([17, Theorem 8]) *Suppose that $d \geq n$. Then the algebra $\mathcal{E}_n(q)$ is isomorphic to the subalgebra $\mathrm{Y}_{d,n}(q)^{(\mathrm{br})}$ of $\mathrm{Y}_{d,n}(q)$.*

Note also the similarity of the condition $d \geq n$ of Theorem 1.17 and Proposition 1.7.

Now, a Markov trace $\rho : \bigcup_{n\geq 0} \mathcal{E}_n(q) \to \mathbb{C}[q^{\pm 1}, z^{\pm 1}, E^{\pm 1}]$ can be defined satisfying the following rules (cf. [2, Theorem 3]):

$$\begin{array}{llcll} \text{(i)} & \rho(ab) &=& \rho(ba) & a, b \in \mathcal{E}_n(q) \\ \text{(ii)} & \rho(1) &=& 1 & 1 \in \mathcal{E}_n(q) \\ \text{(iii)} & \rho(a g_n) &=& z\,\rho(a) & a \in \mathcal{E}_n(q) \quad (\textit{Markov property}) \\ \text{(iv)} & \rho(a e_n) &=& E\,\rho(a) & a \in \mathcal{E}_n(q) \\ \text{(v)} & \rho(a e_n g_n) &=& z\,\rho(a) & a \in \mathcal{E}_n(q)\,. \end{array}$$

Notice the resemblance of the above rules with the five rules of $\mathrm{tr}_{d,D}$ in Theorem 1.2. Further, the trace ρ satisfies similar properties to those of $\mathrm{tr}_{d,D}$ [11]. In [3] the *tied braid monoid* TB_n is defined; it is generated by the braiding generators $\sigma_1, \ldots, \sigma_{n-1}$ and the generating ties $\eta_1, \ldots, \eta_{n-1}$, where η_i connects the i-th and the $i+1$-th strands of a tied braid. Denote by $\bar{\pi} : \mathbb{C}TB_n \to \mathcal{E}_n(q)$ the natural surjection defined by $\sigma_i \mapsto g_i$ and $\eta_i \mapsto e_i$. Then following the procedure of [3] an invariant $\overline{\Theta}$ can be defined as:

Theorem 1.18 ([11, Theorem 8.4]) *For any tied braid $\alpha \in TB_n$, we define*

$$\overline{\Theta}(\widehat{\alpha}) := \left(\frac{1}{z\sqrt{\lambda}}\right)^{n-1} \sqrt{\lambda}^{\epsilon(\alpha)} (\rho \circ \bar{\pi})(\alpha)\,,$$

where $\lambda = \frac{z-(q-q^{-1})E}{z}$ and $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators σ_i in the word α. Then the map $\overline{\Theta}$ is a 3-variable isotopy invariant of oriented tied links.

Note that, for $E = 1$, $\overline{\Theta}$ specializes to the HOMFLYPT polynomial when restricted to classical links.

In [3], a similar invariant has been defined using the presentation $\mathscr{E}_n(q)$. This invariant is re-defined diagrammatically via a skein relation that applies to any crossing in the link diagram. It is proved to be well-defined via the standard Lickorish–Millett method [39]. The same can be done for $\overline{\Theta}$. Specifically, $\overline{\Theta}$ satisfies the following defining skein relation:

$$\frac{1}{\sqrt{\lambda}}\overline{\Theta}(L_+) - \sqrt{\lambda}\,\overline{\Theta}(L_-) = (q - q^{-1})\,\overline{\Theta}(L_{0,\sim}), \tag{1.22}$$

For the well-definedness of Θ one only needs to show that Θ coincides with $\overline{\Theta}$ on classical links and that $\overline{\Theta}$ satisfied the defining rules of Θ as in Theorem 1.13. Indeed, the skein relation (1.22) reduces to the special skein relation of the second rule of Theorem 1.13 and it also satisfies the property of Θ for disjoint unions of knots [11].

1.6 Other Invariants from the Yokonuma–Hecke Algebras

A well-known property of the HOMFLYPT polynomial P, as defined via the Iwahori–Hecke algebra, is that a transformation $g_i \mapsto cg_i$, where $c \in \mathbb{C}$, leads to a change of variables for P. In the case of the Yokonuma–Hecke algebras this is not always true, because we have more possibilities for applying a linear transformation on the braiding generators. Indeed, a transformation of the form $g_i \mapsto cg_i$ would work similarly for the framed link invariants $\Phi_{d,D}$ as for P. However, applying a transformation of the form $g_i \mapsto cg_i + c'e_i g_i$, with $c' \in \mathbb{C}$, does not lead to a change of variables for $\Phi_{d,D}$ but rather to new invariants, potentially topologically non-equivalent to the invariants $\Phi_{d,D}$. Indeed, as we shall see below, this transformation gives rise to an algebra isomorphic to $\mathrm{Y}_{d,n}(q)$ but with a different quadratic relation.

1.6.1 The Old Quadratic Relation

We begin by summarizing the construction of framed and classical links invariants using the other presentation of the Yokonuma–Hecke algebra that we shall denote by $\mathrm{Y}_{d,n}(u)$ and that was used in the papers [12, 29–34]. This old presentation was also used originally for defining the invariants for framed, classical, singular and transverse links described in [10, 11], recall previous sections, but then it was adapted to the presentation $\mathrm{Y}_{d,n}(q)$ that has simpler quadratic relations.

The algebra $\mathrm{Y}_{d,n}(u)$ is generated by the elements $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}$ and $t_1, \ldots, t_n$, satisfying relations (1.1) (with $\widetilde{g}_i$ corresponding to σ_i), (1.2) and the quadratic relations:

$$\widetilde{g}_i^{\,2} = 1 + (u-1)\, e_i + (u-1)\, e_i \widetilde{g}_i \qquad (1 \le i \le n-1). \tag{1.23}$$

The presentation $\mathrm{Y}_{d,n}(q)$ used in this paper and in [10, 11, 35] was obtained in [13] by taking $u := q^2$ and $g_i := \widetilde{g}_i + (q^{-1} - 1)\, e_i \widetilde{g}_i$ (or, equivalently, $\widetilde{g}_i := g_i + (q-1)\, e_i g_i$).

In [26] a quadratic relation with two parameters is considered, which specializes to both the old and the new quadratic relation for the Yokonuma–Hecke algebra.

1.6.2 The Traces $\widetilde{\mathrm{tr}}_d$ and $\widetilde{\mathrm{tr}}_{d,D}$

Theorem 1.1 has been originally proved by J. Juyumaya using the presentation $\mathrm{Y}_{d,n}(u)$ for the Yokonuma-Hecke algebra [29]. He proved that there exists a unique linear Markov trace $\widetilde{\mathrm{tr}}_d$ on $\bigcup_{n\ge 0} \mathrm{Y}_{d,n}(u)$ defined inductively by the four rules of Theorem 1.1, where rule (3) is replaced by the rule:

$$(\widetilde{3})\ \widetilde{\mathrm{tr}}_d(a\widetilde{g}_n) = \widetilde{z}\,\widetilde{\mathrm{tr}}_d(a) \qquad a \in \mathrm{Y}_{d,n}(u) \quad (Markov\ \ property)$$

for some indeterminate $\widetilde{z}$ over $\mathbb{C}$. Since his proof uses the inductive basis of $\mathrm{Y}_{d,n}(u)$, it also works with the new quadratic relations (1.3), thus yielding Theorem 1.1. The E–condition and the E–system as presented in Sect. 1.2 were first defined and used in [31] in order to re-scale $\widetilde{\mathrm{tr}}_d$, and remain the same for $\mathrm{Y}_{d,n}(q)$. In [12, Definition 3] the specialized trace $\widetilde{\mathrm{tr}}_{d,D}$ with parameter $\widetilde{z}$ is defined on $\bigcup_{n\ge 0} \mathrm{Y}_{d,n}(u)$, satisfying the analogous rules: (1), (2), $(\widetilde{3})$ and (4′).

1.6.3 Related Invariants

Now, using the natural $\mathbb{C}$-algebra epimorphism from $\mathbb{C}\mathscr{F}_n$ onto $\mathrm{Y}_{d,n}(u)$ given by $\sigma_i \mapsto \widetilde{g}_i$ and $t_j^k \mapsto t_j^{k (\mathrm{mod}\, d)}$ and abusing notation, one can define the trace $\widetilde{\mathrm{tr}}_d$ on the elements of $\mathbb{C}\mathscr{F}_n$, and thus, in particular, on the elements of $\mathscr{F}_n$. By normalizing and re-scaling the specialized trace $\widetilde{\mathrm{tr}}_{d,D}$, invariants $\Gamma_{d,D}(u, \widetilde{z})$ for oriented framed links are defined in [31, Theorem 8].

As it turned out [31, Proposition 7], the invariants $\Gamma_{d,D}$ satisfy the following skein relation, involving the braiding and the framing generators:

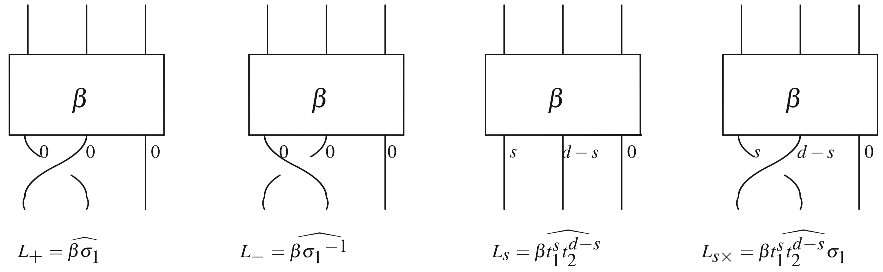

Fig. 1.4 The framed links of the skein relation in open braid form

$$\frac{1}{\sqrt{\widetilde{\lambda}_D}}\Gamma_{d,D}(L_+) - \sqrt{\widetilde{\lambda}_D}\Gamma_{d,D}(L_-) = \frac{1-u^{-1}}{d}\sum_{s=0}^{d-1}\Gamma_{d,D}(L_s) + \frac{1-u^{-1}}{d\sqrt{\widetilde{\lambda}_D}}\sum_{s=0}^{d-1}\Gamma_{d,D}(L_{s\times}) \tag{1.24}$$

where

$$\widetilde{\lambda}_D = \frac{\widetilde{z} - (u-1)E_D}{u\widetilde{z}},$$

and the links L_+, L_-, L_s and $L_{s\times}$ are illustrated in Fig. 1.4. Comparing this skein relation to the corresponding skein relation of $\Phi_{d,D}$ derived from $Y_{d,n}(q)$ (1.15) we can see that terms involving $L_{s\times}$ are missing in (1.15).

Similarly to $\Phi_{d,D}(q,z)$, the invariants $\Gamma_{d,D}(u,\widetilde{z})$ become invariants of oriented classical links, denoted by $\Delta_{d,D}(u,\widetilde{z})$, when the traces $\widetilde{\mathrm{tr}}_d$, $\widetilde{\mathrm{tr}}_{d,D}$ are applied on the classical braid groups B_n and these are the ones studied in [12, 32]. Theorem 1.2 and all its consequences hold also for the specialized trace $\widetilde{\mathrm{tr}}_{d,D}$. In particular, the values of the classical link invariants $\Delta_{d,D}$ depend only on the cardinality $|D|$ of D, so they can be parametrized by the natural numbers, setting $\Delta_d := \Delta_{d,\mathbb{Z}/d\mathbb{Z}}$ for all $d \in \mathbb{Z}_{>0}$. For $d = 1$ we have that $\Delta_1 = \Delta_{1,\{0\}}(u,\widetilde{z}) = \Theta_{1,\{0\}}(q,z)$, the HOMFLYPT polynomial, for $u = q^2$ and $\widetilde{z} = qz$. This can be easily seen by comparing the skein relations for $\Gamma_{1,\{0\}}$ and $\Phi_{1,\{0\}}$. So, both families of invariants Δ_d and Θ_d include the HOMFLYPT polynomial as a special case.

However, the skein relation (1.24) of the invariants $\Gamma_{d,D}$ does not yield a special skein relation for the invariants Δ_d similar to (1.19) of Θ_d. Indeed, if the crossing of L_+ involves two different components, then so does the crossing of $L_{s\times}$ and so the framings in $L_{s\times}$ cannot be collected together. Consequently, Theorem 1.11 and all other results for Θ_d that depend on the special skein relation are not valid for the invariants Δ_d. Clearly, the diagrammatic analysis made for the invariants Θ_d on pairs of P-equivalent links cannot be implemented for the invariants Δ_d. Nevertheless, there are computational indications that the invariants Δ_d are not topologically equivalent to P. Concerning now the properties studied in Sect. 1.2.4, Δ_d has the same behaviour as Θ_d on links with reversed orientation, on split links, on connected sums and on mirror images. However, behaviour of Δ_d under mutation cannot be checked using the methods of Proposition 1.6. Furthermore, there is no reason that the invariants $\Delta_d(u, \widetilde{z})$ and $\Theta_d(q, z)$ are topologically equivalent. In fact, there is computational evidence that they are not [1].

Moreover, using Theorem 1.17, E_D could be taken to be an indeterminate E, and $\widetilde{\mathrm{tr}}_d$ would be well-defined due to the isomorphism of $\mathrm{Y}_{d,n}(u)^{(\mathrm{br})}$ with the algebra of braids and ties $\mathscr{E}_n(u)$ for $d \geq n$; then $\widetilde{\mathrm{tr}}_d$ would coincide with the Markov trace on $\mathscr{E}_n(u)$ defined in [2]. More precisely, in [2, 3], F. Aicardi and J. Juyumaya worked on the algebra of braids and ties $\mathscr{E}_n(u)$, generated by elements $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}, e_1, \ldots, e_{n-1}$, with the braiding generators satisfying the old quadratic relations (1.23). Then they defined a Markov trace $\widetilde{\rho}$ on $\cup_{n\geq 0}\mathscr{E}_n(u)$ [2, Theorem 3] that gave rise to a 3-variable isotopy invariant of tied links, denoted by $\overline{\Delta}$. Our construction of $\overline{\Theta}$ (recall Sect. 1.5.4) is completely analogous to the construction of $\overline{\Delta}$. For $E = 1$, $\overline{\Delta}$ specializes to the HOMFLYPT polynomial when restricted to classical links. In [3], $\overline{\Delta}$ is re-defined diagrammatically via a skein relation that applies to any crossing in the link diagram. The invariant $\overline{\Delta}$ has not been identified topologically. One obstruction to this is the fact that the old quadratic relation is used for the algebra $\mathscr{E}_n(u)$. So, it was impossible to derive a special skein relation that only involves classical links (with no ties). Despite the fact that the algebras $\mathscr{E}_n(u)$ and $\mathscr{E}_n(q)$ are isomorphic, the invariants $\overline{\Delta}$ and $\overline{\Theta}$ are also not necessarily topologically equivalent [1] as we have already observed about the invariants Δ_d and Θ_d.

Restricting now $\overline{\Delta}$ to classical links, similarly to the proof of Theorem 1.13 [11], one can prove that the invariant $\overline{\Delta}$ satisfies the first rule of Theorem 1.13. Given also the isomorphism between the subalgebra $\mathrm{Y}_{d,n}^{(\mathrm{br})}(u)$ of $\mathrm{Y}_{d,n}(u)$ and the algebra of braids and ties $\mathscr{E}_n(u)$ for $d \geq n$, in this case the invariant $\overline{\Delta}$ contains the invariants Δ_d. Consequently, the invariants Δ_d are topologically equivalent to the HOMFLYPT polynomial on knots and on disjoint unions of knots.

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